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# Symmetry considerations for materials of second grade

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ABSTRACT

The symmetry group associated with a material point of second grade is characterized, thereby eludicating the interplay between first- and second-order strain measures in determining its response to deformation.

## Introduction

It is the purpose of this note[1] to record some simple observations on material symmetry for a material point of second grade; that is, a material point X whose response to a deformation $\boldsymbol{\lambda}$ from any configuration $\boldsymbol{\kappa}$ of the body to which it belongs depends both upon the first and second derivatives of $\boldsymbol{\lambda}$ evaluated at $\boldsymbol{\kappa}(X)$. Although symmetry considerations for simple materials were made precise some time ago by Noll [2], only recently has the same been accomplished for higher-grade materials,[2] by Morgan [4, Section 4]. However, in the general case the simplicity associated with second-grade materials is not transparent, nor, in respect of symmetry considerations, are the fundamental differences of such materials from simple bodies self-evident. Emphasis upon second-grade materials is merited by virtue of the work of Toupin [5] and of Mindlin and Tiersten [6] on such bodies, the simplest which admit of polar phenomena.

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[1] We consistently use the terminology of Truesdell and Noll [1] with minor modifications: if $\boldsymbol{\kappa}$ and $\boldsymbol{\mu}$ are any two configurations of a body $\mathscr{B}$ we require that the deformation gradient $\nabla(\boldsymbol{\mu} \circ \boldsymbol{\kappa}^{-1})$ takes values in Invlin $(\mathscr{V})$ with positive determinant. The requirements of frame-indifference are also modified: observers are assumed to agree upon orientation (that is, agree what constitutes "right-handedness") which requires frame changes to involve only *proper* orthogonal tensors.

[2] Cheverton and Beatty [3] also considered this problem, but seem to have confined themselves to (locally) homogeneous configurations which mask the subtlety of response possible in such materials, as will be demonstrated.

**Preliminaries**

Let $\mathcal{B}$ be a (three-dimensional) continuous body of class $C^2$ for which,[3] during any motion $\chi$, the Cauchy stress tensor $\boldsymbol{T}$ and the couple-stress tensor $\boldsymbol{M}$ depend upon $\nabla\chi_\kappa$ and $\nabla\nabla\chi_\kappa$, where $\chi_\kappa$ denotes the motion relative to a (reference) configuration $\boldsymbol{\kappa}$. More precisely, suppose $\boldsymbol{T}$ and $\boldsymbol{M}$ are given at time $t$ by

$$\boldsymbol{T}(\boldsymbol{x}, t) = \boldsymbol{T}_\kappa(\nabla\chi_\kappa(\hat{\boldsymbol{x}}, t), \nabla\nabla\chi_\kappa(\hat{\boldsymbol{x}}, t), \hat{\boldsymbol{x}})$$

and (1)

$$\boldsymbol{M}(\boldsymbol{x}, t) = \boldsymbol{M}_\kappa(\nabla\chi_\kappa(\hat{\boldsymbol{x}}, t), \nabla\nabla\chi_\kappa(\hat{\boldsymbol{x}}, t), \hat{\boldsymbol{x}}),$$

where

$$\boldsymbol{x} = \chi_\kappa(\hat{\boldsymbol{x}}, t), \qquad \hat{\boldsymbol{x}} = \boldsymbol{\kappa}(X),$$

these relations holding for every $X \in \mathcal{B}$. If $\boldsymbol{n}$ denotes a unit normal field to a surface lying in $\chi(\mathcal{B}, t)$ then $\boldsymbol{Tn}$ is the usual traction field and $\boldsymbol{Mn}$ represents the couple per unit area transmitted across the surface. We regard $\boldsymbol{M}$ as a tensor field of rank three with $\boldsymbol{Mn}$ taking skew values. Equivalent to relations (1), but more convenient in respect of considerations of frame-indifference, are the following:

$$\boldsymbol{T}(\boldsymbol{x}, t) = \hat{\boldsymbol{T}}_\kappa(\boldsymbol{F}_\kappa(\hat{\boldsymbol{x}}, t), \boldsymbol{G}_\kappa(\hat{\boldsymbol{x}}, t), \hat{\boldsymbol{x}})$$

and (2)

$$\boldsymbol{M}(\boldsymbol{x}, t) = \hat{\boldsymbol{M}}_\kappa(\boldsymbol{F}_\kappa(\hat{\boldsymbol{x}}, t), \boldsymbol{G}_\kappa(\hat{\boldsymbol{x}}, t), \hat{\boldsymbol{x}}),$$

where

$$\boldsymbol{F}_\kappa = \nabla\chi_\kappa \quad \text{and} \quad \boldsymbol{G}_\kappa = \boldsymbol{F}_\kappa^T \nabla\nabla\chi_\kappa. \tag{3}$$

The tensor field (of rank three) $\boldsymbol{G}_\kappa$ is frame-independent, since a change of frame (cf. [1], Section 17) in which

$$\boldsymbol{x} \to \boldsymbol{x}^* = \boldsymbol{c} + \boldsymbol{Q}(\boldsymbol{x} - \boldsymbol{x}_0)$$

results in

$$\boldsymbol{F}_\kappa \to \boldsymbol{F}_\kappa^* = \boldsymbol{Q}\boldsymbol{F}_\kappa \quad \text{and} \quad \nabla\nabla\boldsymbol{x}_\kappa \to (\nabla\nabla\chi_\kappa)^* = \boldsymbol{Q}\nabla\nabla\chi_\kappa,$$

so that

$$\boldsymbol{G}_\kappa \to \boldsymbol{G}_\kappa^* = \boldsymbol{G}_\kappa$$

by virtue of the orthogonal nature of $\boldsymbol{Q}$. We remark (cf. Duvaut, [7]) that the dependence of the response functions $\hat{\boldsymbol{T}}_\kappa$ and $\hat{\boldsymbol{M}}_\kappa$ upon $\boldsymbol{G}_\kappa$ is equivalent to one upon $\nabla\boldsymbol{C}$, where $\boldsymbol{C}$ denotes the (right) Cauchy–Green tensor $\boldsymbol{F}_\kappa^T\boldsymbol{F}_\kappa$.

If $\boldsymbol{A}$ is a tensor of rank three we define $\boldsymbol{A}^T$ to be that tensor which satisfies, for all

[3] Of course, for a complete *thermo-elastic* theory the free energy, entropy, and heat flux vector would need to be added to $\boldsymbol{T}$ and $\boldsymbol{M}$, and constitutive dependence upon temperature and temperature gradient included. The generalization of our discussion to such a theory is clearly evident, this also being the case for a mechanical theory in which the dependence of $\boldsymbol{T}$ and $\boldsymbol{M}$, is upon *histories* of $\nabla\chi_\kappa$ and $\nabla\nabla\chi_\kappa$.

vectors[4] $\boldsymbol{u}, \boldsymbol{v} \in \mathcal{V}$,

$$(\boldsymbol{A}^T\boldsymbol{u})\boldsymbol{v} = (\boldsymbol{A}\boldsymbol{v})\boldsymbol{u}. \tag{4}$$

The space of third-rank tensors $\boldsymbol{A}$ for which $\boldsymbol{A} = \boldsymbol{A}^T$ will be denoted by $\mathrm{Sym}_3^T(\mathcal{V})$ and the set of all invertible second-rank tensors with positive determinant by $\mathrm{Invlin}^+(\mathcal{V})$. Of course, $\boldsymbol{F}_\kappa$ takes values in $\mathrm{Invlin}^+(\mathcal{V})$ and $\boldsymbol{G}_\kappa$ values in $\mathrm{Sym}_3^T(\mathcal{V})$, the latter because of the symmetry of the second gradient $\nabla\nabla\boldsymbol{\chi}_\kappa$.

## Material symmetry considerations

Let $\boldsymbol{\mu}$ be any configuration of $\mathcal{B}$ such that $\boldsymbol{\mu} \neq \boldsymbol{\kappa}$ and write

$$\boldsymbol{\lambda} = \boldsymbol{\mu} \circ \boldsymbol{\kappa}^{-1}, \qquad \bar{\boldsymbol{x}} = \boldsymbol{\lambda}(\hat{\boldsymbol{x}})$$

so that $\boldsymbol{\lambda}$ is of class $C^2$ and $\bar{\boldsymbol{x}} = \boldsymbol{\mu}(X)$. Denoting the motion $\boldsymbol{\chi}$ relative to $\boldsymbol{\mu}$ by $\boldsymbol{\chi}_\mu$ it follows that

$$\boldsymbol{\chi}_\kappa(\hat{\boldsymbol{x}}, t) = \boldsymbol{\chi}(X, t) = \boldsymbol{\chi}_\mu(\bar{\boldsymbol{x}}, t) = \boldsymbol{\chi}_\mu(\boldsymbol{\lambda}(\hat{\boldsymbol{x}}), t).$$

Suppressing time-dependence we thus have

$$\boldsymbol{\chi}_\kappa = \boldsymbol{\chi}_\mu \circ \boldsymbol{\lambda},$$

whereupon differentiation yields

$$\nabla\boldsymbol{\chi}_\kappa(\hat{\boldsymbol{x}}) = \nabla\boldsymbol{\chi}_\mu(\bar{\boldsymbol{x}})\nabla\boldsymbol{\lambda}(\hat{\boldsymbol{x}}) \tag{5}$$

and

$$\nabla\nabla\boldsymbol{\chi}_\kappa(\hat{\boldsymbol{x}})\boldsymbol{v} = (\nabla\nabla\boldsymbol{\chi}_\mu(\bar{\boldsymbol{x}})\nabla\boldsymbol{\lambda}(\hat{\boldsymbol{x}})\boldsymbol{v})\nabla\boldsymbol{\lambda}(\hat{\boldsymbol{x}}) + \nabla\boldsymbol{\chi}_\mu(\bar{\boldsymbol{x}})\,\nabla\nabla\boldsymbol{\lambda}(\hat{\boldsymbol{x}})\boldsymbol{v}, \tag{6}$$

(6) holding $\forall \boldsymbol{v} \in \mathcal{V}$. Writing $\boldsymbol{F}_\mu = \nabla\boldsymbol{\chi}_\mu$, $\boldsymbol{G}_\mu = \boldsymbol{F}_\mu^T\nabla\nabla\boldsymbol{\chi}_\mu$, and making use of (4), equations (5) and (6) may be written in the equivalent forms

$$\boldsymbol{F}_\kappa(\hat{\boldsymbol{x}}) = \boldsymbol{F}_\mu(\bar{\boldsymbol{x}})\boldsymbol{H}(\hat{\boldsymbol{x}}) \tag{7}$$

and

$$\boldsymbol{G}_\kappa(\hat{\boldsymbol{x}}) = \boldsymbol{H}(\hat{\boldsymbol{x}})^T(\boldsymbol{G}_\mu(\bar{\boldsymbol{x}})^T\boldsymbol{H}(\hat{\boldsymbol{x}}))^T\boldsymbol{H}(\hat{\boldsymbol{x}}) + \boldsymbol{H}(\hat{\boldsymbol{x}})^T\boldsymbol{F}_\mu(\bar{\boldsymbol{x}})^T\boldsymbol{F}_\mu(\bar{\boldsymbol{x}})\boldsymbol{K}(\hat{\boldsymbol{x}}), \tag{8}$$

where

$$\boldsymbol{H} = \nabla\boldsymbol{\lambda} \quad \text{and} \quad \boldsymbol{K} = \nabla\nabla\boldsymbol{\lambda}.$$

If $\hat{\boldsymbol{f}}_\kappa$ and $\hat{\boldsymbol{f}}_\mu$ denote *either* pair of response functions appropriate to configurations $\boldsymbol{\kappa}$ and $\boldsymbol{\mu}$ respectively (so that if $\hat{\boldsymbol{f}}_\kappa = \hat{\boldsymbol{T}}_\kappa$ then $\hat{\boldsymbol{f}}_\mu = \hat{\boldsymbol{T}}_\mu$, while $\hat{\boldsymbol{f}}_\kappa = \hat{\boldsymbol{M}}_\kappa$ implies $\hat{\boldsymbol{f}}_\mu = \hat{\boldsymbol{M}}_\mu$) then (cf. (2))

$$\hat{\boldsymbol{f}}_\kappa(\boldsymbol{F}_\kappa(\hat{\boldsymbol{x}}), \boldsymbol{G}_\kappa(\hat{\boldsymbol{x}}), \hat{\boldsymbol{x}}) = \hat{\boldsymbol{f}}_\mu(\boldsymbol{F}_\mu(\bar{\boldsymbol{x}}), \boldsymbol{G}_\mu(\bar{\boldsymbol{x}}), \bar{\boldsymbol{x}}). \tag{9}$$

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[4] $\mathcal{V}$ denotes the (three-dimensional) space of all vectors.

Writing $\boldsymbol{F}_{\mu}$ as $\boldsymbol{F}$, $\boldsymbol{G}_{\mu}$ as $\boldsymbol{G}$, and suppressing arguments in an obvious manner, (9) yields, on use of (7) and (8),

$$\hat{f}_{\kappa}(\boldsymbol{FH}, \boldsymbol{H}^T(\boldsymbol{G}^T\boldsymbol{H})^T\boldsymbol{H}+\boldsymbol{H}^T\boldsymbol{F}^T\boldsymbol{FK})=\hat{f}_{\mu}(\boldsymbol{F}, \boldsymbol{G}). \tag{10}$$

Furthermore, for a fixed material point X (and consequently fixed values of $\boldsymbol{H}$ and $\boldsymbol{K}$), (10) must hold for all $\boldsymbol{F}\in \text{Invlin}^+(\mathcal{V})$ and all $\boldsymbol{G}\in \text{Sym}_3^T(\mathcal{V})$, since it must hold for all motions. In the event that the material response at X in the configuration $\boldsymbol{\kappa}$ is indistinguishable from that of X in $\boldsymbol{\mu}$ then, clearly,

$$\hat{f}_{\kappa}(\boldsymbol{F}, \boldsymbol{G})=\hat{f}_{\mu}(\boldsymbol{F}, \boldsymbol{G}) \tag{11}$$

$\forall \boldsymbol{F}\in \text{Invlin}^+(\mathcal{V})$, $\forall \boldsymbol{G}\in \text{Sym}_3^T(\mathcal{V})$. It follows from (10) that in such a case

$$\hat{f}_{\kappa}(\boldsymbol{F}, \boldsymbol{G})=\hat{f}_{\kappa}(\boldsymbol{FH}, \boldsymbol{H}^T(\boldsymbol{G}^T\boldsymbol{H})^T\boldsymbol{H}+\boldsymbol{H}^T\boldsymbol{F}^T\boldsymbol{FK}) \tag{12}$$

for all $\boldsymbol{F}$, $\boldsymbol{G}$ as in (11). This motivates the definition of $\text{S}_{\kappa}(\text{X})$, the *symmetry set* of X in configuration $\boldsymbol{\kappa}$, as follows:

$\text{S}_{\kappa}(\text{X})=\{(\boldsymbol{H}, \boldsymbol{K}): \boldsymbol{H}\in \text{Invlin}^+(\mathcal{V}), \boldsymbol{K}\in \text{Sym}_3^T(\mathcal{V})$ with (12) holding for both $\boldsymbol{T}$ and $\boldsymbol{M}\}$.

Clearly, $(\mathbf{1}, \boldsymbol{O})\in \text{S}_{\kappa}(\text{X})$. Further, it is a simple matter to show that if $(\boldsymbol{H}_1, \boldsymbol{K}_1)$ and $(\boldsymbol{H}_2, \boldsymbol{K}_2)\in \text{S}_{\kappa}(\text{X})$ then so does $(\boldsymbol{H}_1\boldsymbol{H}_2, (\boldsymbol{K}_1^T\boldsymbol{H}_2)^T\boldsymbol{H}_2+\boldsymbol{H}_1\boldsymbol{K}_2)$. Defining the operation $(\circ)$ on $\text{S}_{\kappa}(\text{X})$ by

$$(\boldsymbol{H}_1, \boldsymbol{K}_1)\circ(\boldsymbol{H}_2, \boldsymbol{K}_2)=(\boldsymbol{H}_1\boldsymbol{H}_2, (\boldsymbol{K}_1^T\boldsymbol{H}_2)^T\boldsymbol{H}_2+\boldsymbol{H}_1\boldsymbol{K}_2) \tag{13}$$

it is easily shown that $(\circ)$ is associative, $(\mathbf{1}, \boldsymbol{O})$ is an identity element for $\text{S}_{\kappa}(\text{X})$ with each $(\boldsymbol{H}, \boldsymbol{K})$ having an inverse, namely $(\boldsymbol{H}_1^{-1}, -\boldsymbol{H}^{-1}(\boldsymbol{K}^T\boldsymbol{H}^{-1})^T\boldsymbol{H}^{-1})$. Thus we have

PROPOSITION 1. $\text{S}_{\kappa}(\text{X})$ *is a group under the operation defined by* (13).

We accordingly re-label $\text{S}_{\kappa}(\text{X})$ as the *symmetry group* of X in $\boldsymbol{\kappa}$. That subgroup of $\text{S}_{\kappa}(\text{X})$ consisting of elements of the form $(\boldsymbol{H}, \boldsymbol{O})$ might be termed[5] the *homogeneous symmetry group* of X in $\boldsymbol{\kappa}$, $\mathcal{G}_{\kappa}(\text{X})$ say, representing as it does those homogeneous deformations[6] of the whole body which do not alter the material response at X. Of course, $\text{S}_{\kappa}(\text{X})$ is related in a specific manner to $\text{S}_{\mu}(\text{X})$. Indeed, we have

PROPOSITION 2. *If* $(\boldsymbol{H}, \boldsymbol{K})\in \text{S}_{\kappa}(\text{X})$ *then, if* $\boldsymbol{H}_0, \boldsymbol{K}_0$ *denote, respectively,* $\nabla\boldsymbol{\lambda}$ *and* $\nabla\nabla\boldsymbol{\lambda}$ *evaluated at* $\boldsymbol{\kappa}(\text{X})$,

$$(\boldsymbol{H}_0\boldsymbol{H}\boldsymbol{H}_0^{-1}, \boldsymbol{H}_0[\{\boldsymbol{K}-(\boldsymbol{K}_0^T\boldsymbol{H})^T\boldsymbol{H}-\boldsymbol{H}\boldsymbol{H}_0^{-1}\boldsymbol{K}_0\}^T\boldsymbol{H}_0^{-1}]^T\boldsymbol{H}_0^{-1})\in \text{S}_{\mu}(\text{X}). \tag{14}$$

[5] It is this group which Cheverton and Beatty introduced in [3] and which they termed the homogeneous isotropy group. We remark that although non-empty, it may possibly contain only $(\mathbf{1}, \boldsymbol{O})$.

[6] These are deformations of the form

$$\boldsymbol{\lambda}(\hat{\boldsymbol{y}})=\boldsymbol{\lambda}(\hat{\boldsymbol{x}})+\boldsymbol{H}_0(\hat{\boldsymbol{y}}-\hat{\boldsymbol{x}})$$

for all $\hat{\boldsymbol{y}}\in\boldsymbol{\kappa}(\mathcal{B})$, with $\boldsymbol{H}_0\in \text{Invlin}^+(\mathcal{V})$.

The proof of this result is straightforward. An immediate consequence is

COROLLARY 3. *The homogeneous symmetry group* $\mathscr{G}_{\kappa}(X)$ *is conjugate to* $\mathscr{G}_{\mu}(X)$ *whenever*[7] $\boldsymbol{K}_0 = \boldsymbol{O}$.

*Proof.* Since $(\boldsymbol{H}, \boldsymbol{K}) \in \mathscr{G}_{\kappa}(X)$ implies $\boldsymbol{K} = \boldsymbol{O}$, equation (14) yields $(\boldsymbol{H}_0\boldsymbol{H}\boldsymbol{H}_0^{-1}, \boldsymbol{O}) \in S_{\mu}(X)$ and hence $(\boldsymbol{H}_0\boldsymbol{H}\boldsymbol{H}_0^{-1}, \boldsymbol{O}) \in \mathscr{G}_{\mu}(X)$. The interchangeability of $\boldsymbol{\kappa}$ and $\boldsymbol{\mu}$ implies that $(\boldsymbol{H}_0^{-1}\boldsymbol{H}'\boldsymbol{H}_0, \boldsymbol{O}) \in \mathscr{G}_{\kappa}(X)$ whenever $(\boldsymbol{H}', \boldsymbol{O}) \in \mathscr{G}_{\mu}(X)$ so that the result follows. Symbolically we may write

$$\boldsymbol{H}_0\mathscr{G}_{\kappa}(X)\boldsymbol{H}_0^{-1} = \mathscr{G}_{\mu}(X)$$

in such cases.

The following observations give an indication of the essential character of a material point of second grade.

## Remarks

1. Elements of an homogeneous symmetry group must be expected to be proper unimodular tensors[8] since $(\boldsymbol{H}, \boldsymbol{O}) \in \mathscr{G}_{\kappa}(X)$ implies from the group property that $(\boldsymbol{H}^n, \boldsymbol{O}) \in \mathscr{G}_{\kappa}(X)$ for all integers $n$. Indeed, defining $\boldsymbol{\lambda}_n$ to be that homogeneous deformation with gradient $\boldsymbol{H}^n$, the response at X in the configuration $\boldsymbol{\mu}_n = \boldsymbol{\lambda}_n \circ \boldsymbol{\kappa}$ is identical to that of X in $\boldsymbol{\kappa}$, yet by virtue of mass conservation.

$$\rho_{\kappa}(\hat{\boldsymbol{y}}) = (\det \boldsymbol{H})^n \rho_{\mu_n}(\boldsymbol{\lambda}_n(\hat{\boldsymbol{y}})) \qquad \forall \hat{\boldsymbol{y}} \in \boldsymbol{\kappa}(\mathscr{B}).$$

Thus, if $\det \boldsymbol{H} \neq 1$, it is possible to make the density $\rho_{\mu_n}$ *everywhere in a neighborhood* of X arbitrarily small (by choosing $n$ large enough[9]) and yet maintain unchanged response at X, a result clearly at variance with experience. *However, if* $(\boldsymbol{H}, \boldsymbol{K}) \in S_{\kappa}(X)$ *we cannot deduce in a similar manner that* $\boldsymbol{H}$ *be unimodular.* In attempting to do so we notice that $(\boldsymbol{H}^2, (\boldsymbol{K}^T\boldsymbol{H})^T\boldsymbol{H} + \boldsymbol{H}\boldsymbol{K})$,

$$(\boldsymbol{H}^3, (\boldsymbol{K}^T\boldsymbol{H}^2)^T\boldsymbol{H}^2 + \boldsymbol{H}(\boldsymbol{K}^T\boldsymbol{H})^T\boldsymbol{H} + \boldsymbol{H}^2\boldsymbol{K})$$

etc. lie in $S_{\kappa}(X)$ and are forced to the conclusion that, although the density *at* $\boldsymbol{\kappa}(X)$ may be made vanishingly small without change in response thereat, if $\boldsymbol{K}$ is not zero this rarefaction cannot be accomplished in a *neighborhood* of $\boldsymbol{\kappa}(X)$ as evidenced by the second entries in the elements of $S_{\kappa}(X)$ involved. These are, of course, related to density gradients: differentiating

$$\rho_{\kappa} = (\det \boldsymbol{H})\rho_{\mu} \circ \boldsymbol{\lambda}$$

yields, $\forall \boldsymbol{v} \in \mathscr{V}$,

$$\nabla\rho_{\kappa} \cdot \boldsymbol{v} = (\det \boldsymbol{H})((\nabla\rho_{\mu}) \circ \boldsymbol{\lambda}) \cdot \boldsymbol{H}\boldsymbol{v} + \rho_{\kappa}(\boldsymbol{H}^{-1})^T \cdot \boldsymbol{K}\boldsymbol{v}. \tag{15}$$

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[7] In particular this would be true were $\boldsymbol{\lambda}$ homogeneous.

[8] More precisely, if $(\boldsymbol{H}, \boldsymbol{O}) \in \mathscr{G}_{\kappa}(X)$ then $\boldsymbol{H}$ is unimodular with $\det \boldsymbol{H} = +1$.

[9] Without loss of generality we may assume $\det \boldsymbol{H} > 1$ since if $\boldsymbol{H} \in \mathscr{G}_{\kappa}(X)$ so does $\boldsymbol{H}^{-1}$.

Thus, if $(\boldsymbol{H}, \boldsymbol{K}) \in S_\kappa(X)$, the absence of the assumption that $\boldsymbol{H}$ be unimodular is consistent with the possibility that certain deformations involving changes in density leave the response unchanged (by virtue of second-order effects which introduce changes in density gradient). This possibility of first- and second-order effects counterbalancing each other would seem to represent the essential nature of a second grade material. Indeed, we note that

$$(\boldsymbol{H}, \boldsymbol{K}) = (\boldsymbol{H}, \boldsymbol{O}) \circ (\boldsymbol{1}, \boldsymbol{H}^{-1}\boldsymbol{K}) = (\boldsymbol{1}, \hat{\boldsymbol{K}}) \circ (\boldsymbol{H}, \boldsymbol{O}),$$

where

$$\hat{\boldsymbol{K}} = ((\boldsymbol{K}\boldsymbol{H}^{-1})^T \boldsymbol{H}^{-1})^T.$$

It follows that if $\boldsymbol{\lambda}_1$, $\boldsymbol{\lambda}_2$, and $\boldsymbol{\lambda}_3$ are deformations associated with, respectively, $(\boldsymbol{H}, \boldsymbol{O})$, $(\boldsymbol{1}, \boldsymbol{H}^{-1}\boldsymbol{K})$, and $(\boldsymbol{1}, \hat{\boldsymbol{K}})$, so that $\boldsymbol{\lambda}_1$ represents a first-order effect[10] while $\boldsymbol{\lambda}_2$ and $\boldsymbol{\lambda}_3$ are second-order, and if $(\boldsymbol{H}, \boldsymbol{K}) \in S_\kappa(X)$, then $\boldsymbol{\lambda}_1 \circ \boldsymbol{\lambda}_2$ and $\boldsymbol{\lambda}_3 \circ \boldsymbol{\lambda}_1$ leave the response at X unchanged.

2. Elements of the form $(\boldsymbol{1}, \boldsymbol{K})$ comprise a subgroup of $S_\kappa(X)$ since, by (7.9)

$$(\boldsymbol{1}, \boldsymbol{K}_1) \circ (\boldsymbol{1}, \boldsymbol{K}_2) = (\boldsymbol{1}, \boldsymbol{K}_1 + \boldsymbol{K}_2),$$

$(\boldsymbol{1}, -\boldsymbol{K})$ clearly being the inverse of $(\boldsymbol{1}, \boldsymbol{K})$. This implies that if $(\boldsymbol{1}, \boldsymbol{K}) \in S_\kappa(X)$ then $(\boldsymbol{1}, n\boldsymbol{K}) \in S_\kappa(X)$ for all integers $n$. If $\boldsymbol{K} \neq \boldsymbol{O}$, so that there exists a vector $\boldsymbol{u}$ such that[11] $(\boldsymbol{K}\boldsymbol{u})\boldsymbol{u} \neq \boldsymbol{0}$, define for any integer $n$ the deformation $\boldsymbol{\lambda}_n$ on $\boldsymbol{\kappa}(\mathcal{B})$ by

$$\boldsymbol{\lambda}_n(\hat{\boldsymbol{x}} + \boldsymbol{v}) = \hat{\boldsymbol{x}} + \boldsymbol{v} + \tfrac{1}{2}n(\boldsymbol{K}(\boldsymbol{v}\cdot\boldsymbol{u})\boldsymbol{u})(\boldsymbol{v}\cdot\boldsymbol{u})\boldsymbol{u} = \hat{\boldsymbol{x}} + \boldsymbol{v} + (\boldsymbol{v}\cdot\boldsymbol{u})^2 n\boldsymbol{w}$$

say, so that $\nabla\boldsymbol{\lambda}_n(\hat{\boldsymbol{x}}) = \boldsymbol{1}$ and $\nabla\nabla\boldsymbol{\lambda}_n(\hat{\boldsymbol{x}}) = n((\boldsymbol{K}\boldsymbol{u})\boldsymbol{u}) \otimes \boldsymbol{u} \otimes \boldsymbol{u}$. The response of X in $\boldsymbol{\kappa}$ is the same as that of X in $\boldsymbol{\mu}_n (= \boldsymbol{\lambda}_n \circ \boldsymbol{\kappa})$ where $\rho_\kappa(\hat{\boldsymbol{x}}) = \rho_{\mu_n}(\boldsymbol{\lambda}_n(\hat{\boldsymbol{x}}))$ and, from (7.11),

$$\nabla\rho_\kappa(\hat{\boldsymbol{x}})\cdot\boldsymbol{v} = \nabla\rho_{\mu_n}(\boldsymbol{\lambda}_n(\hat{\boldsymbol{x}}))\cdot\boldsymbol{v} + n\rho_\kappa(\hat{\boldsymbol{x}})(\boldsymbol{u}\cdot\boldsymbol{v})((\boldsymbol{K}\boldsymbol{u})\boldsymbol{u}\cdot\boldsymbol{u}) \qquad \forall \boldsymbol{v} \in \mathcal{V}.$$

If $((\boldsymbol{K}\boldsymbol{u})\boldsymbol{u}\cdot\boldsymbol{u} \neq 0$ for any $\boldsymbol{u} \in \mathcal{V}$ this would imply unchanged response at X after deformations involving no change of density but arbitrarily large density gradient component in the direction defined by the vector $\boldsymbol{u}$, the components in perpendicular directions being zero. This we feel to be physically implausible and so make the following

*Postulate.* If $(\boldsymbol{1}, \boldsymbol{K}) \in S_k(X)$ then $(\boldsymbol{K}\boldsymbol{u})\boldsymbol{u}\cdot\boldsymbol{u} = 0 \ \forall \boldsymbol{u} \in \mathcal{V}$.

At this point we remark that couple-stress theories are essentially addressed to material behaviour as manifested in the solid state. In view of the preceding remarks a natural generalization of the concept of solid from that adopted in respect of simple elastic materials would seem to require that deformations of the form

$$\boldsymbol{\lambda}(\hat{\boldsymbol{x}} + \boldsymbol{v}) = \hat{\boldsymbol{x}} + \boldsymbol{v} + \tfrac{1}{2}(\boldsymbol{K}\boldsymbol{v})\boldsymbol{v}$$

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[10] Without loss of generality $\boldsymbol{\lambda}_1$ may be taken to be homogeneous.

[11] Here we note that $\boldsymbol{K}$ is completely determined by the set $\{(\boldsymbol{K}\boldsymbol{w})\boldsymbol{w} : \boldsymbol{w} \in \mathcal{V}\}$ since, by its symmetry, we have

$$2(\boldsymbol{K}\boldsymbol{u})\boldsymbol{v} = (\boldsymbol{K}(\boldsymbol{u}+\boldsymbol{v}))(\boldsymbol{u}+\boldsymbol{v}) - (\boldsymbol{K}\boldsymbol{u})\boldsymbol{u} - (\boldsymbol{K}\boldsymbol{v})\boldsymbol{v}.$$

always result in a change of response, implying that if $(\mathbf{1}, \boldsymbol{K}) \in S_{\boldsymbol{\kappa}}(X)$ and X is "solid" in $\boldsymbol{\kappa}$ then $\boldsymbol{K} = \boldsymbol{O}$. Using this terminology we have

PROPOSITION 3. *If* X *is solid in* $\boldsymbol{\kappa}$ *then for any given* $\boldsymbol{H} \in \text{Invlin}^+(\mathscr{V})$ *not both of* $(\boldsymbol{H}, \boldsymbol{O})$ *and* $(\boldsymbol{H}, \boldsymbol{K})$ *can belong to* $S_{\boldsymbol{\kappa}}(X)$, *where* $\boldsymbol{K} \in \text{Sym}_3^T(\mathscr{V})$, $\boldsymbol{K} \neq \boldsymbol{O}$.

*Proof.* If $(\boldsymbol{H}, \boldsymbol{O})$ *and* $(\boldsymbol{H}, \boldsymbol{K}) \in S_{\boldsymbol{\kappa}}(X)$ then by the group property $(\boldsymbol{H}, \boldsymbol{O})^{-1} \circ (\boldsymbol{H}, \boldsymbol{K}) = (\boldsymbol{H}^{-1}, \boldsymbol{O}) \circ (\boldsymbol{H}, \boldsymbol{K}) = (\mathbf{1}, \boldsymbol{H}^{-1}\boldsymbol{K}) \in S_{\boldsymbol{\kappa}}(X)$ whence, from the solidity assumption $\boldsymbol{H}^{-1}\boldsymbol{K} = \boldsymbol{O}$ so that $\boldsymbol{K} = \boldsymbol{O}$, yielding a contradiction.

3. A second-grade material point X might be described as *isotropic* if there exists a configuration $\boldsymbol{\kappa}$ wherein[12] $\mathscr{O}^+(\mathscr{V}) \subset \mathscr{G}_{\boldsymbol{\kappa}}(X)$. In such a case, $\forall \boldsymbol{F}, \boldsymbol{G}$ as in (11)

$$\hat{\boldsymbol{f}}_{\boldsymbol{\kappa}}(\boldsymbol{F}, \boldsymbol{G}) = \hat{\boldsymbol{f}}_{\boldsymbol{\kappa}}(\boldsymbol{F}\boldsymbol{Q}^T, \boldsymbol{Q}(\boldsymbol{G}^T\boldsymbol{Q}^T)^T\boldsymbol{Q}^T) \qquad \forall \boldsymbol{Q} \in \mathscr{O}^+(\mathscr{V}). \tag{16}$$

However, material frame-indifference requires that $\forall \boldsymbol{F}, \boldsymbol{G}$ as in (11)

$$\hat{\boldsymbol{f}}_{\boldsymbol{\kappa}}(\boldsymbol{Q}\boldsymbol{F}, \boldsymbol{G}) = \boldsymbol{Q}\hat{\boldsymbol{f}}_{\boldsymbol{\kappa}}(\boldsymbol{F}, \boldsymbol{G})\boldsymbol{Q}^T, \tag{17}$$

for $\hat{\boldsymbol{f}}_{\boldsymbol{\kappa}} = \hat{\boldsymbol{T}}_{\boldsymbol{\kappa}}$, and[13]

$$\hat{\boldsymbol{f}}_{\boldsymbol{\kappa}}(\boldsymbol{Q}\boldsymbol{F}, \boldsymbol{G}) = \boldsymbol{Q}(\hat{\boldsymbol{f}}_{\boldsymbol{\kappa}}(\boldsymbol{F}, \boldsymbol{G})^T\boldsymbol{Q}^T)^T\boldsymbol{Q}^T \tag{18}$$

for $\hat{\boldsymbol{f}}_{\boldsymbol{\kappa}} = \hat{\boldsymbol{M}}_{\boldsymbol{\kappa}}$.

Replacing $\boldsymbol{F}$ in (16) by $\boldsymbol{Q}\boldsymbol{F}$ and using (17) or (18) as appropriate, we have

$$\hat{\boldsymbol{T}}_{\boldsymbol{\kappa}}(\boldsymbol{Q}\boldsymbol{F}\boldsymbol{Q}^T, \boldsymbol{Q}(\boldsymbol{G}^T\boldsymbol{Q}^T)^T\boldsymbol{Q}^T) = \boldsymbol{Q}\hat{\boldsymbol{T}}_{\boldsymbol{\kappa}}(\boldsymbol{F}, \boldsymbol{G})\boldsymbol{Q}^T \tag{19}$$

and

$$\hat{\boldsymbol{M}}_{\boldsymbol{\kappa}}(\boldsymbol{Q}\boldsymbol{F}\boldsymbol{Q}^T, \boldsymbol{Q}(\boldsymbol{G}^T\boldsymbol{Q}^T)^T\boldsymbol{Q}^T) = \boldsymbol{Q}(\hat{\boldsymbol{M}}_{\boldsymbol{\kappa}}(\boldsymbol{F}, \boldsymbol{G})^T\boldsymbol{Q}^T)^T\boldsymbol{Q}^T, \tag{20}$$

these relations holding for all $\boldsymbol{F} \in \text{Invlin}^+(\mathscr{V})$, all $\boldsymbol{G} \in \text{Sym}_3^T(\mathscr{V})$ and all $\boldsymbol{Q} \in \mathscr{O}^+(\mathscr{V})$. Thus $\hat{\boldsymbol{T}}_{\boldsymbol{\kappa}}$ and $\hat{\boldsymbol{M}}_{\boldsymbol{\kappa}}$ are isotropic tensor-valued functions of their arguments in the sense of [1] (cf. §8) whenever the material point concerned is isotropic in $\boldsymbol{\kappa}$. Since we cannot in general expect $\mathscr{G}_{\boldsymbol{\kappa}}(X)$ to coincide with $S_{\boldsymbol{\kappa}}(X)$, any additional symmetry of X, which happens to be isotropic in $\boldsymbol{\kappa}$, will place restrictions on response functions over and above those of relations (19) and (20).

4. The approach adopted in this note readily generalizes to higher-grade materials, with, however, symmetry sets having increasingly complex group operations. Since a material of grade $n$ is insensitive to deformation gradients of orders in excess of $n$,

[12] $\mathscr{O}^+(\mathscr{V})$ denotes the group of proper orthogonal tensors on $\mathscr{V}$.

[13] With obvious notation we have, by frame-indifference, $\boldsymbol{M}^*\boldsymbol{n}^* = \boldsymbol{Q}(\boldsymbol{M}\boldsymbol{n})\boldsymbol{Q}^T$ where $\boldsymbol{n}^* = \boldsymbol{Q}\boldsymbol{n}$. Thus, by linearity, $\forall \boldsymbol{v}, \boldsymbol{w} \in \mathscr{V}$,

$$(\boldsymbol{M}^*\boldsymbol{Q}\boldsymbol{v})\boldsymbol{w} = \boldsymbol{Q}(\boldsymbol{M}\boldsymbol{v})\boldsymbol{Q}^T\boldsymbol{w} = \boldsymbol{Q}(\boldsymbol{M}^T(\boldsymbol{Q}^T\boldsymbol{w})\boldsymbol{v}) = \boldsymbol{Q}((\boldsymbol{M}^T\boldsymbol{Q}^T)^T\boldsymbol{v})\boldsymbol{w},$$

so that

$$\boldsymbol{M}^*\boldsymbol{Q} = \boldsymbol{Q}(\boldsymbol{M}^T\boldsymbol{Q}^T)^T.$$

deformations may be partitioned into equivalence classes, any pair in any such class having gradients up to order $n$ which coincide (at the point in question) and hence elicit identical material response. Ljubicic[14] (cf. [4]) was the first to realize that the theory of jets was the most natural mathematical vehicle for a general discussion of such non-simple materials.

5. If, for the sake of example, construction of response functions for a specific second-grade material is desired, it must be noticed that these are restricted not only by frame-indifference, but also by thermodynamic requirements. For elastic materials these place restrictions upon $\hat{\boldsymbol{T}}_\kappa + \hat{\boldsymbol{T}}_\kappa^T$ and $\hat{\boldsymbol{M}}_\kappa + \hat{\boldsymbol{M}}_\kappa^T$ and the specific manner of their dependence upon $\boldsymbol{G}$ (c.f. [5], [8]).

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[14] Presumably motivated by precisely the foregoing observation.