International Journal of Fracture 27 (1985) 113–119.

Macrocrack interaction with semi-infinite microcrack array

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(Received July 21, 1984)

Abstract

The collinear periodical array of microcracks ahead of a semi-infinite crack (macrocrack) is considered. A close form solution in terms of complex stress potentials is given, assuming that a remote, macroscale, stress intensity factor is given. The exact solution of the interaction of a macrocrack with a single microcrack is given.

Results demonstrate that for relatively close location (with respect to crack length) of microcracks to the macrocrack tip, the microcrack spacing becomes important. For microcrack spacing (period) greater than 10 crack lengths the interaction can be taken as for a single microcrack, and, for distance greater than two microcrack lengths, the local stress intensity factor can be taken as equal to that remotely applied (for cases with crack spacing greater than two crack lengths).

In other cases the macro-microcrack interaction is significant.

1. Introduction

Frequently in metals large cracks are observed being surrounded by large numbers of microcracks, concentrated in the vicinity of the crack tip. Many fracture models deal with nucleation and growth of microcracks in the vicinity of the tip of macrocrack. The incorporation of microcrack interaction in the theory of failure is especially important for cases of brittle metals, rocks, and any theory concerning fracture on microscale. The ductile or time dependent fracture mechanisms often also involve the microcracking. For example, high temperature creep rupture mechanism sometimes is represented as grain boundary void (microcracks) nucleation and growth ahead of a macrocrack. However, quantitative effects of macrocrack-microcrack interaction usually are not taken into account.

The aim of this work is to give a quantitative relation for this interaction. The attention here is limited to a particular case which probably can be taken as a statistical average of many possible configurations of plane problems of this type.

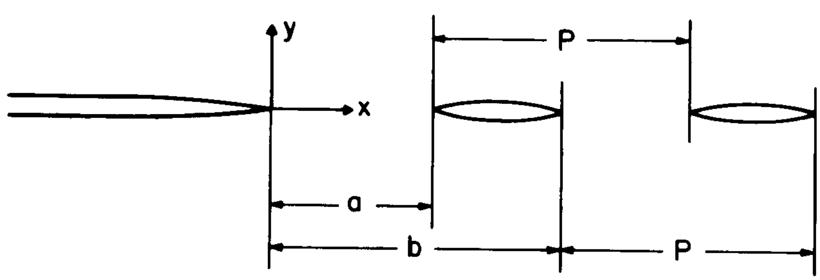

Figure 1. Macrocrack and semi-infinite array of microcracks.

The macrocrack is represented as a semi-infinite crack (small scale approach) and ahead of the crack tip is located the semi-infinite array of periodically distributed collinear cracks (Fig. 1).

The formulation is given in terms of complex potentials, and a close form solution, with some approximation, has been derived.

2. Formulation

Consider an elastic plane with a semi-infinite crack along the negative x axis and an array of cracks located at $a+pk<x<b+pk$, $(b>a>0)$ where p is the period of crack distribution $p>b-a$, and $k=0, 1, 2\ldots$ (Fig. 1). Introducing complex potentials $\phi(z)$, $\psi(z)$ the standard [1] relations for stresses and displacements can be written

$$\begin{aligned} &\sigma_{11}+\sigma_{22}=4\ \mathrm{Re}\ \phi'(z) \\ &\sigma_{22}-\sigma_{11}+2i\sigma_{12}=2(\bar{z}\phi''(z)+\psi'(z)) \\ &2\mu(u_1+iu_2)=\kappa\phi-z\overline{\phi'}-\bar{\psi} \end{aligned} \tag{1}$$

here $\kappa=3-4\nu$ for plane strain case, $\kappa=(3-\nu)/(1+\nu)$ for plane stress case, μ is a shear modulus and ν is the Poisson's ratio.

Consider, for simplicity, mode I type of loading. Symmetry of the problem will give $\sigma_{12}=0$ along $y=0$, so

$$\begin{aligned} &\mathrm{Im}(\bar{z}\phi''+\psi')=0 \qquad \text{on} \quad y=0 \quad \text{or} \\ &\mathrm{Im}(z\phi''+\psi')=0 \qquad \text{on} \quad y=0. \end{aligned} \tag{2}$$

Using the principal of analytical continuation one can write

$$\psi'(z)=-z\phi''(z) \tag{3}$$

(ψ' and ϕ', both vanish at infinity) and the following relations on $y=0$ take place

$$\begin{aligned} &\sigma_{22}=2\ \mathrm{Re}\ \phi'(x) \\ &u_2=\frac{\kappa+1}{2\mu}\mathrm{Im}\ \phi(x), \qquad y=0. \end{aligned} \tag{4}$$

The remote stress is given as

$$\phi'(z)\to\frac{K_1(\infty)}{2\sqrt{2\pi z}} \qquad \text{as} \quad z\to\infty, \tag{5}$$

here $K_1(\infty)$ represents remote stress intensity factor which should be found from the problem on macroscale, and in this case considered as a given value. The boundary conditions for the function $\phi'(z)$ are specified on $y=0$ as

$$\begin{aligned} &\mathrm{Re}\ \phi'(x)=0 \qquad \text{on} \quad x<0 \quad \text{and} \quad a+pk<x<b+pk \\ &\mathrm{Im}\ \phi'(x)=0 \qquad 0<x<a, \quad b+pk<x<a+p(k+1) \\ &k=0,1,2,\ldots \end{aligned} \tag{6}$$

Conditions (5) and (6) determine the analytic function in the upper half of complex plane z up to the restriction on behavior at the points $a+pk$ and $b+pk$ $(k=0,1,2,\ldots)$. Restricting this behavior to a singularity not stronger than integrable and adding the requirement of a similar type of behavior at each point 0, $a+pk$, $b+pk$ $(k=0,1,2,\ldots)$, one would obtain the only possible form of function $\phi'(z)$. That is the form of the homogeneous solution of Keldysh-Sedov problem for half plane [2]. The form of $\phi'(z)$

which satisfies all the above-mentioned conditions is

$$\phi'(z) = \frac{K_{\mathrm{I}}(\infty)}{2\sqrt{2\pi}} \frac{\prod_{k=0}^{\infty}(c_k - z)}{\sqrt{z\prod_{i=0}^{\infty}(a+pi-z)(b+pi-z)}} \tag{7}$$

where constants c_k have to be determined, and by square root it is understood to be the branch that has positive values of $\phi'(x)$ for $0 < x < a$. The function $\phi'(z)$ given by (7) can be defined, through analytical continuation over segments $0 < x < a$ and $b + pk < x < a + p(k+1)$, in the lower half plane. Thus, function $\phi'(z)$ is analytic on z in the plane with cuts.

The constants c_k have to be determined from the supplementary condition of the single valued displacement, that is

$$\int_{a+pk}^{b+pk} \mathrm{Im}\, \phi'(x)\mathrm{d}x = 0, \qquad k = 0, 1, 2, \ldots \tag{8}$$

From the symmetry of the problem, it is obvious that c_k are real, and since they represent zeros of $\phi'(z)$, $a + pk < c_k < b + pk$. c_k represents position of maximal crack opening displacement, so from the geometry of the problem it can be stated

$$a + pk < c_k < (a+b)/2 + pk. \tag{9}$$

The form of solution (7) can be used for arbitrary collinear crack distribution with corresponding adjustment of crack tips location coordinates in (7). In the case of mode II type of loading, relation (2) should be replaced by

$$\mathrm{Re}[2\phi' + z\phi'' + \psi'] = 0, \tag{10}$$

on $y = 0$

which follows from mode II symmetry condition $\sigma_{22} = 0$ on $y = 0$. Then, proceeding in a similar pattern, one would obtain the solution. The mode II solution follows from (7) by multiplying it by $-iK_{\mathrm{II}}(\infty)/K_{\mathrm{I}}(\infty)$.

The problem now is reduced to determination of the constants. It is useful at this point to consider a special case of a single crack ahead of a macrocrack.

3. A single microcrack

In the case of a single microcrack relation (7) reduces to

$$\phi'(z) = \frac{K_{\mathrm{I}}(\infty)}{2\sqrt{2\pi}} \frac{c - z}{\sqrt{z(a-z)(b-z)}}. \tag{11}$$

From condition (8) $k = 0$, c can be evaluated in terms of elliptic integrals of the first and second kind. Thus

$$c = b\frac{E(1 - a/b)}{K(1 - a/b)}, \tag{12}$$

where $K(m)$ and $E(m)$ are complete elliptic integrals of the first and second kind respectively [3]. The ratios of the local stress intensity factors are

$$\begin{aligned} \frac{K_{\mathrm{I}}(0)}{K_{\mathrm{I}}(\infty)} &= \sqrt{\frac{b}{a}}\,\frac{E(1-a/b)}{K(1-a/b)} \\ \frac{K_{\mathrm{I}}(a)}{K_{\mathrm{I}}(\infty)} &= \frac{b}{a}\left(\frac{E(1-a/b)}{K(1-a/b)} - 1\right)\Big/\sqrt{\frac{b}{a} - 1} \\ \frac{K_{\mathrm{I}}(b)}{K_{\mathrm{I}}(\infty)} &= \left(1 - \frac{E(1-a/b)}{K(1-a/b)}\right)\Big/\sqrt{1 - \frac{a}{b}}. \end{aligned} \tag{13}$$

The highest value of the stress intensity factor is at $x = 0$, and $K_I(a) \to K_I(0)$ as $a \to 0$, $K_I(a) \to K_I(b)$ as $a \to \infty$. In Fig. 2 the values of $K(0)/K(\infty)$ versus position of microcrack are given. Finally, it follows

$$c \to (a+b)/2 \text{ as } a \to \infty$$

and c satisfies (9) for $k = 0$.

4. Semi-infinite array of microcracks

Consider solution (7). Each c_k represents the position of the maximal crack opening of the corresponding microcrack. As $k \to \infty$, $c_k \to d + kp$, here $d = (a+b)/2$. Introduce value

$$\Delta_k = (d + pk - c_k)/p, \tag{14}$$

and employ known formula [3]

$$\frac{1}{z\Gamma(z)} = e^{\gamma z} \prod_1^\infty \left(1 + \frac{z}{k}\right) e^{-z/k}$$

rewrite (7) in the form (γ is Euler's constant)

$$\phi'(z) = \frac{K_I(\infty)}{2\sqrt{2\pi}} \frac{c_0 - z}{p} \sqrt{\frac{1}{z} \Gamma\left(\frac{a-z}{p}\right) \Gamma\left(\frac{b-z}{p}\right)} \times e^{\gamma(d-z)/p} \prod_1^\infty \left(\frac{d-z}{pk} - \frac{\Delta_k}{k} + 1\right) e^{-(d-z)/pk} \tag{15}$$

Δ_k, as a function of k, can be expanded

$$\Delta_k = f_0 + f_1 \frac{1}{k} + f_2 \frac{1}{k^2} + \ldots$$

f_0 should be equal to zero since $\Delta_k \to 0$ as $k \to \infty$. Approximating Δ_k as Δ/k for $k > N$, and writing

$$1 + \frac{d-z}{pk} - \frac{\Delta}{k^2} = \left(1 + \frac{1}{k}\delta_1(z)\right)\left(1 + \frac{1}{k}\delta_2(z)\right)$$

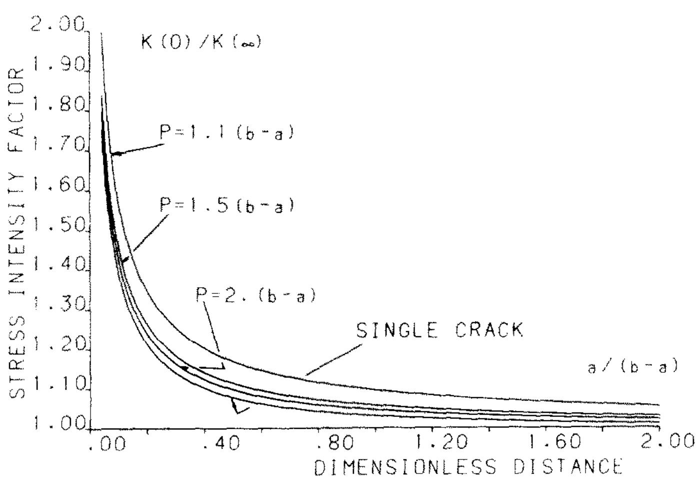

Figure 2. Stress intensity factor at the main crack versus position of the first crack in the microcrack array

with

$$\delta_1(z) = \frac{d-z}{2p} + \sqrt{\left(\frac{d-z}{2p}\right)^2 + \Delta}$$

$$\delta_2(z) = \frac{d-z}{2p} - \sqrt{\left(\frac{d-z}{2p}\right)^2 + \Delta} \tag{16}$$

one can write for $\phi'(z)$

$$\phi'(z) = \frac{K_I(\infty)}{2\sqrt{2\pi}} \frac{z-c_0}{\Delta p} \frac{\sqrt{\frac{1}{z}\Gamma\left(\frac{a-z}{p}\right)\Gamma\left(\frac{b-z}{p}\right)}}{\Gamma(\delta_1(z))\Gamma(\delta_2(z))} \times \frac{1}{N!p^N} \prod_1^N \frac{c_k - z}{\left(\frac{d-z}{pk} - \frac{\Delta}{k^2} + 1\right)} \tag{17}$$

$N+2$ unknown constants c_0, $c_1 \ldots c_N$ and Δ should be found from the system of first $N+2$ equations (8). N can be chosen from considerations of desired accuracy of results. In this study calculations were performed for $N = 0$, 1 and 2.

A simple numerical scheme was employed. The algorithm of the computation is based on choice of Δ (with consequent repetition until (8), case $k = N+2$, is satisfied up to specified accuracy), solving algebraic equation for c_0, c_1, c_2, which arises after substitution of (17) into (8) and performing the integration. Integrations were carried out by Gaussian-Chibyshev scheme, using the fact of square root singularity at the end points of each interval.

The form of $\phi'(z)$ guarantees the right behavior at $z \to \infty$ and approximation (17) gives the required accuracy in the desired region of radius $b + p(N+1)$.

For calculation of the stress intensity factor (meaning ratio of local value to remote one) at the macrocrack tip the difference in the values obtained for $N = 0$, 1 or 2 was so insignificant that for any practical means N can be taken as $N = 0$. So

$$\frac{K_I(0)}{K_I(\infty)} = -\frac{c_0}{\Delta p} \frac{\sqrt{\Gamma\left(\frac{a}{p}\right)\Gamma\left(\frac{b}{p}\right)}}{\Gamma(\delta_1(0))\Gamma(\delta_2(0))}. \tag{18}$$

In Fig. 2 these values are plotted. Knowing the residue of $\Gamma(z)$ at simple poles $z = -n$

$$\operatorname*{res}_{z=-n} \Gamma(z) = (-1)^n/n!,$$

the stress intensity factors at the microcrack tips can be evaluated

$$\frac{K_I(a+pk)}{K_I(\infty)} = \frac{a+pk-c_0}{\Delta p} \frac{\sqrt{\frac{1}{k!}\left|\Gamma\left(\frac{b-a-pk}{p}\right)\right| \frac{1}{a+pk}}}{\Gamma(\delta_1(a+pk))\Gamma(\delta_2(a+pk))} \times \frac{1}{N!p^N} \prod_{i=1}^N \frac{(c_i - a - pk)}{\left(\frac{d-a-pk}{pi} - \frac{\Delta}{i^2} + 1\right)}, \tag{20}$$

for the case of $b + pk$, a and b should replace each other in (20). N can be small (1, 2 for example) if only the first one or two cracks are analyzed.

The ratio $K_I(a+pk)/K_I(\infty)$ is decaying with $k \to \infty$ because here the solution of a homogeneous problem only is given. That means that for k large enough, the local stress intensity factors will not be significantly affected by the presence of a macrocrack, and mostly will be influenced by the local stress field, which is not accounted for in this small scale approach.

During the calculations, the condition (8) was checked for $k > N+2$. Results demonstrated that even for $N=0$, conditions on all intervals are satisfied fairly accurately (up to 10^{-4} with dimensionless length unit equal to the crack length) for almost all values of parameters; exceptions are cases of a being very small with respect to crack length $(b-a)$. In these cases the choice $N=1$ or 2 is more appropriate. Case $N=2$ practically satisfies condition (8) for any k. This indicates that formula (17) does represent the correct features of the solution, and approximation of $c_k(14)$ with $\Delta_k = \Delta/k$ is close to reality. Certainly, the rigorous proof that (17) approaches the solution, as $N \to \infty$, uniformly on z can be done routinely.

5. Results and conclusions

Solution of the problem dealing with macro-microcracks interaction has been presented. Namely, the exact solution for macrocrack interaction with collinear microcrack has been given, as well as an approximate solution for macrocrack interaction with semi-infinite array of periodically distributed collinear array of microcracks. Both solutions give complete stress distribution in the plane through the complex potential $\phi'(z)$ and, defined by (3) in case of mode I or by (10) in case of mode II, potential $\psi'(z)$.

The results presented here are obtained for case $N=2$ in (17). In Fig. 2, the stress intensity factor ratios are given for the cases $p/(b-a)=1.1$, 1.5, 2 and the case of a single crack versus location of the first crack of the array. For the very close positions the numerical values are given in Table 1.

Results demonstrate that in the case of microcrack spacing $p/(b-a)$ greater or equal to 10, the local stress intensity factor at the microcrack can be taken as in the case of a

Table 1. Stress intensity factors $K(0)/K(\infty)$. p is given in units of $(b-a)$

$a/(b-a)$	s.crack	$p=1.1$	$p=1.5$	$p=2.0$	$p=5.0$	$p=10.0$
0.001	6.54887	7.62193	6.91722	6.73294	6.56946	6.55761
0.011	2.66879	3.07970	2.81173	2.74054	2.67625	2.67149
0.021	2.14848	2.46856	2.26081	2.20518	2.15445	2.15065
0.031	1.90118	2.17693	1.99862	1.95059	1.90642	1.90308
0.041	1.74962	1.99750	1.83774	1.79448	1.75442	1.75137
0.051	1.64487	1.87300	1.72641	1.68653	1.64937	1.64651
0.061	1.56716	1.78025	1.64369	1.60639	1.57142	1.56872
0.071	1.50672	1.70782	1.57927	1.54402	1.51080	1.50821
0.081	1.45810	1.64932	1.52739	1.49382	1.46203	1.45954
0.091	1.41799	1.60087	1.48452	1.45239	1.42180	1.41939
0.100	1.38738	1.56375	1.45176	1.42075	1.39110	1.38875
0.110	1.35810	1.52810	1.42038	1.39046	1.36173	1.35944
0.120	1.33279	1.49715	1.39321	1.36425	1.33633	1.33409
0.130	1.31065	1.46997	1.36941	1.34132	1.31413	1.31193
0.140	1.29111	1.44589	1.34837	1.32107	1.29452	1.29237
0.150	1.27373	1.42437	1.32963	1.30304	1.27709	1.27497
0.160	1.25816	1.40501	1.31281	1.28688	1.26146	1.25938
0.170	1.24412	1.38749	1.29763	1.27230	1.24738	1.24533
0.180	1.23140	1.37155	1.28285	1.25907	1.23462	1.23260
0.190	1.21982	1.35697	1.27128	1.24702	1.22300	1.22100
0.200	1.20923	1.34358	1.25977	1.23599	1.21237	1.21040

single microcrack. In the case of the distance of the closest microcrack tip greater than two crack lengths ($a > 2(b - a)$) and spacing greater than 3 crack lengths ($p > 3(b - a)$) the macro-microcrack interaction can be neglected (with accuracy up to 2%) for the computations of stress intensity factor acting at the macrocrack tip. In other configurations this interaction becomes significant and cannot be neglected.

Acknowledgement

This work has been supported by NSF Grant No. MEA8305643. This support is greatly appreciated.

References

[1] N.I. Muskhelishvili, *Some Basic Problems of the Mathematical Theory of Elasticity*, translation by J.R.M. Radok, Noordhoff, Groningen (1975).

[2] N.I. Muskhelishvili, *Singular Integral Equations*, translated by J.R.M. Radok, Noordhoff, Groningen, Holland (1953).

[3] M. Abramowitz and I.A. Stegun, *Handbook of Mathematical Functions*, Dover Publications, Inc., New York (1972).

Résumé

On prend en considération le cas d'une rangée de fissures périodiques et colinéaires qui se forment en avant d'une fissure macroscopique semi-infinie. On propose une solution de forme fermée, exprimée en potentiels complexes de contraintes, et qui suppose l'existence d'un facteur d'intensité de contrainte s'exerçant à une échelle macroscopique. On fournit la solution exacte à l'interaction entre une macro-fissure et une simple microfissure.

Les résultats présentés démontrent que, pour les micro-fissures relativement proches de l'extrémité de la macro-fissure, c'est leur espacement qui est important. Lorsque la période caractérisant l'espacement entre deux micro-fissures est supérieure ou égale à dix fois la longueur de la microfissure, l'interaction peut être assimilée à celle d'une micro-fissure simple. Lorsque la distance qui sépare la première micro-fissure de l'extrémité de la fissure est supérieure à deux fois la longueur de cette micro-fissure, et que la période entre deux micro-fissures successives excède deux fois leur longueur, l'interaction entre micro et macro-fissure peut être négligée pour le calcul du facteur d'intensité de contrainte à extrémité de la macro-fissure.

Dans les autres configurations, l'interaction devient significative et ne peut être négligée.