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DETERMINATION OF RESIDUAL STRESS DISTRIBUTIONS FROM MEASURED STRESS INTENSITY FACTORS

H.J. Schindler
Swiss Federal Laboratories of Materials Testing and Research (EMPA)
Überlandstrasse 129, CH-8600 Duebendorf, Switzerland
Tel: +41-1-823 55 11; Fax: +41-1-821 62 44

If a crack or a machined cut is introduced in a body that contains residual stresses, the latter are released at the newly created surfaces and cause the stress field to be rearranged in the entire body (Fig. 1a). From the change of stress due to progressive cutting as measured at any location (e.g. on the rear surface, as shown in Fig. 1b), it is possible to calculate the stress that acted along the corresponding axis x in the initial, uncracked state. This method of determining residual stress fields, called the crack compliance method, is described in a series of papers by Cheng and Finnie e.g. [1,2]. Since the corresponding analytical formulas are based on principles and relations of linear-elastic fracture mechanics, the stress intensity factor (SIF) $K_{Irs}(a)$ due to the residual stress field at the tip of the introduced crack or cut can be readily obtained by progressive cutting from a single strain measurement as follows [3]:

$$K_{Irs}(a) = \frac{E'}{Z(a)} \frac{d\varepsilon_M}{d\,a} \tag{1}$$

where ε_M denotes the strain measured at an arbitrary location M during the cutting process, a the actual length of the cut, E′ the generalised Young's modulus and Z(a) a geometry-dependent function which reflects the sensitivity of the strain at M with respect to stresses released at the crack tip. Z(a) can be analytically or numerically determined as described in [3]. Hence, functions $K_{Irs}(a)$ can be relatively easily obtained experimentally by using (1).

On the other hand, $K_{Irs}(a)$ depends on the distribution of the normal residual stresses acting prior to cutting, $\sigma_{rs}(x)$ (the axis being chosen such that it coincides with the crack line, or cut plane, respectively), by the general relation

$$K_{Irs}(a) = \int_0^a h(x,a) \cdot \sigma_{rs}(x) \cdot d\,x \tag{2}$$

where h(x,a) denotes the so-called wieght function, which is universal for a given crack geometry [5,6]. Weight functions are known for several crack configurations (see e.g. [7]).

Thus from a known SIF as a function of crack depth, $K_{Irs}(a)$, it is possible to calculate therefrom the residual stress distribution $\sigma_{rs}(x)$ by inversion of (2). A straightforward way to solve this problem is by a step-by-step procedure as follows. The residual stress distribution $\sigma_{rs}(x)$ is approximated by a series of small steps as shown schematically in Fig. 2, so the stress level at each step can be calculated by applying (2) to a hypothetical, incrementally prolonging crack. The average stress of the first increment, σ_0, which represents the average stress acting near the front surface in the range $0<x<a_0$ (where $a_0<<W$) is obtained from the well known relation between the stress and the SIF of a short edge crack [4] as

$$\sigma_0 = \frac{K_{Irs}(a_0)}{1.12 \cdot \sqrt{\pi \cdot a_0}} \tag{3}$$

In order to calculate the average stress level σ_1 of the next step (i.e. the average stress in the range $a_0<x<a_0+\Delta a$), we extend the hypothetical crack by the increment Δa. According to (2) , the following equation holds for the prolonged crack

$$K_{Irs}(a_0+\Delta a) = \sigma_0 \cdot \int_0^{a_0} h(x,a_0+\Delta a) \cdot dx + \sigma_1 \cdot \int_{a_0}^{a_0+\Delta a} h(x,a_0+\Delta a) \cdot dx \tag{4}$$

From (4), σ_1 can be calculated. In the same way, by an additional virtual crack prolongation Δa, the average stress σ_2 in the next interval $a_0+\Delta a<x<a_0+2\cdot\Delta a$ is obtained, and so on. By repeating this procedure of incremental crack extension Δa, the stress profile is determined. Denoting the length of the hypothetical crack after i increments Δa by a_i (i.e., $a_i=a_0+i\cdot\Delta a$), and the average stress in the corresponding interval $a_{i-1}<x<a_i$ by σ_i, the latter can be obtained from the following relation

$$K_{Irs}(a_i) = \sigma_0 \cdot \int_0^{a_0} h(x,a_i) \cdot dx + \sum_{j=1}^{i=1} \sigma_j \cdot \int_{a_{j-1}}^{a_j} h(x,a_i) \cdot dx + \sigma_i \cdot \int_{a_{i-1}}^{a_i} h(x,a_i) \cdot dx \tag{5}$$

The step-shaped approximation converges to the exact solution as $\Delta a \rightarrow 0$. Hence, the accuracy of the approximation can be adjusted to the purpose by choosing a sufficiently small step length Δa.

In principle, (3)-(5) deliver the complete stress distribution across the considered cross section. However, some difficulties usually arise in the region near the rear surface, because, as pointed out in [8], weight functions, which in general are derived by approximation techniques, often fail to be accurate for $W-a<<W$. Thus, the stresses can be inaccurate in the corresponding range of x.

There are two ways to overcome this difficulty: either by using a weight function derived according to [8], which is sufficiently accurate in the whole range of crack depths, or changing the calculation procedure as explained in the following.

For W-a<<W, which practically is fulfilled for about a>3W/4, it can be shown that the SIF can be approximated by the asymptotically exact equation:

$$K_{Irs}(\mathrm{a}) = \frac{3.97M(\mathrm{a})}{(W-\mathrm{a})^{3/2}} + \frac{1.46F(\mathrm{a})}{(W-\mathrm{a})^{1/2}} + 2\cdot\sigma_{rs}(\mathrm{x}=\mathrm{a})\cdot\sqrt{\frac{2}{\pi}q\cdot(W-\mathrm{a})} \tag{6}$$

where M(a) and F(a) denote the stress resultants of ther released residual stress with respect to the neutral axis of the ligament, i.e.,

$$M(\mathrm{a}) = \int_0^{\mathrm{a}} \sigma_{rs}(\mathrm{x})\cdot[0.264\cdot W - \mathrm{x} + 0.736\cdot \mathrm{a}]d\,\mathrm{x} \tag{7}$$

$$F(\mathrm{a}) = \int_0^{\mathrm{a}} \sigma_{rs}(\mathrm{x})\cdot d\,\mathrm{x} \tag{8}$$

The first two terms of (6) are based on dimensional considerations and the corresponding asymptotic solutions for $(W\text{-}a)\rightarrow 0$ as given in [4]. However, they do not account for the singular behaviour of weight functions as $x \rightarrow a$. This special contribution is added by the third term: since the stress $\sigma_{rs}(x=a)$ in the range $(a-q\cdot(W-a))<x<a$, where $q<<1$ (i.e. in a narrow range in the vicinity of the crack tip), can be assumed to be approximately constant, the integration of the singular term of the weight function delivers the third term of (6). The factor q, which characterises the width of the corresponding near-tip region of the crack surface, turned out to be in general about 0.03. However, q can be considered to be an unknown factor that is determined by the condition of a continuous and smooth transition from the stresses resulting from (2) to the ones resulting from (6).

If we define a natural number k such that it represents the number of calculation steps i=k at which the hypothetical crack is long enough for the approximation (6) to become valid, i.e., $a_0+k\cdot\Delta a \cong 3W/4$, the stresses σ_i for i>k shall be determined no longer by (5) but by

$$K_{Irs}(\mathrm{a}_i) = \sum_{j=0}^{i}\left\{\frac{3.97\sigma_j\cdot\Delta \mathrm{a}\cdot[0.264W+0.736\,\mathrm{a}_i-(\mathrm{a}_0+j\cdot\Delta \mathrm{a})]}{(W-\mathrm{a}_i)^{3/2}} + \frac{1.46\sigma_j\Delta \mathrm{a}}{(W-\mathrm{a}_i)^{1/2}}\right\} + 2\sigma_i\sqrt{\frac{2}{\pi}q\cdot(W-\mathrm{a}_i)} \tag{9}$$

where $a_i=a_0+i\cdot\Delta a$

This procedure to obtain stresses from SIFs given as a function of crack length can be validated by applying it to a case of hypothetical residual stress distribution where the exact solution of the SIF is known. As an examaple, consider a radial edge crack of length a in a circular disk of a diameter D containing (in the uncracked state) a stress field given by

$$\sigma_{rs}(x)=\sigma_0\cdot\left(6\frac{x^2}{D^2}-6\frac{x}{d}+1\right) \tag{10}$$

where the x axis coincides with a radial plane. For this case, the SIF can be obtained by the superposition of the exact solutions for the two basic load cases that (10) consists of (constant and parabolic), as given in [9] and [10]. This solution is shown in Fig. 3. The weight function for the corresponding crack configuration is given in [7]. The calculated stress distribution (solid line for the results of (5), dashed line for the results of (9)) is shown in Fig. 4. It is in good agreement with the exact one (thin line), although the incremental steps Δa were chosen relatively large ($\Delta a=D/50$). If desired the accuracy could be easily improved by choosing a smaller Δa. As expected because of the inaccuracies discussed above, the deviations between the stress calculated by means of (5) and the exact one increase as the rear surface is approached. In this range (of about $a>3W/4$), (9) delivers obviously better results than (5).

As a real practical example, the SIF obtained experimentally for a radial edge crack (length a) in a cylinder of a diameter D=140mm by two strain gages (as reported in [12]) is considered (Fig. 5). Applying the procedure described above and using the weight function given by (6) leads to the stress distribution shown in Fig. 6. The agreement with the stress distribution obtained by the usual procedure of the crack c mpliance method (as described in [12]) is reasonable. Again, the agreement in ıe range x>0.75D is improved by using (9) instead of (5).

These examples shov that the procedure as described in this report work satisfyingly. It seems to be numerically stable, so the results converge to the exact solution as the chosen crack increments are decreasing. Compared with superposition - based methods (as used in [1,2,13]) it has the advantage that no pre-assumptions of the stress distribution have to be made, so arbitrary stress distributions, containing e.g. steep gradients, local peaks or non symmetrical components, are expected to be calculable with about the same accuracy as the ones shown above as examples.

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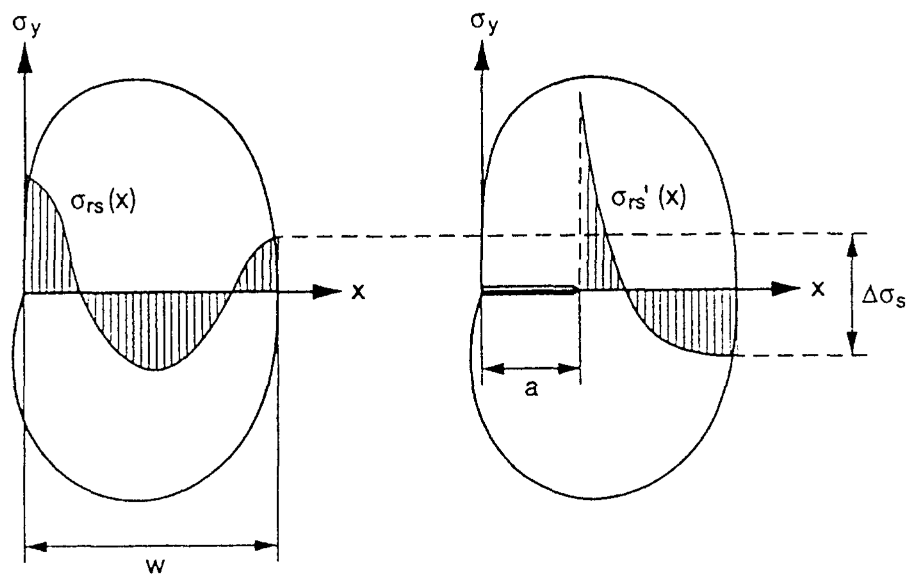

Figure 1. Schematic representation of an arbitrary 2D-body containing residual stresses (left) and the same body with rearranged residual stresses due to an edge crack or cut (right).

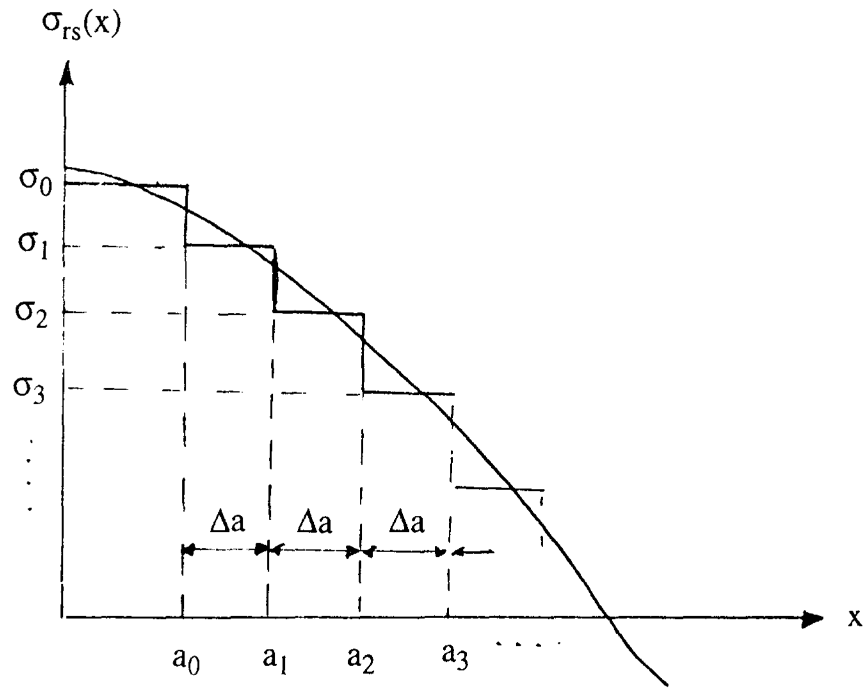

Figure 2. Approximation of the residual stress profile by a step function.

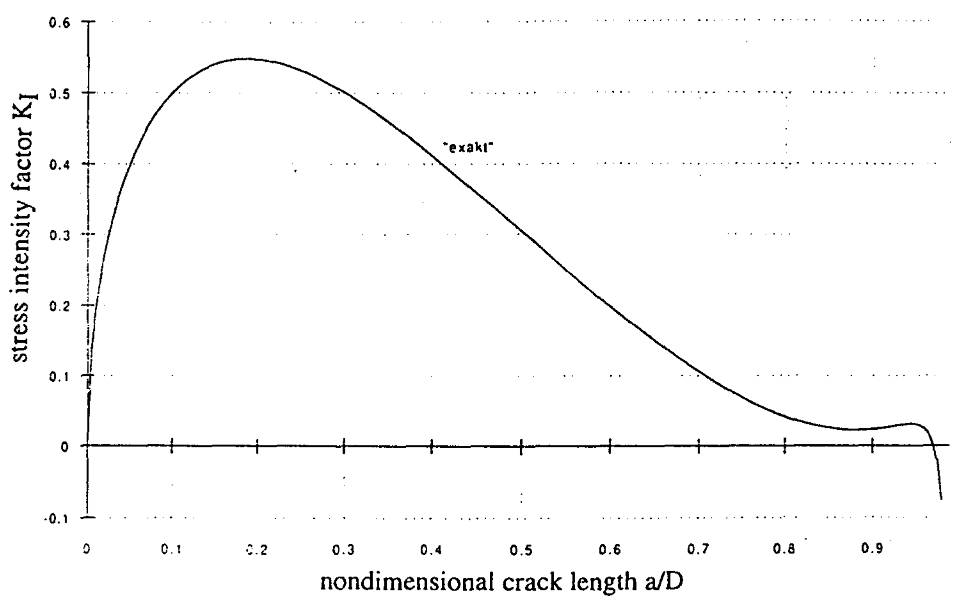

Figure 3. SIF of a radial edge crack in a disk loaded by the stress distribution given by (10).

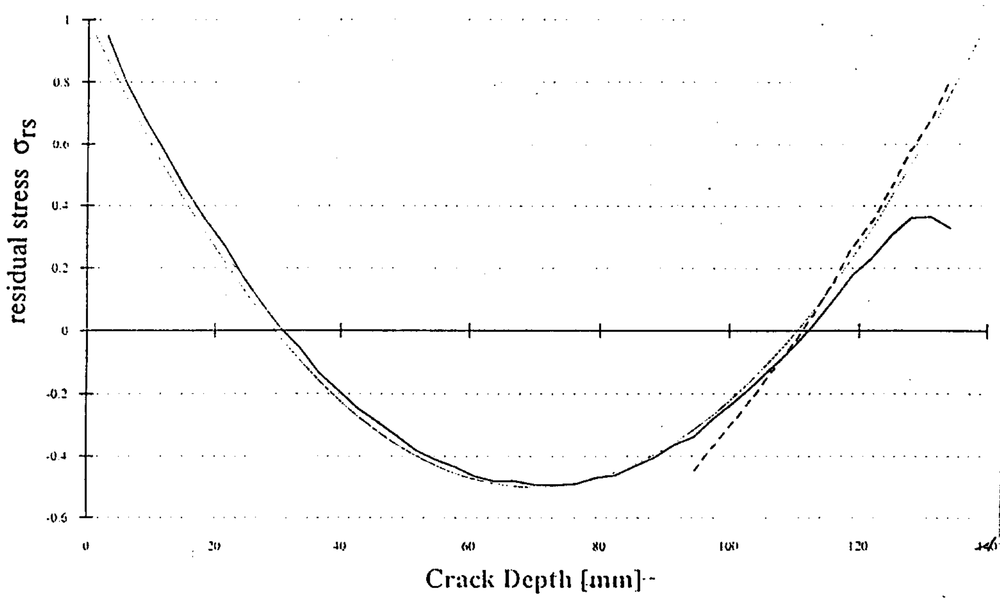

Figure 4. Comparison of calculated stress with the theoretical one as given by (10). The diameter D was assumed to be 140mm.

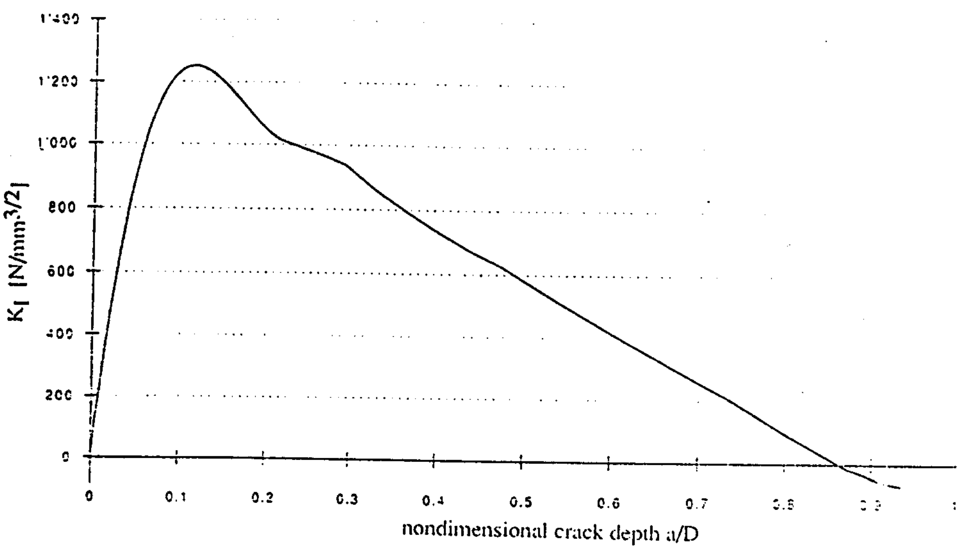

Figure 5. Experimentally determined SIF for a radially edge cracked disk under residual stresses.

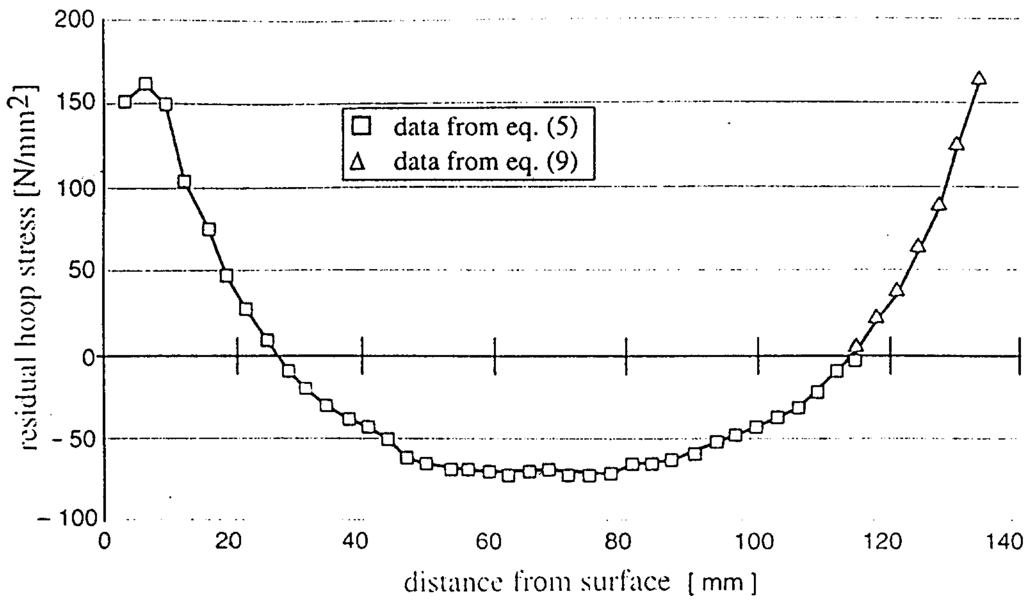

Figure 6. Residual stress derived from the SIF as given in Fig. 5.