International Journal of Fracture 33: 3–16 (1987)

An anisotropic theory of elasticity for continuum damage mechanics

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Received 13 June 1986; accepted 11 September 1986

Abstract

This paper presents the development of an anisotropic elastic damage theory. This is achieved by deriving a modified damage effect tensor $\mathbf{M}(\boldsymbol{D})$ for the effective stress equations capable of including the effect of anisotropic material damage. The modified tensor removes the restriction of a priori knowledge of the directions of principal stresses imposed by a damage effect tensor developed earlier and can now be made for general practical engineering applications of failure analysis. Reduction of the proposed tensor to a scalar for isotropic damage is shown to be possible when it is expressed not only in the principal directions but also in any arbitrary coordinate system, a necessary condition to verify the validity of the proposed tensor. Uniaxial tension and pure torsion are chosen to illustrate the application of the theory as well as associated damage variables that may be experimentally determined using laboratory size specimens. The measured damage variables confirm the presence of anisotropic damage from an initially isotropic material specimen and the magnitude is more pronounced at higher stresses and strains.

1. Introduction

The concept of continuum damage mechanics due to Kachanov [1] has been developed to a stage where the concept can be effectively used to supplement the design of practical engineering structures based on the theory of fracture mechanics [2–23]. Although the damage mechanics provide a measure of material degradation at the micro-mechanics scale due to nucleation and coalescence of voids, cavities, and micro-cracks, the damage variables are introduced to reflect average material degradation in an element at a macro-mechanics scale normally associated with the classical theory of continuum mechanics. This enables the variables to be measured experimentally using laboratory-size specimens recommended for conventional testing standards, thus making possible the application of the concept to solve practical problems using, for example, the finite element analysis [20].

Practising engineers are concerned with the assessment of reliability and life expectancy of engineering components with or without the presence of pre-existing flaws. Accuracy of the assessment based on failure analysis can only be considered acceptable if the analysis chosen yields a reliable stress-strain field at a possible failure site of a structure under service loading conditions. Recent experimental evidences indicate that structural failures are often associated with the development of anisotropic material damage [2,23] even if the initial material properties are isotropic.

Constitutive equations of anisotropic damage have been proposed recently and used to perform a forming limit analysis of metal plates [23]. The damage effect tensor $\mathbf{M}(\boldsymbol{D})$ developed for the effective stress equations was assumed to have a prior knowledge of the principal directions. The assumption was considered acceptable due to the nature of metal forming analysis under investigation. The required knowledge of principal stress directions, however, restricts the application of the effective stress equations developed to general failure analysis.

This paper presents the development of a generalized anisotropic damage theory in elasticity. This is achieved by introducing a modified damage effect tensor $\mathbf{M}(\boldsymbol{D})$ for the effective stress equations which can be applied for general structural analysis. Equations, after coordinate transformation for the tensor, are derived that can be readily incorporated into conventional finite element analysis. Two example cases, namely uniaxial tension and pure torsion, are chosen to illustrate the application of the constitutive equations derived and used to quantify associated damage variables determined experimentally.

2. Damage variables and effective stresses

Based on the theory of continuum damage mechanics, the phenomena of progressive material degradation are introduced in the theory by a number of damage parameters which include effective stress tensor $\tilde{\boldsymbol{\sigma}}$ and damage tensor $\boldsymbol{D}$ of second order, reflecting the damage state of the material under service loading. The damage tensor $\boldsymbol{D}$ may be experimentally determined. The physical implications of the damage variable may be illustrated by considering a damage volume element at macro-scale level shown in Fig. 1. The introduction of the scale level allows the size of the element to be considered to be large enough to contain numerous defects but sufficiently small as a material point that the concept of damage mechanics may be brought within the scope of continuum mechanics.

Let S shown in Fig. 1 be the overall cross-sectional area of the element before loading with its orientation defined by $\underline{n}$. The area S becomes the effective resisting area $\tilde{S}$ after loading due to material degradation caused by the presence of microcracks and cavities, microstress concentrations in the vicinity of discontinuities, and the interactions between the closed defects. If these cracks and cavities are assumed to be uniformly distributed in all directions, S no longer depends on $\underline{n}$ and the isotropic damage variable D may be defined as

$$D = \frac{S - \tilde{S}}{S}. \tag{2.1}$$

The effective Cauchy stress tensor $\tilde{\boldsymbol{\sigma}}$ based on the effective area $\tilde{S}$ is related to the usual Cauchy stress tensor $\boldsymbol{\sigma}$ by

$$\tilde{\boldsymbol{\sigma}} = \boldsymbol{\sigma}\frac{S}{\tilde{S}} = \frac{\boldsymbol{\sigma}}{1 - D}. \tag{2.2}$$

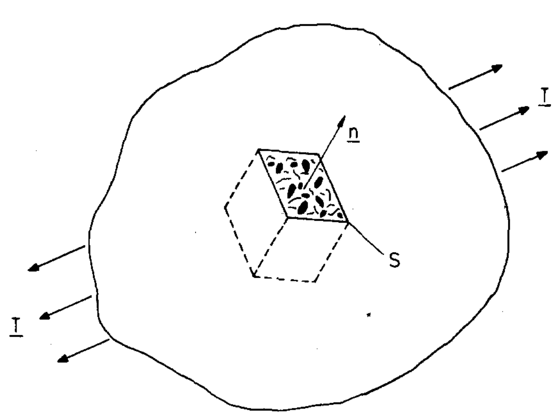

Fig. 1. Damaged material element.

For the anisotropic damage, the effective stress of (2.2) may be expressed in a generalized form as:

$$\tilde{\sigma} = \mathbf{M}(\boldsymbol{D}) : \boldsymbol{\sigma} \tag{2.3}$$

where the symbol (:) means the tensorial product contracted on two indices, and $\mathbf{M}(\boldsymbol{D})$ known as damage effect tensor is a linear symmetric operator represented by a 4th order tensor. In general, $\mathbf{M}(\boldsymbol{D})$ has 21 independent components and may be reduced to a scaler $1/(1-D)$ if the damage effect is isotropic.

Structural analysis with rigorous treatment of material damage through the damage effect tensor $\mathbf{M}(\boldsymbol{D})$ may be mathematically expedient but could prove to be prohibitively time consuming, if not impossible to resolve for practical engineering problems. The question of mathematical tractability is compounded by physical measurement difficulties associated with 21 independent components of the 4th order tensor. The formulation of the damage effect tensor may however be approximated by introducing the concept of proportional loading [12]. The extent to which this approximation yields reliable prediction of material damage from general structural analysis under arbitrary loading condition will be the subject of a separate investigation although initial results have shown promise. When the principal axes of effective stresses $\tilde{\sigma}$ and material damage during loading are assumed to coincide with the conventional stresses $\tilde{\sigma}$, the components of $\mathbf{M}(\boldsymbol{D})$ may be expressed, in the principal coordinate system as

$$M_{ijkl}(\boldsymbol{D}) = A_{ij}(\boldsymbol{D})\delta_{ik}\delta_{jl} \text{ (no summation for } i \text{ and } j). \tag{2.4}$$

If the following notations are chosen,

$$\begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \sigma_4 \\ \sigma_5 \\ \sigma_6 \end{Bmatrix} = \begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{31} \\ \sigma_{12} \end{Bmatrix} \text{ and } \begin{Bmatrix} \epsilon_1 \\ \epsilon_2 \\ \epsilon_3 \\ \epsilon_4 \\ \epsilon_5 \\ \epsilon_6 \end{Bmatrix} = \begin{Bmatrix} \epsilon_{11} \\ \epsilon_{22} \\ \epsilon_{33} \\ 2\epsilon_{23} \\ 2\epsilon_{31} \\ 2\epsilon_{12} \end{Bmatrix} \tag{2.5}$$

(2.3) and (2.4) become

$$\tilde{\sigma}_i = M_{ij}(\boldsymbol{D})\sigma_j \tag{2.6}$$

$$M_{ij}(\boldsymbol{D}) = A_i(\boldsymbol{D})\delta_{ij} \text{ (no summation for } i). \tag{2.7}$$

There are many possible formations of the damage effect tensor M_{ij}. One of the simplest forms is to introduce material damage in the principal directions only [12,23]:

$$[M_{ij}(\mathbf{D})] = \begin{bmatrix} \dfrac{1}{1-D_1} & 0 & 0 \\ 0 & \dfrac{1}{1-D_2} & 0 \\ 0 & 0 & \dfrac{1}{1-D_3} \end{bmatrix} \tag{2.8}$$

where D_1, D_2 and D_3 are the damage variables at their respective principal axes. When the directions of the principal stresses are unknown, the damage effect tensor of (2.8) must be suitably modified. One obvious criterion for developing such a generalized form of the damage effect tensor is that it should be reduced to a scaler for isotropic damage. This reduction should be made possible not only in a principal coordinate system but also in any coordinate system.

One formulation which satisfies the above criteria is developed and expressed, for the first instance, in the principal coordinate system as:

$$[M_{ij}(D)] = \begin{bmatrix} \frac{1}{1-D_1} & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{1-D_2} & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{1-D_3} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{\sqrt{(1-D_2)(1-D_3)}} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{\sqrt{(1-D_3)(1-D_1)}} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{\sqrt{(1-D_1)(1-D_2)}} \end{bmatrix} \tag{2.9}$$

It is obvious that $M_{ij}(\boldsymbol{D})$ of (2.8) is a particular case of (2.9) which can be readily reduced to a scalar for isotropic damage when $D_1 = D_2 = D_3 = D$. The damage variables D_1, D_2 and D_3 in (2.9) refer to, as before, the principal damage components. As $\sigma_4 = \sigma_5 = \sigma_6 = 0$, (2.6) becomes

$$\begin{Bmatrix} \tilde{\sigma}_1 \\ \tilde{\sigma}_2 \\ \tilde{\sigma}_3 \\ \tilde{\sigma}_4 \\ \tilde{\sigma}_5 \\ \tilde{\sigma}_6 \end{Bmatrix} = \begin{Bmatrix} \sigma_1/(1-D_1) \\ \sigma_2/(1-D_2) \\ \sigma_3/(1-D_3) \\ 0 \\ 0 \\ 0 \end{Bmatrix}. \tag{2.10}$$

When $D_1 = D_2 = D_3 = D$, which is the case for isotropic damage, (2.10) is reduced to (2.2) expressed in the principal coordinate systems. A later section will prove that similar reduction is also possible in any coordinate system.

3. Hypothesis of elastic energy equivalence

Lemaitre [24] proposed a hypothesis of strain equivalence for isotropic damage by replacing the conventional stress with the effective stress in the constitutive equation. The hypothesis unfortunately has been proved to lead itself to asymmetry of the stiffness matrix when anisotropic damage is considered. To overcome this inconsistency, the use of the elastic energy equivalence concept was proposed by Sidoroff [12] who postulated that the complementary elastic energy for a damage material is the same in form as that of an undamaged material, except that the stress is replaced by the effective stress in the energy formulation. Since the complementary elastic energy $W^e(\boldsymbol{\sigma}, \boldsymbol{0})$ of an undamaged material ($\boldsymbol{D} = \boldsymbol{0}$) is

$$W^e(\boldsymbol{\sigma}, \boldsymbol{0}) = \tfrac{1}{2}\boldsymbol{\sigma}^T : \mathbf{C}^{-1} : \boldsymbol{\sigma} \tag{3.1}$$

then the complementary energy of a damage material is expressed as

$$W^e(\boldsymbol{\sigma}, \boldsymbol{D}) = W^e(\tilde{\boldsymbol{\sigma}}, \boldsymbol{0}) = \tfrac{1}{2}\tilde{\boldsymbol{\sigma}}^T : \mathbf{C}^{-1} : \tilde{\boldsymbol{\sigma}} = \tfrac{1}{2}\boldsymbol{\sigma} : (\mathbf{M}^T : \mathbf{C}^{-1} : \mathbf{M}) : \boldsymbol{\sigma} \tag{3.2}$$

where $\mathbf{C}$ is the elastic stiffness tensor. Defining the effective stiffness tensor $\tilde{\mathbf{C}}$ as

$$\tilde{\mathbf{C}} = \mathbf{M}^{-1} : \mathbf{C} : \mathbf{M}^{T,-1} \tag{3.3}$$

the linear elastic stress-strain equation for damage material may then be written as

$$\boldsymbol{\epsilon}^e = \frac{\partial W^e(\boldsymbol{\sigma}, \boldsymbol{D})}{\partial \boldsymbol{\sigma}} = (\mathbf{M}^T : \mathbf{C}^{-1} : \mathbf{M}) : \boldsymbol{\sigma} = \tilde{\mathbf{C}}^{-1} : \boldsymbol{\sigma}. \tag{3.4}$$

If the stress components are expressed as the vector shown in (2.5), the elastic matrix for isotropic material is

$$[C]^{-1}=\frac{1}{E}\begin{bmatrix} 1 & -\nu & -\nu & 0 & 0 & 0 \\ & 1 & -\nu & 0 & 0 & 0 \\ & & 1 & 0 & 0 & 0 \\ & & & 2(1+\nu) & 0 & 0 \\ & S & & & 2(1+\nu) & 0 \\ & & & & & 2(1+\nu) \end{bmatrix}. \tag{3.5}$$

Substituting (2.9) and (3.5) into (3.3), we obtain the effective elastic matrix for damaged material in the principal coordinate system as:

$$[\bar{C}]^{-1}=\frac{1}{E}\begin{bmatrix} \frac{1}{(1-D_1)^2} & \frac{-\nu}{(1-D_1)(1-D_2)} & \frac{-\nu}{(1-D_1)(1-D_3)} & 0 & 0 & 0 \\ & \frac{1}{(1-D_2)^2} & \frac{-\nu}{(1-D_2)(1-D_3)} & 0 & 0 & 0 \\ & & \frac{1}{(1-D_3)^2} & 0 & 0 & 0 \\ & & & \frac{2(1+\nu)}{(1-D_2)(1-D_3)} & 0 & 0 \\ S & & & & \frac{2(1+\nu)}{(1-D_3)(1-D_1)} & 0 \\ & & & & & \frac{2(1+\nu)}{(1-D_1)(1-D_2)} \end{bmatrix} \tag{3.6}$$

4. Transformation of damage tensor and damage effect tensor

The proposed damage effect tensor $\mathbf{M}(\boldsymbol{D})$ shown in (2.9) was, for the sake of illustration, expressed initially in the principal stress directions. In terms of a general coordinate system, the tensor may be derived using the law of coordinate transformation.

Let the principal stress axes be x_1, x_2, x_3. In an arbitrary Cartensian rectangular coordinate system of x_1', x_2', x_3', see Fig. 2, the usual convention of direction cosines between these two coordinate systems are adopted:

	x_1	x_2	x_3
x_1'	l_1	m_1	n_1
x_2'	l_2	m_2	n_2
x_3'	l_3	m_3	n_3

The stresses, damage variables and damage effect tensor may be expressed in terms of x_i' coordinates as

$$\sigma_{ij}'=q_{is}q_{jt}\sigma_{st} \tag{4.1}$$

$$D_{ij}'=q_{is}q_{jt}D_{st} \tag{4.2}$$

$$M_{ijkl}'=q_{is}q_{jt}q_{ku}q_{lv}=M_{stuv} \tag{4.3}$$

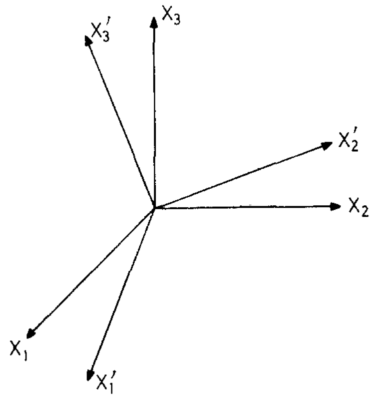

Fig. 2. Cartesian coordinate system.

where

$$[q]=[q_{ij}]=\left[\frac{\partial x_i'}{\partial x_j}\right]=\begin{bmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ l_3 & m_3 & n_3 \end{bmatrix} \tag{4.4}$$

since

$D_{ij}=D_i\delta_{ij}$ (no summation for i)

(4.2) becomes

$$[D_{ij}']=\begin{bmatrix} D_1l_1^2+D_2m_1^2+D_3n_1^2 & D_1l_1l_2+D_2m_1m_2+D_3n_1n_2 & D_1l_1l_3+D_2m_1m_3+D_3n_1n_3 \\ & D_1l_2^2+D_2m_2^2+D_3n_2^2 & D_1l_2l_3+D_2m_2m_3+D_3n_2n_3 \\ S & & D_1l_3^2+D_2m_3^2+D_3n_3^2 \end{bmatrix}. \tag{4.5}$$

For isotropic damage, $D_1=D_2=D_3=D$ and noting that $l_il_j+m_im_j+n_in_j=\delta_{ij}$, (4.5) reduces to

$$[D_{ij}']=\begin{bmatrix} D & 0 & 0 \\ 0 & D & 0 \\ 0 & 0 & D \end{bmatrix}=[D_{ij}]. \tag{4.6}$$

It may be of interest to note that the damage tensor of (4.6) for isotropic damage contains an identical independent component in any reference system and that only a scalar D is needed to represent the damage state.

The 4th order damage effect tensor $\mathbf{M}(\boldsymbol{D})$ expressed in the principal coordinate system has been proposed and shown in (2.9) with a 6×6 matrix. In order to prove that the damage effect tensor is also applicable to any arbitrary coordinate system, the effective stress tensor may be written in the vector form similar to (2.5) so that the transformation law of (4.1) may be expressed as

$$\{\tilde{\sigma}'\}=[Q]\{\tilde{\sigma}\}=[Q]([M]\{\sigma\})=([Q][M][Q]^{-1})\{\sigma'\}=[M']\{\sigma'\} \tag{4.7}$$

where

$[M']$ is defined as $[Q][M][Q]^{-1}$

or

$$M'_{ij}=\frac{l_i^2l_j^2}{W_1}+\frac{m_i^2m_j^2}{W_2}+\frac{n_i^2n_j^2}{W_3}+\frac{2m_in_im_jn_j}{W_{23}}+\frac{2n_il_in_jl_j}{W_{31}}+\frac{2l_im_il_jm_j}{W_{12}}\quad \text{for } i,\ j=1,2,3$$

$$M'_{ii}=\frac{2l_3^2l_{6-i}^2}{W_1}+\frac{2m_3^2m_{6-i}^2}{W_2}+\frac{2n_3^2n_{6-i}^2}{W_3}+\frac{(m_3n_{6-i}+n_3m_{6-i})^2}{W_{23}}+\frac{(n_3l_{6-i}+l_3n_{6-i})^2}{W_{31}}$$

$$+\frac{(l_3m_{6-i}+m_3l_{6-i})^2}{W_{12}}\quad \text{for } i=4,5$$

$$M'_{66}=\frac{2l_1^2l_2^2}{W_1}+\frac{2m_1^2m_2^2}{W_2}+\frac{2n_1^2n_2^2}{W_3}+\frac{(m_1n_2+n_1m_2)^2}{W_{23}}+\frac{(n_1l_2+l_1n_2)^2}{W_{31}}+\frac{(l_1m_2+m_1l_2)^2}{W_{12}}$$

$$M'_{ij}=\frac{l_3l_{6-i}l_j^2}{W_1}+\frac{m_3m_{6-i}m_j^2}{W_2}+\frac{n_3n_{6-i}n_j^2}{W_3}+\frac{m_jn_j(m_3n_{6-i}+n_3m_{6-i})}{W_{23}}$$

$$+\frac{n_jl_j(n_3l_{6-i}+l_3n_{6-i})}{W_{31}}+\frac{l_jm_j(l_3m_{6-i}+m_3l_{6-i})}{W_{12}}\quad \text{for } i=4,5;\ j=1,2,3$$

$$M'_{54}=\frac{2l_1l_2l_3^2}{W_1}+\frac{2m_1m_2m_3^2}{W_2}+\frac{2n_1n_2n_3^2}{W_3}+\frac{(m_2n_3+m_3n_2)(m_1n_3+m_3n_1)}{W_{23}}$$

$$+\frac{(n_2l_3+l_2n_3)(n_3l_1+l_3n_1)}{W_{31}}+\frac{(l_2m_3+m_2l_3)(l_3m_1+m_3l_1)}{W_{12}}$$

$$M'_{6j}=\frac{l_1l_2l_j^2}{W_1}+\frac{m_1m_2m_j^2}{W_2}+\frac{n_1n_2n_j^2}{W_3}+\frac{m_jn_j(m_1n_2+m_2n_1)}{W_{23}}$$

$$+\frac{n_jl_j(n_1l_2+l_1n_2)}{W_{31}}+\frac{l_jm_j(l_1m_2+m_1l_2)}{W_{12}}\quad \text{for } j=1,2,3$$

$$M'_{6j}=\frac{2l_1l_2l_3l_{6-j}}{W_1}+\frac{2m_1m_2m_3m_{6-j}}{W_2}+\frac{2n_1n_2n_3n_{6-j}}{W_3}+\frac{(m_3n_{6-j}+n_3m_{6-j})(m_1n_2+n_1m_2)}{W_{23}}$$

$$+\frac{(n_3l_{6-j}+l_3n_{6-j})(n_1l_2+n_2l_1)}{W_{31}}+\frac{(l_3m_{6-j}+m_3l_{6-j})(l_1m_2+m_1l_2)}{W_{12}}\quad \text{for } j=4,5$$

$$M'_{ji}=2M'_{ij}\quad \text{for } i=4,5,6,\ j=1,2,3$$

$$M'_{ji}=M'_{ij}\quad \text{for } i=5,6,\ j=4,5$$

and

$$W_1=1-D_1;\quad W_2=1-D_2;\quad W_3=1-D_3;\quad W_{23}=\sqrt{(1-D_2)(1-D_3)}\,;$$

$$W_{31}=\sqrt{(1-D_3)(1-D_1)}\,;\quad W_{12}=\sqrt{(1-D_1)(1-D_2)}\,. \tag{4.8}$$

It can be readily verified from the above that the matrix representation $[M]$ of the symmetric 4th order tensor is not symmetric in any reference system except the principal one. For the isotropic case $D_1=D_2=D_3=D$,

$$M'_{ij}=\frac{1}{1-D}\delta_{ij}=M_{ij}$$

which again, reduces to a scalar $\dfrac{1}{1-D}$ and is compatible with the isotropic theory.

5. Example cases

In order to illustrate the application of the concepts of damage mechanics that have been developed in the preceding sections and the damage variables that may be evaluated, the damage analyses for uniaxial tension and pure torsion are performed.

5.1. Uniaxial tension

An important ingredient in the derivation of damage tensor equations is that they should be readily reduced to material parameters measurable from uniaxial test. For instance, by substituting (3.6) into (3.4), the constitutive equations under tension are

$$\begin{aligned}
\epsilon_1^e &= \frac{\sigma_1}{E(1-D_1)^2} = \frac{\sigma_1}{\tilde{E}} \\
\epsilon_2^e &= -\frac{\nu\sigma_1}{E(1-D_1)(1-D_2)} = -\frac{\tilde{\nu}_{12}}{\tilde{E}}\sigma_1 \\
\epsilon_3^e &= -\frac{\nu\sigma_1}{E(1-D_1)(1-D_3)} = -\frac{\tilde{\nu}_{13}}{\tilde{E}}\sigma_1
\end{aligned} \tag{5.1}$$

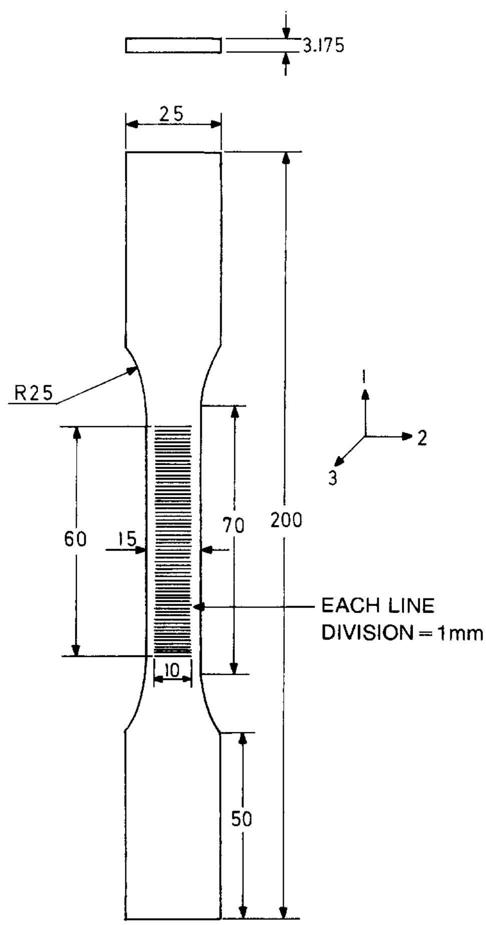

Fig. 3. Tensile specimen used for damage measurement.

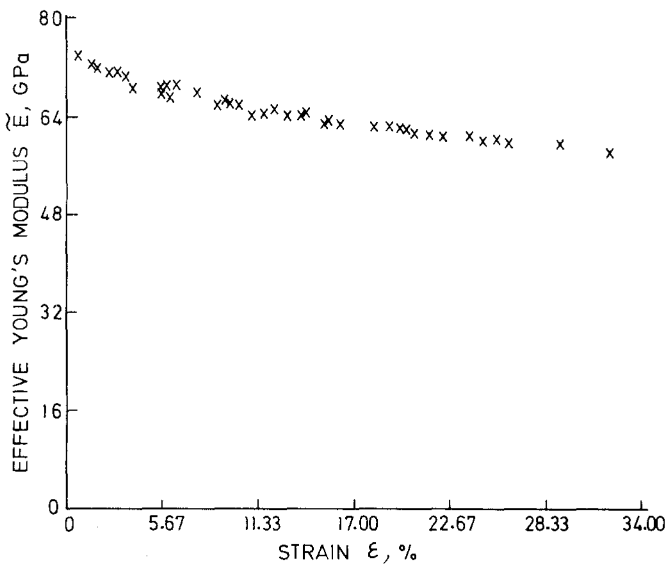

Fig. 4. Effective Young's modulus versus strain.

where

$$\begin{aligned}\tilde{E} &= E(1 - D_1)^2 \\ \tilde{\nu}_{12} &= \nu(1 - D_1)/(1 - D_2) \\ \tilde{\nu}_{13} &= \nu(1 - D_1)/(1 - D_3)\end{aligned} \tag{5.2}$$

are the effective Young's modulus and Poisson's ratios. Accordingly, the damage variables D_1, D_2 and D_3 may be evaluated as

$$D_1 = 1 - \left(\frac{\tilde{E}}{E}\right)^{1/2}; \quad D_2 = 1 - \frac{\nu}{\tilde{\nu}_{12}}(1 - D_1)$$

$$D_3 = 1 - \frac{\nu}{\tilde{\nu}_{13}}(1 - D_1). \tag{5.3}$$

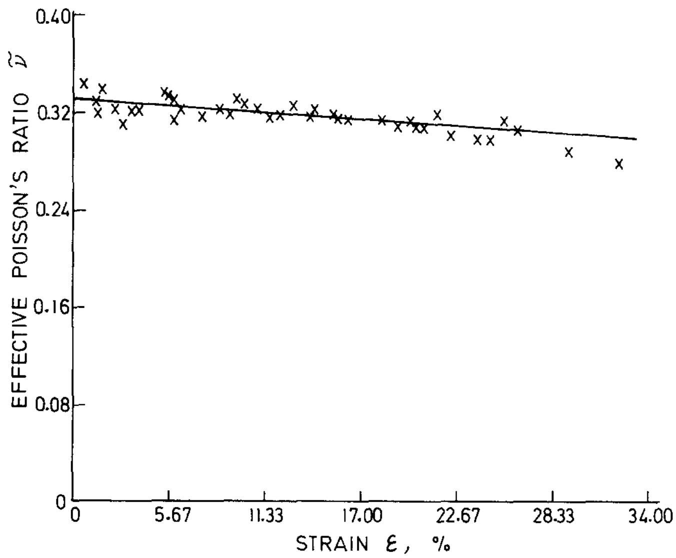

Fig. 5. Effective Poisson's ratio versus strain.

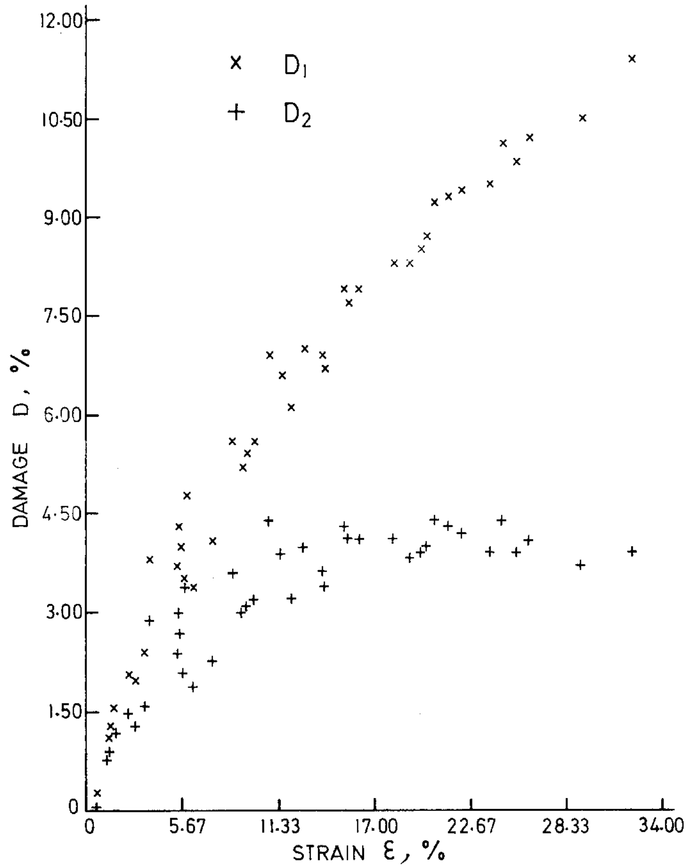

Fig. 6. Damage variables of D_1 and D_2 versus strain.

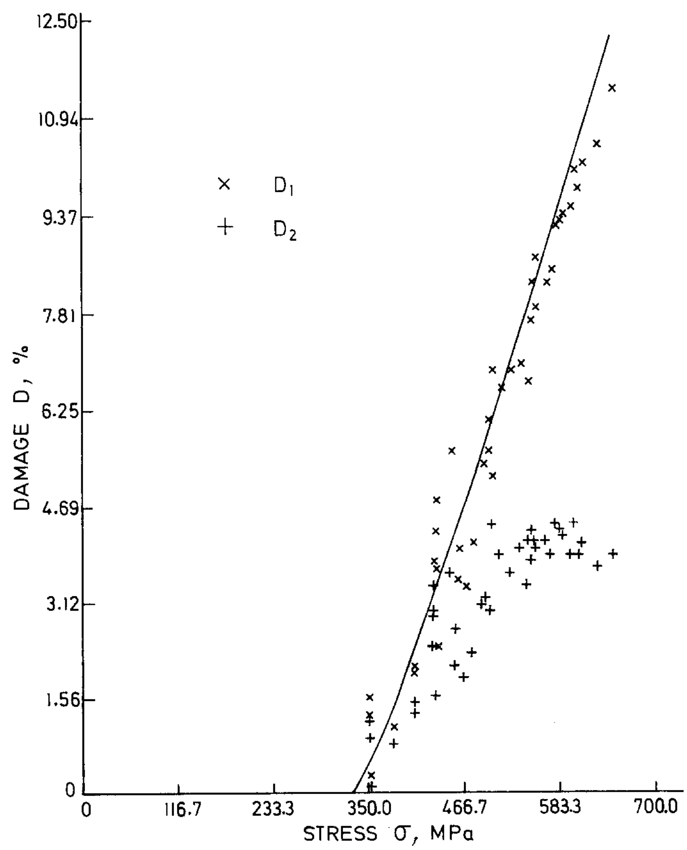

Fig. 7. Damage variables of D_1 and D_2 versus stress.

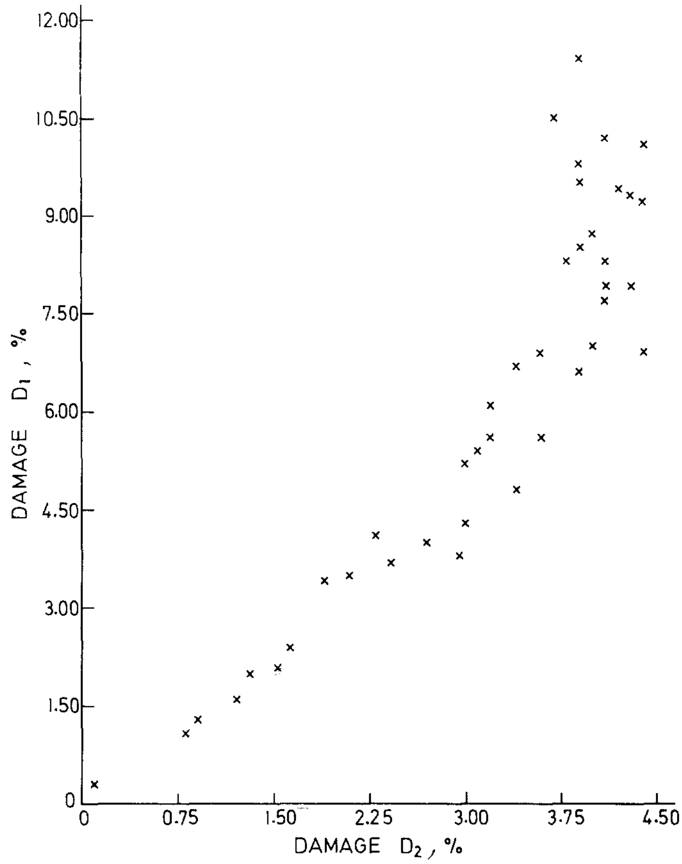

Fig. 8. Damage variables D_1 versus D_2.

A series of tests was performed to measure the above damage variables $\tilde{E}$, $\tilde{\nu}$ and $\boldsymbol{D}$. The measurements were obtained from tensile specimens of aluminium alloy 2024-T3 whose dimensions are shown in Fig. 3. The measured values of $\tilde{E}$ and $\tilde{\nu}$ are depicted in Figs. 4 and 5 respectively, revealing, as expected, gradual material degradation with the increase of strain. The damage variables D_1 and D_2 are evaluated using (5.3) based on the measured values of $\tilde{E}$ and $\tilde{\nu}$ and described respectively in Figs. 6 and 7. The graphs confirm the presence of anisotropic damage which becomes increasingly pronounced at the higher strains and stresses from an initially isotropic material prior to the load application. The D_3-value should on the other hand be identical to D_2 for the isotropic material under uniaxial tension. The degree of material damage in D_1 and D_2 are depicted in Fig. 8 revealing, as expected, that the damage is marked along the loading direction.

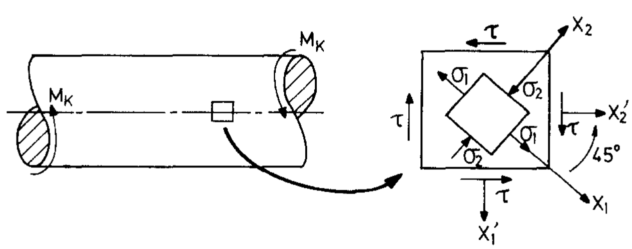

Fig. 9. Pure torsion.

5.2. Pure torsion

For a shaft under pure torsion with the maximum shear stress τ, Fig. 9, the principal stresses are $\sigma_1 = -\sigma_2 = \tau$. For the plane stress problem, only 3×3 matrix representations of **C** and **M** are needed. In the principal coordinate system of x_1, x_2, Fig. 9, the constitutive equation is

$$\begin{Bmatrix} \epsilon_1^e \\ \epsilon_2^e \\ \gamma_{12}^e \end{Bmatrix} = \frac{1}{E} \begin{bmatrix} \frac{1}{(1-D_1)^2} & \frac{-\nu}{(1-D_1)(1-D_2)} & 0 \\ \frac{-\nu}{(1-D_1)(1-D_2)} & \frac{1}{(1-D_2)^2} & 0 \\ 0 & 0 & \frac{2(1+\nu)}{(1-D_1)(1-D_2)} \end{bmatrix} \begin{Bmatrix} \tau \\ -\tau \\ 0 \end{Bmatrix} \tag{5.4}$$

from which we have

$$\epsilon_1^e + \epsilon_2^e = \frac{\tau}{E}\left[\frac{1}{(1-D_1)^2} - \frac{1}{(1-D_2)^2}\right] \tag{5.5}$$

$$\frac{\epsilon_1^e}{-\epsilon_2^e} = \frac{1-D_2}{1-D_1} \frac{\frac{1-D_2}{1-D_1} + \nu}{1 + \frac{1-D_2}{1-D_1}\nu}. \tag{5.6}$$

If we define

$$X = \frac{1}{(1-D_1)^2} - \frac{1}{(1-D_2)^2} \tag{5.7}$$

$$Y = \frac{1-D_2}{1-D_1} \tag{5.8}$$

Eqns. (5.5) and (5.6) become

$$X = \frac{E(\epsilon_1^e + \epsilon_2^e)}{\tau} \tag{5.9}$$

$$Y = \frac{1}{2}\left[-\nu\left(1 + \frac{\epsilon_1^e}{\epsilon_2^e}\right) + \sqrt{\nu^2\left(1 + \frac{\epsilon_1^e}{\epsilon_2^e}\right)^2 - 4\frac{\epsilon_1^e}{\epsilon_2^e}}\right]. \tag{5.10}$$

Then, solving (5.7) and (5.8) gives

$$D_2 = 1 - \sqrt{\frac{Y^2-1}{X}} = 1 - \sqrt{\frac{\tau}{-E\epsilon_2^e}(1+\nu Y)} \tag{5.11}$$

$$D_1 = 1 - \frac{1}{Y}\sqrt{\frac{Y^2-1}{X}} = 1 - \frac{1}{Y}\sqrt{\frac{\tau}{-E\epsilon_2^e}(1+\nu Y)}. \tag{5.12}$$

The effective Young's moduli and Poisson's ratios are thus deduced as

$$\tilde{E}_1 = E(1-D_1)^2 = \frac{1}{Y^2}\frac{\tau}{-\epsilon_2^e}(1+\nu Y)$$

$$\tilde{E}_2 = E(1-D_2)^2 = \frac{\tau}{-\epsilon^e}(1+\nu Y)$$

$$\tilde{\nu}_{12} = \nu\frac{1-D_1}{1-D_2} = \nu/Y$$

$$\tilde{\nu}_{21} = \nu\frac{1-D_2}{1-D_1} = \nu Y.$$

Since

$$l_1 = -m_1 = l_2 = m_2 = \frac{1}{\sqrt{2}}, \quad l_3 = m_3 = n_1 = n_2 = 0, \quad \text{and } n_3 = 1$$

(4.8) is expressed in the x_1', x_2' system as

$$[M_{ij}]' = \frac{1}{4}\begin{bmatrix} \left(\frac{1}{\sqrt{1-D_1}} + \frac{1}{\sqrt{1-D_2}}\right)^2 & \left(\frac{1}{\sqrt{1-D_1}} - \frac{1}{\sqrt{1-D_2}}\right)^2 & 2\left(\frac{1}{1-D_1} - \frac{1}{1-D_2}\right) \\ \left(\frac{1}{\sqrt{1-D_1}} - \frac{1}{\sqrt{1-D_2}}\right)^2 & \left(\frac{1}{\sqrt{1-D_1}} + \frac{1}{\sqrt{1-D_2}}\right)^2 & 2\left(\frac{1}{1-D_1} - \frac{1}{1-D_2}\right) \\ \left(\frac{1}{1-D_1} - \frac{1}{1-D_2}\right) & \left(\frac{1}{1-D_1} - \frac{1}{1-D_2}\right) & 2\left(\frac{1}{1-D_1} + \frac{1}{1-D_2}\right) \end{bmatrix} \tag{5.13}$$

and the effective shear modulus $\tilde{G}$ is evaluated as

$$\frac{1}{\tilde{G}} = \tilde{C}_{33}^{\prime -1} = C_{ij}^{-1} M_{i3} M_{j3} = \frac{1}{E}\left[\frac{1}{(1-D_1)^2} + \frac{1}{(1-D_2)^2} + \frac{2\nu}{(1-D_1)(1-D_2)}\right]$$

or

$$\tilde{G} = \frac{1}{\tilde{C}_{33}^{\prime -1}} = E \Bigg/ \left[\frac{1}{(1-D_1)^2} + \frac{1}{(1-D_2)^2} + \frac{2\nu}{(1-D_1)(1-D_2)}\right]$$

$$= 2(1+\nu)G \Bigg/ \left[\frac{1}{(1-D_1)^2} + \frac{1}{(1-D_2)^2} + \frac{2\nu}{(1-D_1)(1-D_2)}\right]. \tag{5.14}$$

Finally the shearing constitutive equation including anisotropic material damage is expressed as

$$\gamma = \frac{1}{\tilde{G}}\tau. \tag{5.15}$$

It can be observed from the above analysis that the parameters of X and Y shown in (5.9) and (5.10) respectively may be readily evaluated from the measurement of strains, material constants and applied torque. These parameters enable the determination of the damage variables of D_1, D_2, $\tilde{E}_1$, $\tilde{E}_2$, $\tilde{\nu}_{12}$ and $\tilde{\nu}_{21}$. The constitutive equation of (5.15) may then be solved after the evaluation of $\tilde{G}$ of (5.14) based on the computed damage variables.

6. Conclusions

In the development of a general theory of continuum damage mechanics, it is necessary to include the effect of material anisotropic damage at the possible site of fracture. Constitutive equations have been derived to take into account the effect of the anisotropic damage. This is achieved by developing a modified damage effect tensor associated with the effective stress equations which can now be used for general practical engineering application for failure analysis. A necessary condition to verify the validity of the proposed tensor is satisfied when reduction of the tensor to a scalar for isotropic damage is shown to be possible not only in the principal stress directions but also in any arbitrary coordinate system. The effective stress equations have also been presented in such a way that they can be readily incorporated into the finite element analysis.

Two example cases, namely uniaxial tension and pure torsion, have been chosen to illustrate the application of the proposed damage effect tensor. Damage variables have been measured from tensile specimens and confirmed the presence of anisotropic damage from initially isotropic material of aluminium 2024-T3. The anisotropic damage is shown to be pronounced at higher values of stress and strain.

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Résumé

On présente un développement d'une théorie sur l'endommagement élastique anisotrope en déduisant un tenseur modifié décrivant l'effet de l'endommagement pour un système d'équations de contraintes effectives susceptible d'inclure l'effet d'un endommagement dans un matériau anisotrope. Le tenseur modifié supprime la restriction de la connaissance a priori des directions des contraintes principales imposées par un tenseur d'effet d'endommagement développé précédemment; il peut à présent entrer dans les applications pratiques en construction de l'analyse des ruptures.

On montre qu'il est possible de réduire le tenseur proposé à une valeur scalaire dans le cas d'un dommage isotrope, dès lors qu'il est exprimé non seulement suivant les directions principales, mais dans un système de coordonnées arbitraires, ce qui est une condition nécessaire pour en vérifier la validité.

On choisit une traction multiaxiale et une torsion pure pour illustrer l'application de la théorie ainsi que des variables d'endommagement associées, susceptibles d'être déterminées expérimentalement à l'aide d'éprouvettes de laboratoire.

Les variables d'endommagement mesurées confirment la présence d'un dommage anisotrope dans le cas d'une éprouvette d'un matériau initialement isotrope; son amplitude est plus prononcée à des contraintes ou des déformations plus importantes.