International Journal of Fracture **60:** R9–R13, 1993.

DERIVATIONS OF TWO-DIMENSIONAL GREEN'S FUNCTIONS FOR BIMATERIALS BY MEANS OF COMPLEX VARIABLE FUNCTION TECHNIQUE

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The Green's functions discussed below play an important role in solid mechanics. For example, the formulation of the curved crack problem and the boundary integral equation depends on these functions. Using the three-dimensional Green's functions for bimaterials and performing integration along one axis, the two-dimensional Green's functions (TDGFs) are obtainable [1]. In this note, an alternative method for deriving the TDGFs is suggested. The suggested approach and obtained results are compact in form and easy to understand.

It is well known that in the complex variable function method the stresses (σ_x, σ_y, σ_{xy}), the resultant force functions (X,Y), and the displacements (u,v) can be described by two complex potentials $\phi(z)$ and $\psi(z)$ [2]

$$\sigma_x + \sigma_y = 4Re\phi'(z)$$

$$\sigma_y - \sigma_x + i\sigma_{xy} = 2[\bar{z}\phi''(z) + \psi'(z)] \tag{1}$$

$$f = -Y + iX = \phi(z) + z\overline{\phi'(z)} + \overline{\psi(z)} \tag{2}$$

$$2G(u+iv) = \kappa\phi(z) - z\overline{\phi'(z)} - \overline{\psi(z)} \tag{3}$$

where G is the shear modulus of elasticity, $\kappa = 3-4\nu$ for the plane strain problem, and ν is Poisson's ratio.

Suppose the singular point z=t is located in the upper half-plane (Fig. 1), and the upper (lower) half-plane possesses the elastic constants G_1, ν_1, $\kappa_1=3-4\nu_1$ (G_2, ν_2, $\kappa_2 = 3-4\nu_2$), respectively. The complex potentials $\phi_1(z)$ and $\psi_1(z)$ are defined in the upper half-plane S^+, and they may be expressed in the form

$$\phi_1(z) = \phi_{1p}(z) + \phi_{1c}(z)$$

$$\psi_1(z) = \psi_{1p}(z) + \psi_{1c}(z) \tag{4}$$

where $\phi_{1p}(z)$ and $\psi_{1p}(z)$ represent the singular part of $\phi_1(z)$ and $\psi_1(z)$. We express these two functions by

$$\phi_{1p}(z)=a_o\,\mathrm{Log}(z-t)+\sum_{n=1}^{N}\frac{a_n}{(z-t)^n}$$

$$\psi_{1p}(z)=b_o\,\mathrm{Log}(z-t)+\sum_{n=1}^{N}\frac{b_n}{(z-t)^n} \tag{5}$$

Note that these two functions can be defined on the whole plane except for the point z=t. In addition, the complex potentials $\phi_2(z)$ and $\psi_2(z)$ are defined in the lower half-plane S^- in Fig. 1.

The continuation of the displacements and the tractions along the real axis leads to

$$(\phi_1(x)+x\overline{\phi'_1(x)}+\overline{\psi_1(x)})^+=(\phi_2(x)+x\overline{\phi'_2(x)}+\overline{\psi_2(x)})^-$$

$$G_2(\kappa_1\phi_1(x)-x\overline{\phi'_1(x)}-\overline{\psi_1(x)})^+=G_1(\kappa_2\phi_2(x)-x\overline{\phi'_2(x)}-\overline{\psi_2(x)})^- \tag{6}$$

The solution of (6) has been obtained and can be expressed as follows [3]

$$\begin{bmatrix}\phi_1(z)\\ \psi_1(z)\end{bmatrix}=I\begin{bmatrix}\phi_{1p}(z)\\ \psi_{1p}(z)\end{bmatrix}+I_1\begin{bmatrix}\overline{\phi}_{1p}(z)\\ \overline{\psi}_{1p}(z)\end{bmatrix}+I_2\begin{bmatrix}z\overline{\phi}'_{1p}(z)\\ z\overline{\psi}'_{1p}(z)\end{bmatrix}+I_3\begin{bmatrix}z^2\overline{\phi}''_{1p}(z)\\ z^2\overline{\psi}''_{1p}(z)\end{bmatrix} \tag{7}$$

$$\begin{bmatrix}\phi_2(z)\\ \psi_2(z)\end{bmatrix}=J_1\begin{bmatrix}\phi_{1p}(z)\\ \psi_{1p}(z)\end{bmatrix}+J_2\begin{bmatrix}z\phi'_{1p}(z)\\ z\psi'_{1p}(z)\end{bmatrix} \tag{8}$$

where the matrices have the following expressions

$$I=\begin{bmatrix}1&0\\0&1\end{bmatrix},\quad I_1=\begin{bmatrix}0&\beta\\ \alpha&0\end{bmatrix},\quad I_2=\begin{bmatrix}\beta&0\\-\beta&-\beta\end{bmatrix},\quad I_3=\begin{bmatrix}0&0\\-\beta&0\end{bmatrix}$$

$$J_1=\begin{bmatrix}1+\alpha&0\\0&1+\beta\end{bmatrix},\quad J_2=\begin{bmatrix}0&0\\ \beta-\alpha&0\end{bmatrix} \tag{9}$$

and

$$\alpha=\frac{\kappa_1G_2-\kappa_2G_1}{G_2+\kappa_2G_1},\quad \beta=\frac{G_2-G_1}{G_1+\kappa_1G_2} \tag{10}$$

The matrices I_1, I_2 and I_3 represent the reflection ones, and the matrices J_1 and J_2 represent the transmission ones.

For reference several particularly important cases for the singular source are written below:

(a) If a concentrated dislocation with intensity $D = D_1 + iD_2$ is applied at the point z=t (Fig. 2(a)), we have [4]

$$\phi_{1p}(z) = D\,\mathrm{Log}(z-t), \qquad \psi_{1p} = \overline{D}\,\mathrm{Log}(z-t) - D\frac{\bar{t}}{z-t} \tag{11}$$

(b) If a concentrated force with intensity $F_x + iF_y$ is applied at the point z=t (Fig. 2(b)), we have [5]

$$\phi_{1p}(z) = P\,\mathrm{Log}(z-t), \qquad \psi_{1p}(z) = -\kappa_1\overline{P}\,\mathrm{Log}(z-t) - P\frac{\bar{t}}{z-t} \tag{12}$$

where

$$P = P_1 + iP_2 = -\frac{F_x + iF_y}{2\pi(1+\kappa_1)} \tag{13}$$

(c) If a double force F_x is applied at the points z=t and z=t+dx (Fig. 2(c)), after letting $\mathrm{Lim}(-F_x\,dx/(2\pi(1+\kappa_1)) = N_1$, it follows

$$\phi_{1p}(z) = -N_1\frac{1}{z-t}, \quad \psi_{1p}(z) = (\kappa_1 - 1)N_1\frac{1}{z-t} - N_1\frac{\bar{t}}{(z-t)^2} \tag{14}$$

(d) If a double force F_y is applied at the points z=t and z=t+dx (Fig. 2(d)), after letting $\mathrm{Lim}(-F_y dx/(2\pi(1+\kappa_1))= N_2$, it follows

$$\phi_{1p}(z) = -iN_2\frac{1}{z-t}, \quad \psi_{1p}(z) = -i(\kappa_1 + 1)N_2\frac{1}{z-t} - iN_2\frac{\bar{t}}{(z-t)^2} \tag{15}$$

(e) If a double force F_y is applied at the points z=t and z=t+idy (Fig. 2(e)), after letting $\mathrm{Lim}(-F_y dy/(2\pi(1+\kappa_1)) = N_3$, we find

$$\phi_{1p}(z) = N_3\frac{1}{z-t}, \quad \psi_{1p}(z) = (\kappa_1 - 1)N_3\frac{1}{z-t} + N_3\frac{\bar{t}}{(z-t)^2} \tag{16}$$

(f) If a double force F_x is applied at the point z=t and z=t+idy (Fig. 2(f)), after letting $\mathrm{Lim}(-F_x dy/(2\pi(1+\kappa_1)) = N_4$, we find

$$\phi_{1p}(z) = -iN_4\frac{1}{z-t}, \quad \psi_{1p}(z) = i(\kappa_1+1)N_4\frac{1}{z-t} - iN_4\frac{\bar{t}}{(z-t)^2} \tag{17}$$

(g)Letting $N_1 = N_5/(2\kappa_1-1))$ in (14) and $N_3=N_5/(2(\kappa_1-1))$ in (16) and using the principle of superposition (Fig. 2(g)), we have

$$\phi_{1p}(z) = 0, \quad \psi_{1p}(z) = N_5\frac{1}{z-t} \tag{18}$$

Physically, this case corresponds a center of dilation place at the point z=t.

(h) Letting $N_2 = -N_6/(2(\kappa_1+1))$ in (15) and $N_4=N_6/(2(\kappa_1+1))$ in (17) and using the principle of superposition (Fig. 2(h)), we have

$$\phi_{1p}(z) = 0, \quad \psi_{1p}(z) = iN_6\frac{1}{z-t} \tag{19}$$

Physically, this case corresponds a moment applied at the point z=t.

Clearly, it is possible to find other particular kinds of singular solution in the form of (5), for example, for the case of a doublet of dislocation. However, it is not easy to find a physical explanation for any single term in (5), for example, the pairs $\phi_{1p}(z)=0$ and $\psi_{1p}(z)=b_5(z-t)^{-5}$.

Acknowledgements: This research project was supported by the Science Fund of the Chinese Academy of Sciences.

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10 December 1993

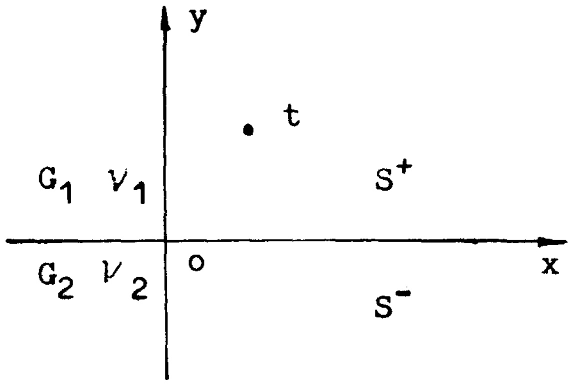

Figure 1. A singular point z=t placed in the upper half-plane of the bimaterials.

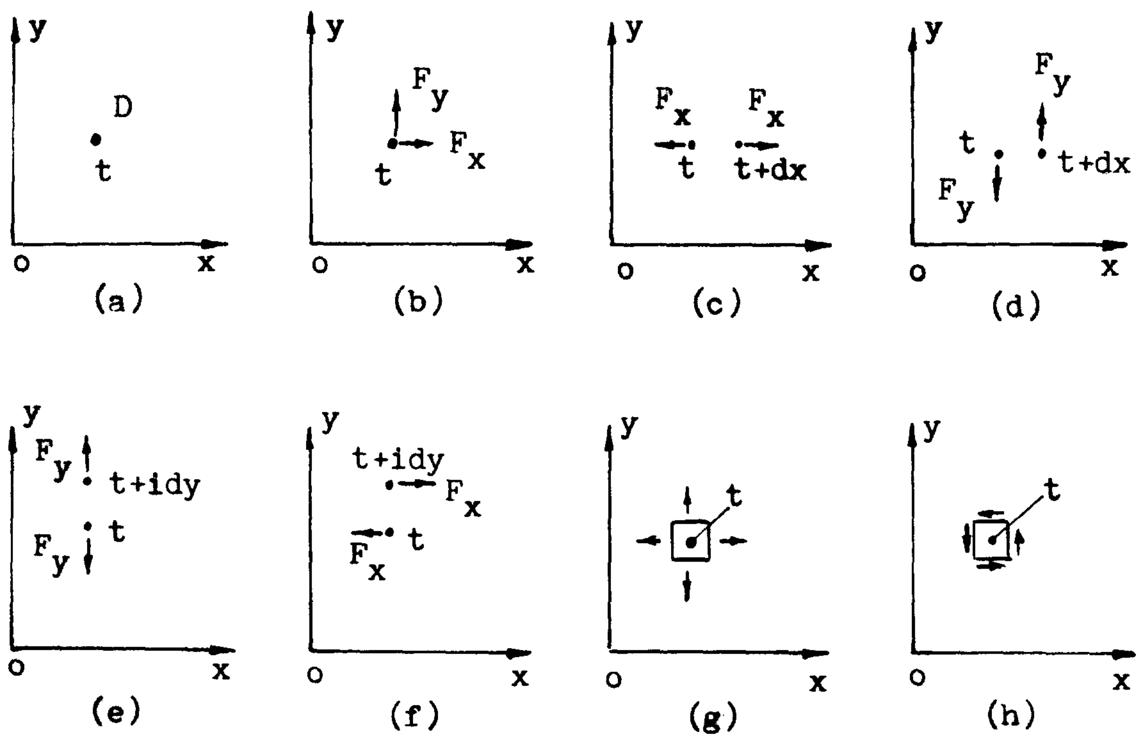

Figure 2. Pictorial representation of singular source at the point z=t, (a) dislocation, (b) concentrated force, (c) double force in x-direction, (d) double force with moment, (e) double force in y-direction, (f) double force with moment, (g) dilation, (h) concentrated moment.