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Near-tip fields and intensity factors for interfacial cracks in dissimilar anisotropic piezoelectric media

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Abstract. A complete form of stress and electric displacement fields in the vicinity of the tip of an interfacial crack, between two dissimilar anisotropic piezoelectric media, is derived by using the complex function theory. New definitions of real-valued stress and electric displacement intensity factors for the interfacial crack are proposed. These definitions are extensions of those for cracks in homogeneous piezoelectric media. Closed form solutions of the stress and electric displacement intensity factors for a semi-infinite crack as well as for a finite crack at the interface between two dissimilar piezoelectric media are also obtained by using the mutual integral.

1. Introduction

Piezoelectric materials have found wide technological applications as transducers, sensors and actuators due to their inherent coupled electromechanical behavior. The strength of the piezoelectric materials is reduced by the presence of defects such as cracks. The subject of cracks in piezoelectric materials has thus received much attention, due to its potential application in various kinds of electromechanical problems. Several basic problems for cracks in homogeneous piezoelectric materials have been solved by Parton [1], Sosa and Pak [2], Pak [3, 4] and Sosa [5]. Recently, Suo et al. [6] examined the problem of an interfacial crack between dissimilar anisotropic piezoelectric media. They used the general representations for stress and electric displacement fields in anisotropic piezoelectricity, as derived by Barnett and Lothe [7], to formulate the interfacial crack problem in terms of four analytic potential functions. If a potential function of the form $z^{-1/2+i\delta}v$ that generates the singular crack-tip fields is assumed as in Suo et al. [6], the traction and charge-free conditions on the surface of the crack lead to an eigenvalue problem associated with the eigenvalue δ and eigenvector v. The eigenvalue δ and the corresponding eigenvector v depend on the material constants. They [6] employed a coordinate system whose base vectors are the orthogonal eigenvectors for mathematical convenience, and found that the general interfacial crack-tip field consists of two pairs of singularities; $r^{-1/2\pm i\varepsilon}$ and $r^{-1/2\pm\kappa}$ at distance r from the crack tip in the eigenvector coordinate system, where ε and κ are real numbers depending on the material constants. They proposed definitions of one complex intensity factor and two real intensity factors that scale the oscillatory fields $(r^{-1/2\pm i\varepsilon})$ and the non-oscillatory fields $(r^{-1/2\pm\kappa})$, respectively, based on the components of the traction vector in the eigenvector coordinate system. Therefore, it is necessary to solve the eigenvalue problem to determine the intensity factors. Furthermore, the definitions of intensity factors of stress and electric displacement for the interfacial crack do not reduce to those of classical stress and electric displacement intensity factors, as the bimaterial continuum degenerates to be a homogeneous one. In order to overcome these difficulties, new definitions of real-valued stress and electric displacement intensity factors are proposed in this paper, based on components of the traction vector in a material-independent global spatial coordinate system, such as the Cartesian coordinates. In defining the intensity factors, a matrix function that plays an important role in representing the coupling and oscillations in the crack-tip fields is introduced. The matrix function is shown to be related explicitly to only one of the generalized Dundurs bimaterial matrices, but not the eigenvectors. The intensity factors can be thus determined without solving the eigenvalue problem. The definitions of the intensity factors may be considered as an extension of the purely elastic version proposed by Wu [8] and Qu and Li [9].

Significant progress has been made in determining complete crack-tip fields for elasticity. Rice [10] has derived the complete form of the stress and displacement fields near the tip of an interfacial crack, between two dissimilar isotropic elastic media, based on analytic functions. The corresponding problem for anisotropic bimaterial has been recently solved by Beom and Atluri [11]. Subsequently the complete crack-tip eigen functions for the anisotropic elastic interfacial crack have been used as the Galerkin basis functions in special finite elements for numerical analysis of finite cracks in finite bodies by Chow et al. [12]. However, a complete form of stress and electric displacement fields in the vicinity of the tip of an interfacial crack, between two dissimilar anisotropic piezoelectric media, has not yet been derived, although a basic singular asymptotic crack-tip solution has been known [6]. The general form of the near tip fields for the interface crack, in the sense of a complete Williams expansion, is derived in this paper using the complex function theory. The near-tip fields are written in terms of two generalized Dundurs bimaterial matrices proposed in this paper.

Conservation integrals such as J, L and M integrals [13, 14] are successfully used in analyzing elastic crack problems. Their path-independent property has been utilized to obtain elegant short-cut solutions for the energy release rates or stress intensity factors for some elastic crack problems [15, 16]. However, the approach of the conservation integrals mentioned above has a limitation, i.e. it can not determine the individual stress intensity factors for an interfacial crack which is inherently under mixed mode. The mutual integral $\mathcal{M}$ proposed by Chen and Shield [17] provides sufficient information for determining the individual stress intensity factors for the interfacial crack problem, if auxiliary solutions are properly introduced, and indeed this mutual integral has been employed to obtain the closed form of individual stress intensity factors for elastic crack problems [18, 19]. On the other hand, studies on conservation integrals for electroelastic materials have been performed by some researchers [3, 20–23] since the work by Cherepanov [24]. Especially, a conservation integral corresponding to the J integral in elasticity was derived for a piezoelectric medium by Pak [3].

It is the purpose of this study to investigate the problem of an interfacial crack between dissimilar *anisotropic piezoelectric* media under electromechanical crack-face loading, and that of an interfacial crack interacting with electromechanical singularities in the material. The problem is formulated using the complex representation derived by Barnett and Lothe [7]. In particular, the one-complex variable approach introduced by Suo [25] is employed in this paper, which simplifies many earlier works. The general form of the near tip fields for the interface crack between dissimilar anisotropic piezoelectric materials, in the sense of a complete Williams expansion, is derived here for the first time using an analysis based on analytic functions. The procedures following those of Rice [10] in solving the near-tip fields for the interfacial crack in dissimilar isotropic elastic media. New definitions of real-valued stress and electric displacement intensity factors are proposed. These definitions are extensions to those for cracks in homogeneous piezoelectric media. The stress and electric displacement intensity factors associated with the interfacial crack in dissimilar anisotropic piezoelectric media are also represented by the mutual integral. The mutual integral, which has the conservation property, is applied to determine the stress and electric displacement intensity factors due to electromechanical crack-facing loading, and due to a body force and a dislocation, for a semi-infinite crack, as well as for a finite crack, at the interface between dissimilar anisotropic piezoelectric media.

2. Formulation

Consider a generalized two-dimensional deformation of an anisotropic piezoelectric solid in which the three components of displacement and the electric potential depend only on the in-plane coordinates, x_1 and x_2. The constitutive equation for a linear piezoelectric material can be written in the following compact form [7]

$$\Sigma_{iJ} = C_{iJMn} v_{M,n}, \tag{2.1a}$$

in which

$$v_M = \begin{cases} u_m, & M = 1,2,3 \\ \phi, & M = 4 \end{cases},$$

$$\Sigma_{iJ} = \begin{cases} \sigma_{ij}, & J = 1,2,3 \\ D_i, & J = 4 \end{cases}, \tag{2.1b}$$

$$C_{iJMn} = \begin{cases} c_{ijmn}, & J, M = 1,2,3 \\ e_{nij}, & J = 1,2,3; M = 4 \\ e_{imn}, & J = 4; M = 1,2,3 \\ -\gamma_{in}, & J, M = 4 \end{cases},$$

where u_m is the displacement, ϕ is the electric potential, σ_{ij} is the stress, D_i is the electric displacement, c_{ijmn} is the electric stiffness, e_{nij} is the piezoelectric constant, γ_{in} is the dielectric permittivity, and the subscript comma (,) denotes a partial derivative with respect to the Cartesian coordinates. In this paper, the repetition of an index in a term denotes a summation with respect to that index over its range 1 to 3 for a lowercase script and 1 to 4 for an uppercase script, unless indicated otherwise; and boldfaced symbols represent vectors or matrices. The displacements and the electric potential satisfy the equation of equilibrium for a homogeneous linear piezoelectric solid

$$C_{iJMn}\ v_{M,ni} = 0. \tag{2.2}$$

A general solution for the displacement and the electric potential fields that satisfy (2.2), and the corresponding stress and electric displacement components, may be written in terms of four analytic functions as [7]

$$v_J = 2\mathrm{Re}\left[\sum_{M=1}^{4} A_{JM} f_M(z_M)\right], \tag{2.3a}$$

$$\Sigma_{1J} = -2\mathrm{Re}\left[\sum_{M=1}^{4} B_{JM} p_M f'_M(z_M)\right], \tag{2.3b}$$

$$\Sigma_{2J} = 2\mathrm{Re}\left[\sum_{M=1}^{4} B_{JM} f'_M(z_M)\right]. \tag{2.3c}$$

Here Re denotes the real part, prime ($'$) implies the derivative with respect to the associated arguments, and $f_M(z_M)$ are analytic in their arguments, $z_M = x_1 + p_M x_2$; and p_M are four distinct complex numbers with positive imaginary parts, which can be solved as the roots of a eighth-order polynomial [7]

$$||\mathbf{Q} + p(\mathbf{R} + \mathbf{R}^T) + p^2\mathbf{T}|| = 0, \tag{2.4}$$

where $||\cdot||$ denotes the determinant of a matrix, superscript T indicates the transpose of a matrix, and the three matrices $\mathbf{Q}, \mathbf{R}$ and $\mathbf{T}$ are defined by $Q_{JM} = C_{1JM1}, R_{JM} = C_{1JM2}$ and $T_{JM} = C_{2JM2}$. When the eigenvalues p_M degenerate, the solutions (2.3a)–(2.3c) need to be modified.

We define two real matrices $\mathbf{L}$ and $\mathbf{M}$, which will appear subsequently in this paper, as

$$\begin{aligned} \mathbf{L} &= -[\mathrm{Im}(\mathbf{AB}^{-1})]^{-1}, \\ \mathbf{M} &= -[\mathrm{Re}(\mathbf{AB}^{-1})], \end{aligned} \tag{2.5}$$

where Im denotes the imaginary part. The matrix $\mathbf{L}$ is symmetric and the matrix $\mathbf{M}$ is anti-symmetric [6]. The matrices $\mathbf{A}$ and $\mathbf{B}$ as they appear in (2.3a)–(2.3c) are not unique, in the sense that any arbitrary constant can be multiplied to the eigenvectors (the column vectors of $\mathbf{A}$ and $\mathbf{B}$); while the two real matrices $\mathbf{L}$ and $\mathbf{M}$ are unique (i.e, they are independent of the normalizing factors for $\mathbf{A}$). Moreover, the two matrices $\mathbf{L}$ and $\mathbf{M}$ have smooth limits even if $\mathbf{A}$ and $\mathbf{B}$ become singular. If a final result involves only the matrices $\mathbf{L}$ and $\mathbf{M}$, but not $\mathbf{A}$ or $\mathbf{B}$, it is also valid for any degenerate cases. The real matrices $\mathbf{L}$ and $\mathbf{M}$ can be calculated directly from the elastic constants without solving the eigenvalue problem [7].

For simplicity, we will present our solutions through the vector function, $\mathbf{f}(z)$, defined as

$$\mathbf{f}(z) = (f_1(z)\ f_2(z)\ f_3(z)\ f_4(z))^T, \tag{2.6}$$

where the argument has the generic form $z = x_1 + px_2(\mathrm{Im}\,\mathrm{p} > 0)$. This one-complex-variable approach has been originally introduced by Suo [25]. Once the solution of $\mathbf{f}(z)$ is obtained for a given boundary value problem, a replacement of z_1, z_2, z_3 or z_4 should be made for each component function to calculate field quantities from (2.3a)–(2.3c).

3. Near tip stress and electric displacement fields

Consider a crack lying along the interface between two dissimilar, anisotropic, homogeneous linear piezoelectric materials with material 1 above and material 2 below as shown in Figure 1. The crack tip lies on the plane $x_2 = 0$ at $x_1 = 0$ and tractions and charge vanish on the crack surfaces. We seek the form of solution in some region $\Omega(= \Omega^{(1)} + \Omega^{(2)})$ surrounding the tip of a traction and charge-free interface crack. Continuity of tractions and charge across all the interface, both the bonded and cracked portions, in Ω requires that

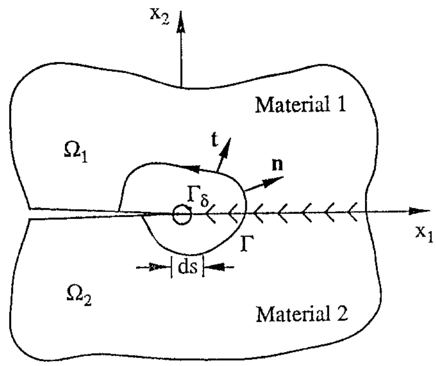


Figure 1. Region near crack tip along piezoelectric bimaterial interface.

$$\mathbf{B}^{(1)}\mathbf{f}'^{(1)}(x_1) + \bar{\mathbf{B}}^{(1)}\bar{\mathbf{f}}'^{(1)}(x_1) = \mathbf{B}^{(2)}\mathbf{f}'^{(2)}(x_1) + \bar{\mathbf{B}}^{(2)}\bar{\mathbf{f}}'^{(2)}(x_1), \tag{3.1}$$

where superscripts 1 and 2 in parentheses indicate that the quantities are for the materials 1 and 2, respectively. Rearranging (3.1), we obtain

$$\mathbf{B}^{(1)}\mathbf{f}'^{(1)}(x_1) - \bar{\mathbf{B}}^{(2)}\bar{\mathbf{f}}'^{(2)}(x_1) = \mathbf{B}^{(2)}\mathbf{f}'^{(2)}(x_1) - \bar{\mathbf{B}}^{(1)}\bar{\mathbf{f}}'^{(1)}(x_1). \tag{3.2}$$

The above equation holds along the entire x_1-axis in Ω. Moreover the functions on the left hand side are analytic in the upper region $\Omega^{(1)}$, whereas those at the right-hand side are analytic in the lower region $\Omega^{(2)}$. By the standard analytic continuation arguments, we see that

$$\mathbf{B}^{(1)}\mathbf{f}'^{(1)}(z) - \bar{\mathbf{B}}^{(2)}\bar{\mathbf{f}}'^{(2)}(z) = \mathbf{B}^{(2)}\mathbf{f}'^{(2)}(z) - \bar{\mathbf{B}}^{(1)}\bar{\mathbf{f}}'^{(1)}(z) = 2\mathbf{h}(z), \tag{3.3}$$

where $\mathbf{h}(z)$ is analytic throughout Ω, including points along all the interface. With the same arguments, the continuity of the displacement and electric potential across the bonded interface gives analytic continuation of different linear combinations of $\mathbf{f}'(z)$ and $\bar{\mathbf{f}}(z)$ across the interface, such that

$$\mathbf{A}^{(1)}\mathbf{f}'^{(1)}(z) - \bar{\mathbf{A}}^{(2)}\bar{\mathbf{f}}'^{(2)}(z) = \mathbf{A}^{(2)}\mathbf{f}'^{(2)}(z) - \bar{\mathbf{A}}^{(1)}\bar{\mathbf{f}}'^{(1)}(z), \tag{3.4}$$

holds everywhere in Ω except on the crack line. The traction and charge-free condition on the surface of the crack leads to a homogeneous Hilbert problem

$$\mathbf{B}^{(1)}\mathbf{f}'^{(1)+}(x_1) + \bar{\mathbf{B}}^{(1)}\bar{\mathbf{f}}'^{(1)-}(x_1) = 0, \quad x_1 < 0. \tag{3.5}$$

We may express the function $\bar{\mathbf{B}}^{(1)}\bar{\mathbf{f}}'^{(1)}(z)$ in terms of $\mathbf{B}^{(1)}\mathbf{f}'^{(1)}(z)$ and $\mathbf{h}(z)$ from (3.3) and (3.4)

$$\bar{\mathbf{B}}^{(1)}\bar{\mathbf{f}}'^{(1)}(z) = (\mathbf{I} + i\beta)^{-1}(\mathbf{I} - i\beta)\mathbf{B}^{(1)}\mathbf{f}'^{(1)}(z) - 2(\mathbf{I} + i\beta)^{-1}(\mathbf{I} + \alpha)\mathbf{h}(z), \tag{3.6}$$

where

$$\begin{aligned} \alpha &= (\mathbf{L}^{(1)} - \mathbf{L}^{(2)})(\mathbf{L}^{(1)} + \mathbf{L}^{(2)})^{-1}, \\ \beta &= (\mathbf{L}^{(1)-1} + \mathbf{L}^{(2)-1})^{-1}(\mathbf{M}^{(1)} - \mathbf{M}^{(2)}). \end{aligned} \tag{3.7}$$

Two bimaterial matrices α and β defined by (3.7) are the generalized Dundurs parameters for an anisotropic piezoelectric bimaterial. These generalized Dundurs parameters are an extension to those for an anisotropic elastic bimaterial proposed by Beom and Atluri [11]. Substituting (3.6) into (3.5), it is found that

$$\begin{aligned} &\mathbf{B}^{(1)}\mathbf{f}'^{(1)+}(x_1) + (\mathbf{I} + i\beta)^{-1}(\mathbf{I} - i\beta)\mathbf{B}^{(1)}\mathbf{f}'^{(1)-}(x_1) \\ &\quad = 2(\mathbf{I} + i\beta)^{-1}(\mathbf{I} + \alpha)\mathbf{h}(x_1), \quad x_1 < 0. \end{aligned} \tag{3.8}$$

It can be shown that the general solution of (3.8) for $\mathbf{f}'^{(1)}(z)$ is given by (see Appendix A for details)

$$\mathbf{f}'^{(1)}(z) = \frac{1}{2\sqrt{2\pi z}}\mathbf{B}^{(1)-1}(\mathbf{I} + i\beta)\mathbf{Y}(z^{i\varepsilon}, z^{\kappa})\mathbf{g}(z) + \mathbf{B}^{(1)-1}(\mathbf{I} + \alpha)\mathbf{h}(z). \tag{3.9}$$

Here

$$\begin{aligned} \varepsilon &= \tfrac{1}{\pi}\tanh^{-1}\eta, \\ \kappa &= \tfrac{1}{\pi}\tan^{-1}\omega, \\ \eta &= [\{(1/4\,\mathrm{tr}(\beta^2))^2 - \|\beta\|\}^{1/2} - 1/4\,\mathrm{tr}(\beta^2)]^{1/2}, \\ \omega &= [\{(1/4\,\mathrm{tr}(\beta^2))^2 - \|\beta\|\}^{1/2} + 1/4\,\mathrm{tr}(\beta^2)]^{1/2}. \end{aligned} \tag{3.10}$$

The matrix function $\mathbf{Y}(\xi, \zeta)$ can be written explicitly in terms of the real bimaterial matrix β as

$$\begin{aligned} \mathbf{Y}(\xi,\zeta) &= \frac{1}{2}\left\{\frac{\omega^2}{\eta^2+\omega^2}(\xi+\xi^{-1}) + \frac{\eta^2}{\eta^2+\omega^2}(\zeta+\zeta^{-1})\right\}\mathbf{I} \\ &+\frac{1}{2}\left\{\frac{i\omega^2}{\eta(\eta^2+\omega^2)}(\xi-\xi^{-1}) - \frac{\eta^2}{\omega(\eta^2+\omega^2)}(\zeta-\zeta^{-1})\right\}\beta \\ &+\frac{1}{2}\frac{1}{\eta^2+\omega^2}\left\{-(\xi+\xi^{-1}) + (\zeta+\zeta^{-1})\right\}\beta^2 \\ &-\frac{1}{2}\left\{\frac{i}{\eta(\eta^2+\omega^2)}(\xi-\xi^{-1}) + \frac{1}{\omega(\eta^2+\omega^2)}(\zeta-\zeta^{-1})\right\}\beta^3, \end{aligned} \tag{3.11}$$

where ξ and ζ are arbitrary functions of z. $\mathbf{Y}(\xi,\zeta)$ given by (3.11) can be shown to have the following properties

$$\mathbf{Y}(1,1) = \mathbf{I}, \tag{3.12a}$$

$$\mathbf{Y}(\xi_1,\zeta_1)\mathbf{Y}(\xi_2,\zeta_2) = \mathbf{Y}(\xi_1\xi_2,\zeta_1\zeta_2). \tag{3.12b}$$

Substitution of (3.9) into (3.6) yields

$$\begin{aligned} \bar{\mathbf{g}}(z) &= \mathbf{g}(z), \\ \bar{\mathbf{h}}(z) &= -\mathbf{h}(z). \end{aligned} \tag{3.13}$$

Using (3.3) and (3.9), we obtain for the other function $\mathbf{f}'^{(2)}(z)$

$$\mathbf{f}'^{(2)}(z) = \frac{1}{2\sqrt{2\pi z}}\mathbf{B}^{(2)-1}(\mathbf{I} - i\beta)\mathbf{Y}(z^{i\varepsilon}, z^{\kappa})\mathbf{g}(z) + \mathbf{B}^{(2)-1}(\mathbf{I} - \alpha)\mathbf{h}(z). \tag{3.14}$$

A Williams type expansion of the near-tip field is generated from (2.3a)–(2.3c), (3.9) and (3.14) by writing $\mathbf{g}(z)$ and $\mathbf{h}(z)$ in terms of local Taylor series expansions, as

$$\begin{aligned} \mathbf{g}(z) &= \sum_{n=0}^{\infty} \mathbf{a}_n z^n, \\ \mathbf{h}(z) &= \sum_{n=0}^{\infty} i\mathbf{b}_n z^n, \end{aligned} \tag{3.15}$$

where $\mathbf{a}_n$ and $\mathbf{b}_n$ are real vectors. Then $\mathbf{a}_0$ represents the strength of the crack tip singularity, which can be defined as an intensity factor of stress and electric displacement. Since $\mathbf{f}'^{(1)}(z)$ and $\mathbf{f}'^{(2)}(z)$ are determined as above, the complete fields of the stress and the electric displacement in the vicinity of the crack tip are evaluated from (2.3a)–(2.3c), which results in

$$\begin{aligned} \tau_1^{(m)} &= -2\mathrm{Re}\left[\mathbf{B}^{(m)}\mathbf{P}^{(m)}\sum_{n=0}^{\infty}\left\{\Phi_n^{(m)}\mathbf{a}_n + \Psi_n^{(m)}\mathbf{b}_n\right\}\right], \\ \tau_2^{(m)} &= 2\mathrm{Re}\left[\mathbf{B}^{(m)}\sum_{n=0}^{\infty}\left\{\Phi_n^{(m)}\mathbf{a}_n + \Psi_n^{(m)}\mathbf{b}_n\right\}\right], \quad (m = 1,2; \text{no sum over } m), \end{aligned} \tag{3.16a}$$

where

$$\tau_1 = (\sigma_{11}\,\sigma_{12}\,\sigma_{13}\,D_1)^T,$$

$$\tau_2 = (\sigma_{21}\,\sigma_{22}\,\sigma_{23}\,D_2)^T,$$

$$\mathbf{P} = \mathrm{diag}(p_1\,p_2\,p_3\,p_4),$$

$$\Phi_n^{(m)} = \frac{1}{4\sqrt{2\pi}}\frac{1}{\eta^2+\omega^2}\begin{bmatrix} \{\omega^2\phi_n^{(m)}(\varepsilon)+\eta^2\phi_n^{(m)}(-i\kappa)\}\widehat{\beta}^{(m)} \\ +\left\{\frac{i\omega^2}{\eta}\varphi_n^{(m)}(\varepsilon)-\frac{\eta^2}{\omega}\varphi_n^{(m)}(-i\kappa)\right\}\widehat{\beta}^{(m)}\beta \\ +\{-\phi_n^{(m)}(\varepsilon)+\phi_n^{(m)}(-i\kappa)\}\widehat{\beta}^{(m)}\beta^2 \\ -\left\{\frac{i}{\eta}\varphi_n^{(m)}(\varepsilon)+\frac{1}{\omega}\varphi_n^{(m)}(-i\kappa)\right\}\widehat{\beta}^{(m)}\beta^3 \end{bmatrix}, \tag{3.16b}$$

$$\Psi_n^{(m)} = i\mathbf{Z}^{(m)n}\,\mathbf{B}^{(m)-1}\{\mathbf{I}-(-1)^m\alpha\},$$

$$\widehat{\beta}^{(m)} = \mathbf{B}^{(m)-1}\{\mathbf{I}-(-1)^m i\beta\},$$

$$\phi_n^{(m)}(\delta) = \mathbf{Z}^{(m)n-1/2+i\delta}+\mathbf{Z}^{(m)n-1/2-i\delta},$$

$$\varphi_n^{(m)}(\delta) = \mathbf{Z}^{(m)n-1/2+i\delta}-\mathbf{Z}^{(m)n-1/2-i\delta},$$

$$\mathbf{Z}^{(m)b} = \mathrm{diag}(z_1^{(m)b}\,z_2^{(m)b}\,z_3^{(m)b}\,z_4^{(m)b}).$$

The singular stress field along the bonded interface near the crack tip is given by

$$\tau_2(x_1) = \frac{1}{\sqrt{2\pi x_i}}\mathbf{Y}(x_1^{i\varepsilon},x_1^{\kappa})\mathbf{g}(x_1). \tag{3.17}$$

Thus, the vector of stress and electric displacement intensity factors which uniquely characterize the singular field can be defined by

$$\mathbf{k} = \lim_{x_1\to 0+}\sqrt{2\pi x_1}\mathbf{Y}(x_1^{-i\varepsilon},x_1^{-\kappa})\tau(x_1), \tag{3.18}$$

where $\mathbf{k} = (K_2\,K_1\,K_3\,K_4)^T$. Since $\mathbf{Y}(x_1^{-i\varepsilon},x_1^{-\kappa})$ and $\tau(_2x_1)$ are real, $\mathbf{k}$ is real. The intensity factor $\mathbf{k}$ may be considered as an extension of the elastic version proposed by Wu [8] and Qu and Li [9]. Although $\mathbf{k}$ defined in (3.18) does not have the proper dimension, it provides a unique characterization of the crack tip state. Stress and electric displacement intensity factors with the same dimension of classical intensity factor, denoted by $\widehat{\mathbf{k}}_l$ also can be defined based on the characteristic length l as suggested by Rice [10] for the isotropic elastic bimaterial case. $\widehat{\mathbf{k}}_l$ is related to $\mathbf{k}$ by $\widehat{\mathbf{k}}_l = \mathbf{Y}(l^{i\varepsilon}, l^{\kappa})\mathbf{k}$. It is noted that the intensity factor $\mathbf{k}$ given in (3.18) for the piezoelectric bimaterial recovers the classical intensity factor $(K_{II}\ K_I\ K_{III}\ K_{IV})^T$ as the bimaterial continuum degenerates to be a homogeneous one. This is in contrast to the stress and electric displacement intensity factor introduced by Suo et al. [6], which is based on components of the traction vector in a coordinate system whose base vectors are orthogonal eigenvectors. In terms of $\mathbf{k}$, the analytic functions generating the singular part of the interface stress and electric displacement can be expressed as

$$\begin{aligned} \mathbf{f}'^{(1)}(z) &= \frac{1}{2\sqrt{2\pi z}}\mathbf{B}^{(1)-1}(\mathbf{I} + i\beta)\mathbf{Y}(z^{i\varepsilon}, z^{\kappa})\mathbf{k}, \\ \mathbf{f}'^{(2)}(z) &= \frac{1}{2\sqrt{2\pi z}}\mathbf{B}^{(2)-1}(\mathbf{I} - i\beta)\mathbf{Y}(z^{i\varepsilon}, z^{\kappa})\mathbf{k}. \end{aligned} \tag{3.19}$$

4. Conservation integral

The generalized J integral for a linear piezoelectric medium that is homogeneous in the x_1 direction is defined by [3, 24]

$$J\{v;\Gamma\} = \int_{\Gamma}(\mathcal{W}n_1 - t_J v_{J,1})ds. \tag{4.1}$$

Here $\mathcal{W}$ is the electric enthalpy density, given by $\mathcal{W} = \frac{1}{2}\Sigma_{iJ}v_{J,i}$, n_i is the unit outward normal vector, t_J is the surface traction and the surface electric displacement, given by $t_J = n_i\Sigma_{iJ}$, Γ is a path connecting any two points on opposite sides of the crack surface and enclosing the crack tip and ds is an element of arc length along Γ as shown in Figure 1. It is well known that the generalized J integral is independent of any path Γ, and has the physical meaning of energy release rate due to crack extension.

As noted in the previous section, the matrices $\mathbf{A}$ and $\mathbf{B}$ in (2.3a)–(2.3c) are not unique. Normalizing the eigenvectors (the column vectors of $\mathbf{A}$ and $\mathbf{B}$) properly, it can be shown that [7]

$$\begin{aligned} \mathbf{L} &= -2i\mathbf{B}\mathbf{B}^T, \\ \mathbf{S} &= i(2\mathbf{A}\mathbf{B}^T - \mathbf{I}), \end{aligned} \tag{4.2}$$

where $\mathbf{A}$ and $\mathbf{B}$ are the normalized matrices, and $\mathbf{S} = \mathbf{M}\mathbf{L}$. For convenience, we use the normalized matrices $\mathbf{A}$ and $\mathbf{B}$ hereafter; $\mathbf{f}(z)$ is the normalized function associated with the normalized matrices $\mathbf{A}$ and $\mathbf{B}$. Recently, Yeh et al. [26] obtained the complex form of the J integral for an anisotropic elastic solid. In a similar way, it can be shown that the generalized J integral given in (4.1) is written in the complex form, for an anisotropic piezoelectric solid, as

$$J\{v;\Gamma_0\} = \mathrm{Re}\left[\sum_{J=1}^{4}\int_{\Gamma_0}\{f_J'(z_J)\}^2 dz_J\right], \tag{4.3}$$

where Γ_0 is a closed contour. Since the complete general solutions for the near tip fields are determined as shown in the previous section, the relation between the generalized J integral and the intensity factors can be derived through the complex formula of the generalized J integral. The generalized J integral in (4.3) is evaluated with near tip fields given by (3.19), resulting in

$$J\{v;\Gamma_\delta\} = \tfrac{1}{4}\mathbf{k}^T\mathbf{U}^{-1}\mathbf{k}. \tag{4.4}$$

Here $\mathbf{U}^{-1} = (\mathbf{L}^{(1)-1} + \mathbf{L}^{(2)-1})(\mathbf{I} + \beta^2)$, and Γ^0 is chosen to be a circle Γ_δ with vanishingly small radius δ as shown in Figure 1. Details required for the above derivations are presented in Appendix B.

Consider two independent equilibrium states of a piezoelectrically deformed bimaterial body, with each displacement and electric potential being denoted by v and $\tilde{v}$, respectively. The mutual integral for the two states, denoted by $\mathcal{M}\{v,\tilde{v};\Gamma\}$ is defined by

$$\mathcal{M}\{v,\tilde{v};\Gamma\} = \int_\Gamma \left(\tilde{\Sigma}_{iJ}v_{J,i}n_1 - t_J\tilde{v}_{J,1} - \tilde{t}_J v_{J,1}\right) ds. \tag{4.5}$$

As noted by Chen and Shield [17] for an elastic material, $\mathcal{M}\{v,\tilde{v};\Gamma\}$ can be written in terms of the J integral as

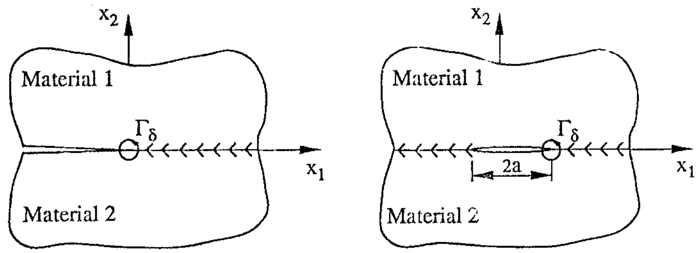
$$\mathcal{M}\{v,\tilde{v};\Gamma\} = J\{v+\tilde{v};\Gamma\} - J\{v;\Gamma\} - J\{\tilde{v};\Gamma\}. \tag{4.6}$$

The $\mathcal{M}$ integral satisfies the same conservation law as that of the J integral. Thus we have the following conservation law

$$\mathcal{M}\{v,\tilde{v};\Gamma_0\} = 0. \tag{4.7}$$

Here an area enclosed by Γ_0 containing the interface bonded perfectly is assumed to be free from any singularities. This conservation law will be applied to the direct calculation of stress and electric displacement intensity factors without actually solving complicated boundary value problems, which will be shown later. Making use of the complex form of the J integral and the relation between J integral and $\mathcal{M}$ integral in (4.6), it can be shown that the complex form of the $\mathcal{M}$ integral is given by

$$\mathcal{M}\{v,\tilde{v};\Gamma_0\} = 2\mathrm{Re}\left[\sum_{J=1}^{4}\int_{\Gamma_0} f_J'(z_J)\tilde{f}_J'(z_J)dz_J\right], \tag{4.8}$$



(a) Semi-infinite crack (b) Finite crack

Figure 2. Auxiliary crack problems.

where overscript tilde(˜) represents the quantities associated with the equilibrium state $\tilde{v}$.

As suggested by Chen and Shield [17] (though they dealt with the crack in a homogeneous isotropic elastic medium), the mutual integral $\mathcal{M}$ as defined by (4.5) can be used to determine the individual stress and electric displacement intensity factors K_1, K_2, K_3 and K_4 for the equilibrium state $\mathbf{u}$, if a solution for another equilibrium state $\tilde{v}$, called the auxiliary solution, is known.

First let us consider an auxiliary problem as shown in Figure 2(a). The crack lies along the negative x_1 axis and the positive x_1 axis is the interface between the material 1 (occupying the upper half) and 2 (occupying the lower half). Each material is assumed to be anisotropic and piezoelectric . An admissible singular solution satisfying the boundary conditions on the bonded portion of the interface and on the cracked portion can be given from (3.19) by

$$\begin{aligned} \tilde{\mathbf{f}}^{(1)}(z) &= \frac{1}{2\sqrt{2\pi z}}\mathbf{B}^{(1)-1}(\mathbf{I}+i\beta)\mathbf{Y}(z^{i\varepsilon}, z^{\kappa})\tilde{\mathbf{k}}, \\ \tilde{\mathbf{f}}^{(2)}(z) &= \frac{1}{2\sqrt{2\pi z}}\mathbf{B}^{(2)-1}(\mathbf{I}-i\beta)\mathbf{Y}(z^{i\varepsilon}, z^{\kappa})\tilde{\mathbf{k}}. \end{aligned} \tag{4.9}$$

Here subscripts 1 and 2 indicate that the potentials are for the materials 1 and 2, respectively. We now choose four independent auxiliary solutions from the field given by (4.9) as follows

$$\begin{aligned} \tilde{\mathbf{f}}^{J(1)}(z) &= \frac{1}{2\sqrt{2\pi z}}\mathbf{B}^{(1)-1}(\mathbf{I}+i\beta)\mathbf{Y}(z^{i\varepsilon}, z^{\kappa})\hat{\mathbf{e}}^J, \\ \tilde{\mathbf{f}}^{J(2)}(z) &= \frac{1}{2\sqrt{2\pi z}}\mathbf{B}^{(2)-1}(\mathbf{I}-i\beta)\mathbf{Y}(z^{i\varepsilon}, z^{\kappa})\hat{\mathbf{e}}^J \quad (J=1,2,3,4), \end{aligned} \tag{4.10}$$

in which $\hat{\mathbf{e}}^J (J = 1,2,3,4)$ is the base vector with the component $\hat{e}^J_M = \delta_{JM}$, where δ_{JM} is the Kronecker delta.

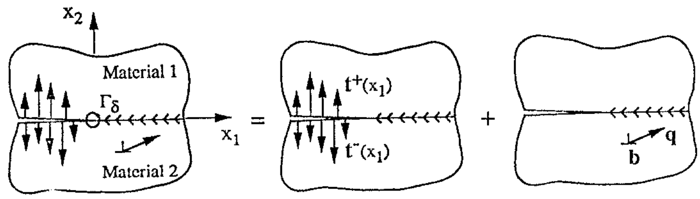


Figure 3. Semi-infinite interfacial crack with singularities and electromechanical crack facing loading.

Next, consider another auxiliary problem as shown in Figure 2(b). The crack lies in the interval $(-a, a)$ on the x_1 axis. The auxiliary solutions for the finite crack are also chosen in a similar way so that (Appendix C)

$$\begin{aligned}\hat{\mathbf{f}}^{J(1)}(z) &= \frac{1}{4\sqrt{\pi a}}\sqrt{\frac{z+a}{z-a}}\mathbf{B}^{(1)-1}\\ &\quad\times(\mathbf{I}+i\beta)\mathbf{Y}\left(\left(2a\frac{z-a}{z+a}\right)^{i\varepsilon}, \left(2a\frac{z-a}{z+a}\right)^{\kappa}\right)\hat{\mathbf{e}}^J,\\ \hat{\mathbf{f}}^{J(2)}(z) &= \frac{1}{4\sqrt{\pi a}}\sqrt{\frac{z+a}{z-a}}\mathbf{B}^{(2)-1}\\ &\quad\times(\mathbf{I}-i\beta)\mathbf{Y}\left(\left(2a\frac{z-a}{z+a}\right)^{i\varepsilon}, \left(2a\frac{z-a}{z+a}\right)^{\kappa}\right)\hat{\mathbf{e}}^J, \quad (J=1,2,3,4).\end{aligned} \tag{4.11}$$

Now, we can introduce the conservation integrals $\mathcal{M}\{v, \tilde{v}^J; \Gamma\}$ and $\mathcal{M}\{v, \hat{v}^J; \Gamma\}$ $(J = 1, 2, 3, 4)$, where $\tilde{v}^J$ and $\hat{v}^J$ are the displacements and the electric potential generated by the complex potentials given by (4.10) and (4.11), respectively. The $\mathcal{M}$ integral is evaluated with $\mathbf{u}$ desired to calculate stress and electric displacement intensity factors under various loading and with the auxiliary field, resulting in

$$\mathcal{M}\{v, \tilde{v}^J; \Gamma_\delta\} = \tfrac{1}{2}U_{JM}^{-1}k_M, \tag{4.12a}$$

$$\mathcal{M}\{v, \hat{v}^J; \Gamma_\delta\} = \tfrac{1}{2}U_{JM}^{-1}k_M. \quad (J = 1, 2, 3, 4) \tag{4.12b}$$

The above equations can be rewritten for $k_M (M = 1, 2, 3, 4)$ as

$$k_M = 2U_{MJ}\mathcal{M}\{v, \tilde{v}^J; \Gamma_\delta\}, \tag{4.13a}$$

$$k_M = 2U_{MJ}\mathcal{M}\{v, \hat{v}^J; \Gamma_\delta\}. \tag{4.13b}$$

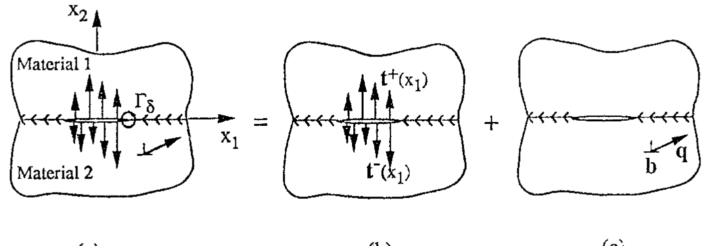


Figure 4. Finite interfacial crack with singularities and electromechanical crack facing loading.

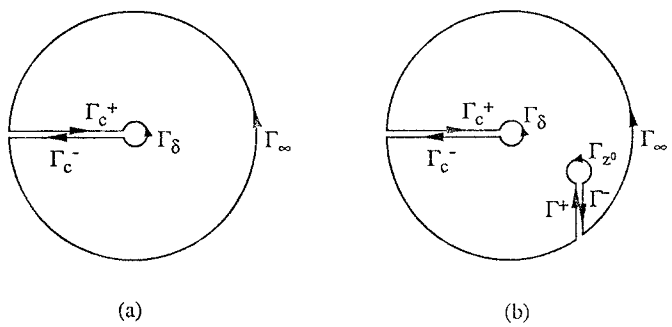


Figure 5. Integration contours for the semi-infinite crack.

It is obvious that the mutual integrals $\mathcal{M}$ in (4.13a) and (4.13b) provide sufficient information for determining the individual stress and electric displacement intensity factors K_1, K_2, K_3 and K_4. The $\mathcal{M}$ integral has the same path-independence as that of the J integral, therefore, (4.13a) and (4.13b) are valid for any path Γ tracing from the lower crack face to the upper crack face.

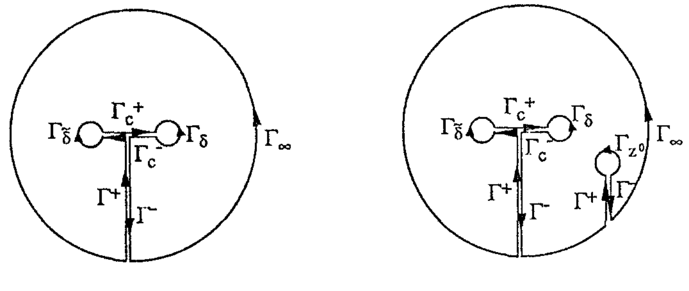
5. Interfacial crack

Two crack configurations in an infinite medium as shown in Figures 5(a) and 6(a), which are of particular importance in the practical application, are considered. Electromechanical tractions are applied on the crack surfaces and electromechanical singularities are embedded in bimaterials. It will be shown that the stress and the electric displacement intensity factors of each problem can be calculated directly by the application of the conservation laws $\mathcal{M}\{v, \tilde{v}^J; \Gamma_0\} = 0$ for the problem of Figure 3(a) and $\mathcal{M}\{v, \hat{v}^J; \Gamma_0\} = 0$ for the problem of Figure 4(a) without actually solving the boundary value problem.

Due to the linearity of the problem, it is sufficient to consider separately the problem under electromechanical tractions acting on the crack surfaces or a singularity. The complete solution of the problem can be obtained by superposition.

5.1. SEMI-INFINITE CRACK

Consider a semi-infinite crack at the interface between two dissimilar anisotropic piezoelectric media as shown in Figure 3(b). Electromechanical tractions $\mathbf{t}^+(x_1)$ and $\mathbf{t}^-(x_1)$ are applied on the upper and lower surfaces of the crack, respectively. To determine the stress and the electric displacement intensity factors, the right hand sides of (4.13a) must be computed as seen earlier. We invoke the conservation law of $\mathcal{M}\{v, \tilde{v}^J; \Gamma_0\} = 0$ for the contour Γ_0 con-



(a) (b)

Figure 6. Integration contours for the finite crack.

sisting of $\Gamma_c^+ + \Gamma_c^- + \Gamma_\infty - \Gamma_\delta$ as shown in Figure 5(a). Here Γ_c^+ and Γ_c^- are the paths on the surfaces of the crack and Γ_∞ is the circular path with an infinitely large radius. The line integral over Γ_∞ makes no contribution to $\mathcal{M}\{v, \tilde{v}^J; \Gamma_0\} = 0$. Thus the conservation law $\mathcal{M}\{v, \tilde{v}^J; \Gamma_0\} = 0$ implies

$$\mathcal{M}\{v, \tilde{v}^J; \Gamma_\delta\} = \mathcal{M}\{v, \tilde{v}^J; \Gamma_c^+ + \Gamma_c^-\}. \tag{5.1}$$

The stress and electric intensity factor is evaluated by using (4.13a) and (5.1), which results in

$$\mathbf{k} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{0} \mathbf{Y}(\xi_0^{-i\varepsilon}, \xi_0^{-\kappa})(\mathbf{I} + i\beta)^{-1}\{(\mathbf{I} - \alpha)\mathbf{t}^+ - (\mathbf{I} + \alpha)\mathbf{t}^-\}\frac{dx_1}{\sqrt{-x_1}} \tag{5.2}$$

where $\xi_0 = -x_1 e^{i\pi}$. In deriving (5.2), (3.12b) and the relations

$$f'_M(z_M) = A_{JM}\Sigma_{2J} + B_{JM}v_{J,1} \quad (\underline{\text{no sum over M}}), \tag{5.3}$$

$$(\mathbf{I} + i\beta)\mathbf{Y}(\xi_0^{i\varepsilon}, \xi_0^{\kappa}) = (\mathbf{I} - i\beta)\mathbf{Y}(\xi_0^{i\varepsilon}, \xi_0^{\kappa}), \tag{5.4}$$

$$\mathbf{U}\mathbf{Y}^T(z^{i\varepsilon}, z^{\kappa})\mathbf{U}^{-1} = \mathbf{Y}(z^{-i\varepsilon}, z^{-\kappa}), \tag{5.5}$$

have been used. It is noted that (5.2) is also valid for the isotropic bimaterial case, since the result involves only the matrices **L** and **M**, but not p_M, **A** or **B** explicitly. For the special case in which $\mathbf{t}^+ = -\mathbf{t}^-$ and $\varepsilon = \kappa = 0$, (5.2) reduces to

$$\mathbf{k} = \sqrt{\frac{2}{\pi}} \int_{-\infty}^{0} \mathbf{t}^+ \frac{dx_1}{\sqrt{-x_1}}. \tag{5.6}$$

Next, let us consider a semi-infinite crack with singularities, such as an electromechanical concentrated force and an electromechanical dislocation as shown in Figure 3(c). Electromechanical singularities $\mathbf{q}(= (q_1\, q_2\, q_3\, q_4)^T)$ and $\mathbf{b}(= (b_1\, b_2\, b_3\, b_4)^T)$ are embedded in the elastic material 2 at the point $z = z^0$. q_1, q_2 and q_3 are the components of a line force and q_4 is the line charge. b_1, b_2 and b_3 are the components of a dislocation and b_4 is the electric potential jump. A contour Γ_0 consisting of $\Gamma_c^+ + \Gamma_c^- + \Gamma^+ + \Gamma^- + \Gamma_\infty - \Gamma_{z^0} - \Gamma_\delta$ as shown in Figure 5(b) is chosen to compute the right hand sides of (4.13a). Here Γ^+ and Γ^- are the interior paths, and Γ_{z^0} is a vanishingly small path enclosing the point $z = z^0$. The line integral over the parts $\Gamma^+ + \Gamma^-$ makes no contribution to $\mathcal{M}\{v, \tilde{v}^J; \Gamma_0\}$. Furthermore, there is no contribution from the infinitely large circle Γ_∞. Thus the conservation law $\mathcal{M}\{v, \tilde{v}^J; \Gamma_0\} = 0$ implies

$$\mathcal{M}\{v, \tilde{v}^J; \Gamma_\delta\} = -\mathcal{M}\{v, \tilde{v}^J; \Gamma_{z^0}\}. \tag{5.7}$$

Potentials for the singularities near the point $z = z^0$ can be written as [4, 7]

$$\mathbf{f}'(z) = \frac{-i}{2\pi(z - z^0)}(\mathbf{B}^{(2)T}\mathbf{b} + \mathbf{A}^{(2)T}\mathbf{q}) + \mathbf{f}^{*\prime}(z), \tag{5.8}$$

where $\mathbf{f}^{*'}(z)$ is analytic at $z = z^0$. Making use of (4.8), (4.13a) and (5.8), it can be shown that

$$k_M = -\sqrt{\frac{2}{\pi}}\mathrm{Re}\left[\sum_{S=1}^{4} \frac{1}{\sqrt{z_S^0}} Y(z_S^{0^{-i\varepsilon}}, z_S^{0^{-\kappa}})_{MN} H_{NR}^{-1} B_{SR}^{(2)^{-1}} (B_{LS}^{(2)} b_L + A_{LS}^{(2)} q_L)\right], \tag{5.9}$$

where $\mathbf{H}^{-1} = (\mathbf{I} + i\beta)\mathbf{U}$. It is noted that (5.9) is not valid for a degenerate case (repeated eigenvalues: for example, isotropic bimaterial case) since the equation involves p_S, $\mathbf{A}$ and $\mathbf{B}$, explicitly. This is in contrast to the result (5.2). For the homogeneous anisotropic case, (5.9) becomes

$$k_M = -\sqrt{\frac{2}{\pi}}\mathrm{Im}\left[\sum_{S=1}^{4} \frac{1}{\sqrt{z_S^0}} B_{MS}(B_{LS} b_L + A_{LS} q_L)\right], \tag{5.10}$$

where Im denotes the imaginary part. The stress intensity factor for the case in which singularities are located on the interface can be obtained by taking the limit $z^0 \to x_1^0 (x_1^0 > 0)$ in (5.9)

$$\mathbf{k} = \frac{1}{\sqrt{2\pi x_1^0}} \mathbf{Y}(x_1^{0^{-i\varepsilon}}, x_1^{0^{-\kappa}}) \left[-2\mathbf{U}\mathbf{b} + (\mathbf{I} + \beta^2)^{-1}\{\beta(\mathbf{I} - \alpha) + (\mathbf{I} - \alpha)\mathbf{S}^{(1)T}\}\mathbf{q}\right]. \tag{5.11}$$

5.2. FINITE CRACK

Consider a finite crack, in the interval $(-a, a)$, between dissimilar anisotropic media as shown in Figure 4(b). Tractions $\mathbf{t}^+(x_1)$ and $\mathbf{t}^-(x_1)$ are applied on the upper and lower surfaces of the crack, respectively. The solution procedure is similar to the case of the semi-infinite crack, which is briefly described as follows. We invoke the conservation law of $\mathcal{M}\{v, \widehat{v}^J; \Gamma_0\} = 0$ for the contour Γ_0 consisting of $\Gamma_c^+ + \Gamma_c^- + \Gamma^+ + \Gamma^- + \widetilde{\Gamma}_\delta + \Gamma_\infty - \Gamma_\delta$ as shown in Figure 6(a). Here $\widetilde{\Gamma}_\delta$ is the vanishingly small circular path enclosing the point $z = -a$. The line integral over the parts $\Gamma^+ + \Gamma^-$ makes no contribution to $\mathcal{M}\{v, \widehat{v}^J; \Gamma_0\}$. Furthermore, there is no contribution from the $\widetilde{\Gamma}_\delta$. Thus the conservation law $\mathcal{M}\{v, \widehat{v}^J; \Gamma_0\} = 0$ implies

$$\mathcal{M}\{v, \widehat{v}^J; \Gamma_\delta\} \mathcal{M}\{v, \widehat{v}^J; \Gamma_c^+ + \Gamma_c^-\} + \mathcal{M}\{v, \widehat{v}^J; \Gamma_\infty\}. \tag{5.12}$$

Evaluating the integrals $\mathcal{M}\{v, \widehat{v}^J; \Gamma_c^+ + \Gamma_c^-\}$ and $\mathcal{M}\{v, \widehat{v}^J; \Gamma_\infty\}$, it can be shown that (Appendix D)

$$\mathbf{k} = \frac{1}{2\sqrt{\pi a}} \int_{-a}^{a} \mathbf{Y}(\zeta_0^{-i\varepsilon}, \zeta_0^{-\kappa})(\mathbf{I} + i\beta)^{-1}\{(\mathbf{I} - \alpha)\mathbf{t}^+ - (\mathbf{I} + \alpha)t^-\}\sqrt{\frac{a + x_1}{a - x_1}}\, \mathrm{d}x_1,$$

$$-\frac{1}{2\sqrt{\pi a}} \mathbf{Y}((2a)^{-i\varepsilon}, (2a)^{-\kappa})(\mathbf{I} + \beta^2)^{-1}\{\beta(\mathbf{I} - \alpha) + (\mathbf{I} - \alpha)\mathbf{S}^{(1)T}\}\mathbf{q}_c, \tag{5.13}$$

where

$$\zeta_0 = \frac{2a(-x_1 e^{i\pi} - a)}{(-x_1 e^{i\pi} + a)} \quad \text{and} \quad \mathbf{q}_c = \int_{-a}^{a} \{\mathbf{t}^+(x_1) + \mathbf{t}^-(x_1)\}\, \mathrm{d}x_1.$$

It is noted that (5.13) is also valid for a degenerate case since the equation does not involve explicitly p_M, **A** and **B**.

Next, let us consider a finite crack with singularities such as a concentrated force and a dislocation as shown in Figure 4(c). Electromechanical singularities **q** and **b** are embedded in the elastic material 2 at the point $z = z^0$. A contour Γ_0 consisting of $\Gamma_c^+ + \Gamma_c^- + \Gamma^+ + \Gamma^- + \widetilde{\Gamma}_\delta + \Gamma_\infty - \Gamma_{z^0} - \Gamma_\delta$ as shown in Figure 6(b) is chosen to compute the right hand sides of (4.13b). The line integral over the parts $\Gamma^+ + \Gamma^-$ makes no contribution to $\mathcal{M}\{v, \widehat{v}^J; \Gamma_0\}$. Furthermore, there is no contribution from $\widetilde{\Gamma}_\delta$. Thus the conservation law $\mathcal{M}\{v, \widehat{v}^J; \Gamma_0\} = 0$ implies that:

$$\mathcal{M}\{v, \widehat{v}^J; \Gamma_\delta\} = -\mathcal{M}\{v, \widehat{v}^J; \Gamma_{z^0}\} + \mathcal{M}\{v, \widehat{v}^J; \Gamma_\infty\}. \tag{5.14}$$

Calculating the integrals $\mathcal{M}\{v, \widehat{v}^J; \Gamma_{z^0}\}$ and $\mathcal{M}\{v, \widehat{v}^J; \Gamma_\infty\}$, it can be shown that **k** is obtained from (4.13b) and (5.14) as (Appendix E)

$$k_M = -\frac{1}{\sqrt{\pi a}}\mathrm{Re}\left[\sum_{S=1}^{4}\sqrt{\frac{z_S^0+a}{z_S^0-a}}\right.$$

$$\left.\times Y(\zeta_S^{0\,-i\varepsilon},\zeta_S^{0\,-\kappa})_{MN}\,H_{NR}^{-1}B_{SR}^{(2)-1}(B_{LS}^{(2)}b_L+A_{LS}^{(2)}q_L)\right]$$

$$+\frac{1}{\sqrt{\pi a}}Y((2a)^{-i\varepsilon},(2a)^{-\kappa})_{MN}U_{NL}b_L-\frac{1}{\sqrt{\pi a}}$$

$$\times Y((2a)^{-i\varepsilon},(2a)^{-\kappa})_{MN}U_{NR}(L^{(1)-1}S^{(1)T}-\beta^T L^{(1)-1})_{RL}q_L, \qquad (5.15)$$

where $\zeta_S^0 = (2a(z_S^0-a)/(z_S^0+a)$. (5.15) is not valid for a degenerate case as noted earlier. For the homogeneous anisotropic case, (5.15) reduces to

$$k_M = -\frac{1}{\sqrt{\pi a}}\mathrm{Im}\left[\sum_{S=1}^{4}\sqrt{\frac{z_S^0+a}{z_S^0-a}}(B_{MS}B_{LS}b_L+B_{MS}A_{LS}q_L)\right]$$

$$+\frac{1}{2\sqrt{\pi a}}(L_{ML}b_L-S_{LM}q_L). \qquad (5.16)$$

The stress intensity factor for the special case in which singularities are located on the interface can be obtained by taking the limit $z^0 \to x_1^0(|x_1^0|>a)$ in (5.15)

$$\mathbf{k} = \frac{1}{2\sqrt{\pi a}}\mathbf{Y}((2a)^{-i\varepsilon},(2a)^{-\kappa})\left\{\mathbf{I}-\sqrt{\left|\frac{x_1^0+a}{x_1^0-a}\right|}\mathbf{Y}\left(\left|\frac{x_1^0-a}{x_1^0+a}\right|^{-i\varepsilon},\left|\frac{x_1^0-a}{x_1^0+a}\right|^{-\kappa}\right)\right\}$$

$$\times[2\mathbf{Ub}-(\mathbf{I}+\mathbf{B}^2)^{-1}\{\beta(\mathbf{I}-\alpha)+(\mathbf{I}-\alpha)\mathbf{S}^{(1)T}\}\mathbf{q}]. \qquad (5.17)$$

6. Concluding remarks

Complete stress and electric displacement fields near the tip of a crack between two dissimilar anisotropic piezoelectric media are obtained in terms of two bimaterial matrices, which may be considered as the generalized Dundurs parameters. New definitions of real-valued stress and electric displacement intensity factors are proposed. The intensity factors of stress and electric displacement for the interfacial crack recover the classical stress and electric displacement intensity factors, as the bimaterial continuum degenerates to be a homogeneous one. Moreover, it is not necessary to solve the eigenvalue problem to determine the stress and electric displacement intensity factors since the intensity factors are related explicitly to the bimaterial matrix β, but not the eigenvectors. Intensity factors of stress and electric displacement with the same dimension of the intensity factors for a crack in a homogeneous piezoelectric material are also discussed. The stress and electric displacement intensity factors associated with an interfacial crack between two dissimilar anisotropic piezoelectric media are represented by the mutual integrals proposed in this paper. Closed form solutions of the stress and electric displacement intensity factors for a semi-infinite crack as well as for a finite crack at the interface between two dissimilar piezoelectric media are obtained by using the mutual integral.

Appendix A. Derivation of (3.9)

Introducing a new function vector $\psi(z)$ defined by

$$\psi(z) = (\mathbf{I} + i\beta)^{-1}\mathbf{B}^{(1)}\mathbf{f}'^{(1)}(z), \tag{A1}$$

(3.8) is rewritten as

$$(\mathbf{I} + i\beta)\psi^{+}(x_1) + (\mathbf{I} - i\beta)\psi^{-}(x_1) = 2(\mathbf{I} + i\beta)^{-1}(\mathbf{I} + \alpha)\mathbf{h}(x_1), \quad -\infty < x_1 < 0. \tag{A2}$$

A homogeneous solution $\chi(z)$ which satisfies the homogeneous Hilbert problem

$$(\mathbf{I} + i\beta)\chi^{+}(x_1) + (\mathbf{I} - i\beta)\chi^{-}(x_1) = 0, \quad -\infty < x_1 < 0, \tag{A3}$$

can be found by considering functions of the form $\chi(z) = z^{-(1/2)+i\delta}\mathbf{v}$, where $\mathbf{v}$ is a eigenvector. Substitution of $\chi(z) = z^{-(1/2)+i\delta}\mathbf{v}$ into (A3) yields

$$(\beta + i\lambda\mathbf{I})\mathbf{v} = 0, \tag{A4}$$

where $\lambda = \tanh\pi\delta$. Solving the eigenvalue problem (A4), we have the four eigenvalues, $\lambda_1 = \eta, \lambda_2 = -\eta, \lambda_3 = -i\omega$ and $\lambda_4 = i\omega$, and the associated eigenvectors, $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ and $\mathbf{v}_4$ [6]. A general expression for the homogeneous solution may be written as

$$\chi(z)\frac{1}{2\sqrt{2\pi z}}\mathbf{V}\mathbf{X}(z^{i\varepsilon}, z^{\kappa})\mathbf{V}^{-1}\mathbf{g}(z), \tag{A5}$$

where $\mathbf{V} = (\mathbf{v}_1\ \mathbf{v}_2\ \mathbf{v}_3\ \mathbf{v}_4)$ and $\mathbf{X}(\xi, \zeta) = \operatorname{diag}(\xi\ \xi^{-1}\ \zeta\ \zeta^{-1})$. Defining a matrix function $\mathbf{Y}(\xi, \zeta)$ as

$$\mathbf{Y}(\xi, \zeta) = \mathbf{V}\mathbf{X}(\xi, \zeta)\mathbf{V}^{-1}. \tag{A6}$$

(A5) is rewritten as

$$\chi(z) = \frac{1}{2\sqrt{2\pi z}}\mathbf{Y}(z^{i\varepsilon}, z^{\kappa})\mathbf{g}(z), \tag{A7}$$

Making use of the following relations

$$\begin{aligned} &\beta\mathbf{V} = -i\mathbf{V}\Lambda, \\ &\beta^4 + (\eta^2 - \omega^2)\beta^2 - \eta^2\omega^2\mathbf{I} = 0, \\ &\Lambda = \eta\mathbf{I}_1 + i\omega\mathbf{I}_2, \end{aligned} \tag{A8}$$

where $\Lambda = \mathrm{diag}(\lambda_1\ \lambda_2\ \lambda_3\ \lambda_4), \mathbf{I}_1 = \mathrm{diag}(1 -1\ 0\ 0)$ and $\mathbf{I}_2 = \mathrm{diag}(0\ 0 -1\ 1)$, it can be shown that the matrix function $\mathbf{Y}(\xi, \zeta)$ defined in (A7) is expressed in terms of β as (3.11). A particular solution of (A2) is also given by

$$\psi(z) = (\mathbf{I} + i\beta)^{-1}(\mathbf{I} + \alpha)\mathbf{h}(z). \tag{A9}$$

Thus the general solution of (A2) for $\psi(z)$ is

$$\psi(z) = \frac{1}{2\sqrt{2\pi z}}\mathbf{Y}(z^{i\varepsilon}, z^{\kappa})\mathbf{g}(z) + (\mathbf{I} + i\beta)^{-1}(\mathbf{I} + \alpha)\mathbf{h}(z). \tag{A10}$$

Finally, we get (3.9) from (A1) and (A10).

Appendix B. Derivation of (4.4)

$J\{v; \Gamma_\delta\}$ can be written as

$$\begin{aligned} J\{v; \Gamma_\delta\} &= \frac{1}{2}\int_{\Gamma_\delta^{(1)}} \{\mathbf{f}'^{(1)T}(z)\mathbf{f}'^{(1)}(z) - \bar{\mathbf{f}}'^{(2)T}(z)\bar{\mathbf{f}}'^{(2)}(z)\}\,\mathrm{d}z \\ &\quad + \frac{1}{2}\int_{\Gamma_\delta^{(2)}} \{\mathbf{f}'^{(2)T}(z)\mathbf{f}'^{(2)}(z) - \bar{\mathbf{f}}'^{(1)T}(z)\bar{\mathbf{f}}'^{(1)}(z)\}\,\mathrm{d}z, \end{aligned} \tag{B1}$$

where Γ_δ consists of $\Gamma_\delta^{(1)}$ and $\Gamma_\delta^{(2)}$ which are the paths contained in the materials 1 and 2, respectively, Using (3.19) and (5.6) together with the relations

$$\begin{aligned} &\mathbf{Y}(z^{i\varepsilon}, z^{\kappa}) = \overline{\mathbf{Y}(\bar{z}^{i\varepsilon}, \bar{z}^{\kappa})}, \\ &(\mathbf{I} + i\beta)^T(\mathbf{L}^{(1)-1} + \mathbf{L}^{(2)-1})(\mathbf{I} + i\beta) = \mathbf{U}^{-1}, \end{aligned} \tag{B2}$$

it can be shown that

$$\mathbf{f}'^{(1)T}(z)\mathbf{f}'^{(1)}(z) - \bar{\mathbf{f}}'^{(2)T}(z)\bar{\mathbf{f}}'^{(2)}(z) = -\frac{i}{4\pi z}\mathbf{k}^T\mathbf{U}^{-1}\mathbf{k}. \tag{B3}$$

From (B1) and (B3), (4.4) is obtained.

Appendix C. Derivation of (4.11)

The solution procedure is similar to the case of a semi-infinite crack presented in Appendix A, which is briefly described as follows. For a finite crack in interval $(-a, a)$, the Hilbert problem (A2) is replaced by

$$(\mathbf{I} + i\beta)\psi^+(x_1) + (\mathbf{I} - i\beta)\psi^-(x_1) = 2(\mathbf{I} + i\beta)^{-1}(\mathbf{I} + \alpha)\mathbf{h}(x_1), \quad -a < x_1 < a. \tag{C1}$$

Considering functions of the form $\chi(z) = \{(z - a)/(z + a)\}^{-(1/2)+i\delta}\mathbf{v}$, it can be shown that a general expression for the homogeneous solution may be written as

$$\chi(z) = \frac{1}{4\sqrt{\pi a}}\sqrt{\frac{z + a}{z - a}}\mathbf{Y}\left(\left(2a\frac{z - a}{z + a}\right)^{i\varepsilon}, \left(2a\frac{z - a}{z + a}\right)^{\kappa}\right)\mathbf{g}(z). \tag{C2}$$

A particular solution of (C1) is also by (A8). Thus the general solution of (C1) for $\psi(z)$ is

$$\psi(z) = \frac{1}{4\sqrt{\pi a}}\sqrt{\frac{z+a}{z-a}}\mathbf{Y}\left(\left(2a\frac{z-a}{z+a}\right)^{i\varepsilon}, \left(2a\frac{z-a}{z+a}\right)^{\kappa}\right)\mathbf{g}(z) + (\mathbf{I}+\alpha)\mathbf{h}(z). \qquad \text{(C3)}$$

Admissible auxiliary solutions can be chosen from (A1) and (C3) with $\mathbf{g}(z) = \widehat{\mathbf{e}}^J$ and $\mathbf{h}(z) = 0$ as (4.11).

Appendix D. Derivation of (5.13)

The integral $\mathcal{M}\{v, \widehat{v}^J; \Gamma_c^+ + \Gamma_c^-\}$ corresponding to each auxiliary field $\widehat{v}^J (J = 1,2,3,4)$ is evaluated by (4.5) with $v^J_{M,1}$ corresponding to the potentials given by (4.11), which results in

$$\begin{aligned}\mathcal{M}\{v, \widehat{v}^J; \Gamma_c^+ + \Gamma_c^-\} &= \frac{1}{2\sqrt{\pi a}}\int_{-a}^{a}(\mathbf{t}^{T+}\mathbf{L}^{(1)-1} - \mathbf{t}^{T-}\mathbf{L}^{(2)-1})(\mathbf{I}+i\beta) \\ &\quad \times\mathbf{Y}(\zeta_0^{i\varepsilon}, \zeta_0^{\kappa})\widehat{\mathbf{e}}^J\sqrt{\frac{a+x_1}{a-x_1}}\,\mathrm{d}x_1.\end{aligned} \qquad \text{(D1)}$$

In obtaining (D1), (2.9) and (5.3) have been used. The potentials near infinity can be written as

$$\begin{aligned}\mathbf{f}'^{(1)}(z) &= \frac{1}{2\pi z}\mathbf{B}^{(1)-1}\mathbf{H}^{-1}\mathbf{B}^{(2)-T}\mathbf{A}^{(2)T}\mathbf{q}_c + \mathrm{O}\left(\frac{1}{z}\right)^2, \\ \mathbf{f}'^{(2)}(z) &= \frac{1}{2\pi z}\mathbf{B}^{(2)-1}\overline{\mathbf{H}}^{-1}\mathbf{B}^{(1)-T}\mathbf{A}^{(1)T}\mathbf{q}_c + \mathrm{O}\left(\frac{1}{z}\right)^2\end{aligned} \qquad \text{(D2)}$$

From (4.8), (4.11) and (D2), it can be shown that

$$\mathcal{M}\{v, \widehat{v}^J; \Gamma_\infty\} = \frac{-1}{2\sqrt{\pi a}}\mathbf{q}_c^T(\mathbf{S}^{(1)}\mathbf{L}^{(1)-1} - \mathbf{L}^{(1)-1}\beta)\mathbf{Y}((2a)^{i\varepsilon})\widehat{\mathbf{e}}^J. \qquad \text{(D3)}$$

Making use of (4.13b), (5.12), (D1) and (D3), we obtain (5.13).

Appendix E. Derivation of (5.15)

Potentials for the singularities near the point $z = z^0$ can be written as (5.8) substituting (4.11) and (5.8) into (4.8), $\mathcal{M}\{v, \widehat{v}^J; \Gamma_{z^0}\}$ corresponding to each auxiliary field is easily evaluated by its residue, to yield

$$\begin{aligned}&\mathcal{M}\{v, \widehat{v}^J; \Gamma_{z^0}\} \\ &= \frac{1}{2\sqrt{\pi a}}\mathrm{Re}\left[\sum_{S=1}^{4}\sqrt{\frac{z_S^0+a}{z_S^0-a}}\mathbf{Y}(\zeta_S^{0i\varepsilon}, \zeta_S^{0\kappa})_{KJ}U_{KN}^{-1}H_{NR}^{-1}B_{SR}^{(2)-1}(B_{LS}^{(2)}b_L + A_{LS}^{(2)}q_L)\right].\end{aligned} \qquad \text{(E1)}$$

The potentials near infinity can be written as

$$\mathbf{f}'^{(1)}(z) = \frac{1}{2\pi z}\{\mathbf{B}^{(1)-1}\mathbf{H}^{-1}\mathbf{b} + \mathbf{B}^{(1)-1}\mathbf{H}^{-1}\mathbf{B}^{(2)-T}\mathbf{A}^{(2)T}\mathbf{q}\} + \mathrm{O}\left(\frac{1}{z}\right)^2,$$
$$\mathbf{f}'^{(2)}(z) = \frac{1}{2\pi z}\{\mathbf{B}^{(2)-1}\overline{\mathbf{H}}^{-1}\mathbf{b} + \mathbf{B}^{(2)-1}\overline{\mathbf{H}}^{-1}\mathbf{B}^{(1)-T}\mathbf{A}^{(1)T}\mathbf{q}\} + \mathrm{O}\left(\frac{1}{z}\right)^2, \quad \text{as } z \to \infty. \tag{E2}$$

Making use of (4.8), (4.11) and (E2), it can be shown that

$$\begin{aligned}\mathcal{M}\{v, \widehat{v}^J; \Gamma_\infty\} &= \frac{1}{2\sqrt{\pi a}}\mathbf{b}^T\mathbf{Y}((2a)^{i\varepsilon}, (2a)^{\kappa})\widehat{\mathbf{e}}^J \\ &-\frac{1}{2\sqrt{\pi a}}\mathbf{q}^T(\mathbf{S}^{(1)}\mathbf{L}^{(1)-1} - \mathbf{L}^{(1)-1}\beta)\mathbf{Y}((2a)^{i\varepsilon}, (2a)^{\kappa})\widehat{\mathbf{e}}^J.\end{aligned} \tag{E3}$$

Since $\mathcal{M}\{v, \widehat{v}^J; \Gamma_{z0}\}$ and $\mathcal{M}\{v, \widehat{v}^J; \Gamma_\infty\}$ are determined as above, $\mathbf{k}$ is obtained from (4.13b) and (5.14) as (5.15).

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