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Polycross designs with complete neighbor balance

John P. Morgan
Department of Mathematics and Statistics, Old Dominion University, Norfolk, Virginia 23529, USA

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Summary

Polycross designs for n clones in n^2 replicates, composed of n $n \times n$ squares, are presented, n being any positive integer. The method depends on whether n is odd or even, and for even n the squares are Latin. In either case, each clone has every other clone as a nearest neighbor exactly n times in each of the four primary directions (N, S, E, W) and $n-2$ times in each of the four intermediate directions (NE, SE, SW, NW). Also, each clone has itself as nearest neighbor $n-1$ times in each intermediate direction.

Introduction

Polycross designs for plant breeding are arrangements of replicates of n clones in a rectangular pattern so that each clone is an immediate neighbor of each other clone an approximately equal number of times. Such an arrangement is an attempt to insure that all $n(n-1)/2$ pairs of distinct clones have an equal chance of crossing. The vagaries of wind direction and other environmental factors make guaranteeing this equality of crosses impossible; the polycross method simply attempts to make conditions as favorable as possible.

If there is a prevailing wind during the pollination season then concern is focused on the downwind neighbors of a clone, for example its NE, E, and SE neighbors for a westerly wind. Freeman (1967) has given designs of this type in 3-rowed arrays for $n \leqq 40$. Regardless of whether or not there is a prevailing wind, maximum protection is afforded by using a design for which the nearest neighbors of a clone are balanced in each of the eight directions individually. The first such completely balanced polycross design was given by Olesen (1976), who presented a formula for polycrosses in n $n \times n$ Latin squares, where n is any even integer such that $n+1$ is prime. In this design, each clone has every other clone as nearest neighbor n times in each of the primary directions and $n-2$ times in each of the intermediate directions. Also each clone has itself as neighbor $n-1$ times in each intermediate direction.

In this paper the result of Olesen (1976) is extended in two ways: it is shown that a completely balanced polycross design in n Latin squares of side n may be obtained for any even n (dropping the restriction that $n+1$ be prime), and that the same neighbor balance properties may be obtained for odd n in n $n \times n$ squares which are not Latin.

The designs for even n

Let the number of clones, n, be even. In order to simplify the statement and derivation of the proposed polycross formula, we introduce the function h(i,j) given by

$$h(i,j) = [(-1)^i \text{int } (j/2)] \pmod n$$

for natural i and j. Here int(.) means the integer part of the enclosed number (e.g. int(3.5) = 3, int(4) = 4). An expression (mod n) means the unique remainder among the set (1,2, . . ., n) upon dividing that expression by n, where n is to be used if the remainder is 0, and where n is added to the remainder if it is negative. For example, with $n = 6$, $h(3,5) = [(-1)^3 \,\mathrm{int}(2.5)] \pmod 6 = (-2) \pmod 6 = 4$. The following important properties of the function h are easily demonstrated:

(i) $h(i-1,j) = h(i+1,j) = -h(i,j)$

(ii) $h(i,i)$ for $i = 1,2, \ldots, n$ is $1,2, \ldots, n$ in some order

(iii) $h(i,i) + h(i,i+1)$ for $i = 1,2, \ldots, n-1$ is $1,2, \ldots, n-1$ in some order.

The construction formula for the n $n \times n$ squares is: place clone c in row i and column j (cell (i,j)) of square k, where

$$c = h(i,i) + h(j,j) + k \pmod n \quad i,j,k = 1,2, \ldots, n. \tag{1}$$

Note that since this expression is symmetric in rows and columns, any property which holds for the columns also holds for the rows. Also, as we now show, each clone occurs once in each row and each column of each square, so that the design forms n Latin squares. For suppose the same clone appears twice in column j of square k, say in rows i_1 and i_2. Then

$$h(i_1,i_1) + h(j,j) + k = h(i_2,i_2) + h(j,j) + k \pmod n$$

which implies that $h(i_1,i_1) = h(i_2,i_2) \pmod n$, which contradicts property (ii). Thus each clone occurs once in each column, and by symmetry the same result is obtained for rows.

Since all equations and indices in this paper are (mod n), this symbol will henceforth be omitted.

We will now show that the proposed design has complete neighbor balance, in that each clone has every other clone as nearest neighbor n times in each primary direction, $n-2$ times in each intermediate direction, and itself as nearest neighbor $n-1$ times in each intermediate direction.

Let clone c occur in row i of square k. Its column is j where by (1) j satisfies

$$h(j,j) = c - k - h(i,i) \tag{2}$$

and its south neighbor is (again from (1))

$$h(i+1,i+1) + h(j,j) + k. \tag{3}$$

By replacing h(j,j) in (3) by (2), the set of south neighbors for clone c in square k is

$$c + h(i+1,i+1) - h(i,i) = c - [h(i,i) + h(i,i+1)] \quad i = 1,2, \ldots, n-1 \tag{4}$$

by property (i) (for $i = n$ c has no south neighbor). Applying property (iii) (4) becomes

$$c - i \text{ with } i = 1,2, \ldots, n-1 \text{ in some order.} \tag{5}$$

Each clone except c occurs in (5) exactly once, so clone c has each other clone as its south neighbor once in square k and hence n times in the set of n squares. Likewise we can show that clone c has each other clone n times as north neighbor, and the symmetry of rows and columns gives the corresponding results for east and west neighbors. Note that this argument shows that each of the n squares is neighbor balanced in the primary directions.

Now, if c is in cell (i,j) of square k, then k satisfies

$$k = c - h(i,i) - h(j,j) \tag{6}$$

and the SE neighbor of c is (for $i \neq n$ and $j \neq n$)

$$h(i+1,i+1) + h(j+1,j+1) + k = c - [h(i,i) + h(i,i+1)] - [h(j,j) + h(j,j+1)] \tag{7}$$

by using (6) in the left-hand side of (7) and applying property (i). In the set of n squares, c occurs in each possible (i,j) exactly once, so the full set of SE neighbors of clone c is (7) evaluated for $i = 1,2, \ldots, n-1$ and $j = 1,2, \ldots, n-1$ (c has no SE neighbor for $i = n$ or $j = n$). By property (iii) these are $c-i-j$ with $i = 1,2, \ldots, n-1$ and $j = 1,2, \ldots, n-1$ both in some order, i.e. each clone $n-2$ times except c $n-1$ times. The neighbors in the other three intermediate directions are shown similarly.

As an example here is the design for $n = 6$ clones. Each clone is neighbored by each other clone 6 times in each primary direction and 4 times in each intermediate direction. Each clone is neighbored by itself 5 times in each intermediate direction.

1 2 6 3 5 4	2 3 1 4 6 5	3 4 2 5 1 6
2 3 1 4 6 5	3 4 2 5 1 6	4 5 3 6 2 1
6 1 5 2 4 3	1 2 6 3 5 4	2 3 1 4 6 5
3 4 2 5 1 6	4 5 3 6 2 1	5 6 4 1 3 2
5 6 4 1 3 2	6 1 5 2 4 3	1 2 6 3 5 4
4 5 3 6 2 1	5 6 4 1 3 2	6 1 5 2 4 3

4 5 3 6 2 1	5 6 4 1 3 2	6 1 5 2 4 3
5 6 4 1 3 2	6 1 5 2 4 3	1 2 6 3 5 4
3 4 2 5 1 6	4 5 3 6 2 1	5 6 4 1 3 2
6 1 5 2 4 3	1 2 6 3 5 4	2 3 1 4 6 5
2 3 1 4 6 5	3 4 2 5 1 6	4 5 3 6 2 1
1 2 6 3 5 4	2 3 1 4 6 5	3 4 2 5 1 6

For a general mathematical treatment of these sets of n Latin squares based on abelian groups, see Morgan (1987).

The designs for odd n

Now let the number of clones, n, be odd. The polycross construction requires the function g(i,j) given by

$$g(i,j) = \begin{cases} (-1)^i\,\mathrm{int}(j/2) & \text{if } j = 1,2,\ldots,m \\ (-1)^i\,\mathrm{int}((j+1)/2) & \text{if } j = m+1,\ m+2,\ \ldots,\ n-1 \\ n & \text{if } j = n \end{cases}$$

where $m = [(-1)^{(n+1)/2}]\,(n+1)/2$ (again all quantities are evaluated (mod n)). This function is easily shown to satisfy

(iv) $g(i-1,j) = g(i+1,j) = -g(i,j)$

(v) $g(i,i)$ for $i = 1,2,\ldots,n$ is $1,2,\ldots,n$ in some order, except that n occurs twice and $n-(m/2)$ does not occur at all

(vi) $g(i,i) + g(i,i+1)$ for $i = 1,2,\ldots,n-1$ is $1,2,\ldots,n-1$ in some order.

The construction formula for the n $n \times n$ squares is: place clone c in row i and column j of square k, where

$$c = g(i,i) + g(j,j) + k \pmod n \qquad i,j,k = 1,2,\ldots,n. \tag{8}$$

For example, the design for n = 7 clones is

1 2 7 3 5 4 1	2 3 1 4 6 5 2	3 4 2 5 7 6 3
2 3 1 4 6 5 2	3 4 2 5 7 6 3	4 5 3 6 1 7 4
7 1 6 2 4 3 7	1 2 7 3 5 4 1	2 3 1 4 6 5 2
3 4 2 5 7 6 3	4 5 3 6 1 7 4	5 6 4 7 2 1 5
5 6 4 7 2 1 5	6 7 5 1 3 2 6	7 1 6 2 4 3 7
4 5 3 6 1 7 4	5 6 4 7 2 1 5	6 7 5 1 3 2 6
1 2 7 3 5 4 1	2 3 1 4 6 5 2	3 4 2 5 7 6 3

4 5 3 6 1 7 4	5 6 4 7 2 1 5
5 6 4 7 2 1 5	6 7 5 1 3 2 6
3 4 2 5 7 6 3	4 5 3 6 1 7 4
6 7 5 1 3 2 6	7 1 6 2 4 3 7
1 2 7 3 5 4 1	2 3 1 4 6 5 2
7 1 6 2 4 3 7	1 2 7 3 5 4 1
4 5 3 6 1 7 4	5 6 4 7 2 1 5

6 7 5 1 3 2 6	7 1 6 2 4 3 7
7 1 6 2 4 3 7	1 2 7 3 5 4 1
5 6 4 7 2 1 5	6 7 5 1 3 2 6
1 2 7 3 5 4 1	2 3 1 4 6 5 2
3 4 2 5 7 6 3	4 5 3 6 1 7 4
2 3 1 4 6 5 2	3 4 2 5 7 6 3
6 7 5 1 3 2 6	7 1 6 2 4 3 7

Although the design has the same neighbor counts as when n is even (as will be shown), the squares are no longer Latin, due to the fact that g does not satisfy a property analogous to (ii) for h. For an experimenter interested in a statistical analysis such as for comparing seed amounts from different parents, loss of the Latin property means that calculations will be somewhat more complicated; this is a small price to pay for neighbor balance. Otherwise the fact that the squares are not Latin is inconsequential. In general, property (v) says that each row (or column) contains one clone twice, $n-2$ clones once, and one clone not at all. Clones occurring twice in a row or column are always at the ends since $g(i,i) = n$ for $i = 1$ and $i = n$. Each clone appears equally often in the design.

To see that the design proposed here has the same neighbor properties as the design for even n, consider column j of square k. If c occurs in this column its row is i where i satisfies (8) or

$$g(i,i) = c-g(j,j)-k. \tag{9}$$

The south neighbor of c in square k is then

$$g(i+1,i+1) + g(j,j) + k = c - [g(i,i) + g(i,i+1)] \tag{10}$$

by using (9) to replace g(j,j) in the left-hand side of (10) and applying (iv). Within the set of all n j^{th} columns in the n squares, c occurs once in each row $i = 1,2,\ldots,n-1$. So its set of south neighbors in the n j^{th} columns is (10) evaluated for $i = 1,2,\ldots,n-1$, which by (vi) is each clone except itself once. Hence in all columns c has each other clone as south neighbor n times. Likewise it has each other clone as north neighbor n times, and symmetry gives the result for east and west neighbors.

The steps to show that there is neighbor balance in the intermediate directions are similar to those used for even n. If c is in cell (i,j) of square k, then k satisfies

$$k = c - g(i,i) - g(j,j) \tag{11}$$

and the SE neighbor of c is (for $i \neq n$ and $j \neq n$)

$$g(i+1,i+1) + g(j+1,j+1) + k = c - [g(i,i) + g(i,i+1)] - [g(j,j) + g(j,j+1)] \tag{12}$$

by using (11) in the left-hand side of (12) and applying (iv). Since in the set of n squares c occurs in each possible cell (i,j) exactly once, the set of SE neighbors of c is (12) evaluated for $i = 1,2,\ldots,n-1$ and $j = 1,2,\ldots n-1$ (c has no SE neighbor for $i = n$ or $j = n$).By (vi) these are $c-i-j$ with $i = 1,2,\ldots,n-1$ and $j = 1,2,\ldots,n-1$ both in some order, i.e. each clone $n-2$ times except c $n-1$ times. Likewise the neighbors in the other three intermediate directions can be shown.

A list of designs for N≦12

The designs for $n = 2,3,\ldots,12$ are displayed ($n = 6$ and $n = 7$ are covered in the examples above). To save space only the first square is given for $n>9$; the other $n-1$ squares may be obtained by adding successively $1,2,\ldots,n-1$ to the first square.

n = 2

12	21
21	12

n = 3

121	232	313
232	313	121
121	232	313

n = 4

1243	2314	3421	4132
2314	3421	4132	1243
4132	1243	2314	3421
3421	4132	1243	2314

n = 5

12431	23542	34153	45214	51325
23542	34153	45214	51325	12431
45214	51325	12431	23542	34153
34153	45214	51325	12431	23542
12431	23542	34153	45214	51325

n = 8

12837465	23148576	34251687
23148576	34251687	45362718
81726354	12837465	23148576
34251687	45362718	56473821
78615243	81726354	12837465
45362718	56473821	67584132
67584132	78615243	81726354
56473821	67584132	78615243

45362718	56473821	67584132
56473821	67584132	78615243
34251687	45362718	56473821
67584132	78615243	81726354
23148576	34251687	45362718
78615243	81726354	12837465
12837465	23148576	34251687
81726354	12837465	23148576

78615243	81726354
81726354	12837465
67584132	78615243
12837465	23148576
56473821	67584132
23148576	34251687
45362718	56473821
34251687	45362718

n = 9

1 2 9 3 7 4 6 5 1	2 3 1 4 8 5 7 6 2	3 4 2 5 9 6 8 7 3
2 3 1 4 8 5 7 6 2	3 4 2 5 9 6 8 7 3	4 5 3 6 1 7 9 8 4
9 1 8 2 6 3 5 4 9	1 2 9 3 7 4 6 5 1	2 3 1 4 8 5 7 6 2
3 4 2 5 9 6 8 7 3	4 5 3 6 1 7 9 8 4	5 6 4 7 2 8 1 9 5
7 8 6 9 4 1 3 2 7	8 9 7 1 5 2 4 3 8	9 1 8 2 6 3 5 4 9
4 5 3 6 1 7 9 8 4	5 6 4 7 2 8 1 9 5	6 7 5 8 3 9 2 1 6
6 7 5 8 3 9 2 1 6	7 8 6 9 4 1 3 2 7	8 9 7 1 5 2 4 3 8
5 6 4 7 2 8 1 9 5	6 7 5 8 3 9 2 1 6	7 8 6 9 4 1 3 2 7
1 2 9 3 7 4 6 5 1	2 3 1 4 8 5 7 6 2	3 4 2 5 9 6 8 7 3
4 5 3 6 1 7 9 8 4	5 6 4 7 2 8 1 9 5	6 7 5 8 3 9 2 1 6
5 6 4 7 2 8 1 9 5	6 7 5 8 3 9 2 1 6	7 8 6 9 4 1 3 2 7
3 4 2 5 9 6 8 7 3	4 5 3 6 1 7 9 8 4	5 6 4 7 2 8 1 9 5
6 7 5 8 3 9 2 1 6	7 8 6 9 4 1 3 2 7	8 9 7 1 5 2 4 3 8
1 2 9 3 7 4 6 5 1	2 3 1 4 8 5 7 6 2	3 4 2 5 9 6 8 7 3
7 8 6 9 4 1 3 2 7	8 9 7 1 5 2 4 3 8	9 1 8 2 6 3 5 4 9
9 1 8 2 6 3 5 4 9	1 2 9 3 7 4 6 5 1	2 3 1 4 8 5 7 6 2
8 9 7 1 5 2 4 3 8	9 1 8 2 6 3 5 4 9	1 2 9 3 7 4 6 5 1
4 5 3 6 1 7 9 8 4	5 6 4 7 2 8 1 9 5	6 7 5 8 3 9 2 1 6
7 8 6 9 4 1 3 2 7	8 9 7 1 5 2 4 3 8	9 1 8 2 6 3 5 4 9
8 9 7 1 5 2 4 3 8	9 1 8 2 6 3 5 4 9	1 2 9 3 7 4 6 5 1
6 7 5 8 3 9 2 1 6	7 8 6 9 4 1 3 2 7	8 9 7 1 5 2 4 3 8
9 1 8 2 6 3 5 4 9	1 2 9 3 7 4 6 5 1	2 3 1 4 8 5 7 6 2
4 5 3 6 1 7 9 8 4	5 6 4 7 2 8 1 9 5	6 7 5 8 3 9 2 1 6
1 2 9 3 7 4 6 5 1	2 3 1 4 8 5 7 6 2	3 4 2 5 9 6 8 7 3
3 4 2 5 9 6 8 7 3	4 5 3 6 1 7 9 8 4	5 6 4 7 2 8 1 9 5
2 3 1 4 8 5 7 6 2	3 4 2 5 9 6 8 7 3	4 5 3 6 1 7 9 8 4
7 8 6 9 4 1 3 2 7	8 9 7 1 5 2 4 3 8	9 1 8 2 6 3 5 4 9

n = 10

1	2	10	3	9	4	8	5	7	6
2	3	1	4	10	5	9	6	8	7
10	1	9	2	8	3	7	4	6	5
3	4	2	5	1	6	10	7	9	8
9	10	8	1	7	2	6	3	5	4
4	5	3	6	2	7	1	8	10	9
8	9	7	10	6	1	5	2	4	3
5	6	4	7	3	8	2	9	1	10
7	8	6	9	5	10	4	1	3	2
6	7	5	8	4	9	3	10	2	1

n = 11

1	2	11	3	10	4	8	5	7	6	1
2	3	1	4	11	5	9	6	8	7	2
11	1	10	2	9	3	7	4	6	5	11
3	4	2	5	1	6	10	7	9	8	3
10	11	9	1	8	2	6	3	5	4	10
4	5	3	6	2	7	11	8	10	9	4
8	9	7	10	6	11	4	1	3	2	8
5	6	4	7	3	8	1	9	11	10	5
7	8	6	9	5	10	3	11	2	1	7
6	7	5	8	4	9	2	10	1	11	6
1	2	11	3	10	4	8	5	7	6	1

n = 12

1	2	12	3	11	4	10	5	9	6	8	7
2	3	1	4	12	5	11	6	10	7	9	8
12	1	11	2	10	3	9	4	8	5	7	6
3	4	2	5	1	6	12	7	11	8	10	9
11	12	10	1	9	2	8	3	7	4	6	5
4	5	3	6	2	7	1	8	12	9	11	10
10	11	9	12	8	1	7	2	6	3	5	4
5	6	4	7	3	8	2	9	1	10	12	11
9	10	8	11	7	12	6	1	5	2	4	3
6	7	5	8	4	9	3	10	2	11	1	12
8	9	7	10	6	11	5	12	4	1	3	2
7	8	6	9	5	10	4	11	3	12	2	1

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