International Journal of Fracture 42: 189–207, 1990.
W.G. Knauss and A.J. Rosakis (eds) Non-linear Fracture.

On some path independent integrals and their use in fracture of nonlinear viscoelastic media

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Received 1 August 1988; accepted 15 August 1988

Abstract. In certain cases it is possible to construct work potentials and J-like path-independent integrals for monolithic or composite nonlinear viscoelastic media. In this paper we discuss some situations in which such quantities exist and are useful in the study of quasi-static initiation and continuation of crack growth. The so-called quasi-elastic approximation and a constitutive equation in the form of a single hereditary integral provide the basis for using J or J-like integrals as fracture characterizing parameters during initiation and the early stages of crack growth. It is also shown that in some cases with significant crack growth the instantaneous crack speed can be characterized in terms of a similar path-independent integral. The problem of characterizing growth of large cracks in viscoelastic media with micro-damage is discussed briefly.

1. Introduction

Rice's introduction of the path-independent J integral [1] provided the basis for a major advancement of the fracture mechanics of ductile metals and other materials exhibiting significant time-independent, nonlinear behavior. Other similar path-independent integrals have been since proposed as fracture characterizing parameters for linear and nonlinear time-dependent materials, such as those reviewed by Kanninen and Popelar [2]. In two-dimensional problems Rice's J integral involves only a contour integration around the crack edge, while other parameters developed for general inelastic behavior include an area integral as well as a contour integral (e.g., Kishimoto et al. [3] and Watanabe and Kurashige [4]). For nonlinear viscous media, an integral which is like J (but with velocities in place of displacements) has been proposed and applied by Ohji et al. [5] and Landes and Begley [6].

A primary objective of these various studies has been to identify a load parameter that enables one to predict when a crack will start to propagate and, following initiation of growth, the amount or rate of growth. In order to be useful, the relationship between crack growth and the parameter should possess at least some degree of transferability. Namely, the same fracture characterization in terms of this parameter should be applicable to the different geometries and loadings of interest; it should be possible to use results from fracture test specimens to predict crack growth in engineering structures. Stress intensity factor serves as such a parameter for linear elastic and linear viscoelastic media.

Beside path-independence, another important feature of the J integral is that it is equal to the decrease in global potential energy (per unit surface area) with self-similar crack growth. This relationship provides the basis for determining instantaneous values of J directly from fracture test specimens, thereby avoiding the need for detailed information on constitutive properties and for possibly involved theoretical calculations to determine J values.

In this paper we discuss some cases in which J-like integrals can be used as fracture characterizing parameters for quasi-static initiation and continuation of crack growth in nonlinear viscoelastic media. Large strains and the effect of distributed microcracking on the growth of much larger cracks are only briefly considered in the concluding remarks.

The central effort in establishing path-independence of J and its relation to potential energy is to demonstrate that a work potential Φ, like strain energy density, can be found which characterizes the stress (σ_{ij}) – strain (ε_{ij}) behavior of the continuum using

$$\sigma_{ij} = \partial\Phi/\partial\varepsilon_{ij} \tag{1}$$

where $\Phi = \Phi(\varepsilon_{ij}, x_k, t)$ and $i, j, k = 1, 2, 3$. The stresses and strains are referred to an orthogonal set of Cartesian coordinates x_i; explicit use of x_k and time t in Φ implies allowance for material nonhomogeneity and aging or certain viscoelastic effects, as discussed below. A significant generalization is achieved, without much additional analytical complexity, by replacing strains in (1) by the quantities ε_{ij}^R,

$$\sigma_{ij} = \partial\Phi/\partial\varepsilon_{ij}^R \tag{2}$$

As above, $\Phi = \Phi(\varepsilon_{ij}^R, x_k, t)$. The quantities ε_{ij}^R are linear functionals of strain which are defined in Section 2; one special case is $\varepsilon_{ij}^R = \partial\varepsilon_{ij}/\partial t$, corresponding to viscous behavior. As discussed by Schapery [7–9], the important aspects of Rice's original J integral carry over to nonlinear viscoelastic media if the constitutive equation is given by (1) or (2). At first we shall not restrict the material nonhomogeneity; but, as is well-known, Φ cannot depend explicitly on the coordinate in the crack plane normal to the crack edge (say x_1) if J, expressed as a contour integral, is to be independent of path.

In Section 2 the basis for using ε_{ij}^R in place of strain and various consequences for fracture theory, as described previously [9], are reviewed. The J-like integral which comes out of this formulation serves as a fracture characterizing parameter for nonlinear elastic, viscoelastic, and viscous media. Power law nonlinear behavior is assumed in Section 3 in order to obtain some explicit results for singular stress fields surrounding crack tips and related implications for fracture characterization; this study serves to extend some of Riedel and Rice's results for nonlinear creeping solids [10] to more general viscoelastic behavior. In Section 4 we discuss the applicability of J theory when stress or strain-reduced time is used in hereditary constitutive integrals.

2. J-like integral based on a single-integral constitutive equation

2.1. Constitutive equation

Outside of the highly damaged and failing material at crack tips, the deformation behavior is assumed to be characterized by a nonlinear viscoelastic constitutive equation in the form of a single hereditary integral for the strain tensor:

$$\varepsilon_{ij} = E_R \int_0^t D(t - \tau, t) \frac{\partial\varepsilon_{ij}^e}{\partial\tau} \, \mathrm{d}\tau \tag{3}$$

The quantity ε_{ij}^e is a second-order tensor which is a material function,

$$\varepsilon_{ij}^e = \varepsilon_{ij}^e(\sigma_{kl}, x_m, t) \tag{4}$$

with all indices taking the values 1, 2, 3. The coefficient E_R is a free constant which will be termed the reference modulus; it is a useful parameter in discussing special material behavior and introducing dimensionless variables.

When ε_{ij}^e is used in (3), the time argument is specified as the variable of integration; that is, t should be replaced by τ where explicitly shown in (4) and in the argument of the stress, $\sigma_{kl} = \sigma_{kl}(x_i, \tau)$. To simplify notation, the arguments of stress and strain will not be written out unless required for clarity. For all cases it will be assumed that $\varepsilon_{ij} = \varepsilon_{ij}^e = \sigma_{ij} = 0$ when $t < 0$ and $D(t - \tau, t) = 0$ when $t < \tau$. To allow for the possibility of a discontinuous change in ε_{ij}^e with time at $t = 0$, the lower integration limit in (3) and succeeding hereditary integrals should be interpreted as 0^-.

The explicit dependence of ε_{ij}^e on x_m in (4) accounts when necessary for material non-homogeneity; t is introduced to allow for aging and time-dependent residual strains (such as those due to thermal expansion in composites [11]. The function $D(t - \tau, t)$ is a creep compliance; it provides creep under constant stress as well as other hereditary effects under time-varying stress in both aging and nonaging materials. The significance of ε_{ij}^e and D will be shown by considering some special cases.

First, however, it will be useful to rewrite (3) and (4) by expressing stress in terms of strain history. Supposing that the inverses exist, and replacing the notation ε_{ij}^e by ε_{ij}^R, (4) may be rewritten as

$$\sigma_{ij} = \sigma_{ij}(\varepsilon_{kl}^R, x_m, t) \tag{5}$$

The quantity ε_{ij}^R is called *pseudo strain*; it is related to the physical strain through the inverse of (3):

$$\varepsilon_{ij}^R \equiv E_R^{-1} \int_0^t E(t - \tau, t) \frac{\partial \varepsilon_{ij}}{\partial \tau} \, d\tau \tag{6}$$

The quantity E is a relaxation modulus; its relationship to D is given by

$$\int_{\tau_0^-}^t E(t - \tau, t) \frac{\partial}{\partial \tau} D(\tau - \tau_0, \tau) \, d\tau = H(t - \tau_0) \tag{7}$$

where $\tau_0 \geqslant 0$ and $H(t - \tau_0)$ is the Heaviside step function (i.e., $H(t - \tau_0) = 0$ and 1 for $t < \tau_0$ and for $t > \tau_0$, respectively). In all cases $\varepsilon_{ij}^R = \varepsilon_{ij}^e$; the superscript R is used when we consider this tensor to be a function of strain history (6), while the superscript e is used when this tensor is viewed as a material function of stress, (4). One can verify that substitution of (3) into (6) yields $\varepsilon_{ij}^R = \varepsilon_{ij}^e$ under the condition that (7) is satisfied.

A linear viscoelastic material without residual stresses which is isotropic, homogeneous and has a constant Poisson's ratio ν is characterized by (3) if we use

$$\varepsilon_{ij}^e = E_R^{-1} [(1 + \nu)\sigma_{ij} - \nu\sigma_{kk}\delta_{ij}] \tag{8}$$

where δ_{ij} is the Kronecker delta, and the standard summation convention is followed in which repeated indices imply summation over their range. Given a uniaxial stress state ($\sigma_{11} \neq 0$ and all other $\sigma_{ij} = 0$) then (3) becomes

$$\varepsilon_{11} = \int_0^t D(t - \tau, t) \frac{\partial \sigma_{11}}{\partial \tau} \mathrm{d}\tau \tag{9}$$

If $\sigma_{11} = \sigma_0 \; H(\tau - t_0)$, where $t_0 \geqslant 0$ and σ_0 is constant, then (9) reduces to $\varepsilon_{11} = D(t - t_0, t)\sigma_0$. Since $\varepsilon_{11}/\sigma_0$ is customarily termed the creep compliance, this name shall be used for D throughout this paper. Similarly, if $\varepsilon_{11} = \varepsilon_0 H(\tau - t_0)$ for a uniaxial stress state, where ε_0 is constant, one finds from (6) that the relaxation modulus, $\sigma_{11}/\varepsilon_0$, is equal to $E(t - t_0, t)$. Linear viscoelastic behavior for a *nonaging* material is characterized when the second argument in D and E in (3) and (6) is dropped, so that $D(t - \tau)$ and $E(t - \tau)$ appear in equations (3) and (6), respectively.

The mechanisms which may require the aging form to be used for D and E (e.g., $D = D(t - \tau, t)$) are not limited to chemical processes. For example, this form accounts for the effect of transient temperatures on the creep compliance and relaxation modulus, and includes the familiar thermorheologically simple behavior of polymers as a special case. It should be noted that the expression $D(t, \tau)$ is sometimes used instead of $D(t - \tau, t)$ in characterizing viscoelastic behavior of an aging material. Although both forms are equally general, the latter is used here as it is a more convenient notation in equations which govern crack growth.

Allowing now for nonlinear, anisotropic and nonhomogeneous material, it can be seen that for the special case of a constant relaxation modulus, $E = E_R$, (6) reduces to $\varepsilon_{ij}^R = \varepsilon_{ij}$. Thus, (5) becomes the constitutive equation for an elastic-like material (in that the current stress depends on the current strain, not past values of strain). An equivalent result is found by using $D = E_R^{-1}$ in (3). Viscous behavior results by using $E = t_v E_R \delta(t - \tau)$ in (6) (where $\delta(t - \tau)$ is the Dirac delta function and t_v is a time constant), or by setting $D = (t - \tau)/t_v E_R$ in (3). In this case the pseudo strain is found to be proportional to the strain rate, $\varepsilon_{ij}^R = t_v \partial \varepsilon_{ij}/\partial t$; thus, the current stress (5) becomes a function of the current strain rate. Equation (3) takes this form after integrating it by parts, then differentiating and inverting the result.

It is desirable to introduce abbreviated notation for the hereditary integrals. Specifically, for any function f of time,

$$\{D \, \mathrm{d}f\} \equiv E_R \int_0^t D(t - \tau, t) \frac{\partial f}{\partial \tau} \mathrm{d}\tau$$

$$\{E \, \mathrm{d}f\} \equiv E_R^{-1} \int_0^t E(t - \tau, t) \frac{\partial f}{\partial \tau} \mathrm{d}\tau \tag{10}$$

Thus, (3) and (6) become, respectively,

$$\varepsilon_{ij} = \{D \, \mathrm{d}\varepsilon_{ij}^e\}, \quad \varepsilon_{ij}^R = \{E \, \mathrm{d}\varepsilon_{ij}\} \tag{11}$$

2.2. Correspondence principle

The close relationship between mechanical states of nonlinear elastic and viscoelastic media with stationary or growing cracks in media defined by (3) or (5) is given in this section. It is stated in the form of a so-called correspondence principle, and serves as the basis for the development of crack growth theory. First, let us introduce a *reference elastic solution* σ_{ij}^R, ε_{ij}^R, u_i^R corresponding to the case in which $D^{-1} = E = E_R$. This solution is specified to satisfy the field equations,

$$\frac{\partial \sigma_{ij}^R}{\partial x_j} = 0 \tag{12}$$

$$\varepsilon_{ij}^R = \frac{1}{2}\left(\frac{\partial u_i^R}{\partial x_j} + \frac{\partial u_j^R}{\partial x_i}\right) \tag{13}$$

$$\sigma_{ij}^R = \sigma_{ij}^R(\varepsilon_{kl}^R, x_m, t) \tag{14}$$

The following correspondence principle was established in [11], in which the instantaneous geometry (including cracks) is the same for both elastic and viscoelastic problems: Let surface traction $T_i = \sigma_{ij} n_j$ be a specified function of time and position (which vanishes when $t < 0$) on all surfaces; n_j is the outer, unit normal vector. Then, the nonlinear viscoelastic solution based on (3) (or (5) and (6)) is

$$\sigma_{ij} = \sigma_{ij}^R, \quad \varepsilon_{ij} = \{D\,\mathrm{d}\varepsilon_{ij}^R\}, \quad u_i = \{D\,\mathrm{d}u_i^R\} \tag{15}$$

where the variables with superscript R satisfy equations (12)–(14) and the traction boundary condition $T_i = \sigma_{ij}^R n_j$ on all surfaces.

The correspondence principle was generalized in [11] to allow for specification of displacement U_i on some or all surfaces. In this case, the specified surface displacement in the elastic problem is $U_i^R \equiv \{E\,\mathrm{d}U_i\}$; as in (15), elastic and viscoelastic stresses throughout the continuum are equal with stationary and growing cracks.

2.3. Pseudo strain energy density

Equation (2), which is a special case of (5), is needed to establish a fracture-characterizing integral which is analogous to Rice's J integral. Since all effects of strain history are contained in the hereditary integral (6), for most cases it follows from the first and second laws of thermodynamics that a potential Φ exists [11]; we shall refer to Φ as "pseudo strain energy density". As proof, consider the case of constant strain rate, starting at some time t_0, for all strain components. Equation (6) becomes for $t > t_0$,

$$\varepsilon_{ij}^R = \varepsilon_{ij} E_R^{-1}(t - t_0)^{-1} \int_{t_0}^{t} E(t - \tau, t)\mathrm{d}\tau \tag{16}$$

For $t \to t_0$, and assuming the "initial" modulus $E(0^+, t_0)$ exists, (16) becomes

$$\varepsilon_{ij}^R = \varepsilon_{ij} E(0^+, t_0)/E_R \tag{17}$$

The stresses (5) at age t_0 are thus functions of the current strains. When this elastic behavior is used along with the assumption that the internal energy density U at age t_0 is a function of only strain and entropy, thermodynamics yields [12]

$$\sigma_{ij} = \partial U/\partial \varepsilon_{ij} \tag{18}$$

If temperature is used in place of entropy as an independent variable, the Helmholtz free energy F replaces U as the potential. Using (17) we obtain (2) by taking

$$\Phi = E(0^+, t_0)U/E_R \quad \text{or} \quad \Phi = E(0^+, t_0)F/E_R \tag{19}$$

Without aging, the potential Φ can be shown to exist in a similar manner by using the long time response to constant strains if the limiting long-time relaxation modulus does not vanish. However, with significant aging (i.e., if the material ages appreciably during the time period required for most of the stress relaxation to occur) we cannot use this strain history to conclude that Φ exists (except at long times) unless the effect of aging enters (5) as a scalar factor. Having argued that stress can be derived from the potential Φ with the special case (17), we can use this potential even when the hereditary integral (6) is needed because the form of the constitutive function (5) does not depend on the relationship between ε_{ij}^R and ε_{ij}.

This approach to establishing (2) is not valid for viscous media since in this case $\varepsilon_{ij}^R = t_v \partial \varepsilon_{ij}/\partial t$ regardless of strain history. Nevertheless, for linear viscous media Onsager's principle provides a symmetric viscosity tensor [12], and thus implies the potential Φ exists; but this principle does not apply to nonlinear viscous media. For *isotropic* linear viscous and viscoelastic media with constant Poisson's ratio, an appeal to thermodynamics is not needed; Φ can be constructed directly from the inverse of (8). This potential is the same as the strain energy density for a linear elastic body except ε_{ij}^R ($\equiv \varepsilon_{ij}^e$) appears in place of ε_{ij}.

Instead of using thermodynamics to argue for the existence of Φ, one could use an experimental approach in which the path-independence of "pseudo work",

$$W^R \equiv \int \sigma_{ij}\, \mathrm{d}\varepsilon_{ij}^R \tag{20}$$

is evaluated for time periods which are short enough that aging is not significant. (Observe from (6) that only the relaxation modulus is needed to calculate ε_{ij}^R from a given strain history.) If this work is independent of path then it follows by a standard argument that (2) is valid, whether the body is viscoelastic or simply viscous; as noted above, the existence of Φ cannot be deduced from thermodynamics for the nonlinear viscous case.

2.4. The J_v integral and crack growth

In analogy with the two-dimensional J integral (plane stress, plane strain, or antiplane strain) for nonlinear elastic behavior, we introduce

$$J_v = \int_{C_1} \left(\Phi \mathrm{d}x_2 - T_i \frac{\partial u_i^R}{\partial x_1} \mathrm{d}s \right) \tag{21}$$

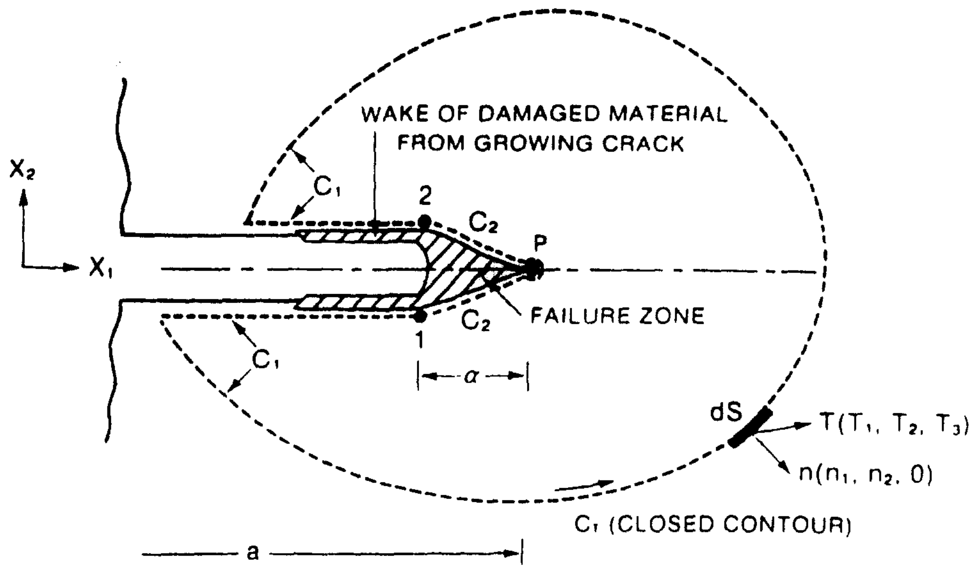

Fig. 1. Cross section of crack in nonlinear viscoelastic material showing contour C_1 (– – –) used in line integral (21). Only the opening mode of displacement is drawn, although the basic formulation allows for shearing deformation and unsymmetric damage. From [9].

where C_1 is the contour in Fig. 1 which starts at point 1 and ends at point 2. (The three-dimensional version [8] could be used, but it is not really needed to make the points of interest here.) If the crack faces are traction-free, then the contour may start and end anywhere along the crack faces to the left of the failure zone (also called the fracture process zone); this zone is, by definition, where the ultimate failure processes occur, and where (2) is not necessarily applicable. If the material is homogeneous with respect to x_1, then J_v is independent of C_1 for all contours C_1 which are outside of the failure zone. Also, in analogy with the elastic case for self-similar crack area increase, ∂A,

$$J_v = -\partial P_v/\partial A \tag{22}$$

where P_v is the potential energy of the reference elastic body expressed in terms of the variables u_i^R and ε_{ij}^R or σ_{ij}.

Equation (21), together with the correspondence principle (15), was used in [9] for some special cases of initiation and continuation of crack growth. Although more general material models of a thin-layer failure zone were treated, here we record only two results on initiation for the case in which the zone exerts a constant normal traction σ_0 on the adjacent continuum. Prior to crack growth, the opening displacement at the left edge of the failure zone (which is essentially the distance between points 1 and 2 in Fig. 1) is

$$\Delta u_{2\alpha} = \{D\mathrm{d}(J_v/\sigma_0)\} \tag{23}$$

and the equation for predicting crack growth initiation time t_i is

$$2\Gamma_i = \{D\,\mathrm{d}J_v\}_i = E_R \int_0^{t_i} D(t_i - \tau, t_i)\frac{\mathrm{d}J_v}{\mathrm{d}\tau}\,\mathrm{d}\tau \tag{24}$$

The right side of (24) is the work input to the left edge of the failure zone and the left side is the work required for rupture of the left edge (per unit of area projected onto the crack plane). For quasi-steady crack growth, in which crack speed is essentially constant during the time interval for growth of an amount equal to the failure zone length α,

$$2\Gamma \;\simeq\; E_R D(k\alpha/\dot{a},\, t)\, J_v \tag{25}$$

The coefficient k depends on the shape of the cusp-like failure zone boundary; but as a fairly good approximation $k \simeq 1/3$. (For linear viscoelasticity and the opening mode of deformation $J_v \sim K_1^2$, where K_1 is the stress intensity factor; equation (25) then takes the form derived by Mueller and Knauss [13] and Schapery [14].) As in (24), the right side is the available work and the left side is the required work (or "fracture energy").

The length α can be related to other crack-tip parameters through the requirement of bounded stresses at the crack tip. An explicit result is given in [9] for power-law nonlinear behavior of the continuum (cf. (29)) with a small-scale failure-zone,

$$\alpha \;=\; \left|\frac{\sigma_n}{\sigma_m}\right|^{1/n} \frac{J_v}{|\sigma_m|\, I_f} \tag{26}$$

where σ_n and σ_m are measures of the yield stress of the continuum (cf. (31)) and the strength of the failure-zone material, respectively. The quantity I_f is a dimensionless function of n; for an aging material it may depend on time. This equation for α, together with (25), provides a means for predicting $\dot{a}$ as a function of the instantaneous J_v, while t_i is obtained from (24) as a function of the history of J_v.

For the special case of a nonaging, isotropic, nonlinear, viscous body, we use $\Phi = \Phi(\varepsilon_{ij}^R)$ and $E = t_v E_R \delta(t - \tau)$; as noted previously this relaxation modulus yields $\varepsilon_{ij}^R = t_v \partial\varepsilon_{ij}/\partial t$, and corresponds to the creep compliance $D = (t - \tau)/t_v E_R$. Equation (24), after an integration-by-parts, and (25) reduce to

$$2\Gamma_i \;=\; \frac{1}{t_v}\int_0^{t_i} J_v \, \mathrm{d}t \tag{27a}$$

and

$$2\Gamma \;=\; \frac{k\alpha}{t_v \dot{a}}\, J_v \tag{27b}$$

If we use $t_v = 1$, then $\varepsilon_{ij}^R = \partial\varepsilon_{ij}/\partial t$; this substitution reduces J_v to the C^* parameter used for characterizing crack speed in viscous media [6]. Equation (27b) provides the physical significance of C^*,

$$C^* \;=\; \frac{2\Gamma\dot{a}}{k\alpha} \simeq 3 \times \frac{2\Gamma\dot{a}}{\alpha} \tag{28}$$

Recalling that 2Γ is the work done on the failure zone per unit projected area of new surface, it is seen that C^* is approximately three-times the mechanical power input to the failure zone per unit area of failure zone; i.e., it is three-times the crack tip power density. For viscoelastic behavior J_v does not have a simple physical interpretation. However, it should be noted that when J_v is expressed in terms of stresses or pseudo strains it is the same as for an elastic body with strain energy density Φ.

Equation (24) for t_i and (25) and (26) for $\dot{a}$ and α are expressed in terms of only one loading parameter J_v which accounts for the effect of all external loads and the geometry of the nonlinear viscoelastic body. If the fracture energies, $2\Gamma_i$ and 2Γ, and the traction in the failure zone are constant (or at least do not depend on the effects of external loading and geometry other than through J_v), these equations show that t_i is a function of only the history of J_v and that $\dot{a}$ depends on only the current value of J_v. In practice it is not necessary to use these equations explicitly. Rather, one may conduct crack growth tests on specimens to obtain these relationships and then use them directly in structural applications. Because J_v controls the crack growth, it is referred to as a fracture characterizing parameter. Determination of instantaneous values of J_v for any given application is aided by recognizing that with a small-scale failure zone it is identical to that for an elastic body with strain energy density Φ when under the same loads as the viscoelastic body. However, it should be added that the analysis leading to (23)–(25) does *not* explicitly use the assumption that α is very small compared to other dimensions of the body; however the validity of approximation (25) is uncertain when α is large.

3. J_v and K_I as fracture characterizing parameters for power-law media

Let us assume now that the material in the immediate neighborhood of a crack tip, but outside of the failure zone, is physically homogeneous and nonaging, and that the local Φ is a homogeneous function of degree $1 + 1/n$

$$\Phi(\lambda \varepsilon_{ij}^R) = |\lambda|^{1+1/n} \Phi(\varepsilon_{ij}^R) \tag{29}$$

where λ is a constant. The value of J_v then determines completely the local stress field if α is sufficiently small. Specifically, in view of (15) the stresses have the same form as for an elastic body; i.e., the so-called Hutchinson–Rice–Rosengren (HRR) singularity solution is valid [2, 15],

$$\sigma_{ij} = \sigma_n \left[\frac{J_v}{\sigma_n r} \right]^{1/(n+1)} \tilde{\sigma}_{ij}(\theta, n), \tag{30}$$

The quantity σ_n is a constant with dimensions of stress; it may be identified with the coefficient used in the uniaxial stress-pseudo strain equation derived from (29)

$$\sigma = \sigma_n |\varepsilon^R|^{1/n} \operatorname{sgn}(\varepsilon^R) \tag{31}$$

where $|\bullet|$ denotes absolute value and sgn $(\bullet)$ is the sign of its argument. Also, r is the radial distance from a material point to the stationary or propagating crack tip, and $\tilde{\sigma}_{ij}$ is a dimensionless function of n and the polar angle θ. The zone of material failure is considered to be very small relative to the size of the HRR field, but it is not limited to the thin layer of Fig. 1.

Equation (29) is of a more general form than the J_2-deformation theory for isotropic power-law materials used to develop the HRR solution. Here, the material may be isotropic or anisotropic; one can show that (29) leads to the HRR form of solution for stresses and displacements. It should be noted that a normalizing constant I_n is explicitly shown in the stresses in [2, 15]; here it is contained in $\tilde{\sigma}_{ij}$, whose dependence on θ is not necessarily the same as for the J_2-theory. Aging could be taken into account by allowing for time-dependence in σ_n, n, and $\tilde{\sigma}_{ij}$.

Because the crack tip zone of material failure is well within the zone over which (30) applies, the mechanical environment of the tip is determined by J_v; thus J_v controls the local failure processes. Both t_i and $\dot{a}$ may be affected by the history of J_v, not just its instantaneous values; but at least their dependence on J_v may be obtained from one geometry (test specimen) and then transferred to other geometries (engineering structures). This is the same justification for using J as a fracture characterizing parameter for nonlinear elastic materials [2]. It should be observed that this justification is sufficient but not necessary; recall that (24) and (25) do not depend fundamentally on the smallness of α, and yet would be transferable criteria if the material parameters which appear are constant or depend on loading and geometry only through J_v.

3.1. Generalization of the viscoelastic constitutive equation

The continuum remote to the crack tip may not satisfy (29). Nevertheless, the value of J_v in (30) can be found from a far-field contour whenever (2) applies and the material is homogeneous with respect to x_1. If, however, one or both of these conditions is not valid for the far-field, additional considerations are needed to determine J_v from far-field information. In particular, consider the situation in which the viscoelastic behavior close to the crack tip is considerably different from that of the far-field, possibly due to different creep compliances for low and high stresses. Let us assume that (29) applies close to the crack tip, and that, for simplicity, far from the tip the material is linearly viscoelastic, isotropic, nonaging, and has a creep compliance $D_l(t)$ which is different from that close to the tip; a constant Poisson's ratio is assumed. One approach to this problem is to use (2) for the entire field (apart from the failure zone) and introduce explicit time-dependence in Φ to account for the difference in compliances, not true material aging. With this approximate constitutive model, J_v is independent of path.

The approach will be illustrated using a two-term constitutive equation,

$$\varepsilon_{ij} = E_R \int_0^t D(t-\tau) \frac{\partial}{\partial \tau} [\partial \Phi_{cn}/\partial \sigma_{ij}] d\tau + E_R \int_0^t D_l(t-\tau) \frac{\partial}{\partial \tau} [\partial \Phi_{cl}/\partial \sigma_{ij}] d\tau \tag{32}$$

where the "complementary" potentials Φ_{cn} and Φ_{cl} are homogeneous functions of degree $1+n$ and 2, respectively, in σ_{ij}. Assuming $n > 1$, at sufficiently high stresses the first term will be dominant; the associated potential $\Phi_n = \Phi_n(\varepsilon_{ij}^R)$ may be found by inversion of the first term in (32) and use of a Legendre transformation, in which $\Phi_n = -\Phi_{cn} + \sigma_{ij}\varepsilon_{ij}^R$. It is readily shown that this high-stress Φ satisfies (29). Similarly, at low stresses another pseudo strain energy density Φ_l can be obtained from Φ_{cl}.

For a uniaxial, constant stress state (32) can be reduced to

$$\varepsilon = E_R D(t) |\sigma/\sigma_n|^n \operatorname{sgn}(\sigma) + D_l(t)\sigma \tag{33}$$

When a power law creep compliance for the nonlinear range is used,

$$E_R D(t) = (t/t_p)^p, \tag{34}$$

and D_l is constant (giving linear elastic response at low stress), then (33) has the form used to characterize creep of many materials [16]. Note that so-called secondary or viscous creep results if $p = 1$. The relaxation modulus corresponding to (34) is infinite at $t = 0$ and zero at $t = \infty$. This behavior does not necessarily preclude the use of thermodynamics to prove the existence of Φ_n and Φ_{cn}; in reality, the power law is usually just an approximation for a limited time period.

Consider next a constitutive equation which has a form that leads to a path-independent integral, say J_v',

$$\varepsilon_{ij}^{Rl} = \frac{\partial}{\partial \sigma_{ij}} (f\Phi_{cn} + \Phi_{cl}) \tag{35}$$

in which ε_{ij}^{Rl} is given by (6), but E is replaced by E_l where E_l is found from the non-aging form of (7) using E_l and D_l; also $f = f(t)$ is as yet an unspecified "aging" function. The associated pseudo strain energy density, $\Phi = -(f\Phi_{cn} + \Phi_{cl}) + \sigma_{ij}\varepsilon_{ij}^{Rl}$, becomes a function of only ε_{ij}^{Rl} and t after (35) is used to eliminate σ_{ij} in favor of these variables. We shall assume that (35) is a good approximation to (32) over the entire stress range provided f is chosen properly. To find f, Laplace transform both (32) and (35), equate the transformed strains and then introduce $s\bar{E}_l = 1/s\bar{D}_l$ (which follows from (7), with s as the transform parameter and the overbar denoting a Laplace transform). Thus,

$$\overline{(f\partial\Phi_{cn}/\partial\sigma_{ij})}\, \bar{D}_l = \overline{(\partial\Phi_{cn}/\partial\sigma_{ij})}\, \bar{D} \tag{36}$$

If $D(t) = D_l(t)$ then (36) correctly yields $f = 1$.

If $D(t) \neq D_l(t)$ it is not possible in general to satisfy all six equations in (36) with one function f. On the other hand, it is possible to do so with proportional stressing,

$$\sigma_{ij} = S\,\sigma_{ij}' \tag{37}$$

where $S = S(t)$ and the σ_{ij}' are independent of time. Equation (36) then yields

$$\overline{fS^n}\, \bar{D}_l = \overline{S^n}\, \bar{D} \tag{38}$$

Thus, with proportional stressing (38) may be used to obtain the $f(t)$ which makes (35) fully equivalent to (32). Proportional stressing is not needed at low stresses because (32) and (35) are equivalent when the nonlinear strain term is negligible.

Instead of using (35) as the basis for a path-independent integral, the original equation (32) can be used if the so-called quasi-elastic approximation is valid for both the nonlinear and linear terms [17]. In this case,

$$\varepsilon_{ij} = \partial\Phi_c/\partial\sigma_{ij} \tag{39a}$$

where

$$\Phi_c \equiv E_R D(t)\, \Phi_{cn} + E_R D_l(t)\, \Phi_{cl} \tag{39b}$$

Obviously (39) is exact when the stresses are constant. For time-varying stresses it is a good approximation if the magnitude of the curvature $\partial^2/(\partial \log t)^2$ or $\partial \log (\bullet)/\partial \log t$ of the potential gradients and ε_{ij}, D, and D_l is small. For this time-dependent elastic characterization $\Phi = -\Phi_c + \sigma_{ij}\varepsilon_{ij}$ is the potential which is used in constructing a path-independent integral. The integral is simply J, in which the actual displacements and strains (instead of pseudo variables) are used.

3.2. A very small nonlinear crack tip zone; opening mode

If the zone of nonlinear behavior is small enough, it will be surrounded by a linear viscoelastic singular stress field. We may consider this outer singular field to be the far-field relative to the region of validity of (30). The form of the stresses is the same as in (30), but $n = 1$. In order to identify the constant σ_n in the linear solution, consider (35) for a uniaxial stress σ,

$$\varepsilon^{Rl} = f\,|\sigma/\sigma_n|^n \operatorname{sgn}(\sigma) + \sigma/E_R \tag{40}$$

where E_R has been introduced so that it cancels out of (40) in the linear range when ε^R is expressed in terms of ε. Comparing the linear term in (40) with (31), we see that by replacing σ_n with E_R in (30) the stresses for the outer singular zone are obtained,

$$\sigma_{ij} = \left[\frac{E_R J_v'}{r}\right]^{1/2} \tilde{\sigma}_{ij}(\theta, 1) \tag{41}$$

In the linear range of (40), the material is elastic with a Young's modulus of E_R and strain ε^{Rl}. The J_v' integral and the stress intensity factors (for single or mixed mode) are related in the same way as for an elastic material. For example, the opening mode stress intensity factor K_{I} for plane strain in the outer singular zone is

$$K_{\mathrm{I}}^2 = E_R J_v'/(1 - \nu^2) \tag{42}$$

Of course for plane stress $(1 - \nu^2)$ is replaced by unity. Equation (41) reduces to

$$\sigma_{ij} = K_{\mathrm{I}} r^{-1/2} \tilde{\sigma}_{ij}(\theta, 1)(1 - \nu^2)^{1/2} \tag{43}$$

It is well-known that σ_{ij} is independent of ν, given K_{I}, and therefore $\tilde{\sigma}_{ij}$ in (43) must be inversely propotional to the last factor.

Since the value of J_v' in (42) is the same in the nonlinear singular zone, (42) can be used to express (30) (with J_v first replaced by J_v') in terms of K_{I}. Notice, however, that f appears in the constitutive equation; comparing (31) and (40), σ_n in (30) has to be replaced by $\sigma_n/f^{1/n}$. Thus,

$$\sigma_{ij} = E_R \left[\frac{K_{\mathrm{I}}^2}{E_R^2 f r}\right]^{1/(n+1)} \tilde{\sigma}_{ij}(\theta, n)(1 - \nu^2)^{1/(1+n)} \tag{44}$$

After making the above substitutions we used $\sigma_n = E_R$, which is permissible since σ_n is a redundant constant. It would be possible to take $E_R = 1$ as well, since E_R always appears with D and D_l; but unless $n = 1$, D would not have dimensions of compliance and f would not be dimensionless.

Equation (30) in terms of the original J_v, with $\sigma_n = E_R$, must agree with (44). Hence,

$$J_v = J_l/f \tag{45}$$

where

$$J_l \equiv K_{\mathrm{I}}^2(1 - v^2)/E_R \tag{46}$$

is the J integral for a linear elastic material and is equal to J_v'.

If the quasi-elastic expression (39) is applicable, we can use it to get another expression for the inner and outer singular stress fields. For uniaxial stress, (39) must reduce to the result (33) given previously for constant stress. In the linear range the modulus is D_l^{-1}, and therefore the quasi-elastic counterpart to (42) is

$$K_{\mathrm{I}}^2 = J/D_l(t)(1 - v^2) \tag{47}$$

Equation (43) is still applicable since the stresses in terms of K_{I} do not depend on the viscoelastic compliance.

In the solution (30) for the nonlinear singular zone, σ_n must be replaced by $\sigma_n/(E_R D)^{1/n}$, as may be seen by comparing (31) with the nonlinear term in (33). Also replace J_v by J and, as before, let $\sigma_n = E_R$; thus

$$\sigma_{ij} = E_R\left[\frac{J}{D\,E_R^2\,r}\right]^{1/(n+1)}\tilde{\sigma}_{ij}(\theta, n) \tag{48}$$

This expression is valid with or without far-field linear behavior. In the former case we may use (47) to express (48) in terms of K_{I},

$$\sigma_{ij} = E_R\left[\frac{D_l\,K_{\mathrm{I}}^2}{D\,E_R^2\,r}\right]^{1/(n+1)}\tilde{\sigma}_{ij}(\theta, n)(1 - v^2)^{1/(n+1)} \tag{49}$$

For agreement between (44) and (49) use $f = D/D_l$. This result is also found by applying the so-called direct method of approximate Laplace transform inversion [17] to (38).

3.3. Transition from small to large-scale nonlinearity

According to (35), the nonlinear viscoelastic strain is dominant when f is large, whereas there is primarily linear behavior when f is small; what constitutes small and large depends of course on the stress level and value of the exponent n. In order to illustrate how behavior changes with time and then to estimate the time scale for the transition, we shall assume D, D_l, and K_{I} obey power laws in time; i.e., use the D in (34) and similar representations for the other two functions,

$$E_R D_l = (t/t_q)^q, \quad K_I = (t/t_k)^k \tag{50}$$

in which $(p, q) \geqslant 0$. Considering only material points which are in the nonlinear singular field (44), we assume power laws as trial solutions for the stress factor S and aging function f,

$$S = (t/t_m)^m, \quad f = (t/t_a)^a \tag{51}$$

Substitution of (34), (50) and (51) into (37), (38), and (44) yields

$$a = p - q \tag{52a}$$

$$m = (q - p + 2k)/(n + 1) \tag{52b}$$

and

$$f = \frac{\Gamma(p + 1)\,\Gamma(mn + 1)}{\Gamma(q + 1)\,\Gamma(p - q + mn + 1)} \frac{D}{D_l} \tag{52c}$$

where $\Gamma(\bullet)$ is the gamma function. When an argument of Γ vanishes or is negative one of the convolution integrals used to predict strain by means of (32) or (35) does not converge, whereas there is no convergence problem with positive arguments. The time exponent for the stress intensity factor, k, which is viewed as a specified quantity, therefore must be such that

$$mn + 1 > 0 \quad \text{and} \quad p - q + mn + 1 > 0 \tag{53}$$

Suppose, for example, in the fully nonlinear range the material is viscous ($p = 1$) and in the linear range it is elastic ($q = 0$). Equation (52) yields

$$f = \frac{n + 1}{2\,kn + 1} \frac{t}{t_p} \tag{54}$$

Recalling (45), we find that this result is the same as reported in [10, p. 123] if we take $k = 0$ (constant K_I) and $t_p = 1$; the latter selection reduces J_v to C^* for a viscous body.

It was found earlier that the quasi-elastic approximation for f is D/D_l, and thus it is the same as (52c) except for a constant factor. This factor is approximately unity with weak time-dependence $0 < (p, q) \ll 1$ or if $q \simeq p$. Since $\Gamma(x)$ is within 13 percent of unity when $1 \leqslant x \leqslant 2$, there is a fairly large set of conditions for which the quasi-elastic approximation is both qualitatively and quantitatively valid. The exponent m, which defines the time-dependence of stress, is different for points in the linear and nonlinear fields. Indeed, neither the proportional stressing assumption nor the power law time-dependence are fully satisfied at any given point in a singular field while it changes from a linear to nonlinear field (or vice-versa). Although this change has not been accounted for in deriving (52c), it is not important at least when the quasi-elastic approximation can be used. Additional support for the present analysis is inferred from the comparisons in [10] for mode III when $q = k = 0$ and $p = 1$.

3.4. *Stability of crack growth*

As observed earlier in this section, the crack growth is controlled by J_v. Although (24) and (25) are based on certain idealizations of crack tip phenomena, it is believed they provide some indication of just how J_v affects crack growth. In turn, (45) shows how far-field stresses affect J_v for a small-scale, nonlinear singular zone. This latter equation, together with (50), (51) and (52a), yields $J_v \sim t^{(q-p+2k)}$. If the time exponent is negative, the high initial J_v will tend to produce a high, initial crack growth rate; but then the decrease of J_v with time may allow crack arrest. For a positive exponent the growth is more likely to be continuous and stable, at least in the early stage. Thus, the greater the time-dependence of the inner field (p) relative to the outer field (q), the more likely the early growth will be unstable, possibly consisting of start-stop steps.

3.5. *Estimate of the characteristic transition time*

When the exponent $a = p - q$ is positive, the factor f increases with time, so that in time the linear viscoelastic component of strain in (35) becomes negligible, after first being dominant. Thus, initially the singular field is linear and controlled by K_{I}, while finally it is the HRR field, and controlled by J_v. The transition from the linear to the nonlinear field takes place of course over a period of time. Following the method used in [10] for constant applied loads, we may estimate a characteristic time, say t_T, for this transition. Initially J_v decreases with time when J_l is constant, according to (45). At long times the first term in (35) is dominant, and therefore the long-time J_v, J_n say, is constant if the applied loads are constant; in this case (45) does not apply because the linear singular field does not exist. The time at which (45) predicts $J_v = J_n$ is the characteristic transition time; thus,

$$J_n = J_l/f(t_T) \tag{55}$$

From (34), (50), and (52c),

$$\frac{t_T}{t_p} = \left[\frac{\Gamma(q+1)\,\Gamma(p-q+mn+1)(t_p/t_q)^q}{\Gamma(p+1)\,\Gamma(mn+1)}\,\frac{J_l}{J_n}\right]^{1/(p-q)} \tag{56}$$

(It is likely that (56) provides a good estimate of t_T even if the loads are not constant ($k \neq 0$) provided that f increases in time ($a > 0$).) With $k = q = 0$ and $t_p = p = 1$, then (56) reduces to the result in [10, Eqn. (42)],

$$t_T = J_l/(n+1)C^* \tag{57}$$

As in [10], we may also determine how the radial position of the intermediate zone between the inner and outer singular fields varies with time. Specifically, equate a measure of the stresses in (43) and (44) to find $r \sim K_{\mathrm{I}}^2 f^{2/(n-1)}$, showing, as expected, that the nonlinear field grows in time if $a > 0$, $n > 1$, and $k \geqslant 0$.

4. Additional constitutive equations derivable from a potential

In the last section the nonlinear viscoelastic constitutive equation (32) was replaced by a time-dependent elastic representation in terms of pseudo strain, (35). For a given time-dependent factor, $f(t)$, this representation provides a J_v integral which is path-independent for proportional and nonproportional stressing. Assuming proportional stressing, an equation for the factor f was developed, (38), whose solution made (35) an exact representation of (32). As a generalization of this method, an additional aging factor, say f_1, could be used with Φ_{c1} and the pseudo strain could be based on a compliance other than D_l. Indeed it may be desirable to use f_1 and D, instead of f and D_l, at small load levels for the case of hardening nonlinearity, $n < 1$, because the far-field behavior is then primarily nonlinear.

The quasi-elastic approximation to convolution integrals reduced (32) to (39), thus providing a path-independent, quasi-elastic J integral. There are other forms of linear and nonlinear viscoelastic constitutive equations which, under certain conditions, can be derived from a potential. For example, this is possible whenever the quasi-elastic approximation can be used in constitutive convolution-integral characterizations of linear isotropic or anisotropic media; in this case, as before, creep or relaxation functions are used in place of elastic constants in elastic-like consitutive equations.

One type of nonlinear constitutive equation which has been used for polymeric materials is similar to that used for linear media but real time is replaced by a stress- or strain-reduced time [18]. In order to show how a reduced time may be included in a strain energy-like potential, let us return to the nonaging version of (3), but replace the times t and τ by reduced times, ψ and ψ',

$$\psi \equiv \int_0^t \mathrm{d}t'/a_\sigma, \quad \psi' \equiv \psi(\tau) \tag{58}$$

where a_σ is a function of stress or strain. Thus,

$$\varepsilon_{ij} = E_R \int_0^\psi D(\psi - \psi') \frac{\partial \varepsilon_{ij}^e}{\partial \psi'} d\psi' \tag{59}$$

We shall comment on two different situations in which (59) leads to a path-independent J integral. As in Section 2, we may use short-time response to argue from thermodynamics that $\varepsilon_{ij}^e = \partial \Phi_c' / \partial \sigma_{ij}$, where Φ_c' is not necessarily a homogeneous function. It should be observed that the inverse form of (59) contains the Knauss and Emri model [19, 20] when the bulk modulus is proportional to the shear modulus; all nonlinearity is due to the effect of dilatation on the time-scale factor, a_σ.

(i) Assume the quasi-elastic approximation is applicable,

$$\varepsilon_{ij} \simeq E_R D(\psi) \varepsilon_{ij}^e \tag{60}$$

and that $\psi \simeq \psi(t, a_\sigma)$. Then, since a potential Φ_c' exists if and only if $\partial \varepsilon_{ij}/\partial \sigma_{kl} = \partial \varepsilon_{kl}/\partial \sigma_{ij}$, we obtain the result that a_σ can depend on stress through only Φ_c'. Alternatively, a_σ and Φ_c' could each depend on the same, single stress invariant. It is encouraging that in a study of fiber-reinforce plastic, the time-scale factor a_σ as well as other nonlinear material functions could be expressed using only one invariant, a local volume average of the octahedral shear stress in plastic matrix [21].

(ii) Do not use the quasi-elastic approximation, but assume proportional stressing and α_σ and Φ_c' are power law functions of one homogeneous stress invariant; also use (34). Again, it can be verified that a time-dependent potential Φ_c' exists.

Obvious generalizations of (59) for which a potential exists are obtained by adding an elastic strain as well as more integrals like (59), but with other material functions of time and stress. Additional realistic representations, including inverse forms, in which stress is a function of strain history, could be discussed. However, it is believed the above examples are sufficient to make the primary point that there is at least a limited basis for using strain energy-like potentials and J or J_v integrals to characterize deformation and fracture behavior of nonlinear viscoelastic media. What is believed needed at this time are experimental studies which directly address this question. Since time appears as a parameter, in these experimental studies isochronal data may be analyzed just as if the material were elastic, except for the possible use of pseudo strain in place of physical strain.

5. Concluding remarks

In this paper we first introduced a single-integral nonlinear viscoelastic constitutive equation, and then expressed it in terms of a strain energy-like potential. This formulation led to a path-independent integral, J_v, which is like the J integral for elastic media except a hereditary integral appears in place of displacement. By varying the form of the kernel function (relaxation modulus) one may characterize the behavior of different types of material, ranging from elastic to viscoelastic to viscous. A Barenblatt model of the failure process at the crack tip resulted in relatively simple relationships between J_v and the time at which growth begins and the crack speed. In the constitutive model all hereditary effects are accounted for by one stress-independent relaxation or creep function. This limitation excludes the type of behavior exhibited by metals at constant stress, in which the response is elastic for low stress and viscous for high stress. However, as shown in Sections 3 and 4, it is possible to introduce an artificial type of aging which greatly extends the characterization, and includes metal-like creep behavior. How this extension can be used to relate a locally path-independent J_v integral to a far-field loading parameter was illustrated in Section 3 for limited crack growth. In [8] the counterpart to this transient analysis was considered; there is quasi-steady state crack growth and the viscoelastic behavior of material near the crack tip is different from that of the far-field.

One version of the aging elastic characterization discussed in Sections 3 and 4 is based on a quasi-elastic approximation of the constitutive equations. If a strain energy-like function $\Phi(\varepsilon_{ij}, x_2, x_3, t)$ exists, then (1) leads to a time-dependent J integral. If Φ is a homogeneous function of degree $1 + 1/n$ in the neighborhood of the crack tip then (48) gives the local stresses, where $D = D(t)$ and the arbitrary constant E_R are quantities which are introduced through the uniaxial stress-strain equation at high stress,

$$\varepsilon = E_R D(t) |\sigma/E_R|^n \operatorname{sgn}(\sigma) \tag{61}$$

As long as the fracture process zone (or failure zone) at the crack tip remains well inside the singular stress field (48), the history of this J characterizes the fracture initiation time and subsequent crack growth. In the immediate vicinity of the tip the quasi-elastic approximation is not expected to be valid with crack growth because of the locally complex stress history, which includes unloading. If, however, (29) characterizes the local nonlinear viscoelastic behavior, then (30) is applicable, in which we may take $\sigma_n = E_R$. For agreement between (30) and (48),

$$J_v = J/E_R D(t) \tag{62}$$

This result provides a link between the far-field linear or nonlinear loading parameter J, and crack-tip loading parameter J_v; if the arbitrary constant E_R has dimensions of modulus, then J_v has the same dimensions as J. Let us further suppose that the crack-tip failure process can be modeled using the thin-layer idealization, Fig. 1. Then we may use (24) (or related generalizations which allow for complex behavior of failure-zone material) for initiation time t_i and (25) for quasi-steady crack-speed, as long as the tip remains well within the original location of the nonlinear singular field (48).

A three-dimensional J_v integral theory was developed in [8], allowing for large strains and a special case of micro-damage in the continuum surrounding a large crack. To establish a J-like theory, including the connection with potential energy (22), one again has to argue that a local strain energy-like potential exists, as in (1) or (2). In [22] it was shown using a certain degree of idealization that a potential exists with continuum damage of a more general type than in [8]. An approximation lead to an aging type of elastic behavior in terms of a potential; it is multivalued with respect to loading and unloading paths, and exhibits aging-like behavior due to the time-dependent damage growth. The situation is analogous to that for a material which obeys deformation plasticity theory with a time-dependent yield stress during loading, and obeys elasticity theory during unloading. The associated J integral exhibits a limited amount of path dependence in transient problems, but is still useful as a loading parameter. Finally, it is noted that the extension to large strains in [8] is approximate in the sense that the constitutive equations are not independent of large rigid body rotations. This problem does not arise with the quasi-elastic constitutive representation when the deformation measure used in the potential is the Green's strain tensor.

Acknowledgement

Sponsorship of this research by the Office of Naval Research is gratefully acknowledged.

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Résumé. Dans certains cas, on peut construire les potentiels de travail et les intégrales indépendantes du parcours, du type intégrale J, pour des milieux visco-élastiques non linéaires monolythiques ou composites.

Dans ce mémoire, on discute de certaines situations où de telles valeurs existent et sont utiles à l'étude de l'amorçage quasi-statique et à la propagation de fissures.

L'approximation dite quasi-élastique, est une équation constitutive sous forme d'une intégrale simple, fournissant la base d'utilisation de l'intégrale J ou des intégrales du même type comme paramètres de caractérisation de la rupture au cours de l'amorçage et des premières étapes de la propagation.

On montre également que dans certains cas de croissance significative de la fissure, la vitesse instantanée de croissance peut être caractérisée par une intégrale similaire indépendante du parcours.

On discute brièvement du problème de la caractérisation de la croissance de grandes fissures dans des milieux visco-élastiques présentant un endommagement microscopique.