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The spinning circular disc with a radial edge crack; an exact solution

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Abstract. A circular disc of radius a, made of homogeneous, isotropic, linearly elastic material, contains a radial edge crack of length b. The disc is rotating with constant angular velocity about an axis through its centre and perpendicular to its plane. The problem of determining the resulting stress and displacement fields throughout the disc is solved (within the two-dimensional linear theory) exactly and in closed form. In particular the stress intensity factor and the crack opening displacement are evaluated for both the plane stress and plane strain cases with any crack length b ($0 < b < 2a$) and any values of the elastic constants.

1. Introduction

The role of two-dimensional linear elastostatics in crack mechanics is well established; see for instance Sih (ed.) [1] where are described the techniques available for determining the stress intensity factor for various specimen geometries and loadings. However, exact closed form solutions corresponding to practical (rather than contrived) loadings are in very short supply. Indeed, the only exact solutions available for an edge crack in *any* two-dimensional body of finite size (subjected to realistic loadings) still seem to be those given by the author [2, 3] for the geometry of the circular disc with a radial edge crack. Such solutions have an importance beyond their intrinsic analytical interest, since they are the "benchmark solutions" against which approximate techniques can be tested.

In the present paper we give a further exact closed form solution for this geometry for the practically important case in which the loading is induced by the disc spinning with constant angular velocity about an axis through its centre and perpendicular to its plane. This problem is solved (within the two-dimensional linear theory) exactly and in closed form. In particular we obtain the stress intensity factor K for any crack length "aspect ratio" (even for crack lengths longer than a radius) and any value of Poisson's ratio; similarly we evaluate the crack opening displacement U at the mouth of the crack. A brief account of the necessary analysis is given in Sections 2, 3, 4 and the results are presented in an easily usable form in Section 5.

The existing literature on this problem is sparse. Delale and Erdogan [4] consider the more general problem of a rotating annulus with a radial crack which may be at an edge or embedded; the method involves the approximate numerical solution of a singular integral equation. However although the authors remark that their method could have been used to find K for the case of a solid disc, they did not investigate this case. Rooke and Tweed [5] have investigated the actual problem solved in the present paper. For the case in which the crack length b is less than the radius a, they used Mellin transforms to reduce the problem to a singular integral equation. They solved this integral equation approximately for a selection of values of b in the range $0 < b < a$ to obtain estimates of K and of the strain energy; this procedure becomes ill conditioned as b approaches a, and is invalid for $b > a$. (It should be noted that the graphs of K/K_0 given in [5] are for the *plane strain* case; moreover, since the loading in the rotating disc problem involves Poisson's ratio explicitly, one cannot immediately deduce the plane stress case from results given.*) Our results for K/K_0, when restricted to the case of plane strain and $b < a$ are in agreement with those in [5]. The problem of determining the crack opening displacement is not considered in either [4] or [5].

2. The rotating elastic disc with an edge crack

The problem to be solved is depicted in Fig. 1. A circular disc of radius a, made of homogeneous, isotropic, linearly elastic material, contains a radial edge crack of length b ($0 < b < 2a$); the disc is spinning about an axis through its centre perpendicular to its plane with *constant* angular speed ω. In the frame of reference Oxy *fixed in the disc*, the disc is in an equilibrium state of plane stress† with its boundary free of tractions and loaded by the "centrifugal body force"

$$\mathbf{F} = \varrho\omega^2\mathbf{r}, \tag{2.1}$$

where ϱ is the uniform density of the disc, and O is the origin of position vectors.

The solution for the case in which the crack is absent is given by Love [7], p. 146. In particular the stress $\tau_{\theta\theta}$ (where r, θ are plane polar co-ordinates centred on O) can

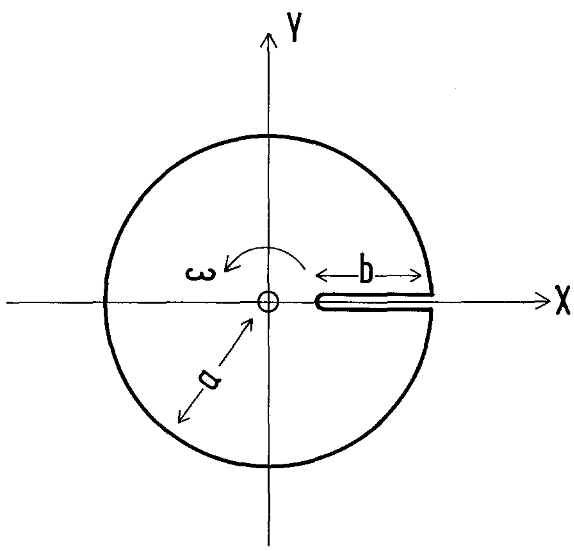

Fig. 1.

* In a recent communication, Sukere [6] has used the work of Rooke and Tweed to obtain an *empirical* formula for the stress intensity factor. In view of the exact solution obtained in the present paper, such a formula is no longer required.

† Corresponding results for the cylinder under plane strain conditions will also be given.

be written

$$\tau_{00} = p_0\left[1 - \alpha\left(\frac{3r^2}{a^2} - 1\right)\right], \tag{2.2}$$

where

$$p_0 = \tfrac{1}{3}\varrho\omega^2 a^2, \tag{2.3}$$

$$\alpha = \begin{cases} \dfrac{1 + 3\nu}{8} & \text{for plane stress,} \\ \dfrac{(1 + 2\nu)}{8(1 - \nu)} & \text{for plane strain,} \end{cases} \tag{2.4}$$

ν being Poisson's ratio for the disc.

When the crack is present the solution will consist of Love's solution plus another solution for the *cracked* disc corresponding to the symmetrical loading

$$\tau_{yy}(x, 0\pm) = -p_0\left[1 - \alpha\left(\frac{3x^2}{a^2} - 1\right)\right], \tag{2.5}$$

$$\tau_{yx}(x, 0\pm) = 0, \tag{2.6}$$

on the crack faces, with no body force; note that Poisson's ratio ν appears explicitly in this loading and that, as a consequence, the stress intensity factor will depend on ν. In a previous paper (Gregory [2]) the author has given an exact closed-form solution for the constant pressure loading

$$\tau_{yy}(x, 0\pm) = -p_0, \tag{2.7}$$

and so it remains to solve for the case of the quadratic loading

$$\tau_{yy}(x, 0\pm) = -p_0\left(\frac{3x^2}{a^2} - 1\right). \tag{2.8}$$

It is convenient to solve this problem in a bipolar co-ordinate system (ξ, η), where $\zeta(=\xi + i\eta)$ is related to $z(=x + iy)$ by

$$\zeta = \log\left[\frac{z - a}{z + a}\right] \tag{2.9}$$

with $0 \leqslant \eta \leqslant 2\pi$. The bi-harmonic equation for the Airy stress function $\chi(x, y)$ is not preserved by (2.9), but if we define the Jeffery potential $\varphi(\xi, \eta)$ by

$$\varphi = \frac{\chi}{J}, \tag{2.10}$$

where

$$J = \left|\frac{\mathrm{d}z}{\mathrm{d}\zeta}\right| = \frac{a}{\cosh \xi - \cos \eta}, \tag{2.11}$$

then $\varphi(\xi, \eta)$ satisfies

$$\left(\frac{\partial^4}{\partial \xi^4} + 2\frac{\partial^4}{\partial \xi^2 \partial \eta^2} + \frac{\partial^4}{\partial \eta^4} - 2\frac{\partial^2}{\partial \xi^2} + 2\frac{\partial^2}{\partial \eta^2} + 1\right)\varphi = 0, \tag{2.12}$$

an equation with constant coefficients. If we also translate the (ξ, η)-region parallel to the axis of ξ so as to bring the image of the crack tip to the point $\xi = 0$, $\eta = \pi$, then the (ξ, η)-region becomes that shown in [2], Fig. 2, with the governing equation (2.12) and the boundary conditions

$$\varphi = \frac{\partial \varphi}{\partial \eta} = 0 \quad \text{on } \eta = \tfrac{1}{2}\pi, \tfrac{3}{2}\pi, \tag{2.13}$$

and

$$\begin{cases} \varphi = \dfrac{-p_0 a}{1 + \cosh(\xi + \xi_0)}, & (2.14) \\[2ex] \dfrac{\partial \varphi}{\partial \eta} = 0 & (2.15) \end{cases}$$

on $\eta = \pi\pm$, $-\infty < \xi < 0$. In (2.14), ξ_0 is related to the crack length b by

$$b = a(1 + \tanh \tfrac{1}{2}\xi_0). \tag{2.16}$$

The above problem for $\varphi(\xi, \eta)$ is clearly of the Wiener–Hopf type (see Noble [8]) but it would be difficult to proceed directly since the left-hand Fourier transform

$$\int_{-\infty}^{0} \frac{\mathrm{a}\, \mathrm{e}^{\mathrm{i}\alpha\xi} \mathrm{d}\xi}{1 + \cosh(\xi + \xi_0)} \tag{2.17}$$

cannot be evaluated. However (2.14) may be written

$$\varphi(\xi, \pi\pm) = -2p_0 a \sum_{n=1}^{\infty} (-1)^{n+1} n\, \mathrm{e}^{n\xi_0}\, \mathrm{e}^{n\xi}, \tag{2.18}$$

where the series in (2.18) is convergent for $\xi \leqslant 0$ provided that $\xi_0 < 0$ i.e., when the crack length b is less than the disc radius a. For the moment we will assume that this is so, but this restriction will be removed later. Thus it suffices to solve for the sequence of "loadings"

$$\begin{cases} \varphi = a\, e^{n\xi}, & (2.19) \\ \dfrac{\partial \varphi}{\partial \eta} = 0, & (2.20) \end{cases}$$

for $n = 1, 2, 3, \ldots$.

The determination of $\varphi_n(\xi, \eta)$ corresponding to the loading (2.19), (2.20) is achieved using the Wiener–Hopf technique. The method is similar to that described in [2] and the extensive details are omitted. The result is

$$\varphi_n(\xi, \eta) = \frac{-ia}{2\pi K_+(ni)} [\cos \eta \, H_1(\xi, \eta) + \sin \eta \, H_2(\xi, \eta)], \tag{2.21}$$

where

$$H_1(\xi, \eta) = \int_{-\infty}^{\infty} \left\{ -\frac{K_+(\alpha) \cosh \alpha(\pi - \eta)}{\alpha - ni} + \frac{\sinh \alpha(\pi - \eta)}{\alpha(\alpha - ni)(\alpha^2 + 1)K_-(\alpha)} \right\} e^{-i\alpha\xi} \, d\alpha \tag{2.22}$$

$$H_2(\xi, \eta) = \int_{-\infty}^{\infty} \left\{ \frac{\cosh \alpha(\pi - \eta) - \coth \frac{1}{2}\pi\alpha \sinh \alpha(\pi - \eta)}{(\alpha - ni)(\alpha^2 + 1)K_-(\alpha)} \right\} e^{-i\alpha\xi} \, d\alpha \tag{2.23}$$

In (2.21)–(2.23) the functions $K_\pm(\alpha)$ are certain analytic functions of the complex variable α which are defined in Appendix 1 by an explicit product formula.

In particular for $\xi > 0$

$$\varphi(\xi, \pi) = \frac{-ia}{2\pi K_+(ni)} \int_{-\infty}^{\infty} \frac{K_+(\alpha) e^{-i\alpha\xi} \, d\alpha}{\alpha - ni}, \tag{2.24}$$

and, by letting $\xi \to 0+$, the stress intensity factor at the crack tip corresponding to the loading (2.19), (2.20) is found to be

$$-\frac{a^{1/2}(\cosh \xi_0 + 1)^{1/2}}{\pi^{1/2} K_+(ni)}. \tag{2.25}$$

This is the stress intensity factor corresponding to the term $e^{n\xi}$ in (2.18) and so summing gives the stress intensity factor due to the required loading (2.8), (2.6); this is

$$K = \frac{2p_0}{\pi^{1/2}} a^{1/2} (\cosh \xi_0 + 1)^{1/2} \sum_{n=1}^{\infty} \frac{(-1)^{n+1} \, n \, e^{n\xi_0}}{K_+(ni)}. \tag{2.26}$$

In terms of the crack length b (see (2.16)) this becomes

$$K = \frac{(2a)^{3/2} p_0}{(\pi b)^{1/2}(2a - b)^{1/2}} \sum_{n=1}^{\infty} \frac{(-1)^{n+1} n}{K_+(ni)} \left[\frac{b}{2a - b} \right]^n \tag{2.27}$$

This formula can be used to find K due to the loading (2.8), (2.6) for $b < a$; the series converge rapidly for small b/a. All that is required to apply it are the (real) values $K_+(n\mathrm{i})$, $n = 1, 2, \ldots$. The first ten of these are given in Appendix 1. Using a ten term series gives three figure accuracy for K for b in the range $0 < b/a < 0.45$. An integral formula, valid for all b will be obtained in Section 4.

3. The crack opening displacement

In order to calculate the opening at the mouth of the crack in the spinning disc, it is necessary to obtain the displacement fields corresponding to the sequence of loadings (2.19), (2.20). The calculation of displacement fields corresponding to a given Jeffery potential $\varphi(\xi, \eta)$ is described by Coker and Filon [9] p. 168 and involves the determination of a displacement potential $Q(\xi, \eta)$. The displacement potential $Q_n(\xi, \eta)$ corresponding to the Jeffery potential $\varphi_n(\xi - \xi_0, \eta)$ given by (2.21) is

$$Q_n(\xi, \eta) = \frac{C_n \cos\eta}{\cosh\xi - \cos\eta}\left[H_3(\xi - \xi_0, \eta) - \frac{\pi\, \mathrm{e}^{\xi_0 - \xi}\cos\eta}{(n+1)K_+(\mathrm{i})}\right] + \frac{C_n \sin\eta}{\cosh\xi - \cos\eta}\left[H_4(\xi - \xi_0, \eta) - \frac{\pi \mathrm{e}^{\xi_0 - \xi}\sin\eta}{(n+1)K_+(\mathrm{i})}\right], \tag{3.1}$$

where

$$C_n = \frac{a^2}{\pi K_+(n\mathrm{i})}, \tag{3.2}$$

$$H_3(\xi, \eta) = \int_{-\infty}^{\infty}\left\{-\frac{K_+(\alpha)\sinh\alpha(\pi - \eta)}{\alpha - n\mathrm{i}} + \frac{\cosh\alpha(\pi - \eta)}{\alpha(\alpha - n\mathrm{i})(\alpha^2 + 1)K_-(\alpha)}\right\}\mathrm{e}^{-\mathrm{i}\alpha\xi}\,\mathrm{d}\alpha, \tag{3.3}$$

$$H_4(\xi, \eta) = \int_{-\infty}^{\infty}\left\{\frac{\sinh\alpha(\pi - \eta) - \coth\frac{1}{2}\pi\alpha\cosh\alpha(\pi - \eta)}{(\alpha - n\mathrm{i})(\alpha^2 + 1)K_-(\alpha)}\right\}\mathrm{e}^{-\mathrm{i}\alpha\xi}\,\mathrm{d}\alpha. \tag{3.4}$$

In (3.3), (3.4) the contour of integration is taken below the pole at $\alpha = 0$.

The displacement $u_\eta(\xi, \eta)$ is then given by

$$Eu_\eta = \frac{1 - \nu}{J}\frac{\partial}{\partial\eta}(J\varphi) + \frac{1}{J}\frac{\partial Q}{\partial\xi}, \tag{3.5}$$

for the case of plane stress, where E is Young's modulus and ν Poisson's ratio. The crack opening displacement U is given by

$$U = -2u_\eta(-\infty, \pi), \tag{3.6}$$

and on performing the necessary analysis, the crack opening corresponding to the loading (2.19), (2.2) is found to be

$$-\frac{4a}{EK_+(n\mathrm{i})}\left[\frac{1}{nK_+(0)}+\frac{\mathrm{e}^{\xi_0}}{(n+1)K_+(\mathrm{i})}\right]. \tag{3.7}$$

By summing we can now find the crack opening U due to the required loading (2.8), (2.6). This is

$$U = \frac{8p_0a}{E}\left[\frac{1}{K_+(0)}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\mathrm{e}^{n\xi_0}}{K_+(n\mathrm{i})}+\frac{\mathrm{e}^{\xi_0}}{K_+(\mathrm{i})}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}n\mathrm{e}^{n\xi_0}}{(n+1)K_+(n\mathrm{i})}\right]. \tag{3.8}$$

This can be written in terms of the crack length b by substituting

$$\mathrm{e}^{\xi_0} = \frac{b}{2a-b}. \tag{3.9}$$

The formula (3.8) can be used to find U due to the loading (2.8), (2.6) for $b < a$; the series converge rapidly for small b/a. The (real) values of $K_+(n\mathrm{i})$, $n = 0, 1, 2, \ldots$ are given in Appendix 1 up to $n = 10$. Using ten term series gives three figure accuracy for U for b in the range $0 < b/a < 0.5$. An integral formula, valid for all b will be obtained in Section 4.

4. Formulae for K, U valid for all b

The formulae (2.26), (3.8) give the stress intensity factor K and the crack opening displacement U due to the loading

$$\tau_{yy}(x, 0\pm) = -p_0\left(\frac{3x^2}{a^2}-1\right), \tag{4.1}$$

$$\tau_{yx}(x, 0\pm) = 0, \tag{4.2}$$

on the crack faces $a - b < x \leqslant a$, $y = 0\pm$. (The formula (3.8) for U is for plane stress; to get the plane strain formula, multiply by $(1 - \nu^2)$). These formulae are valid only for b in the range $0 < b < a$, since the three infinite series involved are actually divergent for $\xi_0 \geqslant 0$, that is $b \geqslant$ a. Consider, for instance, the series

$$S_1(\xi_0) = \sum_{n=1}^{\infty}\frac{(-1)^{n+1}n\mathrm{e}^{n\xi_0}}{K_+(n\mathrm{i})}, \tag{4.3}$$

which appear in (2.26) and which converge only for $\xi_0 < 0$. However for $\xi_0 < 0$, $S_1(\xi_0)$ can be written as the integral

$$S_1(\xi_0) = \int_{-\infty}^{\infty}\frac{\alpha\,\mathrm{e}^{-\mathrm{i}\alpha\xi_0}}{2K_+(\alpha)\sinh\pi\alpha}\,\mathrm{d}\alpha, \tag{4.4}$$

where $K_+(\alpha)$ is defined by (A1.1). This can be shown by closing the contour of integration in (4.4) around the poles of the integrand (at $\alpha = n\mathrm{i}, n \geqslant 1$) in the upper half-plane. But the function $S_1(\xi_0)$ defined by (4.4) exists for *all real* ξ_0 and thus represents an analytic continuation of the series formula (4.3). Thus (4.4) can be used instead of (4.3) in (2.26) to give a formula for K valid for all crack lengths $0 < b < 2a$.

Similar remarks apply to the series

$$S_2(\xi_0) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}\, \mathrm{e}^{n\xi_0}}{K_+(n\mathrm{i})}, \tag{4.5}$$

$$S_3(\xi_0) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} n \mathrm{e}^{n\xi_0}}{(n+1)K_+(n\mathrm{i})}, \tag{4.6}$$

which appear in (3.8). These may be written in the form

$$S_2(\xi_0) = \int_{-\infty}^{\infty} \frac{\mathrm{i}\, \mathrm{e}^{-\mathrm{i}\alpha\xi_0}}{2K_+(\alpha)\sinh \pi\alpha}\, \mathrm{d}\alpha, \tag{4.7}$$

$$S_3(\xi_0) = \int_{-\infty}^{\infty} \frac{\mathrm{i}\alpha\, \mathrm{e}^{-\mathrm{i}\alpha\xi_0}}{2(\alpha + \mathrm{i})K_+(\alpha)\sinh \pi\alpha}\, \mathrm{d}\alpha, \tag{4.8}$$

valid for all ξ_0, $-\infty < \xi_0 < \infty$. Thus (4.7), (4.8) can be used instead of (4.5), (4.6) in (3.8) to give a formula for U valid for all crack lengths $0 < b < 2a$.

5. Results

The final values of K and U for the spinning disc can now be deduced from (*i*) the values K_1, U_1 due to the loading (2.7) and given explicitly in [2], and (*ii*) the values K_2, U_2 due to the loading (2.8), and derived in the present paper. Our results will be normalized by K_0, U_0, where

$$K_0 = p_0(b/2)^{1/2} \tag{5.1}$$

$$U_0 = \begin{cases} 4p_0 b/E \text{ for plane stress,} \\ 4p_0 b(1 - \nu^2)/E \text{ for plane strain,} \end{cases} \tag{5.2}$$

$$p_0 = \tfrac{1}{3}\varrho\omega^2 a^2. \tag{5.3}$$

(K_0, U_0 are associated with the problem of a Griffith crack of length $2b$ expanded by constant internal pressure p_0.)

Thus recalling (2.5)

$$\frac{K}{K_0} = \frac{K_1}{K_0} - \alpha(\nu)\frac{K_2}{K_0}, \tag{5.4}$$

Table 1. Values of K_1/K_0, K_2/K_0, U_1/U_0, U_2/U_0 for $0 \leqslant b/a < 2$.

b/a	K_1/K_0	K_2/K_0	U_1/U_0	U_2/U_0
0.00	1.122	2.243	1.454	2.909
0.10	1.211	2.008	1.640	2.983
0.20	1.314	1.786	1.860	3.050
0.30	1.431	1.578	2.121	3.111
0.40	1.567	1.382	2.435	3.165
0.50	1.727	1.200	2.817	3.212
0.60	1.915	1.031	3.287	3.250
0.70	2.140	0.875	3.874	3.281
0.80	2.413	0.732	4.619	3.303
0.90	2.750	0.602	5.583	3.316
1.00	3.172	0.485	6.860	3.320
1.10	3.715	0.382	8.598	3.314
1.20	4.433	0.291	11.045	3.297
1.30	5.416	0.212	14.639	3.268
1.40	6.825	0.147	20.215	3.227
1.50	8.972	0.094	29.526	3.172
1.60	12.539	0.054	46.787	3.102
1.70	19.305	0.025	84.335	3.014
1.80	35.466	0.008	192.362	2.908
1.90	100.312	0.001	779.877	2.782

where

$$\frac{K_1}{K_0} = \frac{2}{\pi^{1/2}K_+(\mathrm{i})}\left[2 - \frac{b}{a}\right]^{-3/2}, \tag{5.5}$$

$$\alpha(\nu) = \begin{cases} \dfrac{1 + 3\nu}{8} & \text{for plane stress,} \\[2ex] \dfrac{1 + 2\nu}{8(1 - \nu)} & \text{for plane strain,} \end{cases} \tag{5.6}$$

and the value of $K_+(\mathrm{i})$ is given in Appendix 1. The value of K_2/K_0 can be deduced from (2.27) and (4.4); some numerical details are given in Appendix 1. Table 1 gives numerical values of K_1/K_0 and K_2/K_0 for various b/a. These values are *independent of Poisson's ratio* ν.

Similarly

$$\frac{U}{U_0} = \frac{U_1}{U_0} - \alpha(\nu)\frac{U_2}{U_0}, \tag{5.7}$$

where (from [2])

$$\frac{U_1}{U_0} = \frac{1}{K_+(\mathrm{i})}\left\{\frac{1}{K_+(0)}\left[2 - \frac{b}{a}\right]^{-1} + \frac{b/2a}{K_+(\mathrm{i})}\left[2 - \frac{b}{a}\right]^{-2}\right\}, \tag{5.8}$$

and $\alpha(\nu)$ is given by (5.6); values of $K_+(0)$, $K_+(\mathrm{i})$ are given in Appendix 1. The value of U_2/U_0 can be deduced from (3.8), (4.7), (4.8), and numerical values are given in Table 1. These values are *independent of Poisson's ratio* ν.

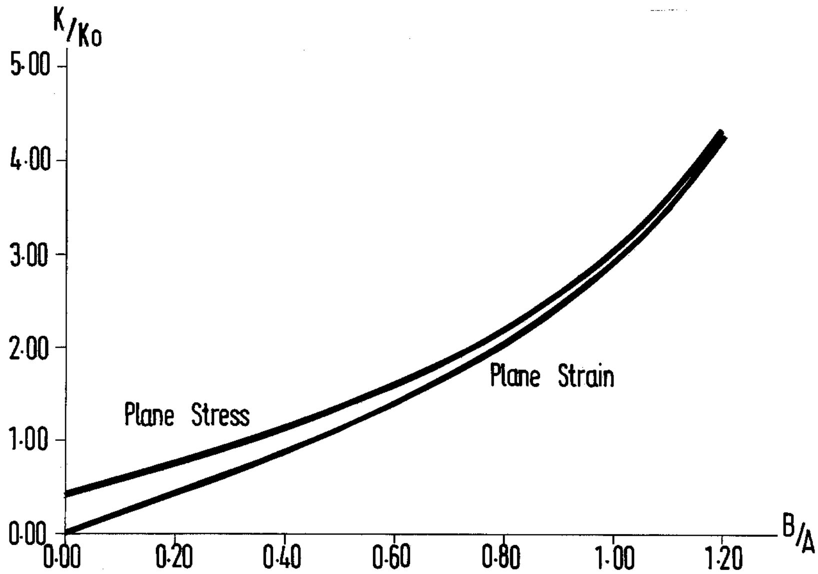

Fig. 2. K/K_0 for plane stress and strain with $\nu = 0.5$.

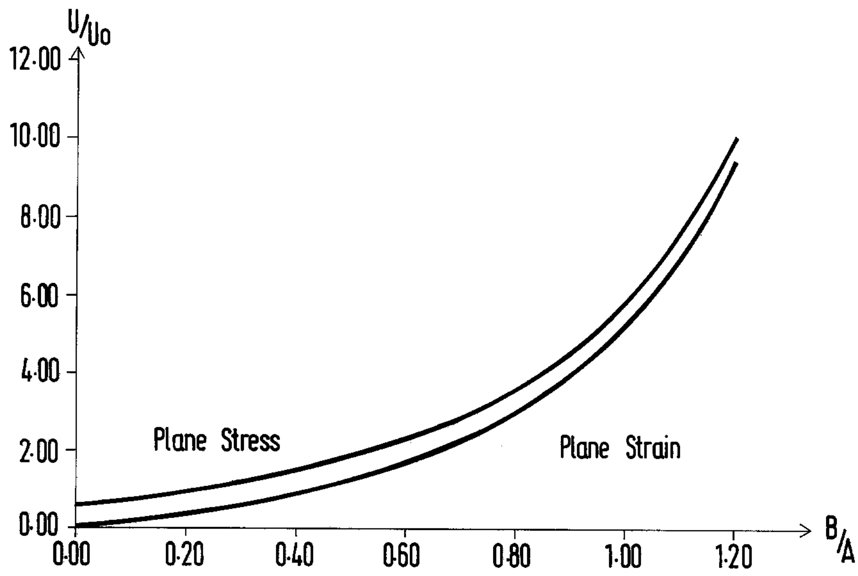

Fig. 3. U/U_0 for plane stress and strain with $\nu = 0.5$.

The values of K/K_0 in plane *strain*, when restricted to the case $b < a$, are in agreement with the results of Rooke and Tweed [5].

The crack opening U may be significant when the disc is made of (e.g.,) polymer whose Young's modulus is much smaller than that of metals. Suppose for example that the disc has radius 20 cm, contains an edge crack 16 cm long, is rotating at 60 revs/sec, and has the material properties $E = 2 \times 10^8\,\mathrm{N\,m^{-2}}$, $\nu = 0.45$, $\varrho = 600\,\mathrm{kg\,m^{-3}}$. Then the calculated value of U is 1.33 cm. (The maximum displacement which would occur in the same disc, but with the crack absent, is 0.047 cm.)

Appendix 1. The function $K_+(\alpha)$ and numerical procedures

The function $K_+(\alpha)$ is defined (see [2], Appendix 1) by

$$K_+(\alpha) = \left(\frac{\pi^2 - 4}{2\pi}\right)^{1/2} 2^{i\alpha} \prod_{n=1}^{\infty} \left\{ \frac{\left(1 + \frac{\alpha}{\alpha_n}\right)\left(1 - \frac{\alpha}{\bar{\alpha}_n}\right)}{1 + \frac{\alpha}{in}} \right\}, \tag{A1.1}$$

where $\{\alpha_n\}$ $(n \geqslant 1)$ are the complex roots of

$$\sinh^2 \tfrac{1}{2}\pi\alpha - \alpha^2 = 0 \tag{A1.2}$$

lying in the first quadrant and arranged in order of increasing modulus. The $\{\alpha_n\}$ were determined numerically by a complex Newton iteration method using as starting values the asymptotic formula for α_n valid for large n. Although (A1.1) can be used to evaluate $K_+(\alpha)$, the terms of this infinite product are only $1 + O(1/n^2)$ as $n \to \infty$ and so convergence is rather slow. It is more efficient to use the modified product

$$K_+(\alpha) = \left(\frac{\pi^2 - 4}{2\pi}\right)^{1/2} 2^{i\alpha} \frac{\sin z}{z} \prod_{n=1}^{\infty} \left\{ \frac{\left(1 + \frac{\alpha}{\alpha_n}\right)\left(1 - \frac{\alpha}{\bar{\alpha}_n}\right)}{\left(1 + \frac{\alpha}{in}\right)\left(1 + \frac{2i\alpha - \alpha^2}{4n^2}\right)} \right\} \tag{A1.3}$$

where $z = \pi(\alpha^2 - 2i\alpha)^{1/2}/2$, whose terms are $1 + O(\log^2 n/n^3)$. If this product is combined with an integral estimate of the truncation error in stopping the product after N terms, then the truncation error is reduced to $O(\log^2 N/N^3)$ compared with $O(1/N)$ for the original product (A1.1). Values of $K_+(ni)$ for $n = 0, 10$ are given in Table 2.

Values of $K_+(ni)$ for $n = 1{,}100$ were used in the series S_1, S_2, S_3 in (4.3), (4.5), (4.6). $K_+(\alpha)$ was also evaluated for α real and numerical integrations were performed to evaluate the integral expressions for S_1, S_2, S_3 in (4.4), (4.7), (4.8). These numerical values were all found to be consistent in the common region of validity. The values of K_2/K_0, U_2/U_0 in Table 1 were calculated using the integral expressions for S_1, S_2 S_3 and are correct to the number of places given.

Table 2. Values of $K_+(ni)$ to 4 figures

n	$K_+(ni)$
0	0.9665
1	0.3557
2	0.1947
3	0.1265
4	0.0905
5	0.0688
6	0.0545
7	0.0446
8	0.0373
9	0.0319
10	0.0276

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Résumé. Un disque circulaire de rayon "a", en un matériau homogène, isotrope et élastique linéaire, comporte une fissure radiale de bord de longueur "b". Le disque tourne à vitesse angulaire constante autour d'un axe normal passant par son centre dans les limites de théorie linéaire à deux dimensions. On résout le problème de la détermination des champs de contraintes et déplacements dans le disque, sous une forme exacte ou sous une forme fermée. En particulier, le facteur d'intensité de contraintes et la COD sont évalués en condition d'état plan de tension et d'état plan de déformation pour toute longueur de fissure comprise entre 0 et $2a$ et pour toute valeur de constantes élastiques.