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Stresses near the edge of bonded dissimilar materials described by two stress intensity factors

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Abstract. Stresses near the free edges of the interface of bonded dissimilar materials can be described by the sum of a regular stress term and one or two stress singularity terms. A method for the calculation of the corresponding stress intensity factors from finite element results is presented which is useful to determine two stress intensity factors together. Results for some geometries show that all three terms may contribute significantly to the stress distribution near the free edge of the interface for thermal stress.

1. Introduction

In Fig. 1 the general configuration at the interface of bonded dissimilar materials is shown which is characterized by the angles θ_1 and θ_2. The stresses near the free edge of the interface can be described by

$$\sigma_{ij}(r, \theta) = \sum_{k=0}^{N} \frac{K_k}{(r/L)^{\omega_k}} f_{ijk}(\theta), \tag{1}$$

where r and θ are defined in Fig. 1 and L is a characteristic length of the component. The angular functions f_{ijk} and the stress intensity factors K_k are defined in such a way, that $f_{\theta k}(\theta = 0) = 1$ $(k = 0, 1, \ldots, N)$. ω_k and f_{ijk} depend on the elastic constants E_1, ν_1, E_2, ν_2 and on the angles θ_1 and θ_2, but are independent of the loading conditions. They can be calculated analytically by solving the general stress function taking into account the boundary conditions

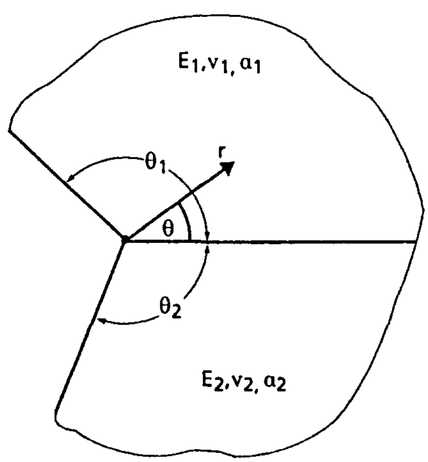

Fig. 1. General configuration at the free edge of a joint.

at $\theta = 0$, θ_1 and θ_2 [1]. The resulting ω_k can be positive real, negative real or complex. For complex values of ω_k (1) is no longer useful and separate considerations are necessary [1, 2]. For $\omega_k > 0$ a stress singularity exists. One solution for ω_k is $\omega_0 = 0$ leading to a stress term, which is independent of r

$$\sigma_{ij0}(\theta) = K_0 f_{ij0}(\theta). \tag{2}$$

Instead of K_0, the designation σ_0 is used in the following text. This term can be calculated analytically for any combination of angles θ_1, θ_2 and any loading condition [3].

The other stress intensity factors K_k ($k = 1, 2, \ldots, N$) have to be calculated numerically from the stress distribution for small values of r, for instance by using the finite element method. They depend on the material properties, the applied loading condition and the geometry of the component.

If, besides $\omega_0 = 0$, more than one value of $|\omega_k|$ is less than about 0.5, then all of them may contribute to the stress field near the edge of the interface. This will be shown in this paper and the corresponding values of K_k are calculated for some examples.

2. Method of determination of several stress intensity factors

For two $\omega_k > 0$ Knesl et al. [4] calculated the two stress intensity factors for mechanical loading. They first obtained K_1 from the stress distribution for small r neglecting the second term and then K_2 from the stress distribution at larger r taking into account the first term. Theocaris [5] used a similar method. In many cases this method cannot be applied, because the term with $k = 2$ contributes significantly to the stress distribution also very close to the free edge of the interface as will be shown in the examples in Section 3. Therefore it is necessary to develop a method by which two or more stress intensity factors can be determined simultaneously. In the following such a method is presented.

Equation (1) can be rewritten by using (2) as

$$\sigma_{ij}(r, \theta) = \sum_{k=1}^{N} \frac{K_k}{(r/L)^{\omega_k}} \cdot f_{ijk}(\theta) + \sigma_{ij0}(\theta), \tag{3}$$

where ω_k, σ_{ij0} and f_{ijk} can be obtained analytically. By the finite element method the stresses $\sigma_{ij}^{FE}(r, \theta)$ can be calculated. Then a quantity Π_{ij} is defined as

$$\Pi_{ij} = \sum_{l=1}^{M} \left\{ \sigma_{ij}^{FE}(r_l, \theta_l) - \sigma_{ij0}(\theta_l) - \sum_{k=1}^{N} \frac{K_k}{(r_l/L)^{\omega_k}} f_{ijk}(\theta_l) \right\}^2. \tag{4}$$

M is the number of points used for determining K_k. According to the least squares method, the minimum of Π_{ij} with respect to the values of K_k has to be found. It is given by

$$\frac{\partial \Pi_{ij}}{\partial K_k} = 0 \tag{5}$$

leading to N equations

$$\sum_{k=1}^{N} K_k \sum_{l=1}^{M} \frac{1}{(r_l/L)^{\omega_k}} f_{ijk}(\theta_l) \cdot \frac{1}{(r_l/L)^{\omega_q}} f_{ijq}(\theta_l)$$

$$= \sum_{l=1}^{M} [\sigma_{ij}^{FE}(r_l, \theta_l) - \sigma_{ij0}(\theta_l)] \frac{1}{(r_l/L)^{\omega_q}} f_{ijq}(\theta_l), \tag{6}$$

with $q = 1, 2, \dots, N$. The values of K_k are obtained by solving these equations.

3. Results

The method given in Section 2 can be used for any geometry, material combinations and loading conditions. To show the applicability of the method two geometries are chosen somewhat arbitrarily. With these examples the general behavior of joints can be discussed.

Cooling down of the joint by ΔT is considered as one loading condition. The stresses are proportional to ΔT. All results for σ_{ij} and K_k are given for $\Delta T = -1°\text{C}$.

A second loading is mechanical loading with a constant load distribution applied at the upper and lower surfaces.

The selected geometries are shown in Fig. 2. For convenience symmetric joints are considered. The angles are $\theta_1 = 165°$, $\theta_2 = -55°$ (combination A) and $\theta_1 = 115°$, $\theta_2 = -45°$ (combination B).

The stress exponents ω_k have been calculated for the material parameters $E_1 = 280\,\text{GPa}$, $\nu_1 = 0.26$, $\nu_2 = 0.3$ and variable E_2. The values of ω_k for which $|\text{Re}(\omega)| < 0.5$ are plotted in Fig. 3 versus E_2/E_1. The complex exponents are described by $\omega = s + ip$.

Different ranges of E_2/E_1 can be distinguished:

Combination A:

$E_2/E_1 < 0.0182$ $\quad$ $\omega_1 < 0$, $\omega_2 < 0$ (no singularity term),

$0.0182 < E_2/E_1 < 8.51$ $\quad$ $\omega_1 > 0$, $\omega_2 < 0$ (one singularity term),

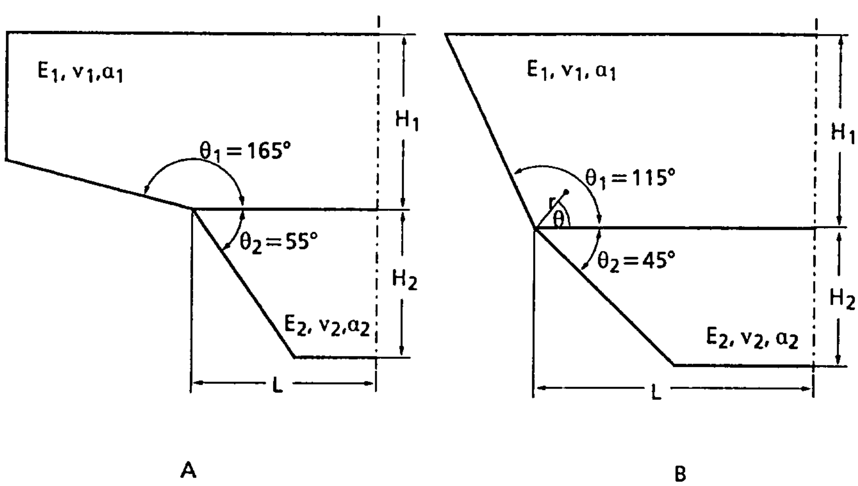

Fig. 2. Geometries of investigated joints (A: $L/H_1 = 1.016$, $L/H_2 = 1.233$, B: $L/H_1 = 1.424$, $L/H_2 = 1.986$).

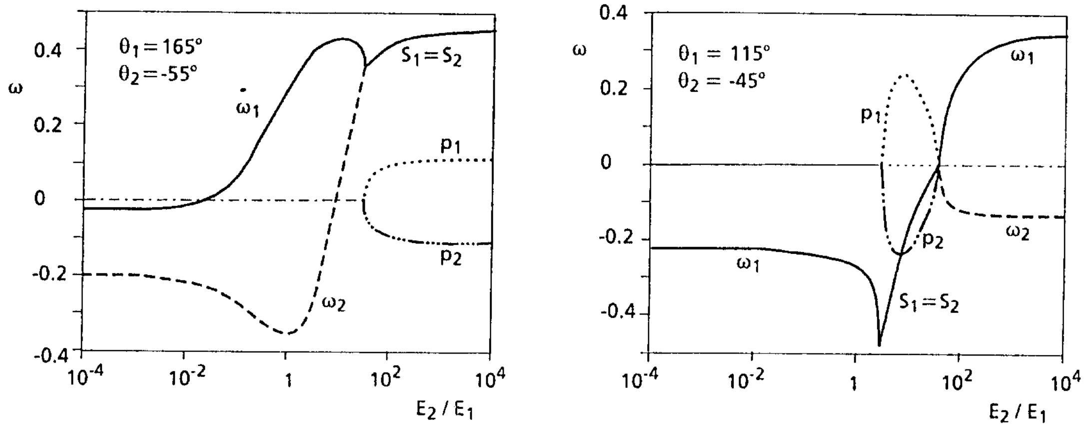

Fig. 3. Stress exponent ω versus E_2/E_1 (E_1 = 280 GPa, $\nu_1 = 0.26$, $\nu_2 = 0.3$).

$8.51 < E_2/E_1 < 30.35$ $\quad\omega_1 > 0$, $\omega_2 > 0$ (two singularity terms),
$E_2/E_1 > 30.35$ $\quad\omega_1$ and ω_2 complex ($s_1 = s_2 > 0$, $p_1 > 0$, $p_2 < 0$).

Combination B:
$E_2/E_1 < 3.931$ $\quad\omega_1 < 0$, no $|\mathrm{Re}(\omega_2)| < 0.5$ (no singularity term),
$3.931 < E_2/E_1 < 43.61$ $\quad\omega_1$ and ω_2 complex, $s_1 = s_2 < 0$, $p_1 > 0$, $p_2 < 0$,
$E_2/E_1 > 43.61$ $\quad\omega_1 > 0$, $\omega_2 < 0$ (one singularity term).

The angular functions and the σ_{ij0} terms can be represented as

$$f_{jrk}(\theta) = \{A_{jk}(2+\omega_k)\sin(\omega_k\theta) + B_{jk}(2+\omega_k)\cos(\omega_k\theta) - C_{jk}(2-\omega_k)\sin[(2-\omega_k)\theta] - D_{jk}(2-\omega_k)\cos[(2-\omega_k)\theta]\}/\{(2-\omega_k)(B_{jk}+D_{jk})\}, \tag{7a}$$

$$f_{j\theta k}(\theta) = \{A_{jk}\sin(\omega_k\theta) + B_{jk}\cos(\omega_k\theta) + C_{jk}\sin[(2-\omega_k)\theta] + D_{jk}\cos[(2-\omega_k)\theta]\} /(B_{jk}+D_{jk})\}, \tag{7b}$$

$$f_{jr\theta k}(\theta) = -\{A_{jk}\omega_k\cos(\omega_k\theta) - B_{jk}\omega_k\sin(\omega_k\theta) + C_{jk}(2-\omega_k)\cos[(2-\omega_k)\theta] - D_{jk}(2-\omega_k)\sin[(2-\omega_k)\theta]\}/\{(2-\omega_k)(B_{jk}+D_{jk})\}, \tag{7c}$$

$$\sigma_{jr0}(\theta) = 2(A_{j0}\theta + B_{j0} - C_{j0}\sin(2\theta) - D_{j0}\cos(2\theta)), \tag{8a}$$

$$\sigma_{j\theta 0}(\theta) = 2(A_{j0}\theta + B_{j0} + C_{j0}\sin(2\theta) + D_{j0}\cos(2\theta)), \tag{8b}$$

$$\tau_{jr\theta 0}(\theta) = -2(\tfrac{1}{2}A_{j0} + C_{j0}\cos(2\theta) - D_{j0}\sin(2\theta)), \tag{8c}$$

with $j = 1, 2$ for the two materials.

In agreement with (2), $\sigma_0 = 2(B_{j0} + D_{j0})$.

It can be shown that the regular stress terms σ_{ij0} are proportionate to $\alpha_1(1+\nu_1) - \alpha_2(1+\nu_2)$ for plane strain [6]. For the following calculations $\alpha_1 = 2.5 \times 10^{-6}$/K and $\alpha_2 = 18.95 \times 10^{-6}$/K have been chosen. The stresses in the joint have been calculated with the FE-code ABAQUS. The stress intensity factors have been determined from the FE-results of σ_θ

at $\theta = 0$. Different values of M in (6) according to different ranges of r/L have been used. The lower bound was fixed to $(r/L)_{\min} = 1.6 \times 10^{-5}$.

Results of K_k as a function of the selected upper bound $(r/L)_{\max}$ for combination A with $E_2/E_1 = 21.42$ and for combination B with $E_2/E_1 = 50$ are shown in Fig. 4 for thermal loading. In both cases the values of K_k are calculated for two terms ($N = 2$). There is a range where the determined K-values are nearly constant. If the upper limit of the selected range in r/L is too high, there will be deviations because terms with $N > 2$ are important. If the upper limit is too low there will be deviations, because of inaccuracies in the FE-results [7]. The range of $1.6 \times 10^{-5} < r/L < 5 \times 10^{-4}$ was used to determine the stress intensity factors.

In Fig. 5 the parameters ω_1, ω_2, K_1, K_2 and σ_0 are plotted versus E_2/E_1 for combination A in the range where ω_1 and ω_2 are real. It can be seen that σ_0 approaches infinity for $\omega_1 \to 0$ and for $\omega_2 \to 0$. This increase in σ_0 is counteracted by an increase of K_1 or K_2 with different signs of σ_0 and K_k. Finite stresses are only possible if for $\omega = 0$ the ratio K/σ_0 is finite. This is shown in Fig. 6 where K_1/σ_0 is plotted versus ω_1 and K_2/σ_0 versus ω_2. For $\omega_1 = 0$ there is $K_1/\sigma_0 = -1$ and for $\omega_2 = 0$ there is $K_2/\sigma_0 = -1$.

For the already mentioned examples the stress distribution is considered in detail. K_1 and K_2, the exponents ω_1 and ω_2, and the stress σ_0 are given in Table 1. Figures 7 and 8 show the stress distributions near the free edge of the interface for $\tau_{r\theta}$ along $\theta = 0$, σ_r along $\theta = \theta_1$ and $\theta = \theta_2$ and σ_r, σ_θ, $\tau_{r\theta}$ along $\theta = -22.5°$. The calculated stresses by means of FEM and analytical form are given. The stresses from the analytical form are the sum of

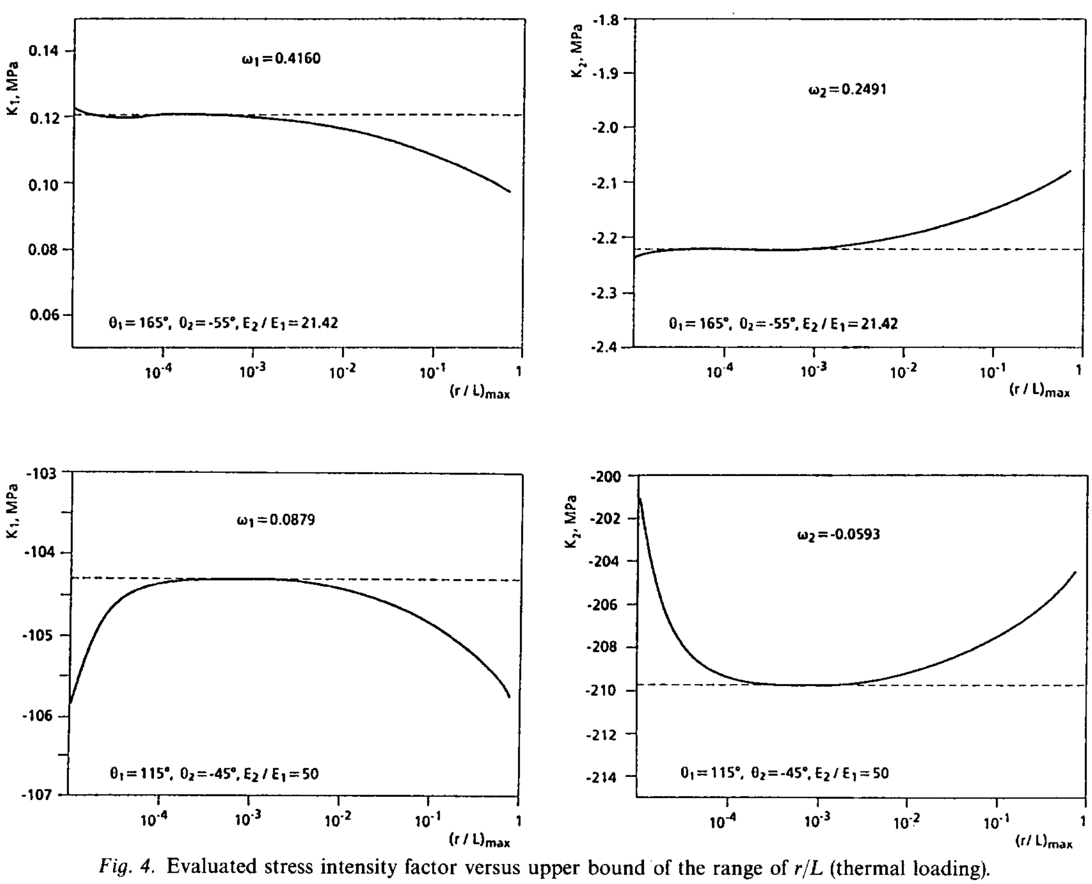

Fig. 4. Evaluated stress intensity factor versus upper bound of the range of r/L (thermal loading).

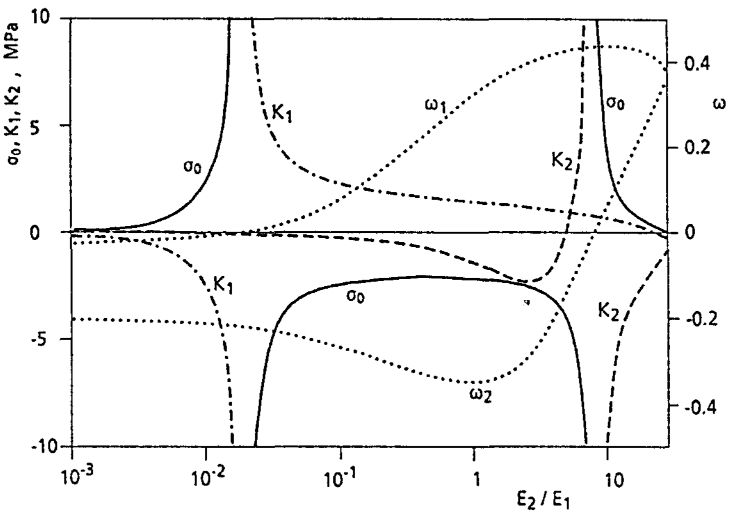

Fig. 5. ω_1, ω_2, K_1, K_2, σ_0 versus E_1/E_2 (combination A, thermal loading).

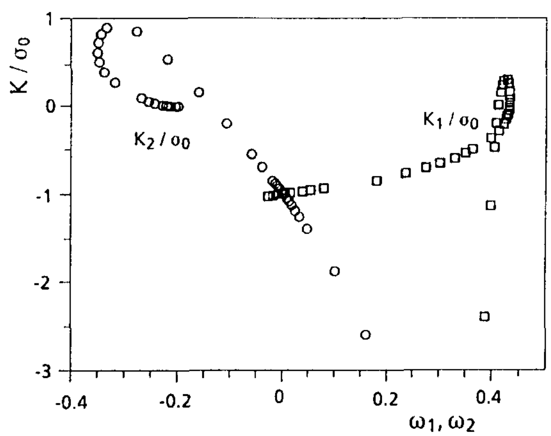

Fig. 6. K_1/σ_0 versus ω_1 and K_2/σ_0 versus ω_2 (combination A, thermal loading).

Table 1. Parameters for two examples

θ_1	θ_2	E_2/E_1	ω_1	ω_2	K_1 [MPa]	K_2 [MPa]	σ_0 [MPa]
165°	−55°	21.42	0.4160	0.2491	0.121	−2.223	0,499
115°	−45°	50.00	0.0879	−0.0593	−104.2	−210.3	306.1

three terms

$$\sigma_{ij} = \sigma_{ij0} + \sigma_{ij1} + \sigma_{ij2} = \sigma_{ij0} + \frac{K_1}{(r/L)^{\omega_1}} f_{ij1} + \frac{K_2}{(r/L)^{\omega_2}} f_{ij2}. \tag{9}$$

The figures show separately the terms σ_{ij0}, σ_{ij1} and σ_{ij2}.

For the example shown in Fig. 7 the two stress exponents are positive ($\omega_1 = 0.4160$, $\omega_2 = 0.2491$). The stress intensity factors have different signs ($K_1 = 0.121$ MPa, $K_2 = -2.223$ MPa). The contribution of σ_{ij0} is small compared to that of σ_{ij1} and σ_{ij2}. It can

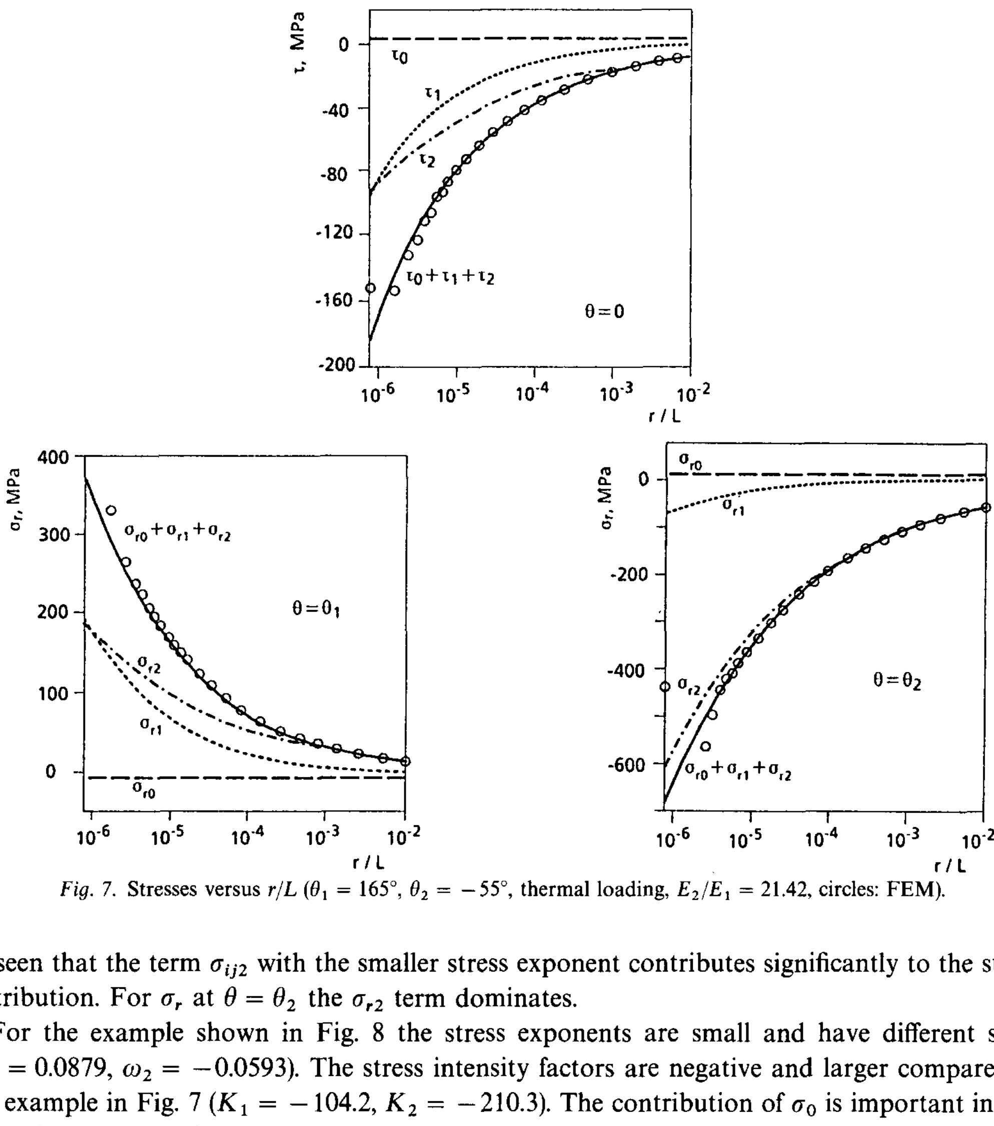

Fig. 7. Stresses versus r/L ($\theta_1 = 165°$, $\theta_2 = -55°$, thermal loading, $E_2/E_1 = 21.42$, circles: FEM).

be seen that the term σ_{ij2} with the smaller stress exponent contributes significantly to the stress distribution. For σ_r at $\theta = \theta_2$ the σ_{r2} term dominates.

For the example shown in Fig. 8 the stress exponents are small and have different signs ($\omega_1 = 0.0879$, $\omega_2 = -0.0593$). The stress intensity factors are negative and larger compared to the example in Fig. 7 ($K_1 = -104.2$, $K_2 = -210.3$). The contribution of σ_0 is important in this example. The nonsingular term σ_{ij2} (ω_2 negative) contributes significantly to the stress distribution even at very small values of r/L.

Figure 9 shows results for combination A with mechanical loading. A constant load distribution is applied at the upper and lower surfaces, leading to a nominal stress σ_n at the interface. The regular stress term σ_{ij0} is zero. In contrast to the thermal loading the stress intensity factors K_1 and K_2 are finite for $\omega_1 = 0$ and $\omega_2 = 0$. For $E_2/E_1 = 21.42$, the stress terms τ_1 and τ_2 are shown in Fig. 10. In contrast to the thermal loadings condition the contribution of the term τ_2 is small.

The Figs. 7, 8, and 10 show also a comparison of the results from the analytical relation and from FEM. The stress intensity factors K_1 and K_2 for the analytical relation have been determined – as already mentioned – from σ_θ at $\theta = 0$. The comparison shows that for all stress components and different θ the agreement between analytical results and FE-calculations is excellent. Only very close to the free edge deviations occur due to the inaccuracies in the FE-method [7]. Thus it is proved that the method is accurate.

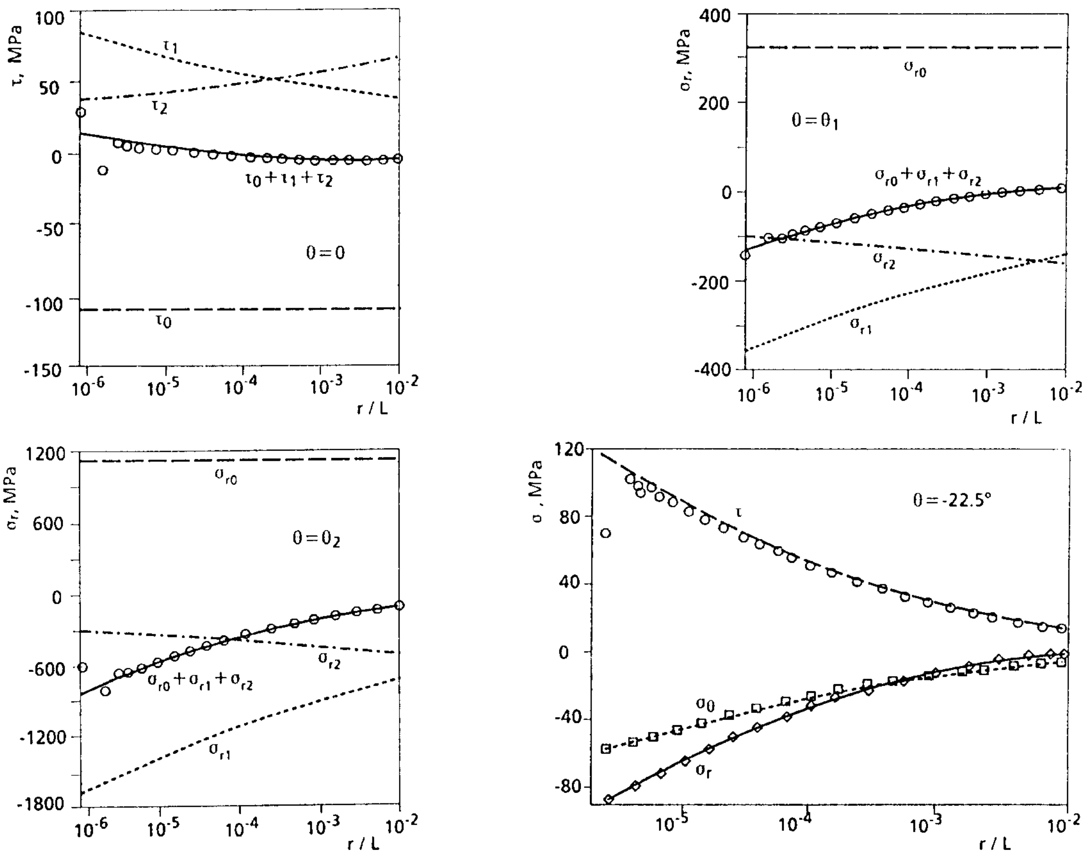

Fig. 8. Stress versus r/L ($\theta_1 = 115°$, $\theta_2 = -45°$, thermal loading, $E_2/E_1 = 50$, circles: FEM).

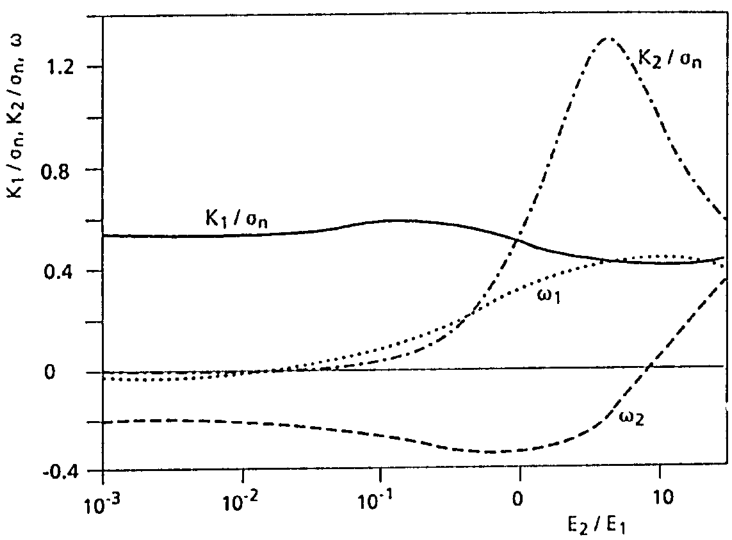

Fig. 9. ω_1, ω_2, K_1/σ_n and K_2/σ_n versus E_2/E_1 (combination A, mechanical loading).

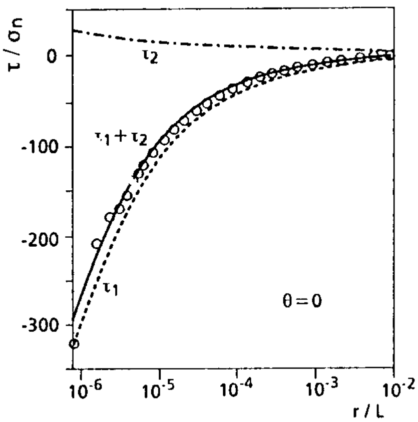

Fig. 10. Stress versus r/L ($\theta_1 = 165°$, $\theta_2 = -55°$, mechanical loading, $E_2/E_1 = 21.42$, circles: FEM).

4. Conclusions

1. A method based on finite element results was given, by which two or more stress intensity factors can be determined at the same time. There is a good agreement between the calculated stresses near the free edge of the interface of the bonded dissimilar materials from the analytical form with the determined stress intensity factors and those from FEM.
2. The stress distribution also very close to the free edge can be considerably influenced by the second term with $\omega_2 < \omega_1$. Even for $\omega_2 < 0$, i.e. no singularity, the second term may contribute significantly to the stresses. Also the term σ_{ij0}, which is independent of the distance from the free edge, can be very important for cooling stresses.
3. For a given geometry K_1, K_2 and σ_0 approach infinity for specific material combinations. The corresponding stress exponent ω_1 and ω_2 are zero and the ratios K_1/σ_0 and K_2/σ_0, respectively are -1. Thus the stresses are finite because the σ_0-term is balanced by the K_1- or K_2-term.
4. With the described procedure it is possible to select material combinations for a given geometry or a geometry (angles θ_1 and θ_2) for given material combinations with minimized stresses.

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