Chapter 2
Vector Spaces

This chapter explores one of the fundamental topics in linear algebra. It starts by defining vector spaces, highlighting their importance as mathematical structures with essential qualities such as closure under addition and scalar multiplication. Subspaces are introduced as vector space subsets with their vector space features, followed by an in-depth analysis of linear dependence and independence of vectors, which are critical for constructing bases. The ideas of span and basis are emphasized as critical tools for understanding the structure of vector spaces, with dimension serving as a quantitative measure of their complexity. Finally, the chapter looks into vector space sums and the particular case of the direct sum, providing a more in-depth understanding of vector space composition.

2.1 Introduction

In Chap. 1, we have called an element of Euclidean space $\mathbb{R}^n$ a "vector". From this chapter onwards, we will be using the term "vector" with a broader meaning. An element of a *vector space* is called a vector. Roughly speaking, a vector space is a collection of objects which are closed under vector addition and scalar multiplication and are subjected to some reasonable rules. The rules are chosen so that we can manipulate the vectors algebraically. We can also consider a vector space as a generalization of the Euclidean space. In this chapter, we will be discussing vector spaces in detail.

Definition 2.1 (*Vector space*) A vector space or linear space V over a field $\mathbb{K}$ is a non-empty set together with two operations called vector addition (denoted by '$+$') and scalar multiplication (as the elements of $\mathbb{K}$ are called scalars) satisfying certain conditions:

$(V1)$ $v_1 + v_2 \in V$ for all $v_1, v_2 \in V$.
$(V2)$ $\lambda v \in V$ for all $\lambda \in \mathbb{K}$ and $v \in V$.

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$(V1)$ and $(V2)$ respectively imply that V is closed under both vector addition and scalar multiplication. The following properties are familiar as we have seen these in Chap. 1, associated with another algebraic structure, *Group*.

$(V3)$ $(v_1 + v_2) + v_3 = v_1 + (v_2 + v_3)$ for all $v_1, v_2, v_3 \in V$.
$(V4)$ there is an element $\mathbf{0} \in V$ such that $v + \mathbf{0} = v$ for all $v \in V$.
$(V5)$ for each $v \in V$ there exists an element $-v \in V$ such that $v + (-v) = \mathbf{0}$.

Thus we can say that V under vector addition must be a group. Now $(V6)$ imply that $(V, +)$ is not just any group, it must be an Abelian group.

$(V6)$ $v_1 + v_2 = v_2 + v_1$ for all $v_1, v_2 \in V$.

Along with closure properties and $(V, +)$ being an Abelian group, the following properties also must be satisfied for V to be a vector space over the field $\mathbb{K}$ under the given operations.

$(V7)$ $\lambda(v_1 + v_2) = \lambda v_1 + \lambda v_2$ for all $\lambda \in \mathbb{K}$ and $v_1, v_2 \in V$.
$(V8)$ $(\lambda + \mu)v = \lambda v + \mu v$ for all $\lambda, \mu \in \mathbb{K}$ and $v \in V$.
$(V9)$ $(\lambda\mu)v = \lambda(\mu v)$ for all $\lambda, \mu \in \mathbb{K}$ and $v \in V$.
$(V10)$ $1v = v$ for all $v \in V$.

Now let us get familiar with some of the important vector spaces that we will see throughout this book. Let us start with a basic one.

Example 2.1 Consider V as the set of all real numbers, $\mathbb{R}$ under usual addition as vector addition and usual multiplication as scalar multiplication, the scalars being taken from the field $\mathbb{R}$ itself. In Chap. 1, we have seen that $(\mathbb{R}, +)$ is an Abelian group. Scalar multiplication in this case is the usual multiplication of real numbers, which is closed. Properties $(V7) - (V10)$ are easy to verify. Thus $\mathbb{R}$ over $\mathbb{R}$ is a vector space. Similarly, we can show that $\mathbb{C}$ over $\mathbb{C}$ is a vector space. What about $\mathbb{C}$ over $\mathbb{R}$ and $\mathbb{R}$ over $\mathbb{C}$?

Example 2.2 Let $\mathbb{K}$ be any field. Then $\mathbb{K}^n$ is a vector space over $\mathbb{K}$, where n is a positive integer and

$$\mathbb{K}^n = \left\{\left(x_1, x_2, \ldots, x_n\right) \mid x_1, x_2, \ldots, x_n \in \mathbb{K}\right\}$$

Addition and scalar multiplication are defined component-wise as we have seen in the previous chapter:

$$\left(x_1, x_2, \ldots, x_n\right) + \left(y_1, y_2, \ldots, y_n\right) = \left(x_1 + y_1, x_2 + y_2, \ldots, x_n + y_n\right)$$

$$\lambda\left(x_1, x_2, \ldots, x_n\right) = \left(\lambda x_1, \lambda x_2, \ldots, \lambda x_n\right), \ \lambda \in \mathbb{K}$$

In particular, $\mathbb{R}^n$ is a vector space over $\mathbb{R}$ and $\mathbb{C}^n$ is a vector space over $\mathbb{C}$ (Verify). Is $\mathbb{R}^n$ a vector space over $\mathbb{C}$?

Example 2.3 The collection of all $m \times n$ matrices, $\mathbb{M}_{m\times n}(\mathbb{K})$, with the usual matrix addition and scalar multiplication is a vector space over $\mathbb{K}$.

Example 2.4 If $\mathbb{F}$ is a sub-field of a field $\mathbb{K}$, then $\mathbb{K}$ is a vector space over $\mathbb{F}$, with addition and multiplication just being the operations in $\mathbb{K}$. Thus, in particular, $\mathbb{C}$ is a vector space over $\mathbb{R}$ and $\mathbb{R}$ is a vector space over $\mathbb{Q}$.

Example 2.5 Let $\mathbb{P}_n[a, b]$ denote the set of all polynomials of degree less than or equal to n defined on $[a, b]$ with coefficients from the field $\mathbb{K}$. For $p, q \in \mathbb{P}_n[a, b]$, and $\lambda \in \mathbb{K}$ the addition and scalar multiplication are defined by

$$(p + q)(x) = p(x) + q(x) = (a_n + b_n)x^n + \cdots + (a_1 + b_1)x + (a_0 + b_0)$$

where $p(x) = a_n x^n + \cdots + a_1 x + a_0$ and $q(x) = b_n x^n + \cdots + b_1 x + b_0$ and

$$(\lambda p)(x) = \lambda\,(p(x)) = (\lambda a_n)x^n + \cdots + (\lambda a_1)x + (\lambda a_0)$$

$\mathbb{P}_n[a, b]$ along with zero polynomial forms a vector space over $\mathbb{K}$. Denote by $\mathbb{P}[a, b]$ the collection of all polynomials defined on $[a, b]$ with coefficients from $\mathbb{K}$. Then $\mathbb{P}[a, b]$ is a vector space over $\mathbb{K}$ with respect to the above operations for polynomials.

Example 2.6 Let $C[a, b]$ denote the set of all real-valued continuous functions on the interval $[a, b]$. If f and g are continuous functions on $[a, b]$, then the vector addition and scalar multiplication are defined by

$$(f + g)(x) = f(x) + g(x) \text{ and } (\lambda f)(x) = \lambda f(x)$$

where $\lambda \in \mathbb{R}$. Then $C[a, b]$ is a vector space with respect to the above operations over the field $\mathbb{R}$.

Example 2.7 Let $\mathbb{K}$ be any field. Let V consist of all sequences $\{a_n\}$ in $\mathbb{K}$ that have only a finite number of non-zero terms a_n. If $\{a_n\}$ and $\{b_n\}$ are in V and $\lambda \in \mathbb{K}$, define

$$\{a_n\} + \{b_n\} = \{a_n + b_n\} \text{ and } \lambda\{a_n\} = \{\lambda a_n\}$$

With the above operations V forms a vector space over $\mathbb{K}$.

Example 2.8 $V = \{0\}$ over the field $\mathbb{K}$ is a vector space called the *zero space*.

Now, we will establish some of the basic properties of vector spaces.

Theorem 2.1 *Let V be a vector space over a field $\mathbb{K}$. Then the following statements are true.*

(a) $0v = \boldsymbol{0}$ *for each* $v \in V$.
(b) $\lambda\boldsymbol{0} = \boldsymbol{0}$ *for each* $\lambda \in \mathbb{K}$.
(c) For $v \in V$ *and* $\lambda \in \mathbb{K}$, *if* $\lambda v = \boldsymbol{0}$, *then either* $\lambda = 0$ *or* $v = \boldsymbol{0}$.
(d) If v_1, v_2, *and* v_3 *are vectors in a vector space* V *such that* $v_1 + v_3 = v_2 + v_3$, *then* $v_2 = v_3$.
(e) $(-\lambda)v = -(\lambda v) = \lambda(-v)$ *for each* $\lambda \in \mathbb{K}$ *and each* $v \in V$.

Proof (a) For $v \in V$, by $(V2)$, $0v \in V$. By $(V5)$, for $0v \in V$ there exists $(-0v)$ such that $0v + (-0v) = \mathbf{0}$. And by using $(V8)$,

$$0v = (0 + 0)v = 0v + 0v \Rightarrow 0v = \mathbf{0}$$

(b) For $\lambda \in \mathbb{K}$ by $(V2)$ $\lambda\mathbf{0} \in V$. By $(V5)$, for $\lambda\mathbf{0} \in V$ there exists $(-\lambda\mathbf{0})$ such that $\lambda\mathbf{0} + (-\lambda\mathbf{0}) = \mathbf{0}$. And by using $(V7)$,

$$\lambda\mathbf{0} = \lambda(\mathbf{0} + \mathbf{0}) = \lambda\mathbf{0} + \lambda\mathbf{0} \Rightarrow \lambda\mathbf{0} = \mathbf{0}$$

(c) Let $\lambda v = \mathbf{0}$. From (1), if $\lambda = 0$, then $\lambda v = \mathbf{0}$. Now suppose that $\lambda \neq 0$, then there exists $\frac{1}{\lambda} \in \mathbb{K}$ and $\frac{1}{\lambda}(\lambda v) = \frac{1}{\lambda}\mathbf{0} \Rightarrow v = \mathbf{0}$.

(d) Suppose that $v_1, v_2, v_3 \in V$ be such that $v_1 + v_3 = v_2 + v_3$. Since $v_3 \in V$, by $(V5)$ there exists $-v_3 \in V$ such that $v_3 + (-v_3) = \mathbf{0}$. Then

$$\begin{aligned} v_1 + v_3 = v_2 + v_3 &\Rightarrow (v_1 + v_3) + (-v_3) = (v_2 + v_3) + (-v_3) \\ &\Rightarrow v_1 + (v_3 + (-v_3)) = v_2 + (v_3 + (-v_3)) \text{ (using } (V3)) \\ &\Rightarrow v_1 = v_2 \text{ (using } (V6)) \end{aligned}$$

(e) By $(V5)$, we have $\lambda v + (-(\lambda v)) = \mathbf{0}$. Also $\lambda v + (-\lambda)v = (\lambda + (-\lambda))v = 0$. By the uniqueness of additive inverse, this implies that $(-\lambda)v = -(\lambda v)$. In particular, $(-1)v = -v$. Now by $(V9)$,

$$\lambda(-v) = \lambda[(-1)v] = [\lambda(-1)]v = (-\lambda)v$$

From the next section, we will use 0 for zero vector, instead of $\mathbf{0}$.

2.2 Subspaces

For vector spaces, there may exist subsets which themselves are vector spaces under the same operations as defined in the parent space. Such subsets of a vector space are called subspaces. We will define the subspace of a vector space as follows.

Definition 2.2 (*Subspace*) A subset W of a vector space V over a field $\mathbb{K}$ is called a *subspace* of V if W is a vector space over $\mathbb{K}$ with the operations of addition and scalar multiplication defined on V.

If V is a vector space, then V and $\{0\}$ are subspaces of V called *trivial subspaces*. The latter is also called the *zero subspace* of V. A subspace W of V is called a proper subspace if $V \neq W$. Otherwise it is called an improper subspace (if it exists). Can you find any subspaces for the vector space $\mathbb{R}$ over $\mathbb{R}$ other than $\mathbb{R}$ and $\{0\}$? By definition, a subspace is a vector space in its own right. To check whether a subset is a subspace, we don't have to verify all the conditions $(V1)-(V10)$. The following theorem gives the set of conditions that are to be verified.

Theorem 2.2 *Let V be a vector space over a field $\mathbb{K}$. A subset W of V is a subspace if and only if the following three conditions hold for the operations defined in V.*

(a) $0 \in W$.
(b) $w_1 + w_2 \in W$ *whenever* $w_1, w_2 \in W$.
(c) $\lambda w \in W$ *whenever* $\lambda \in \mathbb{K}$ *and* $w \in W$.

Proof Suppose that W is a subspace of V. Then W is a vector space with the operation addition and scalar multiplication defined on V. Therefore (b) and (c) are satisfied. And by the uniqueness of identity element in a vector space $0 \in W$.

Conversely suppose that the conditions (a), (b), and (c) are satisfied. We have to show that W is a vector space with the operations defined on V. Since W is a subset of the vector space V, the conditions $(V3)$, $(V5)-(V10)$ are automatically satisfied by the elements in W. Therefore W is a subspace of V.

Certainly, we can observe that Condition (a) in the above theorem need not be checked separately, as it can be obtained from Condition (c) with $\lambda = 0$. But Condition (a) can be used to identify subsets which are not subspaces as shown in the following example.

Example 2.9 Let $V = \mathbb{R}^2 = \{(x_1, x_2) \mid x_1, x_2 \in \mathbb{R}\}$. We have seen that $\mathbb{R}^2$ is a vector space over $\mathbb{R}$. Consider $W_1 = \{(x_1, x_2) \mid x_1 + x_2 = 0\}$ and $W_2 = \{(x_1, x_2) \mid x_1 + x_2 = 1\}$. Then W_1 is a subspace of V. For,

(a) Clearly, the additive identity $(0, 0)$ is in W_1.
(b) Take two elements $(x_1, x_2), (y_1, y_2) \in W_1$. Then $x_1 + x_2 = 0$ and $y_1 + y_2 = 0$. This implies that $(x_1, x_2) + (y_1, y_2) = (x_1 + y_1, x_2 + y_2) \in W_1$ as $x_1 + x_2 + y_1 + y_2 = 0$.
(c) Take $(x_1, x_2) \in W_1$ and $\lambda \in \mathbb{R}$. Then $x_1 + x_2 = 0$. This implies that $\lambda(x_1, x_2) = (\lambda x_1, \lambda x_2) \in W_1$ as $\lambda x_1 + \lambda x_2 = \lambda(x_1 + x_2) = 0$.

But W_2 is not a subspace of $\mathbb{R}^2$ as zero vector does not belong to W_2. Now let us discuss the geometry of W_1 and W_2 a bit. W_1 and W_2 represent two lines on the plane as shown in the figure (Fig. 2.1).

Later, we will see that the only non-trivial proper subspaces of $\mathbb{R}^2$ are straight lines passing through origin.

Example 2.10 Let $V = \mathbb{M}_{n\times n}(\mathbb{K})$ and $W = \left\{A \in \mathbb{M}_{n\times n}(\mathbb{K}) \mid A^T = A\right\}$. That is, W is the set of all $n \times n$ symmetric matrices over $\mathbb{K}$. We will check whether the conditions in Theorem 2.2 are satisfied or not.

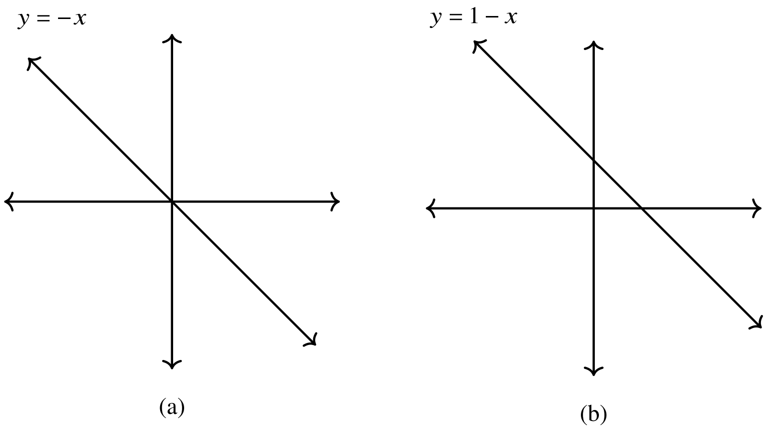

Fig. 2.1 Observe that W_1 depicted in (**a**) (straight line passing through origin) is a subspace and W_2 depicted in (**b**) (straight line not passing through origin) is not a subspace

(a) The zero matrix is equal to its transpose and hence belongs to W.
(b) By the properties of symmetric matrices, the sum of two symmetric matrices is again a symmetric matrix. That is, $A + B \in W$ whenever $A, B \in W$.
(c) Also $\lambda A \in W$ whenever $A \in W$ and $\lambda \in \mathbb{K}$, since $(\lambda A)^T = \lambda A^T = \lambda A$ as $A^T = A$.

Therefore, the set of all $n \times n$ symmetric matrices over $\mathbb{K}$ is a subspace of $\mathbb{M}_{n\times n}(\mathbb{K})$. What about the set of all $n \times n$ skew-symmetric matrices over $\mathbb{K}$?

Example 2.11 Let $V = \mathbb{P}_2[a, b]$. Consider $W = \{p \in \mathbb{P}_2[a, b] \mid p(0) = 0\}$.

(a) Since $p(0) = 0$ for zero polynomial, zero polynomial belongs to W.
(b) Take $p, q \in W$, then $p(0) = q(0) = 0$ and hence $(p + q)(0) = p(0) + q(0) = 0$. Thus $p + q \in W$ whenever $p, q \in W$.
(c) Let $p \in W$ and $\lambda \in \mathbb{R}$, then $(\lambda p)(0) = \lambda p(0) = 0$. That is, $\lambda p \in W$ whenever $p \in W$ and $\lambda \in \mathbb{R}$.

Therefore $\{p \in \mathbb{P}_2[a, b] \mid p(0) = 0\}$ is a subspace of $\mathbb{P}_2[a, b]$. Now, consider the subset $\tilde{W} = \{p \in \mathbb{P}_2[a, b] \mid p(0) = 1\}$. Is it a subspace of $\mathbb{P}_2[a, b]$? It is not!! (Why?)

Remark 2.1 To check whether a subset of a vector space is a subspace, we verify only the closure properties of vector addition and scalar multiplication in the given set. Therefore Theorem 2.2 can also be stated as follows:

- A subset W of a vector space V is a subspace of V if and only if $\lambda w_1 + \mu w_2 \in W$, whenever $w_1, w_2 \in W$ and $\lambda, \mu \in \mathbb{K}$
- A subset W of a vector space V is a subspace of V if and only if $\lambda w_1 + w_2 \in W$, whenever $w_1, w_2 \in W$ and $\lambda \in \mathbb{K}$.

Example 2.12 In the previous chapter, we have seen that the collection of all solutions to the system $Ax = 0$ satisfies the conditions in Remark 2.1 where $A \in \mathbb{M}_{m\times n}(\mathbb{K})$ and hence they form a subspace of $\mathbb{K}^n$. That is, the solutions of a homogeneous system form a vector space under the operations defined on $\mathbb{K}^n$. But the solutions of a non-homogeneous system does not form a vector space as zero vector is never a solution for a non-homogeneous system.

The next theorem gives a method to construct new subspaces from known subspaces.

Theorem 2.3 *Let W_1 and W_2 be two subspaces of a vector space V over a field $\mathbb{K}$, then their intersection $W_1 \cap W_2 = \{w \mid w \in W_1$ and $w \in W_2\}$ is a subspace of V.*

Proof Since W_1 and W_2 are subspaces of V, $0 \in W_1$ and $0 \in W_2$. Therefore $0 \in W_1 \cap W_2$. Now let $v, w \in W_1 \cap W_2$, then

$$\begin{aligned} v, w \in W_1 \cap W_2 &\Rightarrow v, w \in W_1 \text{ and } v, w \in W_2 \\ &\Rightarrow v + w \in W_1 \text{ and } v + w \in W_2 \text{ as } W_1 \text{ and } W_2 \text{ are subspaces} \\ &\Rightarrow v + w \in W_1 \cap W_2 \end{aligned}$$

For $\lambda \in \mathbb{K}$ and $w \in W_1 \cap W_2$,

$$\begin{aligned} w \in W_1 \cap W_2 &\Rightarrow w \in W_1 \text{ and } w \in W_2 \\ &\Rightarrow \lambda w \in W_1 \text{ and } \lambda w \in W_2 \text{ as } W_1 \text{ and } W_2 \text{ are subspaces} \\ &\Rightarrow \lambda w \in W_1 \cap W_2 \end{aligned}$$

Therefore $W_1 \cap W_2$ is a subspace of V.

The above result can be extended to any number of subspaces. As we have shown that the intersection of subspaces is again a subspace, it is natural to ask whether the union of subspaces is again a subspace. It is clear that the union of two subspaces need not be a subspace of V (Fig. 2.2).

The following theorem gives a scenario in which union of two subspaces of a vector space is again a subspace of the same.

Theorem 2.4 *Let V be a vector space over the field $\mathbb{K}$ and let W_1 and W_2 be subspaces of V. Then $W_1 \cup W_2$ is a subspace of V if and only if either $W_1 \subseteq W_2$ or $W_2 \subseteq W_1$.*

Proof Let W_1 and W_2 be subspaces of V. Suppose that either $W_2 \subseteq W_1$ or $W_1 \subseteq W_2$. Then $W_1 \cup W_2$ is either W_1 or W_2. In either cases, $W_1 \cup W_2$ is a subspace of V. Conversely, suppose that $W_1 \cup W_2$ is a subspace of V, $W_1 \not\subseteq W_2$ and $W_2 \not\subseteq W_1$. Then there exists at least one element $w_1 \in W_1$ such that $w_1 \notin W_2$ and $w_2 \in W_2$ such that $w_2 \notin W_1$. As $W_1, W_2 \subseteq W_1 \cup W_2$ both $w_1, w_2 \in W_1 \cup W_2$. Since $W_1 \cup W_2$ is a subspace of V, $w_1 + w_2 \in W_1 \cup W_2$. Then either $w_1 + w_2 \in W_1$ or $w_1 + w_2 \in W_2$.

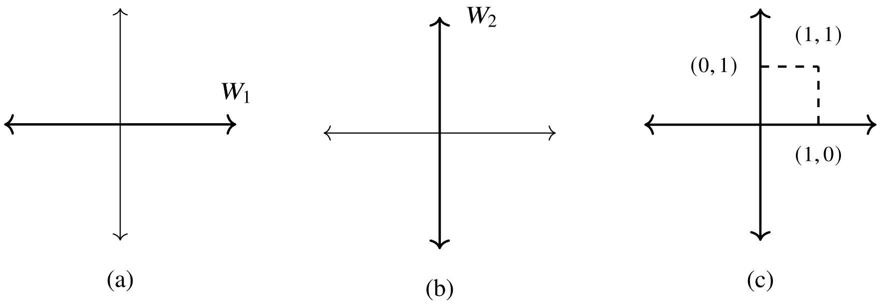

Fig. 2.2 Consider $V = \mathbb{R}^2$, take $W_1 = x - axis$ and $W_2 = y - axis$ (depicted as (**a**) and (**b**) respectively). Then W_1 and W_2 are subspaces of V but $W_1 \cup W_2$ is not a subspace of V, since $(1,0) \in W_1,\ (0,1) \in W_2$, but $(1,0) + (0,1) = (1,1) \notin W_1 \cup W_2$, as we can observe from (**c**)

Suppose $w_1 + w_2 \in W_1$. Since $w_1 \in W_1$ and W_1 is a subspace, $-w_1 \in W_1$ and hence $(-w_1) + w_1 + w_2 = (-w_1 + w_1) + w_2 = w_2 \in W_1$ which is a contradiction. Now suppose $w_1 + w_2 \in W_2$. Since $w_2 \in W_2$ and W_2 is a subspace, $-w_2 \in W_2$ and hence $w_1 + w_2 + (-w_2) = w_1 + (w_2 - w_2) = w_1 \in W_2$ which is again a contradiction. Therefore our assumption is wrong. That is, $W_1 \cup W_2$ is a subspace of V if and only if either $W_1 \subseteq W_2$ or $W_2 \subseteq W_1$.

Example 2.13 Let V be the vector space $\mathbb{R}^3$ over $\mathbb{R}$. Consider $W_1 = \big\{(x_1, x_2, 0) \mid x_1, x_2 \in \mathbb{R}\big\}$ and $W_2 = \{(0, x_2, 0) \mid x_2 \in \mathbb{R}\}$. Clearly, $W_1 \cup W_2 = W_1$ is a subspace. Observe that $W_2 \subset W_1$.

2.3 Linear Dependence and Independence

Let V be a vector space over a field $\mathbb{K}$. Let $v_1, v_2, \ldots, v_n \in V$ and $\lambda_1, \lambda_2, \ldots, \lambda_n \in \mathbb{K}$. Then the vector

$$v = \lambda_1 v_1 + \lambda_2 v_2 + \cdots + \lambda_n v_n$$

is called a *linear combination* of the vectors and the scalars $\lambda_1, \lambda_2, \ldots, \lambda_n$ are called the *coefficients* of the linear combination. If all the coefficients are zero, then $v = 0$, which is trivial. Now suppose that there exists a non-trivial representation for 0, that is, there exists scalars $\lambda_1, \lambda_2, \ldots, \lambda_n$ not all zero such that a linear combination of the given vectors equals zero . Then we say that the vectors $v_1, v_2, \ldots, v_n$ are *linearly dependent*. In other words, the vectors $v_1, v_2, \ldots, v_n$ are linearly dependent if and only if there exist scalars $\lambda_1, \lambda_2, \ldots, \lambda_n$ not all zero such that

$$\lambda_1 v_1 + \lambda_2 v_2 + \cdots + \lambda_n v_n = 0$$

The vectors $v_1, v_2, \ldots, v_n$ are *linearly independent* if they are not linearly dependent. That is,

$$\text{if } \lambda_1 v_1 + \lambda_2 v_2 + \cdots + \lambda_n v_n = 0, \text{ then } \lambda_1 = \lambda_2 = \cdots = \lambda_n = 0$$

Clearly, any subset of a vector space V containing zero vector is linearly dependent as 0 can be written as $0 = 1.0$. Since $\lambda v = 0$ implies either $\lambda = 0$ or $v = 0$, any singleton subset of V containing a non-zero vector is linearly independent.

Example 2.14 Consider the vector space $V = \mathbb{R}^2$ and the subset $S_1 = \{(1, 0), (1, 1)\}$. To check whether S_1 is linearly dependent or not, consider a linear combination of vectors in S_1 equals zero for some scalars λ_1 and λ_2. Then

$$\begin{aligned} \lambda_1(1, 0) + \lambda_2(1, 1) = (0, 0) &\Rightarrow (\lambda_1 + \lambda_2, \lambda_2) = (0, 0) \\ &\Rightarrow \lambda_1 + \lambda_2 = 0, \lambda_2 = 0 \\ &\Rightarrow \lambda_1 = 0, \lambda_2 = 0 \end{aligned}$$

That is, there does not exist non-trivial representation for zero vector in $\mathbb{R}^2$ using vectors of S_1. Thus S_1 is linearly independent. Note that $(1, 0)$ cannot be obtained by scaling $(1, 1)$ or vice verse.

Now consider a subset $S_2 = \{(1, 0), (2, 0)\}$ of $\mathbb{R}^2$ and a linear combination of the vectors in S_2 equals zero. Then

$$\begin{aligned} \lambda_1(1, 0) + \lambda_2(2, 0) = (0, 0) &\Rightarrow (\lambda_1 + \lambda_2, \lambda_2) = (0, 0) \\ &\Rightarrow \lambda_1 + 2\lambda_2 = 0 \end{aligned}$$

Then there are infinitely many possibilities for λ_1 and λ_2. For example, $\lambda_1 = 2$ and $\lambda_2 = -1$ is one such possibility. Clearly, $2(1, 0) + (-1)(2, 0) = (0, 0)$. Thus the zero vector in $\mathbb{R}^2$ has a non-trivial representation using the vectors of S_2. Thus S_2 is linearly dependent. Note that $(2, 2) = 2(1, 1)$ is a scaled version of $(1, 1)$ (Fig. 2.3).

Using the above geometrical idea, try to characterize the linearly independent sets in $\mathbb{R}$ and $\mathbb{R}^2$. Also observe that the equation, $\lambda_1(1, 0) + \lambda_2(1, 1) = (\lambda_1 + \lambda_2, \lambda_2) = (0, 0)$ formed by vectors in S_1, from the above example, can be written in the form of a system of homogeneous equation as $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$. We have seen in Section 1.7 that a $n \times n$ homogeneous system $Ax = 0$ has a non-trivial solution when the coefficient matrix A has rank less than n. In this case, $rank\left(\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\right) = 2$. Therefore the system does not have a non-trivial solution. That is, $\lambda_1 = \lambda_2 = 0$. Now, for vectors in S_2, observe that the coefficient matrix $A = \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}$ has rank 1, which implies that there exists a non-trivial representation for the zero vector. Using this idea, can we say something about the linear dependency/independency of a collection of vectors in $\mathbb{R}^2$? Is it possible to generalize this idea to $\mathbb{R}^n$? Think!!!

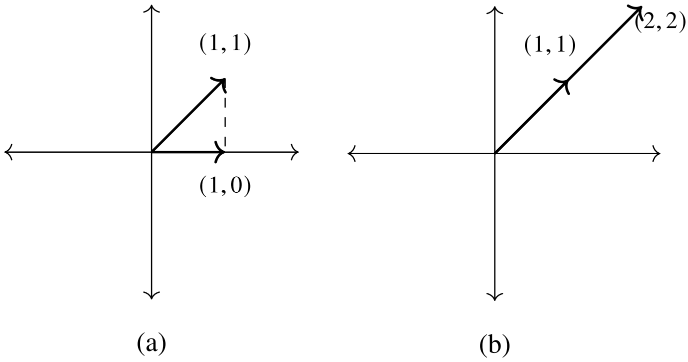

Fig. 2.3 Examples for **a** linearly independent vectors in $\mathbb{R}^2$ and **b** linearly dependent vectors in $\mathbb{R}^2$. Observe that the linearly independent vectors lie on two distinct straight lines passing through origin and the linearly dependent vectors lie on the same line passing through origin. We will soon prove that a set of two vectors is linearly dependent if and only if one vector is a scalar multiple of the other

Remark 2.2 We can say that the number of linearly independent vectors in a collection S of m vectors of $\mathbb{K}^n$ is the rank of the $n \times m$ matrix A formed by the vectors in S as columns. As the rank of a matrix and its transpose is the same, we may redefine the rank of a matrix as the number of linearly independent rows or columns of that matrix.

Example 2.15 Consider the vector space $V = \mathbb{P}_2[a, b]$ and the subset $S_1=\{1, x, x^2\}$. Now, for $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{K}$,

$$\lambda_1.1 + \lambda_2 x + \lambda_3 x^2 = 0 \Rightarrow \lambda_1 = \lambda_2 = \lambda_3 = 0$$

Thus S_1 is linearly independent.

Now consider the subset $S_2 = \{1 - x, 1 + x^2, 3 - 2x + x^2\}$ of $\mathbb{P}_2[a, b]$. As

$$2(1 - x) + 1(1 + x^2) = 3 - 2x + x^2$$

S_2 is linearly dependent.

As we have seen in the previous example, consider the matrices $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ and $\begin{bmatrix} 1 & 1 & 3 \\ -1 & 0 & -2 \\ 0 & 1 & 1 \end{bmatrix}$. Is there any relation between the rank of these matrices and the linear dependency/independency of vectors in S_1 and S_2 given in Example 2.15?

The following results are some of the important consequences of definitions of linear dependence and independence.

Theorem 2.5 *Let V be a vector space over a field $\mathbb{K}$ and $W = \{w_1, w_2, \ldots, w_n\}$ be a subset of V, where $n \geq 2$. Then W is linearly dependent if and only if at least one vector in W can be written as a linear combination of the remaining vectors in W.*

Proof Suppose that W is linearly dependent. Then there exists scalars $\lambda_1, \lambda_2, \ldots, \lambda_n \in \mathbb{K}$, not all zero such that

$$\lambda_1 w_1 + \lambda_2 w_2 + \cdots + \lambda_n w_n = 0$$

Without loss of generality, assume that $\lambda_1 \neq 0$. Then since $\lambda_1 \in \mathbb{K}$, $\frac{1}{\lambda_1} \in \mathbb{K}$ and hence

$$w_1 = -\frac{\lambda_2}{\lambda_1} w_2 - \frac{\lambda_3}{\lambda_1} w_3 - \cdots - \frac{\lambda_n}{\lambda_1} w_n$$

Conversely suppose that one vector in W can be written as a linear combination of the remaining vectors in W. Without loss of generality, take $w_1 = \lambda_2 w_2 + \cdots + \lambda_n w_n$. Then $w_1 - \lambda_2 w_2 + \cdots + \lambda_n w_n = 0$. That is, there exists a non-trivial representation for zero. Therefore W is linearly dependent.

Corollary 2.1 *A subset of a vector space V containing two non-zero vectors is linearly dependent if and only if one vector is a scalar multiple of the other.*

Proof Suppose that $\{v_1, v_2\} \subseteq V$ be linearly dependent. Then there exists scalars $\lambda_1, \lambda_2 \in \mathbb{K}$ not both zero such that $\lambda_1 v_1 + \lambda_2 v_2 = 0$. Without loss of generality, let $\lambda_1 \neq 0$. Then $v_1 = -\frac{\lambda_2}{\lambda_1} v_2$. The converse part is trivial.

Theorem 2.6 *Let V be a vector space over a field $\mathbb{K}$, and let $W_1 \subseteq W_2 \subseteq V$. If W_1 is linearly dependent, then W_2 is linearly dependent and if W_2 is linearly independent, then W_1 is linearly independent.*

Proof Suppose that W_1 is linearly dependent and $W_1 \subseteq W_2$. Then there exists $v_1, v_2, \ldots, v_n \in W_1$ and $\lambda_1, \lambda_2, \ldots, \lambda_n \in \mathbb{K}$, not all 0 such that $\lambda_1 v_1 + \lambda_2 v_2 + \cdots + \lambda_n v_n = 0$. Since $W_1 \subseteq W_2$, W_2 is linearly dependent.

Now suppose that W_2 is linearly independent. Then from above W_1 is linearly independent. For if W_1 is linearly dependent, then W_2 must be linearly dependent.

Thus we can say that any super set of a linearly dependent set is linearly dependent and any subset of a linearly independent set is linearly independent.

2.4 Basis and Dimension

In this section, we will study the basic building blocks of vector spaces known as basis. A basis of a vector space is a subset of the vector space which can be used to uniquely represent each vector in the given space. We will start by the following definition.

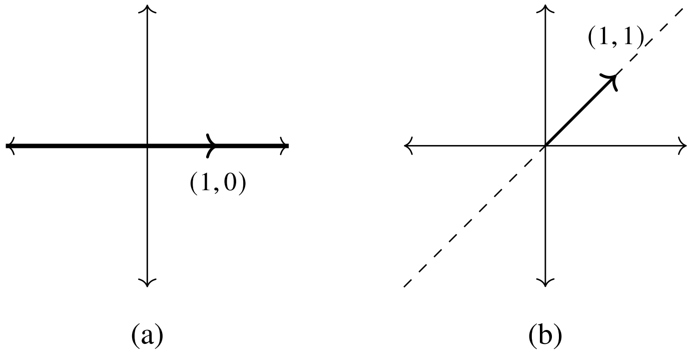

Fig. 2.4 Observe that both $Span(S_1)$ and $Span(S_2)$ are straight lines passing through origin

Definition 2.3 (*Span of a set*) Let $S = \{v_1, v_2, \ldots, v_n\}$ be a subset of a vector space V. Then the span of S, denoted by $span(S)$, is the set consisting of all linear combinations of the vectors in S. That is,

$$span(S) = \{\lambda_1 v_1 + \lambda_2 v_2 + \cdots + \lambda_n v_n \mid \lambda_1, \lambda_2, \ldots, \lambda_n \in \mathbb{K}\}$$

For convenience, we define $span\{\phi\} = \{0\}$. A subset S of a vector space V spans (or generates) V if $span(S) = V$. If there exists a finite subset S of V such that $span(S) = V$, then V is called finite-dimensional vector space. Otherwise it is called infinite-dimensional vector space.

Example 2.16 Consider $S_1 = \{(1, 0)\}$ and $S_2 = \{(1, 1)\}$ in $\mathbb{R}^2$. Then (Fig. 2.4)

$$Span(S_1) = \{\lambda(1, 0) \mid \lambda \in \mathbb{R}\} = \{(\lambda, 0) \mid \lambda \in \mathbb{R}\} = x - \text{axis}$$

and

$$Span(S_2) = \{\lambda(1, 1) \mid \lambda \in \mathbb{R}\} = \{(\lambda, \lambda) \mid \lambda \in \mathbb{R}\}$$

In fact, span of any non-zero vector of the form $(x_1, 0)$ in $\mathbb{R}^2$ will be the $x-$axis and span of any non-zero vector of the form (x_1, x_1) in $\mathbb{R}^2$ will be the line $y = x$. In general, we can say that span of any single non-zero vector in $\mathbb{R}^2$ will be a straight line passing through that vector and the origin. This can be generalized to $\mathbb{R}^n$ also. Now consider the set $S_3 = \{(1, 0, 0), (0, 1, 0)\}$ in $\mathbb{R}^3$. Then (Fig. 2.5)

$$\begin{aligned} Span(S_3) &= \{\lambda_1(1, 0, 0) + \lambda_2(0, 1, 0) \mid \lambda_1, \lambda_2 \in \mathbb{R}\} \\ &= \{(\lambda_1, \lambda_2, 0) \mid \lambda_1, \lambda_2 \in \mathbb{R}\} = x - y \text{ plane} \end{aligned}$$

Theorem 2.7 *Let V be a vector space over a field $\mathbb{K}$. Let $S = \{v_1, v_2, \ldots, v_n\}$ be a subset of V, then $span(S)$ is a subspace of V and any subspace of V that contains S must also contain span (S).*

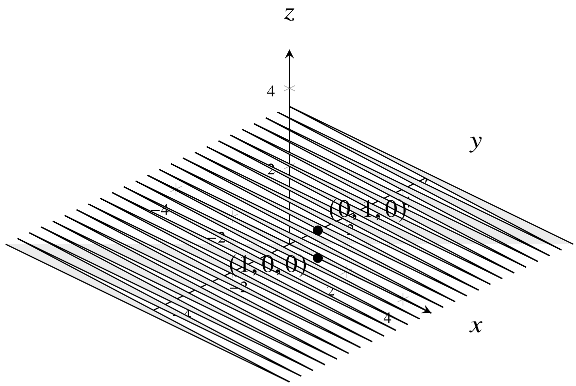

Fig. 2.5 The span of $S_3 = \{(1,0,0),(0,1,0)\}$ in $\mathbb{R}^3$ is the entire $x - y$ plane

Proof Clearly, $0 = 0v_1 + 0v_2 + \cdots + 0v_n \in span(S)$. Let $v, w \in span(S)$. Then there exists $\lambda_1, \lambda_2, \ldots, \lambda_n, \mu_1, \mu_2, \ldots, \mu_n \in \mathbb{K}$ such that $v = \lambda_1 v_1 + \lambda_2 v_2 + \cdots + \lambda_n v_n$ and $v = \mu_1 v_1 + \mu_2 v_2 + \cdots + \mu_n v_n$. Then

$$u + v = (\lambda_1 + \mu_1)v_1 + (\lambda_2 + \mu_2)v_2 + \cdots + (\lambda_n + \mu_n)v_n \in span(S)$$

and for $\mu \in \mathbb{K}$,

$$\mu v = \mu(\lambda_1 v_1 + \lambda_2 v_2 + \cdots + \lambda_n v_n) = (\mu\lambda_1)v_1 + (\mu\lambda_2)v_2 + \cdots + (\mu\lambda_n)v_n \in span(S)$$

Therefore $span(S)$ is a subspace of V. Now let W be any subspace of V containing $S = \{v_1, v_2, \ldots, v_n\}$. Then for any scalars $\lambda_1, \lambda_2, \ldots, \lambda_n \in \mathbb{K}$, as W is a subspace of V, $\lambda_1 v_1 + \lambda_2 v_2 + \cdots + \lambda_n v_n \in W$. That is, $span(S) \subseteq W$.

Remark 2.3 Consider a matrix $A \in \mathbb{M}_{m\times n}(\mathbb{K})$. We can view each row(column) as a vector in $\mathbb{K}^n(\mathbb{K}^m)$. The span of the row vectors of A is called *row space* of A and the span of the column vectors of A is called *column space* of A.

Definition 2.4 (*Basis*) Let V be a vector space over a field $\mathbb{K}$. If a set $B \subseteq V$ is linearly independent and $span(B) = V$, then B is called a basis for V. If the basis has some specific order, then it is called an ordered basis.

Theorem 2.8 *Let V be a finite-dimensional vector space over a field $\mathbb{K}$ and S=$\{v_1, v_2, \ldots, v_n\}$ spans V. Then S can be reduced to a basis B of V.*

Proof Let V be a finite-dimensional vector space over a field $\mathbb{K}$ and $S = \{v_1, v_2, \ldots, v_n\}$ spans V. Let $S_\sigma = \{v_{\sigma_1}, v_{\sigma_2}, \ldots, v_{\sigma_k}\}$ denote the set of all non-zero elements of S. Now, we will construct a linearly independent set B from S, with $span(B) = S$. Pick the element $v_{\sigma_1} \in S_\sigma$ to B. If $v_{\sigma_2} = \lambda v_{\sigma_1}$, for some $\lambda \in \mathbb{K}$, then $v_{\sigma_2} \notin B$, otherwise $v_{\sigma_2} \in B$. Now consider $v_{\sigma_3} \in S_\sigma$. If $v_{\sigma_3} = \lambda_1 v_{\sigma_1} + \lambda_2 v_{\sigma_2}$ for some $\lambda_1, \lambda_2 \in \mathbb{K}$, then $v_{\sigma_3} \notin B$, otherwise $v_{\sigma_3} \in B$. Proceeding like this, after σ_k steps we will get a linearly independent set with $span(B) = V$.

Corollary 2.2 *Every finite-dimensional vector space V has a basis.*

Proof Let V be a finite-dimensional vector space. Then there exists a finite subset S of V with $span(S) = V$. Then by S can be reduced to a basis.

Example 2.17 Consider the set

$$B = \{e_1 = (1, 0, 0, \ldots, 0), e_2 = (0, 1, 0, \ldots, 0), \ldots, e_n = (0, 0, 0, \ldots, 1)\}$$

in $\mathbb{K}^n$ over $\mathbb{K}$. We will show that B is a basis for $\mathbb{K}^n$. Let us consider an element $a = (a_1, a_2, \ldots, a_n) \in \mathbb{K}^n$ arbitrarily, then we have $a = a_1e_1 + a_2e_2 + \cdots + a_ne_n$. That is, every element in $\mathbb{K}^n$ can be written as a linear combination of elements in B with coefficients from $\mathbb{K}$. Thus B spans $\mathbb{K}^n$ over $\mathbb{K}$. Also

$$a_1e_1 + a_2e_2 + \cdots + a_ne_n = 0 \Rightarrow a_1 = a_2 = \cdots = a_n = 0$$

That is, B is linearly independent. Therefore B is a basis for $\mathbb{K}^n$ over $\mathbb{K}$ and is called the standard ordered basis for $\mathbb{K}^n$ over $\mathbb{K}$.

Example 2.18 Consider the set $B = \left\{E_{11} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, E_{12} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, E_{21} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, E_{22} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\right\}$ in $\mathbb{M}_{2\times 2}(\mathbb{K})$ over the field $\mathbb{K}$. Consider an element $\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \in \mathbb{M}_{2\times 2}(\mathbb{K})$. Then

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = a_{11}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + a_{12}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + a_{21}\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + a_{22}\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

That is, B spans $\mathbb{M}_{2\times 2}(\mathbb{K})$ over the field $\mathbb{K}$. Also

$$\lambda_1\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \lambda_2\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + \lambda_3\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + \lambda_4\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$\Rightarrow \begin{bmatrix} \lambda_1 & \lambda_2 \\ \lambda_3 & \lambda_4 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \Rightarrow \lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = 0$. That is, B is linearly independent. Therefore B is a basis for $\mathbb{M}_{2\times 2}(\mathbb{K})$ over $\mathbb{K}$.

Example 2.19 Consider the set $B = \{1, x, \ldots, x^n\}$ in $\mathbb{P}_n[a, b]$ over $\mathbb{R}$. Then B is linearly independent as

$$\lambda_1.0 + \lambda_1 x + \cdots + \lambda_n x^n = 0 \Rightarrow \lambda_0 = \lambda_1 = \cdots = \lambda_n = 0$$

and clearly B spans $\mathbb{P}_n[a, b]$. Therefore B is a basis for $\mathbb{P}_n[a, b]$ over $\mathbb{R}$

Example 2.20 Now consider a subset $S = \{(1, 1, 2), (2, 1, 1), (3, 2, 3), (-1, 0, 1)\}$ of $\mathbb{R}^3$ over $\mathbb{R}$. We know that $span(S)$ is a subspace of $\mathbb{R}^3$. Can you find a basis for $span(S)$? To find a basis for $span(S)$, we have to find a linearly independent subset $\tilde{S}$ of $\mathbb{R}^3$ such that $span(S) = span(\tilde{S})$. We may observe that the $span(S)$ is the same as the row space of the matrix $A = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & 1 \\ 3 & 2 & 3 \\ -1 & 0 & 1 \end{bmatrix}$. Thus to find a basis for $span(S)$, it is enough to find the linearly independent rows of A. We can reduce A to the row reduced form as $A = \begin{bmatrix} 1 & 1 & 2 \\ 0 & -1 & -3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. From this we can say that the set $\tilde{S} = \{(1, 1, 2), (0, -1, -3)\}$ forms a basis for $span(S)$.

Theorem 2.9 *Let V be a vector space over a field $\mathbb{K}$. If $B = \{v_1, v_2, \ldots, v_n\}$ is a basis for V, then any $v \in V$ can be uniquely expressed as a linear combination of vectors in B.*

Proof Let B be a basis of V and $v \in V$. Suppose that v can be expressed as a linear combination of vectors in B as

$$v = \lambda_1 v_1 + \lambda_2 v_2 + \cdots + \lambda_n v_n$$

and as

$$v = \mu_1 v_1 + \mu_2 v_2 + \cdots + \mu_n v_n$$

where $\lambda_i, \mu_i \in \mathbb{K}$ for all $i = 1, 2, \ldots, n$. Subtracting the second expression from first, we get

$$0 = (\lambda_1 - \mu_1)v_1 + (\lambda_2 - \mu_2)v_2 + \cdots + (\lambda_n - \mu_n)v_n$$

Since B is linearly independent, this implies that $\lambda_i - \mu_i = 0$ for all $i = 1, 2, \ldots, n$. That is, $\lambda_i = \mu_i$ for all $i = 1, 2, \ldots, n$.

Theorem 2.10 *Let V be a finite-dimensional vector space over a field $\mathbb{K}$ and B be a basis of V. Then basis is a minimal spanning set in V. That is, if B is a basis of V, there does not exist a proper subset of B that spans V.*

Proof Let V be a finite-dimensional vector space over a field $\mathbb{K}$ and $B = \left\{ v_1, v_2, \ldots, v_n \right\}$ be a basis of V. Let S be a proper subset of B that spans V. Since $S \subset B$ and $S \neq B$, there exists at least one element v such that $v \in B$ and $v \notin S$. Rearrange the elements of B so that the first k elements are also elements of S and the remaining $n - k$ elements belong to B only. Now take any element $v_{k+i} \in B$ where $i \in \{1, 2, \ldots n - k\}$. Since $span(S) = V$ and $v_{k+i} \in V$, there exists $\lambda_1, \lambda_2, \ldots, \lambda_k \in \mathbb{K}$ such that $v_{k+i} = \lambda_1 v_1 + \lambda_2 v_2 + \cdots + \lambda_k v_k$. This can also be written as $v_{k+i} = \lambda_1 v_1 + \lambda_2 v_2 + \cdots + \lambda_k v_k + 0v_{k+1} + \cdots + 0v_n$. Also as $v_{k+i} \in B$, v_{k+i} can be represented as a linear combination of elements of B by taking 1 as the coefficient to v_{k+i} and 0 as the coefficient for all elements in B other than v_{k+i}. This is a contradiction to the fact that representation for any element with respect to a basis must be unique.

Theorem 2.11 *Let V be a finite-dimensional vector space and S be a minimal spanning set of V, then S is a basis.*

Proof Let $S = \{v_1, v_2, \ldots, v_n\}$ be a minimal spanning set of V. To prove that S is a basis, it is enough to show that S is linearly independent. Suppose that it is linearly dependent, then by Theorem 2.5, at least one element say $v_i \in S$ can be written as a linear combination of the remaining vectors. Then $S \setminus \{v_i\}$ is a spanning set for V. This is a contradiction to the fact that S is a minimal spanning set.

Theorem 2.12 *Let V be a vector space over a field $\mathbb{K}$ and $B = \{v_1, v_2, \ldots, v_n\}$ be a basis of V. Let $W = \{w_1, w_2, \ldots, w_m\}$ be a linearly independent set in V, then $m \leq n$.*

Proof Since $B = \{v_1, v_2, \ldots, v_n\}$ is a basis of V, B spans V and B is linearly independent. Since $w_1 \in V$, by the previous theorem w_1 has a unique representation using the vectors in B, say

$$w_1 = \lambda_1 v_1 + \lambda_2 v_2 + \cdots + \lambda_n v_n \tag{2.1}$$

Now we can express one of the v_i, say v_k, in terms of w_1 and the remaining v_i's. That is,

$$v_k = \mu w_1 + \mu_1 v_1 + \cdots + \mu_{k-1} v_{k-1} + \mu_{k+1} v_{k+1} + \cdots + \mu_n v_n \tag{2.2}$$

where $\mu = \frac{-1}{\lambda_k}$ and $\mu_j = \frac{-\lambda_j}{\lambda_k}$, $j \neq k$.

Now we will show that the set $B_1 = \{w_1, v_1, v_2, \ldots, v_{k-1}, v_{k+1}, \ldots, v_n\}$ obtained by replacing v_k by w_1 is a basis for V. That is, we will prove that B_1 is linearly independent and B_1 spans V. Suppose that they are linearly dependent. Then by Theorem 2.5 at least one of the vectors in B_1 can be written as a linear combination of the remaining vectors. Since (2.1) is the unique representation for w_1, we cannot express w_1 in terms of $v_1, v_2, \ldots, v_{k-1}, v_{k+1}, \ldots, v_n$. Therefore some $v_l \in B_1$ can be written as a linear combination of the remaining vectors in B_1. That is, there exist scalars $\alpha, \alpha_1, \ldots, \alpha_{l-1}, \alpha_{l+1}, \ldots, \alpha_{k-1}, \alpha_{k+1}, \ldots, \alpha_n \in \mathbb{K}$ such that

$$v_l = \alpha w_1 + \alpha_1 v_1 + \cdots + \alpha_{l-1} v_{l-1} + \alpha_{l+1} v_{l+1} + \cdots + \alpha_{k-1} v_{k-1} + \alpha_{k+1} v_{k+1} + \cdots + \alpha_n v_n$$

Now substituting (2.1) in the above equation we get that v_l can be expressed as a linear combination of vectors in B, which is a contradiction as B is linearly independent. Therefore B_1 is linearly independent. Since v_k can be expressed as in (2.2), $span(B_1) = span(B) = V$. Therefore B_1 is a basis of V. We repeat this process by replacing some $v_j \in B_1$, by w_2, and so on.

Now if $m \leq n$, $B_m = \{w_1, w_2, \ldots, w_m, v_{i_1}, v_{i_2}, \ldots, v_{i_{m-n}}\}$ is a basis for V. If $m > n$, $B_n = \{w_1, w_2, \ldots, w_n\}$ is a basis for V. Then $w_{n+1} \in W$ can be written as a linear combination of vectors in B_n, which is a contradiction to the fact that W is linearly independent. Therefore $m \leq n$.

Basis of a vector space is not unique. For example, consider $\mathbb{R}^2$. Clearly $B_1 = \{(1, 0), (0, 1)\}$ is a basis for $\mathbb{R}^2$ as any vector $(x_1, x_2) \in \mathbb{R}^2$ can be written as $(x_1, x_2) = x_1(1, 0) + x_2(0, 1), x_1, x_2 \in \mathbb{R}$ and B_1 is linearly independent. Now consider the set $B_\lambda = \{(1, 0), (0, \lambda)\}$. Then B_λ is a basis for $\mathbb{R}^2$ for any $\lambda \neq 0 \in \mathbb{R}$ as any vector $(x_1, x_2) \in \mathbb{R}^2$ can be written as $(x_1, x_2) = x_1(1, 0) + \frac{x_2}{\lambda}(0, \lambda), x_1, x_2 \in \mathbb{R}$ and B_λ is linearly independent for any $\lambda \neq 0 \in \mathbb{R}$. The following corollary shows that any two bases for a vector space have the same cardinality.

Corollary 2.3 *For a finite-dimensional vector space V over $\mathbb{K}$, any two bases for V have the same cardinality.*

Proof Let $B_1 = \{v_1, v_2, \ldots, v_n\}$ and $B_2 = \{w_1, w_2, \ldots, w_m\}$ be any two bases for V. Consider B_1 as a basis and B_2 as a linearly independent set, then by the above theorem, $m \leq n$. Now consider B_2 as a basis and B_1 as a linearly independent set, then $n \leq m$. Therefore $m = n$.

Corollary 2.4 *Let V be a vector space over a field $\mathbb{K}$ and B be a basis of V. Then basis is a maximal linearly independent set in V. That is, if B is a basis of V, there does not exist a linearly independent set S such that $B \subset S \subset V$.*

Proof Let B be a basis of V and S be a linearly independent set in V. By the Theorem 2.12, the cardinality of S is less than or equal to cardinality of B. Therefore there does not exist a linearly independent set S such that $B \subset S \subset V$.

In the above corollary, we have shown that every basis is a maximal linearly independent set. Now we will prove that the converse is also true.

Theorem 2.13 *Let V be a finite-dimensional vector space over a field $\mathbb{K}$. Let $S = \{v_1, v_2, \ldots, v_n\}$ be a maximal linearly independent set in V, then S is a basis.*

Proof Let S be a maximal linearly independent set in V. To show that S is a basis, it is enough to prove that $span(S) = V$. Suppose that this is not true. Then there exists a non-zero vector $v \in V$ such that $v \notin span(S)$. Now consider the set $S_1 = S \cup \{v\}$. We will show that S_1 is linearly independent, which will be a contradiction to the fact that S is maximal. Now let $\lambda, \lambda_1, \lambda_2, \ldots, \lambda_n \in \mathbb{K}$ be such that $\lambda v + \lambda_1 v_1 + \lambda_1 v_2 + \cdots + \lambda_n v_n = 0$. If $\lambda = 0$, then as S is linearly independent $\lambda_1 = \lambda_2 = \cdots = \lambda_n = 0$. If $\lambda \neq 0$, as $v \notin span(S)$, the expression $\lambda v + \lambda_1 v_1 + \lambda_1 v_2 + \cdots + \lambda_n v_n = 0$ is not possible. Therefore S_1 is linearly independent.

Theorem 2.14 *Let V be a finite-dimensional vector space over a field $\mathbb{K}$ and S be a linearly independent subset of V. Then S can be extended to a basis.*

Proof Let V be a finite-dimensional vector space over a field $\mathbb{K}$. Let $B=\{v_1, v_2, \ldots, v_n\}$ be a basis of V. Let S be a linearly independent subset of V. Now $S \cup B$ is a spanning set of V. By Theorem 2.8, it can be reduced to a basis. If $|S| = n$, then by Theorem 2.12, S is a maximal linearly independent set and hence a basis. Suppose that $|S| < n$. Then take the vector $v_1 \in B$. If $v_1 \notin span(S)$, then $S_1 = S \cup \{v_1\}$ is a linearly independent set. If $|S_1| = n$, then as above S_1 is a basis. If $v_1 \in span(S)$, discard v_1. Then choose $v_2 \in V$ and proceed in the same way. By repeating this process, we obtain a basis for V which is an extension of S.

The following theorem summarizes the results from Theorems 2.10–2.14.

Theorem 2.15 *Let V be a finite-dimensional vector space over a field $\mathbb{K}$ and $B = \{v_1, v_2, \ldots, v_n\}$. Then the following are equivalent:*

(a) B is a basis of V.
(b) B is a minimal spanning set.
(c) B is a maximal linearly independent set.

In Corollary 2.3, we have seen that any basis for a vector space has the same cardinality. Therefore, we can uniquely define a quantity to express the cardinality of a basis for a vector space.

Definition 2.5 (*Dimension*) Let V be a vector space over a field $\mathbb{K}$ and B be basis of V. The number of elements of B is called dimension of V. It is denoted by $dim(V)$. For convenience, the dimension of $\{0\}$ is defined as 0.

Example 2.21 From Example 2.17, it is easy to observe that $\mathbb{K}^n$ over $\mathbb{K}$ has dimension n.

Example 2.22 From Example 2.18, $\mathbb{M}_{2\times 2}(\mathbb{K})$ over $\mathbb{K}$ has dimension 4. In general, $\mathbb{M}_{n\times n}(\mathbb{K})$ over $\mathbb{K}$ has dimension n^2.

Example 2.23 From Example 2.19, $\mathbb{P}_n[a, b]$ over $\mathbb{R}$ has dimension $n + 1$.

What about the dimension of $\mathbb{P}[a, b]$? Does there exist a finite set which is linearly independent and spans $\mathbb{P}[a, b]$? If such a finite set does not exist, such vector spaces are called infinite-dimensional vector spaces. Can you give another example for an infinite-dimensional vector space? What about $C[a, b]$? Now, the following remark discusses some interesting facts about the importance of field $\mathbb{K}$, while considering a vector space $V(\mathbb{K})$.

Remark 2.4 One set can be a vector space over different fields and their dimension may vary with the field under consideration. For example $\mathbb{C}$ = the set of all complex numbers is a vector space over both the fields $\mathbb{R}$ and $\mathbb{C}$. Since every element $a + bi \in \mathbb{C}$ can be written as

$$a + bi = (a + bi)1$$

where $a + bi \in \mathbb{C}$ (field under consideration) and $1 \in \mathbb{C}$ (set under consideration), $\{1\}$ is a basis for $\mathbb{C}(\mathbb{C})$ and $dim_{\mathbb{C}}(\mathbb{C}) = 1$. If $\mathbb{R}$ is the field under consideration, then $a + bi \in \mathbb{C}$ can be written as

$$a + bi = a(1) + b(i)$$

where $a, b \in \mathbb{R}$ and $1, i \in \mathbb{C}$. Therefore $\{1, i\}$ is a basis for $\mathbb{C}$ $(\mathbb{R})$ and $dim_{\mathbb{R}}(\mathbb{C}) = 2$.

Theorem 2.16 *Let V be a finite-dimensional vector space, then*

(a) Every spanning set of vectors in V with cardinality the same as that of $dim(V)$ is a basis of V.
(b) Every linearly independent set of vectors in V with cardinality the same as that of $dim(V)$ is a basis of V.

Proof (a) Let V be a finite-dimensional vector space with $dim(V) = n$. Then by Corollary 2.3, any basis of V have cardinality n. Let S be subset of V with $span(S) = V$ and $|S| = n$. By Theorem 2.8 any spanning set can be reduced to a basis. Therefore S is a basis for V.

(b) Let V be a finite-dimensional vector space with $dim(V) = n$. Let S be a linearly independent subset of V with $|S| = n$. By Theorem 2.14 any linearly independent set S can be extended to a basis. Therefore S is a basis for V.

Theorem 2.17 *Let V be a finite-dimensional vector space over a field $\mathbb{K}$. Let W be a subspace V. Then W is finite-dimensional and $dim(W) \le dim(V)$. Moreover, if $dim(W) = dim(V)$, then $V = W$.*

Proof Let W be a subspace of V. Then W is a vector space with the operations defined on B. Consider a basis B for W. Then B is a linearly independent set in V. Then by Theorem 2.12, $dim(W) \le dim(V)$. If $dim(W) = dim(V)$, then by the previous theorem, B is a basis for V also and hence $V = W$.

Example 2.24 Consider the vector space $\mathbb{R}^2$ over $\mathbb{R}$. Let W be a subspace of $\mathbb{R}^2$. Since $dim(\mathbb{R}^2) = 2$ the only possible dimensions for W are 0, 1, and 2. If $dim(W) = 0$, then $W = \{0\}$ and if $dim(W) = 2$, then $W = \mathbb{R}^2$. Now let $dim(W) = 1$. Then W is spanned by some non-zero vector. Therefore W is given by $W = \{\lambda v \mid \lambda \in \mathbb{R}\}$ for some $v \neq 0 \in \mathbb{R}^2$. That is, W is a line passing through origin. Hence the only subspaces of $\mathbb{R}^2$ are the zero space, lines passing through origin, and $\mathbb{R}^2$ itself. Similarly, the only subspaces of $\mathbb{R}^3$ are the zero space, lines passing through origin, planes passing through origin, and $\mathbb{R}^3$ itself.

2.5 Sum and Direct Sum

In the previous section, we have seen that the union of two subspaces need not necessarily be a subspace. Therefore analogous to union of subsets in set theory, we define a new concept called the sum of subspaces and analogous to disjoint union of subsets we introduce direct sums.

Theorem 2.18 *Let $W_1, W_2, \ldots, W_n$ be subspaces of a vector space over a field $\mathbb{K}$, then their sum $W_1 + W_2 + \cdots + W_n = \{w_1 + w_2 + \cdots + w_n \mid w_i \in W_i\}$ is a subspace of V and it is the smallest subspace of V containing $W_1, W_2, \ldots, W_n$.*

Proof Since $W_1, W_2, \ldots, W_n$ are subspaces of V, $0 \in W_i$ for all $i = 1, 2, \ldots, n$. Then

$$0 = 0 + 0 + \cdots + 0 \in W_1 + W_2 + \cdots + W_n$$

Now let $v, w \in W_1 + W_2 + \cdots + W_n$ and $\lambda \in \mathbb{K}$, then $v = v_1 + v_2 + \cdots + v_n$ and $w = w_1 + w_2 + \cdots + w_n$ where $v_i, w_i \in W_i$ for all $i = 1, 2, \ldots, n$. As each W_i is a subspace of V, $v_i + w_i \in W_i$ and $\lambda v_i \in W_i$ for all $i = 1, 2, \ldots, n$. Hence

$$v + w = \sum_{i=1}^{n} (v_i + w_i) \in W_1 + W_2 + \cdots + W_n$$

and

$$\lambda v = \sum_{i=1}^{n} \lambda v_i \in W_1 + W_2 + \cdots + W_n$$

Therefore $W_1 + W_2 + \cdots + W_n$ is a subspace of V. Since $w_i \in W_i$ can be written as $w_i = 0 + \cdots + 0 + w_i + 0 + \cdots + 0 \in W_1 + W_2 + \cdots + W_n$, $W_1 + W_2 + \cdots + W_n$ contains each W_i. Now to prove that $W_1 + W_2 + \cdots + W_n$ is the smallest subspace containing $W_1, W_2, \ldots, W_n$, we will show that any subspace of V containing $W_1, W_2, \ldots, W_n$ contains $W_1 + W_2 + \cdots + W_n$. Let W be any subspace containing $W_1, W_2, \ldots, W_n$. Let $w = w_1 + w_2 + \cdots + w_n \in W_1 + W_2 + \cdots + W_n$ where $w_i \in W_i$ for all $i = 1, 2, \ldots, n$. Since W is a subspace of V and W contains $W_1, W_2, \ldots, W_n$, $w \in W$.

Example 2.25 Let $V = \mathbb{R}^2$. Consider $W_1 = \{(x_1, x_2) \mid x_1 = x_2, x_1, x_2 \in \mathbb{R}\}$ and $W_2 = \{(x_1, x_2) \mid x_1 = -x_2, x_1, x_2 \in \mathbb{R}\}$. Then W_1 and W_2 are subspaces of V (Fig. 2.6).

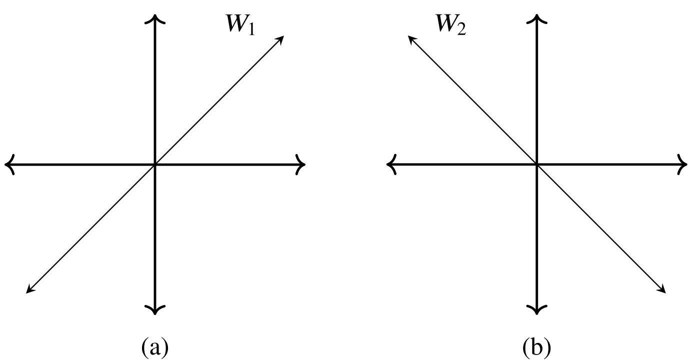

Fig. 2.6 Observe that both W_1 and W_2 depicted in (**a**) and (**b**) respectively are straight lines passing through origin and hence are subspaces of $\mathbb{R}^2$

Any vector $(x_1, x_2) \in \mathbb{R}^2$ can be written as a linear combination of elements of W_1 and W_2 as follows:

$$(x_1, x_2) = \left(\frac{x_1 + x_2}{2}, \frac{x_1 + x_2}{2}\right) + \left(\frac{x_1 - x_2}{2}, \frac{x_2 - x_1}{2}\right) \in W_1 + W_2$$

As $W_1 + W_2$ is a subspace of $\mathbb{R}^2$, this implies that $W_1 + W_2 = \mathbb{R}^2$. Also observe that the representation of any vector as the sum of elements in W_1 and W_2 is unique here.

Example 2.26 Let $V = \mathbb{M}_{2\times 2}(\mathbb{R})$. Consider

$$W_1 = \left\{\begin{bmatrix} a_{11} & a_{12} \\ 0 & a_{22} \end{bmatrix} \mid a_{11}, a_{12}, a_{22} \in \mathbb{R}\right\}$$

and

$$W_2 = \left\{\begin{bmatrix} a_{11} & 0 \\ a_{21} & a_{22} \end{bmatrix} \mid a_{11}, a_{21}, a_{22} \in \mathbb{R}\right\}$$

Then W_1 and W_2 are subspaces of V (Verify). Also any vector in $\mathbb{M}_{2\times 2}(\mathbb{R})$ can be expressed as a sum of elements in W_1 and W_2. But here this expression is not unique. For example,

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 0 & 4 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 3 & 0 \end{bmatrix} \in W_1 + W_2$$

and

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 3 & 4 \end{bmatrix} \in W_1 + W_2$$

If the elements can be expressed uniquely, then it has particular importance and is called *direct sum*. That is, the sum $W_1 + W_2$ is called *direct sum* denoted by $W_1 \oplus W_2$ if every element $w \in W_1 + W_2$ can be uniquely written as $w = w_1 + w_2$, where $w_1 \in W_1$ and $w_2 \in W_2$. That is, if $w = v_1 + v_2$, where $v_1 \in W_1$ and $v_2 \in W_2$, then $v_1 = w_1$ and $v_2 = w_2$.

Definition 2.6 (*Direct sum*) Let V be a vector space over a field $\mathbb{K}$ and $W_1, W_2, \ldots, W_n$ be subspaces of V. If every element in V can be uniquely represented as a sum of elements in $W_1, W_2, \ldots, W_n$, then V is called the direct sum of $W_1, W_2, \ldots, W_n$ and is denoted by $V = W_1 \oplus W_2 \oplus \cdots \oplus W_n$.

Suppose we have a vector space V over a field $\mathbb{K}$ and subspaces $W_1, W_2, \ldots, W_n$ of V. Then it is not easy to check whether every element in V has a unique representation as the sum of elements of $W_1, W_2, \ldots, W_n$. The following theorem provides a solution for this.

Theorem 2.19 *Let V be a vector space over a field $\mathbb{K}$ and $W_1, W_2, \ldots, W_n$ be subspaces of V. Then $V = W_1 \oplus W_2 \oplus \cdots \oplus W_n$ if and only if the following conditions are satisfied:*

(a) $V = W_1 + W_2 + \cdots + W_n$
(b) zero vector has only the trivial representation.

Proof Let $V = W_1 \oplus W_2 \oplus \cdots \oplus W_n$. Then by the definition of direct sum both (a) and (b) hold. Conversely, suppose that both (a) and (b) hold. Let $v \in V$ have two representations namely,

$$v = v_1 + v_2 + \cdots + v_n \tag{2.3}$$

and

$$v = w_1 + w_2 + \cdots + w_n \tag{2.4}$$

where $v_i, w_i \in W_i$ for all $i = 1, 2, \ldots, n$. Then subtracting (2) from (1) gives

$$0 = (v_1 - w_1) + (v_2 - w_2) + \cdots + (v_n - w_n)$$

and as zero has trivial representation only, $v_i - w_i = 0$ for all $i = 1, 2, \ldots, n$ which implies $v_i = w_i$ for all $i = 1, 2, \ldots, n$. That is, every vector has a unique representation. Therefore $V = W_1 \oplus W_2 \oplus \cdots \oplus W_n$.

Example 2.27 Consider $V = \mathbb{R}^2$ and take W_1 and W_2 as in Example 2.25. Then $V = W_1 \oplus W_2$. We already know that $V = W_1 + W_2$. It is enough to prove that the zero vector has only the trivial representation. Let $(x_1, x_1) \in W_1$ and $(x_2, -x_2) \in W_2$ be such that $(x_1, x_1) + (x_2, -x_2) = (0, 0)$. This implies that $(x_1 + x_2, x_1 - x_2) = (0, 0)$ and hence $x_1 = x_2 = 0$. Thus zero vector has only the trivial representation.

The following theorem gives a necessary and sufficient condition to check whether the sum of two subspaces is a direct sum or not.

Theorem 2.20 *Let V be a vector space over a field $\mathbb{K}$. Let W_1 and W_2 be two subspaces of V, then $V = W_1 \oplus W_2$ if and only if $V = W_1 + W_2$ and $W_1 \cap W_2 = \{0\}$.*

Proof Let $V = W_1 \oplus W_2$, then by the definition of direct sum $V = W_1 + W_2$. If $w \in W_1 \cap W_2$, then

$$w \in W_1 \cap W_2 \Rightarrow w \in W_1 \text{ and } w \in W_2 \Rightarrow -w \in W_2$$

Now $0 = w + (-w) \in W_1 + W_2$. Since $V = W_1 \oplus W_2$, this implies that $w = 0$. That is, $W_1 \cap W_2 = \{0\}$.

Conversely, suppose that $V = W_1 + W_2$ and $W_1 \cap W_2 = \{0\}$. Let $0 = w_1 + w_2$ where $w_1 \in W_1$ and $w_2 \in W_2$ be a non-trivial representation of the zero vector. Now $0 = w_1 + w_2 \Rightarrow -w_1 = w_2 \in W_1$, since W_1 is a subspace. As $W_1 \cap W_2 = \{0\}$, this implies that $w_1 = w_2 = 0$.

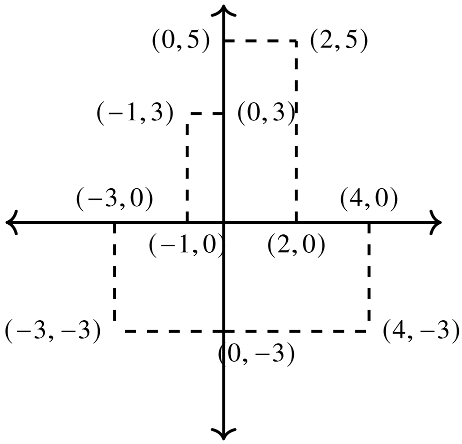

Fig. 2.7 Observe that any vector in $\mathbb{R}^2$ can be written as a sum of elements of W_1 and W_2. Also observe that $W_1 \cap W_2 = \{0\}$

Example 2.28 Let $V = \mathbb{P}_3[a, b]$. Let

$$W_1 = \{a_0 + a_2x^2 \mid a_0, a_2 \in \mathbb{R}\}$$

and

$$W_2 = \{a_1x + a_3x^3 \mid a_1, a_3 \in \mathbb{R}\}$$

Any element in $\mathbb{P}_3[a, b]$ is of the form $a_0 + a_1x + a_2x^2 + a_3x^3$. Then clearly $\mathbb{P}_3[a, b] = W_1 + W_2$. Also $W_1 \cap W_2 = \{0\}$, as polynomials in W_1 and W_2 have different orders. Therefore $\mathbb{P}_3[a, b] = W_1 \oplus W_2$.

Example 2.29 Let $V = \mathbb{R}^2$. Let $W_1 = \{(x_1, 0) \mid x_1 \in \mathbb{R}\}$ and $W_2 = \{(0, x_2) \mid x_2 \in \mathbb{R}\}$.

Then any vector $(x_1, x_2) \in \mathbb{R}^2$ can be written as $(x_1, x_2) = (x_1, 0) + (0, x_2) \in W_1 + W_2$. Since $W_1 + W_2$ is a subspace of $\mathbb{R}^2$, we get $V = W_1 + W_2$. Also $W_1 \cap W_2 = \{0\}$. Therefore $\mathbb{R}^2 = W_1 \oplus W_2$ (Fig. 2.7).

The examples discussed deal with subspaces of finite dimensional vector spaces. Now let us give you an example from an infinite-dimensional vector space.

Example 2.30 Let $V = C[a, b]$. Take

$$W_1 = \{f(x) \mid f(-x) = -f(x)\}$$

and

$$W_2 = \{f(x) \mid f(-x) = f(x)\}$$

W_1 and W_2 are respectively the collection of all odd functions and even functions. (Verify that they are subspaces of $C[a, b]$.) Now, for any $f \in C[a, b]$, consider $f_1(x) = \frac{f(x)-f(-x)}{2}$ and $f_2(x) = \frac{f(x)+f(-x)}{2}$. We have

$$f_1(-x) = \frac{f(-x) - f(-(-x))}{2} = \frac{-(f(x) - f(-x))}{2} = -f_1(x)$$

and

$$f_2(-x) = \frac{f(-x) + f(-(-x))}{2} = \frac{f(-x) + f(x)}{2} = f_2(x)$$

Thus $f_1 \in W_1$ and $f_2 \in W_2$. Clearly, $f = f_1 + f_2$ and hence $C[a, b] = W_1 + W_2$. Also observe that $W_1 \cap W_2 = \{0\}$. For if $f \in W_1 \cap W_2$, $f(-x) = -f(x) = f(x) \forall x \in [a, b]$. This gives $f(x) = 0$ for all $x \in [a, b]$. Thus we can conclude that $C[a, b] = W_1 \oplus W_2$.

Observe that the above proposition discusses the case of two subspaces only. When asking about a possible direct sum with more than two subspaces, it is not enough to check that the intersection of any two of the subspaces is $\{0\}$. For example, consider the subspaces of $\mathbb{R}^3$ given by $W_1 = \{(x_1, 0, 0) \mid x_1 \in \mathbb{R}\}$, $W_2 = \{(0, x_2, x_3) \mid x_2, x_3 \in \mathbb{R}\}$, $W_3 = \{(x_1, x_1, 0) \mid x_1 \in \mathbb{R}\}$. Clearly, $\mathbb{R}^3 = W_1 + W_2 + W_3$ and $W_1 \cap W_2 = W_1 \cap W_3 = W_2 \cap W_3 = \{0\}$ (verify). But $\mathbb{R}^3 \neq W_1 \oplus W_2 \oplus W_3$ as $(0, 0, 0) = (0, 0, 0) + (0, 0, 0) + (0, 0, 0)$ and $(0, 0, 0) = (1, 0, 0) + (0, 1, 0) + (-1, -1, 0)$.

Now we will discuss the dimension of the sum of two subspaces of a finite-dimensional vector space.

Theorem 2.21 *Let V be a finite-dimensional vector space over a field $\mathbb{K}$ and W_1, W_2 be two subspaces of V, then*

$$dim(W_1 + W_2) = dim(W_1) + dim(W_2) - dim(W_1 \cap W_2)$$

Proof Let W_1, W_2 be two subspaces of finite-dimensional vector space V. Then $W_1 \cap W_2$ is also a subspace of V. Let $\{u_1, u_2, \ldots, u_l\}$ be a basis for $W_1 \cap W_2$. Since $W_1 \subseteq W_1 \cap W_2$, $\{u_1, u_2, \ldots, u_l\}$ is a linearly independent set in W_1, and hence it can be extended to a basis $\{u_1, u_2, \ldots, u_l, v_1, v_2, \ldots, v_m\}$ of W_1. Similarly, let $\{u_1, u_2, \ldots, u_l, w_1, w_2, \ldots, w_n\}$ be a basis of W_2. Clearly

$$B = \{u_1, u_2, \ldots, u_l, , v_1, v_2, \ldots, v_m, w_1, w_2, \ldots, w_n\}$$

is a spanning set of $W_1 + W_2$. Now will show that B is a basis for $W_1 + W_2$. It is enough to show that B is linearly independent. Let $\lambda_1, \ldots, \lambda_l, \mu_1, \ldots, \mu_m, \xi_1, \ldots, \xi_n \in \mathbb{K}$ be such that

$$\lambda_1 u_1 + \cdots + \lambda_l u_l + \mu_1 v_1 + \cdots + \mu_m v_m + \xi_1 w_1 + \cdots + \xi_n w_n = 0 \qquad (2.5)$$

This implies

$$\xi_1 w_1 + \cdots + \xi_n w_n = -\lambda_1 u_1 - \cdots - \lambda_l u_l - \mu_1 v_1 - \cdots - \mu_m v_m \in W_1 \cap W_2$$

as $\{u_1, u_2, \ldots, u_l, v_1, v_2, \ldots, v_m\}$ is basis for W_1 and $\{w_1, w_2, \ldots, w_n\} \subseteq W_2$. Now $\{u_1, u_2, \ldots, u_l\}$ is a basis for $W_1 \cap W_2$ implying there exist scalars $\alpha_1, \alpha_2, \ldots, \alpha_l \in \mathbb{K}$ such that

$$\xi_1 w_1 + \cdots + \xi_n w_n = \alpha_1 u_1 + \alpha_2 u_2 + \cdots + \alpha_l u_l$$

Since $\{u_1, u_2, \ldots, u_l, w_1, w_2, \ldots, w_n\}$ is a basis for W_2, the above equation implies that $\xi_1 = \cdots = \xi_n = \alpha_1 = \cdots = \alpha_l = 0$. Then (3) changes to $\lambda_1 u_1 + \cdots + \lambda_l u_l + \mu_1 v_1 + \cdots + \mu_m v_m = 0$. Since $\{u_1, u_2, \ldots, u_l, v_1, v_2, \ldots, v_m\}$ is a basis of W_1, this implies that $\lambda_1 = \cdots = \lambda_l = \mu_1 = \cdots = \mu_m = 0$. That is, B is linearly independent. Thus we have shown that B is a basis for $W_1 + W_2$. Now

$$\begin{aligned} dim(W_1 + W_2) &= l + m + n \\ &= (l + m) + (l + n) - l \\ &= dim(W_1) + dim(W_2) - dim(W_1 \cap W_2) \end{aligned}$$

Example 2.31 Consider the vector space $\mathbb{M}_{2\times 2}(\mathbb{R})$ over the field $\mathbb{R}$. Let

$$W_1 = \left\{ \begin{bmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{bmatrix} \mid a_{11}, a_{12}, a_{22} \in \mathbb{R} \right\}$$

and

$$W_2 = \left\{ \begin{bmatrix} a_{11} & -a_{12} \\ a_{12} & 0 \end{bmatrix} \mid a_{11}, a_{12} \in \mathbb{R} \right\}$$

Verify that W_1 and W_2 are subspaces of $\mathbb{M}_{2\times 2}(\mathbb{R})$. Since $\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$ is a basis for W_1, $dim(W_1) = 3$ and as $\left\{ \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \right\}$ is a basis for W_2, $dim(W_2) = 2$. Now

$$W_1 \cap W_2 = \left\{ \begin{bmatrix} a_{11} & 0 \\ 0 & 0 \end{bmatrix} \mid a_{11} \in \mathbb{R} \right\}$$

Since $\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \right\}$ is a basis for $W_1 \cap W_2$, $dim(W_1 \cap W_2) = 1$. Thus

$$dim(W_1 + W_2) = dim(W_1) + dim(W_2) - dim(W_1 \cap W_2) = 4$$

Hence $W_1 + W_2 = \mathbb{M}_{2\times 2}(\mathbb{R})$.

Example 2.32 Consider the vector space $\mathbb{P}_4[a, b]$. Let

$$W_1 = \{\lambda_0 + \lambda_2 x^2 + \lambda_4 x^4 \mid \lambda_0, \lambda_2, \lambda_4 \in \mathbb{R}\}$$

and

$$W_2 = \{\lambda_1 x + \lambda_3 x^3 \mid \lambda_1, \lambda_3 \in \mathbb{R}\}$$

Since $\{1, x^2, x^4\}$ is a basis for W_1, $dim(W_1) = 3$ and as $\{x, x^3\}$ is a basis for W_2, $dim(W_2) = 2$. Clearly $dim(W_1 \cap W_2) = 0$ (How?) and hence

$$dim(W_1 + W_2) = dim(W_1) + dim(W_2) - dim(W_1 \cap W_2) = 5$$

As $W_1 + W_2 = \mathbb{P}_4[a, b]$ and $W_1 \cap W_2 = \{0\}$, we have $\mathbb{P}_4[a, b] = W_1 \oplus W_2$.

Theorem 2.22 *Let V be a finite-dimensional vector space over a field $\mathbb{K}$. Let $W_1, W_2, \ldots, W_n$ be subspaces of V, such that $V = W_1 + W_2 + \cdots + W_n$ and $dim(V) = dim(W_1) + dim(W_2) + \cdots + dim(W_n)$. Then $V = W_1 \oplus W_2 \oplus \cdots \oplus W_n$.*

Proof Let V be a finite-dimensional vector space with $W_1, W_2, \ldots, W_n$ as subspaces of V. Consider a basis B_i for each $i = 1, 2, \ldots, n$ and let $B = \cup_{i=1}^{n} B_i$. Since $V = W_1 + W_2 + \cdots + W_n$, B spans V. Now suppose that B is linearly dependent. Then at least one of the vectors can be written as a linear combination of other vectors. Then $dim(V) < dim(W_1) + dim(W_2) + \cdots + dim(W_n)$, which is a contradiction. Therefore B is linearly independent and hence B is a basis of V. Now let $0 = w_1 + w_2 + \cdots + w_n$ where $w_i \in W_i$. Since B_i is a basis for W_i, each $w_i \in W_i$ can be expressed uniquely as a sum of elements in B_i. i.e., 0 can be written as a linear combination of elements of B. As B is a basis for V, this implies that the coefficients are zero. That is, $w_i = 0$ for all $i = 1, 2, \ldots, n$. Therefore $V = W_1 \oplus W_2 \oplus \cdots \oplus W_n$.

2.6 Exercises

1. Show that the collections given in Examples 2.2–2.7 are vector spaces with respect to the given operations.
2. Consider the vector space $\mathbb{R}^2$ with usual addition and multiplication over $\mathbb{R}$. Give an example for a subset of $\mathbb{R}^2$ which is
   (a) closed under addition but not closed under scalar multiplication.
   (b) closed under scalar multiplication but not closed under addition.
3. Does $\mathbb{R}^2$ over $\mathbb{R}$ with operations defined by
$$(x_1, x_2) + (y_1, y_2) = (x_1 + x_2, y_1 + y_2) \text{ and } \lambda(x_1, x_2) = (\lambda x_1, 0)$$
   form a vector space?
4. Check whether the following vectors are linearly dependent or not.
   (a) $\{(1, 2), (2, 1)\}$ in $\mathbb{R}^2$ over $\mathbb{R}$.
   (b) $\{(1, 2, 1), (2, 1, 1), (1, 1, 2)\}$ in $\mathbb{R}^3$ over $\mathbb{R}$.
   (c) $\{(i, -i), (-1, 1)\}$ in $\mathbb{C}^2$ over $\mathbb{R}$.
   (d) $\{(i, -i), (-1, 1)\}$ in $\mathbb{C}^2$ over $\mathbb{C}$.

(e) $\{1+x, 1+x^2\}$ in $\mathbb{P}_2[a,b]$ over $\mathbb{R}$.
(f) $\{2, x-2, 1+x+x^2, x^3-x^2\}$ in $\mathbb{P}_3[a,b]$ over $\mathbb{R}$.
(g) $\left\{\begin{bmatrix}1&2\\0&1\end{bmatrix}, \begin{bmatrix}1&0\\2&1\end{bmatrix}, \begin{bmatrix}0&1\\2&1\end{bmatrix}, \begin{bmatrix}1&1\\2&0\end{bmatrix}\right\}$ in $\mathbb{M}_2(\mathbb{R})$ over $\mathbb{R}$.
(h) $\left\{\begin{bmatrix}1&-1\\1&0\end{bmatrix}, \begin{bmatrix}1&1\\0&1\end{bmatrix}, \begin{bmatrix}3&-1\\2&1\end{bmatrix}\right\}$ in $\mathbb{M}_2(\mathbb{R})$ over $\mathbb{R}$.

5. Let $\{v_1, v_2\}$ be a linearly independent subset of a vector space V over a field $\mathbb{K}$. Then show that $\{v_1+v_2, v_1-v_2\}$ is linearly independent only if characteristic of $\mathbb{K}$ is not equal to 2.
6. Check whether the following subsets of $\mathbb{R}^2$ are subspaces of $\mathbb{R}^2$ over $\mathbb{R}$. If yes, find its dimension and write down a basis.

(a) $\{(x_1, x_2) \in \mathbb{R}^2 \mid x_2 = 1\}$
(b) $\{(x_1, x_2) \in \mathbb{R}^2 \mid x_1 + x_2 = 0\}$
(c) $\{(x_1, x_2) \in \mathbb{R}^2 \mid \frac{x_1}{x_2} = 1\}$
(d) $\{(x_1, x_2) \in \mathbb{R}^2 \mid x_1, x_2 \leq 0\}$
(e) $\{(x_1, x_2) \in \mathbb{R}^2 \mid x_1^2 + x_2^2 = 0\}$.

7. Check whether the following subsets of $\mathbb{M}_2(\mathbb{K})$ are subspaces of $\mathbb{M}_2(\mathbb{K})$ over $\mathbb{K}$. If yes, find its dimension and write down a basis.

(a) $\left\{\begin{bmatrix}a_{11}&a_{12}\\a_{21}&a_{22}\end{bmatrix} \in \mathbb{M}_2(\mathbb{K}) \mid a_{11}+a_{12}=0\right\}$
(b) $\left\{\begin{bmatrix}a_{11}&a_{12}\\a_{21}&a_{22}\end{bmatrix} \in \mathbb{M}_2(\mathbb{K}) \mid a_{11}+a_{12}=1\right\}$
(c) $\{A \in \mathbb{M}_2(\mathbb{K}) \mid det(A) = 0\}$
(d) $\{A \in \mathbb{M}_2(\mathbb{K}) \mid det(A) \neq 0\}$
(e) $\left\{\begin{bmatrix}a_{11}&a_{12}\\a_{21}&a_{22}\end{bmatrix} \in \mathbb{M}_n(\mathbb{K}) \mid a_{11}=a_{22}\right\}$.

8. Check whether the following subsets of $\mathbb{P}_2(\mathbb{R})$ are subspaces of $\mathbb{P}_2(\mathbb{R})$ over $\mathbb{R}$. If yes, find its dimension and write down a basis.

(a) $\{p(x) \in \mathbb{P}_2(\mathbb{R}) \mid p(0) = 0\}$
(b) $\{p(x) \in \mathbb{P}_2(\mathbb{R}) \mid p(0) = 1\}$
(c) $\{p(x) \in \mathbb{P}_2(\mathbb{R}) \mid p(0) = p(1) = 0\}$
(d) $\{p(x) \in \mathbb{P}_2(\mathbb{R}) \mid p(x) \geq 0\}$
(e) $\{p(x) \in \mathbb{P}_2(\mathbb{R}) \mid p(x) = p(-x)\}$.

9. State whether the following statements are true or false.

(a) A non-trivial vector space over the fields $\mathbb{R}$ or $\mathbb{C}$ always has an infinite number of elements.
(b) The set of all rational numbers $\mathbb{Q}$ is a vector space over $\mathbb{R}$ under usual addition and multiplication.
(c) $\{(x_1, x_2) \mid x_1^2 + x_2^2 = 0, x_1, x_2 \in \mathbb{C}\}$ is a subspace of $\mathbb{C}^2$ over $\mathbb{C}$.

(d) There exists a non-trivial subspace of $\mathbb{R}$ over $\mathbb{R}$ under usual addition and multiplication.
(e) $\{(i, 1), (-1, i)\}$ is a linearly independent set in $\mathbb{C}^2$ over $\mathbb{C}$.
(f) If W_1, W_2, W are subspaces of a vector space V such that $W_1 + W = W_2 + W$, then $W_1 = W_2$.
(g) If W_1, W_2 are subspaces of $\mathbb{R}^7$ with $dim\,(W_1) = 4$ and $dim\,(W_2) = 4$, then $dim\,(W_1 \cap W_2) = 1$.

10. Show that $\mathbb{R}$ with usual addition and multiplication over $\mathbb{Q}$ is an infinite-dimensional vector space. (**Hint:** Use the fact that π is a transcendental number.)
11. Find the row space and column space of $\begin{bmatrix} 1 & -1 \\ 2 & 0 \end{bmatrix}$.
12. Show that the rows of a 3×2 matrix are linearly dependent.
13. Show that the columns of a 3×5 matrix are linearly dependent.
14. Which of the following collection of vectors span $\mathbb{R}^2$ over $\mathbb{R}$?

(a) $\{(1, 1)\}$
(b) $\{(1, 2), (0, 4)\}$
(c) $\{(0, 0), (1, -1), (3, 2)\}$
(d) $\{(2, 4), (4, 8)\}$
(e) $\{(3, 2), (1, 4), (4, 6)\}$.

15. Which of the following collection of vectors span $\mathbb{R}^3$ over $\mathbb{R}$?

(a) $\{(1, 1, 0), (0, 1, 1)\}$
(b) $\{(0, 2, 0), (1, 0, 0), (1, 2, 0)\}$
(c) $\{(0, 0, -1), (0, 1, -1), (-1, 1, -1)\}$
(d) $\{(0, 4, 2), (0, 8, 4), (1, 12, 6)\}$
(e) $\{(1, 3, 2), (1, 2, 3), (3, 2, 1), (2, 1, 3)\}$.

16. Which of the following collection of vectors span $\mathbb{P}^2[a, b]$ over $\mathbb{R}$?

(a) $\{x^2 + 1, x^2 + x, x + 1\}$
(b) $\{x + 1, x - 1, x^2 - 1\}$
(c) $\{x^2 + x + 1, 2x - 1\}$
(d) $\{2x^2 - x + 1, x^2 + x, 2x - 3, x^2 - 5\}$
(e) $\{x + 1, 2x + 2, x^2 + x\}$.

17. Let W_1, W_2 be subsets of a vector space V over the field $\mathbb{K}$. Show that

(a) $span\,(W_1 \cap W_2) \subseteq span\,(W_1) \cap span\,(W_2)$.
(b) $span\,(W_1) \cup span\,(W_2) \subseteq span\,(W_1 \cup W_2)$.

Does the converse hold in both a) and b)?

18. Let W_1, W_2 be subspaces of a vector space V over the field $\mathbb{K}$. Show that $span\,(W_1 + W_2) = span\,(W_1) + span\,(W_2)$.
19. Let $V_1 = \{v_1, v_2, \ldots, v_n\}$, $V_2 = \{v_1, v_2, \ldots, v_n, v\}$ be subsets of a vector space V. Then $span\,(V_1) = span\,(V_2)$ if and only if $v \in span\,(V_1)$.

20. Check whether the given collection of vectors form a basis for corresponding vector spaces.

    (a) $\{(2, 1, 1), (1, 2, 1), (1, 1, 2)\}$ for $\mathbb{R}^3$ over $\mathbb{R}$.
    (b) $\{1, x - 1, (x - 1)^2\}$ for $\mathbb{P}_2(\mathbb{R})$ over $\mathbb{R}$.
    (c) $\{1, x^2 - 1, 2x^2 + 5\}$ for $\mathbb{P}_2(\mathbb{R})$ over $\mathbb{R}$.
    (d) $\left\{\begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix}, \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 2 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\right\}$ for $\mathbb{M}_2(\mathbb{R})$ over $\mathbb{R}$.

21. Determine which of the given subsets forms a basis for $\mathbb{R}^3$ over $\mathbb{R}$. Express the vector $(1, 2, 3)$ as a linear combination of the vectors in each subset that is a basis.

    (a) $\{(1, 1, 1), (1, 1, 0), (1, 0, 0)\}$
    (b) $\{(1, 2, 1), (2, 1, 1), (1, 1, 2)\}$
    (c) $\{(2, 3, 1), (1, -2, 0), (1, 5, 1)\}$.

22. Check whether the sets given in Questions $14 - 16$ form a basis for the respective vector spaces. If not, find the dimension of their span.
23. Find the dimension of span of the following collection of vectors:

    (a) $\{(1, -2), (-2, 4)\}$ in $\mathbb{R}^2$ over $\mathbb{R}$.
    (b) $\{(-2, 3), (1, 2), (5, 6)\}$ in $\mathbb{R}^2$ over $\mathbb{R}$.
    (c) $\{(0, 3, 1), (-1, 2, 3), (2, 3, 0), (-1, 2, 4)\}$ in $\mathbb{R}^3$ over $\mathbb{R}$.
    (d) $\{1 + x, x^2 + x + 1\}$ in $\mathbb{P}_2[a, b]$ over $\mathbb{R}$.
    (e) $\{1 - x, x^2, 2x^2 + x - 1\}$ in $\mathbb{P}_2[a, b]$ over $\mathbb{R}$.
    (f) $\left\{\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ -1 & 1 \end{bmatrix}\right\}$ in $\mathbb{M}_2(\mathbb{R})$ over $\mathbb{R}$.
    (g) $\left\{\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}\right\}$ in $\mathbb{M}_2(\mathbb{R})$ over $\mathbb{R}$.

    Also, find a basis for the linear space spanned by the vectors.
24. Consider two subspaces of $\mathbb{R}^4$ given by

$$W_1 = \{(x_1, x_2, 2x_1, x_1 + x_2) \in \mathbb{R}^4 \mid x_1, x_2 \in \mathbb{R}\}$$

    and

$$W_2 = \{(x_1, 2x_1, x_2, x_1 - x_2) \in \mathbb{R}^4 \mid x_1, x_2 \in \mathbb{R}\}$$

    Find

    (a) $W_1 + W_2$ and $W_1 \cap W_2$.
    (b) $dim\,(W_1 + W_2)$ and $dim\,(W_1 \cap W_2)$.

25. Let V be a finite-dimensional vector space over a field $\mathbb{K}$ and W_1 be a subspace of V. Prove that there exists a subspace W_2 of V such that $V = W_1 \oplus W_2$.
26. Let V be a vector space over a field $\mathbb{K}$ and $W_1, W_2, \ldots, W_n$ be subspaces of V with $W_i \cap W_j = \{0\}\ \forall\, i \neq j$ and $W_1 + W_2 + \cdots + W_n = V$. Is the sum a direct sum?

27. Let $W_1 = \{A \in \mathbb{M}_n(\mathbb{K}) \mid A_{ij} = 0 \ \forall i \geq j\}$, $W_2 = \{A \in \mathbb{M}_n(\mathbb{K}) \mid A_{ij} = 0 \ \forall i \leq j\}$, and $W_3 = \{A \in \mathbb{M}_n(\mathbb{K}) \mid A_{ij} = 0 \ \forall\, i \neq j\}$. Then show that $\mathbb{M}_n(\mathbb{K}) = W_1 \oplus W_2 \oplus W_3$.
28. Let

$$W_1 = \{A \in \mathbb{M}_n(\mathbb{K}) \mid A^T = A\}$$

and

$$W_2 = \{A \in \mathbb{M}_n(\mathbb{K}) \mid A^T = -A\}$$

Then show that $\mathbb{M}_n(\mathbb{K}) = W_1 \oplus W_2$.

Solved Questions related to this chapter are provided in Chap. 9.