Chapter 3
Estimation of a Normal Mean Vector Under Unknown Scale

3.1 Equivariance

In this chapter, we consider estimation of the mean of a multivariate normal distribution when the scale is unknown. Let

$$X \sim \mathcal{N}_p(\theta, I/\eta) \quad \text{and} \quad \eta S \sim \chi^2_n,$$

where θ and η are both unknown. For estimation of θ, the loss function is scaled quadratic loss $L(\delta; \theta, \eta) = \eta\|\delta(x, s) - \theta\|^2$.

The first three sections cover issues of Bayesianity, admissibility and minimaxity among estimators which are both orthogonally and scale equivariant. The remaining sections consider these issues among all estimators.

Consider a group of transformations,

$$X \to \gamma\Gamma X, \quad \theta \to \gamma\Gamma\theta, \quad S \to \gamma^2 S, \quad \eta \to \eta/\gamma^2, \tag{3.1}$$

where $\Gamma \in \mathcal{O}(p)$, the group of $p \times p$ orthogonal matrices, and $\gamma \in \mathbb{R}_+$. Equivariant estimators for this group (3.1) satisfy

$$\hat{\theta}(\gamma\Gamma x, \gamma^2 s) = \gamma\Gamma\hat{\theta}(x, s). \tag{3.2}$$

The following result gives the form of such equivariant estimators.

Theorem 3.1 *Equivariant estimators for the group* (3.1) *are of the form*

$$\hat{\theta}_\psi = \left\{1 - \psi(\|X\|^2/S)\right\} X, \quad \textit{where} \quad \psi : \mathbb{R}_+ \to \mathbb{R}.$$

Proof Let the orthogonal matrix $\Gamma \in \mathcal{O}(p)$ satisfy

$$\Gamma = (x/\|x\| \; z_2 \; \ldots \; z_p)^{\mathrm{T}} \quad \text{and} \quad \Gamma x = (\|x\| \; 0 \; \ldots \; 0)^{\mathrm{T}} = \|x\| e_1, \tag{3.3}$$

where unit vectors $z_2, \ldots, z_p \in \mathbb{R}^p$ satisfy

$$z_i^{\mathrm{T}} z_j = 0 \text{ for } i \neq j, \; z_i^{\mathrm{T}} x = 0 \text{ for } i = 2, \ldots, p$$

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and $e_1 := (1, 0, \dots, 0)^{\mathrm{T}} \in \mathbb{R}^p$. Further let $\gamma = 1/\sqrt{s}$. Then, by (3.2), the equivariant estimator $\hat{\theta}(x, s)$ satisfies

$$\begin{aligned}\hat{\theta}(x,s) &= \frac{1}{\gamma}\Gamma^{\mathrm{T}}\hat{\theta}(\gamma\Gamma x, \gamma^2 s) = \sqrt{s}\Gamma^{\mathrm{T}}\hat{\theta}(\{\|x\|/\sqrt{s}\}e_1, 1)\\ &= \frac{\hat{\theta}_1(\{\|x\|/\sqrt{s}\}e_1, 1)}{\|x\|/\sqrt{s}}x + \sqrt{s}\sum_{i=2}^{p}\hat{\theta}_i\Big(\frac{\|x\|}{\sqrt{s}}e_1, 1\Big)z_i,\end{aligned} \tag{3.4}$$

where $\hat{\theta}_i$ is the ith component of $\hat{\theta}$.

For the orthogonal matrix $\Gamma_1\Gamma$ where $\Gamma_1 = \mathrm{diag}(1, -1, 1, \dots, 1)$, we have

$$\Gamma_1\Gamma = (x/\|x\| \;\; -z_2 \; z_3 \; \dots \; z_p)^{\mathrm{T}} \quad \text{and} \quad \Gamma_1\Gamma x = \|x\|e_1.$$

Hence the estimator (3.4) should be also expressed by

$$\begin{aligned}\hat{\theta}(x,s) &= \frac{1}{\gamma}(\Gamma_1\Gamma)^{\mathrm{T}}\hat{\theta}(\gamma(\Gamma_1\Gamma)x, \gamma^2 s)\\ &= \frac{\hat{\theta}_1(\{\|x\|/\sqrt{s}\}e_1, 1)}{\|x\|/\sqrt{s}}x - \sqrt{s}\hat{\theta}_2\Big(\frac{\|x\|}{\sqrt{s}}e_1, 1\Big)z_2 + \sqrt{s}\sum_{i=3}^{p}\hat{\theta}_i\Big(\frac{\|x\|}{\sqrt{s}}e_1, 1\Big)z_i.\end{aligned} \tag{3.5}$$

By (3.4) and (3.5), $\hat{\theta}_2(\{\|x\|/\sqrt{s}\}e_1, 1) = 0$. Similarly, $\hat{\theta}_i(\{\|x\|/\sqrt{s}\}e_1, 1) = 0$ for $i = 3, \dots, p$. Therefore, in (3.4), we have

$$\hat{\theta}(x,s) = \frac{\hat{\theta}_1(\{\|x\|/\sqrt{s}\}e_1, 1)}{\|x\|/\sqrt{s}}x,$$

where the coefficient of x is a function of $\|x\|^2/s$. This completes the proof. □

Let $f(t) = \{(2\pi)^{p/2}\Gamma(n/2)2^{n/2}\}^{-1}\exp(-t/2)$. Then the joint probability density of X and S is given by

$$\begin{aligned}&\eta^{p/2+n/2}s^{n/2-1}f(\eta\{\|x-\theta\|^2 + s\})\\ &= \frac{\eta^{p/2}}{(2\pi)^{p/2}}\exp\Big(-\frac{\eta\|x-\theta\|^2}{2}\Big) \times \frac{\eta^{n/2}s^{n/2-1}}{\Gamma(n/2)2^{n/2}}\exp(-\eta s/2).\end{aligned}$$

Also, the generalized Bayes estimator of θ with respect to a prior of the form

$$Q(\theta, \eta; \nu, q) = \eta^{\nu}\eta^{p/2}q(\eta\|\theta\|^2) \tag{3.6}$$

for $\nu \in \mathbb{R}$ is given by

$$\hat{\theta}_{q,\nu}(x,s) = \frac{\iint \theta\eta^{(2p+n)/2+\nu+1}f(\eta\{\|x-\theta\|^2+s\})q(\eta\|\theta\|^2)\mathrm{d}\theta\mathrm{d}\eta}{\iint \eta^{(2p+n)/2+\nu+1}f(\eta\{\|x-\theta\|^2+s\})q(\eta\|\theta\|^2)\mathrm{d}\theta\mathrm{d}\eta}. \tag{3.7}$$

The value of the estimator $\hat{\theta}_{q,\nu}(x,s)$ evaluated at $x = \gamma\Gamma x$ and $s = \gamma^2 s$ where $\Gamma \in \mathcal{O}(p)$, the group of $p \times p$ orthogonal matrices, and $\gamma \in \mathbb{R}_+$, is given by

$$\hat{\theta}_{q,\nu}(\gamma\Gamma x, \gamma^2 s) = \frac{\iint \theta\eta^{(2p+n)/2+\nu+1} f(\eta\{\|\gamma\Gamma x - \theta\|^2 + \gamma^2 s\}) q(\eta\|\theta\|^2) \mathrm{d}\theta \mathrm{d}\eta}{\iint \eta^{(2p+n)/2+\nu+1} f(\eta\{\|\gamma\Gamma x - \theta\|^2 + \gamma^2 s\}) q(\eta\|\theta\|^2) \mathrm{d}\theta \mathrm{d}\eta}. \tag{3.8}$$

By the change of variables $\theta = \gamma\Gamma\theta_*$ and $\eta_* = \gamma^2\eta$, this may be rewritten as

$$\begin{aligned}\hat{\theta}_{q,\nu}(\gamma\Gamma x, \gamma^2 s) &= \gamma\Gamma \frac{\iint \theta_*\eta_*^{(2p+n)/2+\nu+1} f(\eta_*\{\|x - \theta_*\|^2 + s\}) q(\eta_*\|\theta_*\|^2) \mathrm{d}\theta_* \mathrm{d}\eta_*}{\iint \eta_*^{(2p+n)/2+\nu+1} f(\eta_*\{\|x - \theta_*\|^2 + s\}) q(\eta_*\|\theta_*\|^2) \mathrm{d}\theta_* \mathrm{d}\eta_*} \\ &= \gamma\Gamma\hat{\theta}_{q,\nu}(x,s). \end{aligned} \tag{3.9}$$

By (3.2), $\hat{\theta}_{q,\nu}(x,s)$ is equivariant.

In the next section, we show that the case $\nu = -1$ is special within this class.

3.2 Proper Bayes Equivariant Estimators

In this section we first show that the risk of an estimator that is equivariant under the group (3.1), depends only on the one dimensional parameter $\lambda = \eta\|\theta\|^2 \in \mathbb{R}_+$. We then consider Bayes estimators among the class of equivariant estimators relative to proper priors on λ. We show that such estimators are admissible among equivariant estimators and are also generalized Bayes estimators relative to $Q(\theta, \eta; \nu, q)$ with $\nu = -1$ given by (3.6).

Theorem 3.2 *The risk function of an equivariant estimator for the group* (3.1)*,*

$$\hat{\theta}_\psi = \left\{1 - \psi(\|X\|^2/S)\right\} X$$

depends only on $\lambda = \eta\|\theta\|^2 \in \mathbb{R}_+$.

Proof As in (3.3), let the orthogonal matrix Γ be of the form

$$\Gamma^{\mathrm{T}} = (\theta/\|\theta\| \; z_2 \; \ldots \; z_p)^{\mathrm{T}} \quad \text{and} \quad \Gamma^{\mathrm{T}}\theta = (\|\theta\| \; 0 \; \ldots \; 0)^{\mathrm{T}}. \tag{3.10}$$

By the change of variables, $y = \eta^{1/2}\Gamma^{\mathrm{T}} x$ and $v = \eta s$, we have

$$
\begin{aligned}
&\mathrm{R}(\hat{\theta}_\psi;\theta,\eta)\\
&=\iint \eta\left\|\{1-\psi(\|x\|^2/s)\}x-\theta\right\|^2 s^{n/2-1}\eta^{(p+n)/2} f(\eta\{\|x-\theta\|^2+s\})\mathrm{d}x\mathrm{d}s\\
&=\iint \left\|\{1-\psi(\|y\|^2/v)\}\Gamma y-\eta^{1/2}\theta\right\|^2 v^{n/2-1} f(\{\|\Gamma y-\eta^{1/2}\theta\|^2+v\})\mathrm{d}y\mathrm{d}v\\
&=\iint \left\|\{1-\psi(\|y\|^2/v)\}y-\Gamma^{\mathrm{T}}\eta^{1/2}\theta\right\|^2 v^{n/2-1} f(\{\|y-\Gamma^{\mathrm{T}}\eta^{1/2}\theta\|^2+v\})\mathrm{d}y\mathrm{d}v\\
&=\iint \left\{\left(\{1-\psi(\|y\|^2/v)\}y_1-\eta^{1/2}\|\theta\|\right)^2+\{1-\psi(\|y\|^2/v)\}^2\sum_{i=2}^p y_i^2\right\}\\
&\qquad\times v^{n/2-1} f\left(\left(y_1-\eta^{1/2}\|\theta\|\right)^2+\sum_{i=2}^p y_i^2+v\right)\mathrm{d}y\mathrm{d}v,
\end{aligned}
$$

where the last equality follows from (3.10). This completes the proof. □

By Theorem 3.2, the risk function may be expressed as

$$\mathrm{R}(\hat{\theta}_\psi;\theta,\eta)=\tilde{\mathrm{R}}(\hat{\theta}_\psi;\eta\|\theta\|^2). \tag{3.11}$$

Now assume that $\lambda=\eta\|\theta\|^2\in\mathbb{R}_+$ has the prior density $\bar{\pi}(\lambda)$, which, in this section, we assume to be proper, that is, $\int_0^\infty \bar{\pi}(\lambda)\mathrm{d}\lambda=1$. For an equivariant estimator $\hat{\theta}_\psi$, we define the Bayes equivariant risk as

$$\tilde{\mathrm{r}}(\hat{\theta}_\psi;\bar{\pi})=\int_0^\infty \tilde{\mathrm{R}}(\hat{\theta}_\psi;\lambda)\bar{\pi}(\lambda)\mathrm{d}\lambda. \tag{3.12}$$

In this book, the estimator $\hat{\theta}_\psi$ which minimizes $\tilde{\mathrm{r}}(\hat{\theta}_\psi;\bar{\pi})$, is called a (relative to $\bar{\pi}(\lambda)$). In the following, let

$$\pi(\lambda)=\frac{\Gamma(p/2)}{\pi^{p/2}}\lambda^{1-p/2}\bar{\pi}(\lambda) \tag{3.13}$$

so that $\pi(\|\mu\|^2)$ is a proper probability density on $\mathbb{R}^p$, that is,

$$\int_{\mathbb{R}^p}\pi(\|\mu\|^2)\mathrm{d}\mu=1. \tag{3.14}$$

Let the Bayes equivariant estimator, which minimizes $\tilde{\mathrm{r}}(\hat{\theta}_\psi;\bar{\pi})$, be denoted by $\hat{\theta}_\pi$. Theorem 3.3 below shows that $\hat{\theta}_\pi$ is equivalent to the generalized Bayes estimator of

θ with respect to the joint prior density $\eta^{-1}\eta^{p/2}\pi(\eta\|\theta\|^2)$, and that it is admissible among equivariant estimators.

Theorem 3.3 (Maruyama and Strawderman 2020) *Assume that $\bar{\pi}(\lambda)$ is proper.*

1. *The Bayes equivariant risk, $\tilde{\mathrm{r}}(\hat{\theta}_\psi;\bar{\pi})$ given by (3.12) is*

$$\tilde{\mathrm{r}}(\hat{\theta}_\psi;\bar{\pi}) = \int_{\mathbb{R}^p} \psi(\|z\|^2)\Big\{\psi(\|z\|^2) - 2\Big(1 - \frac{z^{\mathrm{T}} M_2(z,\pi)}{\|z\|^2 M_1(z,\pi)}\Big)\Big\} \times \|z\|^2 M_1(z,\pi)\mathrm{d}z + p,$$

where

$$\begin{aligned} M_1(z,\pi) &= \iint \eta^{(2p+n)/2} f(\eta\{\|z-\theta\|^2+1\})\pi(\eta\|\theta\|^2)\mathrm{d}\theta\mathrm{d}\eta, \\ M_2(z,\pi) &= \iint \theta\eta^{(2p+n)/2} f(\eta\{\|z-\theta\|^2+1\})\pi(\eta\|\theta\|^2)\mathrm{d}\theta\mathrm{d}\eta. \end{aligned} \tag{3.15}$$

2. *Given $\bar{\pi}(\lambda)$, the minimizer of $\tilde{\mathrm{r}}(\hat{\theta}_\psi;\bar{\pi})$ with respect to ψ is*

$$\psi_\pi(\|z\|^2) = \arg\min_\psi \tilde{\mathrm{r}}(\hat{\theta}_\psi;\bar{\pi}) = 1 - \frac{z^{\mathrm{T}} M_2(z,\pi)}{\|z\|^2 M_1(z,\pi)}, \tag{3.16}$$

and the Bayes risk difference under $\bar{\pi}(\lambda)$ is

$$\begin{aligned} &\tilde{\mathrm{r}}(\hat{\theta}_\psi;\bar{\pi}) - \tilde{\mathrm{r}}(\hat{\theta}_\pi;\bar{\pi}) \\ &= \int_{\mathbb{R}^p} \left\{\psi(\|z\|^2) - \psi_\pi(\|z\|^2)\right\}^2 \|z\|^2 M_1(z,\pi)\mathrm{d}z. \end{aligned} \tag{3.17}$$

3. *The Bayes equivariant estimator*

$$\hat{\theta}_\pi = \left\{1 - \psi_\pi(\|X\|^2/S)\right\} X$$

with ψ_π by (3.16), is equivalent to the generalized Bayes estimator of θ with respect to the joint prior density $\eta^{-1}\eta^{p/2}\pi(\eta\|\theta\|^2)$ where $\pi(\lambda)$ is given by (3.13).

4. *The Bayes equivariant estimator $\hat{\theta}_\pi$ is admissible within the class of estimators equivariant under the group (3.1).*

Proof (Parts 1 and 2) The Bayes equivariant risk given by (3.12) is

$$\begin{aligned}\tilde{\mathrm{r}}(\hat{\theta}_\psi;\bar{\pi}) &= \int_{\mathbb{R}^p} \tilde{R}(\hat{\theta}_\psi;\|\mu\|^2)\pi(\|\mu\|^2)\mathrm{d}\mu \\ &= \int_{\mathbb{R}^p} \tilde{\mathrm{R}}(\hat{\theta}_\psi;\eta\|\theta\|^2)\eta^{p/2}\pi(\eta\|\theta\|^2)\mathrm{d}\theta = \int_{\mathbb{R}^p} \mathrm{R}(\hat{\theta}_\psi;\theta,\eta)\eta^{p/2}\pi(\eta\|\theta\|^2)\mathrm{d}\theta,\end{aligned}$$

where the third equality follows from (3.11). Further, expanding terms, $\tilde{\mathrm{r}}(\hat{\theta}_\psi;\bar{\pi})$ may be expressed as

$$\begin{aligned}\tilde{\mathrm{r}}(\hat{\theta}_\psi;\bar{\pi}) = &\int_{\mathbb{R}^p} \mathrm{E}\left[\eta\|X\|^2\psi^2(\|X\|^2/S)\right]\eta^{p/2}\pi(\eta\|\theta\|^2)\mathrm{d}\theta \\ &- 2\int_{\mathbb{R}^p} \mathrm{E}\left[\eta\|X\|^2\psi(\|X\|^2/S)\right]\eta^{p/2}\pi(\eta\|\theta\|^2)\mathrm{d}\theta \\ &+ 2\int_{\mathbb{R}^p} \mathrm{E}\left[\eta\psi(\|X\|^2/S)X^{\mathrm{T}}\theta\right]\eta^{p/2}\pi(\eta\|\theta\|^2)\mathrm{d}\theta \\ &+ \int_{\mathbb{R}^p} \mathrm{E}\left[\eta\|X-\theta\|^2\right]\eta^{p/2}\pi(\eta\|\theta\|^2)\mathrm{d}\theta.\end{aligned} \tag{3.18}$$

Note that, by the propriety of the prior given by (3.14), the third term is equal to p, that is,

$$\int_{\mathbb{R}^p} \mathrm{E}\left[\eta\|X-\theta\|^2\right]\eta^{p/2}\pi(\eta\|\theta\|^2)\mathrm{d}\theta = \int_{\mathbb{R}^p} p\pi(\|\mu\|^2)\mathrm{d}\mu = p. \tag{3.19}$$

The first and second terms of (3.18) with $\psi^j(\|x\|^2/s)$ for $j=2,1$ respectively, may be rewritten as

$$\begin{aligned}&\int_{\mathbb{R}^p} \mathrm{E}\left[\eta\|X\|^2\psi^j(\|X\|^2/S)\right]\eta^{p/2}\pi(\eta\|\theta\|^2)\mathrm{d}\theta \\ &= \iiint \eta\|x\|^2\psi^j(\|x\|^2/s)\eta^{(2p+n)/2}s^{n/2-1}f(\eta\{\|x-\theta\|^2+s\})\pi(\eta\|\theta\|^2)\mathrm{d}\theta\mathrm{d}x\mathrm{d}s \\ &= \iiint \eta s\|z\|^2\psi^j(\|z\|^2)\eta^{(2p+n)/2}s^{(p+n)/2-1}f(\eta\{\|\sqrt{s}z-\theta\|^2+s\}) \\ &\quad\times\pi(\eta\|\theta\|^2)\mathrm{d}\theta\mathrm{d}z\mathrm{d}s \quad (z=x/\sqrt{s},\ J=s^{p/2}) \\ &= \iiint \eta s\|z\|^2\psi^j(\|z\|^2)\eta^{(2p+n)/2}s^{(2p+n)/2-1}f(s\eta\{\|z-\theta_*\|^2+1\}) \\ &\quad\times\pi(\eta s\|\theta_*\|^2)\mathrm{d}\theta_*\mathrm{d}z\mathrm{d}s \quad (\theta_*=\theta/\sqrt{s},\ J=s^{p/2}) \\ &= \iiint \|z\|^2\psi^j(\|z\|^2)\eta_*^{(2p+n)/2}f(\eta_*\{\|z-\theta_*\|^2+1\}) \\ &\quad\times\pi(\eta_*\|\theta_*\|^2)\mathrm{d}\theta_*\mathrm{d}z\mathrm{d}\eta_* \quad (\eta_*=\eta s,\ J=1/\eta)\end{aligned}$$

$$= \int_{\mathbb{R}^p} \|z\|^2 \psi^j(\|z\|^2) M_1(z, \pi) \mathrm{d}z, \tag{3.20}$$

where $z = x/\sqrt{s}$, J is the Jacobian, and

$$M_1(z, \pi) = \iint \eta^{(2p+n)/2} f(\eta\{\|z - \theta\|^2 + 1\}) \pi(\eta\|\theta\|^2) \mathrm{d}\theta \mathrm{d}\eta.$$

Similarly, the third term of (3.18) may be rewritten as

$$\int_{\mathbb{R}^p} \mathrm{E}\left[\eta\psi(\|X\|^2/S) X^{\mathrm{T}}\theta\right] \eta^{p/2} \pi(\eta\|\theta\|^2) \mathrm{d}\theta = \int_{\mathbb{R}^p} \psi(\|z\|^2) z^{\mathrm{T}} M_2(z, \pi) \mathrm{d}z, \tag{3.21}$$

where

$$M_2(z, \pi) = \iint \theta \eta^{(2p+n)/2} f(\eta\{\|z - \theta\|^2 + 1\}) \pi(\eta\|\theta\|^2) \mathrm{d}\theta \mathrm{d}\eta.$$

Hence, by (3.19), (3.20) and (3.21), we have

$$\begin{aligned} \tilde{\mathrm{r}}(\hat{\theta}_\psi; \bar{\pi}) = \int_{\mathbb{R}^p} \big\{ &\psi^2(\|z\|^2)\|z\|^2 M_1(z, \pi) \\ &-2\psi(\|z\|^2)\{\|z\|^2 M_1(z, \pi) - z^{\mathrm{T}} M_2(z, \pi)\}\big\} \mathrm{d}z + p. \end{aligned} \tag{3.22}$$

Then the Bayes equivariant solution, or minimizer of $\tilde{\mathrm{r}}(\hat{\theta}_\psi; \bar{\pi})$, is

$$\psi_\pi(\|z\|^2) = \arg\min_\psi \tilde{\mathrm{r}}(\hat{\theta}_\psi; \bar{\pi}) = 1 - \frac{z^{\mathrm{T}} M_2(z, \pi)}{\|z\|^2 M_1(z, \pi)} \tag{3.23}$$

and hence the corresponding Bayes equivariant estimator is

$$\hat{\theta}_\pi = \frac{z^{\mathrm{T}} M_2(z, \pi)}{\|z\|^2 M_1(z, \pi)} x, \tag{3.24}$$

where $z = x/\sqrt{s}$. Parts 1 and 2 follow from (3.22), (3.23) and (3.24).

[Part 3] Note that for $\Gamma \in \mathcal{O}(p)$, the group of $p \times p$ orthogonal matrices, $M_2(\Gamma z, \pi) = \Gamma M_2(z, \pi)$. Hence, as in (3.8) and (3.9), $M_2(z, q)$ is proportional to z and the length of $M_2(z, q)$ is $z^{\mathrm{T}} M_2(z, q)/\|z\|$, which implies that

$$M_2(z, \pi) = \frac{z^{\mathrm{T}} M_2(z, q)}{\|z\|} \frac{z}{\|z\|}. \tag{3.25}$$

By (3.25),

$$
\begin{aligned}
\hat{\theta}_\pi &= \frac{z^{\mathrm{T}} M_2(z,\pi)}{\|z\|^2 M_1(z,\pi)} x = \sqrt{s}\frac{z z^{\mathrm{T}} M_2(z,q)}{\|z\|^2 M_1(z,q)} = \sqrt{s}\frac{M_2(z,\pi)}{M_1(z,\pi)} \\
&= \sqrt{s}\frac{\iint \theta \eta^{(2p+n)/2} f(\eta\{\|x/\sqrt{s}-\theta\|^2+1\})\pi(\eta\|\theta\|^2)\mathrm{d}\theta\mathrm{d}\eta}{\iint \eta^{(2p+n)/2} f(\eta\{\|x/\sqrt{s}-\theta\|^2+1\})\pi(\eta\|\theta\|^2)\mathrm{d}\theta\mathrm{d}\eta}.
\end{aligned}
$$

By the change of variables $\theta_* = \sqrt{s}\theta$ and $\eta_* = \eta/s$, we have

$$
\hat{\theta}_\pi = \frac{\iint \theta_* \eta_*^{(2p+n)/2} f(\eta_*\{\|x-\theta_*\|^2+s\})\pi(\eta_*\|\theta_*\|^2)\mathrm{d}\theta_*\mathrm{d}\eta_*}{\iint \eta_*^{(2p+n)/2} f(\eta_*\{\|x-\theta_*\|^2+s\})\pi(\eta_*\|\theta_*\|^2)\mathrm{d}\theta_*\mathrm{d}\eta_*},
$$

which is the generalized Bayes estimator of θ with respect to $\eta^{-1}\eta^{p/2}\pi(\eta\|\theta\|^2)$, as in (3.7).

[Part 4] Since the quadratic loss function is strictly convex, the Bayes solution is unique, and hence Part 4 follows. □

As in (3.9), the generalized Bayes estimator of θ with respect to $Q(\theta, \eta; \nu, \pi)$ for any $\nu \in \mathbb{R}$, given by (3.6), is equivariant under the group (3.1). Part 3 of Theorem 3.3, however, applies only to the special case of

$$
\nu = -1. \tag{3.26}
$$

This is the main reason that we focus on the case of $\nu = -1$ in this book. It should be noted, however, that Theorem 3.3 implies neither admissibility or inadmissibility of generalized Bayes estimators within the class of equivariant estimators, if $\nu \neq -1$.

3.3 Admissible Bayes Equivariant Estimators Through the Blyth Method

Even if $\bar{\pi}(\lambda)$ on $\mathbb{R}_+$ (and hence $\pi(\|\mu\|^2)$ on $\mathbb{R}^p$) is improper, that is

$$
\int_{\mathbb{R}^p} \pi(\|\mu\|^2)\mathrm{d}\mu = \int_0^\infty \bar{\pi}(\lambda)\mathrm{d}\lambda = \infty,
$$

the estimator $\hat{\theta}_\pi$ discussed in the previous section can still be defined if $M_1(z,\pi)$ and $M_2(z,\pi)$ given by (3.15) are both finite. The admissibility of such $\hat{\theta}_\pi$ within the class of equivariant estimators can be investigated through Blyth (1951) method.

3.3.1 A General Admissibility Equivariance Result for Mixture Priors

Suppose

$$\bar{\pi}(\lambda) = \int_0^\infty \frac{\lambda^{p/2-1} g^{-p/2}}{2^{p/2}\Gamma(p/2)} \exp\Big(-\frac{\lambda}{2g}\Big)\Pi(\mathrm{d}g)$$

or equivalently

$$\pi(\|\mu\|^2) = \int_0^\infty \frac{g^{-p/2}}{(2\pi)^{p/2}} \exp\Big(-\frac{\|\mu\|^2}{2g}\Big)\Pi(\mathrm{d}g), \tag{3.27}$$

where $\int_0^\infty \Pi(\mathrm{d}g) = \infty$. Then, for (3.15), we have

$$\begin{aligned} M_1(z,\pi) &= \iint \eta^{(2p+n)/2} f(\eta\{\|z-\theta\|^2+1\})\pi(\eta\|\theta\|^2)\mathrm{d}\theta\mathrm{d}\eta \\ &= \frac{1}{q_1(p,n)} \iiint \eta^{(2p+n)/2} \exp\Big(-\frac{\eta\{\|z-\theta\|^2+1\}}{2}\Big) \\ &\quad \times \frac{1}{(2\pi)^{p/2} g^{p/2}} \exp\Big(-\frac{\eta\|\theta\|^2}{2g}\Big)\Pi(\mathrm{d}g)\mathrm{d}\theta\mathrm{d}\eta \\ &= \frac{1}{q_1(p,n)} \iint \frac{\eta^{(p+n)/2}}{(g+1)^{p/2}} \exp\Big(-\frac{\eta\{\|z\|^2/(g+1)+1\}}{2}\Big)\Pi(\mathrm{d}g)\mathrm{d}\eta \\ &= \frac{\Gamma((p+n)/2+1)}{q_1(p,n)2^{-(p+n)/2-1}} \int_0^\infty \frac{(g+1)^{-p/2}\Pi(\mathrm{d}g)}{\{1+\|z\|^2/(g+1)\}^{(p+n)/2+1}}, \end{aligned} \tag{3.28}$$

where the third equality follows from Lemma A.1, and

$$q_1(p,n) = (2\pi)^{p/2}\Gamma(n/2)2^{n/2}. \tag{3.29}$$

Similarly, for (3.15), we have

$$M_2(z,\pi) = \frac{\Gamma((p+n)/2+1)}{q_1(p,n)2^{-(p+n)/2-1}} \int_0^\infty \frac{gz}{g+1} \frac{(g+1)^{-p/2}\Pi(\mathrm{d}g)}{\{1+\|z\|^2/(g+1)\}^{(p+n)/2+1}}. \tag{3.30}$$

Then, by (3.16), (3.28) and (3.30), the (improper or generalized) Bayes equivariant estimator is

$$\begin{aligned} \hat{\theta}_\pi &= \big\{1-\psi_\pi(\|z\|^2)\big\}\, x \\ &= \left(1 - \frac{\int_0^\infty (g+1)^{-p/2-1}\{1+\|z\|^2/(g+1)\}^{-(p+n)/2-1}\Pi(\mathrm{d}g)}{\int_0^\infty (g+1)^{-p/2}\{1+\|z\|^2/(g+1)\}^{-(p+n)/2-1}\Pi(\mathrm{d}g)}\right)x, \end{aligned} \tag{3.31}$$

where $\|z\|^2 = \|x\|^2/s$. For some $k_i^2(g)$, assume the propriety of $k_i^2(g)\Pi(\mathrm{d}g)$ as $\int_0^\infty k_i^2(g)\Pi(\mathrm{d}g) < \infty$. Then

$$\bar{\pi}_i(\lambda) = \int_0^\infty \frac{\lambda^{p/2-1}}{g^{p/2}2^{p/2}\Gamma(p/2)} \exp\Bigl(-\frac{\lambda}{2g}\Bigr) k_i^2(g)\Pi(\mathrm{d}g) \tag{3.32}$$

is also proper. Let $\hat{\theta}_{\pi i} = \{1 - \psi_{\pi i}(\|x\|^2/s)\}x$ be the proper Bayes equivariant estimator under $\bar{\pi}_i(\lambda)$. By (3.17), the Bayes risk difference between $\hat{\theta}_\pi$ and $\hat{\theta}_{\pi i}$ under $\bar{\pi}_i$ is

$$\begin{aligned} &\tilde{\mathrm{r}}(\hat{\theta}_\pi; \bar{\pi}_i) - \tilde{\mathrm{r}}(\hat{\theta}_{\pi i}; \bar{\pi}_i) \\ &= \int_{\mathbb{R}^p} \left\{\psi_\pi(\|z\|^2) - \psi_{\pi i}(\|z\|^2)\right\}^2 \|z\|^2 M_1(z, \pi_i)\mathrm{d}z. \end{aligned} \tag{3.33}$$

For $w = \|z\|^2$, the integrand of (3.33) is expressed as

$$\begin{aligned} &\{\psi_\pi(\|z\|^2) - \psi_{\pi i}(\|z\|^2)\}^2 \|z\|^2 M_1(z, \pi_i) \\ &= w\Biggl(\frac{\int_0^\infty (g+1)^{-p/2-1}\{1 + w/(g+1)\}^{-(p+n)/2-1}\Pi(\mathrm{d}g)}{\int_0^\infty (g+1)^{-p/2}\{1 + w/(g+1)\}^{-(p+n)/2-1}\Pi(\mathrm{d}g)} \\ &\quad - \frac{\int_0^\infty (g+1)^{-p/2-1}\{1 + w/(g+1)\}^{-(p+n)/2-1}k_i^2(g)\Pi(\mathrm{d}g)}{\int_0^\infty (g+1)^{-p/2}\{1 + w/(g+1)\}^{-(p+n)/2-1}k_i^2(g)\Pi(\mathrm{d}g)}\Biggr)^2 \\ &\quad \times \frac{\Gamma((p+n)/2+1)2^{(p+n)/2+1}}{q_1(p,n)} \int_0^\infty \frac{(g+1)^{-p/2}k_i^2(g)\Pi(\mathrm{d}g)}{\{1 + w/(g+1)\}^{(p+n)/2+1}}. \end{aligned} \tag{3.34}$$

As in Sect. 2.4.2, with the sequence $k_i^2(g) = i/(g+i)$, we have the following result on admissibility within the class of equivariant estimators.

Theorem 3.4 (Maruyama and Strawderman 2020) *The estimator $\hat{\theta}_\pi$ is admissible within the class of equivariant estimators if*

$$\int_0^\infty \frac{\Pi(\mathrm{d}g)}{g+1} < \infty.$$

Proof Under the above assumption, $k_i^2(g) = i/(g+i)$ gives an increasing sequence of proper priors since

$$\int_0^\infty k_i^2(g)\Pi(\mathrm{d}g) = i\int_0^\infty \frac{\Pi(\mathrm{d}g)}{g+i} \le i\int_0^\infty \frac{\Pi(\mathrm{d}g)}{g+1} < \infty,$$

for fixed i. Applying the inequality (Part 3 of Lemma A.3) to (3.34), we have

$$\begin{aligned}
&\frac{q_1(p,n)}{\Gamma((p+n)/2+1)2^{(p+n)/2+1}}\{\psi_\pi(\|z\|^2)-\psi_{\pi i}(\|z\|^2)\}^2\|z\|^2 M_1(z,\pi_i) \\
&\leq 2w\Bigg(\frac{\{\int_0^\infty (g+1)^{-p/2-1}\{1+w/(g+1)\}^{-(p+n)/2-1}\Pi(\mathrm{d}g)\}^2}{\int_0^\infty (g+1)^{-p/2}\{1+w/(g+1)\}^{-(p+n)/2-1}\Pi(\mathrm{d}g)} \\
&\quad + \frac{\{\int_0^\infty (g+1)^{-p/2-1}\{1+w/(g+1)\}^{-(p+n)/2-1}k_i^2(g)\Pi(\mathrm{d}g)\}^2}{\int_0^\infty (g+1)^{-p/2}\{1+w/(g+1)\}^{-(p+n)/2-1}k_i^2(g)\Pi(\mathrm{d}g)}\Bigg),
\end{aligned}$$

where $q_1(p,n)$ is given by (3.29). Further, applying the Cauchy-Schwarz inequality (Part 1 of Lemma A.3) to the first and second terms, we have

$$\begin{aligned}
&\frac{q_1(p,n)}{\Gamma((p+n)/2+1)2^{(p+n)/2+1}}\{\psi_\pi(\|z\|^2)-\psi_{\pi i}(\|z\|^2)\}^2\|z\|^2 M_1(z,\pi_i) \\
&\leq 4\|z\|^2\int_0^\infty \frac{(g+1)^{-p/2-2}\Pi(\mathrm{d}g)}{\{1+\|z\|^2/(g+1)\}^{(p+n)/2+1}}.
\end{aligned}$$

Hence, we have

$$\begin{aligned}
&\frac{q_1(p,n)}{\Gamma((p+n)/2+1)2^{(p+n)/2+1}}\int_{\mathbb{R}^p}\{\psi_\pi(\|z\|^2)-\psi_{\pi i}(\|z\|^2)\}^2\|z\|^2 M_1(z,\pi_i)\mathrm{d}z \\
&\leq 4\int_{\mathbb{R}^p}\int_0^\infty \frac{\|z\|^2}{\{1+\|z\|^2/(g+1)\}^{(p+n)/2+1}}\frac{\Pi(\mathrm{d}g)}{(g+1)^{p/2+2}}\mathrm{d}z \\
&= 4\frac{\pi^{p/2}}{\Gamma(p/2)}\int_0^\infty\int_0^\infty \frac{t^{p/2}}{(1+t)^{(p+n)/2+1}}\frac{\Pi(\mathrm{d}g)}{g+1}\mathrm{d}t \\
&= 4\frac{\pi^{p/2}}{\Gamma(p/2)}\mathrm{B}(p/2+1,n/2)\int_0^\infty\frac{\Pi(\mathrm{d}g)}{g+1} < \infty,
\end{aligned}$$

where the equalities follow from Part 1 of Lemma A.2 and Part 3 of Lemma A.2, respectively.

Then by the dominated convergence theorem, we have

$$\lim_{i\to\infty}\left\{\tilde{\mathrm{r}}(\hat{\theta}_\pi;\bar{\pi}_i)-\tilde{\mathrm{r}}(\hat{\theta}_{\pi i};\bar{\pi}_i)\right\}=0$$

which, by the Blyth method, implies the admissibility of $\hat{\theta}_\pi$ within the class of equivariant estimators. □

As in Sect. 2.4.3, suppose $\Pi(\mathrm{d}g)$ in (3.27) has a regularly varying density of the form

$$\pi(g;a,b,c)=\frac{1}{(g+1)^a}\left(\frac{g}{g+1}\right)^b\frac{1}{\{\log(g+1)+1\}^c}. \tag{3.35}$$

Then, by (3.31), the corresponding generalized Bayes estimator is of the form

$$\hat{\theta}_\pi = \left(1 - \frac{\int_0^\infty (g+1)^{-p/2-1}\{1+w/(g+1)\}^{-(p+n)/2-1}\pi(g;a,b,c)\mathrm{d}g}{\int_0^\infty (g+1)^{-p/2}\{1+w/(g+1)\}^{-(p+n)/2-1}\pi(g;a,b,c)\mathrm{d}g}\right)x. \tag{3.36}$$

As a corollary of Theorem 3.4, using the argument in the admissibility proofs of Sect. 2.4.3, we have the following result.

Corollary 3.1 *The generalized Bayes estimator $\hat{\theta}_\pi$ given by (3.36) is admissible within the class of equivariant estimators if*

$$\textit{either } \{a>0,\ b>-1,\ c\in\mathbb{R}\} \textit{ or } \{a=0,\ b>-1,\ c>1\}.$$

3.3.2 On the Boundary Between Equivariant Admissibility and Inadmissibility

For the class of densities $\pi(g;a,b,c)$ given by (3.35), with either $-p/2+1<a<0$ or $\{a=0$ and $c>1\}$, Corollaries 3.3 and 3.4 in Sect. 3.6 show the inadmissibility of the corresponding generalized Bayes estimator by finding an improved estimator among the class of equivariant estimators. Hence, together with Corollary 3.1, the issue of admissibility/inadmissibility within the class of equivariant estimators for all values of a and c except for the cases $\{a=0$ and $|c|\le 1\}$, has been settled. The following result addresses this case.

Theorem 3.5 (Maruyama and Strawderman 2020) *Assume the measure $\Pi(\mathrm{d}g)$ in (3.27) has the density $\pi(g;a,b,c)$ given by (3.35) with*

$$a=0,\ b>-1,\ -1<c\le 1.$$

Then the corresponding generalized Bayes estimator is admissible within the class of equivariant estimators.

Proof See Appendix A.6. □

Our proof unfortunately does not cover the case $c=-1$, although we conjecture that admissibility holds within the class of equivariant estimators as well. The proof of Theorem 3.5 is based on Maruyama and Strawderman (2020), where $b\ge 0$ was assumed. In this book, we also include the case $-1<b<0$.

While this section considers admissibility only within the class of equivariant estimators, the next section broadens the discussion and considers admissibility among all estimators.

3.4 Admissibility Among All Estimators

3.4.1 The Main Result

In this section, we consider admissibility of generalized Bayes estimators among all estimators for a broad class of mixture priors. In particular, we consider the following class of joint prior densities:

$$\pi_*(\theta, \eta) = \frac{1}{\eta} \times \eta^{p/2}\pi(\eta\|\theta\|^2)$$

where

$$\pi(\|\mu\|^2) = \int_0^\infty \frac{g^{p/2}}{(2\pi)^{p/2}} \exp\Big(-\frac{\|\mu\|^2}{2g}\Big)\pi(g; a, b, 0)\mathrm{d}g, \tag{3.37}$$

and where $\pi(g; a, b, c)$ is given in (3.35). We note that all such priors are improper because each is non-integrable in η for any given θ. Then, as in (3.36), the corresponding generalized Bayes estimator is

$$\{1 - \phi(\|x\|^2/s)/\{\|x\|^2/s\}\}x$$

where

$$\phi(w) = w\frac{\int_0^\infty (g+1)^{-p/2-1}\{1 + w/(g+1)\}^{-p/2-n/2-1}\pi(g; a, b, 0)\mathrm{d}g}{\int_0^\infty (g+1)^{-p/2}\{1 + w/(g+1)\}^{-p/2-n/2-1}\pi(g; a, b, 0)\mathrm{d}g}. \tag{3.38}$$

Here is the main theorem of this section.

Theorem 3.6 (Maruyama and Strawderman 2021, 2023a) *The generalized Bayes estimator under $\pi_*(\theta, \eta)$ is admissible among all estimators if*

$$\max(-p/2 + 1, 0) < a < n/2 + 2, \ b > -1, \ c = 0.$$

Remark 3.1 As far as we know, Theorem 3.6 is the only known result on admissibility of generalized Bayes estimators of the form $\big\{1 - \phi(\|x\|^2/s)/(\|x\|^2/s)\big\}\, x$. As in Corollary 3.5 in Sect. 3.7, the generalized Bayes estimator under $\pi_*(\theta, \eta)$ is also minimax if

$$-p/2 + 1 < a \le \frac{(p-2)(n+2)}{2(2p+n-2)}, \ b \ge 0, \ c = 0.$$

Strawderman (1973) considered the truncated proper prior on η, $\eta^c I_{(\gamma,\infty)}$ with $c < -1$ and $\gamma > 0$ instead of the invariant prior on η. Under this prior, a class of proper Bayes, and hence admissible estimators dominating the usual unbiased estimator for $p \ge 5$ was found. However, because of the truncation of the prior on η, such estimators are not scale equivariant of the form $\left\{1 - \phi(\|x\|^2/s)/(\|x\|^2/s)\right\} x$, but instead have the form $\left\{1 - \phi(\|x\|^2/s, s)/(\|x\|^2/s)\right\} x$.

Recall $\pi(g; a, b, c)$ given by (3.37) is proper for $a > 1$ and $c \in \mathbb{R}$. In order to prove the result, we construct a sequence of proper priors $\pi_i(\theta, \eta)$ converging to $\pi_*(\theta, \eta)$ of the form

$$\pi_i(\theta, \eta) = \frac{h_i^2(\eta)}{\eta} \int_0^\infty \frac{\eta^{p/2}}{(2\pi)^{p/2} g^{p/2}} \exp\Bigl(-\frac{\eta}{2g}\|\theta\|^2\Bigr)\pi(g) k_i^2(g) \mathrm{d}g \tag{3.39}$$

where

$$h_i(\eta) = \frac{\log(i+1)}{\log(i+1) + |\log \eta|},$$
$$k_i(g) = \begin{cases} 1 - \dfrac{\log(g+1)}{\log(g+1+i)} & \max(-p/2+1, 0) < a \le 1, \\ 1 & 1 < a < n/2+2. \end{cases}$$

Note that $\log(1+1) < 1 < \log(2+1)$. For this technical reason, the sequence starts at $i = 2$. Properties of $h_i(\eta)$ and $k_i(g)$ are provided in Lemmas 3.1 and A.6. In particular, we emphasize that $h_i^2(\eta)/\eta$ and $\pi(g)k_i^2(g)$ are both proper by Part 2 of Lemma 3.1 and Part 5 of Lemma A.6, respectively, which implies that $\pi_i(\theta, \eta)$ given by (3.39) is proper.

Lemma 3.1 *Let*

$$h_i(\eta) = \frac{\log(i+1)}{\log(i+1) + |\log \eta|}.$$

1. $h_i(\eta)$ *is increasing in* i *and* $\lim_{i\to\infty} h_i(\eta) = 1$ *for all* $\eta > 0$.
2. $\displaystyle\int_0^\infty \eta^{-1} h_i^2(\eta) \mathrm{d}\eta = 2\log(i+1)$.

Proof ***(Part 1)*** This part is straightforward given the form of $h_i(\eta)$.

[Part 2] Let $j = \log(i+1)$. The results follow from the integrals,

$$\int_0^\infty \frac{h_i^2(\eta)}{\eta} \mathrm{d}\eta = \int_0^1 \frac{j^2 \mathrm{d}\eta}{\eta(j - \log\eta)^2} + \int_1^\infty \frac{j^2 \mathrm{d}\eta}{\eta(j + \log\eta)^2}$$
$$= \left[\frac{j^2}{j - \log\eta}\right]_0^1 + \left[\frac{-j^2}{j + \log\eta}\right]_1^\infty = 2j.$$

□

3.4.2 A Proof of Theorem 3.6

We start by developing expressions for Bayes estimators and risk differences which are used to prove Theorem 3.6. We make use of the following notation. For any function $\psi(\theta,\eta)$, let

$$m(\psi(\theta,\eta)) = \iint \psi(\theta,\eta)\frac{\eta^{p/2}}{(2\pi)^{p/2}}\exp\Big(-\eta\frac{\|x-\theta\|^2}{2}\Big)\frac{\eta^{n/2}s^{n/2-1}}{\Gamma(n/2)2^{n/2}}\exp\Big(-\frac{\eta s}{2}\Big)\mathrm{d}\theta\mathrm{d}\eta.$$

Then, under the loss (1.3), the generalized Bayes estimator under the improper prior $\pi_*(\theta,\eta)$ is

$$\hat{\theta}_* = \frac{m(\eta\theta\pi_*(\theta,\eta))}{m(\eta\pi_*(\theta,\eta))}$$

and the proper Bayes estimator under the proper prior $\pi_i(\theta,\eta)$ is

$$\hat{\theta}_i = \frac{m(\eta\theta\pi_i(\theta,\eta))}{m(\eta\pi_i(\theta,\eta))}.$$

The Bayes risk difference under π_i is

$$\Delta_i = \int_{\mathbb{R}^p}\int_0^\infty \Big\{\mathrm{E}\Big[\eta\|\hat{\theta}_*-\theta\|^2\Big] - \mathrm{E}\Big[\eta\|\hat{\theta}_i-\theta\|^2\Big]\Big\}\pi_i(\theta,\eta)\mathrm{d}\theta\mathrm{d}\eta.$$

Note that $\|\hat{\theta}_*-\theta\|^2 - \|\hat{\theta}_i-\theta\|^2 = \|\hat{\theta}_*\|^2 - \|\hat{\theta}_i\|^2 - 2\theta^{\mathrm{T}}(\hat{\theta}_*-\hat{\theta}_i)$. Then Δ_i can be re-expressed as

$$\begin{aligned}\Delta_i &= \iiiint \eta\left(\|\hat{\theta}_*\|^2 - \|\hat{\theta}_i\|^2 - 2\theta^{\mathrm{T}}(\hat{\theta}_*-\hat{\theta}_i)\right) \\ &\quad\times \frac{\eta^{p/2}}{(2\pi)^{p/2}}\exp\Big(-\eta\frac{\|x-\theta\|^2}{2}\Big)\frac{\eta^{n/2}s^{n/2-1}}{\Gamma(n/2)2^{n/2}}\exp\Big(-\frac{\eta s}{2}\Big)\pi_i(\theta,\eta)\mathrm{d}x\mathrm{d}s\mathrm{d}\theta\mathrm{d}\eta \\ &= \iint \Big\{m(\eta\pi_i)(\|\hat{\theta}_*\|^2 - \|\hat{\theta}_i\|^2) - 2m(\eta\theta^{\mathrm{T}}\pi_i)(\hat{\theta}_*-\hat{\theta}_i)\Big\}\mathrm{d}x\mathrm{d}s \\ &= \iint \|\hat{\theta}_*-\hat{\theta}_i\|^2 m(\eta\pi_i(\theta,\eta))\mathrm{d}x\mathrm{d}s. \qquad (3.40)\end{aligned}$$

Next, we rewrite $\hat{\theta}_*$, $\hat{\theta}_i$ and $\|\hat{\theta}_*-\hat{\theta}_i\|^2 m(\eta\pi_i(\theta,\eta))$, the integrand of (3.40). By Lemma A.1, we have

$$
\begin{aligned}
m(\eta\pi_i) &= \iiint \eta \frac{\eta^{p/2}}{(2\pi)^{p/2}} \exp\Big(-\eta \frac{\|x-\theta\|^2}{2}\Big) \frac{\eta^{n/2}s^{n/2-1}}{\Gamma(n/2)2^{n/2}} \exp\Big(-\frac{\eta s}{2}\Big) \\
&\quad \times \frac{\eta^{p/2}}{(2\pi)^{p/2}g^{p/2}} \exp\Big(-\frac{\eta}{2g}\|\theta\|^2\Big) \frac{h_i^2(\eta)}{\eta} \pi(g) k_i^2(g) \mathrm{d}\theta \mathrm{d}g \mathrm{d}\eta \\
&= \frac{s^{n/2-1}}{q_1(p,n)} \iint F(g,\eta;w,s) h_i^2(\eta)\pi(g)k_i^2(g)\mathrm{d}g\mathrm{d}\eta, \qquad (3.41)
\end{aligned}
$$

where $w = \|x\|^2/s$, $q_1(p,n) = (2\pi)^{p/2}\Gamma(n/2)2^{n/2}$, and

$$
F(g,\eta;w,s) = \frac{\eta^{p/2+n/2}}{(g+1)^{p/2}} \exp\Big(-\frac{\eta s}{2}\Big(1+\frac{w}{g+1}\Big)\Big).
$$

Similarly we have

$$
m(\eta\theta\pi_i) = \frac{s^{n/2-1}}{q_1(p,n)} \iint \frac{gx}{g+1} F(g,\eta;w,s)h_i^2(\eta)\pi(g)k_i^2(g)\mathrm{d}g\mathrm{d}\eta. \qquad (3.42)
$$

By (3.41) and (3.42), the Bayes estimator under π_i is

$$
\hat{\theta}_i = \frac{m(\theta\eta\pi_i)}{m(\eta\pi_i)} = \Big(1 - \frac{\phi_i(w,s)}{w}\Big)x, \qquad (3.43)
$$

where

$$
\phi_i(w,s) = w \frac{\iint (g+1)^{-1}F(g,\eta;w,s)h_i^2(\eta)\pi(g)k_i^2(g)\mathrm{d}g\mathrm{d}\eta}{\iint F(g,\eta;w,s)h_i^2(\eta)\pi(g)k_i^2(g)\mathrm{d}g\mathrm{d}\eta}. \qquad (3.44)
$$

With $h_i \equiv 1$ and $k_i \equiv 1$ in (3.44), we have

$$
\phi_*(w,s) = w \frac{\iint (g+1)^{-1}F(g,\eta;w,s)\pi(g)\mathrm{d}g\mathrm{d}\eta}{\iint F(g,\eta;w,s)\pi(g)\mathrm{d}g\mathrm{d}\eta} \qquad (3.45)
$$

and our target generalized Bayes estimator given by

$$
\hat{\theta}_* = \Big(1 - \frac{\phi_*(w,s)}{w}\Big)x. \qquad (3.46)
$$

Note that

$$
\int_0^\infty F(g,\eta;w,s)\mathrm{d}\eta = \frac{\Gamma(p/2+n/2+1)}{(g+1)^{p/2}} \Big(\frac{2s^{-1}}{1+w/(g+1)}\Big)^{p/2+n/2+1}
$$

which implies

$$
\frac{\phi_*(w,s)}{w} = \frac{\int_0^\infty (g+1)^{-p/2-1}\{1+w/(g+1)\}^{-p/2-n/2-1}\pi(g)\mathrm{d}g}{\int_0^\infty (g+1)^{-p/2}\{1+w/(g+1)\}^{-p/2-n/2-1}\pi(g)\mathrm{d}g}. \qquad (3.47)
$$

In the following development, however, we utilize (3.45) not (3.47) as the expression of $\phi_*(w, s)$.

By (3.41), (3.43) and (3.46), we have

$$\begin{aligned}\frac{q_1(p,n)}{\|x\|^2 s^{n/2-1}} \left\|\hat{\theta}_* - \hat{\theta}_i\right\|^2 m(\eta\pi_i) &= \frac{q_1(p,n)}{s^{n/2-1}}\Big(\frac{\phi_*(w,s)}{w} - \frac{\phi_i(w,s)}{w}\Big)^2 m(\eta\pi_i) \\ &= A(w, s; i),\end{aligned}$$

where

$$A(w, s, i) = \left(\frac{\iint \frac{F\pi}{g+1} \mathrm{d}g\mathrm{d}\eta}{\iint F\pi \mathrm{d}g\mathrm{d}\eta} - \frac{\iint \frac{Fh_i^2\pi k_i^2}{g+1} \mathrm{d}g\mathrm{d}\eta}{\iint Fh_i^2\pi k_i^2 \mathrm{d}g\mathrm{d}\eta} \right)^2 \iint Fh_i^2\pi k_i^2 \mathrm{d}g\mathrm{d}\eta. \tag{3.48}$$

Applying the inequality (Part 3 of Lemma A.3) to (3.48), we have

$$\begin{aligned}&\frac{1}{3}\left(\frac{\iint (g+1)^{-1}F\pi\mathrm{d}g\mathrm{d}\eta}{\iint F\pi\mathrm{d}g\mathrm{d}\eta} - \frac{\iint (g+1)^{-1}Fh_i^2\pi k_i^2\mathrm{d}g\mathrm{d}\eta}{\iint Fh_i^2\pi k_i^2\mathrm{d}g\mathrm{d}\eta}\right)^2 \\ &\le \left(\frac{\iint (g+1)^{-1}F\pi\mathrm{d}g\mathrm{d}\eta}{\iint F\pi\mathrm{d}g\mathrm{d}\eta} - \frac{\iint (g+1)^{-1}Fh_i^2\pi\mathrm{d}g\mathrm{d}\eta}{\iint Fh_i^2\pi\mathrm{d}g\mathrm{d}\eta}\right)^2 \\ &\quad + \left(\frac{\iint (g+1)^{-1}Fh_i^2\pi\mathrm{d}g\mathrm{d}\eta}{\iint Fh_i^2\pi\mathrm{d}g\mathrm{d}\eta} - \frac{\iint (g+1)^{-1}Fh_i^2\pi k_i^2\mathrm{d}g\mathrm{d}\eta}{\iint Fh_i^2\pi\mathrm{d}g\mathrm{d}\eta}\right)^2 \\ &\quad + \left(\frac{\iint (g+1)^{-1}Fh_i^2\pi k_i^2\mathrm{d}g\mathrm{d}\eta}{\iint Fh_i^2\pi\mathrm{d}g\mathrm{d}\eta} - \frac{\iint (g+1)^{-1}Fh_i^2\pi k_i^2\mathrm{d}g\mathrm{d}\eta}{\iint Fh_i^2\pi k_i^2\mathrm{d}g\mathrm{d}\eta}\right)^2.\end{aligned}$$

Hence we have

$$\frac{A(w, s; i)}{3} \le A_1(w, s; i) + A_2(w, s; i) + A_3(w, s; i),$$

where

$$\begin{aligned}A_1(w, s; i) &= \left\{\iint \left|\frac{1}{\iint F\pi\mathrm{d}g\mathrm{d}\eta} - \frac{h_i^2}{\iint Fh_i^2\pi\mathrm{d}g\mathrm{d}\eta}\right| \frac{F\pi\mathrm{d}g\mathrm{d}\eta}{g+1}\right\}^2 \iint Fh_i^2\pi\mathrm{d}g\mathrm{d}\eta, \\ A_2(w, s; i) &= \frac{\left(\iint (g+1)^{-1}Fh_i^2\pi(1-k_i^2)\mathrm{d}g\mathrm{d}\eta\right)^2}{\iint Fh_i^2\pi\mathrm{d}g\mathrm{d}\eta}, \\ A_3(w, s; i) &= \frac{\left(\iint (g+1)^{-1}Fh_i^2\pi k_i^2\mathrm{d}g\mathrm{d}\eta\right)^2}{(\iint Fh_i^2\pi\mathrm{d}g\mathrm{d}\eta)^2 \iint Fh_i^2\pi k_i^2\mathrm{d}g\mathrm{d}\eta} \left(\iint Fh_i^2\pi(1-k_i^2)\mathrm{d}g\mathrm{d}\eta\right)^2.\end{aligned}$$

In Sects. A.7.1–A.7.3, we prove that

$$\lim_{i\to\infty} \iint \|x\|^2 s^{n/2-1} A_\ell(\|x\|^2/s, s; i)\mathrm{d}x\mathrm{d}s = 0, \text{ for } \ell = 1, 2, 3,$$

which implies that $\Delta_i \to 0$ as $i \to \infty$. Thus the corresponding generalized Bayes estimator is admissible among all estimators, as was to be shown.

3.5 Simple Bayes Estimators

Interestingly, and perhaps somewhat surprisingly, suitable choices of the constants a and b (with $c = 0$) lead to admissible minimax generalized Bayes estimators of a simple form. Further, this form represents a relatively minor adjustment to the form of the James–Stein estimator. Here are the details. Consider the case $b = n/2 - a$ in (3.38). For the numerator of (3.38), we have

$$\begin{aligned}
&\int_0^\infty \frac{(g+1)^{-p/2-a-1}\{g/(g+1)\}^b \mathrm{d}g}{\{1+w/(g+1)\}^{p/2+n/2+1}} = \int_0^\infty \frac{g^{n/2-a}\mathrm{d}g}{(g+1+w)^{p/2+n/2+1}} \\
&= \frac{1}{(1+w)^{p/2+a}} \int_0^\infty \frac{t^{n/2-a}\mathrm{d}g}{(1+t)^{p/2+n/2+1}} = \frac{\mathrm{B}(n/2+1-a, p/2+a)}{(1+w)^{p/2+a+2}}.
\end{aligned}$$

Similarly, for the denominator of of (3.38), we have

$$\begin{aligned}
&\int_0^\infty \frac{(g+1)^{-p/2-a}\{g/(g+1)\}^b \mathrm{d}g}{\{1+w/(g+1)\}^{p/2+n/2+1}} \\
&= \int_0^\infty (1+g)\frac{(g+1)^{-p/2-1-a}\{g/(g+1)\}^b \mathrm{d}g}{\{1+w/(g+1)\}^{p/2+n/2+1}} \\
&= \frac{\mathrm{B}(n/2+1-a, p/2+a)}{(1+w)^{p/2+a}} + \frac{\mathrm{B}(n/2+2-a, p/2-1+a)}{(1+w)^{p/2-1+a}} \\
&= \frac{\mathrm{B}(n/2+1-a, p/2+a)}{(1+w)^{p/2+a}}\Big(1 + \frac{n/2+1-a}{p/2-1+a}(w+1)\Big).
\end{aligned}$$

Thus the generalized Bayes estimator is of the form

$$\hat{\theta}_\alpha^{\mathrm{SB}} = \Big(1 - \frac{\alpha}{\|x\|^2/s + \alpha + 1}\Big)x,$$

where $\alpha = (p/2 - 1 + a)/(n/2 + 1 - a)$. This estimator was discovered and studied in Maruyama and Strawderman (2005). By Theorem 3.6, provided

$$\alpha > \frac{p-2}{n+2} \quad \Leftrightarrow \quad a > 0,$$

$\hat{\theta}_\alpha^{\mathrm{SB}}$ is admissible among all estimators. Also, by Theorem 3.5, $\hat{\theta}_\alpha^{\mathrm{SB}}$ with $\alpha = (p - 2)/(n + 2)$ is admissible within the class of equivariant estimators. Additionally, by

Corollary 3.5, in Sect. 3.7 below, minimaxity of $\hat{\theta}_\alpha^{\mathrm{SB}}$ holds for

$$0 < \alpha \le 2\frac{p-2}{n+2} \quad \Leftrightarrow \quad -p/2+1 < a \le \frac{(p-2)(n+2)}{2(2p+n-2)}.$$

3.6 Inadmissibility

3.6.1 A General Sufficient Condition for Inadmissibility

This section is devoted to the question of inadmissibility of shrinkage estimators of the form $\hat{\theta}_\phi = (1-\phi(w)/w)x$ where $w = \|x\|^2/s$. Note that such estimators are equivariant. By (1.48) in Chap. 1, with $\psi(w) = \phi(w)/w$, the SURE for an estimator of the form $\hat{\theta}_\phi$ is

$$\hat{\mathrm{R}}_\phi = p + \frac{(n+2)\{\phi(w) - 2c_{p,n}\}\phi(w)}{w} - 4\phi'(w)\{1+\phi(w)\}, \qquad (3.49)$$

where $c_{p,n} = (p-2)/(n+2)$. For a competing estimator of the form

$$\hat{\theta}_{\phi+\nu} = \Big(1 - \frac{\phi(w)+\nu(w)}{w}\Big)x,$$

the difference in the SURE between $\hat{\theta}_\phi$ and $\hat{\theta}_{\phi+\nu}$ is

$$\hat{\mathrm{R}}_\phi - \hat{\mathrm{R}}_{\phi+\nu} = \nu(w)\{\Delta_1(w;\phi) + \Delta_2(w;\phi,\nu)\} \qquad (3.50)$$

where

$$\begin{aligned} \Delta_1(w;\phi) &= 2(n+2)\frac{c_{p,n} - \phi(w)}{w} + 4\phi'(w), \\ \Delta_2(w;\phi,\nu) &= -(n+2)\frac{\nu(w)}{w} + 4\nu'(w) + 4\frac{\nu'(w)}{\nu(w)}\{1+\phi(w)\}. \end{aligned}$$

Our approach to finding an estimator dominating $\hat{\theta}_\phi$ is to find a non-zero solution $\nu(w)$ to the differential inequality $\hat{\mathrm{R}}_\phi - \hat{\mathrm{R}}_{\phi+\nu} \ge 0$. Here is the result.

Theorem 3.7 (Maruyama and Strawderman 2017) *Let* $c_{p,n} = (p-2)/(n+2)$. *Suppose*

$$\begin{aligned} &\limsup_{w\to\infty} \phi(w) \le c_{p,n} \\ \textit{and} \quad &\liminf_{w\to\infty} \log w \left\{(n+2)\{c_{p,n} - \phi(w)\} + 2w\phi'(w)\right\} > 2(1+c_{p,n}). \end{aligned} \qquad (3.51)$$

Then the estimator $\hat{\theta}_\phi = (1-\phi(w)/w)x$ *with* $w = \|x\|^2/s$ *is inadmissible.*

Proof By (3.51), there exist

$$w_1 > \exp(1) \text{ and } 0 < \epsilon < 1 \tag{3.52}$$

such that for all $w \geq w_1$,

$$\phi(w) - c_{p,n} \leq \frac{1 + c_{p,n}}{6}\epsilon$$

and

$$\begin{aligned} &\log w \left\{(n+2)\{c_{p,n} - \phi(w)\} + 2w\phi'(w)\right\} - 2(1 + c_{p,n})(1+\epsilon) \geq 0, \\ &\text{or equivalently} \quad \Delta_1(w;\phi) - 4\frac{(1+c_{p,n})(1+\epsilon)}{w\log w} \geq 0, \end{aligned} \tag{3.53}$$

Let $q(w; w_2)$ be the cumulative distribution function of $Y + w_2$, where $w_2 > w_1$ will be precisely determined later and Y is a Gamma random variable with the probability density function $y\exp(-y)I_{(0,\infty)}(y)$, that is,

$$q(w; w_2) = \begin{cases} 0 & \text{for } 0 \leq w < w_2 \\ \int_0^{w-w_2} y\exp(-y)\mathrm{d}y & \text{for } w \geq w_2. \end{cases}$$

Then $q(w; w_2)$ is non-decreasing, differentiable with $q'(w)|_{w=w_2} = 0$ and $q(\infty) = 1$.

Let $\nu(w)$ for the competing estimator be given by

$$\nu(w; w_2) = \frac{q(w; w_2)}{(\log w)^{1+\epsilon/2}}, \tag{3.54}$$

with ϵ satisfying (3.52) and (3.53). Then, for all $w \geq w_2$, we have

$$\begin{aligned} &\Delta_2[w;\phi,\nu(w;w_2)] + 4\frac{(1+c_{p,n})(1+\epsilon)}{w\log w} \\ &= -(n+2)\frac{q(w;w_2)}{w(\log w)^{1+\epsilon/2}} - \frac{4(1+\epsilon/2)q(w;w_2)}{w(\log w)^{2+\epsilon/2}} + \frac{4q'(w;w_2)}{(\log w)^{1+\epsilon/2}} \\ &\quad + 4\left\{\frac{q'(w;w_2)}{q(w;w_2)} - \frac{1+\epsilon/2}{w\log w}\right\}\{1+\phi(w)\} + 4\frac{(1+c_{p,n})(1+\epsilon)}{w\log w}. \end{aligned}$$

Note that $q'(w; w_2) \geq 0$, $q(w; w_2) \leq 1$, $(\log w)^{2+\epsilon/2} \geq (\log w)^{1+\epsilon/2}$ and

$$\begin{aligned} 4(1+\epsilon/2)\{1+\phi(w)\} &\leq 4(1+\epsilon/2)\Big(1 + c_{p,n} + \frac{1+c_{p,n}}{6}\epsilon\Big) \\ &= (1+c_{p,n})\Big(4 + 2\epsilon + \frac{2}{3}(1+\epsilon/2)\epsilon\Big) \leq (4+3\epsilon)(1+c_{p,n}). \end{aligned}$$

Hence

$$\begin{aligned}
&\Delta_2[w;\phi,\nu(w;w_2)] + 4\frac{(1+c_{p,n})(1+\epsilon)}{w\log w} \\
&\geq -\frac{4(1+c_{p,n})(1+3\epsilon/4)}{w\log w} - \frac{4(1+\epsilon/2)+n+2}{w(\log w)^{1+\epsilon/2}} + 4\frac{(1+c_{p,n})(1+\epsilon)}{w\log w} \\
&= \frac{(1+c_{p,n})\epsilon}{w\log w}\Big(1 - \frac{4(1+\epsilon/2)+n+2}{(1+c_{p,n})\epsilon}\frac{1}{(\log w)^{\epsilon/2}}\Big) \\
&\geq \frac{(1+c_{p,n})\epsilon}{w\log w}\Big(1 - \frac{4(1+\epsilon/2)+n+2}{\epsilon}\frac{1}{(\log w)^{\epsilon/2}}\Big).
\end{aligned} \tag{3.55}$$

Now let

$$w_2 = \max\Big\{\exp\Big(\Big\{\frac{4(1+\epsilon/2)+n+2}{\epsilon}\Big\}^{2/\epsilon}\Big), w_1\Big\}.$$

Then, by (3.53) and (3.55), we have

$$\begin{aligned}
&\Delta_1(w;\phi) + \Delta_2[w;\phi,\nu(w;w_2)] \\
&= \Big\{\Delta_1(w;\phi) - \frac{(1+c_{p,n})(1+\epsilon)}{(1/4)w\log w}\Big\} + \Big\{\Delta_2[w;\phi,\nu(w;w_2)] + \frac{(1+c_{p,n})(1+\epsilon)}{(1/4)w\log w}\Big\} \\
&\geq 0,
\end{aligned} \tag{3.56}$$

for all $w \geq w_2$. Hence, by (3.50), (3.54) and (3.56),

$$\hat{\mathrm{R}}_\phi - \hat{\mathrm{R}}_{\phi+\nu} = \nu(w)\{\Delta_1(w;\phi) + \Delta_2(w;\phi,\nu(w;w_2))\} \begin{cases} = 0 & \text{for } w < w_2 \\ \geq 0 & \text{for } w \geq w_2, \end{cases}$$

which completes the proof. □

As a corollary of Theorem 3.7, we have the following result.

Corollary 3.2 *The estimator $\hat{\theta}_\phi$ is inadmissible if $\phi(w)$ satisfies either*

$$\limsup_{w\to\infty} \phi(w) < \frac{p-2}{n+2} \quad \textit{and} \quad \lim_{w\to\infty} w\phi'(w) = 0 \tag{3.57}$$

or

$$\begin{aligned}
&\lim_{w\to\infty} \phi(w) = \frac{p-2}{n+2}, \quad \lim_{w\to\infty} w\log w\frac{\phi'(w)}{\phi(w)} = 0, \\
&\textit{and} \quad \liminf_{w\to\infty} \log w\Big\{\frac{p-2}{n+2} - \phi(w)\Big\} > \frac{2(p+n)}{(n+2)^2}.
\end{aligned} \tag{3.58}$$

3.6.2 Inadmissible Generalized Bayes Estimators

In this subsection, we apply the results of the previous subsection to a class of generalized Bayes estimators. As in Sect. 2.5, we assume that $\Pi(\mathrm{d}g)$ in (3.27) has a regularly varying density $\pi(g) = (g+1)^{-a}\xi(g)$ where $\xi(g)$ satisfies AS.1 and AS.2 given in the end of Sect. 2.1. The corresponding generalized Bayes estimator is of the form $(1 - \phi(w)/w)x$ where

$$\phi(w) = w\frac{\int_0^\infty (g+1)^{-p/2-1-a}\{1+w/(g+1)\}^{-(p/2+n/2+1)}\xi(g)\mathrm{d}g}{\int_0^\infty (g+1)^{-p/2-a}\{1+w/(g+1)\}^{-(p/2+n/2+1)}\xi(g)\mathrm{d}g},$$

In addition to AS.1 and AS.2, we assume the following mild assumptions on the asymptotic behaviors on $\xi(g)$;

A.S.5 $\limsup_{g\to\infty}\left\{(g+1)\log(g+1)\frac{\xi'(g)}{\xi(g)}\right\}$ is bounded,

A.S.6 $\xi(g)$ is ultimately monotone i.e., $\xi(g)$ is monotone on (g_0, ∞) for some $g_0 > 0$.

Under AS.1, AS.2, AS.5 and AS.6, we have the following result on the properties of $\phi(w)$.

Lemma 3.2 *Suppose* $-p/2+1 < a < n/2+1$. *Assume AS.1, AS.2, AS.5 and AS.6. Then* $\phi(w)$ *satisfies the following;*

1. $\lim_{w\to\infty} \frac{\int_0^\infty (g+1)^{-p/2-a}\{1+w/(g+1)\}^{-(p/2+n/2+1)}\xi(g)\mathrm{d}g}{w^{-p/2+1-a}\xi(w)\mathrm{B}(p/2-1+a, n/2-a+2)} = 1.$

2. $\lim_{w\to\infty} \phi(w) = \frac{p/2-1+a}{n/2+1-a}.$

3. $\lim_{w\to\infty} w\frac{\phi'(w)}{\phi(w)} = 0.$

Proof See Sect. A.10. □

By (3.57) of Corollary 3.2 and Parts 2 and 3 of Lemma 3.2, we have the following result.

Theorem 3.8 *Assume AS.1, AS.2, AS.5 and AS.6. Then the generalized Bayes estimator, with respect to the regularly varying density* $\pi(g) = (g+1)^{-a}\xi(g)$*, is inadmissible if* $-p/2+1 < a < 0$.

As in Sect. 2.4.3, suppose $\Pi(\mathrm{d}g)$ in (3.27) has a regularly varying density $\pi(g; a, b, c)$ as given in (3.35). It is easily seen that $\xi(g) = \{g/(g+1)\}^b\{\log(g+1)+1\}^{-c}$, for $b > -1$ and $c \in \mathbb{R}$, satisfies AS.5 and AS.6 as well as AS.1, AS.2. Hence we have the following corollary.

Corollary 3.3 *Assume*

$$-p/2+1 < a < 0,\ b > -1,\ c \in \mathbb{R},$$

in $\pi(g; a, b, c)$*. Then the corresponding generalized Bayes estimator is inadmissible.*

When $\lim_{w\to\infty} \phi(w) = (p-2)/(n+2)$, recall that a sufficient condition for inadmissibility is given by (3.58) of Corollary 3.2. The following lemma on the behavior of $\phi(w)$ is helpful for providing an inadmissibility result for $\pi(g; a, b, c)$ when $a = 0$.

Lemma 3.3 *Let $a = 0$, $b > -1$, and $c \neq 0$ in $\pi(g; a, b, c)$. Then*

$$\lim_{w\to\infty} \log w \left\{ \frac{p-2}{n+2} - \phi(w) \right\} = -c \frac{2(p+n)}{(n+2)^2}, \tag{3.59}$$

$$\textit{and} \quad \lim_{w\to\infty} w \log w \frac{\phi'(w)}{\phi(w)} = 0. \tag{3.60}$$

Proof See Sect. A.11. □

Then, by Parts 2 and 3 of Lemma 3.2, Lemma 3.3, and (3.58) of Corollary 3.2, we have the following result.

Corollary 3.4 *Assume*

$$a = 0, \ b > -1, \ c < -1$$

in $\pi(g; a, b, c)$. Then the corresponding generalized Bayes estimator is inadmissible.

Note that Corollaries 3.3 and 3.4 correspond to Corollary 2.1 for the known scale case.

3.7 Minimaxity

3.7.1 A Sufficient Condition for Minimaxity

In this section, we study the minimaxity of shrinkage estimators of the form

$$\hat{\theta}_\phi = \left(1 - \frac{\phi(w)}{w}\right) x$$

where $w = \|x\|^2/s$ and $\phi(w)$ is differentiable. The risk function of the estimator is

$$\mathrm{R}(\hat{\theta}_\phi; \theta, \eta) = p - 2 \sum_{i=1}^{n} \mathrm{E}\left[\eta \frac{\phi(W)}{W} X_i (X_i - \theta_i)\right] + \eta\, \mathrm{E}\left[S \frac{\phi^2(W)}{W}\right]. \tag{3.61}$$

As in (3.49) the SURE for an estimator $\hat{\theta}_\phi$ is give by $\mathrm{R}(\hat{\theta}_\phi; \theta, \eta) = \mathrm{E}[\hat{\mathrm{R}}_\phi(W)]$, where

$$\hat{\mathrm{R}}_\phi(w) = p + \frac{\{(n+2)\phi(w) - 2(p-2)\}\phi(w)}{w} - 4\phi'(w)\{1 + \phi(w)\}. \tag{3.62}$$

Hence, for a nonnegative $\phi(w)$, we have the following equivalence,

$$\frac{w\{\hat{\mathrm{R}}_\phi(w) - p\}}{\phi(w)\{1+\phi(w)\}} \le 0 \quad \Leftrightarrow \quad \frac{2(p-2)-(n+2)\phi(w)}{1+\phi(w)} + 4w\frac{\phi'(w)}{\phi(w)} \ge 0.$$

This implies the following result, which is Lemma 4.1 of Wells and Zhou (2008).

Theorem 3.9 *Assume that for $p \ge 3$ and a constant $\gamma \ge 0$, the differentiable function $\phi(w)$ satisfies the conditions: for any $w \ge 0$*

$$\frac{w\phi'(w)}{\phi(w)} \ge -\gamma \ \ \textit{and} \ \ 0 \le \phi(w) \le 2\frac{p-2-2\gamma}{n+2+4\gamma}.$$

Then, the estimator $\hat{\theta}_\phi$ is minimax.

Kubokawa (2009) proposed an alternative expression for the risk function which differs from the SURE estimator given by (3.62). We will use the result below to strengthen Theorem 3.9.

Theorem 3.10 (Kubokawa 2009) *The risk function is* $\mathrm{R}(\hat{\theta}_\phi; \theta, \eta) = p + \eta\,\mathrm{E}\,[(S/W)\mathcal{I}(W)]$, *where*

$$\mathcal{I}(w) = \phi^2(w) + 2\phi(w) - (n+p)\int_0^1 z^{n/2}\phi(w/z)\mathrm{d}z.$$

Proof Unlike the development of (3.62), we apply both Lemmas 1.1 and 1.2 to the second term on the right hand side of (3.61). Define a function $\Phi(W)$ by

$$\Phi(w) = \frac{1}{2w}\int_0^1 z^{n/2}\phi(w/z)\mathrm{d}z = \frac{w^{n/2}}{2}\int_w^\infty \frac{\phi(t)}{t^{n/2+2}}\mathrm{d}t,$$

where the third expression results from the transformation $t = w/z$. Using Lemma 1.2, we obtain

$$\eta\,\mathrm{E}^{S\mid X}\,[\Phi(W)S] = \mathrm{E}^{S\mid X}\Big[n\Phi(W) + 2S\frac{\partial}{\partial s}\Phi(W)\Big] = \mathrm{E}^{S\mid X}\Big[\frac{\phi(W)}{W}\Big], \tag{3.63}$$

where $\mathrm{E}^{S\mid X}[\cdot]$ denotes the conditional expectation with respect to S given X. Note that all the expectations are finite since $\phi(w)$ is bounded.

By (3.63), we can rewrite the cross product term in (3.61) as

$$\eta\sum_{i=1}^p \mathrm{E}\Big[\frac{\phi(W)}{W}X_i(X_i-\theta_i)\Big] = \eta^2\sum_{i=1}^p \mathrm{E}\,[S\Phi(W)X_i(X_i-\theta_i)]\,. \tag{3.64}$$

Note

$$\frac{\partial}{\partial x_i}x_i\Phi(\|x\|^2/s) = \Phi(\|x\|^2/s) + 2\,\frac{x_i^2}{s}\Phi'(w)\Big|_{w=\|x\|^2/s}, \tag{3.65}$$

where

$$\Phi'(w) = \frac{1}{2}\Big(\frac{n}{2}w^{n/2-1}\int_w^\infty \frac{\phi(t)}{t^{n/2+2}}\mathrm{d}t - \frac{\phi(w)}{w^2}\Big) = \frac{1}{2}\Big(n\frac{\Phi(w)}{w} - \frac{\phi(w)}{w^2}\Big). \quad (3.66)$$

By Lemma 1.1, (3.65) and (3.66), we have

$$\eta\sum_{i=1}^{p}\mathrm{E}^{X\,|\,S}\left[\Phi(W)X_i(X_i-\theta_i)\right] = \mathrm{E}^{X\,|\,S}\Big[(p+n)\Phi(W) - \frac{\phi(W)}{W}\Big]$$

and, by (3.64)

$$\eta\sum_{i=1}^{p}\mathrm{E}\Big[\frac{\phi(W)}{W}X_i(X_i-\theta_i)\Big] = \eta\,\mathrm{E}\Big[S\Big\{(p+n)\Phi(W) - \frac{\phi(W)}{W}\Big\}\Big].$$

The proof is completed by combining the appropriate terms above. □

Suppose $\phi(w)$ is differentiable in Theorem 3.10. Then we have

$$\begin{aligned}\phi(w/z) - z^\gamma\phi(w) &= \frac{z^\gamma}{w^\gamma}\left\{(w/z)^\gamma\phi(w/z) - w^\gamma\phi(w)\right\}\\ &= \frac{z^\gamma}{w^\gamma}\int_w^{w/z}\Big\{\frac{\mathrm{d}}{\mathrm{d}t}t^\gamma\phi(t)\Big\}\mathrm{d}t = \frac{z^\gamma}{w^\gamma}\int_w^{w/z}t^{\gamma-1}\phi(t)\Big\{\gamma + t\frac{\phi'(t)}{\phi(t)}\Big\}\mathrm{d}t,\end{aligned}$$

and hence

$$\begin{aligned}\mathcal{I}(w) &= \phi^2(w) + 2\phi(w) - (n+p)\int_0^1 z^{n/2}\left\{\phi(w/z) - z^\gamma\phi(w) + z^\gamma\phi(w)\right\}\mathrm{d}z\\ &\le \phi^2(w) + 2\phi(w) - (n+p)\phi(w)\int_0^1 z^{n/2+\gamma}\mathrm{d}z\\ &= \phi^2(w) + 2\phi(w) - 2\frac{n+p}{n+2+2\gamma}\phi(w)\\ &= \phi(w)\Big(\phi(w) - 2\frac{p-2-2\gamma}{n+2+2\gamma}\Big),\end{aligned}$$

where the inequality follows if $\phi(w) \ge 0$ and $w\phi'(w)/\phi(w) + \gamma \ge 0$. Then we have the following result.

Theorem 3.11 (Kubokawa 2009) *Assume that for $p \ge 3$ and a constant $\gamma \ge 0$, the differentiable function $\phi(w)$ satisfies the conditions for any $w \ge 0$,*

$$\frac{w\phi'(w)}{\phi(w)} \ge -\gamma \quad \text{and} \quad 0 \le \phi(w) \le 2\frac{p-2-2\gamma}{n+2+2\gamma}.$$

Then, the estimator $\hat{\theta}_\phi$ is minimax.

Note that the result given by Theorem 3.11 is slightly stronger than that in Theorem 3.9 since

$$2\frac{p-2-2\gamma}{n+2+4\gamma} \le 2\frac{p-2-2\gamma}{n+2+2\gamma}.$$

For this reason we will use Theorem 3.11, to consider the minimaxity of generalized Bayes estimator in Sect. 3.7.2.

3.7.2 Minimaxity of Some Generalized Bayes Estimators

Suppose $\pi(g) = (g+1)^{-a}\xi(g)$ where $\xi(g)$ satisfies AS.1–AS.4 as in Sect. 2.5.1. In this section, we investigate minimaxity of the corresponding generalized Bayes estimators with

$$\phi(w) = w\frac{\int_0^\infty (g+1)^{-p/2-a-1}\{1+w/(g+1)\}^{-(p/2+n/2+1)}\xi(g)\mathrm{d}g}{\int_0^\infty (g+1)^{-p/2-a}\{1+w/(g+1)\}^{-(p/2+n/2+1)}\xi(g)\mathrm{d}g}.$$

Recall that, in Sect. 2.5.1, $\Xi(g)$, $\Xi_1(g)$, $\Xi_2(g)$ and Ξ_{2*} were defined based on $\xi(g)$ and that the properties of these functions are summarized in Lemma 2.1. These results imply to the following properties for $\phi(w)$.

Lemma 3.4 *Suppose* $-p/2+1 < a < n/2+1-\Xi_{2*}$. *Then*

$$\phi(w) \le \frac{p-2+2a+2\Xi_{2*}}{n+2-2a-2\Xi_{2*}} \quad \textit{and} \quad w\frac{\phi'(w)}{\phi(w)} \ge -\Xi_{2*}. \tag{3.67}$$

Proof Section A.12. □

Hence by Theorem 3.11 and Lemma 3.4, we have the following result.

Theorem 3.12 *The generalized Bayes estimator is minimax if*

$$\frac{p+2+2a+2\Xi_{2*}}{n-2-2a-2\Xi_{2*}} \le 2\frac{p-2-2\Xi_{2*}}{n+2+2\Xi_{2*}}.$$

For $\xi(g) = \{g/(g+1)\}^b/\{\log(g+1)+1\}^c$ with $b \ge 0$, the following corollary follows from Lemma 2.2 and Theorem 3.12.

Corollary 3.5 *For* $\pi(g; a, b, c)$ *given by* (3.35) *with* $b \ge 0$, *the corresponding generalized Bayes estimator is minimax if either*

$$-p/2+1 < a \le \frac{(p-2)(n+2)}{2(2p+n-2)}, \quad c \le 0$$

or

$$-p/2+1<a<+\frac{(p-2)(n+2)}{2(2p+n-2)},\ c>0,$$
$$\frac{(p-2+2a)\{1+\log(b/c+1)\}+2c}{(n+2-2a)\{1+\log(b/c+1)\}-2c}\le 2\frac{(p-2)\{1+\log(b/c+1)\}-2c}{(n+2)\{1+\log(b/c+1)\}+2c}.$$

Suppose

$$\xi(g)=\Big(\frac{g}{g+1}\Big)^b \ \text{ for } -1<b<0,$$

as considered in Sect. 2.5.2. For this case, the behavior of the corresponding $\phi(w)$ is summarized in the next result.

Lemma 3.5 *Let $-1<b<0$. Then $\phi(w)$ of the corresponding generalized Bayes estimator satisfies*

$$\phi(w)\le\frac{p-2+2a}{n+2-2a+b(p+n)} \ \text{ and } \ w\frac{\phi'(w)}{\phi(w)}\ge\frac{(p+2a)b}{2(b+1)}.$$

Proof Section A.13. □

Thus Theorem 3.11 and Lemma 3.5, give minimaxity under the following conditions.

Theorem 3.13 *The generalized Bayes estimator is minimax if $-1<b<0$ and*

$$\frac{p-2+2a}{n+2-2a+b(p+n)}\le 2\frac{(p-2)(b+1)+b(p+2a)}{(n+2)(b+1)-b(p+2a)}.$$

3.8 Improvement on the James–Stein Estimator

In this section we extend the discussion in Sect. 2.6 to the case of unknown variance. As in (1.49) and Theorem 1.8, the James–Stein estimator

$$\hat\theta_{\mathrm{JS}}=\Big(1-\frac{p-2}{n+2}\frac{S}{\|X\|^2}\Big)X$$

dominates the estimator X for $p\ge 3$. Using the expression for the risk of $\hat\theta_\phi$ given by (3.49), the risk difference is given by

$$\begin{aligned}\Delta(\lambda)&=\mathrm{R}(\hat\theta_{\mathrm{JS}};\theta,\eta)-\mathrm{R}(\hat\theta_\phi;\theta,\eta)\\&=\mathrm{E}\Big[-(n+2)\frac{\{\phi(W)-c_{p,n}\}^2}{W}+4\{1+\phi(W)\}\phi'(W)\Big],\end{aligned}$$

where $c_{p,n}=(p-2)/(n+2)$, $\lambda=\eta\|\theta\|^2$ and $W=\|X\|^2/S$. Conditions on ϕ which ensure that $\Delta(\lambda)\ge 0$ are provided in the following theorem.

Theorem 3.14 (Kubokawa 1994) *The shrinkage estimator* $\hat{\theta}_\phi$ *improves on the James–Stein estimator* $\hat{\theta}_{\mathrm{JS}}$ *if* ϕ *satisfies the following conditions: (i)* $\phi(w)$ *is non-decreasing in* w*; (ii)* $\lim_{w\to\infty}\phi(w) = (p-2)/(n+2)$ *and* $\phi(w) \geq \phi_0(w)$ *where*

$$\phi_0(w) = w\frac{\int_0^\infty (g+1)^{-p/2-1}\{1+w/(g+1)\}^{-p/2-n/2-1}\mathrm{d}g}{\int_0^\infty (g+1)^{-p/2}\{1+w/(g+1)\}^{-p/2-n/2-1}\mathrm{d}g}.$$

Proof Let $U = \eta\|X\|^2$ and $V = \eta S$, and let $f_p(u;\lambda)$ and $f_n(v)$ be density functions of $\chi_p^2(\lambda)$ and χ_n^2, respectively. Then $U \sim \chi_p^2(\lambda)$ where $\lambda = \eta\|\theta\|^2$ and $V \sim \chi_n^2$. The expected value of a function $\psi(\|x\|^2/s)$ may be expressed as

$$\begin{aligned}
&\mathrm{E}[\psi(\|X\|^2/S)] = \mathrm{E}[\psi(\{\eta\|X\|^2\}/\{\eta S\})] \\
&= \iint \psi(u/v) f_p(u;\lambda) f_n(v)\mathrm{d}u\mathrm{d}v = \iint \psi(w) v f_p(wv;\lambda) f_n(v)\mathrm{d}v\mathrm{d}w \\
&= \int_0^\infty \psi(w)\Big\{\int_0^\infty v f_p(wv;\lambda) f_n(v)\mathrm{d}v\Big\}\mathrm{d}w \\
&= \int_0^\infty \psi(w)\sum_{i=0}^\infty \frac{(\lambda/2)^i}{e^{\lambda/2} i!}\Big\{\int_0^\infty v\frac{(wv)^{p/2+i-1}\exp(-wv/2)}{\Gamma(p/2+i)2^{p/2+i}}\frac{v^{n/2-1}\exp(-v/2)}{\Gamma(n/2)2^{n/2}}\mathrm{d}v\Big\}\mathrm{d}w \\
&= \int_0^\infty \psi(w) j_{p,n}(w;\lambda)\mathrm{d}w,
\end{aligned}$$

where

$$j_{p,n}(w;\lambda) = \sum_{i=0}^\infty \frac{(\lambda/2)^i}{e^{\lambda/2} i!}\frac{w^{p/2-1}(1+w)^{-p/2-n/2}}{\mathrm{B}(p/2+i, n/2)}\Big(\frac{w}{w+1}\Big)^i.$$

Then, arguing as in (2.45) and (2.46), the first term of $\Delta(\lambda)$ may be expressed as written as

$$\begin{aligned}
&-\mathrm{E}\Big[W^{-1}(\phi(W) - c_{p,n})^2\Big] \\
&= 2\int_0^\infty \{\phi(w) - c_{p,n}\}\phi'(w)\Big\{\int_0^\infty \frac{j_{p,n}(w/(g+1);\lambda)}{g+1}\mathrm{d}g\Big\}\mathrm{d}w
\end{aligned}$$

and hence $\Delta(\lambda)$ may be written as

$$\begin{aligned}\Delta(\lambda) &= 2\int_0^\infty \phi'(w)\Big((n+2)\{\phi(w)-c_{p,n}\}\int_0^\infty \frac{j_{p,n}(w/(g+1);\lambda)}{g+1}\mathrm{d}g \\ &\quad + 2\{1+\phi(w)\}j_{p,n}(w;\lambda)\Big)\mathrm{d}w \\ &= 2\int_0^\infty \phi'(w)\Big((n+2)\{\phi(w)-c_{p,n}\} + 2\{1+\phi(w)\}J_{p,n}(w;\lambda)\Big) \\ &\quad \times \Big\{\int_0^\infty \frac{j_{p,n}(w/(g+1);\lambda)}{g+1}\mathrm{d}g\Big\}\mathrm{d}w,\end{aligned}$$

where

$$J_{p,n}(w;\lambda) = \frac{j_{p,n}(w;\lambda)}{\int_0^\infty (g+1)^{-1} j_{p,n}(w/(g+1);\lambda)\mathrm{d}g}.$$

Further, as in (2.47), $J_{p,n}(w;\lambda) \geq J_{p,n}(w;0)$ holds where

$$\begin{aligned}J_{p,n}(w;0) &= \frac{j_{p,n}(w;0)}{\int_0^\infty (g+1)^{-1} j_{p,n}(w/(g+1);0)\mathrm{d}g} \\ &= \frac{(1+w)^{-p/2-n-2}}{\int_0^\infty (g+1)^{-p/2}\{1+w/(g+1)\}^{-p/2-n/2}\mathrm{d}g}.\end{aligned} \tag{3.68}$$

Hence we have $\Delta(\lambda) \geq 0$ if $\phi'(w) \geq 0$ and

$$(n+2)\{\phi(w)-c_{p,n}\} + 2\{1+\phi(w)\}J_{p,n}(w;0) \geq 0,$$

which is equivalent to $\phi(w) \geq \phi_0(w)$ where

$$\phi_0(w) = \frac{p-2-2J_{p,n}(w;0)}{n+2+2J_{p,n}(w;0)} \tag{3.69}$$

$$= \frac{(p-2)\displaystyle\int_0^\infty \frac{(g+1)^{-p/2}\mathrm{d}g}{\{1+w/(g+1)\}^{p/2+n/2}} - \frac{2}{(1+w)^{p/2+n/2}}}{(n+2)\displaystyle\int_0^\infty \frac{(g+1)^{-p/2}\mathrm{d}g}{\{1+w/(g+1)\}^{p/2+n/2}} + \frac{2}{(1+w)^{p/2+n/2}}}. \tag{3.70}$$

For the denominator of (3.70), an integration by parts gives

$$
\begin{aligned}
&(n+2)\int_0^\infty \frac{(g+1)^{-p/2}\mathrm{d}g}{\{1+w/(g+1)\}^{p/2+n/2}} + \frac{2}{(1+w)^{p/2+n/2}} \\
&= (n+2)\int_0^\infty \frac{(g+1)^{n/2}\mathrm{d}g}{(1+g+w)^{p/2+n/2}} + \frac{2}{(1+w)^{p/2+n/2}} \\
&= 2\int_0^\infty (g+1)^{n/2+1}\Big\{\frac{(p+n)/2}{(1+g+w)^{p/2+n/2+1}}\Big\}\mathrm{d}g \\
&= (p+n)\int_0^\infty \frac{(g+1)^{-p/2}\mathrm{d}g}{\{1+w/(g+1)\}^{p/2+n/2+1}}. \qquad (3.71)
\end{aligned}
$$

Similarly, for the numerator of (3.70), an integration by parts gives

$$
\begin{aligned}
&(p-2)\int_0^\infty \frac{(g+1)^{-p/2}\mathrm{d}g}{\{1+w/(g+1)\}^{p/2+n/2}} - \frac{2}{(1+w)^{p/2+n/2}} \\
&= 2\int_0^\infty (g+1)^{-p/2+1}\Big\{\frac{w}{(g+1)^2}\frac{(p+n)/2}{\{1+w/(g+1)\}^{p/2+n/2+1}}\Big\}\mathrm{d}g \\
&= (p+n)w\int_0^\infty \frac{(g+1)^{-p/2-1}\mathrm{d}g}{\{1+w/(g+1)\}^{p/2+n/2+1}}. \qquad (3.72)
\end{aligned}
$$

By (3.70), (3.71) and (3.72), we have

$$
\phi_0(w) = w\frac{\int_0^\infty (g+1)^{-p/2-1}\{1+w/(g+1)\}^{-p/2-n/2-1}\mathrm{d}g}{\int_0^\infty (g+1)^{-p/2}\{1+w/(g+1)\}^{-p/2-n/2-1}\mathrm{d}g}, \qquad (3.73)
$$

which completes the proof of Theorem 3.14. □

By (3.68), we have

$$
J_{p,n}(w;0) = \frac{1}{\int_0^\infty (g+1)^{n/2}\{(1+w)/(1+w+g)\}^{p/2+n/2}\mathrm{d}g},
$$

which is decreasing in w and approaches 0 as $w \to \infty$. It then follows directly from the first line of (3.69) that

$$
\phi_0'(w) \geq 0, \ \lim_{w\to\infty} \phi_0(w) = (p-2)/(n+2),
$$

and hence the function $\phi_0(w)$ satisfies conditions (i) and (ii) of Theorem 3.14. It follows that the estimator associated with $\phi_0(w)$ is a minimax estimator improving on the James–Stein estimator. Further, comparing $\phi_0(w)$ with (3.36), we see that

$$
\Big(1 - \frac{\phi_0(\|X\|^2/S)}{\|X\|^2/S}\Big)X
$$

can be characterized as the generalized Bayes estimator under $\pi(g; a, b, c)$ in (3.35) with $a = b = c = 0$, or equivalently, the joint Stein (1974) prior given by (1.23),

$$\eta^{-1} \times \eta^{p/2}\pi_S(\eta\|\theta\|^2) = \eta^{-1} \times \eta^{p/2} \left\{\eta\|\theta\|^2\right\}^{1-p/2} = \|\theta\|^{2-p}, \tag{3.74}$$

where π_S is given by (1.14).

Additionally, by (3.73), $\phi_0(w) \le w$ and hence the the truncated function

$$\phi_{\mathrm{JS}}^{+} = \min\{w, (p-2)/(n+2)\}$$

corresponding to the James–Stein positive-part estimator

$$\hat{\theta}_{\mathrm{JS}}^{+} = \max\Big(0, 1 - \frac{p-2}{n+2}\frac{S}{\|X\|^2}\Big)X,$$

also satisfies conditions (i) and (ii) of Theorem 3.14, which implies that the James–Stein positive-part estimator dominates the James–Stein estimator, See Baranchik (1964) and Lehmann and Casella (1998) for the original proof of the domination.

It seems that the choice $a = b = c = 0$ in $\pi(g; a, b, c)$ is the only one which satisfies the conditions (i) and (ii) of Theorem 3.14. Recall, however, that we have concentrated on priors with $\nu = 1$ in (3.6) when deriving minimaxity and admissibility results in this chapter. As a choice of prior with $\nu \neq -1$ in (3.26), suppose the joint improper prior

$$\eta^{\alpha(n+p)/2-1} \times \int_0^\infty \frac{\eta^{p/2}}{(2\pi)^{p/2}g^{p/2}} \exp\Big(-\frac{\eta\|\theta\|^2}{2g}\Big)\frac{1}{(g+1)^{\alpha(p-2)/2}}\mathrm{d}g,$$

for $\alpha > 0$. The choice $\alpha = 0$ corresponds to the joint Stein prior (3.74). Then the generalized Bayes estimator is given by

$$\hat{\theta}_\alpha = \left(1 - \frac{\int_0^\infty (g+1)^{-(\alpha+1)(p/2-1)-2}\{1 + w/(g+1)\}^{-(\alpha+1)(p/2+n/2)-1}\mathrm{d}g}{\int_0^\infty (g+1)^{-(\alpha+1)(p/2-1)-1}\{1 + w/(g+1)\}^{-(\alpha+1)(p/2+n/2)-1}\mathrm{d}g}\right)x.$$

The following result is due to Maruyama (1999).

Theorem 3.15 (Maruyama 1999) *The generalized Bayes estimator $\hat{\theta}_\alpha$ for $\alpha > 0$ dominates the James–Stein estimator $\hat{\theta}_{\mathrm{JS}}$. Further $\hat{\theta}_\alpha$ approaches the James–Stein positive-part estimator $\hat{\theta}_{\mathrm{JS}}^{+}$ as $\alpha \to \infty$.*

Proof Appendix A.14. □

We do not know whether $\hat{\theta}_\alpha$, for $\alpha > 0$, is admissible within the class of equivariant estimators.

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