Alireza Abbasimoshaei
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Machine Dynamics

Kinematics and Dynamics

Springer

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Preface

Welcome to the world of dynamic machines! This book aims to provide a comprehensive understanding of the principles, analysis, and design of dynamic machines, covering a wide range of topics from displacement and velocity analysis to cams, gear trains, and balancing. Whether you are a student, an engineer, or a curious enthusiast, this book is designed to be your guide to unlocking the fascinating realm of machine dynamics.

The study of dynamic machines plays a vital role in various fields, including mechanical engineering, robotics, automotive systems, and many more. Understanding the behavior and performance of machines in motion is essential for optimizing their design, ensuring their reliability, and achieving desired functionality. Through this book, we delve into the fundamental principles and techniques that govern the dynamics of machines, enabling you to analyze, evaluate, and design dynamic systems with confidence.

The idea for this book was born in 2017. Originally intended as a supplement to my lectures, it was to fill a gap I had noticed: a book without a lot of text and with a sufficient number of practical examples to motivate students. It was also intended to help lecturers find easily useful examples with sufficient explanations. During my lectures, I learned that the areas to be covered in such a book had to be much more extensive than originally expected. As I outlined the topics, I realized that the book would be greatly enhanced by the contributions of specialists in various fields. Not only did the explicitly named authors contribute to the book, but also former and current colleagues were recruited for dedicated collaboration during the project. The initial inquiry with Springer, our preferred publisher, resulted in an impressively positive response and subsequently in constructive cooperation at all times.

Our journey begins with an introduction to the motivation behind studying machine dynamics, setting the stage for the subsequent chapters. We explore the definitions and types of motion, providing a solid foundation for understanding the principles and concepts that follow. The discussion then progresses to various components and mechanisms, such as levers, cams, gear trains, and balancing, which form the building blocks of dynamic machines. Each topic is carefully explained, accompanied by illustrations, examples, and practical applications to enhance your comprehension.

One of the strengths of this book lies in its balance between theoretical principles and practical applications. We have strived to strike a harmonious chord between theory and practice, allowing you to grasp the underlying concepts while appreciating their real-world implications. Theoretical derivations and mathematical analyses are presented in a clear and concise manner, supported by visual aids and diagrams to aid in visualization. Moreover, practical examples and case studies are included throughout the book, illustrating the application of the concepts in real-life scenarios.

As authors, our aim has been to create a valuable resource that not only imparts knowledge, but also ignites your curiosity and passion for dynamic machines. We have drawn upon our collective expertise and experience in the field to ensure that the content is accurate, up-to-date, and relevant. We hope that this book serves as a trusted companion, empowering you to unravel the intricacies of dynamic machines and inspiring you to explore new horizons in your own endeavors.

I would like to express my gratitude to the colleagues who have provided valuable feedback and suggestions during the development of this book. Their insights have undoubtedly enhanced its quality and clarity. I also extend my appreciation to the publishers and the editorial team for their support and guidance throughout the publication process.

We encourage you to embark on this dynamic journey with an open mind, a thirst for knowledge, and a desire to unravel the mysteries of machines in motion. May this book serve as a stepping stone, unlocking new perspectives, and empowering you to become proficient in the captivating realm of dynamic machines. This book can be used as a reference book in the field of dynamic machines. This book provides a comprehensive explanation of mechanical systems and a wide range of examples to familiarize students with the concepts and practical applications of various mechanical systems.

Thanks to the support of colleagues and my supervisors, especially Prof. Dr. Jochen Steil, Prof. Dr. Dardel, Prof. Dr. Farshid Najafi, and Prof. Dr. Majid Mohammadi Moghaddam, I was able to explore different design processes and find out what kinds of mechanisms and examples will be useful for students to write their thesis and implement the concept and useful designs in their projects. A big thanks also to Springer, Behzad Saeedi and Ege Hassürücü, for their support in preparing this book.

As a last word, I should thank the person who gave me a lot of support in all directions, Prof. Dr.-Ing. Thorsten A. Kern, to whom I owe a lot and who encouraged me to pursue and finish this book.

Best regards,

Hamburg, Germany Alireza Abbasimoshaei
June 2023

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Symbols

This list includes the most relevant symbols used throughout the book.

Symbol	Description	Unit
a	acceleration	$\frac{m}{s^2}$
A	area	m^2
α	angle, Euler rotation (around the x-axis)	degree, radian
β	angle, Euler rotation (around the y-axis)	degree, radian
c	spring constant	–
d	damping/friction	$\frac{N}{m \cdot s}$
d	distance, diameter	m
F	mechanism DoF	–
f_{tot}	sum of all joint degrees of freedom of a mechanism	–
$f_{i,\ldots,g}$	degree of freedom of the ith joint in a mechanism	–
f_{id}	sum of identical condition	–
f_{id}	sum of all identical links in a mechanism	–
F	bearing-/movement-DOF of a mechanism	–
F	force	–
ΔF	force difference	N
ϕ	roll angle, rotation (around z-axis)	degree, radian
φ	angle	degree
g	number of joints in a mechanism	–
g	number of joints	–
γ	angle, Euler rotation (around the z-axis)	degrees, radians
h	height	m

(continued)

Symbol	Description	Unit
I	moment of inertia	m^4
j, i	imaginary unit, $i = \sqrt{-1} \in \mathbb{C}$	–
k	spring constant, mechanical stiffness	$N\,m^{-1}$
k	number of chains in a mechanism	–
l	length	m
m	mass	kg
M	torque	Nm
n	number of bodies	–
ω, Ω	angular velocity	$\frac{rad}{s}$
ψ	yaw angle, rotation around x-axis	degree, radian
r	distance, radius	m
t	time/point in time	s
τ	torque	Nm
θ	pitch angle, rotation about the y-axis	degree, radians
v	velocity	$m\,s^{-1}$
x	distance, displacement, translation, position	m
$\mathbf{x} = (x, y, z)$	Cartesian coordinates	–
Δx	position displacement	m

Indices and Distinctions

The usage of the most relevant indices and distinctions used throughout the book is shown using the replacement character ■.

Index	Description
$\blacksquare_0$	base or reference value
$\blacksquare_E$	referring to the real or VR environment
$\blacksquare_{max}$	maximum value
$\blacksquare_{min}$	minimum value
$\blacksquare_{rot}$	referring to a rotational value
$\delta\blacksquare$	small change, differential
$\Delta\blacksquare$	discretized element
$\blacksquare(t)$	time-depending value
$\dot{\blacksquare}$	derivative with respect to time

Chapter 1
Motivation and Introduction to Machine Dynamics

This chapter focuses on providing an introductory overview of machine dynamics. The chapter begins with an introduction, setting the stage for understanding the importance and relevance of machine dynamics in the study of mechanical systems. Then key terms and concepts are defined, laying the foundation for further exploration. These definitions help establish a common understanding of the terminology used in machine dynamics.

Next, the chapter discusses different types of motion, giving readers an understanding of the various ways machines can move. It also introduces the concept of levers and categorizes them into different types based on their characteristics and functionality.

Furthermore, the concept of degree of freedom is explained, highlighting its importance in analyzing and understanding the motion capabilities of mechanical systems. Additionally, the chapter provides an overview of the four-bar linkage, a commonly used mechanism, discussing its structure, functionality, and applications.

Overall, this chapter serves as a comprehensive introduction to machine dynamics, providing readers with the necessary background and motivation to delve deeper into the subject and explore the dynamics of mechanical systems.

1.1 Introduction

Mechanics is a part of experimental science related to motion, force, and time. Mechanics divides into two parts: one part is static, that time does not affect it, and the other part is dynamic, that time has some effect on it. Dynamics itself consists of kinematic and kinetic, namely motion analysis and force analysis. These two analyses are usually performed together. But in machine dynamics with the assumption of rigid components, these two analyses can be performed separately.

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Machine dynamics is the study and analysis of the forces acting on the components of a machine and the movements resulting from these forces.

Movement in machines creates forces. Unwanted forces disrupt machine movements or failure and breakdown of its components. The analysis of these forces and the reduction of unwanted forces are issues addressed in machine dynamics. To analyze the forces caused by motion, we first need to analyze the motion itself, regardless of its cause and the kinematics of the mechanism or machine. After the kinematic analysis, the forces are analyzed that is the kinetic analysis of the mechanism, and finally, a solution is thought to reduce or balance the forces.

This chapter introduces the different mechanisms and explains the basic concepts. In the following, we will discuss other topics.

1.2 Definitions

In exploring mechanisms, we frequently come across expressions such as mechanism, machine, interface, etc., whose definitions are as follows.

A mechanism is a combination of rigid and resistant objects interconnected to have a specific relative motion to each other. Crankshaft, piston, and piston assemblies in an internal combustion engine are examples of a mechanism.

A machine is a mechanism or a set of mechanisms that transmit power from a power source to the consumer or the ultimate mechanical resistance. The internal combustion engine is an example of a machine.

A lever or interface is the simplest member of a mechanism. By attaching it to adjacent members to move relative to each other, a specific task or action is performed. The joint is the junction of two levers that can move relative to each other. A kinematic diagram is a diagram in which the dimension or dimensions of an effective lever in a particular mechanism motion are plotted. Figure 1.1 shows the kinematic diagram of an internal combustion engine.

When a kinematic lever diagram is drawn, no information about the actual shape is available, so any point on the page can be a part of that lever.

The geometric shape of the relationship between two members of a mechanism that leads to the relatively consistent motion of the two members is called a pair. If the connection of two members has a contact surface like a pin connection, this

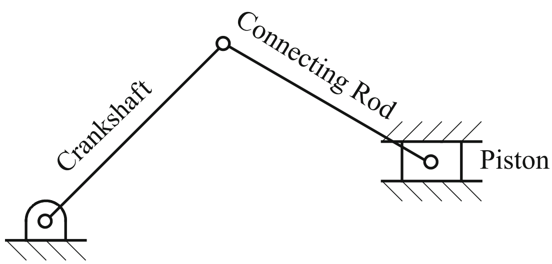


Fig. 1.1 Kinematic diagram of an internal combustion engine

connection is called a lower pair. If the connection is made at a point or along a line, the connection is known as a higher pair. Motion linkage in bearings and contact of two gear teeth are examples of higher pairs. A pair that only allows relative rotation is called a turning or rolling pair, and a pair that only allows sliding is called a sliding pair. A rolling pair is a higher or lower pair, depending on having a pin connection or ball bearing. A sliding pair is a lower pair, like the motion of a piston on a wall of a cylinder.

A kinematic chain is obtained by connecting some levers with some pairs. If the connection of these levers eliminates the possibility of movement, then a locking chain (structure) appears. If the levers are connected so that the relative motion of the levers is always the same regardless of the number of cycles completed, the resulting chain is called a bound chain. In addition, the levers can be connected in a way that an unbound chain is created, meaning a chain in which the pattern of movement changes depending on the amount of friction in the joints at any moment. If one of the levers of a bound chain is fixed, the resulting set will be a mechanism.

Note A mechanism (bound cinematic chain) may consist of several cinematic chains, with the output of the first chain being the input of the second chain, and so on.

If a fixed lever is allowed to move in a mechanism and another lever becomes fixed instead, the resulting mechanism is called reversing the previous mechanism.

Note In reversing a mechanism, the relative motion of the levers does not change, but the absolute motions (relative to the ground) change.

When the components of a mechanism start moving simultaneously from a given position set and return to the original position set after passing through all possible positions, these components have gone through a movement cycle. The time required to complete a movement cycle is called periodic time. The set of situations that the components of the mechanism pass through simultaneously at a particular moment of motion is called a state (phase) of motion.

The ability to move or the number of degrees of freedom is one of the most fundamental concepts of cinematic science. By definition, the mobility of a mechanism is the minimum number of independent parameters required to determine the position of each lever of the mechanism.

In studying the mechanisms, it is necessary to examine how the motion is transferred from one member to another.

Movement is transmitted in three ways:

a) Direct contact between two members, such as the contact between a cam and a follower or two gears
b) Transmission via an intermediate lever or a connecting rod
c) By a flexible intermediate such as a belt or chain

1.3 Types of Motion

The change in position of an object (material particle or rigid object) relative to another object over time is called motion.

1.3.1 Plane Motion

An object moves in a plane if all its points move in planes parallel to a base plane. Movement on the plane can be one of three types: transition, rotation, and a combination of transition and rotation:

- If the object moves in a way that all the straight lines located on it always have parallel positions to each other, the object will have a transfer. There are two types of transmission: direct line transfer and curve line transfer, which refer to the movement along a straight line and a curved line, respectively.
- In rotation, the distance of all the points on the object will remain constant relative to the line perpendicular to the plane of motion.
- Many objects have a motion composed of rotation and transmission simultaneously.

1.3.2 Spiral Movement

A point that rotates at a constant distance from an axis and moves simultaneously along this axis has a helical motion, such as the movement of a nut along a screw.

1.3.3 Spherical Motion

If the motion of an object is such that each point moves at a fixed distance around a fixed point, the object will have a spherical motion.

1.3.4 Spatial Motion

If an object rotates around three non-parallel axes and moves in three independent directions, it has general spatial motion.

1.4 Types of Levers

There are four types of levers in a mechanism:

- The base lever, to which the reference coordinate is attached and the mechanism's movement is analyzed relative to it (ground)
- The input lever, which is usually connected to the base lever, and kinematic quantities are given to it
- The output lever, which is often connected to the base lever and from which kinematic quantities are taken
- The interface lever, which acts as the interface between the input and output levers

In terms of oscillation range (movement), we also have two types of levers:

- Crank lever, which can rotate 360 degrees during movement
- The oscillator or rocker lever, which can travel part of a circular path but not the whole circle during movement, meaning it oscillates at an angle of fewer than 360 degrees

For example, in Fig. 1.2, for full rotation of lever two or crank, lever four or oscillator oscillates only in the range of $\Delta\theta$.

Also, in another classification, levers are divided into two types, simple and compound:

- If the lever has a maximum of two joints, it is called a simple lever.
- If the lever has more than two joints, it is called a compound lever.

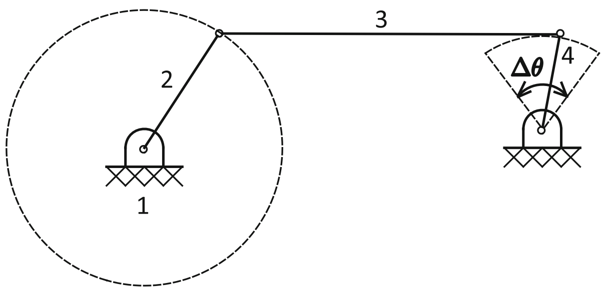


Fig. 1.2 A normal four linkage mechanism

1.5 Connection and Its Types

A connection or joint is between two levers, through which motion or force is transmitted from one lever to another. Joints allow relative movement in some directions while restricting movement in other directions. The types of allowed movements depend on the freedom degrees of the joint. Freedom degrees of the joint are equal to the independent coordinates needed to determine the unique position of one part relative to other parts bound by the joint. In plane mechanisms, four types of joints are typically distinguishable: rotary (or hinged or pin), sliding (or linear or prismatic), rolling (each with one degree of freedom), and camshaft or sprocket (with two degrees of freedom). Figure 1.3 shows these joints.

The rotary joint allows only a relative angular movement to the two interfaces, while in a sliding joint, there is only a relative transition motion between the two

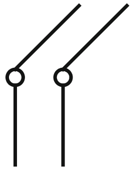
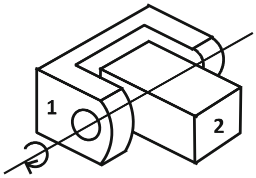
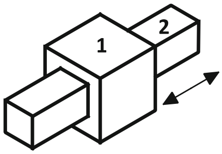
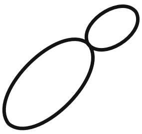
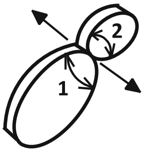
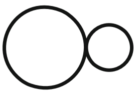
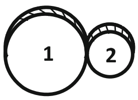
Freedom Degrees	Schematic Shape	Actual Shape	Joint Type (Symbol)
1 DOF (Just Rotation)			Rotary (R)
1 DOF (Just Sliding)			Prismatic (Linear) (P)
2 DOF (Rolling and Sliding)			Camshaft or Sprocket
1 DOF (Rolling without Sliding)			TRolling Touch

Fig. 1.3 Different joint types

interfaces. In one definition, if two objects do not have angular velocities relative to each other, the sliding motion is complete.

Note In a direct contact mechanism, sliding occurs when the point of contact of two levers is located along their centerline.

Note To have a joint and rolling contact, the linear velocities of the members must be the same at the point of contact with each other, and the point of contact must be located on the centerline.

Note Locating the point of contact on the centerline is necessary, but it is not enough because the joint may only be rolling at one moment and not at other times.

If there are rolling and sliding conditions in a joint, it is called a camshaft or sprocket or sliding–rolling joint. Then, two quantities are needed to determine the position of one lever relative to the other.

The relative motion of two interfaces by three degrees of freedom connections will be three types of motion, angular, transitional, or a combination of the two of them. These connections are for spatial or three-dimensional mechanisms. Figure 1.4 shows some common connections with three degrees of freedom.

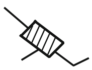
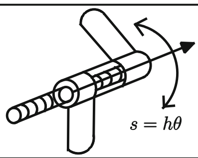
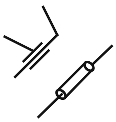
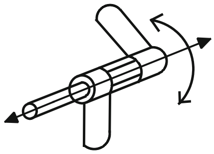
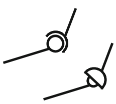
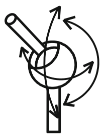
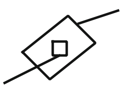
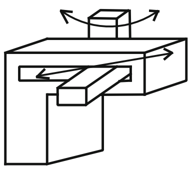
Name	Schematic Symbol	Practical Example	Connection Ability (Degrees of Freedom)	Text Symbol
Screw Joint Helical Joint Screw Pair			2	***H***
Cylindrical Joint Cylindrical Pair			2	***C***
Spherical Joint Ball and Socket joint Spherical Pair			3	***S***
Planar Joint Planar Pair			3	P_L

Fig. 1.4 Some common three degrees of freedom connections

1.6 Degree of Freedom

A rigid body in a planar motion (two-dimensional) can have a maximum of three degrees of freedom in motion, which are two transient movements in horizontal and vertical directions and one rotational movement around the axis perpendicular to the plane of motion. When a set of rigid members are connected, the members' connection to each other causes them to be bound and reduces the number of degrees of freedom of the system.

Note Since the system is usually connected to the ground, the ground is also considered as a member of the system.

Suppose the system has n members considering the ground, assuming that the system has a planar movement. In that case, the maximum possible degree of freedom for the system is $3(n-1)$ because all members except the ground have the possibility of three independent movements. On the other hand, each joint with one degree of freedom takes two degrees of freedom of the system. If we have f_1 joints with one degree of freedom, the degree of freedom of the system is reduced by $2f_1$. Similarly, each joint with two degrees of freedom takes one degree of freedom of the system, and if we have f_2 joints with two degrees of freedom, the degree of freedom of the system is reduced by f_2. Thus, the mobility of the set is determined as follows from the Gruebler's equation:

$$DOF = 3(n-1) - 2f_1 - f_2 \tag{1.1}$$

where DOF is the ability to move or the number of degrees of freedom and n is the total number of levers, including the ground lever.

Example What is the number of degrees of freedom of Fig. 1.5 mechanism?

1) 2 2) 3 3) 5 4) 6

Solution By considering the bulldozer chamber as ground (member 1), the mechanism has 12 members and 15 connections with one degree of freedom. Therefore, according to Gruebler's equation for $n = 12$, $f_1 = 15$ and $f_2 = 0$, we have

$$DOF = 3(n-1) - 2f_1 - f_2 = 3 \times (12-1) - 2 \times 15 - 0 = 3$$

Note that members (3 and 4), (5 and 6), and (8 and 9) of the cylinder and piston are the bulldozer actuator jacks, which control the 3 degrees of freedom of the mechanism.

Option (2) is correct.

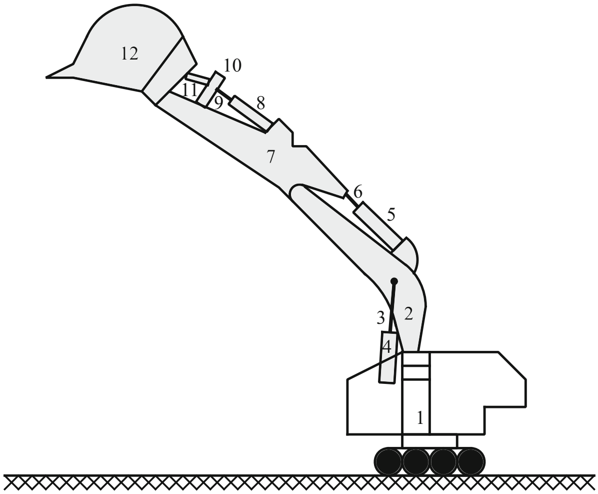


Fig. 1.5 Bulldozer arm mechanism

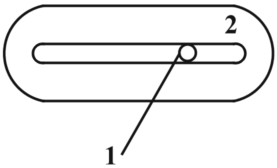


Fig. 1.6 Slider joint

If a point on one member is restricted to move within the groove created in the other member, it forms a connection with two degrees of freedom. In Fig. 1.6, the end point of member 1 is bound to move inside the groove of member 2.

In this system, if we keep member 2 fixed, member 1 can rotate around the connection point, and the connection point itself can be moved inside the groove, meaning two degrees of freedom remain.

Note Springs and jacks do not affect the degree of freedom of the system and can be removed.

Example What is the degree of freedom of the mechanism shown in Fig. 1.7?

1) 1 2) 2 3) 3 4) 4

Solution According to the stated note, spring and jack do not affect the degree of freedom. By removing these two members from the mechanism, a set of 5-rod levers is obtained with 4 rotating hinges and a sliding joint (which are joints of one degree

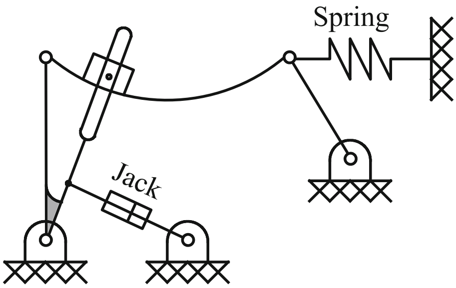


Fig. 1.7 A combined mechanism

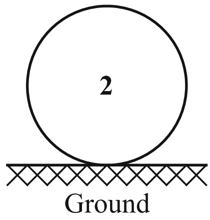


Fig. 1.8 A roller on the ground

of freedom), and a rolling and sliding joint (a joint of two degrees of freedom). Thus:

$$n = 5\ ,\ f_1 = 5\ ,\ f_2 = 1$$

Using the Gruebler's criterion, we have

$$\mathrm{DOF} = 3\,(n-1) - 2f_1 - f_2 = 3 \times (5-1) - 2 \times 5 - 1 = 1$$

Option (1) is correct.

In camshaft or contact connections where pressure is possible between the contact points of the two members (common mode), if there is no slide between the two members and we have a pure rolling mode, the contact connection will be one degree of freedom because it only allows rolling between two levers. But if the members can slide on top of each other, the connection will be two degrees of freedom because, in addition to rolling, a degree of freedom caused by sliding is also added to the system. For example, in Fig. 1.8, if member 2 moves on the ground without sliding (just rolling), the connection will be one degree of freedom. If, in addition to rolling, member 2 can slide on the ground, the connection will be two degrees of freedom.

Example In Fig. 1.9, if the wheels move in two horizontal directions without sliding, this system:

1) Is a mechanism of one degree of freedom?
2) Is a chain of two degrees of freedom?

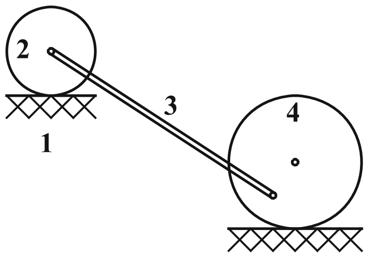


Fig. 1.9 Two wheels on two surfaces

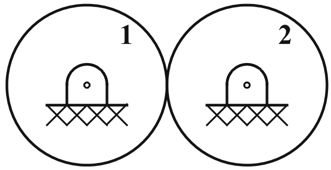


Fig. 1.10 Two contact members without pressure

3) Is a structure?
4) Is a chain system without constraint?

Solution According to the figure, all constraints are of one degree of freedom because we have two hinged joints and two points of contact with the ground that form two rolling joints (without sliding). On the other hand, including the ground, the system has four members, and we have

$$n = 4\ ,\ \ f_1 = 4\ ,\ \ f_2 = 0$$

From the Gruebler's equation, we have

$$\text{DOF} = 3\,(n - 1) - 2f_1 - f_2 = 3 \times (4 - 1) - 2 \times 4 - 0 = 1$$

Option (1) is correct.

If, as in Fig. 1.10, there is no possibility of pressure between the two contact members, meaning discs 1 and 2 are not able to move horizontally and push each other because they are hinged to the ground in the center, in this case, if there is rolling contact, we will have a connection of two degrees of freedom that only takes the possibility of sliding (degree of freedom is limited). Still, if the contact is sliding, the connection does not create any restrictions, and in fact, bounding one of the disks cannot prevent the other disk from moving, meaning this type of connection is not considered a constraint.

Note If a joint connects k levers at one point, it must be considered $(k - 1)$ joints.

Example In Fig. 1.11 mechanism for speed analysis:

1) Only having ω_2 is enough.
2) In addition to ω_2 value, ω_9 or V_D is also needed.

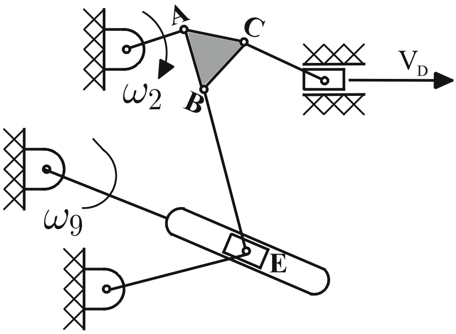


Fig. 1.11 Combination of sliders and rotational joints

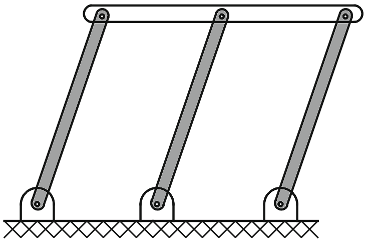


Fig. 1.12 More than two parallel mechanism levers

3) In addition to ω_2 value, ω_9 is also needed.
4) In addition to ω_2 value, ω_9 and V_D are also needed.

Solution The system shown has 9 members and 11 joints of one degree of freedom. In counting the joints according to the stated note, the rotating joint E that connects three levers should be considered as 2 joints.

$$n = 9\ ,\ \ f_1 = 11\ ,\ \ f_2 = 0$$

$$\text{DOF} = 3\,(n-1) - 2f_1 - f_2 = 3 \times (9-1) - 2 \times 11 - 0 = 2$$

Therefore, the degree of freedom of the system is 2 and indicates that two inputs are needed for speed analysis, and given that ω_2 is known, another degree of freedom independent of ω_2, such as ω_9 or V_D, is needed.

Option (2) is correct.

Gruebler's equation must be used with caution, as it does not apply to several specific mechanisms. These exceptional cases usually occur when more than two mechanism levers are parallel. Figure 1.12 shows one of these cases.

In Fig. 1.12,$n = 5$, $f_1 = 6$, $f_2 = 0$. By applying the Gruebler's equation to this mechanism, we have

$$\text{DOF} = 3\,(5-1) - 2(6) = 0$$

So it results in zero degrees of freedom. But the degree of freedom of this mechanism is one.

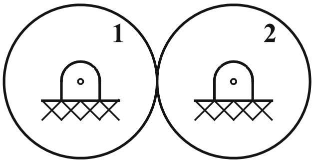


Fig. 1.13 Rollers without pressure

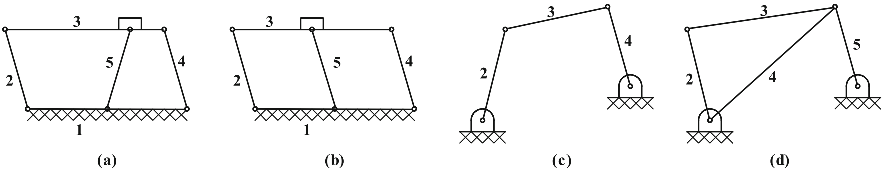


Fig. 1.14 Some normal four linkage mechanisms

Also, for the mechanism of Fig. 1.13 with the assumption of rolling between the two disks $n = 5$ and $f_1 = 6$ and from the Gruebler's equation, we have

$$\mathrm{DOF} = 3\,(3 - 1) - 2(3) = 0$$

But the system can move and has one degree of freedom.

Note If the mobility (degree of freedom) is zero or negative, we have a structural set.

Note If the mobility is zero, the structure is statically determinate, and if the mobility is negative, the structure is statically indeterminate.

Example Which option correctly describes types of levers in Fig. 1.14?

$$1)\ \begin{cases} \text{a} - \text{structure} \\ \text{b} - \text{mechanism} \\ \text{c} - \text{mechanism} \\ \text{d} - \text{structure} \end{cases} \qquad 2)\ \begin{cases} \text{a} - \text{structure} \\ \text{b} - \text{structure} \\ \text{c} - \text{mechanism} \\ \text{d} - \text{mechanism} \end{cases}$$

$$3)\ \begin{cases} \text{a} - \text{mechanism} \\ \text{b} - \text{mechanism} \\ \text{c} - \text{structure} \\ \text{d} - \text{structure} \end{cases} \qquad 4)\ \begin{cases} \text{a} - \text{mechanism} \\ \text{b} - \text{structure} \\ \text{c} - \text{structure} \\ \text{d} - \text{mechanism} \end{cases}$$

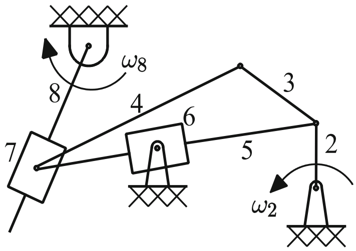


Fig. 1.15 A mechanism with two sliders

Solution Lever (b) is one of the exceptions for which the Gruebler's equation is not true, and as stated, this lever has one degree of freedom and is, therefore, a mechanism. For other levers, we have

$$\mathrm{a} := 5,\ f_1 = 6,\ f_2 = 0 \implies \mathrm{DOF} = 0$$

$$\mathrm{c} := 4,\ f_1 = 4,\ f_2 = 0 \implies \mathrm{DOF} = 1$$

$$\mathrm{d} := 5,\ f_1 = 6,\ f_2 = 0 \implies \mathrm{DOF} = 0$$

Therefore, lever (c) that has a degree of freedom greater than zero is a mechanism, and levers (a) and (d) are structures. Note that in counting constraints for the lever (d), the joints that connect three levers at one point should be considered as two joints.

Option (1) is correct.

Note If the number of inputs of a lever is less than the number of degrees of freedom, the movement of the lever is unpredictable, and if the number of inputs is more than the number of degrees of freedom, the lever is locked in part of its movement.

Example Which statement is correct about Fig. 1.15?

1) The lever has one unpredictable movement.
2) For input ω_2, the set has one degree of freedom.
3) The lever will be locked in part of its movement.
4) The set will have two different movements based on ω_2 or ω_9.

Solution For this lever, we have $n = 8$, $f_1 = 10$, $f_2 = 0$. It should be noted that the junction of the members (2, 3, 5) and also (4, 5, 7) each count as two joints. So, according to the Gruebler's equation, we have

$$\mathrm{DOF} = 3\,(n-1) - 2f_1 - f_2 = 3 \times (8-1) - 2 \times 10 = 1$$

Therefore, the set will have one degree of freedom for one entry of ω_2. According to the stated point, if the number of inputs is more than one, the set will be locked in a part of the movement.
Option (2) is correct.

1.7 Four-Bar Linkage

One of the simplest and most common mechanisms is the four-bar linkage. The design of this lever can be seen in Fig. 1.16. Lever 1 is frame or ground in this form, which is usually fixed. Lever 2 is the actuator, and its movement may be full rotation or oscillation. If lever 2 is in full rotation, the mechanism converts the rotational movement into an oscillating movement. If the crankshaft oscillates, the mechanism will create oscillating movement.

Note When lever 2 has a full rotation, there is no risk of locking the linkage; otherwise, care must be taken in choosing the ratio of lengths so that the mechanism is not located at the dead points and does not stop moving at the end positions.

Dead points are created when the line of action of the driving force is along with lever 4. The dashed line shows this position in Fig. 1.17.

Note If the four-bar mechanism is designed so that lever 2 has a full rotation and lever 4 is actuated, it will create dead points, and to pass the mechanism through these points, it must be equipped with a flywheel.

A four-bar linkage may take different positions, as shown in Fig. 1.18.

The movement of the four-bar linkage is often expressed by the terms crank and oscillator, indicating that crank 2 has a full rotation and lever 4 oscillates according to Fig. 1.18A. Same way, the term double crank means that both levers 2 and 4 have a full rotation, as shown in Fig. 1.18B and C. The term double oscillator indicates that both levers 2 and 4 oscillate, as shown in Fig. 1.18D.

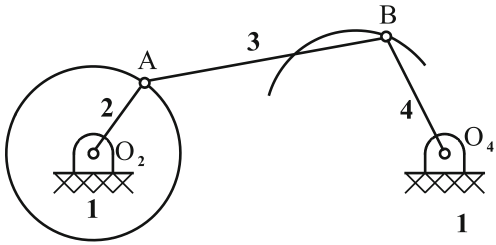


Fig. 1.16 A normal four-bar linkage

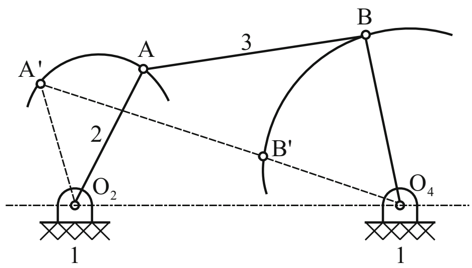


Fig. 1.17 Position for deadpoint creation

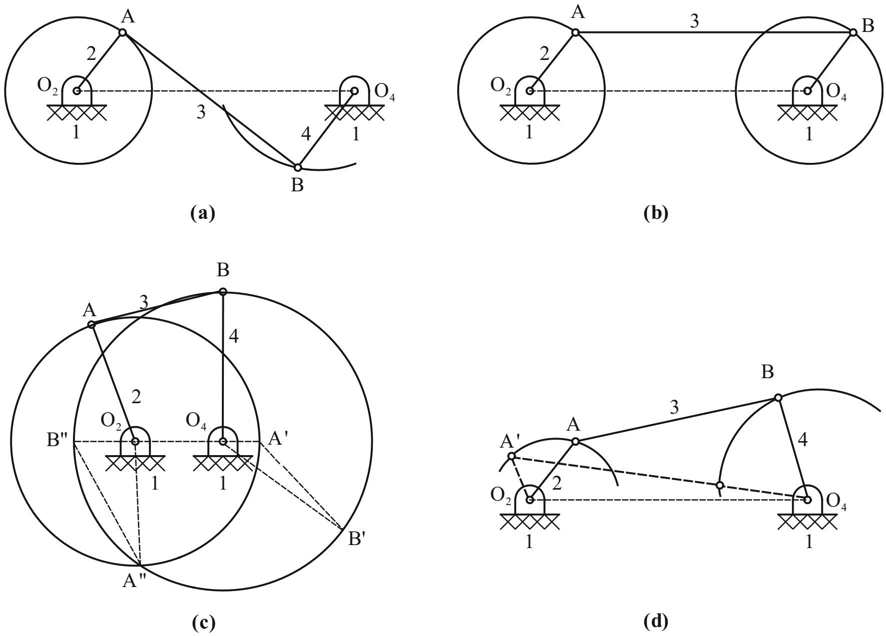


Fig. 1.18 Different positions of a four-bar linkage

Grashof's law can determine the mode in which the four-bar linkage operates: crank and oscillator, double crank, or double oscillator. This rule states that if the sum of the lengths of the longest and shortest levers is less than the sum of the lengths of the other two levers, then:

- If the shortest lever is a crank lever and one of the adjacent levers is fixed, two different crank and oscillator mechanisms are obtained.
- If the shortest lever is a fixed lever, a double crank mechanism emerges.
- If the lever facing the shortest lever is a fixed lever, the mechanism will be a double oscillator.

Also, suppose the sum of the lengths of the longest and shortest levers is greater than the sum of the lengths of the other two levers. In that case, only double oscillator mechanisms will be obtained. Also, if the sum of the lengths of the longest and shortest levers is equal to the sum of the lengths of the other two levers, the possible modes will be similar to the above. But in this case, the centerline of the levers may be aligned and change the direction of rotation of the moving lever unless this change is somehow prevented. The lever of Fig. 1.18B is such that the levers are aligned along the O_2O_4 centerline. In this case, if the set inertia does not cause lever 4 to pass through this point, the direction of rotation of the moving lever might reverse.

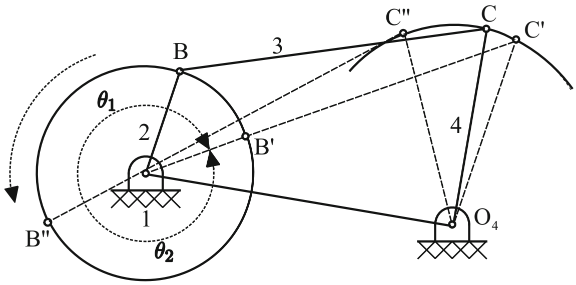


Fig. 1.19 A crank-pendulum bar mechanism

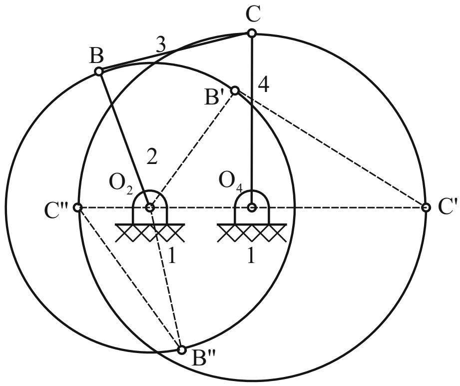


Fig. 1.20 A mechanism with double rotary cranks

To move a crank-pendulum bar mechanism as Fig. 1.19, these relationships must be true:

$$\begin{aligned} O_2B + BC + O_4C &> O_2O_4 \\ O_2B + O_2O_4 + O_4C &> BC \\ O_2B + BC - O_4C &< O_2O_4 \\ BC - O_2B + O_4C &> O_2O_4 \end{aligned} \tag{1.2}$$

For a mechanism with double rotary cranks or crank–crank, according to Fig. 1.20, the ratio of the length of the bar is as follows:

$$\begin{aligned} BC &> O_2O_4 + O_4C - O_2B \\ BC &< O_4C - O_2O_4 + O_2B \end{aligned} \tag{1.3}$$

These equations can be proved according to triangles $O_2B^{'}C^{'}$ and $O_2B^{''}C^{''}$.

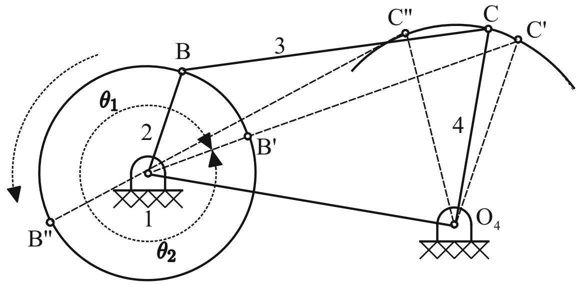


Fig. 1.21 A normal four-bar mechanism

Example Which of the following equations is not a prerequisite for the operation of the shown four-bar mechanism in Fig. 1.21?

1) $O_2B + O_2O_4 + O_4C > BC$ 2) $O_2B + BC - O_4C < O_2O_4$
3) $O_2O_4 + O_4C - O_2B < BC$ 4) $BC - O_2B + O_4C > O_2O_4$

Solution It is clear that equation (1.3) is not correct.

In a quadrilateral, the sum of the lengths of the three sides is always greater than the fourth side, and therefore equation (1.1) is correct. In this mechanism, the sum of the longest and shortest levers (BC and O_2B, respectively) is less than the sum of the other two levers, and equation (1.2) is also correct.

But for equation (1.3), since BC and O_2B are the longest and shortest levers, respectively, we have

$$\begin{pmatrix} BC > O_2O_4 \\ O_4C > O_2B \end{pmatrix} \Rightarrow BC + O_4C > O_2O_4 + O_2B$$

Which is the same as equation (4), so this equation is also correct.

Option (3) is correct.

1.8 Equivalent Mechanisms

When acceleration analysis of a direct contact mechanism is desired, the problem can be simplified by replacing an equivalent multi-rod mechanism. An equivalent mechanism is a mechanism whose members' angular velocity and acceleration are instantly equal to the angular velocity and acceleration of the members of the main mechanism.

We proceed as follows to find an equivalent mechanism to a direct contact mechanism (Fig. 1.22), such as camshaft and follower. Assume that points P_2 and P_3 are the points of contact between members 2 and 3, and C_2 and C_3 are also the centers of curvature of members 2 and 3 at point P, respectively. We only need to connect the centers of curvature of two members at point P (connection of points C_2

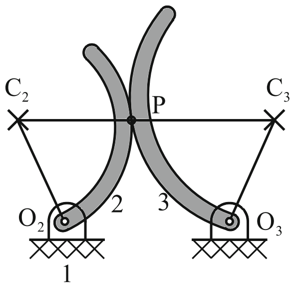


Fig. 1.22 A direct contact mechanism

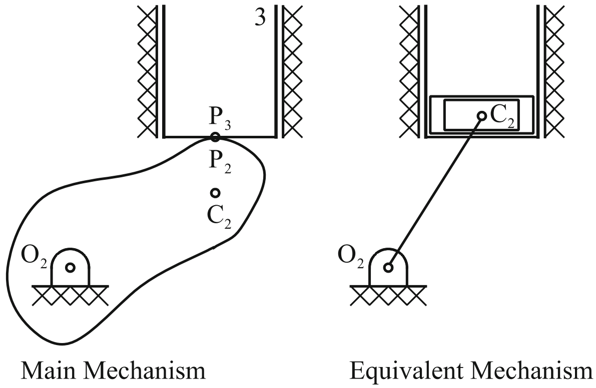


Fig. 1.23 Camshaft and follower

and C_3) and then draw a line segment from the center of rotation of each object to the center of its curvature (connection of O_2 to C_2 and O_3 to C_3). Thus the equivalent mechanism to the main mechanism is created.

Although the camshaft always rotates, the follower may move back and forth along a straight line instead of rotating (as shown in Fig. 1.23). In this case, we can create an equivalent mechanism by replacing the slider located in the center of curvature of path P_3 on member 2 and moving inside the groove on the equivalent of member 3.

Note The direction of the groove should be parallel to the direction of the straight line of the slider's path.

Note The slider is located at the center of the curvature of the bent member, not at the point of contact with the member with a straight line.

Example Figure 1.24 shows a three-bar direct contact mechanism. Which option is the four-bar mechanism equivalent to it with only low-level connections?

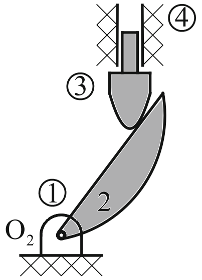
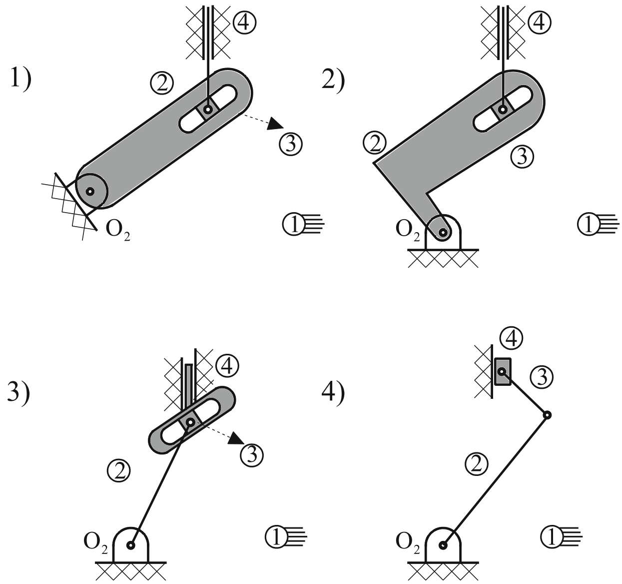


Fig. 1.24 A three-bar direct contact mechanism



Solution If P_2 and P_3 are the points belonging to members 2 and 3 at the point of contact, respectively, the path of P_3 on member 2 is a straight line. Simply place the intermediate slider member inside member 2 in the groove parallel to the corresponding straight line and at the center of curvature of P_2 path on member 3, the point equivalent to C_3. Note that the slider always moves on the member that has a straight line, so options (3) and (4) are wrong. On the other hand, since the location of the slider is at the center of curvature of the P_2 path on member 3 and not at point P_3, option (1) is also wrong.

Option (2) is correct.

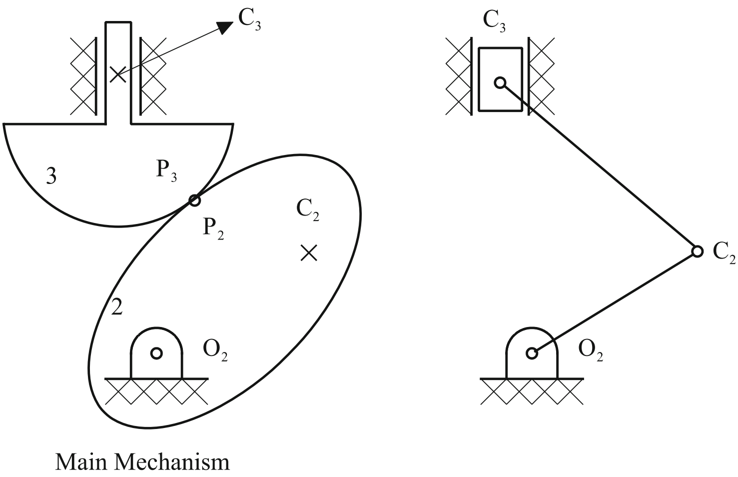


Fig. 1.25 Equivalent mechanism with one slider

Sometimes both surfaces in contact with each other are curves, but since one of the members is bound to a reciprocating motion, a slider must be used to provide this motion in the equivalent mechanism. In this case, unlike the previous case, the slider does not play the role of an intermediary member but is considered one of the members. Figure 1.25 shows this mode.

Note If the surfaces in contact are both circles, or one of them is a circle, and the other is a straight line (a circle with infinite radius), the equivalent mechanism is no longer instantaneous and is permanent.

Note Mechanisms are equivalent so that objects do not have relative velocity toward each other, and the Coriolis acceleration is zero.

Some Examples of "Motivation and Introduction to Machine Dynamics"

1. In the mechanism of Fig. 1.26, how many inputs are needed if we want the mechanism to move from the existing state to the new state in which the output block P has been moved to the right by one unit?

   1) 1 2) 2 3) 3 4) 4

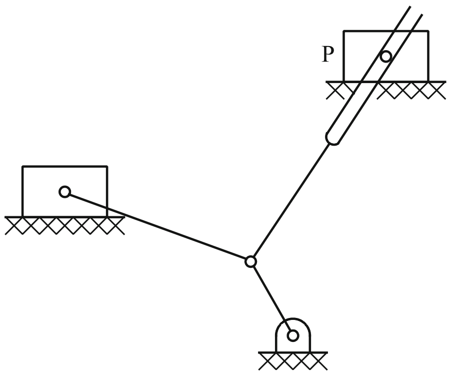


Fig. 1.26 A mechanism with three sliders

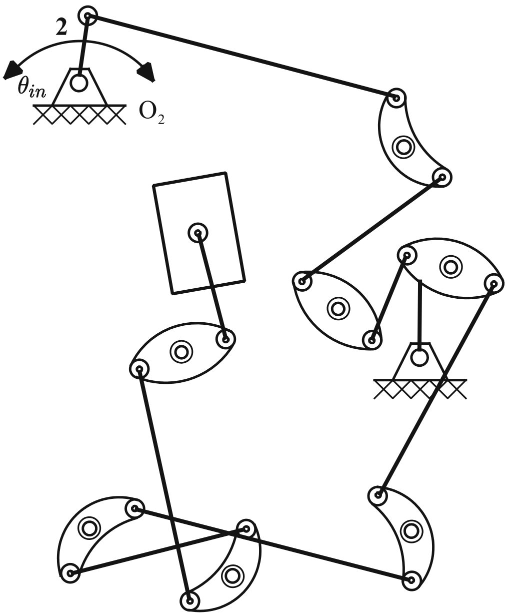


Fig. 1.27 Room lighting switch

2. Input movement is provided for the eighteen-bar mechanism of connecting and disconnecting the room lighting switch through lever 2 (Fig. 1.27), which is connected to the circular movement of the room door. Which statement about the degree of freedom of this mechanism is appropriate? (Two of the levers can be adjusted.)

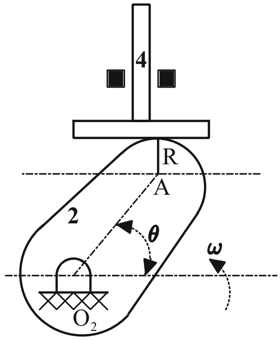
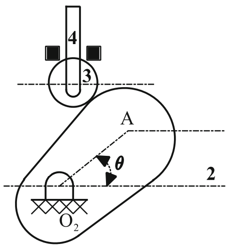


Fig. 1.28 Camshaft mechanism

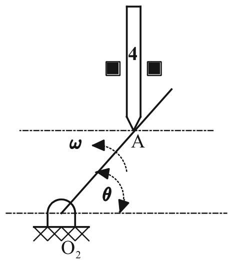
1) After adjustment, the degree of freedom of the mechanism is always one.
2) The degree of freedom of the mechanism depends on the adjusted conditions.
3) The degree of freedom of the mechanism can be more than one.
4) The degree of freedom of the mechanism varies between one and two.

3. Determine the mechanism equivalent to the camshaft mechanism in Fig. 1.28.

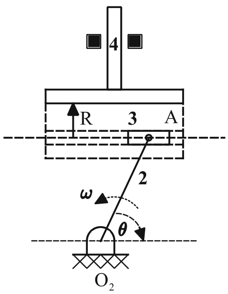
1)



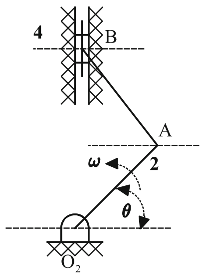
2)



3)



4)



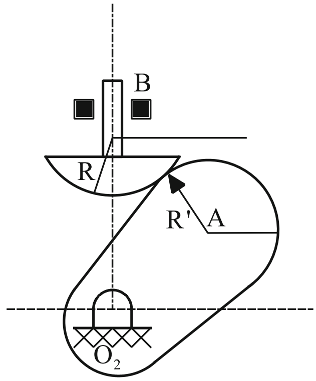
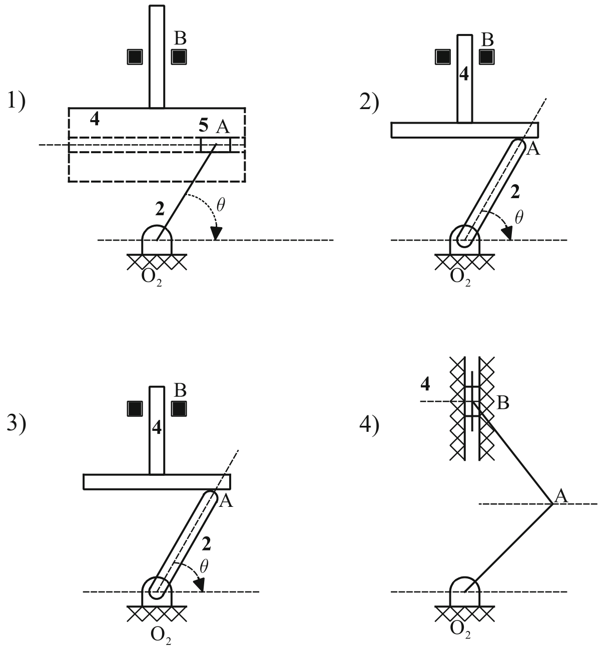


Fig. 1.29 Camshaft mechanism

4. Determine the mechanism equivalent to the camshaft mechanism in Fig. 1.29.



5. In the following mechanism in Fig. 1.30:

    1) Velocity C can only be obtained if V_A is known.
    2) If V_A and V_B are known, velocity C can be obtained.
    3) In addition to V_A, ω_2 must also be known to obtain V_C.
    4) Knowing V_A and V_B is not enough to determine V_C.

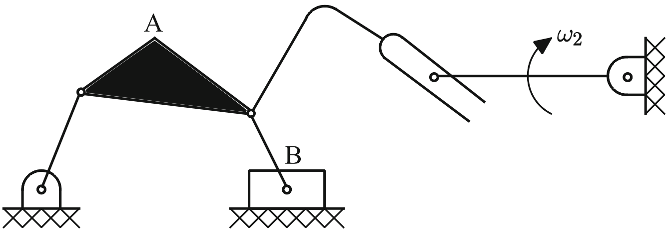


Fig. 1.30 Mechanism with two sliders

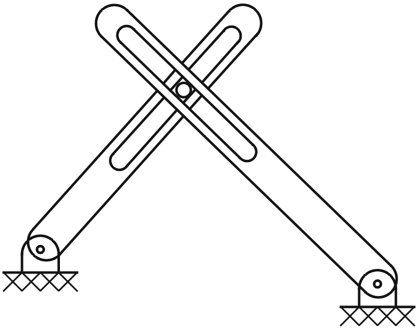


Fig. 1.31 A pin connecting two grooved bars

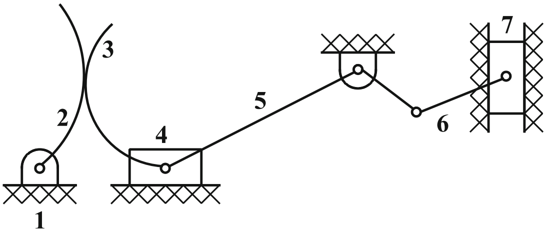


Fig. 1.32 Mechanism to be situated in a desired position

6. A pin connects two grooved bars with the ability to slide on both bars (Fig. 1.31). What is the degree of freedom of the mechanism?

   1) −1 2) 0 3) +1 4) +2

7. For the mechanism in Fig. 1.32 to be situated in the desired position, ...

   1) Bar 2 must be given a specific movement.
   2) This mechanism cannot have a movement.
   3) Bar 2 and slider 7 must have specific movements.
   4) Connections 2, 3, and another bar must have specific movements.

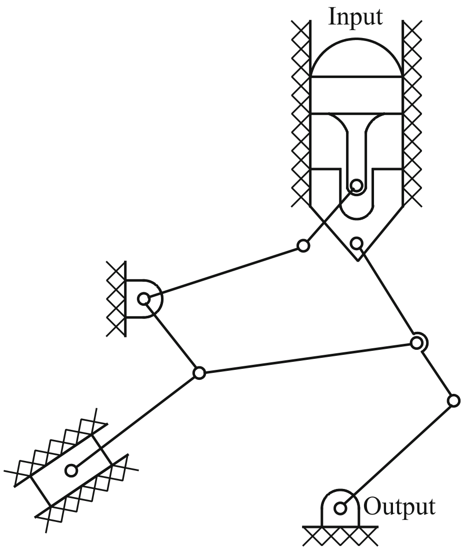


Fig. 1.33 Mechanism with a slider as an input

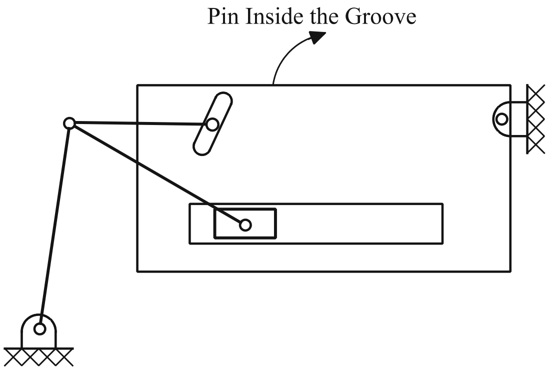


Fig. 1.34 Mechanism with a pin inside a groove

8. According to Fig. 1.33, what is the degree of freedom of this mechanism?

   1) 1 2) 2 3) 3 4) 4

9. What is the number of degrees of freedom of the mechanism according to Fig. 1.34?

   1) 1 2) 2 3) 3 4) 4

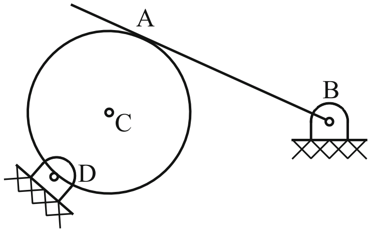
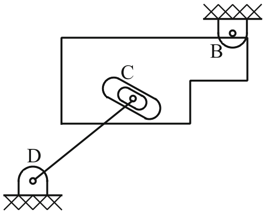
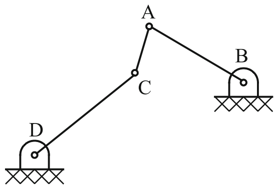
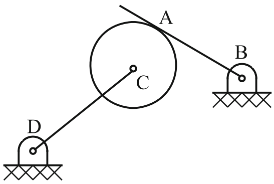


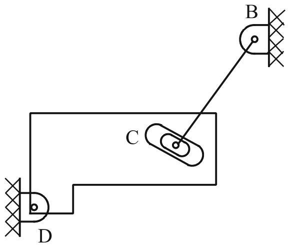
Fig. 1.35 Equivalent mechanism

10. What is the equivalent mechanism of the mechanism in Fig. 1.35?

1)


2)


3)


4)


11. What is the number of degrees of freedom of the mechanism in Fig. 1.36?

    1) 1 2) 2 3) 3 4) 4

12. What is the mechanism's degree of freedom or mobility shown in Fig. 1.37?

    1) 3 2) 2 3) 1 4) 0

13. In the mechanism in Fig. 1.38, if ω_2 is known, considering knowing the length and position of the levers at this moment, which of the given statements is correct?

    1) Velocity of block 6 or angular velocity of lever 5 can be specified in terms of ω_2.
    2) All instantaneous centers of this mechanism can be specified.
    3) Options 1 and 2 are correct.
    4) It is impossible to specify the velocity of block 6 or the angular velocity of lever 5 in terms of ω_2.

Fig. 1.36 Mechanism with one roller and three sliders

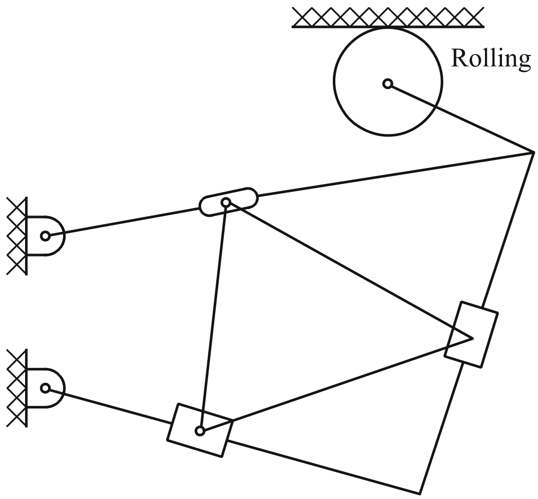


Fig. 1.37 A mechanism with a fork joint

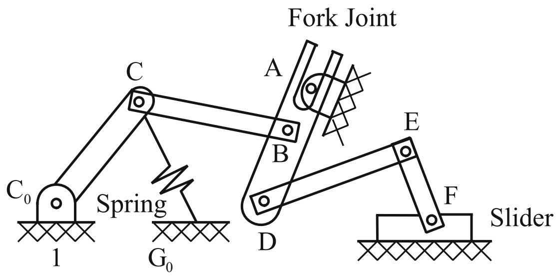
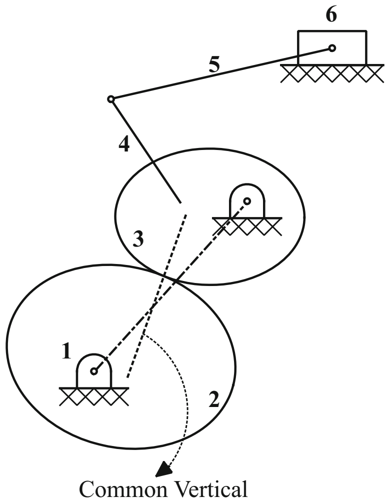


Fig. 1.38 A combined mechanism



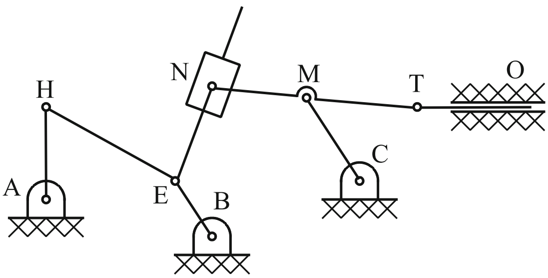


Fig. 1.39 Mechanism for links position detection

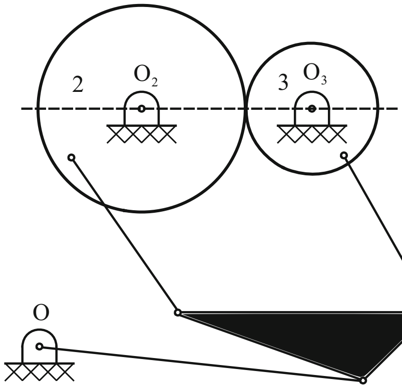


Fig. 1.40 Mechanism with gears

14. In the mechanism in Fig. 1.39, for determining the movement situation, at least how many members are needed to determine the complete position of the mechanism?

    1) Zero member
    2) One member
    3) Two members
    4) Three members

15. Which statement is correct about the mechanism in Fig. 1.40, including gears 2 and 3?

    1) If an actuator force is applied to one of the mechanism members, the mechanism will not move.
    2) If an actuator force is applied to one of the mechanism members, the mechanism will have a very specific motion.
    3) An actuator force must be applied to the two members of the mechanism so that the mechanism has a very specific motion.
    4) No specific comment can be made about the possibility of movement in the mechanism due to the actuator force.

16. What is the number of degrees of freedom in the mechanism in Fig. 1.41?

    1) 1
    2) 2
    3) 3
    4) 4

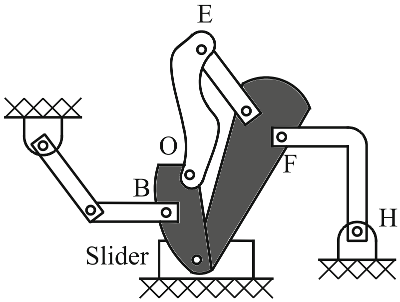


Fig. 1.41 Mechanism with rotational input

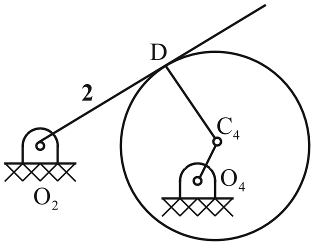
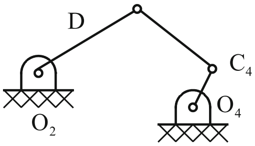
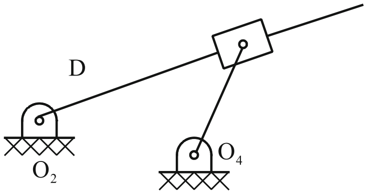
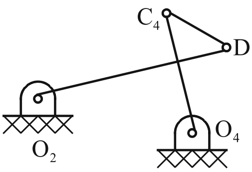


Fig. 1.42 Equivalent mechanism linkage

17. Which of the mechanisms is equivalent to the mechanism in Fig. 1.42?

1) 

2) 

3) 

4) Option 2 and 3 are correct.

18. In the system in Fig. 1.43, if the wheels move in two horizontal directions without sliding, which statement is correct?
    1) It is a mechanism with one degree of freedom.
    2) It is a chain with two degrees of freedom.
    3) It is a structure.
    4) It is an unrestricted chain system.

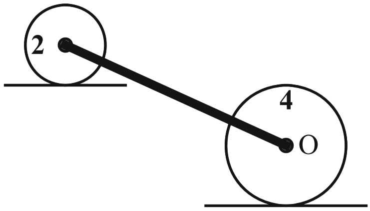


Fig. 1.43 Wheels move in two horizontal directions without sliding

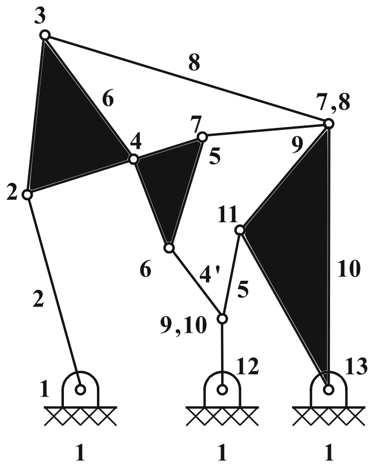


Fig. 1.44 13 links mechanism

19. What is the degree of freedom of the formed mechanism in Fig. 1.44?

1) 2 2) 3 3) 4 4) 1

Answers for the Examples of "Motivation and Introduction to Machine Dynamics"

1. Option (1) is correct.

   The system has 6 members, 6 joints of one degree of freedom, and one joint of two degrees of freedom. Note that the middle joint that connects the three members counts as two joints. So, according to Gruebler's equation, we have

   $$3\,(n-1) - 2f_1 - f_2 = 3\,(6-1) - (2 \times 6) - 1 = 2$$

   Therefore, the degree of freedom of the system is two, and this means that for a movement with a known output, we need two specific input movements.
2. Option (1) is correct.

   Two levers are installed to adjust the mechanism to increase the mechanism's compatibility with the conditions of the key and the door of the room. They can be considered two fixed and rigid levers after adjusting them to the desired goal.

Using Gruebler's equation and the parameters n, f_1, and f_2 in the lever, we have

$$n = 18\ ,\ \ f_1 = 25\ ,\ \ f_2 = 0$$

$$\text{DOF} = 3\,(n-1) - 2f_1 - f_2 = 3 \times (18-1) - 2 \times 25 = 1$$

3. Option (4) is correct.
   Point A is the center of the curve of contact of the camshaft with the follower. Because both members have non-flat surfaces at the contact point of the follower and the camshaft, the contact of the intermediate member between the rod O_2A and the slider B must be non-sliding, which is provided in option (4) with a rod. While in options (1) and (2), slider B has a sliding contact with the rod O_2A or the intermediate member, which cannot be accepted.
4. Option (3) is correct.
   Since the contact surface of the slider with the camshaft is flat, there must be a groove parallel to the slider surface at the center of the curve of the camshaft.
5. Option (4) is correct.
   The system, including land, has 7 members. Seven connections have one degree of freedom, and one connection has two degrees of freedom. Therefore:

$$\text{DOF} = 3\,(n-1) - 2f_1 - f_2 = 3\,(7-1) - 2\,(7) - 1 = 3$$

   Therefore, the degree of freedom of the system is more than two, and by knowing the two velocity parameters in general, no other velocity parameter such as V_C can be obtained.
6. Option (4) is correct.
   Solution 1: If we keep one of the two bars fixed, the other bar will still be able to rotate. That is, a degree of freedom remains unchecked. Therefore, the desired mechanism has two degrees of freedom.
   Solution 2: We have a four-bar linkage so that the pin forms two degrees of freedom from the other two levers. On the other hand, the other two levers form two degrees of freedom and create a joint of one degree of freedom with the ground. So:

$$n = 4\ ,\ \ f_1 = 2\ ,\ \ f_2 = 3$$

   From Gruebler's equation, we have

$$\text{DOF} = 3\,(n-1) - 2f_1 - f_2 = 3\,(3) - 2(2) - 3 = 2$$

7. Option (1) is correct.
   This mechanism has 7 members, 7 hinged connections, and a slider (sliding connection with two degrees of freedom). Since we have three members in contact on the hinge on member 4, we consider this hinge as two joints;

therefore, $f_1 = 8$ and $f_2 = 1$. From Gruebler's equation, we have

$$\mathrm{DOF} = 3\,(n-1) - 2f_1 - f_2 = 3\,(7-1) - (2\times 8) - 1 = 1$$

Therefore, the desired mechanism has one degree of freedom, and the status of the mechanism is determined by knowing the movement of the desired member.

8. Option (1) is correct.
   We have two input members that are bound to move together in a groove in the vertical direction. Therefore, we consider them as a member. Thus we will have a mechanism with 10 members and $f_1 = 13$ and $f_2 = 0$. Using Gruebler's criterion, we have

$$\mathrm{DOF} = 3\,(n-1) - 2f_1 - f_2 = 3\,(10-1) - (2\times 13) = 1$$

9. Option (2) is correct.
   The connection of the pin inside the groove is a connection of two degrees of freedom because the rod, assuming that the plate in which the groove is located is fixed, can rotate around an axis perpendicular to the plate and move in the direction of the groove. Using Gruebler's criterion, we have

$$n = 6\,,\ \ f_1 = 6\,,\ \ f_2 = 1$$

$$\mathrm{DOF} = 3\,(n-1) - 2f_1 - f_2 = 3\,(6-1) - 2\times 6 - 1 = 2$$

10. Option (1) is correct.
    At point A, the path of the point belonging to the rod on the disk is the same as the curve of the disk, which is the center of curvature of the path at point C. Also, the path of the point belonging to the disk on rod AB is a straight line, so we put the slider located at point C and the groove parallel to the rod as the intermediate between the two members.
11. Option (1) is correct.
    The disk's motion on the rolling surface is pure, and therefore, it is a constraint with one degree of freedom. In addition, it should be noted that in 4 nodes of the mechanism, we have a connection of 3 members, and each should be considered two joints. So from Gruebler's criterion, we have

$$n = 12\,,\ \ f_1 = 16\,,\ \ f_2 = 0$$

$$\mathrm{DOF} = 3\,(n-1) - 2f_1 - f_2 = 3\,(12-1) - (2\times 16) = 1$$

12. Option (1) is correct.
    Since spring is not a rigid member, it does not affect the degree of freedom, so we do not consider it a member. The fork connection in A is a connection of two degrees of freedom because the rod has both rotational and transverse

motions in the direction of the groove. Using Gruebler's criterion, we have

$$n = 7\ ,\ \ f_1 = 7\ ,\ \ f_2 = 1$$

$$\text{DOF} = 3\,(n-1) - 2f_1 - f_2 = 3\,(7-1) - (2 \times 7) - 1 = 3$$

13. Option (4) is correct.

Members 2 and 3 can slide relative to each other, as shown, so only one constraint is created at the point of contact (equality of the velocity components of the two points of contact in the direction of the common perpendicular), and therefore, we have a joint of two degrees of freedom. Therefore,

$$n = 6\ ,\ \ f_1 = 6\ ,\ \ f_2 = 1$$

$$\text{DOF} = 3\,(n-1) - 2f_1 - f_2 = 3\,(6-1) - (2 \times 6) - 1 = 2$$

Therefore, we need two inputs to determine the mechanism. Option (2) is also wrong because if it is correct, we can get the velocity of different points, which conflicts with the mechanism having two degrees of freedom.

14. Option (2) is correct.

The number of members to determine the status of the mechanism is the same as the degree of freedom of the mechanism. To determine the number of members, one must pay attention to the location of the joints. For example, the shape created from the connections of the joints H, E, and N, which consists of a rigid triangular body with a welded rod to it, is only one member. So we have

$$n = 8\ ,\ \ f_1 = 10\ ,\ \ f_2 = 0$$

$$\text{DOF} = 3\,(n-1) - 2f_1 - f_2 = 3\,(8-1) - (2 \times 10) = 1$$

15. Option (2) is correct.

Assuming movement with sliding between two gears, we have

$$n = 7\ ,\ \ f_1 = 8\ ,\ \ f_2 = 1$$

$$\text{DOF} = 3\,(n-1) - 2f_1 - f_2 = 3\,(7-1) - (2 \times 8) - 1 = 1$$

Since the degree of freedom of the mechanism is one, according to the definition of the degree of freedom that is the number of inputs required to specify the mechanism, option (2) is correct.

16. Option (2) is correct.

According to Gruebler's criterion, we have

$$n = 9\ ,\ \ f_1 = 11\ ,\ \ f_2 = 0$$

$$\text{DOF} = 3\,(n-1) - 2f_1 - f_2 = 3\,(9-1) - (2 \times 11) = 2$$

17. Option (2) is correct.
    If D_2 and D_4 are points belonging to members 2 and 4 at the point of contact, the path of D_4 on member 2 is a straight line, so the intermediate member is a slider in point C that is the center of the path curvature D_2 on the member 4.
18. Option (1) is correct.
    From Gruebler's criterion, we have

    $$n = 4\,,\ \ f_1 = 4\,,\ \ f_2 = 0$$

    $$\mathrm{DOF} = 3\,(n-1) - 2f_1 - f_2 = 3\,(4-1) - (2 \times 4) = 1$$

19. Option (4) is correct.
    Including land, the system has 10 members. There are also 13 constraints of one degree of freedom. Then:

    $$\mathrm{DOF} = 3\,(n-1) - 2f_1 - f_2 = 3\,(10-1) - (2 \times 13) = 1$$

    Note that three members are connected in two joints, and these joints are each considered two. While there are eleven joints in total, we consider $11+2 = 13$ constraints of one degree of freedom.

Chapter 2
Displacement and Velocity Analysis

2.1 Introduction

The motion of a lever is expressed in terms of the linear displacements, linear velocities, and linear accelerations of its individual particles. However, the motion of a lever can also be determined based on the angular displacements, angular velocities, and angular accelerations of moving lines with the desired rigid lever. No matter what method is used to analyze the leverage, it is always necessary to determine the angular positions of the members before analyzing the velocity. Similarly, we need the angular velocities of the members before acceleration analysis. The kinematic analysis of a lever should always be performed as follows: position analysis, velocity analysis, and acceleration analysis. In addition to displacement analysis, several methods for determining velocities in mechanisms will be presented in this chapter.

2.2 Velocity Equations for the Curve Motion

Measuring and describing the motion of objects relative to a stationary coordinate system is called absolute motion analysis. The motion analysis will be relative if this analysis is performed on a moving device. If we denote the position of a particle moving on a straight line from the origin of coordinates with x, then we can write

$$\overline{V} = \frac{\Delta x}{\Delta t} \tag{2.1}$$

where Δx is the displacement in meters, Δt is the time interval in seconds, and $\overline{V}$ is the average velocity in meters per second.

A. Abbasimoshaei, T. A. Kern, *Machine Dynamics*,
https://doi.org/10.1007/978-981-99-6010-1_2

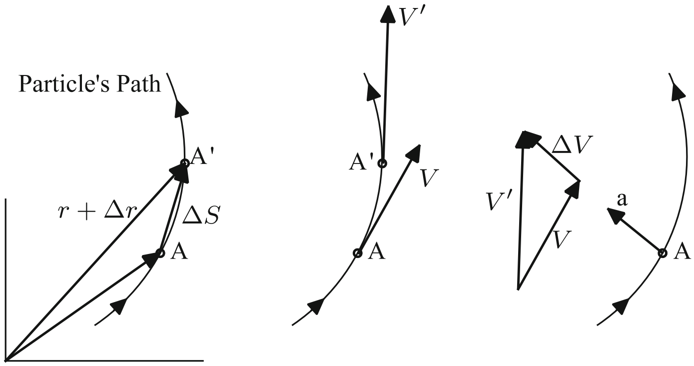


Fig. 2.1 A particle that moves continuously on a plane curve

Also, if V is the instantaneous velocity in meters per second, we have

$$V = \lim_{\Delta t \to 0} \frac{\Delta x}{\Delta t} = \frac{dx}{dt} = \dot{x} \tag{2.2}$$

Note Velocity is the rate at which location changes relative to time. If the displacement is positive, the velocity is positive, and if it is negative, the velocity is negative.

The motion of a particle that travels along a curved path is called a curved transmission. Consider a particle that moves continuously on a plane curve, as shown in Fig. 2.1.

The average velocity of a particle between A and A' is defined as $\overline{v} = \Delta r/\Delta t$, in which r represents the location vector, and Δr represents the displacement of the particle over time Δt. Instantaneous velocity v, by definition, is the limit of average velocity when the time interval Δt converges to zero. Therefore,

$$v = \frac{dr}{dt} = \dot{r} \tag{2.3}$$

Note The value of v, a scalar quantity, is called speed.

In the orthogonal coordinate system $(x - y)$, the curve motion of the particle is determined by summing the x and y components of vectors of location, velocity, and acceleration. For this type of coordinate system, we have

$$r = x\hat{i} + y\hat{j} \tag{2.4}$$

$$\overrightarrow{v} = \overrightarrow{\dot{r}} = \dot{x}\hat{i} + \dot{y}\hat{j} \tag{2.5}$$

In the vertical-tangential coordinate system $(n - t)$, where the unit vector e_n is defined in the direction n and the unit vector e_t is defined in the direction t, we can write

$$\overrightarrow{v} = v\hat{e}_t \tag{2.6}$$

Note that t specifies the direction of motion and n the direction perpendicular to the motion path.

Also, in the polar coordinate system $(r - \theta)$, where the unit vector e_r is in the positive direction r, and the unit vector e_θ is in the positive direction θ, the velocity vector equation is as follows:

$$\overrightarrow{v} = \dot{r}\hat{e}_r + r\dot{\theta}\hat{e}_\theta \tag{2.7}$$

Note Component v in direction r represents the rate of increase of vector r in direction θ due to the rotation.

Figure 2.2 shows the orthogonal and polar coordinate systems with unit vectors on the $x - y$ plane.

These equations can also be generalized for spatial curve motion. Figures 2.2 and 2.3 show the three coordinate systems of orthogonal $(x - y - z)$, cylindrical $(r - \theta - z)$, and spherical $(R - \theta - \varnothing)$ with unit vectors.

For orthogonal coordinates in the three-dimensional motion, we only need to add coordinate z and its derivatives to the equations of two-dimensional motion:

$$\overrightarrow{R} = x\hat{i} + y\hat{j} + z\hat{k} \tag{2.8}$$

$$v = \dot{R} = \dot{x}\hat{i} + \dot{y}\hat{j} + \dot{z}\hat{k} \tag{2.9}$$

Note that we show the location vector with the capital letter R in three-dimensional motion instead of r.

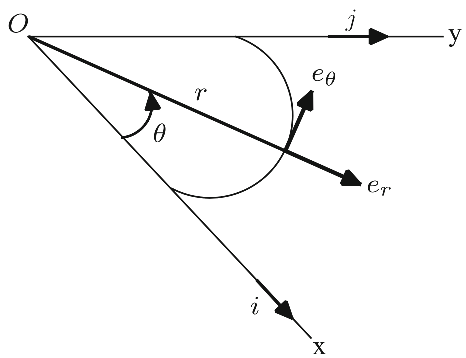


Fig. 2.2 Orthogonal and polar coordinate systems with unit vectors on the $x - y$ plane

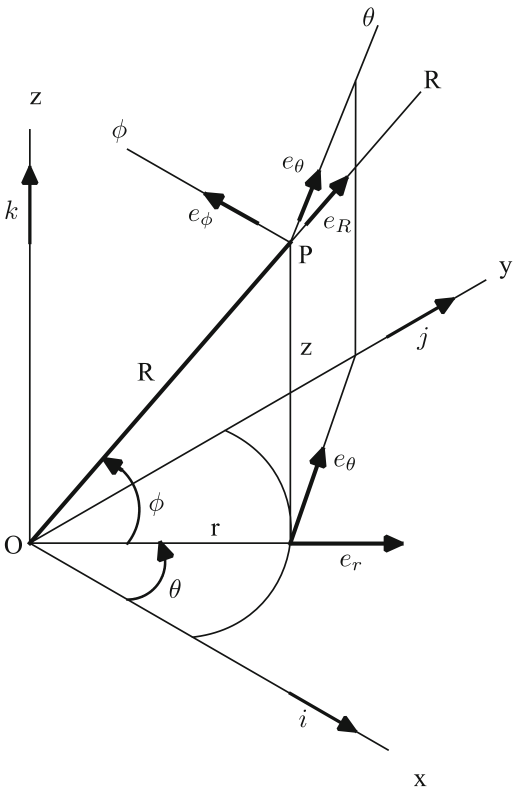


Fig. 2.3 Cylindrical and spherical coordinate systems in 3D

In the case of cylindrical coordinates, we only need to add coordinate z to the equations of motion in polar coordinates. Therefore,

$$\overrightarrow{v} = \dot{r}\hat{e}_r + r\dot{\theta}\hat{e}_\theta + \dot{z}\hat{k} \tag{2.10}$$

Spherical coordinates also use a radial distance and two angles to determine the position of a particle. Unit vector e_R is in the direction of motion in which R increases but θ and $\varnothing$ are constant. Unit vector $e_\varnothing$ is in the direction of motion in which $\varnothing$ increases, but R and θ remain constant. Finally, unit vector $e_\varnothing$ is in the direction of motion in which $\varnothing$ increases and R and θ are constant. So we have

$$\overrightarrow{v} = \dot{R}\hat{e}_R + R\dot{\theta}\cos\varnothing\hat{e}_\varnothing + R\dot{\varnothing}\hat{e}_\varnothing \tag{2.11}$$

Example In Fig. 2.4, the piston shaft of the hydraulic jack moves to the left at a constant velocity of v. We denote $\overline{OA}$ by r. The values of $\dot{r}$ and $\dot{\theta}$ are

1) $\dot{r} = -v\cos\theta, \dot{\theta} = \frac{v}{r}\cos\theta$ 2) $\dot{r} = v\sin\theta, \dot{\theta} = \frac{v}{r}\cos\theta$
3) $\dot{r} = -v\cos\theta, \dot{\theta} = \frac{v}{r}\sin\theta$ 4) $\dot{r} = -v\cos\theta, \dot{\theta} = \frac{-v}{r}\sin\theta$

Fig. 2.4 A hydraulic jack

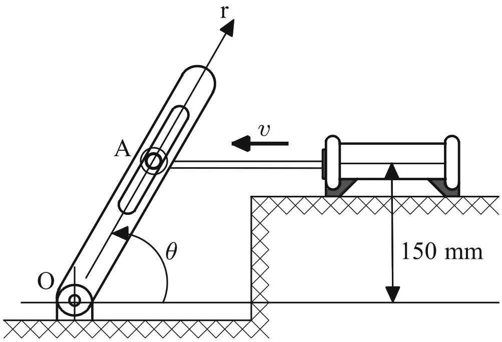
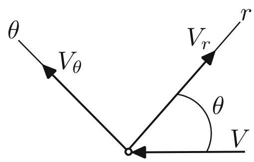


Fig. 2.5 Polar coordinate



Solution The motion is of the plane type. Using the motion equations in polar coordinate, we can plot the known velocity v of point A in directions r and θ (Fig. 2.5).

$$\vec{v} = v_r\hat{e}_r + v_\theta\hat{e}_\theta = -v\cos\theta\hat{e}_r + v\sin\theta\hat{e}_\theta$$

$$\vec{v} = \dot{r}\hat{e}_r + r\dot{\theta}\hat{e}_\theta \quad \text{(velocity equation for polar coordinates)}$$

From the above two equations, we have

$$\dot{r}\hat{e}_r + r\dot{\theta}\hat{e}_\theta = -v\cos\theta\hat{e}_r + v\sin\theta\hat{e}_\theta \Rightarrow \begin{cases} \dot{r} = -v\cos\theta \\ \dot{\theta} = \frac{v}{r}\sin\theta \end{cases}$$

Option (3) is correct.

2.3 Angular Motion

Angular velocity and acceleration are the first and second derivatives of angular displacement θ of a line relative to time t, respectively. In machine analysis, the angular motion of each lever is expressed by the angular motion of a hypothetical line connected to it.

Note The angular motion of a lever may be similar to or different from the angular motion of the radius of the path curvature of individual particles of the lever.

An important concept in mechanisms is that only transmission is possible for a particle that is a point of infinitesimally small size, and the particle cannot rotate. Angular motion is the motion of a line, and since a particle is a point and not a line, angular motion is inconceivable for it. This concept must be well understood to understand the relative motion between particles. In the following, various methods of velocity analysis are examined.

2.4 Analytical Method of Velocity Analysis

It is not possible or appropriate to use a fixed coordinate system to study all motions. There are many geometric problems in which motion analysis is easier using measurements obtained from a moving coordinate system. By combining these measurements with the absolute motion of the moving coordinate system, the desired absolute motion can be determined. This method is called relative motion analysis.

Note An object only has relative motion relative to another object when their absolute motions are different from each other.

The position of an object like A relative to an object like B is equal to the absolute position A minus the absolute position B. A similar interpretation is used for velocity and acceleration. Thus, for velocity, we can write

$$V_{A/B} = V_A - V_B \tag{2.12}$$

or in other words

$$V_A = V_B + V_{A/B} \tag{2.13}$$

This study relative to the moving device is limited to devices with a transmission motion and no rotational motion. If the moving device also has a rotation velocity of ω, we will have

$$V_A = V_B + \omega \times r + V_{rel} \tag{2.14}$$

V_{rel} has no rotation velocity relative to the moving device, and to find it, stop the device's rotation and find $V_{A/B}$.

Note The difference between the relative velocities in rotating and non-rotating axes is in $\omega \times r$.

Note The relative velocity of the points that match at the point of contact of two rolling members is zero.

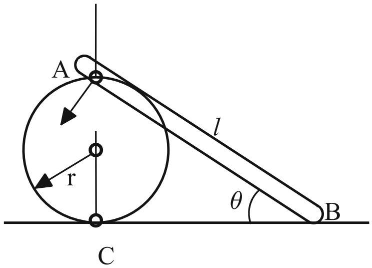


Fig. 2.6 A disk with pure rolling movement

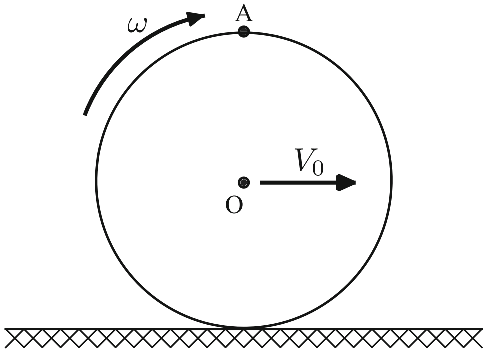


Fig. 2.7 A disk on the ground

Example A disk with pure rolling moves in Fig. 2.6, and the bar AB is jointed to it at point A. Obtain velocity B if the rotational velocity of the disk is ω.

1) Zero
2) $2r\omega$
3) $(2r + l\cos\theta)\omega$
4) The information is not enough.

Solution At this point, the velocity of point A is horizontal and equal to $2r\omega$. The velocity of point B is also always horizontal due to ground contact. So since the direction of the velocity of two different points of a rigid body is the same, it can be said that the object has no angular velocity and only has a transmission motion at that moment. Therefore the velocity vectors of all its points are equal to each other. So $V_B = V_A = 2r\omega$.

Option (2) is correct.

Note For two points located on a lever of a mechanism, $V_{rel} = 0$.

Example The disk's center shown in Fig. 2.7 moves at a velocity $V_0 = 1m/s$. If the disk has a radius $R = 10cm$ and an angular velocity $\omega = 20rad/s$, what is the velocity of point A at the top of the disk at the shown moment in meters per second?

1) 1
2) 2
3) 3
4) 4

Solution If we set the center of the moving device to O, we will have

$$V_A = V_O + \omega \times r + V_{rel}$$

Given that the points O and A are on one object, $V_{rel} = 0$ and we have

$$V_A = V_O + \omega \times |OA| \Rightarrow V_A = 1\hat{i} + (20 \times 0.1)\,\hat{i} = 3\hat{i}$$

Note that the disk has a sliding motion on the ground, and its motion is not pure rolling. In the case of pure rolling, the velocity of point O would be $2m/s$. Also, if the velocity of the point of contact with the ground is calculated, this velocity will not be zero.

Option (3) is correct.

2.5 Graphical Method for Velocity Analysis

Velocity polygons are good tools for determining the velocity of mechanisms. These polygons can be solved by drawing, analyzing, or combining the two. The main method used in graphical lever analysis is to work with one or two points at a known velocity to find one of the unknown velocities using the relation between the velocities of two points belonging to one lever in equation (2.14). Rotational joints form the transmission points because the mentioned points belong to two different members. Thus, the velocity of the rotating joint can be obtained by considering it as a point on one of the members to which it is attached.

In the mechanisms studied in machine dynamics and the method used, because the two points under analysis belong to the same lever, $V_{rel} = 0$, and this equation is simplified as follows:

$$\vec{V}_B = \vec{V}_A + \vec{\omega} \times \vec{r}_{B/A} \tag{2.15}$$

Equation 2.15 can be graphically represented as a vector triangle, as shown in Fig. 2.8.

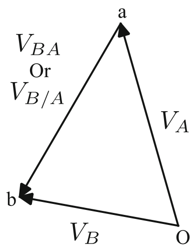


Fig. 2.8 Vector triangle

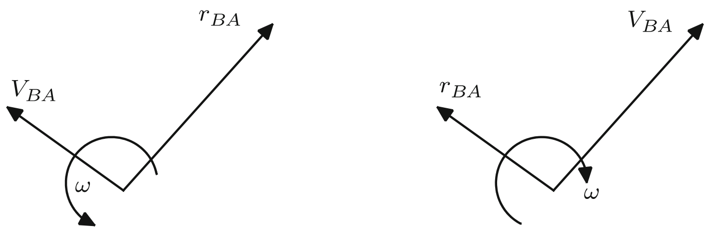


Fig. 2.9 Finding direction of the third vector

Note This triangle can always be solved by knowing the direction and size of one of the three vectors and the direction of the other two vectors. This is a common situation in planar velocity analysis.

According to Fig. 2.9, the vectors used in equation $\vec{\omega} \times \vec{r}_{B/A}$ are reciprocally orthogonal. Because we know the lines that the vectors must be in line with, the main issue is determining the direction of the lines and the size of each of the vectors. By knowing the direction of the two vectors, the direction of the third vector can be found by the rule of the right hand by observing the known directions.

For example, suppose lever 2 moves in the four-bar mechanism in Fig. 2.10, and its uniform angular velocity ω_2 is known. We want to find the velocity V_B of point B and the angular velocities ω_3 and ω_4. The known geometric parameters are also shown in the figure.

Since each vector has size m and direction d, we can easily reflect the information and unknowns of a vector equation in a table. Two scalar (numerical) unknowns

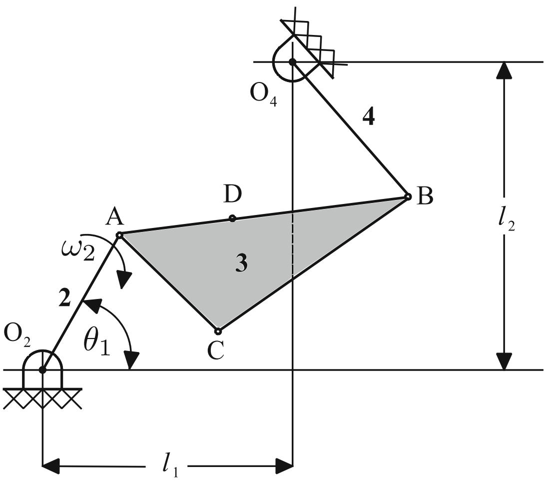


Fig. 2.10 Four-bar mechanism

(I).	$\boldsymbol{V_B}$ =	$\boldsymbol{V_A}$ +	$\boldsymbol{V_{BA}}$
Length (***m***)		✓	
Direction (***d***)	✓	✓	✓

Fig. 2.11 Velocity calculation parameters

can be obtained from a vector equation. The velocity equation can be written in Fig. 2.11.

wherein

$$V_A = \text{along the perpendicular to } O_2A \text{ of known size } V_A = |O_2A|\,\omega_2$$

$$V_{BA} = \text{along the perpendicular to } BA \text{ of unknown size}$$

$$V_B = \text{along the perpendicular to } O_4A \text{ with an unknown size}$$

We can plot a velocity polygon with only two unknowns of the vector equation. First, we consider the desired origin O_v and draw it with its size and direction V_A. On the other hand, the directions of V_B and V_{BA} are known. From the origin, we determine the direction of V_B with a dashed line. On the other hand, according to the equation $V_{BA} = V_B - V_A$, so the V_{BA} vector must start from the end of vector A and be connected to the end of vector B. So with the V_{BA} direction, we draw a dashed line from the end of the vector V_A in this direction. The intersection of the V_B and V_{BA} directions completes the polygons. Now we add the direction of the arrows V_B and V_{BA} so that the sum of the polygons matches the sum of the sentences of the velocity Equation (I). We will mark the tip of the V_B vector with B. The steps are shown in Fig. 2.12.

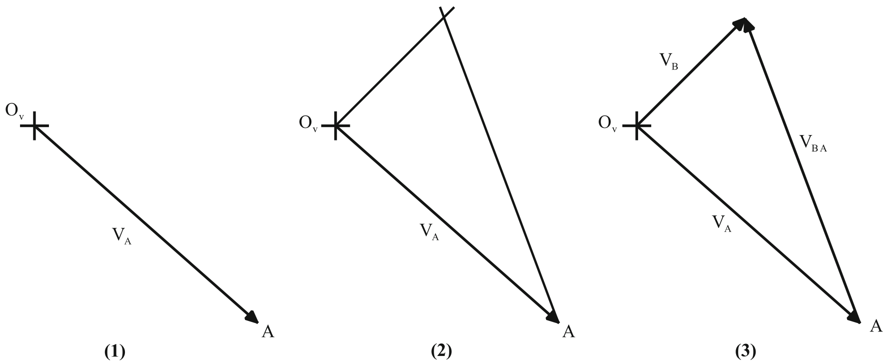


Fig. 2.12 Steps of finding the relative speed

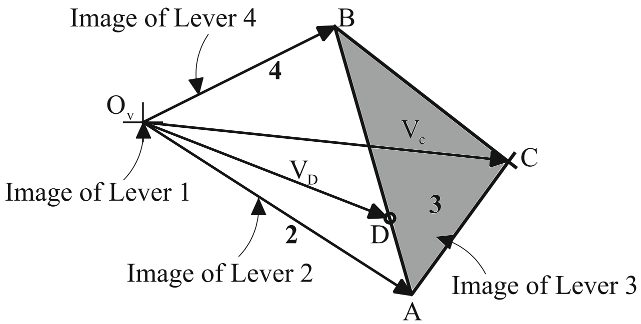


Fig. 2.13 Image of velocities

According to the figures, ω_3 and ω_4 can be obtained from V_{BA} and V_B, respectively. The size of ω_{32} and ω_{43} can also be determined. We have

$$\omega_3 = \frac{V_{BA}}{|BA|}\ ,\ \omega_4 = \frac{V_B}{|O_4B|}\ ,\ \omega_{32} = \omega_3 - \omega_2\ ,\ \omega_{43} = \omega_4 - \omega_3$$

We should use Equations (II) and (III), which express the relation between V_C, V_A, and V_B to determine V_C.

$$\text{(II) } V_C = V_A + V_{CA}$$

$$\text{(III) } V_C = V_B + V_{CB}$$

The V_{CA} and V_{CB} extensions are known according to the shape of the mechanism and are perpendicular to the sides CA and CB, respectively. Equation (II) is used, and the extension of the vector V_{CA} from point A is plotted in the following figure. Then, according to Equation (III), the extension of the vector V_{CB} is plotted from point 3. The intersection of the extensions V_{CA} and V_{CB} completes the polygon. The intersection of point C gives V_C. It should be noted that the sum of the vectors of polygons must be compatible with equations (II) and (III) (Fig. 2.13).

The hatched triangle ABC is called the image (projection) of the velocity of lever 3 and is similar to lever 3. By determining the position of any given point D of lever 3 on the velocity image, its velocity can be obtained. According to the figure, the vector drawn from O_v to D is V_D. The image of the velocity of lever 1 at origin is O_v because lever 1 is fixed, and its velocity is zero. The images of velocities of levers 2 and 4 are the lines O_vA and O_vB, respectively, which correspond to the O_2A and O_4B of the mechanism, respectively.

Note Having velocity images of all levers of a mechanism allows calculating the linear velocity of all lever points.

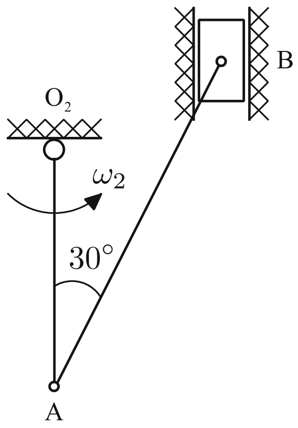


Fig. 2.14 Crank-slider mechanism

From the above analysis, we have

$$\omega_3 = \frac{V_{BA}}{|BA|}, \ \omega_3 = \frac{V_{CA}}{|CA|} = \frac{V_{CB}}{|CB|} = \frac{V_{DA}}{|DA|}$$

In other words, all the relative velocities of the points on a lever are proportional to the distances between these points.

Example Given the mechanism in Fig. 2.14 for ω_{AB} angular velocity, which of the following is true? (O_2A is parallel to the motion path B.)

1) The direction of ω_{AB} is the same as the direction of ω_2, but $|\omega_{AB}| > |\omega_2|$.
2) When O_2A is in line with AB, ω_{AB} is zero.
3) ω_{AB} equals twice the velocity B divided by the length AB.
4) ω_{AB} equals the difference of the absolute velocities A and B divided by the length AB.

Solution We draw the velocity polygon (Fig. 2.15) with a free scale using the relative velocity equations in the mechanisms.

$$V_A = V_A + V_{BA}$$

$$\omega_{BA} = \frac{V_{BA}}{|BA|}$$

From the polygon, velocity is $|V_B| = \frac{|V_{BA}|}{2}$, and by placing it in the above equation, we have

$$\omega_{BA} = \frac{V_{BA}}{|BA|} = \frac{2V_B}{|BA|}$$

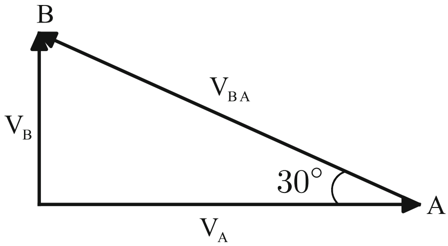


Fig. 2.15 Velocity polygon

So option (3) is correct. But about other options,

$$\omega_{BA} = \frac{V_{BA}}{|BA|} = \frac{2V_B}{|BA|}$$

Therefore, option (1) depends on the numerical value of $|O_2A|$. Option (2) is also incorrect. If the two levers, O_2A and AB, are in the same direction since the directions of velocities A and B will be different, there will also be angular velocities of ω_{AB}, and it will not be zero. But for option (4), we write from the polygon of velocity

$$\omega_{AB} = \frac{V_{BA}}{|BA|} = \frac{V_B - V_A}{|BA|} = \frac{V_B - V_B \cot 30}{|BA|} \neq \frac{2V_B}{|BA|}$$

It is observed that the statement of the option (4) cannot always be correct.

It should be noted that this question can be easily solved by finding the instantaneous center of rotation A and B.

Option (3) is correct.

Example If the velocity of point A is 2.5(m/s), what is the velocity of point B? (Fig. 2.16)

1) 1.5 m/s 2) 2.5 m/s 3) 7.5 m/s 4) 5 m/s

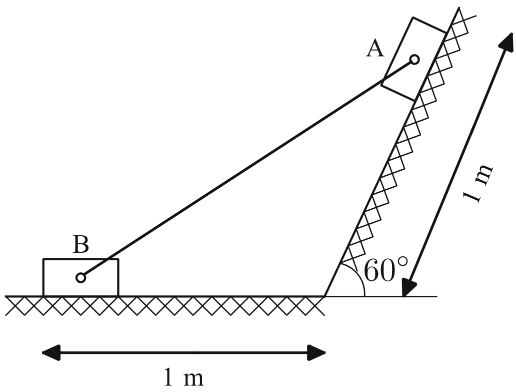


Fig. 2.16 Two points connected with one link

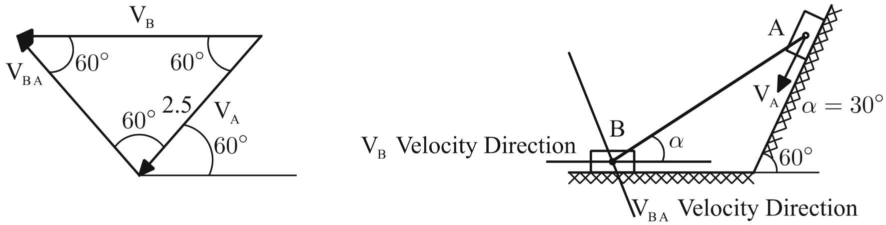


Fig. 2.17 Velocity triangle

Solution We have

$$V_B = V_A + V_{BA}$$

Given that the directions of V_B and V_{BA} are known, then the equation has two unknowns, magnitude of V_B and V_{BA}. By drawing the velocity triangle (Fig. 2.17), since the resulting triangle is equilateral,

$$V_B = 2.5 \text{ m/s}$$

Option (2) is correct.

2.6 Instantaneous Center of Rotation Method for Velocity Analysis

2.6.1 Instantaneous Center of Velocity

In the following discussion, another concept is used to determine the linear velocity of the mechanism particles, which is the concept of the instantaneous center of velocity. This concept is based on the fact that at a given moment, the velocities of a pair of matching points located on two moving levers are equal relative to a fixed lever, and therefore their relative velocities relative to each other are zero. At this point, each lever has only a pure rotation around the matching points relative to the other lever. A special case of this is when one lever is moving, and the other is fixed. Thus, the absolute velocity of a pair of matching points of these two levers is zero, and at this moment, the moving lever rotates around these matching points relative to the fixed lever. In both cases, the set of matching points is called the instantaneous center of velocity (or instantaneous center). Thus an instantaneous center is a point that

(a) Is located on both objects.
(b) The object has no relative velocity in it.

(c) At a given moment, one object can be imagined rotating around it relative to another object.

Note If a hinge connects two levers, their instantaneous center is the point of contact in the hinge because the velocity vector of the points of contact in the hinge is the same.

Even if the two levers are not directly connected, there will be an instantaneous center (geometric position) for them in any desired state.

Note As the mechanism passes through different positions of a motion cycle, the position of the instantaneous center of one lever relative to the other lever changes over time due to the polygonal deformation of velocity, except for levers that have a pure rotation whose instantaneous center is fixed points.

The velocities of all points of the levers, which act as sliders in a single crank and slider mechanism and only have transmission, are parallel to each other. Also, their perpendiculars are parallel, and their intersection is at infinity. Thus the instantaneous center of a lever in the transmission is at an infinite distance from the lever and perpendicular to the transmission path. Therefore, if, as shown in Fig. 2.18, the slider (member 3) slides on a non-curved member (member 2), the instantaneous center of the two members is at an infinite distance from the point of sliding and on a line perpendicular from the sliding point to the member on which the slider slides (drawn dashed line).

Note If two members have sliding contact with each other, the instantaneous center of the two members is somewhere on the common perpendicular line of the two members drawn at the point of contact (Fig. 2.19A and B), and if one of the two members is a slider, the instantaneous center is on the center of curvature of the other member (Fig. 2.19C).

Note If two members have a pure rolling contact, their instantaneous center is at the same point of contact, because in pure rolling, the velocity vector of the contact points of the two members is the same.

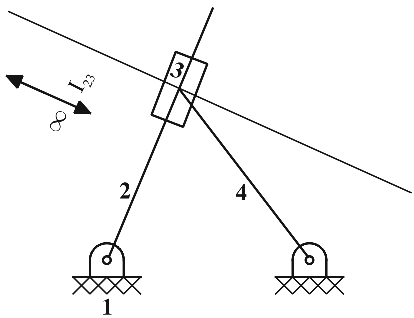


Fig. 2.18 Slider on the connecting points of two links

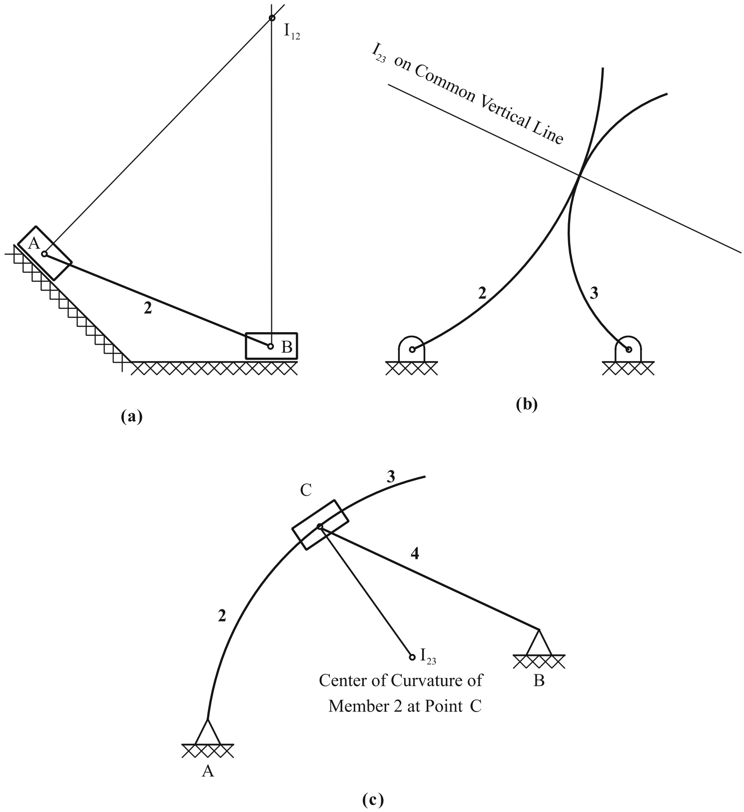


Fig. 2.19 Common perpendicular line of the two members drawn at the point of contact

Example In a five-bar mechanism, according to Fig. 2.20, what are the locations of instantaneous centers (moments) between members (4 and 5), (3 and 4), and (1 and 5)?

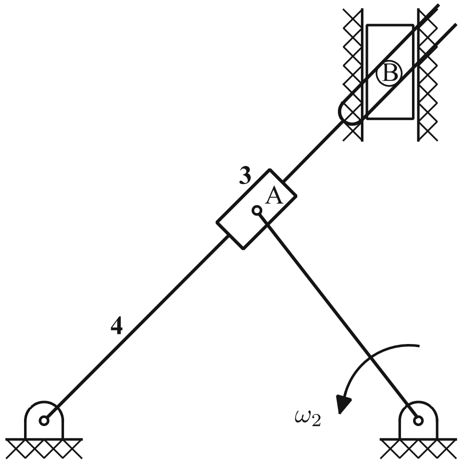
1) The instantaneous center makes no sense in this type of mechanism.
2) The instantaneous centers of 45, 34, and 15 are at infinity.
3) The instantaneous center 45 is at point B, 34 at point A, and 15 on the perpendicular line at infinity.
4) The instantaneous center 45 is at point B, 34 on the line perpendicular to the member at infinity, and 15 at infinity.

Solution The members 5 on 4, 3 on 4, and 5 on 1 have a straight sliding motion, so the instantaneous centers are at infinity.

Option (2) is correct.

The desired instantaneous center of lever 2 relative to lever 1 is denoted by 21 or 12, and the instantaneous center of lever 4 relative to lever 3 is displayed with 43

Fig. 2.20 Five-bar mechanism



or 34. In general, the instantaneous center of lever A relative to lever B is displayed with AB or BA. Sometimes it is also indicated by I_{AB} or I_{BA}.

Note The desired instantaneous center of lever A relative to lever B is on the desired instantaneous center of lever B relative to lever A.

2.6.2 Kennedy's Theorem

Kennedy's theorem for three independent objects with a generally planar motion states that their three instantaneous centers are on a common straight line. In a mechanism consisting of n levers, there are $n - 1$ instantaneous centers for each assumed lever. Thus for n lever, there will be $n(n - 1)$ instantaneous centers. But since the position of each instantaneous center is assigned to two centers, the total number of positions (N) is obtained from the following equation:

$$N = \frac{n(n-1)}{2} \tag{2.16}$$

To determine the instantaneous centers of a mechanism, we only need to write the numbers of all members on the perimeter of a circle at separate points. Then, if the instantaneous center of both members is known, connect the points related to those two members with a line so that with a simple look, it is determined which instantaneous centers are known and which are unknown.

Example In the mechanism shown in Fig. 2.21, according to the coordinate axes specified in the figure at point B, which square locates the instantaneous center I_{36}?

1) First 2) Second 3) Third 4) Fourth

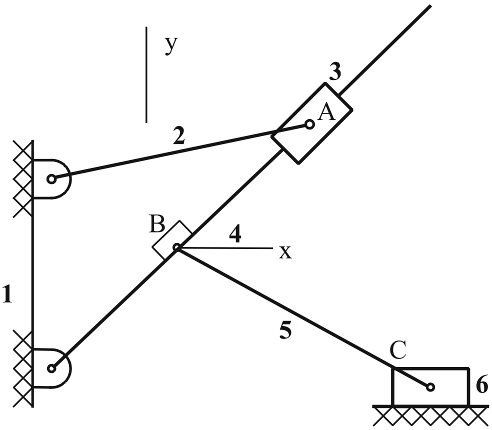


Fig. 2.21 Mechanism with three sliders

Solution According to the existing rotational and sliding joints, the instantaneous centers I_{12}, I_{34}, I_{45}, I_{56}, and I_{16} are known.

We draw a circle chart according to the available information (Fig. 2.22A). We use Kennedy's theorem for centers that are a little harder to find. In Fig. 2.22B, a dashed line is used to find the instantaneous center I_{13} that completes two triangles. Triangle 3.2.1 represents three centers (12, 23, and 13) of levers 1, 2, and 3, which are on a straight line according to Kennedy's theorem. The intersection of the two lines of the mechanism determines the center 13 that should be on both of these lines. The corresponding dashed line must be converted to a full line (Fig. 2.22C) to show the unknown center. Figure 2.22D shows the next step in which the position of center 46 is determined using triangles 6.4.1 and 6.5.4.

$$I_{13} \xrightarrow{\text{At the intersection of the connecting lines}} \begin{cases} I_{12} - I_{23} \\ I_{14} - I_{34} \end{cases}$$

$$I_{64} \xrightarrow{\text{At the intersection of the connecting lines}} \begin{cases} I_{16} - I_{14} \\ I_{65} - I_{54} \end{cases}$$

$$I_{63} \xrightarrow{\text{At the intersection of the connecting lines}} \begin{cases} I_{34} - I_{64} \\ I_{16} - I_{13} \end{cases}$$

We see that I_{63} is in the second area (Fig. 2.23).

Option (2) is correct.

Example In the mechanism of Fig. 2.24, where is the location of the instantaneous center between members 1 and 3?

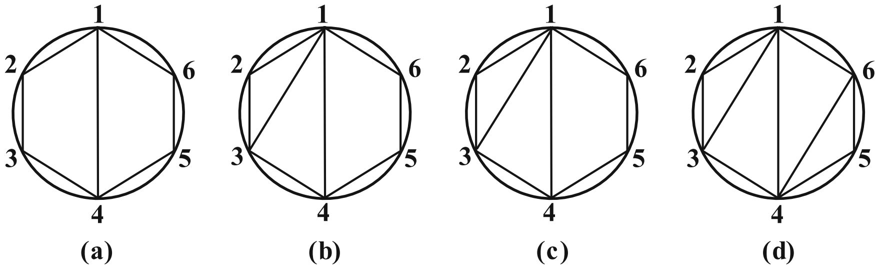


Fig. 2.22 Instantaneous centers finding

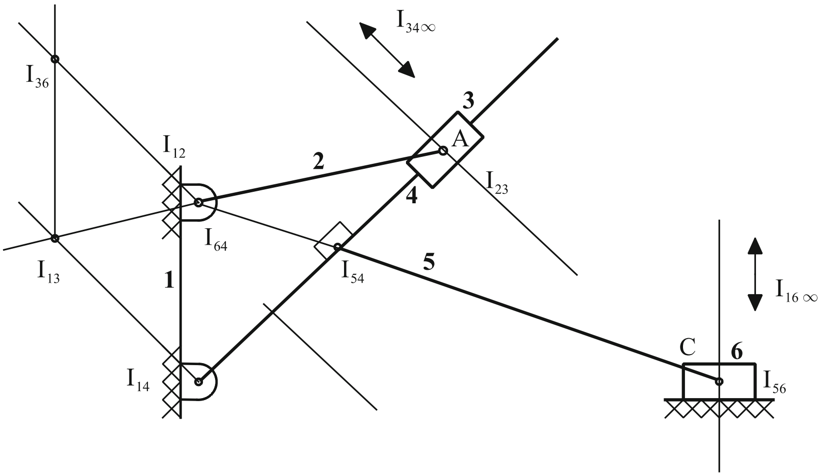


Fig. 2.23 Instantaneous centers

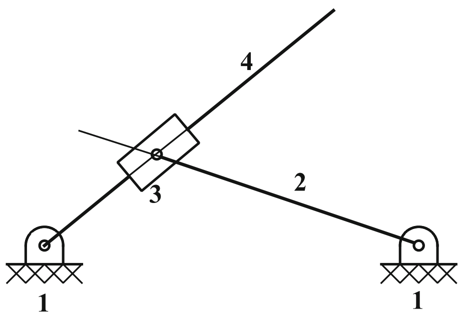


Fig. 2.24 Mechanism with slider in between

1) Along member 2 but not at infinity
2) Along member 2 and at infinity
3) On member 2
4) On the instantaneous center of members 1 and 2

Solution We know from Kennedy's theorem that for three members (1, 3, and 4), if the instantaneous centers of I_{34} and I_{14} are known, the instantaneous center of I_{13} will be somewhere on the connecting line of points I_{34} and I_{14}. For members 3 and 4, the instantaneous center is on a line perpendicular to the axis on which the slider slides and is at infinity (Fig. 2.25).

The line connecting the instantaneous centers 14 and 34 starts from the hinge connecting the lever 4 to the ground (I_{14}) and continues until the perpendicular and member 4 and infinity. On the other hand, the instantaneous center 13 will be along the line connecting the instantaneous centers 12 and 23, which is the extension of member 2. According to Fig. 2.26, it can be seen that the instantaneous center 13 is along the member 2 but is not at infinity.

Option (1) is correct.

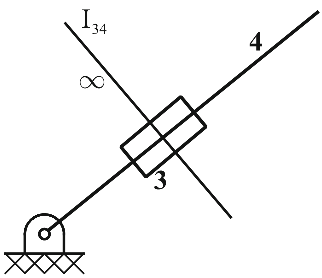


Fig. 2.25 Instantaneous centers of the mechanism with slider in between

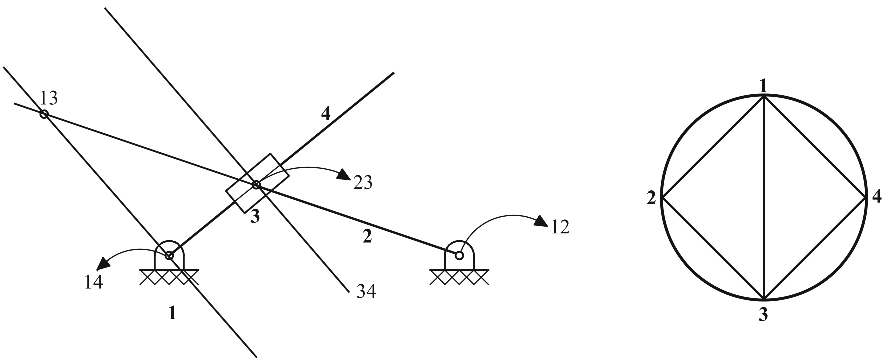


Fig. 2.26 Instantaneous center 13 is along member 2

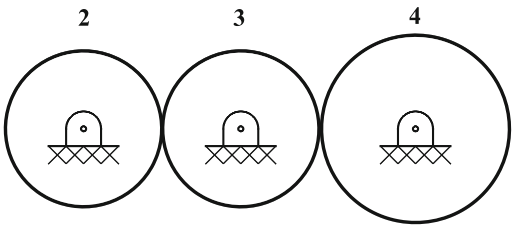


Fig. 2.27 A three-gear system

Note The instantaneous center of two levers is a point that has the same velocity if placed on either of them.

Example Where is the instantaneous center of rotation (24) in the three-gear system shown in Fig. 2.27?

1) In the middle of the center line of circles (2) and (4)
2) Point of contact of circles (2) and (3)
3) Intersection of the center line with the common internal tangent of circles (2) and (4)
4) Intersection of the center line with the common external tangent of circles (2) and (4)

Solution According to Kennedy's theorem, the instantaneous center 24 (I_{24}) is on the line connecting I_{12} and I_{14}. On the other hand, according to Fig. 2.28 and the point expressed, if the instantaneous center is a point of member 2 (or its extension), its velocity must be equal to when it is a point of member 4 (or its extension). The location of the instantaneous center is found by forming the velocity triangles of two objects and obtaining the point of common velocity.

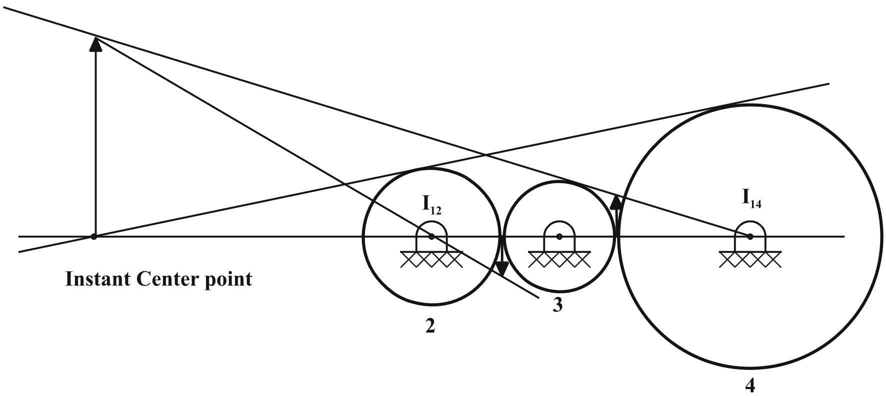


Fig. 2.28 Instantaneous centers finding of the three-gear system

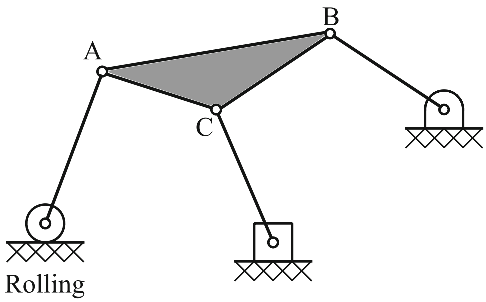


Fig. 2.29 A rolling, slider, rotational joint mechanism

Option (4) is correct.

If the degree of freedom of the mechanism is more than one, determining all instantaneous centers is possible when the velocity characteristics of all its n members are known. For example, suppose the number of inputs is less than the number of degrees of freedom. In that case, the movement of the lever cannot be fully predicted, and only some of its instantaneous centers can be determined.

Note For a system of one degree of freedom, it is always possible to determine all instantaneous centers.

Example Which statement is correct for the shown mechanism in Fig. 2.29?

1) According to the available information, all instantaneous centers can be determined.
2) For this mechanism, more than three instantaneous centers of rotation are at infinity.
3) For such a mechanism, the instantaneous center of rotation cannot be defined.
4) According to the available information, some instantaneous centers can be determined.

Solution For this mechanism $n = 7$, $f_1 = 8$, and $f_2 = 0$ from Gruebler's equation, we have

$$\text{DOF} = 3\,(n - 1) - 2f_1 - f_2 = 3\,(7 - 1) - (2 \times 8) = 2$$

Only some instantaneous centers can be determined because the number of degrees of freedom is more than one.

Option (4) is correct.

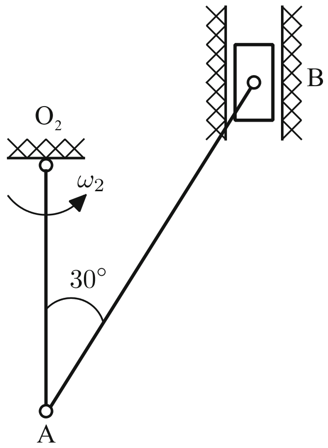


Fig. 2.30 Rotating slider mechanism

2.6.3 Determining the Velocity with the Help of Instantaneous Centers

The Kennedy theorem can be used as a suitable tool to directly determine the absolute velocity of any point in a mechanism without determining the velocity of the midpoints (such as the velocity polygon method).

Example According to the mechanism for angular velocity ω_{AB} shown in Fig. 2.30, which of the following statements is true? (O_2A is parallel to path B.)

1) The direction of ω_{AB} is equal to the direction of ω_2, but $|\omega_{AB}| > |\omega_2|$.
2) When O_2A is in line with AB, ω_{AB} is zero.
3) ω_{AB} equals twice the velocity B divided by the length AB.
4) ω_{AB} is equal to the difference of the absolute velocities A and B divided by the length AB.

Solution Because the velocity direction is known at two points, A and B, the instantaneous center I_{13} is also known, and it seems that object 3 is pinned around this point (Fig. 2.31).

$$V_A = |CA|\,\omega_{AB} \quad V_A = |O_2A|\,\omega_2 \qquad V_B = |CB|\,\omega_{AB}$$

$$\Longrightarrow \omega_{AB} = \frac{V_B}{|CB|} = \frac{V_B}{|AB|\sin 30} = \frac{2V_B}{|AB|}$$

The reason for the incorrectness of options (1), (2), and (4) was mentioned in solving this example in the drawing method for the velocity analysis.

Option (3) is correct.

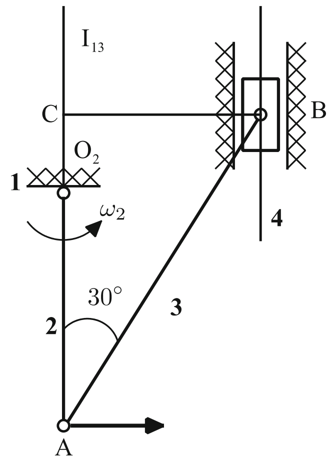


Fig. 2.31 Velocity direction analyses

2.7 Using the Transmission Line for Velocity Analysis

The distance between the points on a rigid object is always the same and does not change. Therefore, the velocity component of any two desired points of a rigid object in line with the connection of these two points is equal to each other. For example, if point A has the velocity component V_{At} along the line AB in Fig. 2.32, point B must also have the same velocity component in this direction. It is said that the velocity component of V_{At} is transmitted exactly to point B along line AB and line AB is called the transmission line.

The use of the transmission line is not limited to one member. When two rigid members are in contact with each other, the points of contact of both members must have the same velocity component in the direction of the common perpendicular at the point of contact. In this case, the common perpendicular of the two members at the point of contact is called the velocity transmission line between the two members. Using this concept in analyzing some problems is a much simpler and faster method than using velocity vector equations and other methods.

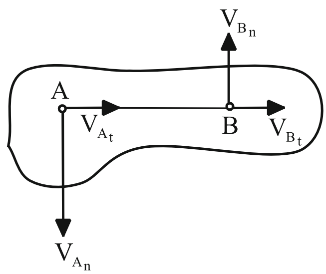


Fig. 2.32 Velocity along the connection line

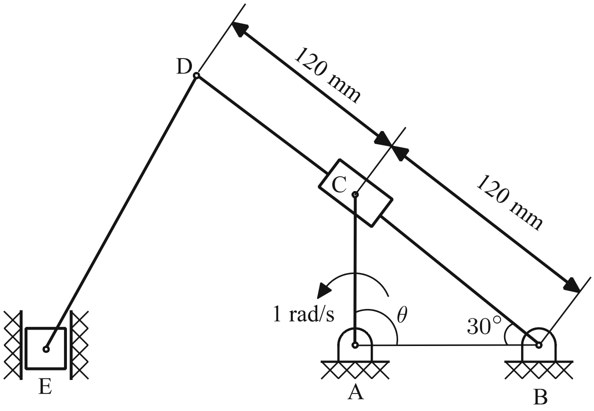


Fig. 2.33 A crankshaft bar mechanism

Example If the angular velocity of the AC crankshaft is constant and equal to $1(rad/s)$ (Fig. 2.33), at the moment when $\theta = 90°$, the angular velocity of the bar DB is equal to

1) $\frac{1}{4}$ rad/s 2) $\frac{1}{6}$ rad/s 3) $\frac{1}{8}$ rad/s 4) Neither

Solution The velocity of point C is known, and the line perpendicular to BD at point C acts as the transmission line (Fig. 2.34). Members 2 and 3 must have the same velocity component along the transmission line. We denote this component with V'. We have

$$\left.\begin{aligned} V_C &= |AC|\,\omega_4 \\ |AC| &= 120\sin 30 = 60\text{ mm} \\ \omega_4 &= 1\text{ rad/s} \end{aligned}\right\} \Longrightarrow V_C = 60\text{ mm/s}$$

$$V' = V_C\cos 60 = 30\text{ mm/s}$$

$$\omega_2 = \frac{V'}{|BC|} = \frac{30}{120} = \frac{1}{4}\text{ rad/s}$$

Option (1) is correct.

Example Bar 3 makes an angle of 30° with the horizon surface and the ramp, and bar 5 is perpendicular to the ramp (Fig. 2.35). If $V_A = 1$ cm/s, then

1) $V_B = V_A = V_{A/B}$
2) $V_B = 1\cos 30°\sin 30°$
3) $V_B = 1\cos 30°\cos 30°$
4) $V_B = 1$ Parallel to the ramp

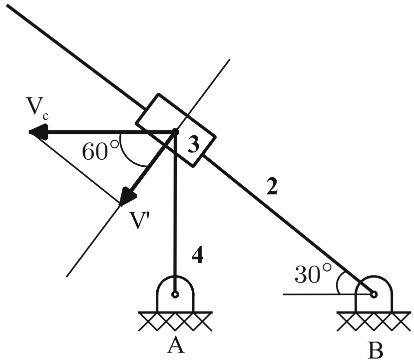


Fig. 2.34 Two bars and one slider

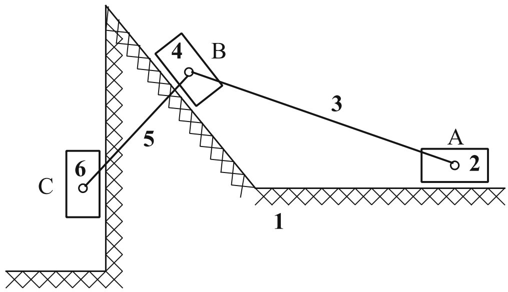


Fig. 2.35 Objects on the surface, connecting by two links

Solution Method 1: If we consider bar 3 as a velocity transmission line between its end points A and B, the velocity components of points A and B in the direction of bar 3 must be the same. On the other hand, points A and B also belong to sliders 2 and 4, so their velocity direction is parallel to their slide surface. Since the angle between the slide surfaces with bar 3 is both the same and equal to 30 degrees, therefore,

$$|V_A| \cos 30 = |V_B| \cos 30 \Longrightarrow |V_B| = |V_A| = 1 \text{ cm/s}$$

It is clear that velocity B is also parallel to the ramp.

Method 2: The instantaneous center of rotation of levers 2 and 4 with joints A and B forms an equilateral triangle (Fig. 2.36), and it can be written as

$$V_A = V_B = V_{A/B} = V_{B/A} = 1 \text{ cm/s}$$

Option (4) is correct.

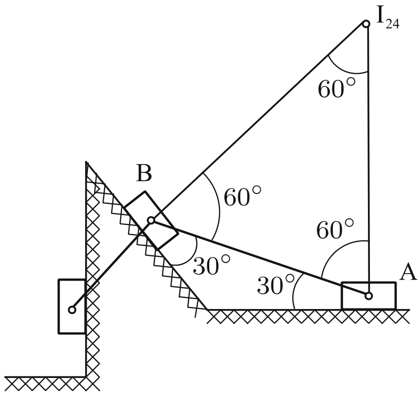


Fig. 2.36 Equilateral triangle

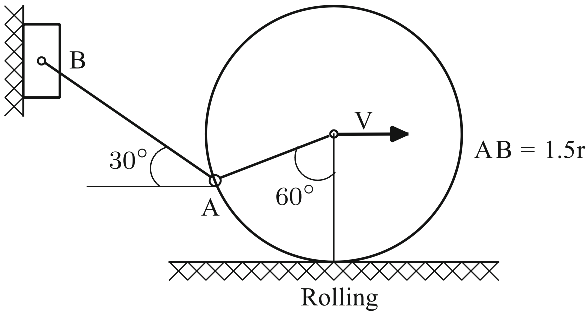


Fig. 2.37 A roller movement

Some Examples of "Displacement and Velocity Analysis"

1. In the shown mechanism in Fig. 2.37, the velocity of the roller center is V. What is the velocity of slider B?
   1) $\frac{1}{2}V$ 2) 0 3) V 4) $\frac{3}{2}V$
2. In the formed four-bar mechanism in Fig. 2.38, if the point S is assumed to belong to the AB interface, the velocity value of this point relative to the O_4B interface is equal to
   1) 0 2) $\overline{O_4S}.\omega_4$ 3) $\overline{SB}(\omega_4 - \omega_2)$ 4) $\overline{SB}\omega_4$
3. The eight-bar lever in Fig. 2.39 is under ω_2. Given the degree of freedom, which of the following equations is true for the value of the rotational velocity of the levers?

   1) $\omega_2 = \omega_3 = \omega_4$
   2) $\omega_5 = \omega_6 = \omega_7 = \omega_8 = 0$
   3) $\omega_2 = \omega_3 = \omega_4 = \omega_5 = \omega_6 = \omega_7 = \omega_8$
   4) $\omega_3 = \omega_5 = \omega_6 = \omega_7 = \omega_8$

4. In the six-bar mechanism in Fig. 2.40 for the positions $\theta_2 = \theta_6$ and $2O_4P_4 = O_4Q_4$, the velocity of slider 5 relative to slider 3 is
   1) 0 2) 0.5 3) 2 4) 1

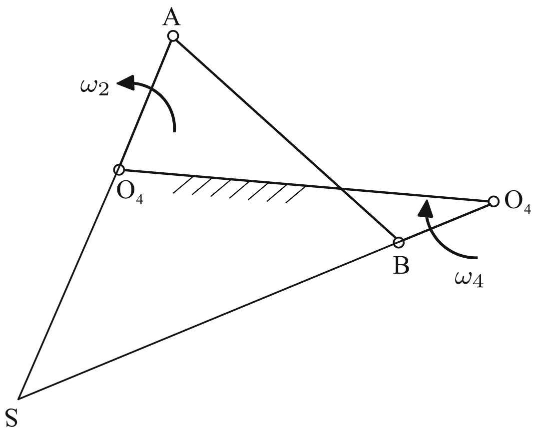


Fig. 2.38 A four-bar mechanism for velocity analysis

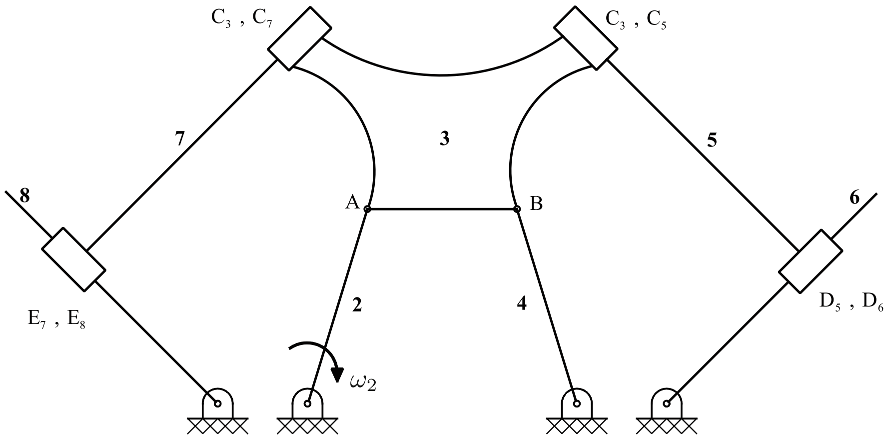


Fig. 2.39 An eight-bar lever

5. In the mechanism of Fig. 2.41, where is the instantaneous center of rotation of slider B relative to the ground (frame)?

1) Along the (p) and at ∞	2) At point Q
3) At point P	4) Along the (b) and at ∞

6. In the mechanism in Fig. 2.42, if the bar AB with a length of 10 cm moves with a clockwise rotational velocity of 1 rad/s, determine the velocity of the joint B and its direction at the shown moment. Do the rollers have both rolling and sliding movements?

   1) 8.7 cm/s to the right
   2) 5 cm/s to the left

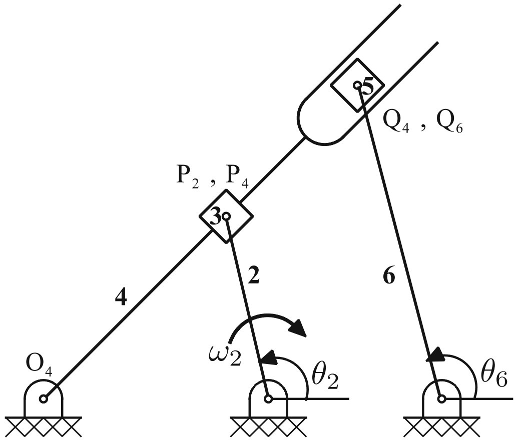


Fig. 2.40 A six-bar mechanism

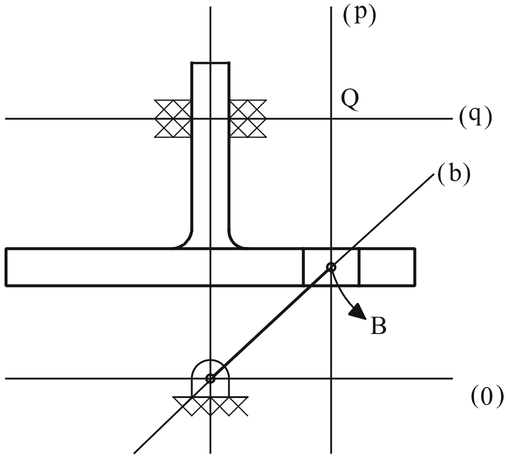


Fig. 2.41 Crank-slider

3) 5 cm/s to the right
4) It cannot be determined because the degree of freedom of the mechanism becomes 2.

7. In the six-bar lever, according to Fig. 2.43, if the velocity of point D, V_D is known, which of the following statements is correct?

1) At this point, the velocity of slider 6 is smaller than V_D.
2) At this point, the velocity of slider 6 is the same as the velocity of point D.
3) It is evident from the figure that the velocity of slider 6 is greater than V_D.
4) Because bar 4 carries a slider and is along the slider 6, the lever locks at this point.

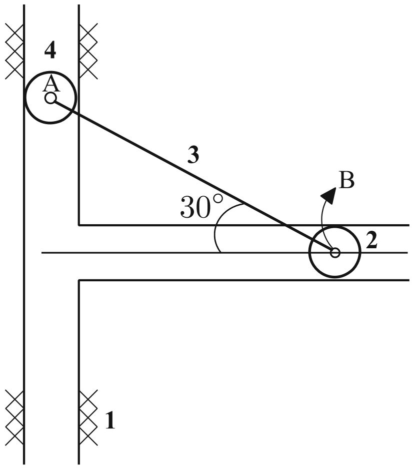


Fig. 2.42 Rollers with rolling and sliding movements

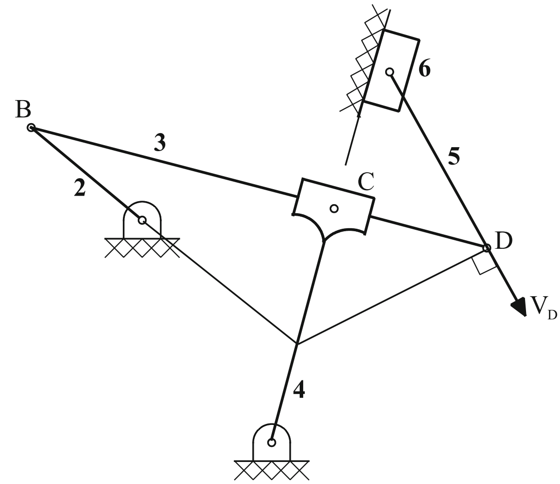


Fig. 2.43 A six-bar lever mechanism

8. In the four-bar mechanism in Fig. 2.44, if point C is the center of curvature of bar 2 at the point of contact with 3, which option is correct for instantaneous center 13?
   1) Point A 2) Point B 3) Point C 4) Point D
9. In the six-bar mechanism, according to Fig. 2.45, if V_Q is known for this moment, which of the following statements is correct?
   1) The velocities of sliders 4 and 6 are fractions of the velocity vector V_Q.
   2) The velocity of slider 4 depends on V_Q, but the velocity of slider 6 is independent.
   3) The velocity of slider 4 is equal to V_Q, and the velocity of slider 6 is determined by it.

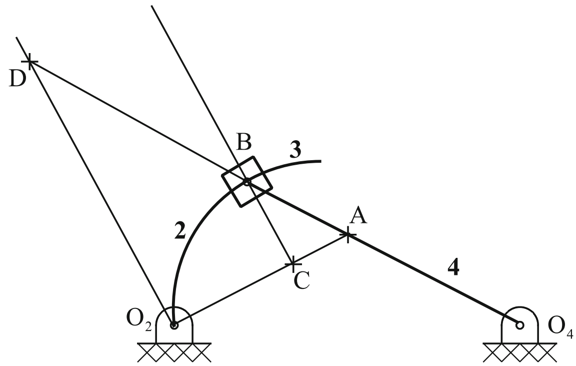


Fig. 2.44 A four-bar mechanism with a curved link

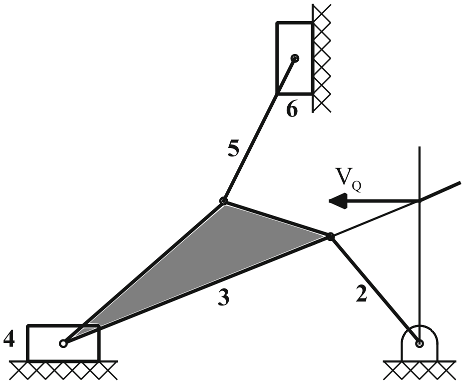


Fig. 2.45 A six-bar mechanism with two sliders

4) Sliders 4 and 6 have a rotational motion around the center at infinity, and their velocity has nothing to do with V_Q.

10. The value of ω_2 in the mechanism in Fig. 2.46 is equal to 10 rad/s and counterclockwise. Find the value of ω_4.

1) 8.3, counterclockwise	2) 10.7, moving counterclockwise
3) 15.9, counterclockwise	4) 20.6, clockwise

11. In a four-bar mechanism, according to Fig. 2.47, if bar 2 provides input movement, where is the center that slider 3 rotates around?

1) In terms of the type of lever, at point B
2) At the intersection of bar 4 with a line perpendicular to the groove from point O_2
3) At the intersection of bar 4 with a line perpendicular to the groove from point B
4) In terms of the lever type, it only has a sliding motion, and this center does not exist.

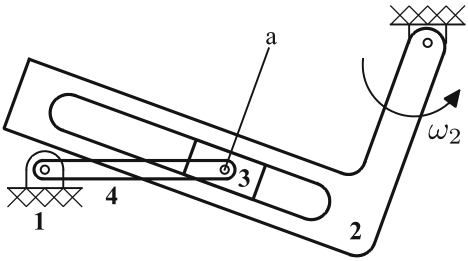


Fig. 2.46 A mechanism with counterclockwise input

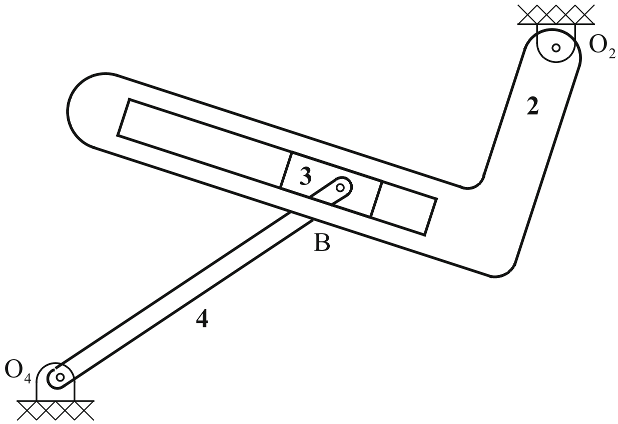


Fig. 2.47 A four-bar mechanism with rotating link in between

12. According to Fig. 2.48, the ascent velocity of the follower at the shown moment in centimeters per second is equal to N = 120 rpm, and the dimensions are in centimeters.
    1) 31.416 2) 37.25 3) 37.7 4) 43.53
13. In the mechanism in Fig. 2.49, if ω_2 is known, the mechanism has 21 instantaneous centers. How many of them can be determined?
    1) All 2) 9 3) 11 4) 10
14. In Fig. 2.50, obtain ω_{AB} in rad/s.
    1) 56.6 CCW 2) 56.6 CW 3) 34.1 CCW 4) 28.3 CCW
15. In the OAC slider-crank mechanism (Fig. 2.51), which answer is correct for the slider's velocity?
    1)$|V_C| = \omega_2.\overline{BC}$ 2)$|V_C| = \frac{\overline{OA}.\overline{BC}}{\overline{AB}}.\omega_2$
    3)$|V_C| = \frac{\overline{OA}.\overline{AC}}{\overline{BC}}.\omega_2$ 4)$|V_C| = \frac{\overline{BC}.\overline{AC}}{\overline{OA}}.\omega_2$
16. In the shown mechanism in Fig. 2.52, which statement is correct for the location of the instantaneous center of the velocity of members 4 and 5?

    1) It is along member 4.
    2) It is at point A.

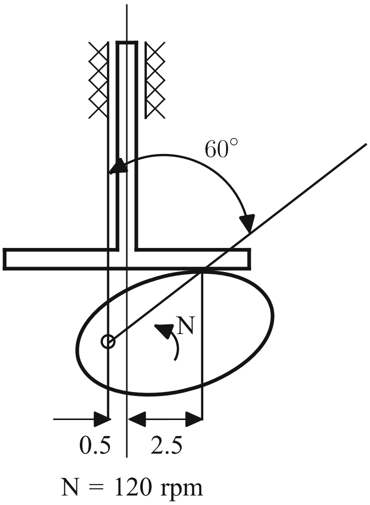


Fig. 2.48 A cam-follower mechanism

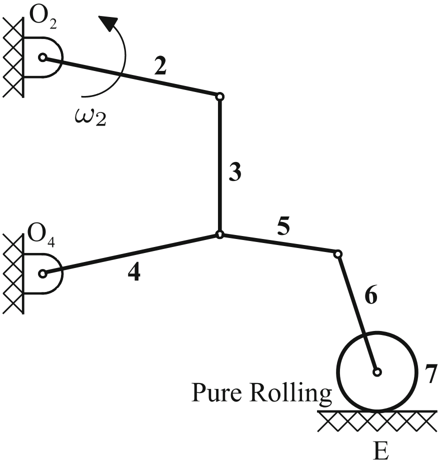


Fig. 2.49 Mechanism with 21 instantaneous centers

3) It is along the line perpendicular to member 4 from point *A* and at an infinite distance.
4) It is along the line perpendicular to member 4 from point *A* and at a finite distance.

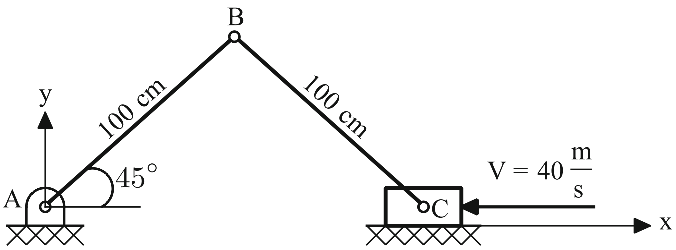


Fig. 2.50 Link AB rotational velocity

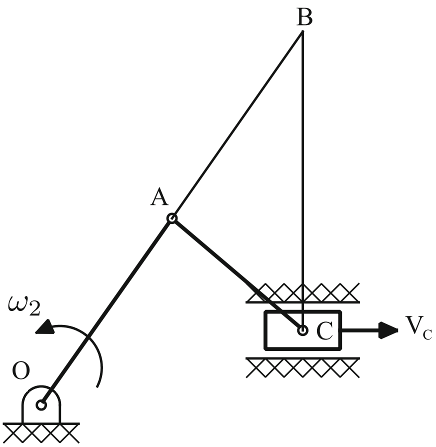


Fig. 2.51 A slider-crank mechanism

17. The bar AB rotates clockwise (Fig. 2.53) with angular speed $\omega = 3$ rad/s. The speed of slider D and angular speed of bar DB are

1) $\dot{\theta} = 3$ rad/s, $V_D = 2$ m/s 2) $\dot{\theta} = 3$ rad/s, $V_D = 4$ m/s
3) $\dot{\theta} = \frac{20}{3}$ rad/s, $V_D = 2$ m/s 4) $\dot{\theta} = \frac{20}{3}$ rad/s, $V_D = 4$ m/s

18. According to Fig. 2.54, the parallel plates are in contact with the cylinder, moving at certain velocities without sliding. Which of the following is <u>incorrect</u>?

1) Point $V_O = \frac{V_1}{4}$ 2) Point$\omega = \frac{3}{2}\frac{V_2}{R}$
3) Point $\omega = \frac{3}{4}\frac{V_1}{R}$ 4) Point $\omega = \frac{|V_1|-|V_2|}{2R}$

19. According to Fig. 2.55, what is the ascent velocity of the follower at the shown moment in centimeters per second? (N = 120rpm and sizes are in centimeters.)
1) 25.133 2) 29.25 3) 37.7 4) 45.533

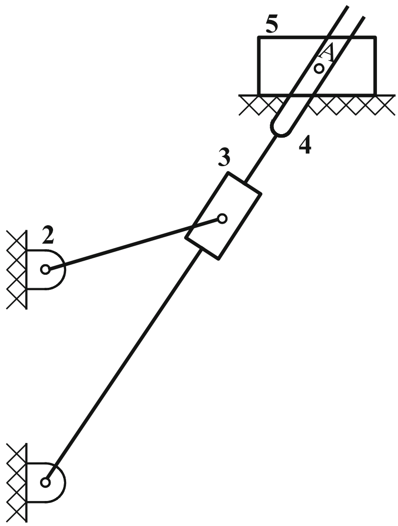


Fig. 2.52 A mechanism with some sliding members

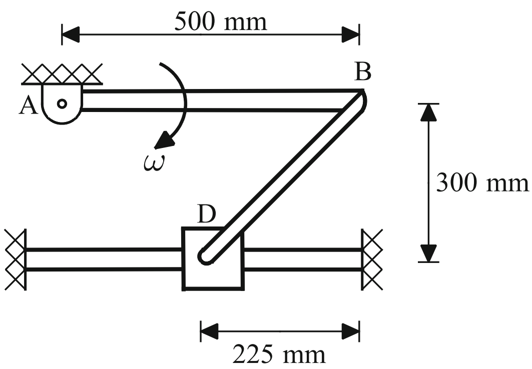


Fig. 2.53 Bar AB rotates clockwise as input

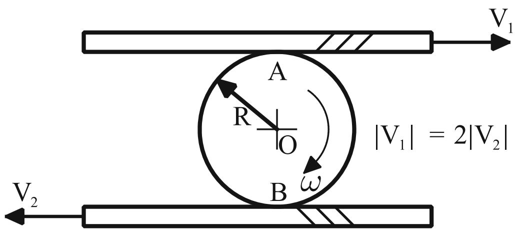


Fig. 2.54 Two parallel plates in contact with the cylinder

20. According to Fig. 2.56, what is the magnitude of the angular velocity of bar 3 at this moment in terms of radians per second? 1) 4.45 2) 6.36 3) 7.22 4) 8
21. In the mechanism shown in Fig. 2.57, the velocity of point *A* is given. Point *A* is marked on the center of the slider. The velocity of point *B* is equal to 1) 1.9 m/s 2) 2.9 m/s 3) 3.9 m/s 4) 4.9 m/s

Fig. 2.55 Cam-slider mechanism

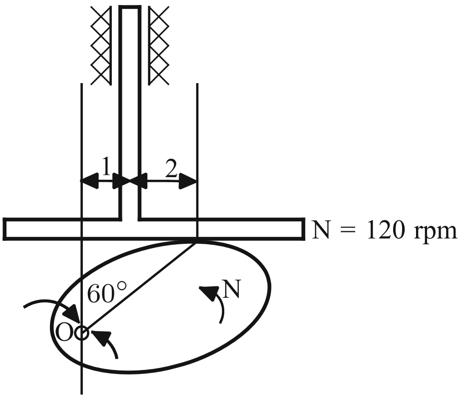


Fig. 2.56 Rod 3 has a slider

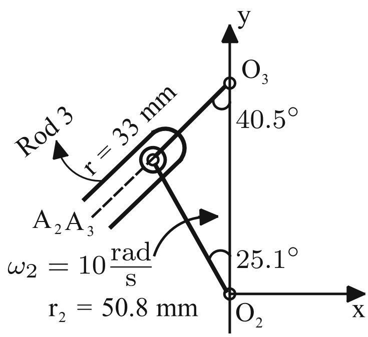
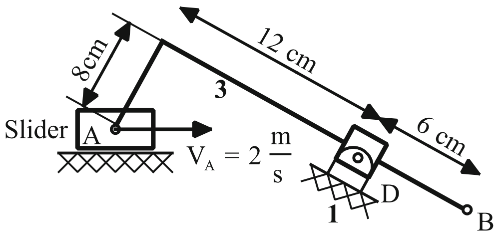


Fig. 2.57 Point A is sliding on the surface



22. What equation is correct for the four-bar mechanism in Fig. 2.58?

$$O_2A = 1 \quad O_4B = 4 \quad O_2O_4 = 4$$

1) $\omega_2 = 4\omega_4$ 2) $\omega_4 = 4\omega_2$ 3) $\omega_2 = 2\omega_4$ 4) $\omega_4 = 2\omega_2$

23. Given the angular velocity of the arm OB (member 2) (Fig. 2.59), which of the following equations is correct for finding the angular velocity of the arm BD (member 3)?

1) $\overline{V}_D = \overline{V}_B + \overline{V}_{D/B}, \omega_{BD} = \frac{V_{D/B}}{\overline{DB}}$

2) $\overline{V}_{C_3} = \overline{V}_B + \overline{V}_{C_3/B}, \omega_{BD} = \frac{V_{C_3/B}}{\overline{BC_3}}$

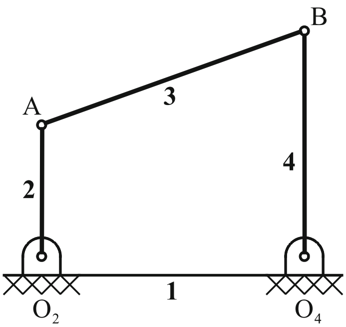


Fig. 2.58 Four-link mechanism

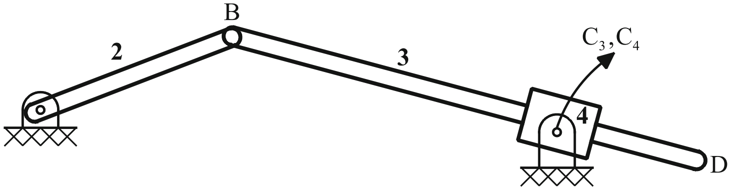


Fig. 2.59 Angular velocity of the arm OB is given

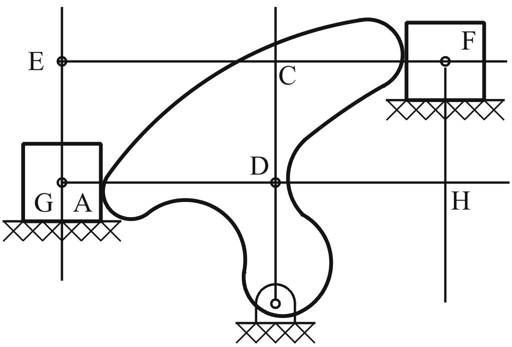


Fig. 2.60 Two sliding motions in two sides

3) $\overline{V}_{C_3} = \overline{V}_B + \overline{V}_{C_4/B}, \omega_{BD} = \frac{V_{C_4/B}}{\overline{C_4B}}$

4) $\overline{V}_{C_4} = \overline{V}_{C_3} + \overline{V}_{C_4/C_3}, \omega_{BD} = \frac{V_{C_4/C_3}}{\overline{C_4C_3}}$

24. What are the two points of the instantaneous centers of the non-primary rotation (other than 12, 13, and 14) in the mechanism of Fig. 2.60?
    1) *A* and *B* 2) *C* and *D* 3) *F* and *G* 4) *F* and *E*

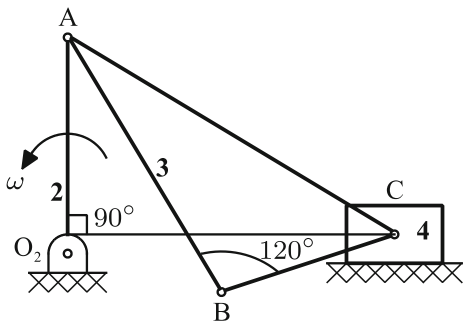


Fig. 2.61 A crank mechanism

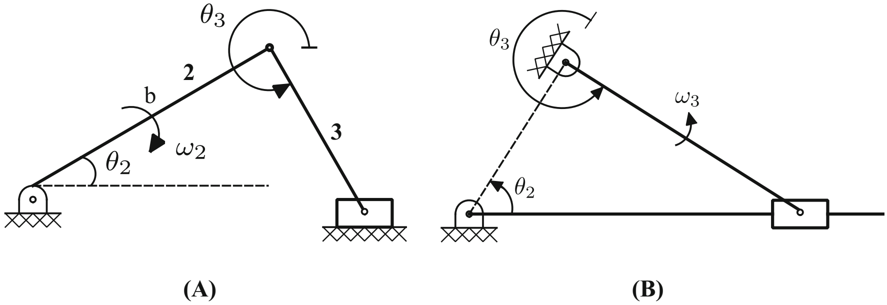


Fig. 2.62 Two crank-slider mechanisms

25. In the crank mechanism of Fig. 2.61, point B is a point on interface 3. In the shown state in the figure, what is the velocity of point B? ($AB = BC$)

    1) Zero
    2) Half the velocity of point A.
    3) Equal to the velocity of point A.
    4) Twice the velocity of point A.

26. If the angular velocity equation $\omega_3 = \frac{-b}{c}\frac{\cos\theta_2}{\cos\theta_3}\omega_2$ is correct for the sliding crank mechanism of Fig. 2.62A, which of the following equations about the rapid return mechanism of Fig. 2.62B is correct?

    1) $\omega_2 = \dfrac{-b}{c}\dfrac{\cos\theta_2}{\cos\theta_3}\omega_2$
    2) $\omega_3 = \dfrac{\omega_1}{1+\frac{b}{c}\frac{\cos\theta_2}{\cos\theta_3}}$
    3) $\omega_1 = \dfrac{\omega_3}{1+\frac{b}{c}\frac{\cos\theta_2}{\cos\theta_3}}$
    4) $\omega_3 = \dfrac{-b}{c}\dfrac{\cos\theta_2}{\cos\theta_3}\omega_2$

27. For the shown mechanism in Fig. 2.63, if the velocity of point D is known, which group of the following equations is sufficient to obtain the angular velocity of member 2?

    1) $V_E = V_{C_4} + V_{E/C_4}$ $\quad V_B = V_E + V_{B/E}$ $\quad V_E = V_D + V_{E/D}$

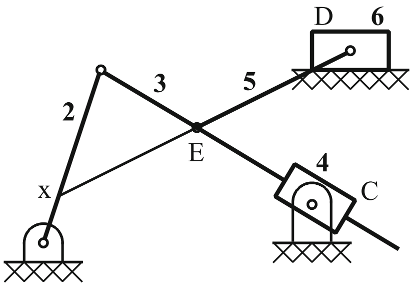


Fig. 2.63 Consider the velocity of point D is known

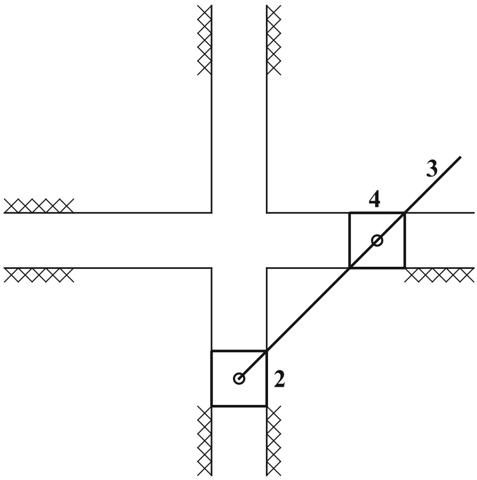


Fig. 2.64 An elliptical compass

2) $V_{C_4} = V_{C_3} + V_{C_4/C_3}$ $V_B = V_{C_4} + V_{B/C_4}$ $V_E = V_D + V_{E/D}$
3) $V_B = V_{C_3} + V_{B/C_3}$ $V_E = V_D + V_{E/D}$ $V_B = V_x + V_{B/x}$ $V_{C_3} = V_{C_4} + V_{C_3/C_4}$
4) $V_B = V_{C_3} + V_{B/C_3}$ $V_E = V_D + V_{E/D}$ $V_{C_3} = V_x + V_{C_3/x}$ $V_{C_3} = V_{C_3/C_4}$

28. Which of the following statements is true about the elliptical compass (Fig. 2.64)?

    1) It is a lever with four bars, one degree of freedom, and four instantaneous centers.
    2) It is a lever with six instantaneous centers, three of which are at infinity.
    3) The elliptical compass cannot be used to draw a circle.
    4) The elliptical compass can be used to draw a circle if the degree of freedom of the mechanism is changed.

29. In the mechanism in Fig. 2.65, the instantaneous centers 2 and 4 are

    1) Placed on component 3
    2) Along component 3 and at infinity
    3) Along component 3 but not at infinity
    4) Not defined for this mechanism

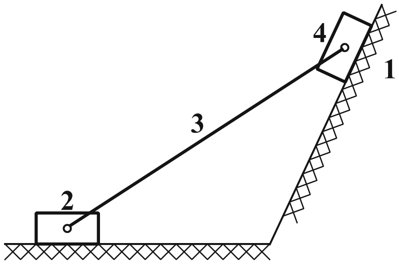


Fig. 2.65 Objects 2 and 4 are on the surface

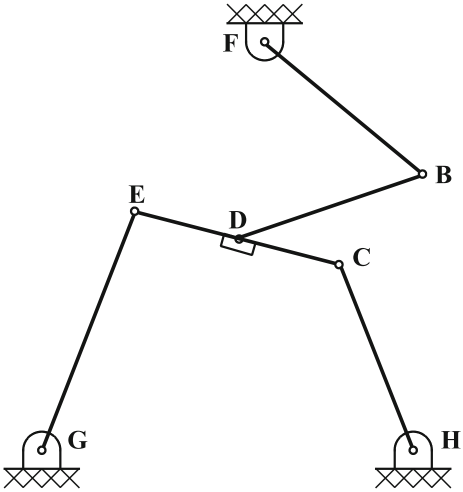


Fig. 2.66 A six-bar mechanism

30. Which of the following equations is not true about the six-bar mechanism in Fig. 2.66?

1) $\begin{cases} -V_{C/H} - V_{E/C} + V_{G/E} = 0 \\ -V_{B/F} - V_{D/B} - V_{E/D} + V_{G/E} = 0 \end{cases}$

2) $\begin{cases} -V_{C/H} - V_{E/C} + V_{E/G} = 0 \\ -V_{B/F} - V_{D/B} - V_{E/D} + V_{E/G} = 0 \end{cases}$

3) $\begin{cases} V_D = V_E + V_{D/E} \\ V_D = V_C + V_{D/C} \end{cases}$

4) $\begin{cases} V_C = V_H + V_{C/H} \\ V_C = V_D + V_{C/D} \end{cases}$

31. According to Fig. 2.67, a pin with diameter d is installed around disk A, and four grooves with width d have been created on the other disk. Disc A rotates evenly every second, and the motion is transferred to Disc B. In this case,

1) Disc B rotates only half a turn each time Disc A rotates.

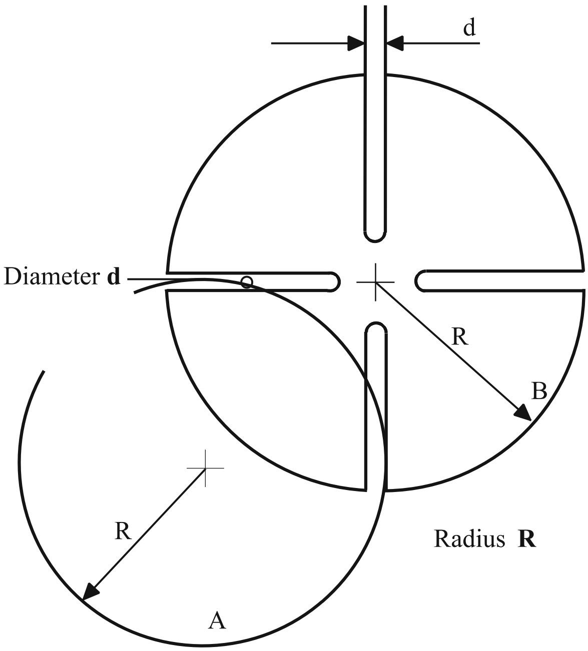


Fig. 2.67 Disc A rotates evenly every second

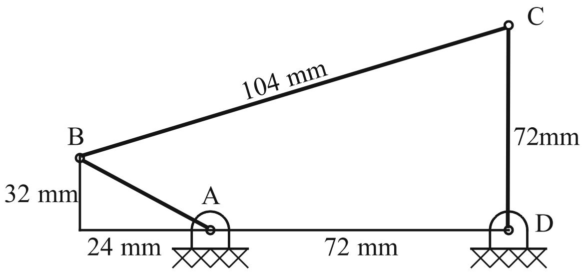


Fig. 2.68 Angular velocity of the member AB is equal to 20rad/s

2) Disc B Continues to rotate for only half a second for each rotation of disc A.
3) When disk B rotates, its angular velocity at any moment is equal to the angular velocity of disk A.
4) None.

32. In the previous problem, when the pins are in line with the centers of disks A and B, the angular velocity of disk B is equal to

1) 2π rad/s 2) $\frac{2\pi}{\sqrt{2}-1}$ rad/s 3) 4π rad/s 4) $\frac{1}{\sqrt{2}-1}$ rad/s

33. In the mechanism of Fig. 2.68, if the angular velocity of the member AB is equal to $20 rad/s$, the angular velocity of the member BC will be equal to

1) 5 rad/s 2) 10 rad/s 3) 15 rad/s 4) None

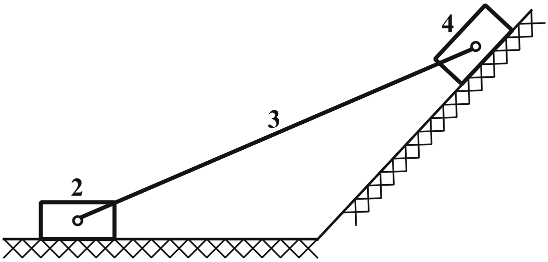


Fig. 2.69 Link 3 connects objects 2 and 4

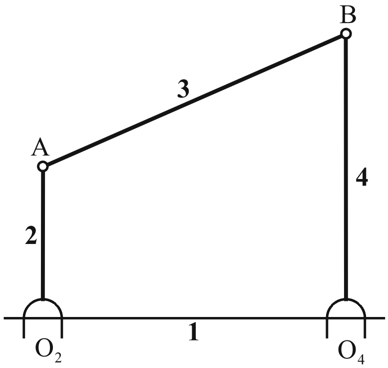


Fig. 2.70 A normal four-bar mechanism

34. In the mechanism shown in Fig. 2.69, the instantaneous centers 2 and 4 are
    1) Located on component 3
    2) Along component 3 but not at infinity
    3) Along component 3 and at infinity
    4) Not defined for this mechanism
35. Which equation is true in the four-bar mechanism in Fig. 2.70?

$$O_2A = 1 \quad O_4B = 4 \quad O_2O_4 = 4$$

1) $\omega_2 = 2\omega_4$ 2) $\omega_4 = 4\omega_2$ 3)$\omega_4 = 0$ 4)$\omega_2 = 4\omega_4$

Answers for the Examples of "Displacement and Velocity Analysis"

1. Option (2) is correct.
   The angular velocity of the roller is equal to $\omega = \frac{V}{R}$. If its rolling point is called O, we have

$$V_A = V_O + \omega \times r = 0 + \left(\frac{V}{R}\right) r = V$$

$\overrightarrow{V}_A$ is perpendicular to OA and has no component in the direction OA. Since OA and AB are also aligned, $\overrightarrow{V}_A$ has no component in the direction AB. Therefore, $\overrightarrow{V}_B$ also has no component in the direction AB. Note that the direction of velocity B must be in the direction of sliding (vertical direction). And since the component of this velocity in the direction AB is zero, $\overrightarrow{V}_B$ itself is also zero. That is, the velocity of slider B is zero.

2. Option (1) is correct.

   O_4 is the instantaneous center 14, B is the instantaneous center 34, A is the instantaneous center 23, and O_2 is the instantaneous center 12. It is easy to conclude from Kennedy's theorem that S is the instantaneous center 13, and therefore its instantaneous velocity is zero.

3. Option (4) is correct.

   To have a certain movement, the degree of freedom of the mechanism must be one, because otherwise, the calculation of all velocity parameters will not be possible. The contact of the sliders and bars is with weld, and it is possible to move through the sliders. Members 2, 3, and 4 form a four-bar mechanism that, with ω_2, the values of ω_3 and ω_4 will also be known. The rotation of member 3 moves bars 5 and 7 at the same rotational velocity (ω), and so bars 6 and 8 will rotate at the same rotational velocity. So we have

$$\omega_3 = \omega_5 = \omega_6 = \omega_7 = \omega_8$$

4. Option (2) is correct.

   First method:

   The velocity of member 5 in the direction perpendicular to member 4 is twice the velocity of member 3 in the direction perpendicular to member 4. It is equal to $O_4Q_4 = \omega_4$ and $O_4P_4 = \omega_4$, respectively. On the other hand, the velocities of both sliders 5 and 2 are parallel. Since the length of member 6 is twice the length of member 2 (according to the similarity of the triangles), the velocity of member 5 in the direction perpendicular to member 6 is twice the velocity of member 3 in the direction perpendicular to member 2. Therefore, their image along member 4 has the same ratio.

   Second method:

$$V_{P_2} = V_{P_4} + V_{P_2/P_4}$$

$$V_{Q_6} = V_{Q_4} + V_{Q_6/Q_4}$$

   It can be shown that these two vector equations form two similar triangles with a similarity ratio of 2. Therefore $V_{Q_6} = 2V_{P_2}$ and therefore $V_{Q_5} = 2V_{P_3}$.

5. Option (4) is correct.

   Using Kennedy's theorem, we can find the instantaneous center of rotation of slider B relative to the ground. According to Fig. 2.71,

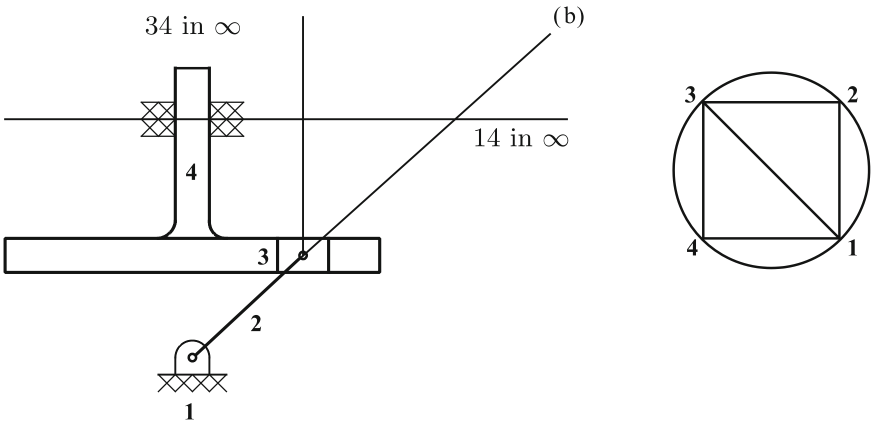


Fig. 2.71 Finding the instantaneous rotation centers

I_{12} is connected to I_{23}, which is a line in the direction (b). The junction of I_{14} and I_{34} is also at infinity. Thus the instantaneous center I_{13} is located along (b) and at infinity.

6. Option (2) is correct.

$$V_A = V_B + \omega \times R_{BA}$$

$$\Longrightarrow V_A \hat{j} = V_B \hat{i} + (1\hat{k}) \times 10(\frac{\sqrt{3}}{2}\hat{i} - \frac{1}{2}\hat{j})$$

Since the component $\hat{j}$ of the velocity B is zero, we consider only the component $\hat{i}$, and from the above equation, we have

$$V_B = -5\hat{i}$$

7. Option (3) is correct.

According to the transmission line discussion, the velocity component of the two ends of member 5 in the direction of connection of these two points is the same. On the other hand, point D has only one velocity component in the direction of member 5. Still, the other end of member 5 (the head connected to lever 6) and this component of velocity have another component in the direction perpendicular to member 5. Therefore, part of the velocity of slider 6 is equal to the total velocity of point D. It can be said that the total velocity of slider 6 is greater than the velocity D.

8. Option (1) is correct.

Instantaneous center 13 is located at the intersection of the connecting line between instantaneous centers 23–12 (O_2C) and instantaneous centers 34–43 (member 4). This intersection is point A.

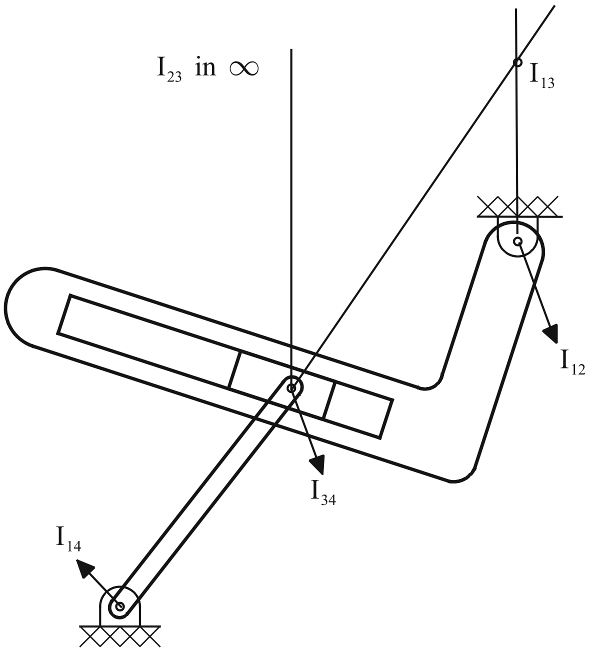


Fig. 2.72 Line connecting instantaneous centers of links 2 and 3 and links 1 and 2

9. Option (3) is correct.
   Point Q is located along the virtual extension of member 2. This point is located along the connecting line of I_{12} and I_{14}. On the other hand, this point is along the connecting line of I_{34} and I_{23}. Thus, this point is the instantaneous center of rotation 24 (I_{24}) according to Kennedy's theorem. So its velocity is equal to the velocity of slider 4.
10. Option (2) is correct.
    Due to the lack of dimensions and sizes, an exact solution cannot be provided. But considering Fig. 2.72, the distance of I_{12} to slider 3 is approximately equal to the distance of I_{14} to this slider. On the other hand, I_{12}, I_{14}, and slider 3 are approximately in the same direction. Therefore, the velocity of slider 3 perpendicular to this direction is calculated from the following equations:

$$\left.\begin{array}{l} V_3 = \omega_4 \times r_4 \\ V_3 = \omega_2 \times r_2 \end{array}\right\} \Longrightarrow \omega_4 \simeq \omega_2$$

11. Option (2) is correct.
    Instantaneous center I_{13} is located along the line connecting instantaneous centers I_{23} and I_{12}, as well as instantaneous centers I_{14} and I_{34}.
12. Option (3) is correct.

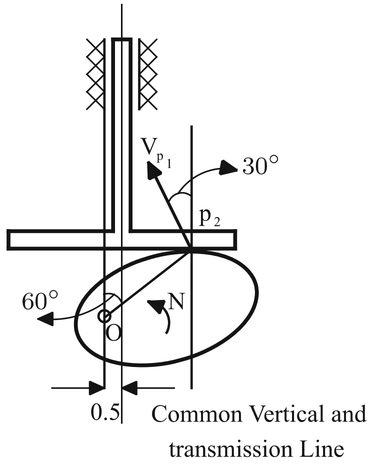


Fig. 2.73 Camshaft and follower at the point of contact

Assuming that P_1 and P_2 are the points belonging to the camshaft and the follower at the point of contact (Fig. 2.73), respectively:

$$V_{P_1} = |OP_1|\omega_1 \quad \omega_1 = \frac{120 \times 2\pi}{60} = 12.57\ \text{rad/s}$$

$$|OP_1| = \frac{3}{\sin 60} = 3.46\text{cm}$$

$$V_{P_1} = 3.46 \times 12.57 = 43.49\text{cm/s}$$

The image of the velocities of points P_1 and P_2 along the common vertical must be equal and in the same direction. And on the other hand, because the direction of V_{P_2} is also in the direction of the common vertical,

$$V_{P_2} = V_{P_1} \cos 30 = 43.49 \cos 30 = 37.7\text{cm/s}$$

Test method: The desired answer is the multiplication of the horizontal distance of the point of contact from the point O $(3cm)$ with ω_1.

$$V_{P_2} = 3 \times 12.57 = 37.7$$

10. Option (3) is correct.

By knowing w and ω_2, the velocity characteristics of members 2, 3, and 4 can be obtained. Therefore, all instantaneous centers related to the 4-3-2-1 mechanism, which are 6, can be calculated.

Also, due to the pure rotation of the disk, the instantaneous center I_{17} is the point of contact of the disk with the ground. Also, I_{67}, I_{56}, I_{35}, and I_{45} are quite clear. Then the other 5 instantaneous centers of the system are also identified. Because the system has two degrees of freedom, it is impossible to determine all instantaneous centers by knowing only one velocity quantity (ω_2). So $6 + 5 = 11$ instantaneous centers can be determined.

11. Option (4) is correct.

The velocity component of points B and C along the BC transmission line must be the same, so

$$V_{C/BC} = 40 \cos 45^\circ = \frac{40}{\sqrt{2}} = 28.28$$

This component is the velocity in the direction of BC, from C to B. But the velocity vector of point B is perpendicular to AB. Therefore due to the perpendicularity of BC to AB, it is in the direction of BC (away from C).

$$V_{B/BC} = V_B = 28.28 \text{ m/s}$$

On the other hand,

$$V_B = |AB|\,\omega_{AB} = \frac{100}{100}\omega_{AB} \Longrightarrow \omega_{AB} = V_B = 28.3$$

12. Option (2) is correct.

Point B is the instantaneous center of rotation and the instantaneous center of the bar AC with the ground, and it looks like this bar is pinned to the point B, so

$$\begin{pmatrix} V_A = |AB|\,\omega_{AC} \\ V_A = |OA|\,\omega_2 \end{pmatrix} \Longrightarrow \omega_{AC} = \frac{|OA|\,\omega_2}{|AB|}$$

$$V_C = |BC|\,\omega_{AC} = \frac{|BC|\,|OA|}{|AB|}\omega_2$$

Note that the dot symbol does not mean internal multiplication in this question.

13. Option (3) is correct.

Member 5 slides on member 4, and this slide is a straight line. So the instantaneous center is on the line perpendicular to member 4 and is at infinity.

14. Option (3) is correct.

You can see the directions of the velocities in Fig. 2.74.

$$|V_B| = |AB|\,\omega \qquad |V_B| = 0.5 \times 3 = 1.5 \text{ m/s} \qquad V_D = V_B + V_{D/B}$$

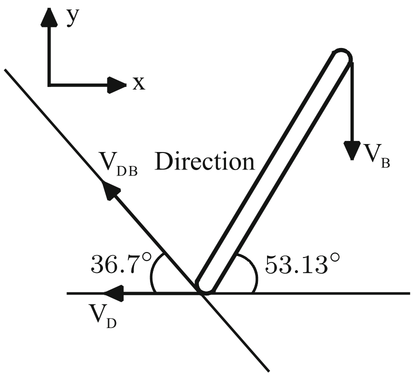


Fig. 2.74 Directions of velocities

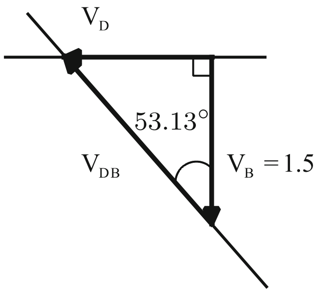


Fig. 2.75 Velocity triangle

According to the velocity triangle drawn in Fig. 2.75, we can write

$$\left|V_{D/B}\right| = \frac{|V_B|}{\cos 53.13} = 2.5 \text{ m/s}$$

$$\dot{\theta} = \frac{\left|V_{D/B}\right|}{|DB|} = \frac{2.5}{0.375} = \frac{20}{3} \text{ rad/s}$$

$$V_D = |V_B| \tan 53.13 = 2 \text{ m/s}$$

15. Option (4) is correct.

The planes have non-slide contact with the disk, so their velocity is the same as the disk velocity at points A and B (Fig. 2.76). By obtaining the center of rotation C,

$$\omega = \frac{\left|V_{A/B}\right|}{|AB|} = \frac{3V_2}{2R} = \frac{3V_1}{4R}$$

$$V_1 = |CA|\,\omega \Longrightarrow |CA| = \frac{4R}{3} \Longrightarrow |CO| = \frac{R}{3}$$

$$V_0 = |CO|\,\omega = \frac{R}{3} . \frac{3}{4} \frac{V_1}{R} = \frac{V_1}{4}$$

So options (1), (2), and (3) are correct and option (4) is incorrect.

Fig. 2.76 Velocities of points A and B

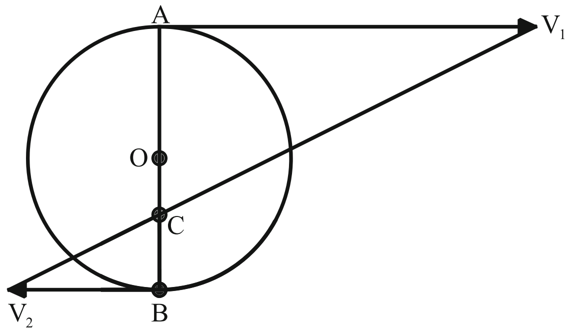
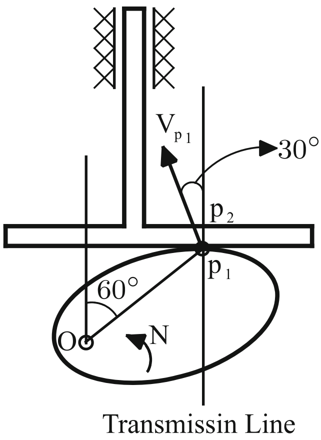


Fig. 2.77 Transmission line and common vertical direction



16. Option (3) is correct.

    Assuming that P_1 and P_2 are the points belonging to the camshaft and the follower (Fig. 2.77) at the point of contact, respectively:

$$V_{P_1} = |OP_1|\,\omega_1$$

$$|OP_1| = \frac{3}{\sin 60} = 3.46\ \text{cm} \qquad \omega_1 = \frac{120 \times 2\pi}{60} = 12.57\ \text{rad/s}$$

$$V_{P_1} = 3.46 \times 12.57 = 43.49\ \text{cm/s}$$

    The image of the velocities of points P_1 and P_2 in the common vertical direction must be equal and in the same direction. And on the other hand, because the direction of V_{P_2} is also in the common vertical direction,

$$V_{P_2} = \left|V_{P_1}\right| \cos 30 = 43.49 \cos 30 = 37.7\ \text{cm/s}$$

17. Option (2) is correct.

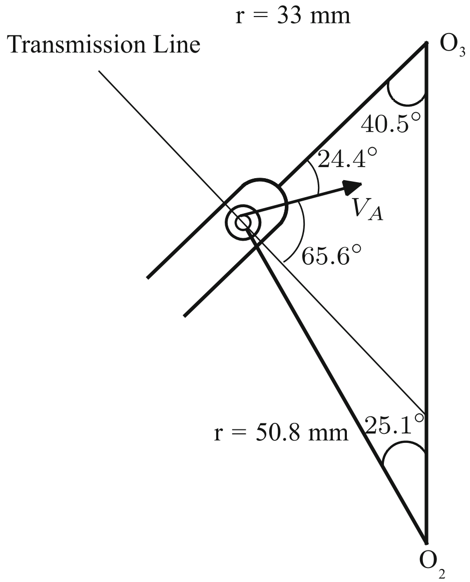


Fig. 2.78 The velocity of the center of the roller

The velocity of point A_2 in the center of the roller is as in Fig. 2.78:

$$\left|V_{A_2}\right| = r_2.\omega_2 \qquad \left|V_{A_2}\right| = 50.8 \times 10 = 508 \text{ mm/s}$$

The image of the velocity of point A_2 in the common vertical direction (transmission line) gives us the velocity of point A_3 belonging to bar 3 at the point of contact.

$$\left|V_{A_3}\right| = \left|V_{A_2}\right| \cos 65.6 = 508 \cos 65.6 = 209.85 \text{ mm/s}$$

$$\omega_3 = \frac{\left|V_{A_3}\right|}{r_3} = \frac{209.85}{33} = 6.36 \text{ rad/s}$$

18. None of the options is correct.

If point D belongs to member 3, its velocity direction is known. On the other hand, points D and A belong to a rigid body, so the velocity components are the same along the connecting line (Figs. 2.79 and 2.80) (see the text of the transmission line section).

$$V_D \cos 33.6 = V_A \Longrightarrow V_D = 2.4 \text{ m/s}$$

$$V_{D/A} = V_D \sin 33.6 = 1.33 \text{ m/s}$$

$$\omega_3 = \frac{V_{D/A}}{|AD|} = \frac{1.33}{0.144} = 9.24 \text{ rad/s } c\omega$$

$$V_B = V_D + V_{B/D}$$

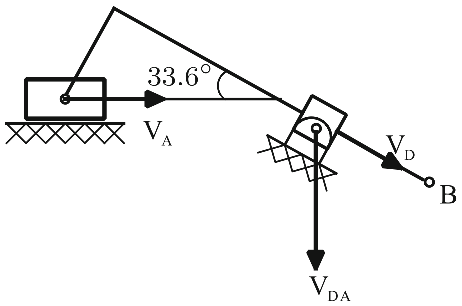


Fig. 2.79 Velocity components of points A and D

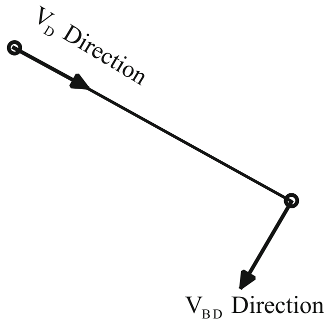


Fig. 2.80 Velocity of point D relative to point B

$$V_{B/D} = |DB|\,\omega = 0.06 \times 9.24 = 0.55 \text{ m/s}$$

$$V_B = \sqrt{V_D^2 + V_{B/D}^2} = \sqrt{2.4^2 + 0.55^2} = 2.46 \text{ m/s}$$

19. Option (1) is correct.

    The velocities of points A and B from member 3 are in the same direction, so ω_3 is zero.

$$V_{B/A} = |AB|\,\omega_3 = 0$$

$$V_B = V_A + V_{B/A} \Longrightarrow V_B = V_A$$

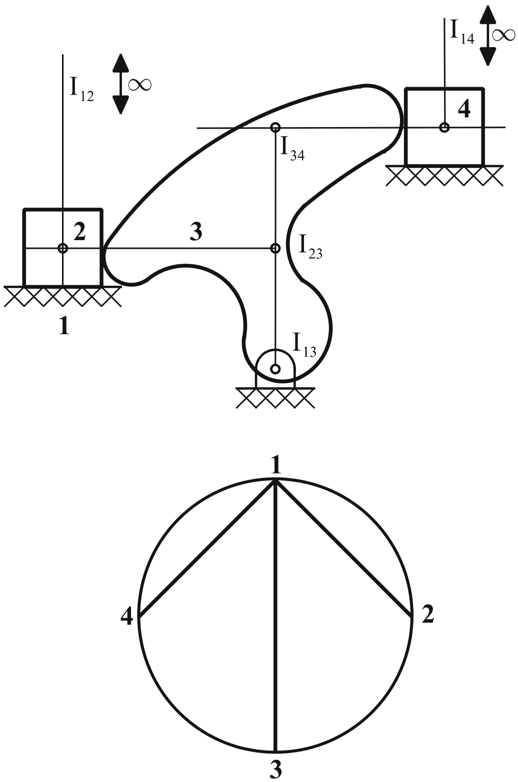
$$|O_2A|\,\omega_2 = |O_4B|\,\omega_4\omega_2 = 4\omega_4$$

20. Option (2) is correct.

    In the given mechanism, the velocity of point B and the velocity direction of point C_3 are known. At the same time, the velocity direction of $V_{C_3/B}$ is also known, so by solving the following equation, the size of $V_{C_3/B}$ and V_{C_3} can be obtained.

$$V_{C_3} = V_B + V_{C_3/B}$$

Fig. 2.81 Auxiliary circle



By obtaining $V_{C_3/B}$,

$$\omega_{BD} = \frac{V_{C_3/B}}{|BC_3|}$$

So option (2) is correct.

Option (1) is wrong because we know nothing about point D, the equations of option (3) are also wrong, and option (4) does not give us any specific information.

21. Option (2) is correct.

First, note that the velocities of components A and B must be equal at the instantaneous center I_{AB}. Therefore, I_{23} and I_{34} must be located on the common vertical (3,2) and (4,3), respectively.

By drawing an auxiliary circle (Fig. 2.81) and considering the direct contact of 2 to 3 and 3 to 4, we can conclude

$$I_{34} \xrightarrow{\text{At the intersection}} \begin{cases} I_{14} - I_{13} \text{ The connecting line} \\ \qquad \text{Common vertical} \end{cases}$$

$$I_{23} \xrightarrow{\text{At the intersection}} \begin{cases} I_{12} - I_{13} \text{ The connecting line} \\ \qquad \text{Common vertical} \end{cases}$$

So C and D are the instantaneous centers.

22. Option (3) is correct.

The direction of velocity at points A and C is clear. Because the two velocities are parallel, the instantaneous center of I_{13} is at infinity, meaning that all points on member 3 have the same velocity:

$$V_B = V_A$$

23. Option (3) is correct.

We must first note the difference between the two sets. The difference is in the fixed position of hinge 23 and the consequent fixing of bar 2. On the other hand, due to the similarity of the geometry of the two sets, the relative angular velocity between the members must remain constant. The given equation of angular velocity in relative terms is as follows:

$$(\omega_3 - \omega_1) = -\frac{b}{c}\frac{\cos\theta_2}{\cos\theta_3}(\omega_2 - \omega_1)$$

Because for the first mechanism, $\omega_1 = 0$.

In the second mechanism, $\omega_2 = 0$, so from the placement of $\omega_2 = 0$ in the above equation, we have

$$\omega_3 - \omega_1 = +\frac{b}{c}\frac{\cos\theta_2}{\cos\theta_3}\omega_1 \Longrightarrow \omega_1 = \frac{\omega_3}{1 + \frac{b}{c}\frac{\cos\theta_2}{\cos\theta_3}}$$

24. Option (3) is correct.

If C_3 and C_4 are points corresponding to C belonging to members 3 and 4, respectively:

In option (1) between E and C_4, in option (2) between B and C_4, and in option (4) between x and C_3, where the points belong to two different objects, the written velocity equations are not correct, but if the two points coincide, such as C_4 and C_3, the equation between the velocities is correct. The equations of the option (3), while regarding this note, use the points about which we have information, such as points B, D, and C_3, the velocity of which is known to us, while the last equation is an additional equation and there is no need for it.

25. Option (2) is correct.

$$n = 4 \Longrightarrow \text{Number of instantaneous centers} = \frac{n(n-1)}{2} = \frac{4(4-1)}{2} = 6$$

$$n = 4,\ f_1 = 4,\ f_2 = 0 \Longrightarrow \text{DOF} = 3\,(n-1) - 2f_1 - f_2$$

$$= 3\,(4-1) - (-2 \times 4) = 1$$

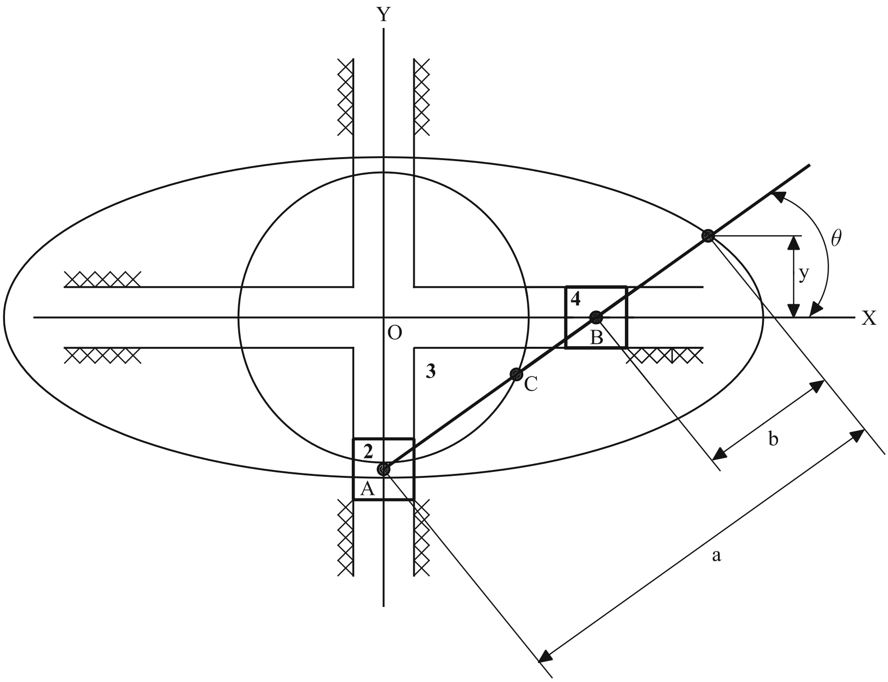


Fig. 2.82 Circle is a kind of ellipse

Option (3) is wrong because the circle is a kind of ellipse (Fig. 2.82). It should be noted that if the tip of the pen is in the middle of the distance between joints 23 and 34 (meaning when the pen P is at point C), the equation of the circle is obtained, and there is no need to change the degree of freedom of the mechanism.

26. Option (3) is correct.

According to Kennedy's theorem, I_{24} is at the intersection of the connecting line of I_{23} and I_{34} with the connecting line of I_{12} and I_{14}, but I_{14} and I_{12} are at infinity, so the instantaneous center I_{24} is along component 3 and at infinity (Fig. 2.83).

27. Option (1) is correct.

Options (3) and (4) represent the velocity equations between two points on a rigid body, so they are correct. In option (2), since points F, H, and G are fixed,

$$V_{B/F} = V_B \qquad V_{E/G} = V_E \qquad V_{C/H} = V_C$$

$$-V_{C/H} - V_{E/C} + V_{E/G} = -V_C - V_E + V_C + V_E = 0$$

$$-V_{B/F} - V_{D/B} - V_{E/D} + V_{E/G} = -V_B - V_D + V_B - V_E + V_D + V_E = 0$$

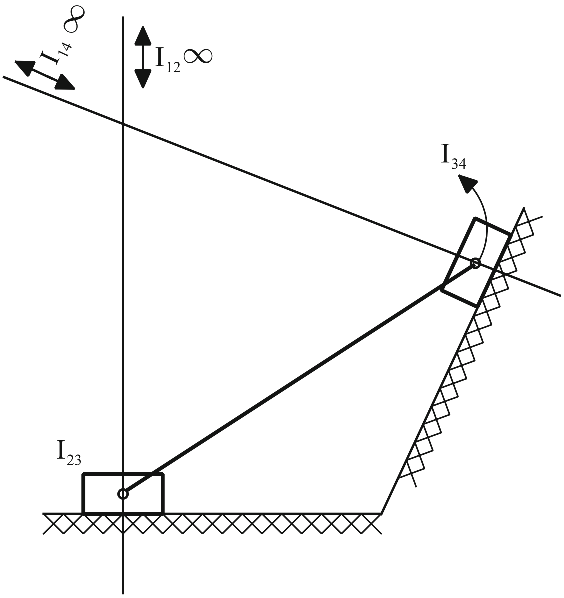


Fig. 2.83 Intersection of the connecting lines

Option (2) equations are also correct, so option (1) is wrong. Option (1) equations can be checked like option (2). Note that these equations are vectors.

28. Option (4) is correct.

    In the Genoa wheel mechanism, for one rotation of disc A, disc B rotates a quarter, and then disc B continues to rotate for a quarter of a second.

29. Option (2) is correct.

    The pin is a moving point pin on the coordinate system connected to disk B (Fig. 2.84). When the pin is aligned with the disc centers A and B, its relative velocity relative to the groove is zero. So if P_1 and P_2 are points belonging to disks 1 and 2 at the point of contact, then

$$V_P = V_{P_1} = V_{P_2} + V_{P_1/P_2} \qquad V_{P_1/P_2} = 0$$

$$V_{P_1} = V_{P_2} \qquad R_{\omega_1} = |C_2P|\,\omega_2$$

$$|C_1C_2| = \sqrt{2}R \qquad |C_2P| = \left(\sqrt{2} - 1\right) R$$

$$\omega_1 = 2\pi \text{ rad/s}$$

$$R_{\omega_1} = \left(\sqrt{2} - 1\right) R\omega_2 \Longrightarrow \omega_2 = \frac{2\pi}{\sqrt{2} - 1} \text{ rad/s}$$

30. Option (1) is correct.

Fig. 2.84 P is the contact point

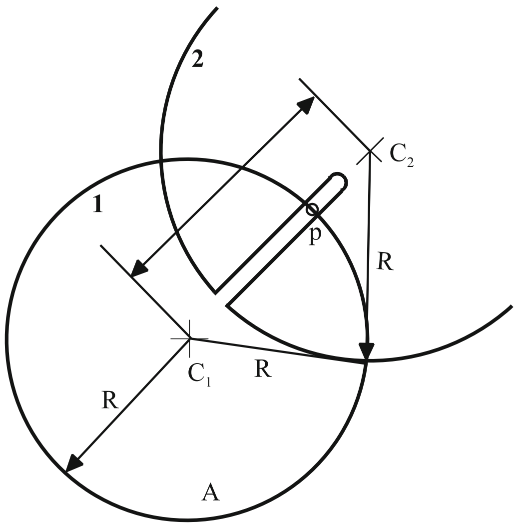
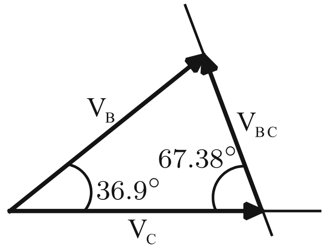


Fig. 2.85 Points B and C velocities



The direction of velocity V_C is known. Using vector equations and the geometry of the object (Fig. 2.85),

$$|AB| = 40 \text{ mm}$$

$$V_B = |AB|\,\omega = 40 \times 20 = 800 \text{ mm/s}$$

$$V_B = V_C + V_{B/C}$$

$$\frac{\sin 67.38}{|V_B|} = \frac{\sin 36.9}{\left|V_{B/C}\right|}$$

$$\left|V_{B/C}\right| = 520.36 \text{ mm/s}$$

$$\omega_{BC} = \frac{\left|V_{B/C}\right|}{|B_C|} = \frac{520.36}{104} = 5 \text{ rad/s}$$

31. Option (3) is correct.

According to Kennedy's theorem, I_{24} is at the intersection of the connecting line of I_{23} and I_{34} with the connecting line of I_{12} and I_{14}. But I_{12} and I_{14} are at infinity. So I_{24} is along lever 3 and at infinity.

32. Option (4) is correct.

The velocities of points A and B, from member 3, are in the same direction, so $\omega_3 = 0$. So we have

$$\vec{V}_B = \vec{V}_A + \vec{V}_{B/A},\ V_{B/A} = |AB|\,\omega_3 = 0 \Longrightarrow V_B = V_A$$

$$\Longrightarrow |O_2A|\,\omega_2 = |O_4B|\,\omega_4 \Longrightarrow \omega_2 = 4\omega_4$$

Chapter 3
Acceleration Analysis

This chapter delves into the study of acceleration in different motions. It presents an overview of the subject along with various analytical and drawing methods for its examination. The chapter begins with an introduction, highlighting the significance of acceleration analysis, followed by an exploration of acceleration equations specifically tailored for curved motions. Additionally, it discusses an analytical method employed for the study of acceleration, providing valuable insights. Furthermore, a drawing method is presented, which serves as a practical tool for verifying acceleration calculations. Overall, this chapter offers a comprehensive and detailed account of acceleration analysis, laying the foundation for further understanding and application in relevant fields.

3.1 Introduction

In the previous chapters, we learned how to determine the instantaneous velocity of any point of the mechanism. In this chapter, we will study the determination of acceleration at different points of the mechanism. Because of the effect of acceleration on inertial forces, which in turn affect the resulting stresses in the components of a machine, bearing loads, vibration, noise, etc., acceleration in a mechanism is of particular importance. Analysis of acceleration in a mechanism can be performed by summing the relative accelerations. The method is similar to that used for relative velocities.

A. Abbasimoshaei, T. A. Kern, *Machine Dynamics*,
https://doi.org/10.1007/978-981-99-6010-1_3

3.2 Acceleration Equations for Curved Motions

If we denote the velocity by V, the average acceleration of the particle between two points A and A' on a curve is defined as $\Delta v/\Delta t$, which is a vector in the Δv direction. The value of the average acceleration is equal to the value of Δv divided by the value of Δt. The instantaneous acceleration of the particle a is by definition the limit of the average acceleration when the time interval Δt tends to zero. This means:

$$a = \frac{\Delta v}{\Delta t} \tag{3.1}$$

So, according to the definition of the derivation, we can write

$$a = \frac{dv}{dt} = \dot{v} \tag{3.2}$$

The acceleration equation for the plane motion of the curved line in the orthogonal coordinate system $(x - y)$ is

$$a = \dot{v} = \ddot{r} = \ddot{x}\hat{i} + \ddot{y}\hat{j} \tag{3.3}$$

where r is the location vector.

If ρ is the radius of curvature of the object's path and v is its velocity, in the vertical-tangent coordinate system $(n - t)$, we have

$$a = \frac{v^2}{\rho}\hat{e}_n = \dot{v}\hat{e}_t, \tag{3.4}$$

where $a_t = \dot{v}$ represents the tangential acceleration and $a_n = \frac{v^2}{\rho}$ represents the vertical acceleration. The vertical component of the acceleration a_n is always in the direction of the center of curvature of the motion path. Note that the tangential component of the acceleration a_t is in the positive direction of t as the value of v increases, and in the negative direction of t as it decreases.

Note A circular motion is a special case of motion on a plane curved line where the radius of curvature ρ is a fixed distance and is equal to the radius r of the circle.

The acceleration equation for this type of motion in the polar coordinate system $(r - \theta)$ is

$$a = \left(\ddot{r} - r\dot{\theta}^2\right)\hat{e}_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\hat{e}_\theta \tag{3.5}$$

where $a_r = \ddot{r} - r\dot{\theta}^2$, $a_\theta = r\ddot{\theta} + 2\dot{r}\dot{\theta}$, and $|a| = \sqrt{a_r^2 + a_\theta^2}$.

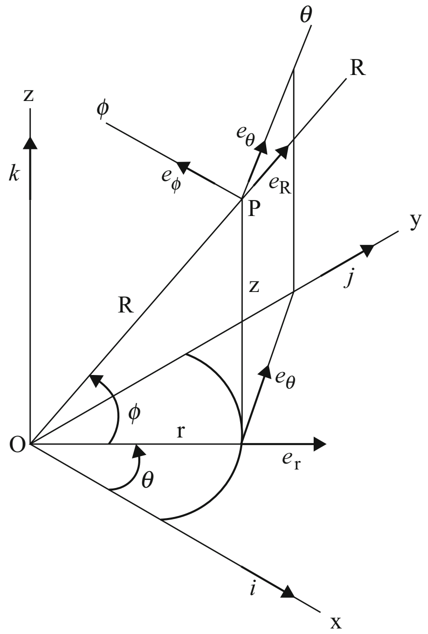


Fig. 3.1 Three coordinate systems

Figure 3.1 shows three coordinate systems, orthogonal (x− y −z), cylindrical (r −θ−z), and spherical (R−θ − Φ) along with the unit vectors.

The acceleration equation for spatially curved line motion in orthogonal ($x - y - z$), cylindrical ($r - \theta - z$), and spherical ($R - \theta - \varphi$) coordinate systems is

$$a = \dot{v} = \ddot{R} = \ddot{x}\hat{i} + \ddot{y}\hat{j} + \ddot{z}\hat{k} \qquad \text{(orthogonal coordinate system)}$$

$$a = \left(\ddot{r} - r\dot{\theta}^2\right)\hat{e}_r + \left(r\ddot{\theta} + 2\dot{r}\dot{\theta}\right)\hat{e}_\theta + \ddot{z}\hat{k} \qquad \text{(cylindrical coordinate system)}$$

$$a = a_R\hat{e}_R + a_\theta\hat{e}_\theta + a_\varphi\hat{e}_\varphi \qquad \text{(spherical coordinate system)}$$

which in the last equation we have

$$a_R = \ddot{R} - R\dot{\varphi}^2 - R\dot{\theta}^2\cos^2\varphi$$

$$a_\theta = \frac{\cos\varphi}{R}\frac{d}{dt}(R^2\dot{\theta}) - 2R\dot{\theta}\dot{\varphi}\sin\varphi$$

$$a_\varphi = \frac{1}{R}\frac{d}{dt}\left(R^2\dot{\varphi}\right) + R\dot{\theta}^2\sin\varphi\cos\varphi$$

3.3 Analytical Method for the Study of Acceleration

In the following, different methods of acceleration analysis are investigated.

To determine the equation of relative acceleration from the equation of relative velocity $V_A = V_B + V_{A/B}$, we take a derivative with respect to time and have

$$a_A = a_B + a_{A/B} \tag{3.6}$$

This equation states that the acceleration of point A is equal to the sum of the acceleration vectors of point A and B from the point of view of a non-rotating observer who is moving along with B.

Example Car A is driving on a curved path with a radius of curvature of 60 m with a constant speed of 50 km/h. When car A is in the position shown in Fig. 3.2, car B is 30 m away from the intersection and has an acceleration of 1.2 m/s^2 in the south direction (toward the intersection). Calculate the acceleration of car A from the point of view of the passenger of car B at the given time.

1) 2.8 m/s^2, 17° 2) 3.1 m/s^2, 31° 3) 4.3 m/s^2, 22° 4) 2.7 m/s^2, 57°

Solution Since the velocity of car A is constant, the only component of acceleration is the acceleration perpendicular to the direction of motion (in the direction of OA) and equal to $\frac{V_A^2}{R}$. Thus:

$$\vec{a}_A = \frac{V_A^2}{R}\frac{\overrightarrow{AO}}{AO} = \left(\frac{50}{3.6}\right)^2 \times \frac{1}{60} \times \left(-\sin 30\hat{i} + \cos 30\hat{j}\right) = -1.6\hat{i} + 1.6\sqrt{3}\hat{j} \tag{3.7}$$

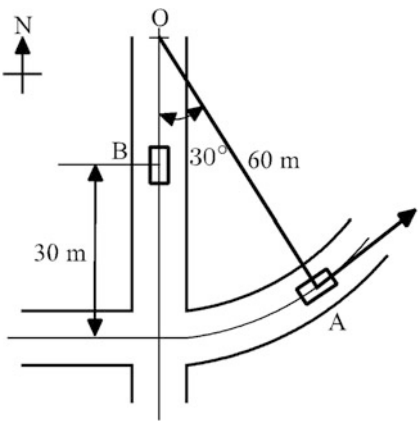


Fig. 3.2 Cars A and B

The acceleration of car B is also 1.2 m/s^2 downward. This means:

$$\vec{a}_B = -1.2\hat{j}$$

The acceleration of A from the point of view of B is equal to the relative acceleration vector of $a_{A/B}$. Therefore,

$$\vec{a}_{A/B} = \vec{a}_A - \vec{a}_B = -1.6\hat{i} + 3.97\hat{j}$$

$$|a_{A/B}| = \sqrt{1.6^2 + 3.97^2} = 4.28 \text{ m/s}^2$$

The angle of this vector with respect to the vertical line (on which B lies) is denoted by θ and is equal to

$$\tan\theta = \frac{1.6}{3.97} = 0.403 \Rightarrow \theta = 21.95°$$

Option (3) is correct.

Considering the relative accelerations of $a^n_{A/B}$ and $a^t_{A/B}$, the above equation is written as follows:

$$a_A = a_B + a^n_{A/B} + a^t_{A/B} \tag{3.8}$$

or

$$a_A = a_B + \omega \times (\omega \times r) + \alpha \times r \tag{3.9}$$

where ω and α are the angular velocity and acceleration, respectively.

Note The equations of relative acceleration depend on absolute angular velocity and absolute angular acceleration.

Example A roller with a radius of r on the ground with a constant angular velocity of ω is in a pure rolling state. In the case shown in Fig. 3.3, the angular acceleration of the bar AB is desired.

1) $\alpha_{AB} = 0$ 2) $\alpha_{AB} = \frac{\omega^2}{2}$ 3) $\alpha_{AB} = \omega^2$ 4) $\alpha_{AB} = 2\omega^2$

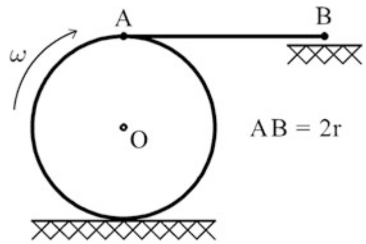


Fig. 3.3 A roller with pure rolling movement

Solution Points B and O have only horizontal motion, so their acceleration in the vertical direction is zero. If we denote the vertical direction by y and considering the equation of relative acceleration between points O and A, we have

$$a_A = a_O + a_{A/O} \Rightarrow (a_A)_y = (a_O)_y + (a_{A/O})_y = 0 - r\omega^2 = -r\omega^2 \qquad \text{(I)}$$

Now, if we consider two points A and B belonging to the rigid rod, we have

$$\vec{a}_A = a_B + a_{A/B} = \vec{a}_B + \vec{\omega}_{AB} \times \left(\vec{\omega}_{AB} \times \vec{r}_{AB}\right) + \alpha_{AB} \times \vec{r}_{AB} \qquad \text{(II)}$$
$$\Rightarrow (a_A)_y = (a_B)_y + 0 - |AB|\,\alpha_{AB} = 0 + 0 - 2r\alpha_{AB} = -2r\alpha_{AB}$$

We note that according to the direction of $\vec{\omega}_{AB}$ and $\vec{r}_{A/B}$, the expression $\vec{\omega}_{AB} \times \left(\vec{\omega}_{AB} \times \vec{r}_{AB}\right)$ in the above equation has no components in the direction y.

$$(I)\,,(II) \Rightarrow 2r\alpha_{AB} = r\omega^2 \Rightarrow \alpha_{AB} = \frac{\omega^2}{2}$$

Option (2) is correct.

Equation 3.9 was calculated for the non-rotating coordinate system. The equation of relative acceleration under the general condition that the motion is with respect to the rotating axes is

$$a_A = a_B + \dot{\omega} \times r + (\omega \times r) + 2\omega \times V_{rel} + a_{rel} \qquad (3.10)$$

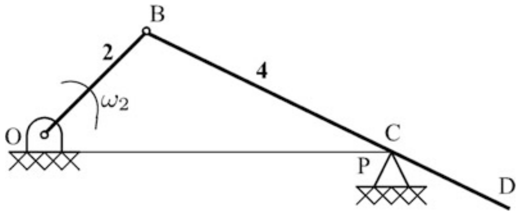
The resulting equation is a general vector equation giving the absolute acceleration of particle A in terms of its relative acceleration a_{rel}, from the point of view of a moving coordinate system rotating with an angular velocity of ω and an angular acceleration of $\dot{\omega}$.

Note The expression $2\omega \times V_{rel}$, also called Coriolis acceleration, shows the difference in acceleration of point A compared to other points from the viewpoint of a rotating and a non-rotating coordinate system.

Example In which of the following situations is the Coriolis force less than zero (it is assumed that the absolute and relative velocities are not equal to zero)?

1) Acceleration is zero.
2) Velocity and angular acceleration vectors are parallel.
3) Relative velocity and angular velocity vectors are parallel.
4) Angular acceleration is zero.

Fig. 3.4 C is the contact point with support



Solution The Coriolis force is obtained by multiplying the mass of the particle by its Coriolis acceleration. The Coriolis acceleration is also obtained from the external multiplication of the apparent relative velocity vector and the angular velocity of the device in which the relative velocity is measured.

$$a_c = 2\omega \times V_{rel}$$

The external multiplication of two parallel vectors is zero. Thus, for a_c to be zero, it is sufficient that ω is parallel to V_{rel}.

Option (3) is correct.

Example In the mechanism shown in Fig. 3.4, the angular velocity of limb 2 is ω_2 and constant. From which equation can the angular acceleration of limb 4 be derived? (The point C on the rod BD is next to the support P, and the rod is always in contact with this support.)

1) $a_C = a_B + a^n_{C/B} + a^t_{C/B}, a_C = a_P + a^{rel}_{C/P} + a^{Cr}_{C/P}$
2) $a_D = a_C + a^n_{D/C} + a^t_{D/C}, a_P = a_C + a^{cr}_{P/C} + a^{rel}_{P/C}$
3) $a_D = a_B + a^n_{D/B} + a^t_{D/B}, a_C = a_P + a^{rel}_{C/P} + a^{Cr}_{C/P}$
4) $a_C = a_B + a^n_{C/B} + a^t_{C/B}, a_P = a_C + a^{rel}_{P/C} + a^{Cr}_{P/C}$

Solution The acceleration of point B is known, so one of our points in setting up the equations must be point B. We know nothing about point D, so options 2 and 3 are wrong. Point P behaves like a particle moving relative to the rotating coordinate system associated with the link BD. Its relative motion path according to point C is in line with the rod, which is a straight line, so the radius of curvature of the path is infinite, and this means that the relative vertical acceleration is zero, according to the equations of the rotating axes, option (4) is correct. Option (1) is wrong, because if we put the observer on the ground, it sees the path of motion of C as a curved line whose radius of curvature is not known to us, i.e., the relative vertical acceleration is not known.

Option (4) is correct.

3.4 Drawing Method for Checking Acceleration

Similar to the determination of velocity, the acceleration of the points of a mechanism can also be determined by drawing acceleration polygons and acceleration images. The vector polygon associated with Eq. 3.9 is shown schematically in Fig. 3.5. When the lever velocity analysis is performed, the angular velocities of all elements are known, so the radial component $\vec{a}^{\,r}_{B/A} = -\omega^2 r_{B/A}$ can always be calculated and drawn. Thus, if one of the other vectors is known and the orientation of the other two is also known, the polygon can be drawn using a method similar to the vector triangle used in velocity analysis.

The angular acceleration for the known element is determined in a similar way to the angular velocity, except that the tangential component of the relative acceleration is used instead of the linear velocity. To determine the value of $\vec{a}$, we need to know the tangential components of the relative acceleration between each of the two selected points on the element. For example, the equation of relative tangential acceleration for two points A and B can be written as $\vec{a}^{\,t}_{B/A} = \vec{a} \times \vec{r}_{B/A}$. Since we need to know the lines along which the vectors are aligned, the main problem is to determine the direction of the lines and the magnitude of each vector. If we know the directions of one of the vectors, we can determine the third direction using the right-hand rule (Fig. 3.6).

For example, consider the mechanism in Fig. 3.7, where the angular velocity ω_2 and angular acceleration a_2 and all geometric parameters are assumed to be known.

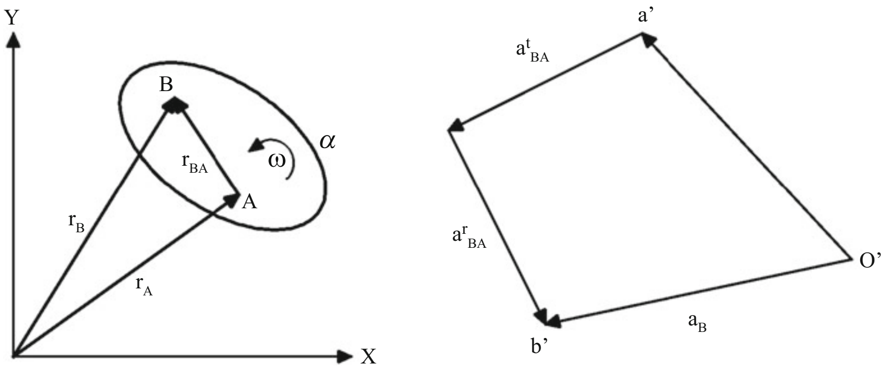


Fig. 3.5 Drawing method for checking acceleration

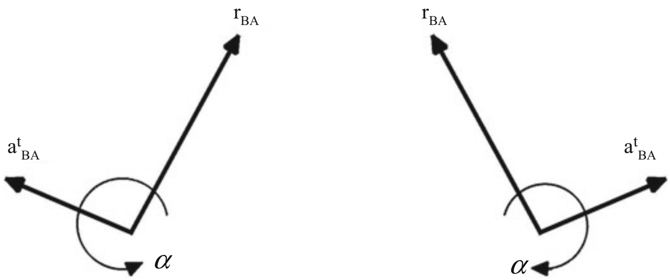


Fig. 3.6 Tangential acceleration direction

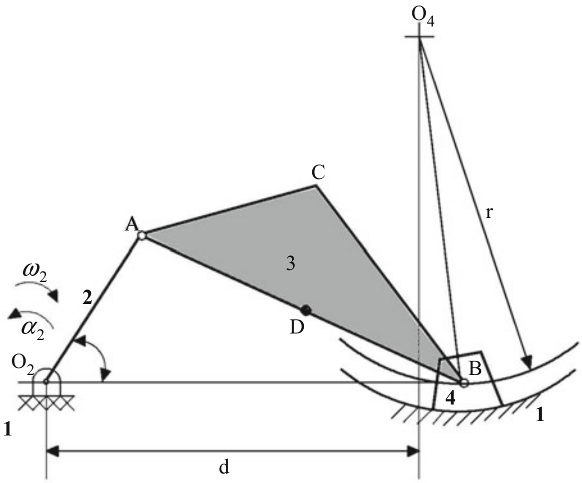


Fig. 3.7 Mechanism with curved output

To determine other unknown parameters, the velocity polygon must first be drawn. In this example:

	V_B =	V_A +	V_{BA}
Length (m)		✓	
Direction (d)	✓	✓	✓

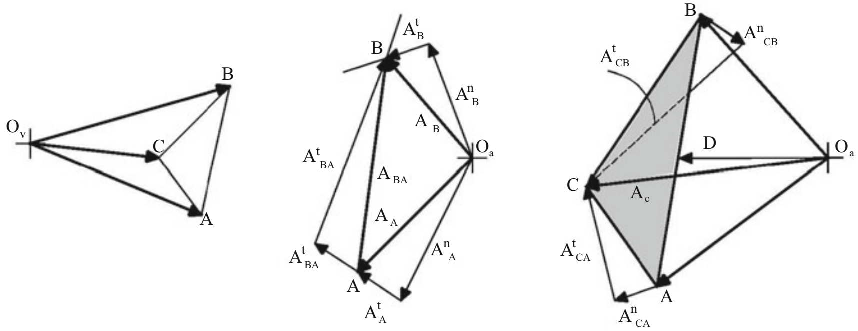


Fig. 3.8 Velocity and acceleration polygons

Using a vector equation and two unknowns, the magnitude of V_B and V_{BA} can be determined. By solving two vector equations as below, the magnitude and direction of V_C, the magnitude of V_{CA}, and the magnitude of V_{CB} can then be determined.

	$V_C =$	$V_A +$	V_{CA}
Length (m)		✓	
Direction (d)		✓	✓

	$V_C =$	$V_B +$	V_{CB}
Length (m)		✓	
Direction (d)		✓	✓

With the velocity values, we can now draw the acceleration polygon and determine the unknown acceleration values (Fig. 3.8). We have:

$$A_B = A_A + A_{BA}$$

$\Rightarrow$	A^n_B	$+ A^t_B$	$= A^n_A$	$+ A^t_A$	$+ A^n_{BA}$	$+ A^t_{BA}$
Length (m)	✓		✓	✓	✓	
Direction (d)	✓	✓	✓	✓	✓	✓

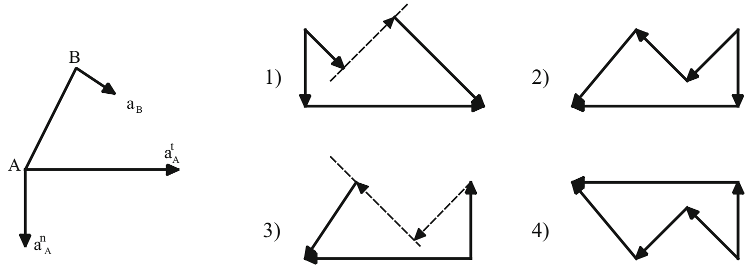
We have a vector equation and two unknowns A^t_B and A^t_{BA}, which are easy to determine. In the same way, we obtain the following equations:

$$A_C = A_A + A^n_{CA} + A^t_{CA}$$
$$A_C = A_B + A^n_{CB} + A^t_{CB}$$

The velocity and acceleration polygons are as follows:

Note The origin of the velocity polygon is given by O_V, and the origin of the acceleration polygon is given by O_a.

Example What is the acceleration polygon for the rod AB of the below mechanism?



	$A_B =$	$A^n_A +$	$A^t_A +$	$A^n_{BA} +$	A^t_{BA}
Length (m)	√	√	√		
Direction (d)	√	√	√	√	√

Solution From the diagram, it can be seen that the extent of the normal and tangential vector components of point B with respect to A is known. On the other hand, the acceleration vectors of points A and B are completely known (magnitude and direction). Therefore, the vector equation can be solved with two unknowns (the value of the normal and tangential acceleration components of B with respect to A), and the unknowns are obtained by drawing the acceleration polygons.

Option (1) is correct.

Instead of drawing a table to find out the known and unknown quantities and directions, we can briefly represent them with + and − signs above the velocity and acceleration equations. In this case, the + sign means the parameter is known and the − sign means it is unknown. The sign on the left represents the direction, and the sign on the right represents the magnitude. For example, $\pm A_D$ means that the direction of the acceleration of point D is known and its magnitude is unknown.

It should be noted that the drawing method is also valid for the case when the Coriolis acceleration component is added and the procedure is similar.

Some Examples of "Acceleration Analysis"

1. What is the acceleration of the point P from link 2, when link 1 is stationary and link 2 rotates about it with a constant angular velocity of ω? (R_1, R_2 are the radii of links 1 and 2, respectively) (Fig. 3.9)

   1) $R_2\omega^2$ 2) $R_1\omega^2$ 3) $(R_1 + R_2)\omega^2$ 4) $\frac{R_1 R_2}{R_1+R_2}\omega^2$

2. If we write down all the relative acceleration equations for the six-bar mechanism, which of the following statements is true for the acceleration components (Fig. 3.10)?

Fig. 3.9 Link 1 is stationary

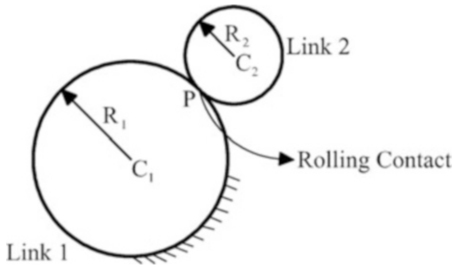
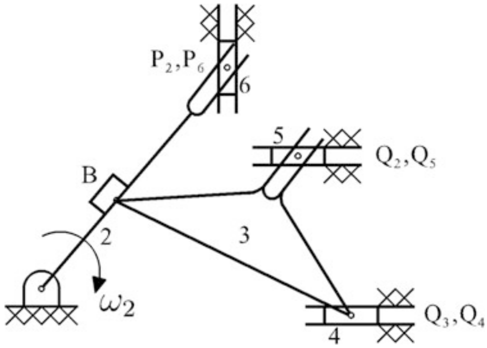


Fig. 3.10 A six-bar mechanism



1) There are two non-zero components of the Coriolis acceleration and two zero components of the relative normal acceleration.
2) All components of Coriolis acceleration and normal acceleration are non-zero.
3) There are three components of Coriolis acceleration that are non-zero and three components of relative normal acceleration that are zero.
4) All components of the relative acceleration can be zero or non-zero depending on the type of joint.

3. The diagram in Fig. 3.11 shows the direction of velocity and acceleration of two points A and B relative to a fixed reference point and their values. What are the velocity (in meters per second) and acceleration (in meters per second squared) of point B relative to point A?

1) 30 and 253 2) 30 and 224 3) 28 and 115 4) 41 and 224

4. The object shown is rolling on the ground without sliding with an angular velocity of ω and an angular acceleration of α (Fig. 3.12). The acceleration of the point A_2 is:

1) $r\alpha$ perpendicular to the motion path of point A_2
2) $r\alpha$ tangential to the motion path of point A_2
3) $r\omega^2$ perpendicular to the motion path of point A_2
4) $r\omega^2$ tangential to the motion path of point A_2

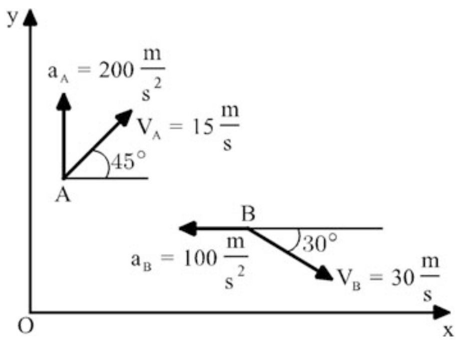


Fig. 3.11 Velocity and acceleration of two points A and B

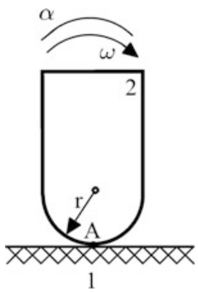


Fig. 3.12 Rolling on the ground without sliding

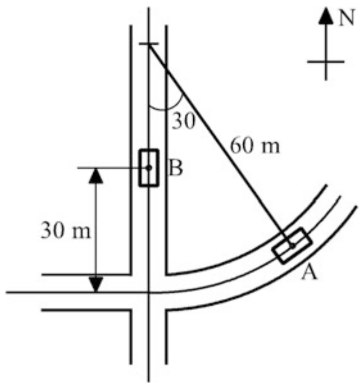


Fig. 3.13 Car A and car B

5. Car *A* is moving at a constant velocity of 50 kilometers per hour, and car *B* is moving at 1.2 meters per second. From the point of view of the passenger of car *B*, what is the acceleration of car *A* in meters per second squared (Fig. 3.13)?

   1) 1.2 2) 4.3 3) 6.3 4) 8.7

6. Car *A* is moving on a curved path with a radius of curvature of 60 meters at a constant velocity of 50 kilometers per hour. When car *A* is in the position shown in Fig. 3.14, car *B* is 30 meters from the intersection and has an acceleration of 1.2 meters per second squared toward the south (toward the intersection). Calculate the acceleration of car *A* from the point of view of the passenger of car *B* at the shown moment.

Fig. 3.14 Car A with a constant velocity

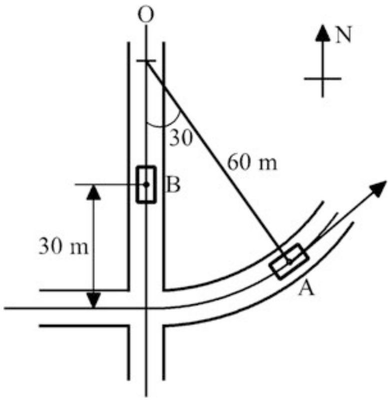
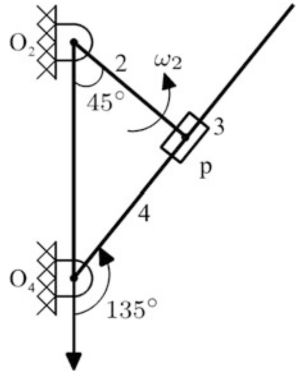


Fig. 3.15 A lever



1) 2.8 meters per second squared, 17°
2) 3.1 meters per second squared, 31°
3) 4.3 meters per second squared, 22°
4) 2.7 meters per second squared, 57°

7. Which statement about Coriolis acceleration is true about this lever (Fig. 3.15)?

   1) Since $V_{P_2} = V_{P_4}$, then $A^C = 0$.
   2) Since $V_{P_2} = V_{P_4}$, then $A^C = 0$.
   3) Since $V_{P_3} = 0$, then $A^C = \omega_2 \times V_{P_4/P_2}$.
   4) Since $V_{P_2} = V_{P_2/P_4}$, then $A^C = \omega_4 \times V_{P_4/P_2}$.

8. In the shown mechanism, the acceleration of point B is (Fig. 3.16): (the magnitude of AC is twice that of OA)

   1) $\frac{V_C^2\sqrt{3}}{3OA}$ 2) $\frac{V_C^2\sqrt{3}}{OA}$ 3) $\frac{2V_C^2\sqrt{3}}{3OA}$ 4) $\frac{V_C^2\sqrt{3}}{2OA}$

9. Which statement about the shown mechanism is correct (Fig. 3.17)?

   1) The acceleration of point A_4 relative to point A_3 has only a tangential component.
   2) The acceleration of point A_4 relative to point A_3 has a tangential component and a Coriolis component.

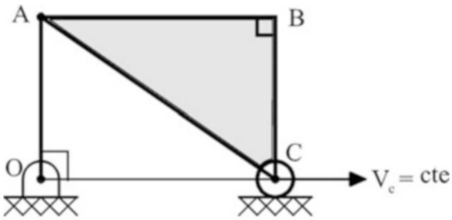


Fig. 3.16 Calculate the acceleration of point B

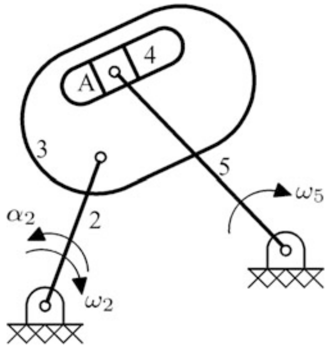


Fig. 3.17 A slider mechanism

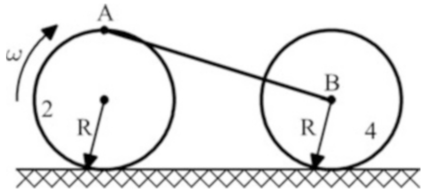


Fig. 3.18 Two rollers on a horizontal surface

3) The acceleration of point A_4 relative to point A_3 has a vertical component and a Coriolis component.
4) Because of the constant angular velocity of rod 5, point A_4 has no acceleration relative to point A_3.

10. If rollers 2 and 4 move on a horizontal surface without sliding, and roller 2 has a constant angular velocity of ω, the angular acceleration of roller 4 is equal to what option (Fig. 3.18)? $AB = 2R$

1) $\sqrt{2}R\omega^2$ 2) $\sqrt{3}R\omega^2$ 3) $\frac{\sqrt{3}}{3}R\omega^2$ 4) $\frac{\sqrt{2}}{2}R\omega^2$

11. In the shown mechanism, the angular velocity of element 2 is ω_2 and constant. From which equation can the angular acceleration of element 4 be derived? (The point C on the rod BD is next to the support P, and the rod is always in contact with this support) (Fig. 3.19).

1) $a_C = a_B + a^n_{C/B} + a^t_{C/B}, a_C = a_P + a^{rel}_{C/P} + a^{cr}_{C/P}$
2) $a_D = a_C + a^n_{D/C} + a^t_{D/C}, a_P = a_C + a^{cr}_{P/C} + a^{rel}_{P/C}$
3) $a_D = a_B + a^n_{D/B} + a^t_{D/B}, a_C = a_P + a^{rel}_{C/P} + a^{cr}_{C/P}$
4) $a_C = a_B + a^n_{C/B} + a^t_{C/B}, a_P = a_C + a^{rel}_{P/C} + a^{cr}_{P/C}$

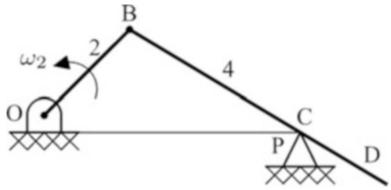


Fig. 3.19 Link 2 has a constant velocity

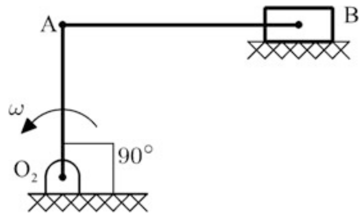


Fig. 3.20 Calculate the angular acceleration of AB

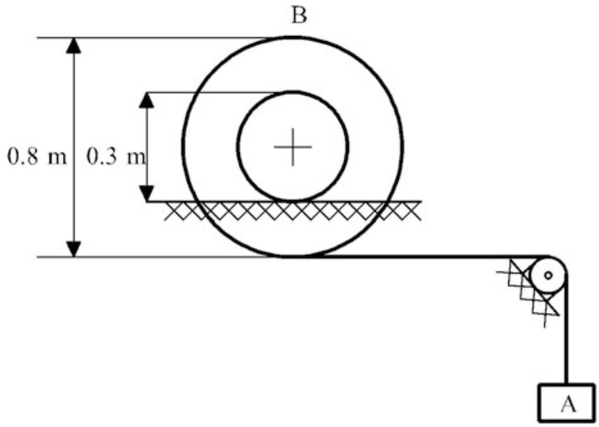


Fig. 3.21 Weight A is moving downward

12. In Fig. 3.20, $O_2A = AB$. O_2A rotates with constant velocity of ω. The angular acceleration of AB at this moment is:

    1) $\omega^2 \Uparrow$ 2) $2\omega^2 \Uparrow$ 3) $3\omega^2 \Uparrow$ 4) 0

13. In the mechanism in Fig. 3.21, if weight A is moving downward at a constant velocity of $2(m/s)$, the velocity of point B on the circumference of the disk is ....

    1) 2 m/s and to the right
    2) 4.4 m/s and to the right
    2) 2 m/s and to the left
    3) 4.4 m/s and to the left

14. A wheel is rolling on a flat ground, acceleration and velocity of point P are given, what is the radius of curvature of point P (Fig. 3.22)?

    1) $r_P = 179.6$ mm 2) $r_P = 153.3$ mm
    3) $r_P = 127$ mm 4) $r_P = 89.8$ mm

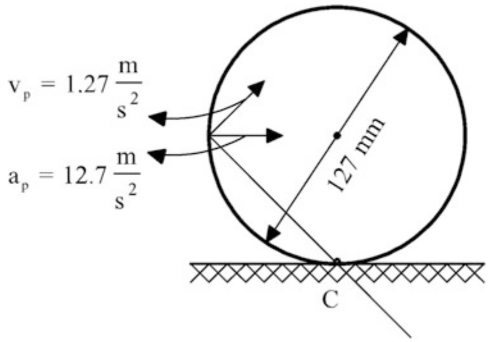


Fig. 3.22 Wheel is rolling on a flat ground

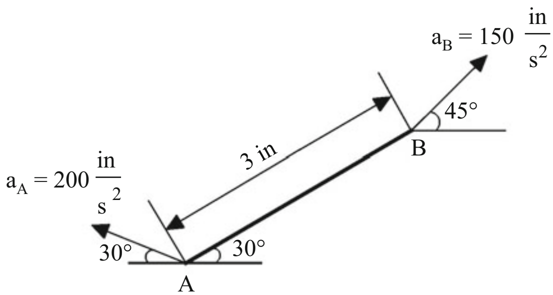


Fig. 3.23 A link with two different accelerations

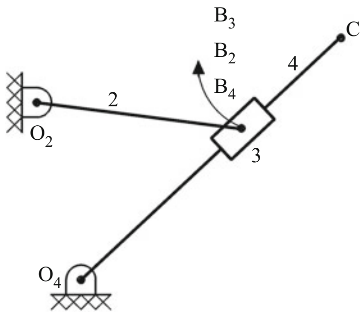


Fig. 3.24 Angular velocity of element 2 is known

15. The acceleration of the end points of AB link is given in Fig. 3.23. What is the angular acceleration of this link?

1) $\alpha = 450\ \text{rad/s}^2$ (CCW) 2) $\alpha = 45\ \text{rad/s}^2$ (CCW)
3) $\alpha = 45\ \text{rad/s}^2$ (CW) 4) $\alpha = 450\ \text{rad/s}^2$ (CW)

16. In Fig. 3.24, knowing the angular velocity of element 2 and assuming it is constant, the acceleration of point C results from which equation?

1) $\overline{a}_C = \overline{a}_{B_3} + \overline{a}^t_{C/B_3} + \overline{a}^n_{C/B_3} + \overline{a}^{cr}_{C/B_3}$
2) $\overline{a}_C = \overline{a}_{B_4} + \overline{a}^t_{C/B_4}$ and $\overline{a}^n_{B_3} = \overline{a}^t_{B_4} + \overline{a}^n_{B_4} + \overline{a}^n_{B_3/B_4} + \overline{a}^{cr}_{B_4/B_3}$
3) $\overline{a}_C = \overline{a}_{B_4} + \overline{a}_{C/B_4}$ and $\overline{a}^n_{B_4} + \overline{a}^t_{B_4} = \overline{a}^n_{B_3} + \overline{a}^{rel}_{B_4/B_3}$
4) $\overline{a}_C = \overline{a}_{B_4} + \overline{a}_{C/B_4}$ and $\overline{a}^n_{B_3} = \overline{a}^t_{B_4} + \overline{a}^n_{B_4} + \overline{a}^{rel}_{B_3/B_4} + \overline{a}^{cr}_{B_3/B_4}$

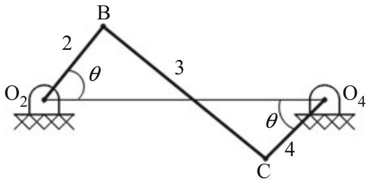


Fig. 3.25 Element 2 has a constant angular velocity

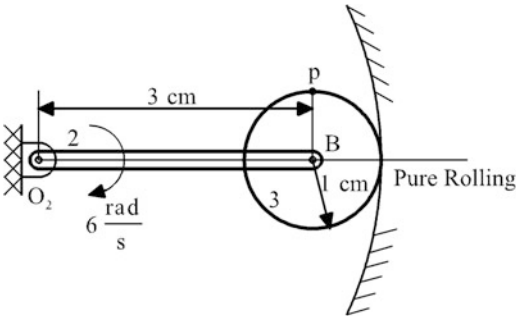


Fig. 3.26 Pure rolling movement

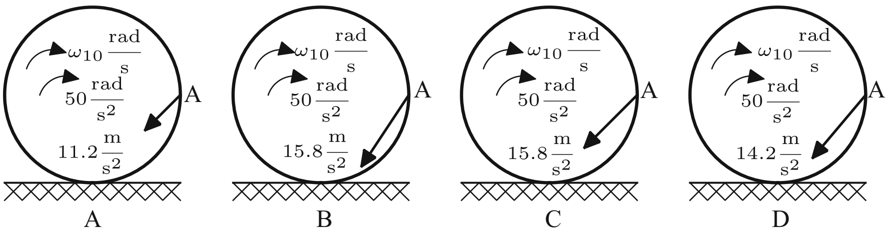


Fig. 3.27 Rolling on a horizontal surface

17. Which answer is correct for the angular velocity and acceleration of element 3 in Fig. 3.25 for the mechanism shown when element 2 has a constant angular velocity of ω_2?

1) $\omega_3 = 0, \alpha_3 \neq 0$ 2) $\omega_3 = 0, \alpha_3 = 0$
3) $\omega_3 \neq 0, \alpha_3 = 0$ 4) $\omega_3 \neq 0, \alpha_3 \neq 0$

18. What is the acceleration of point P in the mechanism in Fig. 3.26 in cm/s^2?

1) 108 2) 324 3) 341 4) 432

19. A disk with a radius of 10 cm rolls on a horizontal surface without sliding, as shown in Fig. 3.27. Which option correctly indicates the relative acceleration of point A with respect to the center of rotation?

1) *A* 2) *B* 3) *C* 4) *D*

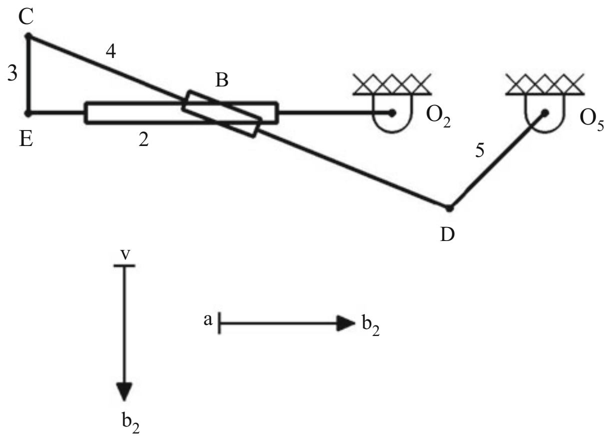


Fig. 3.28 Determine the acceleration of point C

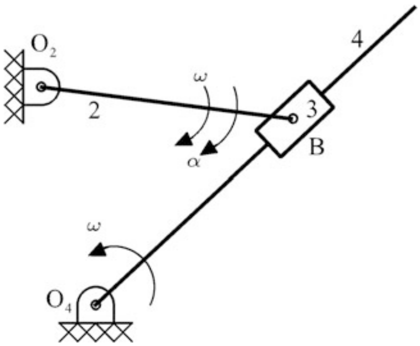
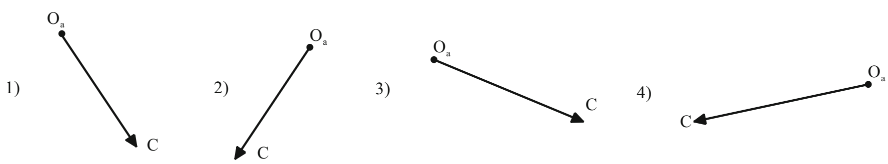


Fig. 3.29 Input velocity and acceleration and output velocity are known

20. Using the scale of velocity and acceleration of point b_2, determine the acceleration of point C by drawing the velocity and acceleration diagram (Fig. 3.28).



21. If ω_2 and ω_4 and α_2 are known, which of the following vector equations should be used to analyze the acceleration of the mechanism in Fig. 3.29?

    1) $a_{B_3} = a_{B_4} + a_{B_3/B_4}$
    2) $a_{B_4} = a_{B_3} + a_{B_4/B_3}$
    3) $a_{B_4} = a_{B_3} + a^{rel}_{B_4/B_3} + (2\omega_3) \times (V_{B_4/B_3})$
    4) $a_{B_3} = a_{B_4} + a^{rel}_{B_3/B_4} + (2\omega_4) \times (V_{B_3/B_4})$

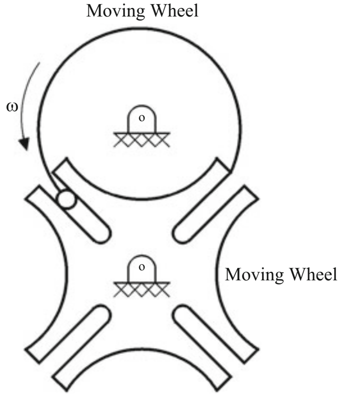


Fig. 3.30 Geneva wheel mechanism

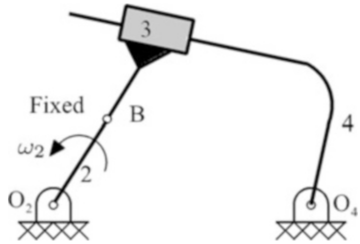


Fig. 3.31 Four-bar linkage with a slider

22. Which statement about the determination of the angular acceleration of the moving wheel in the Geneva wheel mechanism is correct if the moving wheel rotates with constant angular velocity (Fig. 3.30)?

    1) The Coriolis acceleration is non-zero in all situations where the moving wheel is not fixed and should be considered.
    2) The Coriolis acceleration is zero at the beginning and end of the contact of the moving wheel and at the contact along both centers of the wheels.
    3) The Coriolis acceleration is considered only during the stationary phase of the moving wheel.
    4) The Coriolis acceleration is not considered in this mechanism.

23. Which of the following equations is more appropriate for calculating α_4 (Fig. 3.31)?

    1) $a_{B_4} = a_{O_4} + a^n_{B_4/O_4} + a^t_{B_4/O_4}$
    2) $a_{B_2} = a^n_{B_4/O_4} + a^t_{B_4/O_4} + a^t_{B_2/O_4} + 2V_{B_2/B_4}\omega_4$
    3) $a_{B_2} = a_{B_4} + a^n_{B_2/O_4} + a^t_{B_2/O_4} + 2V_{B_2/B_4}\omega_4$
    4) $a_{B_2} = a_{O_4} + a^n_{B_4/O_2} + a^t_{B_4/O_2}$

24. Which of the following equations about velocity and acceleration of points A_2 and A_3 is correct (Fig. 3.32)?

Fig. 3.32 Output is a slider

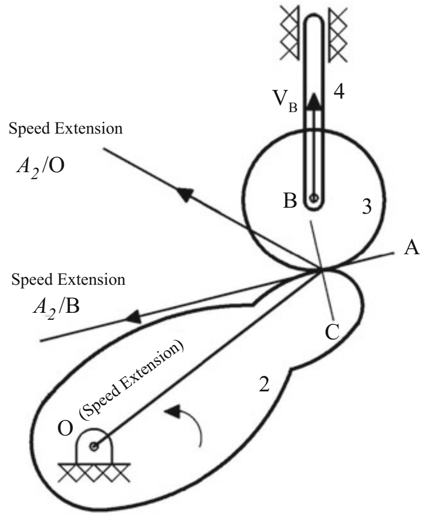
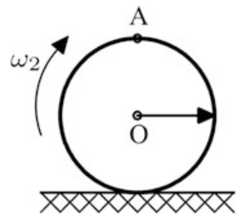


Fig. 3.33 A disk rolling on a surface



1) $V^n_{A_2} = V_{A_3}, a_{A_3} = a_{A_2} + a^{cr}_{A_3/A_2} + a^t_{A_3/A_2} + a^n_{A_3/A}$, all three values of vertical, tangential, and Coriolis acceleration are present.
2) $V_{A_2} = V_{A_3}, a_{A_3} = a_{A_2} + a_{A_3/A_2}$, direction of a_{A_3/A_2} on the line BC.
3) $V_{A_3} = V_{A_2} + V_{A_3/A_2}, a_{A_3} = a_{A_2} + a_{A_3/A_2}$, V_{A_3/A_2} is perpendicular to line BC.
4) $V_B = V_{A_2} + V_{B/A_2}, V_{A_2} = V_{A_3}, a_{A_2} = a_{A_3}, a_B = a_{A_2} + a_{B/A_2}$.

25. The Coriolis acceleration is caused by ....

1) Change of the direction of motion of the object
2) Change of the magnitude of the velocity of the object
3) Change of the rotational velocity of the object
4) Rotation of the plane of motion of the object

26. What is the value in rad/s of the angular velocity of a disk with a radius of 20cm and the velocity of the center $V_0 = 2$m/s, if the velocity of the point A above it is 4 m/s (Fig. 3.33)?

1) 1 2) 4 3) 8 4) 10

Answers for the Examples of "Acceleration Analysis"

1. Option (2) is correct.

   The link 2 rotates about its center of rotation P with an angular velocity of ω. Therefore, its linear velocity and centripetal acceleration are

$$V_{C_2} = R_2\omega \Rightarrow a_{C_2} = \frac{(V_{c_2})^2}{R_1 + R_2} = \frac{(R_2\omega)^2}{R_1 + R_2}$$

   The relative acceleration of a_{P/C_2} is in the direction of the center C_2 and has a value of $R_2\omega$. This is because if we hold C fixed, P rotates about C_2 and has a centripetal acceleration in that direction. Therefore:

$$a_P = a_{C_2} + a_{\frac{P}{C_2}} = \frac{R_2^2\omega^2}{R_1 + R_2} - R_2\omega^2 = \frac{\left(R_2^2 - R_1R_2 - R_2^2\right)}{R_1 + R_2}\omega^2$$

$$\Rightarrow a_p = \frac{-R_1R_2}{R_1 + R_2}\omega^2$$

2. Option (1) is correct.

   At points B and C, there is a rotary joint, so we can say that $V_{B_2} = V_{B_3}$ and $V_{C_2} = V_{C_3}$ and also $A_{B_2} = A_{B_3}$ and $A_{C_2} = A_{C_3}$. Thus, at points B and C, both velocity and relative acceleration are zero, and therefore, the expressions $2\omega \times V_{rel}$ (in terms of relative normal acceleration) and A_{rel} (in terms of Coriolis acceleration) are zero at these two points. For joints 5 and 6, the motion is a sliding motion, and the direction of the velocity of points P_2 and P_6 and Q_3 and Q_5 is different. Therefore, the relative velocity at points P and Q is non-zero. Accordingly, the relative acceleration at these points is also non-zero.
3. Option (2) is correct.

$$V_{BA} = V_B - V_A$$

$$= 30\left(\cos(-30)\,\hat{i} + \sin(-30)\,\hat{j}\right) - 15\left(\cos(45)\,\hat{i} + \sin(45)\,\hat{j}\right)$$

$$= 15.37\hat{i} - 25.6\hat{j}$$

$$|V_{BA}| = 29.86\frac{m}{s}$$

$$A_{BA} = A_B - A_A = \left(-100\hat{i}\right) - \left(200\hat{j}\right) \Rightarrow |A_{BA}| = 223.6\frac{m}{s^2}$$

4. Option (3) is correct.

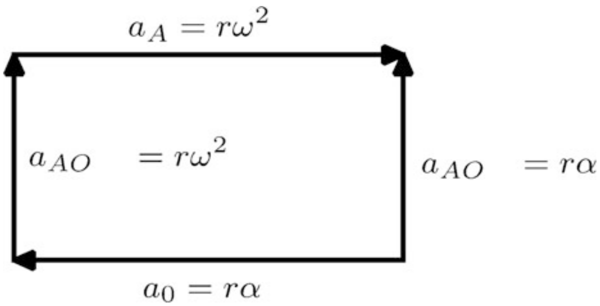


Fig. 3.34 Acceleration diagram

A_2 is the point Alocated on the lever 2. The acceleration of the center of instant rotation A as the point of the object is given by the following equation:

$$a_A = a_O + a_{\frac{A}{O}}$$

where O is the center of the circular arc and the components of the term of relative acceleration are $\left(a_{\frac{A}{O}}\right)_n = r\omega^2$, whose direction is from A to O and $\left(a_{\frac{A}{O}}\right)_t = r\alpha$, whose direction is to the right, which corresponds to the angular acceleration of the line $A0$ around the point O in the counterclockwise direction. We have summarized these theorems in the diagram in Fig. 3.34.

It can be seen that $a_A = r\omega^2$ and is directed upward.

5. Option (2) is correct.

Since car B is moving in a straight line, the acceleration of A with respect to point B or with respect to the coordinate system associated with B is the same.

$$\vec{a}_A = \frac{V^2}{R}\left(-sin30\hat{i} + cos30\hat{j}\right) = \frac{\left(\frac{50}{3.6}\right)^2}{60}\left(-\frac{\hat{i}}{2} + \frac{\sqrt{3}}{2}\hat{j}\right) = -1.6\hat{i} + 2.87\hat{j}$$

This question is phrased as if B is moving at a constant velocity of 1.2 (because of the unit meters per second), but the correct option is shown in the key as if the answer is obtained by assuming that B is moving downward at an accelerated velocity of 1.2:

$$\vec{a}_B = -1.2\hat{j},\ \vec{a}_{\frac{A}{B}} = \vec{a}_A - \vec{a}_B = -1.6\hat{i} + 3.98\hat{j}$$

$$\left|\vec{a}_{\frac{A}{B}}\right| = \sqrt{1.6^2 + 3.98^2} = 4.29$$

It should be noted that this question is applied to the dynamics course and not to the autodynamics. In other words, the position of this question in the autodynamics course is not appropriate.

6. Option (3) is correct.

Car A has a centripetal acceleration relative to the ground. From this follows:

$$a_A = \frac{v_A^2}{R} \Rightarrow a_A = \frac{v_A^2}{R}(-\sin 30\hat{i} + \cos 30\hat{j})$$

$$= \frac{(\frac{50}{3.6})^2}{60}\left(-\frac{1}{2}\hat{i} + \frac{\sqrt{3}}{2}\hat{j}\right) \Rightarrow \vec{a}_A = 3.215(\frac{-1}{2}\hat{i} + \frac{\sqrt{3}}{2}\hat{j})$$

$$\vec{a}_A = \vec{a}_B + \vec{a}_{\frac{A}{B}} \Rightarrow \vec{a}_{\frac{A}{B}} = \vec{a}_A - \vec{a}_B = 3.215(\frac{-1}{2}\hat{i} + \frac{\sqrt{3}}{2}\hat{j}) - (-1.2\hat{j})$$

$$\vec{a}_A = -1.6\hat{i} + 3.98\hat{j} \Rightarrow \left|\vec{a}_A\right| = \sqrt{(-1.6)^2 + (3.984)^2} = 4.3 \text{ m/s}^2$$

$$(\beta \text{ is the angle of } \vec{a}_A \text{ with y}) \ \tan\beta = \frac{a_{Ay}}{a_{Ax}} \Rightarrow \beta = \tan^{-1}\frac{1.6}{3.98} = 22°$$

7. Option (2) is correct.

The slider 3 slides on the element 4. The points P_2 and P_3 coincide, and the Coriolis acceleration of the points P_2 or P_3 with respect to the element 4 is

$$a_c = 2\omega_4 \times V_{P_2/P_4}$$

Since the common vertical connection of elements 3 and 4 is perpendicular to element 4 and in line with element 2, it is necessary that the velocity of P_4 and P_3 in line with element 2 be equal, and since the velocity of points P_2 or P_3 in line with element 2 is zero, the velocity of P_4 perpendicular to element 4 is also zero. Then $\omega_4 = 0$ and the Coriolis acceleration becomes zero.

8. Option (1) is correct.

From the acceleration equation between A and C, we obtain

$$\vec{a}_C = \vec{a}_A = \left(\vec{\omega}_{AC} \times \vec{\omega}_{AC} \times \vec{r}_{AC}\right) + \vec{a}_{AC} \times \vec{r}_{AC}$$

Since the directions of the velocity of points A and C are the same and both have no vertical velocity component, $\vec{\omega}_{AC} = 0$. But since V_C is constant, $a_C = 0$. Therefore,

$$a_A + \vec{a}_{AC} \times \vec{r}_{AC} = 0 \Rightarrow \{a_{Ay} + a_{AC} \times \overline{OC} = 0 a_{Ax} + a_{AC} \times \overline{OA} = 0$$

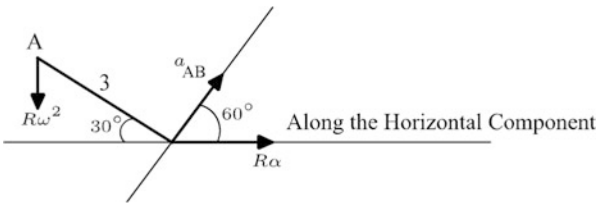


Fig. 3.35 Acceleration vectors

On the other hand, $a_{Ay} = \frac{-V_A^2}{\overline{OA}}$; therefore,

$$a_{AC} = \frac{-V_A^2}{\overline{OA}\,\overline{OC}} = \frac{-V_A^2}{\left(\overline{OA}\right)\left(\sqrt{3}\overline{OA}\right)} = -\frac{V_A^2}{\sqrt{3}\left(\overline{OA}\right)^2}$$

Now we write the acceleration equation between B and C:

$$\vec{a}_B = \vec{a}_C + \vec{a}_{BC} \times \vec{r}_{BC} = 0 - \left(\overline{BC}\alpha_{AC}\hat{i}\right)$$

Since $\overline{BC} = \overline{OA}$ and $V_A = V_C$, we thus have

$$\vec{a}_B = \frac{\sqrt{3}V_C^2}{3OA}$$

9. Option (2) is correct.
   According to the explanation given in the discussion of the rotating coordinate system, the point A_4 behaves like a particle moving in the rotating device associated with the element 3, so it has a Coriolis acceleration. On the other hand, A_4 moves in the direction of the groove with respect to element 3, and since the groove is a straight line, the radius of curvature is infinite, and consequently, the vertical component of the relative acceleration is zero.
10. Option (3) is correct (Fig. 3.35).

$$a_B = a_A + a_{BA}$$

$$\Rightarrow R\alpha_4\hat{i} = -R\omega^2\hat{j} + 2R\alpha_3\cos 60\hat{i} + 2R\alpha_3\sin 60\hat{j}$$

$$\Rightarrow \begin{cases} \text{direction } \hat{i} : R\alpha_4 = R\alpha_3 \Rightarrow \alpha_4 = \alpha_3 & (1) \\ \text{direction } \hat{j} : R\omega^2 = \sqrt{3}R\alpha_3 & (2) \end{cases}$$

$$(1)\,,\,(2) \Rightarrow \alpha_4 = \frac{\sqrt{3}}{3}\omega^2$$

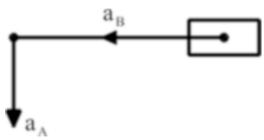


Fig. 3.36 Points A and B acceleration

All options have an additional R, which makes the dimension of the options different from the dimension of the angular acceleration.

11. Option (4) is correct.

    Since the acceleration of point B is known, one of our points must be point B when we write the equations, and we do not have any information about point D. Therefore, options (2) and (3) are incorrect. Point P behaves like a particle moving relative to the rotating coordinate system associated with the element BD. Its relative path of motion with respect to point C is in line with the rod, which is a straight line. Thus, the radius of curvature of the path is infinite, which means that the relative vertical acceleration is zero. According to the equations for rotating axes, option (4) is correct. Option (1) is incorrect because when we place the observer on the ground, he sees the motion path of C as a curved line whose radius is unknown to us, i.e., the relative vertical acceleration is unknown.

12. Option (1) is correct.

    Point B moves along a horizontal path, so the vertical component of acceleration at B is zero. Point A also has a net rotation about O_2, and since ω is constant, point A has only the vertical component of acceleration (Fig. 3.36).

$$a_A = |O_2A|\,\omega^2$$

$$\alpha = \frac{\left|a^t_{\frac{A}{B}}\right|}{|AB|} = \frac{|O_2A|\,\omega^2}{|O_2A|} = \omega^2$$

13. Option (4) is correct.

    The velocity is the same at all points on the rope, including the point of contact with the disk, and the disk is rolling on the ground, i.e., $V_D = 0$; thus:

$$\omega = \frac{V_C}{|DC|} = \frac{2}{0.25} = 8$$

$$V_B = |DB|\,\omega = 0.55 \times 8 = 4.4 \text{ m/s}$$

    According to the direction of ω, the velocity of the point V_B is directed to the left (Fig. 3.37).

14. Option (1) is correct.

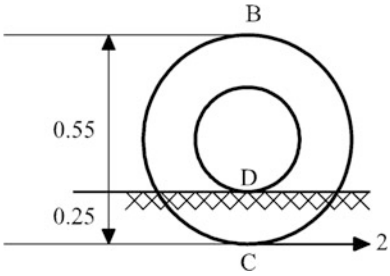


Fig. 3.37 Point D has zero velocity

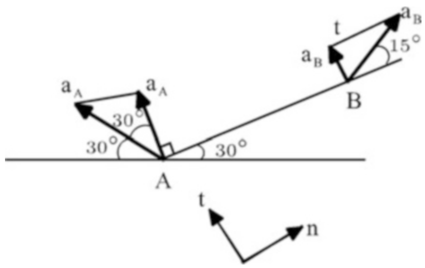


Fig. 3.38 A and B accelerations

The center of curvature is on the line perpendicular to the velocity, and the acceleration component in the direction of the radius of curvature is equal to $\frac{V^2}{\rho}$.

$$a^n = a\cos 45 = \frac{\sqrt{2}}{2} \times 12.7 = 8.98 \text{ m/s}^2$$

$$a^n = \frac{V^2}{\rho} = \frac{1.27^2}{\rho}\frac{1.61}{\rho} = 8.98\rho = 179.3 \text{ mm}$$

15. Option (3) is correct (Fig. 3.38).

$$\alpha = \frac{a^t_{B/A}}{|AB|} = \frac{a^t_B - a^t_A}{3}$$

$$\alpha = \frac{150sin15 - 200cos30}{3} = -44.8 \approx -45 \text{ rad/s}^2$$

A negative number means $a^t_B < a^t_A$, which means angular acceleration is clockwise.

16. Option (4) is correct.

Point C is a point belonging to the rigid element 4; thus:

$$\overline{a}_C = \overline{a}_{B4} + \overline{a}_{C/B_4}$$

If we assume that the coordinate system is associated with element 4, B_3 is like a moving particle in this device. Now if we put an observer on the element 4, he will see the motion path of B_3 in line with the rod 4, so the radius of curvature of the path is infinite, and in other words, $\overset{n}{\bar{a}}_{B_3/B_4}$ is zero.

$$\overset{n}{\bar{a}}_{B_3/B_4} = \frac{|V_{rel}|^2}{\rho = \infty}$$

The velocity and acceleration of points B_2 and B_3 are equal, and since rod 2 rotates about point O_2 and the angular velocity of 2 is constant, the tangential acceleration of B_3 is zero.

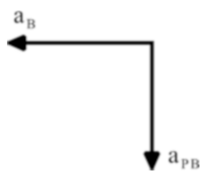
According to the equations for rotating axes, option 4 is correct. It should be noted that option 3 is incorrect because this equation assumes that the rotating axes are connected to element 3, while an observer located on element 3 sees the motion path of B_4 as a curved line whose radius of curvature is unknown to us.

17. Option (1) is correct.

    Points B and C are the points belonging to a rigid body. The direction of the velocity is the same in these two points, so the motion is translational, which means that $\omega_3 = 0$. Since ω is constant, the acceleration of points C and B is only in the direction n, and since the accelerations are in different directions, α exists (Fig. 3.39).

18. Option (3) is correct.

    Point B is moving on a circular path around point O_2. If we call the point of contact with the ground O_1, we have:



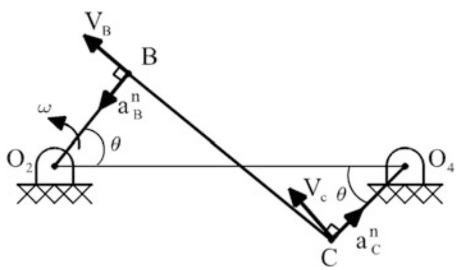


Fig. 3.39 Different points acceleration and velocity

$$V_B = |O_2B|\,\omega_2 = 3 \times 6 = 18 \text{ cm/s}$$

$$V_B = |O_1B|\,\omega_3 \rightarrow \omega_3 = 18 \text{ rad/s}$$

$$a_B = \frac{V^2}{\rho} = \frac{18^2}{3} = 108 \text{ cm/s}^2$$

$$a_P = a_B + a_{P/B} a_{P/B} = |PB|\,\omega_3^2 = 324 \text{ cm/s}^2$$

$$a_P = \sqrt{a_B^2 + a_{P/B}^2} = 341.5 \text{ cm/s}^2$$

19. Option (2) is correct.

    If the disk is rolling on the horizon, the center of rotation is the point of contact (Fig. 3.40), so:

$$a_{A/O} = a^n_{A/O} + a^t_{A/O}$$

$$a^n_{A/O} = r\omega^2 = \sqrt{2}R\omega^2 = \sqrt{2} \times 0.1 \times 10^2 = 14.14 \text{ m/s}^2$$

$$a^t_{A/O} = r\alpha = \sqrt{2}R\alpha = \sqrt{2} \times 0.1 \times 50 = 7.07 \text{ m/s}^2$$

$$a_{A/O} = \sqrt{14.14^2 + 7.07^2} = 15.8 \text{ m/s}^2$$

    According to the size of the components, option (2) is correct.

20. Option (3) is correct.

    Using the equations for relative velocity and acceleration, the velocity and acceleration of point C can be obtained using two points D and E (Fig. 3.41):

$$V_C = V_D + V_{C/D}, a_C = a_D + a_{C/D}$$

$$V_C = V_E + V_{C/E}, a_C = a_E + a_{C/E}$$

    The desired answer is obtained from the first equation. According to the drawn acceleration diagram, options (2) and (4) are automatically removed. On the other hand, between options (1) and (3), we find that the horizontal slope of a_C is lower relative to $a_{C/D}$ (CD extension). Therefore, option (1) is also incorrect.

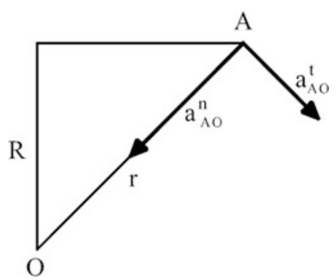


Fig. 3.40 Acceleration of point A

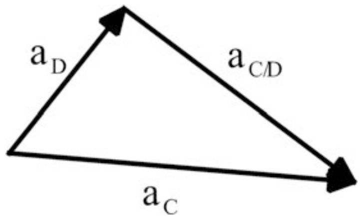


Fig. 3.41 Acceleration triangle

21. Option (4) is correct.

    Point B_3 behaves as a moving point in the rotating coordinate system associated with the element 4, and the observer on 4 sees the motion path of B_3 in the direction of motion of the rod, so that the radius of curvature is infinite and the relative vertical acceleration is zero because:

    $$a_n = \frac{V_{rel}^2}{\rho = \infty} = 0$$

    Using the equations for rotating axes, option 4 is correct. Option 3 is a correct equation, but because the point B_4 in moving on a curved path relative to B_3 and the radius of curvature of the path is unknown, it is not an appropriate equation. Options (1) and (2) also do not include the Coriolis acceleration term.
22. Option (2) is correct.

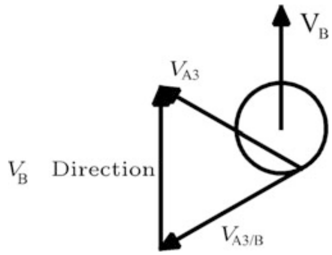
    The gripper behaves like a moving point moving in a rotating coordinate system associated with the moving wheel. Therefore, it has a Coriolis acceleration. According to the corresponding equation, the Coriolis acceleration is zero when the angular velocity of the axes or V_{rel} is zero. Therefore, the Coriolis acceleration is zero at the beginning and at the end of the contact when the angular velocity of the moving wheel is zero and when the gripper is along the two centers of the wheel where V_{rel} is zero.

    $$a^{cr} = 2\omega \times V_{rel}$$

23. Option (2) is correct.

    If we assume that the coordinate system is connected to the element 4, the point B_2 in this system moves in the direction of the rod 4 relative to B_4, so the radius of curvature is infinite, i.e., the relative vertical acceleration is zero, so according to the equations of rotating axes, option (2) is correct. Option (1) does not give us any specific information, option (4) states that $a_{B_2} = a_{B_4}$, which is incorrect, and option (3) does not take into account the fact that the vertical component of the relative acceleration is zero.
24. Option (2) is correct.

Fig. 3.42 Velocity triangle



According to the following equations and the formation of the velocity triangle (Fig. 3.42):

$$V_{A_3} = V_B + V_{A_3/B}$$

$$V_{A_2} = V_O + V_{A_2/O} = V_{A_2/O}$$

We see that V_{A_3} is equal to $V_{A_2/O}$, that is, it is equal to V_{A_2}, or in other words, the points A_2 and A_3 roll over each other. On the other hand:

$$a_{A3} = a_{A2} + a_{A3/A2}$$

If we consider the rotating axes associated with element 2, the motion path of point A_3 on 2 is the cam curve, so $a^n_{A_3/A_2}$ is located in the direction of the radius of curvature, i.e., on the line BC, and $a^t_{A_3/A_2}$ is also zero, since the relative tangential velocity is zero at all times due to the rolling contact.

$$a^t_{A_3/A_2} = \frac{d\,|V_{rel}|}{dt} = 0$$

25. Option (1) is correct.

For any arbitrary vector such as Q in two fixed and rotating coordinate systems, we can write

$$\frac{d\vec{Q}}{dt})_{Fix} = \frac{d\vec{Q}}{dt})_{rot} + \vec{\omega} \times \vec{A}$$

In the same way, $\frac{d^2\vec{Q}}{dt^2})_{Fix}$ can be calculated. Now if $\vec{Q} = \vec{r}$, we have $\frac{d^2\vec{r}}{dt^2})_{Fix} = \frac{d^2\vec{r}}{dt^2})_{rot} + 2\vec{\omega} \times \frac{d\vec{r}}{dt} + \frac{d\vec{\omega}}{dt} \times \vec{r} + \vec{\omega} \times (\vec{\omega} \times \vec{r})$

In the above equation, the expression $2\vec{\omega} \times \frac{d\vec{r}}{dt}$ is called Coriolis acceleration, caused by the motion of the particle in the rotating coordinate system.

26. Option (4) is correct.

$$V_A = V_0 + \omega \times R \rightarrow 4 = 2 + \omega \times 0/2 \rightarrow \omega = 10$$

Chapter 4
Force Analysis of Mechanisms

This chapter explores the subject of force analysis in mechanical systems. The chapter begins with providing an overview of the importance of understanding and analyzing forces in mechanisms. It then delves into the concept of inertia force and torque, followed by a discussion on the determination of forces in mechanisms.

The chapter proceeds to present various force analysis methods for linkages and covers the determination of the center of mass and moment of inertia in mechanisms. Furthermore, the chapter discusses dynamically equivalent masses and explores the role of flywheels in mechanical systems. Finally, it concludes with an examination of gyroscopic effects in mechanisms.

Overall, this chapter provides a comprehensive and detailed analysis of force-related aspects in mechanical systems, offering valuable insights and methodologies for force analysis in various applications.

4.1 Introduction

For the strength-based design of the components of a machine or mechanism, the forces and torques acting on the individual links must be determined. Inertia forces, which arise due to acceleration, are ignored in the analysis of static forces exerted on machine components. However, if inertia forces are incorporated, it turns into a dynamic analysis. The weights of machine components are often ignored in static analyses since they are small compared to the applied static forces. In high-speed machines, the accelerations and, then, the inertia forces can be very large relative to the static forces doing useful work.

A. Abbasimoshaei, T. A. Kern, *Machine Dynamics*,
https://doi.org/10.1007/978-981-99-6010-1_4

4.2 Inertia Force and Torque

The relationship between force and motion in kinetic studies is expressed by Newton's second law:

$$\overrightarrow{F} = m.\overrightarrow{a} \tag{4.1}$$

where $\overrightarrow{F}$ represents the force exerted on the particle, m is the mass of the particle, and $\overrightarrow{a}$ is the particle acceleration. For a rigid body with a distributed mass, the center of mass moves in such a way as if the whole body mass is concentrated at particles located at this point:

$$\sum \overrightarrow{F} = m.\overrightarrow{a}_G \tag{4.2}$$

The rotational motion of a body is expressed using Euler's equation of motion:

$$\sum \overrightarrow{M}_G = I_G.\overrightarrow{\alpha} \tag{4.3}$$

where $\sum \overrightarrow{M}_G$ is the resultant torque of the force system exerted on the body around its center of mass, I_G is the inertia matrix with respect to a fixed coordinate system with its origin located at the center of mass, and $\overrightarrow{\alpha}$ shows the angular acceleration of the body relative to an identical fixed frame. On the other hand, if the body rotates around the fixed point P, the torques can be calculated about this point, and the inertia matrix can be expressed with respect to an identical frame with its origin located at P. In this case, Euler's equation of motion can be expressed as follows:

$$\sum \overrightarrow{M}_P = I_P.\overrightarrow{\alpha} \tag{4.4}$$

Note Rotation about a fixed point is the only case where the inertia matrix may be considered relative to a point other than the center of mass.

In practice, it is easiest to describe the inertia matrix about a frame fixed on a rigid body in all cases, except for the rotation of a symmetric body around a fixed axis of symmetry. Three orthogonal axes are always fixed to a body relative to which the inertia matrix becomes diagonal. These axes are called the principal axes of inertia and constitute the main reference frame. If the inertia matrix I_G is expressed relative to the main reference frame and the angular velocity $\overrightarrow{\omega}$ and angular acceleration $\overrightarrow{\alpha}$ are expressed relative to the same frame, Eq. (4.3) can be rewritten as follows:

$$\sum \overrightarrow{M}_G = I_G.\overrightarrow{\alpha} + \overrightarrow{\omega} * I_G.\overrightarrow{\omega} \tag{4.5}$$

This relationship is a formulation of Euler's equation of motion appropriate for spatial motion.

In dynamic analysis, the solution method and free-body diagram concept used in Eqs. (4.2)–(4.5) are similar to the one used in static equilibrium. If the right-hand sides of Eqs. (4.2) and (4.3) are set to 0, these equations will be identical to static equilibrium equations. Nevertheless, inertia terms on the right-hand sides of dynamic equilibrium problems make them more difficult to solve.

The body's motion in the system is known in most machine design problems. Therefore, the acceleration of the center of mass and the angular acceleration of every link are either known or can be determined using kinematic methods. Accordingly, the right-hand side of the dynamic equilibrium equations can be considered known quantities, and the equations are solved algebraically, similar to solving the static equilibrium equations.

4.3 Determination of Forces

For the force analysis of a complete mechanism, the free-body diagram of each link must be drawn to display the forces exerted on it. To determine the directions of these forces, several laws from statics are reviewed in the following:

1. A rigid body under the effect of two forces will be in static equilibrium only if the forces are equal in magnitude and are in opposite direction (Fig. 4.1).
2. In a rigid body acted upon by three forces and in static equilibrium, the lines of action of the three forces are concurrent at a point like k. Therefore, if the lines of action of two forces are known, the line of action of the third force must pass through its point of application and the point of concurrence k (Fig. 4.2).

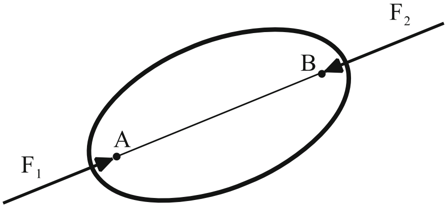


Fig. 4.1 Rigid body under the effect of two forces

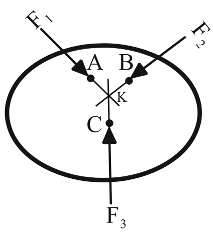


Fig. 4.2 A rigid body acted upon by three forces

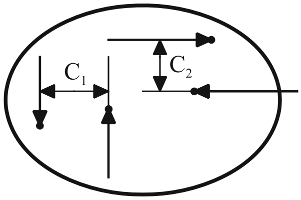


Fig. 4.3 A rigid body under the effect of a couple

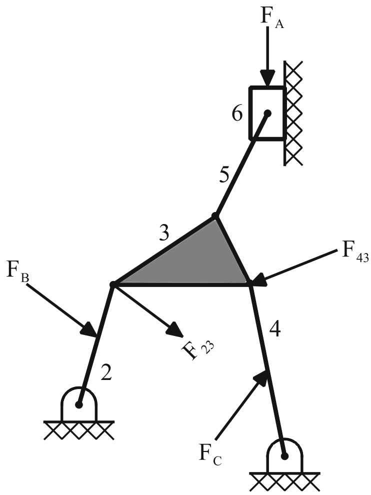


Fig. 4.4 A mechanism with static equilibrium

Note If the number of forces is larger than three, it can be reduced to three by finding the resultants of the force vectors.

3. A rigid body under the effect of a couple will be in static equilibrium if an equal and opposite couple acts in the same plane (Fig. 4.3).

The advantage of using inertia forces is that dynamics problems can be treated statically. In both analyses, the vector equations can be solved both analytically and graphically to determine the unknown forces.

Example Which of the following is true about the static equilibrium of the mechanism shown in Fig. 4.4?

1) $\sum \mathbf{F}=\mathbf{F_A}+\mathbf{F_B}+\mathbf{F_C}=\mathbf{0}$
2) $\sum \mathbf{F}=\mathbf{F_A}+\mathbf{F_B}+\mathbf{F_C}+\mathbf{F_{23}}+\mathbf{F_{43}}=\mathbf{0}$
3) $\sum \mathbf{F}=\mathbf{F_A}+\mathbf{F_B}+\mathbf{F_C}+\mathbf{F_{16}}+\mathbf{F_{12}}+\mathbf{F_{14}}=\mathbf{0}$
4) $\sum \mathbf{F}=\mathbf{F_A}+\mathbf{F_B}+\mathbf{F_C}+\mathbf{F_{12}}+\mathbf{F_{16}}+\mathbf{F_{23}}+\mathbf{F_{43}}=\mathbf{0}$

Solution Since the system is assumed to be in static equilibrium, the members have zero acceleration and the inertia forces are zero. Therefore, the sum of the vector forces exerted on the system must be equal to 0 ($\sum F = 0$). The ground exerts a force on the system at the joint between the system and the ground (Link 1). Since

Links 2, 4, and 6 are connected to the ground, the forces $\mathbf{F_{12}}$, $\mathbf{F_{14}}$, and $\mathbf{F_{16}}$ must be considered in addition to the external forces $\mathbf{F_A}$, $\mathbf{F_B}$, and $\mathbf{F_C}$. The forces between the other members (e.g., $\mathbf{F_{23}}$ and $\mathbf{F_{43}}$) are considered internal forces and must not be taken into account in the overall equilibrium of the system. Hence, choice (3) is correct.

4.4 Force Analysis Methods for Linkages

Two-force analysis methods are often used for mechanisms: a) the superposition method and b) the matrix method. Among superposition methods, two techniques have found widespread use. The first method involves the direct use of the inertia force and torque and is the best analytical method. The second method excludes the inertia torque analysis by deviating the inertia force by an amount shown by “e.”

4.4.1 Superposition Method

The superposition principle can be used in the force analysis of a rigid body in static equilibrium. This principle states that the effect of the resultant force is equivalent to the sum of the effects of individual forces. This method can be used to analyze a linkage acted on by several forces by considering the effect of each force individually. Subsequently, the effects of the individual forces are added together to find the total effect of all the forces on each joint of the linkage. The superposition principle can also be used to combine the static and inertia force analysis results obtained independently.

Note Although this method is simple to use, the involved analysis must be performed numerous times, which leads to difficulty. Moreover, it will lose its accuracy in the presence of friction forces.

Note The friction problem does not usually arise in linkages with revolute joints since friction forces can be ignored in this case. However, the superposition method is not suitable if friction is considered in prismatic joints, such as cylinders and pistons.

The overall solution process is as follows:

1. The position, velocity, and acceleration kinematics are solved to determine the translational accelerations of all the bodies with significant mass and the angular accelerations of all the bodies with a considerable moment of inertia.
2. The inertia force and torque exerted on each body are calculated based on D’Alembert’s Principle.
3. The inertia force and torque are applied to each member as an external force and torque.

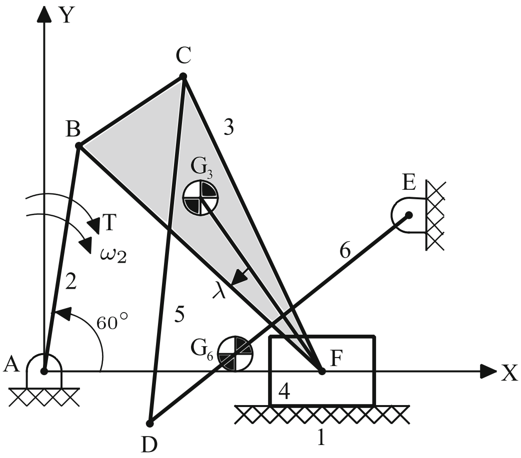


Fig. 4.5 A mechanism with an input of a constant angular speed

4. The free-body diagram of each member includes the forces applied to the member and all the reaction forces applied by other members on the member in question.
5. Three force and torque equations are written for each member, and the equations (including the unknown forces) are solved.

For instance, the members of the mechanism in Fig. 4.5 have the following known inertia properties. Find the driving torque that must be exerted on the crank (Link 2) to maintain a constant clockwise angular speed of 60 rpm (ignore the friction in all the joints and mass and moments of inertia of Link 5).

$$\mathrm{DC} = 2.25\,\mathrm{in},\ \mathrm{BF} = 2.75\,\mathrm{in},\ \mathrm{DG_6} = 1.5\,\mathrm{in},\ \mathrm{FG_3} = 1.82\,\mathrm{in},\ \mathrm{AB} = 1.5\,\mathrm{in},\ \mathrm{DE} = 3\,\mathrm{in},\ \mathrm{BC} = 0.75\,\mathrm{in},\ \mathrm{CF} = 2.75\,\mathrm{in},\ k_3 = 0.87\,\mathrm{in},\ k_6 = 0.87\,\mathrm{in},\ \lambda = 7.84^\circ,\ \alpha_2 = 0,$$

$$\omega_2 = 60\,\mathrm{rpm}\,(\mathrm{CW}),\ m_3 g = 0.5\,\mathrm{lb},\ m_4 g = 1.0\,\mathrm{lb},\ m_6 g = 0.5\,\mathrm{lb}.$$

It is of note that the velocity and acceleration are assumed to be known.

(Along 132° - relative to the x-axis)	$a_{G_3} = 43.2\mathrm{in/s^2}$
(In the negative x - direction)	$a_{G_4} = 13.9\ \mathrm{in/s^2}$
(Along 63° - relative to the x-axis)	$a_{G_6} = 20\ \mathrm{in/s^2}$

$$\alpha_3 = 19.1\ \mathrm{rad/s^2}\ (\mathrm{CCW})\ ,\ \alpha_6 = 13.3\ \mathrm{rad/s^2}\ (\mathrm{CCW})$$

In addition, we do not seek to find the acceleration of the mass center of link 2 since there is insufficient information for determining it. In this respect, as long as the angular speed of link 2 is constant, the angular acceleration of the mass center of link 2 is not required as long as we are after the torque T only and we do not need to find the reaction force of A. The inertia force exerted on G_3 is calculated as follows:

$$\text{(Along 48°- relative to the x direction-axis)} \qquad \vec{F}_{13} = -m_3.\vec{a}_{G_3} = \frac{0.5}{32.2} \cdot \frac{43.2}{12} = 0.56 \text{ lb}$$

Here, the direction of F_{13} is determined by adding 180° to the direction of a_{G_3}. Similarly, the inertia force exerted on G_6 is calculated as follows:

$$\text{(Along 117°- relative to the x direction-axis)} \qquad \vec{F}_{16} = -m_3.\vec{a}_{G_6} = \frac{0.5}{32.2} \cdot \frac{20}{12} = 0.26 \text{ lb}$$

The inertia force applied to the translating mass 4 is as follows:

$$\text{(In the x-direction)} \qquad \vec{F}_{14} = -m_4.\vec{a}_{G_4} = \tfrac{1.0}{32.2} \cdot \tfrac{13.9}{12} = 0.36 \text{ lb}$$

Also, the inertia force exerted on link 3 is computed as follows:

$$\vec{M} = -I_3.\vec{\alpha}_3 = -m_3.k_3^2.\vec{\alpha}_3 = \frac{0.5}{32.2} \cdot \left(\frac{0.87}{12}\right)^2 \cdot 19.1 = 0.00156 \text{ ft-lb}$$
$$= 0.018 \text{ in-lb(CW)}$$

Now, we can move to step 4 of the previously mentioned procedure. The free-body diagrams of the mechanism are shown in Fig. 4.6:

Step 5 in the above procedure involves writing the dynamic equilibrium equations for each member. Beginning with link 2 gives

$$\sum M_A = 0 \Rightarrow T + F_{B_x} * 1.5 \sin 60° = F_{B_y} * 1.5 \cos 60°$$

Here, the torques about point A are arbitrary since this eliminates the components F_A and F_{12}. Therefore, the force equilibrium leads to two equations, which are then solved for the components of F_A. They will not be written here since they are not desired.

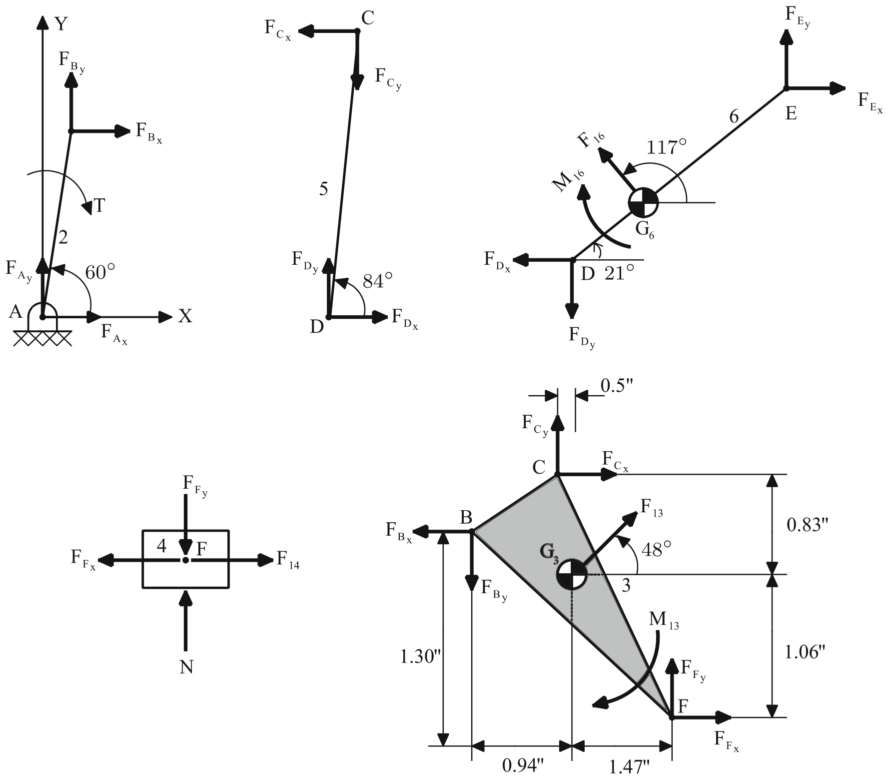


Fig. 4.6 Free-body diagrams of the mechanism

Moving to Link 3:

$$\sum F_x = 0 \Rightarrow F_{C_x} + F_{F_x} * 0.056 \cos 48^\circ = F_{B_x}$$

$$\sum F_y = 0 \Rightarrow F_{C_y} + F_{F_y} * 0.056 \cos 48^\circ = F_{B_y}$$

$$\sum F_{G_3} = 0 \Rightarrow F_{C_x} * 0.83 + F_{C_y} * 0.5 + 0.0187 = F_{B_x} * 0.24 + F_{B_y} * 0.94 + F_{F_x} * 1.06 + F_{F_y} * 1.47$$

Here, the choice of the point about which the torque is considered is slightly different. Using G_3 simplifies the equation by eliminating F_{13}.

Considering link 4 gives

$$\sum F_x = 0 \Rightarrow F_{F_x} = 0.036\ \text{N}$$

$$\sum F_y = 0 \Rightarrow F_{F_y} = 0.03\ \text{N}$$

The torque equation is not written for this link since it is constrained with respect to rotation.

Link 5 is a two-force member because its mass and moments of inertia are ignored. Therefore, $F_C = F_D$, and the two forces have the opposite direction.

Extending the forces along the member requires satisfying the torque equation. As a result:

$$\sum F_x = 0 \Rightarrow F_{C_x} = F_{D_x}$$

$$\sum F_y = 0 \Rightarrow F_{C_y} = F_{D_y}$$

$$\sum M_D = 0 \Rightarrow F_{C_y} * 2.25 \cos 84^\circ = F_{C_x} * 2.25 \sin 84^\circ$$

or

$$F_{C_y} = F_{C_x} \tan 84^\circ,\ F_{D_y} = F_{D_x} \tan 84^\circ$$

Finally, for link 6, we have

$$\sum M_E = 0 \Rightarrow 0.013 + F_{D_x} * 3 \sin 21^\circ + 0.026 * 1.5 \sin 96^\circ = F_{D_y} * 3 \cos 21^\circ$$

or

$$F_{C_x} = F_{D_x} = 0.002\ \text{lb}$$

Then:

$$F_{C_y} = F_{D_y} = 0.002 \cdot \tan 84^\circ = 0.0193\ \text{lb}$$

Substituting the result in the rotation equations leads to the following:

$$\begin{aligned} 0.03 &= 0.24 F_{B_x} + 0.94 F_{B_y} + 1.06 F_{F_x} + 1.47 F_{F_y} \\ &= 0.24 \cdot 0.0755 + 0.94 F_{B_y} + 1.06 \cdot 0.036 + 1.47 F_{F_y} \end{aligned}$$

or

$$0 = 0.0263 + 0.94F_{B_y} + 1.47F_{F_y}$$

Eliminating F_{F_y} results in the following:

$$0 = 0.0263 + 0.94F_{B_y} + 1.47(F_{B_y} - 0.0609)$$

or

$$F_{B_y} = 0.0262 \text{ lb}$$

The values obtained for F_{B_x} and F_{B_y} can be substituted in the torque equation of link 2 to obtain the following:

$$T + 0.0755 * 1.5 \sin 60^\circ = 0.0262 * 1.5 \cos 60^\circ$$

or

$$T = -0.078 \text{ in-lb}$$

As a result, when the system passes through this position, the torque 0.078 in-lb in the counterclockwise direction is required to prevent the acceleration of link 2 and to maintain it at a constant angular speed.

4.4.2 Matrix Method

Although the superposition method is computationally simple, it is difficult due to the need to repeat the analysis for the linkage. On the other hand, the matrix method requires a single instant of analysis. However, since the results are in the form of a linear system of equations, from which the unknown forces and torques must be found, the superposition principle is still simpler.

As an example of force analysis via the matrix method, consider the following four-bar linkage (Fig. 4.7). Note that the mass centers g2, g3, and g4 of the moving links are not necessarily along the line joining the joints.

Note In the matrix method, the translational position and acceleration of the mass center and the angular acceleration of the moving link should have been determined from previous analyses, similar to the superposition method.

In the matrix method, each link must be separately displayed in a free-body diagram. This process is depicted in Fig. 4.8.

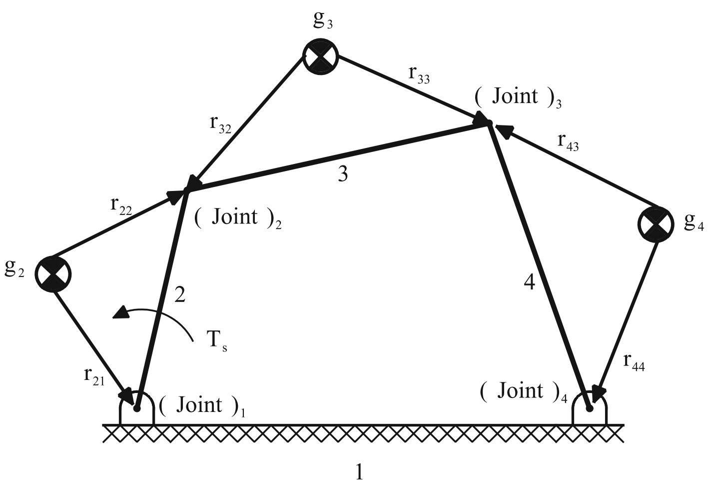


Fig. 4.7 A four-bar linkage for force analysis

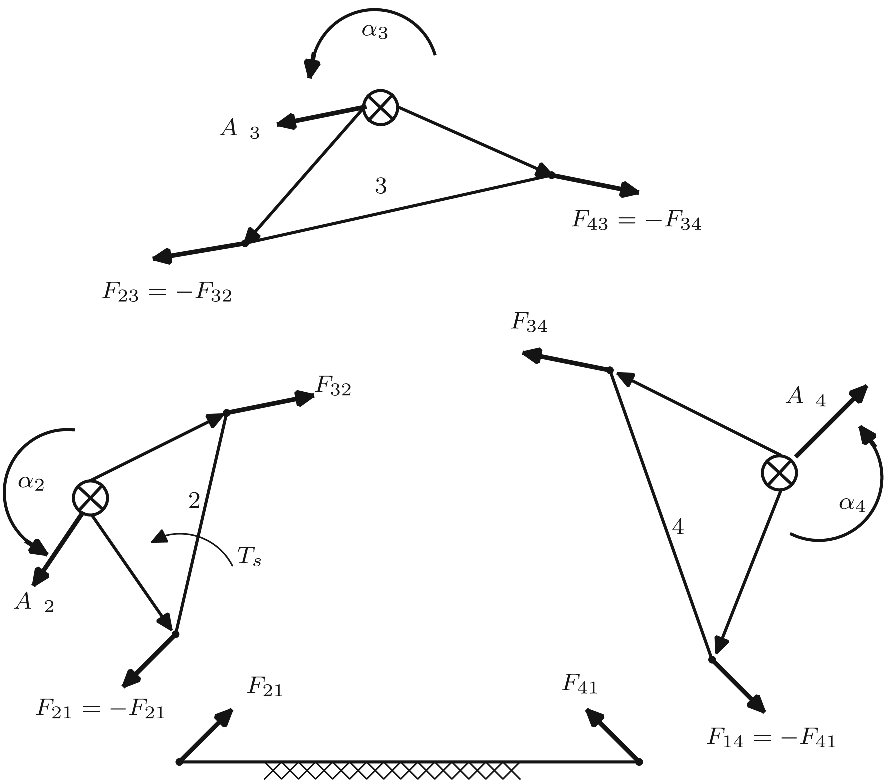


Fig. 4.8 Separate view of each link

The equations of motion of each moving link can be written from the free-body diagrams in the following vector form:

Link 2:

$$F_{32} - F_{21} = m_2.A_{g2}$$

$$r_{22} * F_{32} - r_{21} * F_{21} + T_s = I_2.\alpha_2$$

Link 3:

$$F_{43} - F_{32} = m_3.A_{g3}$$

$$r_{33} * F_{43} - r_{32} * F_{21} = I_3.\alpha_3$$

Link 4:

$$F_{14} - F_{43} = m_4.A_{g4}$$

$$r_{44} * F_{14} - r_{43} * F_{43} = I_4.\alpha_4$$

where:

r_{ij} = vector extending from the center of gravity of link i to joint j
F_{ik} = force exerted by link i on link k (note that $F_{ik} = -F_{ki}$)
g_i = center of gravity of link i
A_{g_i} = acceleration of the center of gravity
α_i = angular acceleration of link i
m_i = mass of link i
I_i = mass moment of inertia of link i about its center of gravity
T_s = driving torque exerted on the input link

The vector force equations are decomposed into components x and y, as follows:

$$F_{32x} - F_{21x} = M_2.A_{g_2}x$$

$$F_{32y} - F_{21y} = M_2.A_{g_2}y$$

$$F_{43x} - F_{32x} = M_3.A_{g_3}x$$

$$F_{43y} - F_{32y} = M_3.A_{g_3}y$$

$$F_{14x} - F_{43x} = M_4.A_{g_4}x$$

$$F_{14y} - F_{43y} = M_4.A_{g_4}y$$

The expansion of the cross product of the vectors using the equation $r \times F = r_x.F_y - r_y.F_x$ gives

$$r_{22x}.F_{32y} - r_{22y}.F_{32x} - r_{21x}.F_{21y} + r_{21y}.F_{21x} = I_2.\alpha_2 - T_s$$

$$r_{33x}.F_{43y} - r_{33y}.F_{43x} - r_{32x}.F_{32y} + r_{32y}.F_{32x} = I_3.\alpha_3$$

$$r_{44x}.F_{14y} - r_{44y}.F_{14x} - r_{43x}.F_{43y} + r_{43y}.F_{43x} = I_4.\alpha_4$$

The above nine equations form a system with nine linear equations and the nine unknowns F_{21x}, F_{21y}, F_{32x}, F_{32y}, F_{43x}, F_{43y}, F_{14x}, F_{14y}, and T_s. These equations can be re-ordered into a matrix form.

There are several other force analysis methods, which will be briefly introduced in terms of their application.

4.4.3 Virtual Work Method

The two-force analysis methods presented so far were based on force equilibrium. However, the virtual work method is based on the principle that the total work done by the external forces exerted on a rigid body at equilibrium is zero for infinitesimal displacements of the body.

According to the concept of work, the work done by a force exerted on a body is

$$\delta U = F.\delta s.\cos\theta \tag{4.6}$$

where F is the force, δs is the displacement, and θ is the angle between them. The work done is the numerical product of the displacement and the force component along the displacement. In other words,

$$\delta U = F.\delta s \tag{4.7}$$

Sometimes, the virtual displacement is in the form of a rotation. Thus, we can write

$$\delta U = T.\delta\theta \tag{4.8}$$

where T is the torque and $\delta\theta$ is the angular displacement.

Note Although virtual displacements are not real, they must be consistent with the constraints in the mechanism under study.

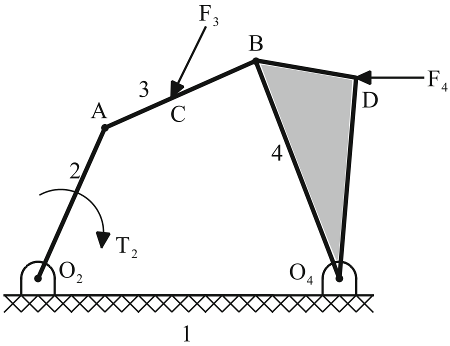


Fig. 4.9 A normal four-bar linkage with torque as input

According to the definition of virtual work, it must be zero for a virtual displacement assumed in a system at equilibrium under external forces and torques. In mathematical terms:

$$\delta U = \sum F_n.\delta s_n + \sum T_n.\delta\theta_n = 0 \tag{4.9}$$

For instance, consider Fig. 4.9, where a four-bar linkage is subjected to forces F_3 and F_4 at points C and D, respectively. Therefore, it is necessary to determine the torques T_2 required for the static equilibrium of the linkage. If we cause a virtual displacement $\delta\theta_2$ in link 2, equations δs_C and δs_D are expressed as functions of $\delta\theta_2$ to determine T_2 from Eq. 4.9.

The virtual work method can also be used for dynamic analyses, provided that the inertia forces and torques are considered among the applied forces and torques. Equation 4.9 can be used for the dynamic case by dividing its terms by dt:

$$\sum F_n.\frac{\delta s_n}{dt} + \sum T_n.\frac{\delta\theta_n}{dt} = 0 \tag{4.10}$$

or

$$\sum F_n.V_n + \sum T_n.\omega_n = 0 \tag{4.11}$$

As a result, the virtual work of the external forces and torques is proportional to the velocities of points of action of the forces exerted on the links.

4.4.4 Force Analysis of the Linkage Using Complex Numbers

Another force analysis method is representing vectors as complex numbers. For instance, the following figure shows a four-bar linkage at a given position of its motion cycle. The torque T_s is applied to the driving link at point O_2. Assuming known translational and angular accelerations, the inertia forces F_0 related to the acceleration values represent the dynamic load exerted on the mechanism. The analysis aims to determine the bearing forces and the axis torque causing the dynamic loads.

In the analysis of the linkage using complex numbers in Fig. 4.10, the inertia force F_{O_3} is the only force (load) affecting the mechanism. Thus, only the bearing forces and torque corresponding to F_{O_3} are calculated. Similar independent analyses with only F_{O_2} and F_{O_4} are performed, and the resultant of the bearing forces and torques is obtained via superposition.

In the analysis performed with only F_{O_3}, the free-body diagram that must be initially examined is link 3 of Fig. 4.10C. Assuming the translational acceleration A_{g_3} (expressed as $A_{g_3}e^{iB_3}$) and the angular acceleration α_3 to be known, the inertia force vector F_{O_3} can be determined as follows:

$$F_{O_3} = (m_3.A_{g_3})e^{i(\beta_3+\pi)}$$

where $(\beta_3 + \pi)$ means that the sense of F_{O_3} is opposite to that of A_{g_3} (the angular sense given by β_3). As shown in Fig. 4.10B, the line of action of F_{O_3} has a distance of $e_3 = I_3.\alpha_3 / F_{O_3}$ from that of A_{g_3} due to the angular acceleration α_3. To simplify calculations, one can express the position of the line of action of F_{O_3} with the distance l_3 (Fig. 4.10C).

$$l_3 = r_{g_3} + \frac{e_3}{\sin(\beta_3 - \theta_3)} = r_{g_3} + \frac{\frac{I_3.\alpha_3}{F_{O_3}}}{\sin(\beta_3 - \theta_3)}$$

Figure 4.10C depicts the three forces exerted on Link 3, where F_{O_3} is a known dynamic load, and F'_{23} and F'_{43} are unknown bearing forces that must be determined. The following relationship represents the static equilibrium:

$$F'_{23} + F'_{43} + F_{O_3} = 0$$

$$F'_{23}(e^{i\gamma'_3}) + F'_{43}(e^{i\theta_4}) + F_{O_3}(e^{i(\beta_3-\pi)}) = 0$$

Equating the real and imaginary parts of the above equation leads to the following:

$$F'_{23}\cos\gamma'_3 + F'_{43}\cos\theta_4 + F_{O_3}\cos(\beta_3 - \pi) = 0 \quad \text{(I)}$$

$$F'_{23}\sin\gamma'_3 + F'_{43}\sin\theta_4 + F_{O_3}\sin(\beta_3 - \pi) = 0 \quad \text{(II)}$$

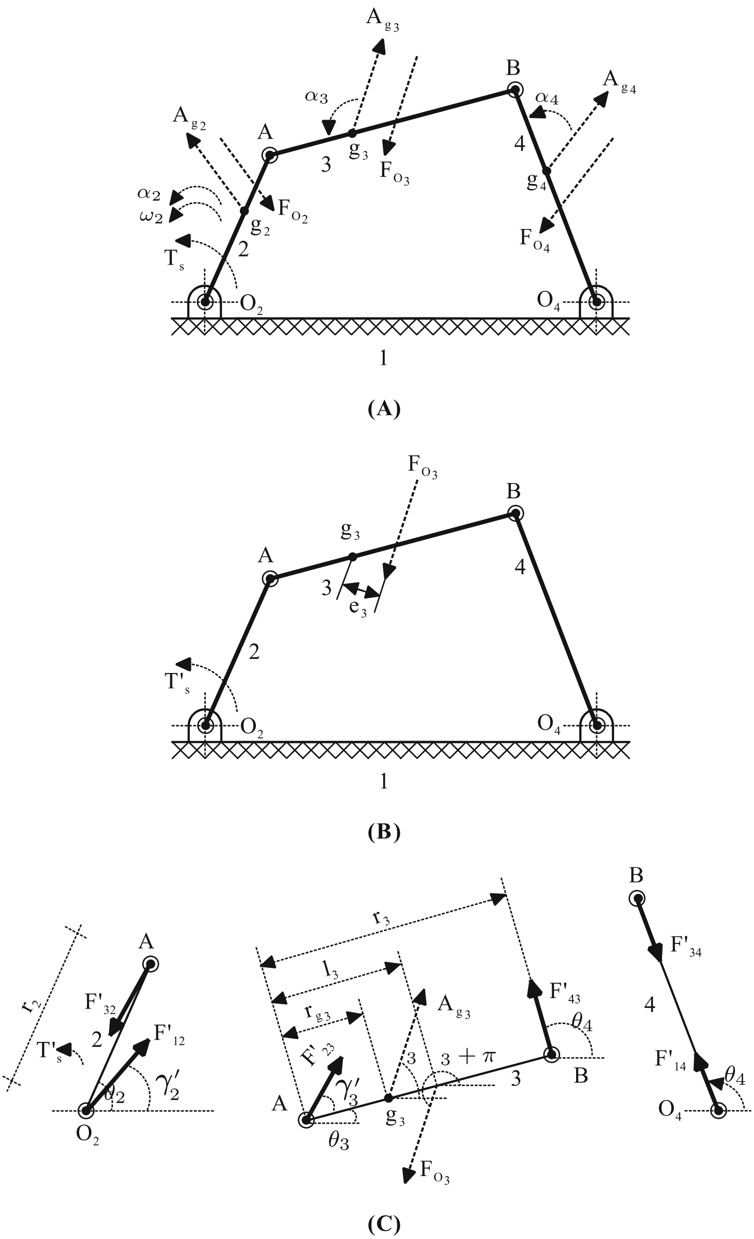


Fig. 4.10 Force analysis of the linkage using complex numbers

As can be seen, three unknowns must be found, i.e., the magnitude of F'_{23} and its direction γ'_3 and the magnitude of F'_{43}. The directions of F'_{43} and θ_4 are known since link 4 is acted on by only two forces when only the effect of F_{O_3} is considered (Fig. 4.10B). Another equation must be added to the above to determine these three unknowns. The additional equation is a torque equation around point A or B. If we consider point A, the sum of the torques about this point must equal zero, as follows:

$$F'_{43} r_3 \sin(\theta_4 - \theta_3) - F_{O_3} l_3 \sin(\beta_3 - \theta_3) = 0$$

$$F'_{43} = F_{O_3} l_3 \frac{\sin(\beta_3 - \theta_3)}{r_3 \sin(\theta_4 - \theta_3)}$$

To determine F'_{43} in the above equation, one can find the real and imaginary components of F'_{23} from Eq. I and II as follows:

$$\Re F'_{23} = F'_{23} \cos\gamma'_3 = -F'_{43} \cos\theta_4 - F_{O_3} \cos(\beta_3 - \pi)$$

$$\Im F'_{23} = F'_{23} \sin\gamma'_3 = -F'_{43} \sin\theta_4 - F_{O_3} \sin(\beta_3 - \pi)$$

where $\Re$ and $\Im$ symbols denote the real and imaginary components of the vector F'_{23}, respectively. The resultant of these components is the vector F'_{23}, the magnitude of which is determined as follows:

$$F'_{23} = \sqrt{(\Re F'_{23})^2 + (\Im F'_{23})^2}$$

The direction of F'_{23} along angle γ_3 is obtained by the following relationship:

$$\tan\gamma'_3 = \frac{\Im F'_{23}}{\Re F'_{23}}$$

Hence, the magnitude and direction of the bearing forces at A and B are determined from the above equations. The forces of the other links can be determined similarly.

4.5 Determining the Center of Mass and Moment of Inertia

To analyze the forces exerted on the links of a mechanism, the mass center of each link must be known. The center of mass of a body is a point on which the weight of the body acts regardless of the position and orientation of the body. If the center of mass of a set of material points is expressed by G, the coordinates of this center of mass relative to an arbitrary origin are determined as follows:

$$x_{\mathrm{G}} = \frac{\sum m_i x_i}{M} \qquad y_{\mathrm{G}} = \frac{\sum m_i y_i}{M} \qquad z_{\mathrm{G}} = \frac{\sum m_i z_i}{M}$$

where m_i represents the mass of each point and x_i, y_i, and z_i are the coordinates of the point. The total mass of the set of material points is equal to M.

Note Most machine members have two axes of symmetry in their plane of motion, at the intersection of which lies their center of gravity.

An experimental method for determining the center of mass of a body is as follows: the body is suspended from a point so that it can freely rotate. Next, a vertical line is drawn from the suspension point along the body. Next, the body is suspended from another point, and another line is drawn. Finally, the center of mass will be at the intersection of these two lines.

Note In some bodies, the point of intersection of the vertical lines (i.e., the center of mass) lies outside the body.

The moment of inertia of a body about a known axis is as follows:

$$I = \sum m_i.r_i^2 \tag{4.12}$$

where m_i is the mass of every point on the body, and r_i is its distance from the above axis. Usually, it is desired to find the moment of inertia about the axis passing through the mass center. If we name this moment of inertia I_G, the moment of inertia about every axis parallel to the above axis is as follows:

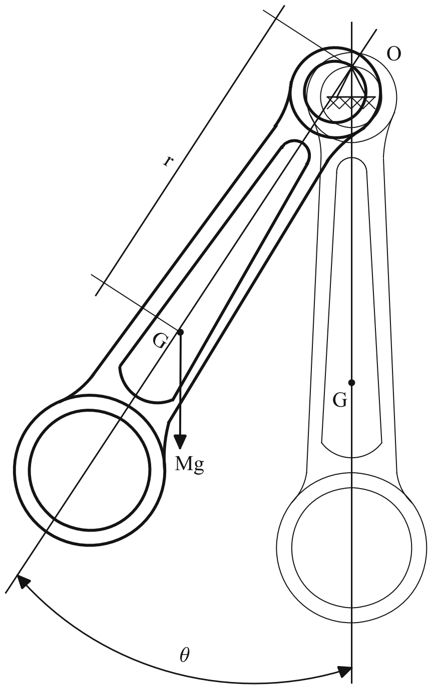
$$I_o = I_G + M.d^2 \tag{4.13}$$

where M is the total mass of the body, and d is the distance between the two parallel axes. This theorem is known as the parallel axis.

Figure 4.11 represents an experimental method used to determine the moment of inertia of a body. As shown in Fig. 4.11, the body has been suspended from point O, which is different from the center of mass, on a sharp edge. If the body is displaced by a small angle θ and released, it will oscillate about point O. The moment of inertia about point G can be determined based on the time required for a specific number of oscillations. The relationship between the torque around point O and the angular accelerations α is expressed as follows:

$$T_o = I_o.\alpha \quad \text{or} \quad -M.g.r.\sin\theta = I_o.\frac{d^2\theta}{dt^2} \tag{4.14}$$

Fig. 4.11 A suspended body



where $r\sin\theta$ is the moment arm corresponding to the force Mg. The negative sign expresses that the torque is opposite to the angle θ. If the angle θ is small, $\sin\theta$ is almost equal to the angle in radians, and the above relationship is simplified as follows:

$$-M.g.r.\theta = I_o.\frac{d^2\theta}{dt^2} \quad \text{or} \quad \frac{d^2\theta}{dt^2} + \frac{M.g.r}{I_o}\theta = 0 \tag{4.15}$$

which represents the differential equation of motion. Solving this equation results in the following:

$$\theta = \theta_{\max}.\cos\sqrt{\frac{M.g.r}{I_o}}t \tag{4.16}$$

where $\theta_{\max}$ corresponds to the time $t = 0$. This equation represents a cosine wave, which completes one cycle when

$$\sqrt{\frac{M.g.r}{I_o}}t = 2\pi \tag{4.17}$$

Solving this equation for t will determine the time required for completing a cycle, i.e., the period.

$$T = 2\pi\sqrt{\frac{I_o}{M.g.r}} \tag{4.18}$$

Solving this equation for I_o:

$$I_o = M.g.r\left(\frac{T}{2\pi}\right)^2 \tag{4.19}$$

Next, the moment of inertia about the center of mass can be determined using the parallel axis theorem. Hence,

$$I_G = I_o - M.r^2 = M.r\left[\left(\frac{T}{2\pi}\right)^2 g - r\right] \tag{4.20}$$

Note The accuracy in the calculation of I_G depends on the accuracy of T and r.

Note To reduce the calculation error in I_G, one needs to select a large T and a small r.

The moment of inertia of a body can also be found by placing it over a table suspended by several threads. To determine the moment of inertia of the body about the axis G–G passing through its center of gravity, it is placed on the table such that the axis G–G is parallel to and directly under the axis O–O. The period of small oscillations can be found by counting the number of oscillations (Fig. 4.12).

The moment of inertia of the body about its center of mass or the axis G–G is as follows:

$$T = 2\pi\sqrt{\frac{I_{po} + I_{to}}{\left(M_p.r_p + M_t.r_t\right) g}} \tag{4.21}$$

where

M_p = weight of the body

M_t = weight of the table

r_p = distance between Point O and the center of mass of the body

r_t = distance between Point O and the center of mass of the table

I_{po} = moment of inertia of the body about the axis O–O

T = oscillation period of the table with the body

T_t = oscillation period of the table without the body

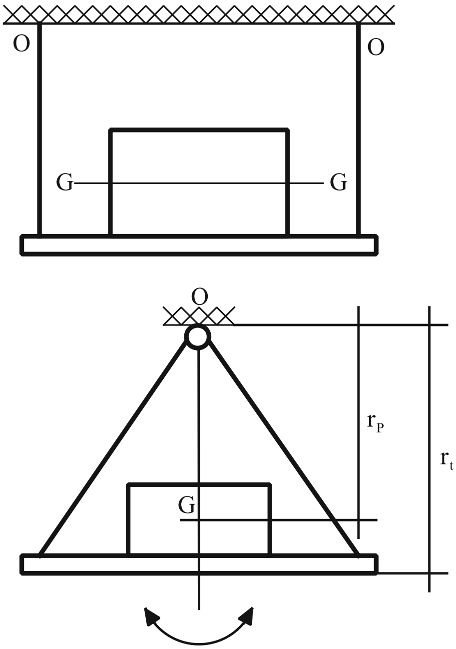


Fig. 4.12 Two suspended objects

r = distance between O–O and the center of mass of the body and table combined

The value of r can be obtained by considering the static torque about O–O as follows:

$$\left(M^{p} + M_{t}\right) r = M_{p}.r_{p} + M_{t}.r_{t} \quad \text{or} \quad r = \frac{M_{p}.r_{p} + M_{t}.r_{t}}{M_{p} + M_{t}} \tag{4.22}$$

Substituting Eq. 4.22 into Eq. 4.21:

$$\frac{T}{2\pi} = \sqrt{\frac{I_{po} + I_{to}}{\left(M_{p}.r_{p} + M_{t}.r_{t}\right) g}} \tag{4.23}$$

Solving this equation for I_{po}:

$$I_{po} = \left(\frac{T}{2\pi}\right)^{2} \left(M_{p}.r_{p} + M_{t}.r_{t}\right) g - I_{to} \tag{4.24}$$

From Eq. 4.19:

$$I_{to} = M_{t}.g.r_{t} \left(\frac{T_{t}}{2\pi}\right)^{2} \tag{4.25}$$

Substituting I_{to} from Eq. 4.25 into Eq. 4.24:

$$I_{po} = \left(\frac{T_t}{2\pi}\right)^2 M_p.g.r_o + \frac{M_t.g.r_t}{4\pi^2}(T^2 - T_t^2) \tag{4.26}$$

On the other hand, from the parallel axis theorem:

$$I_{po} = I_p + M_p.r_p^2 \tag{4.27}$$

where I_p denotes the moment of inertia of the body about the axis G–G passing through the center of mass. Substituting Eq. 4.26 into Eq. 4.27 gives

$$I_p = M_p.g.r_p\left[\left(\frac{T}{2\pi}\right)^2 - \frac{r_p}{g}\right] + \frac{M_t.g.r_t}{4\pi^2}(T^2 - T_t^2) \tag{4.28}$$

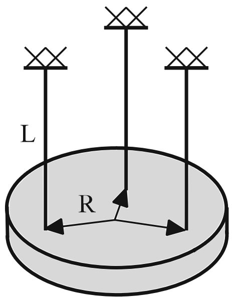
Since the moment of inertia of a body about an arbitrary axis is equal to the product of the sum of the masses of the material points making up the body and the square of the distances between the axis and the material points, it can often be expressed as follows:

$$I = M.k^2 \tag{4.29}$$

where M is the total mass of the body, and k is a constant known as the radius of gyration.

Note If the whole body is concentrated at a distance of k from the axis, the moment of inertia of the set of points will be equal to the moment of inertia of the body.

Example A flywheel of mass m is suspended horizontally from three cables of length L arranged at equal distances on a circle of radius R. If the period of oscillation about a vertical axis passing through the center of the flywheel is $\mathbf{t_n}$, the gyration radius is equal to:



1) $K_o = \frac{t_n}{2\pi R}\sqrt{\frac{L}{g}}$ 3) $K_o = \frac{t_n}{2R}\sqrt{\frac{g}{L}}$

2) $K_o = \frac{t_n R}{2\pi}\sqrt{\frac{g}{L}}$ 4) $K_o = t_n\sqrt{\frac{L}{g}}$

Solution Merely a dimensional study of the choices reveals (2) as the correct choice.

However, to properly solve the problem, based on the stated relationships, the period of a physical pendulum is

$$t_n = 2\pi\sqrt{\frac{I_o}{m.g.r}}$$

where I_o is the moment of inertia about an axis the body rotates around, and r is the distance of the body's mass center from this axis. Also,

$$L = \frac{I_o}{m.r}$$

where L is the length of a simple pendulum equivalent to the physical pendulum.

$$\Rightarrow r = \frac{I_0}{m.L} = \frac{m.R^2}{m.L} = \frac{R^2}{L} \Rightarrow t_n = 2\pi\sqrt{\frac{I_o}{m.g\frac{R^2}{L}}} \Rightarrow t_n = 2\pi\sqrt{\frac{T_o.L}{m.g.R^2}}$$

Moreover, based on the definition of the radius of gyration, $K_o = \sqrt{\frac{I_o}{m}}$; therefore,

$$t_n = 2\pi\sqrt{\frac{L}{g.R^2}}\sqrt{\frac{I}{m}} = 2\pi K_o\sqrt{\frac{L}{g.R^2}}$$

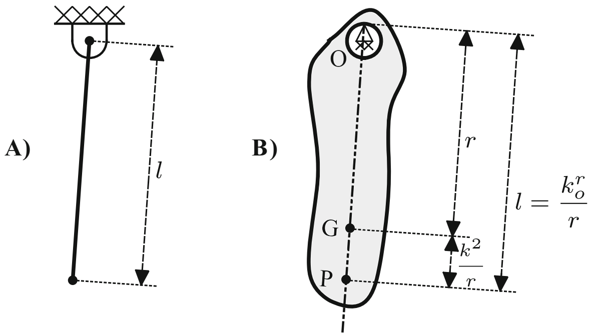
$$\Rightarrow K_o = \frac{t_n.R}{2\pi}\sqrt{\frac{g}{L}}$$

Choice (2) is correct.

4.6 Center of Percussion

Another important concept used in the study of dynamics is the center of percussion. If the pendulum is simple (Fig. 4.13A), its percussion center matches its center of mass at the end of the bar. However, if the pendulum's mass is not lumped, as shown

Fig. 4.13 A simple pendulum



in Fig. 4.13B (i.e., the pendulum is of the compound type), this will no longer be the case. Instead, the mass of the compound pendulum must be considered concentrated at a point such that its period remains constant. This point is called the center of percussion.

The period of a simple pendulum is equal to

$$T = 2\pi\sqrt{\frac{l}{g}} \tag{4.30}$$

For a compound pendulum, Eq. 4.18 gives

$$T = 2\pi\sqrt{\frac{I_o}{M.g.r}} = 2\pi\sqrt{\frac{M.k_o^2}{M.g.r}} = 2\pi\sqrt{\frac{k_o^2}{g.r}} \tag{4.31}$$

where M is the mass of the compound pendulum, and I_o and I_o represent the moment of inertia and gyration of the compound pendulum about Point O. Assume that the period of the simple pendulum is equal to the period of the compound pendulum:

$$l = \frac{k_o^2}{r} \tag{4.32}$$

Hence, the period of the displayed compound pendulum will not change if its mass is concentrated at a point such as P. Point P is called the center of percussion relative to point O.

Note One may not refer to the center of percussion of a body as a single point; instead, it must be considered relative to another point on the body (the point of suspension).

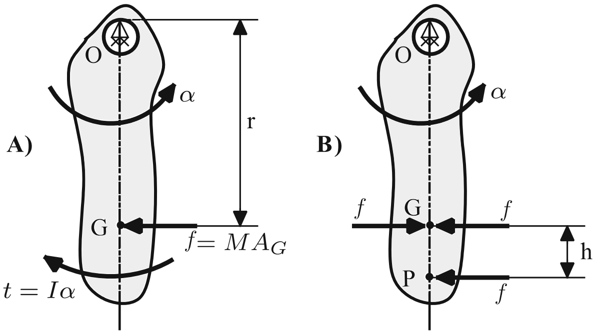


Fig. 4.14 Pendulum

The distance between the center of mass and the center of percussion of a body is equal to

$$l - r = \frac{k^2}{r} \tag{4.33}$$

This distance can be readily computed from the parallel axis theorem.

If a pendulum is given an acceleration of α about its point of suspension, the inertia force and torque may be replaced by a force at the center of percussion. In the pendulum shown in Fig. 4.14A, the inertia force f is exerted at the center of mass G with a sense opposite to $A_G = r\alpha$. In addition, there is an inertia torque t with a direction opposite to α. Figure 4.14B illustrates the compound pendulum equivalent to the pendulum of Fig. 4.14A, where the force f has replaced the force and torque in Fig. 4.14A.

Since the torque is specified using the couple fh, we can write

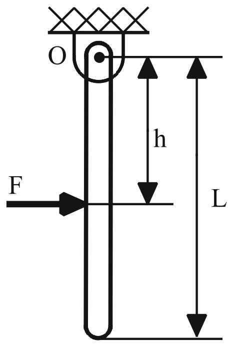
$$t = f.h \qquad \text{or} \qquad I.\alpha = M.A_G.h \Rightarrow h = \frac{I\alpha}{M.A_G} = \frac{M.k^2.\alpha}{M.r.\alpha} = \frac{k^2}{r} \tag{4.34}$$

It can be seen that h equals the distance GP. Since the two forces exerted at G are equal and opposite, they cancel out. Hence, the inertia effect can be expressed only by the force f exerted at the center of percussion. In other words, if a force perpendicular to OG is applied to the pendulum such that its line of action passes through the center of percussion, there will be no reaction force at the suspension point (Fig. 4.15).

Example A bar with a length of L and a mass of m lies on a horizontal surface. A force F is exerted on the bar at a distance of h from the joint. How much should h be for the joint force to be zero (just after the force F is exerted)?

1) L/3 2) L/3 3) L/2 4) L/6

Fig. 4.15 A bar with a length of L



Solution h is the distance between the center of percussion and the rotation axis. Based on Eq. 4.32:

$$h = \frac{k_o^2}{r} = \frac{\frac{L^2}{3}}{\frac{L}{2}} = \frac{2}{3}L$$

Here, the gyration radius of a uniform bar is equal to $\frac{L}{\sqrt{3}}$ because its moment of inertia about the point of rotation is $\frac{1}{3}mL^2$. If this expression is equated with mk_o^2, then $k_o^2 = \frac{L^2}{3}$.

Hence, Choice (2) is correct.

4.7 Dynamically Equivalent Masses

Any rigid link with a mass M and a moment of inertia I undergoing planar motion can be represented by a system composed of two point masses, the inertia of which is kinetically equivalent to that of the link. The following figure shows the inertia force F_0 exerted on one link, which has been displaced by a distance of e from the center of mass g based on the angular acceleration α (Fig. 4.16).

This figure also displays the two point masses M_p and M_Q, the resultant of the inertia forces of which $F_p = M_p.A_p$ and $F_Q = M_Q.A_Q$ must be $F_o = M.A_g$. Therefore,

$$F_p + F_Q = F_o \tag{4.35}$$

Note The mass, center of mass, and moment of inertia equilibrium conditions must hold for Eq. 4.35 to be satisfied.

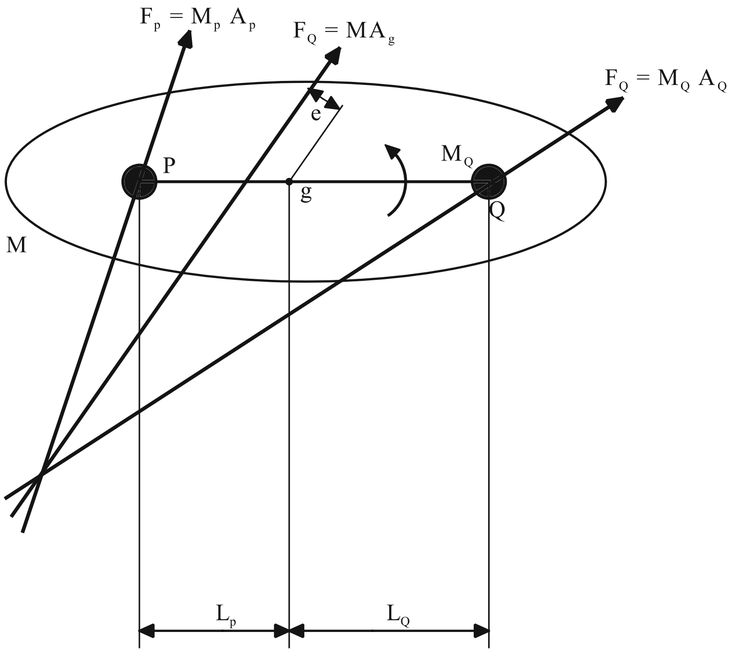


Fig. 4.16 A rigid link with a mass M

We will examine these three conditions in the following:

1. **Mass equilibrium:** The sum of the point masses must be equal to the mass M of the link:

$$M_p + M_Q = M \tag{4.36}$$

2. **Center of mass equilibrium:** The combined center of mass of the two point masses must coincide with the center of mass of the link. To this end, the point masses must lie along a line passing through g. Another outcome of this equilibrium is that the sum of the torques of the point masses about g will be equal to zero:

$$M_p.l_p - M_Q.l_Q = 0 \tag{4.37}$$

3. **Moment of inertia equilibrium:** The sum of the moments of inertia of the point masses about g must equal the moment of inertia I of the link:

$$M_p.l_p^2 - M_Q.l_Q^2 = I \tag{4.38}$$

To replace a link with the above equivalent set of two point masses, we must determine four related quantities, i.e., the masses M_p and M_Q and the distances l_p and l_Q.

The simultaneous solution of Eqs. 4.36 and 4.37 results in the following:

$$M_p = M\frac{l_Q}{l_p + l_Q} \tag{4.39}$$

$$M_Q = M\frac{l_p}{l_p + l_Q} \tag{4.40}$$

Substituting these equations into Eq. 4.38 gives

$$l_p.l_Q = \frac{I}{M} \tag{4.41}$$

Since four unknowns must be determined by three equations, one of the parameters can be selected arbitrarily. Usually, the distance l_p or l_Q is selected arbitrarily, and the other three parameters are determined by Eqs. 4.39, 4.40, and 4.41.

The method of dynamically equivalent masses is commonly used in the analysis of engines and the design of crankshaft balance weights to reduce engine vibration.

Example If the connecting rod is replaced by a dynamic system with two point masses such that the mass and center of mass of the system remain the same, the moment of inertia of the replaced system compared to that of the connecting rod:

1) is less
2) is the same
3) is more
4) may be more or less

Solution We know that the moment of inertia of an element of mass dm rotating at a distance r about an axis is calculated from the integral $I = \int r^2 dm$. If r is taken to be the distance from the mass center, the masses will be identical, and the center of mass remains at the same point; however, the whole mass moves to the furthest points of the connecting rod, increasing the moment of inertia (Fig. 4.17).

Therefore, Choice (3) is correct.

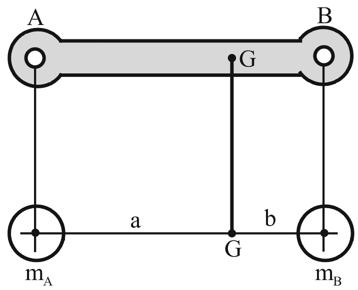


Fig. 4.17 A connecting rod replaced by a dynamic system

4.8 Flywheel

The flywheel is a rotating mass used to store energy in machines. The kinetic energy of a rotating body is $\frac{1}{2}I\omega^2$. Here, I is the mass moment of inertia about the rotation axis, and ω is the angular velocity of the body. If the body's angular velocity increases, energy will be stored in the flywheel. In contrast, if the angular velocity decreases, energy stored in the flywheel will be released. For instance, flywheels can be used in press machines. Press machines require high amounts of power, which must be provided by a motor in case of using no flywheel. However, with a flywheel, a smaller motor can be used since the flywheel stores energy between pressing operations and appropriately releases the stored energy during pressing operations.

Example What is the role of flywheels in machines?

1) Increasing the operating speed
2) Decreasing the operating speed
3) Increasing the power required for operation
4) Decreasing the power required for operation

Solution Choice (4) is correct.

4.8.1 Coefficient of Fluctuation

The coefficient of fluctuation expresses the permissible changes in speed. This coefficient is written as follows:

$$C = \frac{\omega_M - \omega_m}{\omega_{av}} \tag{4.42}$$

where ω_M is the maximum angular speed of the flywheel, ω_m is the minimum angular speed of the flywheel, and ω_{av} is the mean angular speed or nominal speed of the flywheel. This coefficient can also be expressed as follows:

$$C = \frac{V_M - V_m}{V_{av}} \tag{4.43}$$

where V_M is the maximum translational speed of a point on the flywheel, V_m is the minimum translational speed of this point, and V_{av} is the mean speed of this point.

Note The acceptable values of the coefficient of fluctuation range from 0.002 for generators to 0.2 for stone crushers.

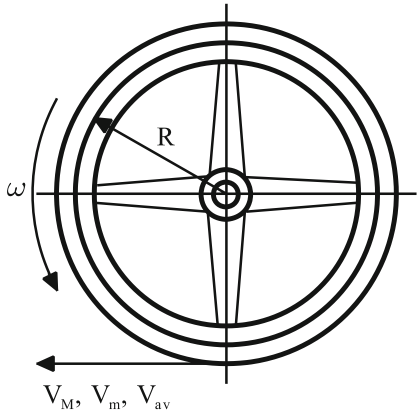


Fig. 4.18 A flywheel

4.8.2 Mass of the Flywheel for a Known Coefficient of Fluctuation of Speed

Imagine the flywheel shown in Fig. 4.18. Assume that the angular speed is variable, such that V_M, V_m, and V_{av} are the maximum, minimum, and mean speeds of the rim, respectively. Therefore:

$$V_{av} = \frac{V_M + V_m}{2} \tag{4.44}$$

Considering the definition of the coefficient of fluctuation of speed, one can write

$$C.V_{av} = V_M - V_m \tag{4.45}$$

Let us assume the entire mass M of the flywheel to be concentrated at a distance equal to the mean radius R of the rim. In this case, the kinetic energy at speeds of V_M and V_m will be equal to

$$K.E_M = \frac{1}{2} M.V_M^2 \tag{4.46}$$

$$K.E_m = \frac{1}{2} M.V_m^2 \tag{4.47}$$

If energy changes are expressed by E:

$$E = \frac{1}{2} M(V_M^2 - V_m^2) \tag{4.48}$$

Multiplying the two sides of Eq. 4.44 and 4.45 gives

$$2C.V_{av}^2 = V_M^2 - V_m^2 \tag{4.49}$$

Substituting Eq. 4.49 into Eq. 4.48:

$$E = M.C.V_{av}^2 \tag{4.50}$$

In the above equation, M is the effective mass of the flywheel at the rim, i.e., the mass of the rim of the flywheel plus the masses of the arms and the inner hub.

Note In a real flywheel, the entire mass is not concentrated at the rim. A flywheel is usually designed such that most of its mass is concentrated at the rim so that its kinetic energy is higher for a known angular speed.

Note In an armed flywheel, the actual mass of the rim is about 90% of the effective mass M.

Since the resultant stresses in the rim and arms are due to the centripetal forces (which are themselves functions of the speed), the translational speed limit of the rim is 30 m/s for cast iron and 40 m/s for steel.

4.8.3 Flywheel of an Internal Combustion Engine

Figure 4.19 displays a single-cylinder engine equipped with a flywheel. The free-body diagram of the flywheel shows the unbalanced torques exerted on and accelerating the flywheel.

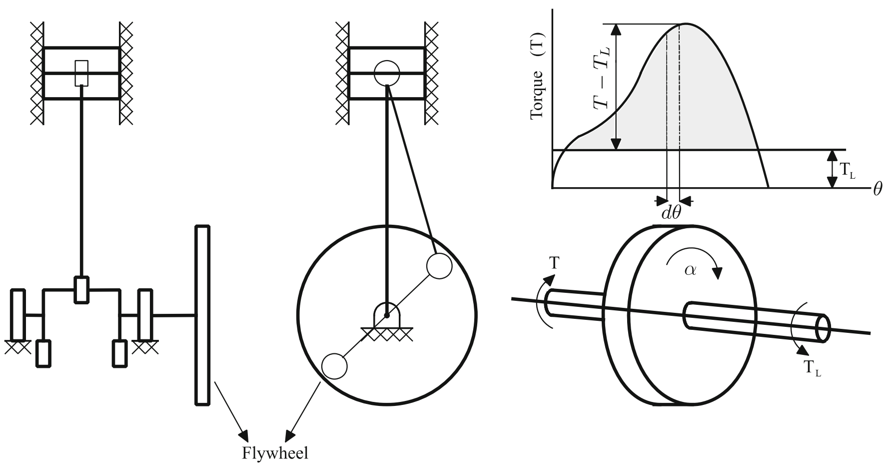


Fig. 4.19 A single-cylinder engine equipped with a flywheel

The following equation of motion can be written for output torques T larger than the load torque T_L:

$$T - T_L = I.\alpha \tag{4.51}$$

where I is the moment of inertia of the flywheel about the crankshaft, and T_L is the resistive torque. Also, α is in the direction of the resultant torque.

Note Under steady-state performance at a specific crankshaft speed, the average torque T_{av} equals torque T_L of the load driven by the engine and the flywheel.

Based on the relationship $\alpha = \omega(\frac{d\omega}{d\theta})$ and after simplifying Eq. 4.51 and integrating:

$$\int_{\theta at\omega_m}^{\theta at\omega_M} (T - T_L)d\theta = I \int_{\omega_m}^{\omega_M} \omega.d\omega = \frac{1}{2} I(\omega_M^2 - \omega_m^2), \tag{4.52}$$

left-hand side of this equation is the work done on the flywheel, shown by the hatched surface under the torque graph. Moreover, the right-hand side equation represents the corresponding change in the flywheel's kinetic energy due to a change in speed. Figure 4.20 shows the torque graph of an engine.

The hatched surfaces on the positive side correspond to parts of the engine cycle where the work done increases the flywheel speed, and those on the negative side

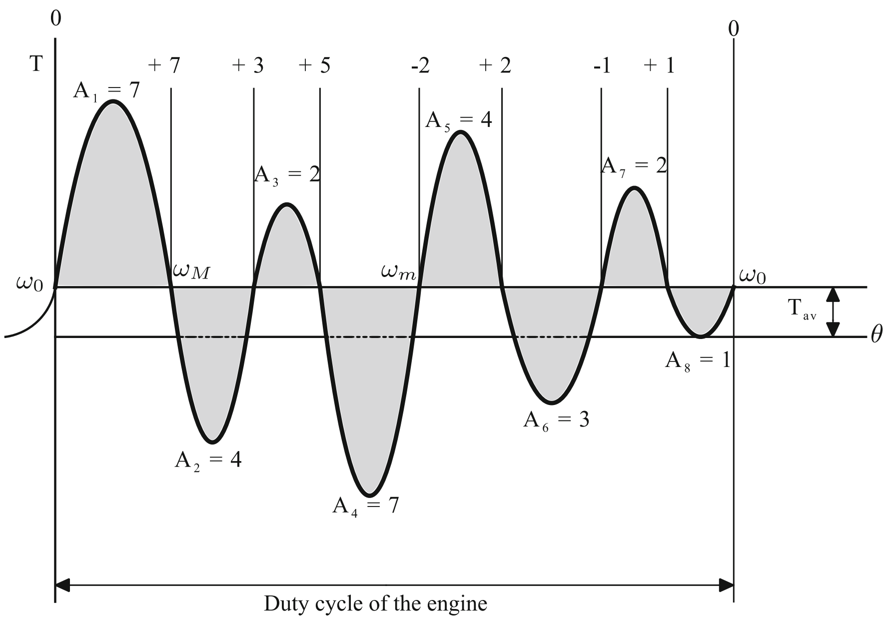


Fig. 4.20 Torque graph of an engine

represent work that decreases the flywheel speed. The limits of the integral in Eq. 4.52 are specified such that to determine the largest change in the flywheel speed. In this equation, ω_M and ω_m, respectively, represent the maximum and minimum angular speeds of the flywheel. If the speed equals the reference value ω_0 at the beginning of the first loop, the positive surface $A_1 = 7$ at the end of the first loop will be larger than ω_0. Moreover, the speed at the end of the second loop (which is negative) is higher than at the end of the first loop and larger than ω_0 because the algebraic sum of the two surface areas is positive: $A_1 + A_2 = 7 - 4 = 3$. As shown in the figure, the sum of the surface areas of all the loops must be zero at the end of every loop since the average torque line is plotted at a position where the sum of the positive surfaces above it is equal to the sum of the negative surface below it.

Note The maximum sum of the surface areas results in the location of ω_M, which is the maximum speed in the positive direction.

Note Usually, ω_M is located after a large positive region, and ω_m is located after a large negative region.

Note The sum of the surface areas between ω_m and ω_M represents the work done by the torque to change the kinetic energy of the flywheel from minimum to maximum.

The integral term in Eq. 4.52 can be expressed by the surface area A:

$$A = \int_{\theta at \omega_m}^{\theta at \omega_M} (T - T_L) d\theta \tag{4.53}$$

which is the algebraic sum of the surface areas of the loops of one cycle and causes the largest change in the flywheel speed.

With known mean torque ($\overline{T}$) and mean engine angular speed ($\overline{\omega}$), the engine power is determined as follows:

$$P = \overline{T}.\overline{\omega} \tag{4.54}$$

Example An electric motor runs at 900 rpm and produces a power of 2 HP. How much is its torque?

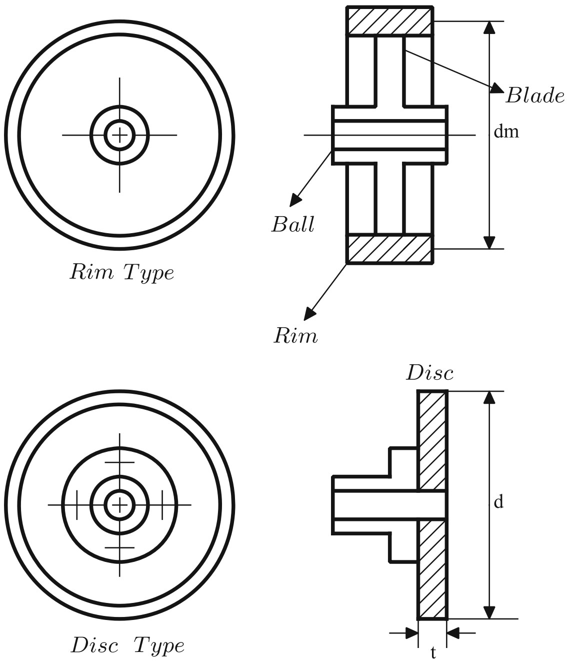
1) 15.8 N.m 2) 18.2 N.m 3) 12 N.m 4) 5.5 N.m

Solution We know that one horsepower is equal to 746 watts ($1HP = 746W$). On the other hand, $P = T.\omega$. Hence,

$$T = \frac{P}{\omega} = \frac{2 \cdot 746}{(900 \cdot \frac{2\pi}{60})} = 15.8N.m$$

It is of note that a unit conversion has been performed in the denominator of this term, resulting in the angular speed in rad/s. Therefore, Choice (1) is correct.

Fig. 4.21 Flywheels used in vehicle engines



Flywheels used in vehicle engines are usually of the solid disc type, and those used in steam engines or drilling press machines are of the rimmed type (Fig. 4.21).

For rimmed flywheels, $I = Mk^2$, where k is the gyration radius. In this equation, k can be considered equal to the mean radius r_m without loss of accuracy.

$$I = M.r_m^2 = \frac{W}{4g} d_m^2 \tag{4.55}$$

Solving this equation for W only provides the weight of the rim. The weights of the hub and the arms contribute a small share to the moment of inertia of the flywheel. As a result, the speed fluctuations will be somehow smaller than the specified value.

4.9 Gyroscopic Effects

In vehicles where the engine consists of rotational components with a high moment of inertia, gyroscopic forces take effect when the vehicle changes direction. The figure below shows a rigid body rotating at the constant angular speed ω about the axis passing through the center of mass. The angular momentum H of the rotating body is expressed by a vector whose magnitude is $I\omega$. Here, I is the moment of

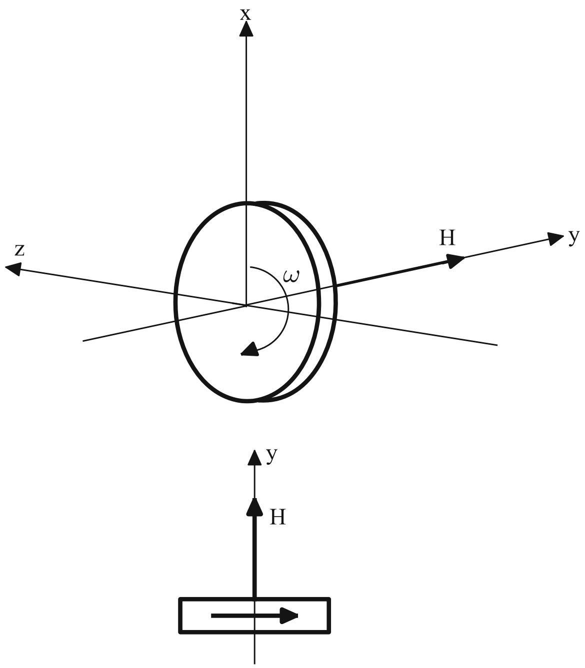


Fig. 4.22 A vector normal to the plane of rotation

inertia of the body about a rotation axis passing through the center of mass. The body's angular momentum, which is in a plane parallel to the plane of motion of the body's individual particles, is expressed by a vector normal to the plane of rotation, as shown in Fig. 4.22. The direction of this vector is obtained via the right-hand rule and based on the angular speed ω of the body. The length of this vector represents the magnitude of the angular momentum.

From dynamics, we know that $H = I.\omega$ and $T = I.\alpha = \frac{d}{dt}(I.\omega)$. Therefore,

$$T = \frac{dH}{dt} \tag{4.56}$$

In the state shown in the figure, a torque exerted in the direction of ω in the plane of motion of the rotating body increases the angular momentum with the given rate. This increase can be expressed by elongating the vector. The rotation axis in the previous discussion was considered fixed. If the rotation axis undergoes an angular displacement similar to the case of motion on a curved planar path, shown in Fig. 4.23 A, gyroscopic forces will be generated. At a constant ω, the magnitude of the angular momentum remains constant for a constant angular displacement $\Delta\theta$ of the rotation axis; however, the angular momentum changes since the direction of motion changes according to the momentum polygon in Fig. 4.23B.

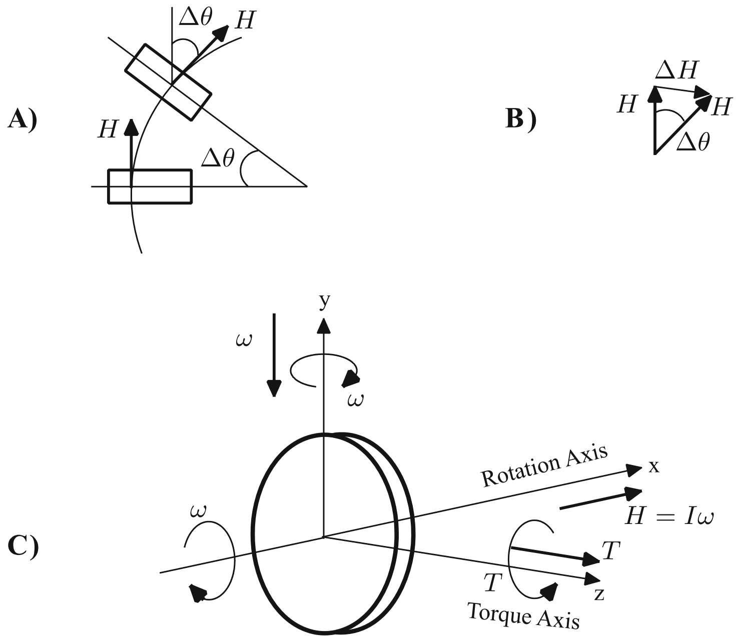


Fig. 4.23 A rotation axis undergoes an angular displacement

A rotation axis undergoes an angular displacement. Thus, the time rate of change of angular momentum is as follows:

$$\frac{dH}{dt} = I.\omega.\omega_P \tag{4.57}$$

From Eq. 4.56:

$$T = I.\omega.\omega_P \tag{4.58}$$

where $\omega_P = d\theta / dt$ is the angular speed of the spin axis or its angular displacement rate. In Fig. 4.23C, the x-axis represents the rotation axis, and the y-axis shows the spin axis. In this case, the z-axis will be the torque axis since the direction of the torque T is normal to the rotation axis and is in the x-z plane.

Eq. 4.58 can be expressed in vector form as follows:

$$\vec{T} = \vec{\omega}_P * I\vec{\omega} \tag{4.59}$$

T is known as the gyroscopic couple and represents the torque exerted on the body about the z-axis, i.e., the torque axis.

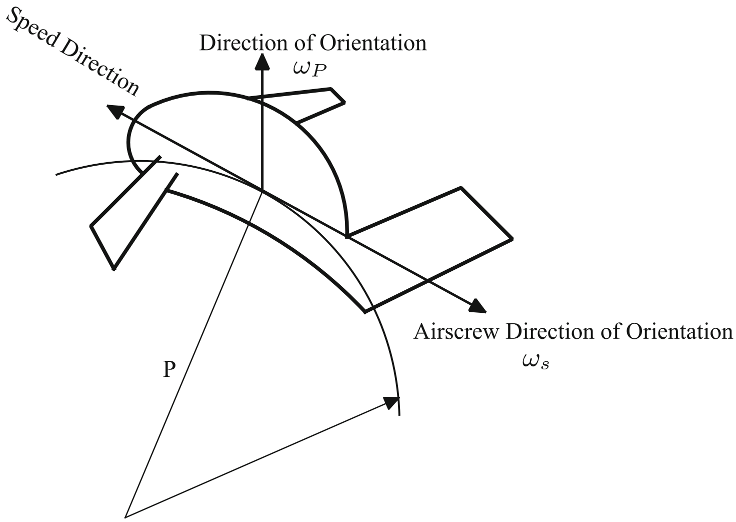


Fig. 4.24 Gyroscopic effect is applied to the airplane

Example If the gyroscopic effect is applied to the airplane, which of the following is true?

1) The gyroscopic effect causes the plane to crash ω_P.
2) The gyroscopic effect generates a torque that does not affect the horizontal equilibrium of the airplane.
3) The gyroscopic effect generates a torque that causes the airplane's nose to rise and its tail to fall.
4) The gyroscopic effect does not influence an airplane moving at a constant speed (Fig. 4.24).

Solution Based on the above equations, one can write the following for the airplane:

$$\overrightarrow{T} = \overrightarrow{\omega}_P * I\overrightarrow{\omega}_s$$

The direction of $\overrightarrow{T}$ is easily determined from the right-hand rule. It can be seen that $\overrightarrow{T}$ is along the radius of curvature of the path and pointing outward. Therefore, Choice (3) is correct (Figs. 4.25, 4.26, 4.27, and 4.28).

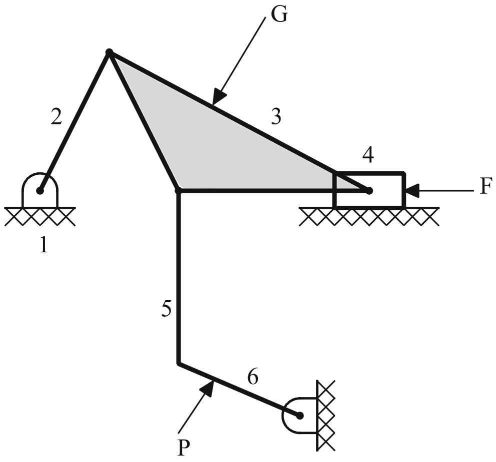


Fig. 4.25 A mechanism with static equilibrium

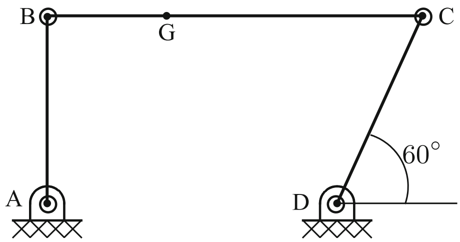


Fig. 4.26 A mechanism moving in the vertical plane

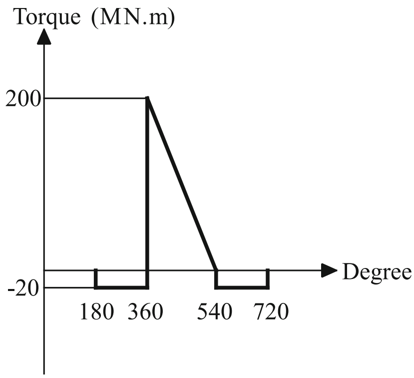


Fig. 4.27 Output torque diagram of a four-stroke single-cylinder engine

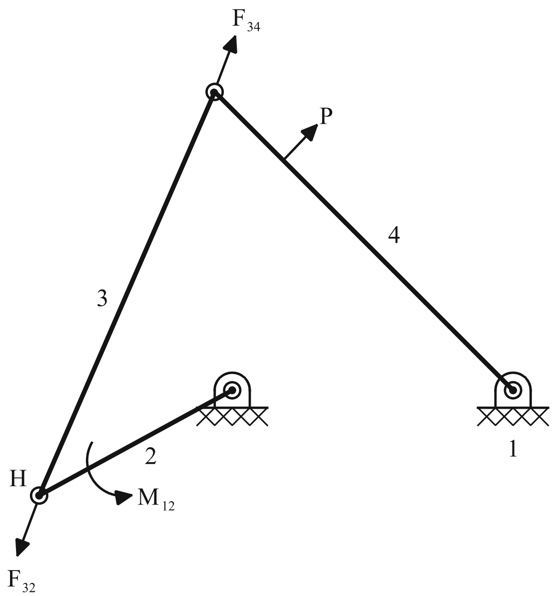


Fig. 4.28 Static forces analysis in a four-bar linkage

Some Examples of "Force Analysis of Mechanisms"

1. Which of the following is true about the static equilibrium of the mechanism shown below?

   1) $F + G + P = 0$
   2) $F_{12} + F_{14} + P + G + F_{16} + F = 0$
   3) $F_{12} + F_{14} + F_{34} + F_{16} + F + G + P = 0$
   4) $F_{12} + F_{34} + F_{53} + F_{65} + F_{23} + F_{25} + G + P + F = 0$

2. The following mechanism moves in the vertical plane. Bar BC is horizontal at the instant shown:
   The mass moment of inertia of CD is negligible. The driving torque T is equal to:

   1) 208 N.m 2) 186 N.m 3) 186 N.m 4) 175 N.m

3. The output torque diagram of a four-stroke single-cylinder engine is shown below. The resisting torque on the crankshaft is constant, and the mean angular speed is 1000 rpm. How many kilowatts is the engine power?

   1) 0.25 2) 2.62 3) 15 4) 1.57

4. Which of the following equations is false in the static forces analysis in a four-bar linkage?

   1) $\sum F = F_{34} + F_{32} + P = 0$ 3) $\sum F_4 = F_{34} + P + F_{14} = 0$
   2) $\sum M_{o4} = R_B.F_{34} + R_C.P = 0$ 4) $\sum M_{O_2} = M_{12} + R_A.F_{32} = 0$

5. One reason for selecting larger rear wheels for tractors is … :

   1) To increase the force of friction
   2) To reduce the engine rpm and, hence, increase the engine power
   3) To increase the tractor's speed for a lower engine rpm
   4) To prevent slipping due to the large couple exerted on the rear wheels

Answers for the Examples of "Force Analysis of Mechanisms"

1. Choice (2) is correct.
   The resultant of the external forces exerted on the system (including the forces applied by the ground, i.e., F_{12}, F_{14}, and F_{16}) must equal zero.
2. Choice (4) is correct.
   The equilibrium equation can be written based on the free-body diagram of each link.
   Torque equilibrium about point A:

$$AB : T - B_x.AB = I_{AB}.\alpha_{AB} \tag{4.60}$$

Torque equilibrium about point G:

$$BC : B_y.BG + C_y.CG = I_{BC}.\alpha_{BC} \tag{4.61}$$

Force equilibrium along the y-direction:

$$BC : B_y - C_y = m_{BC}.a_G \sin 45^\circ \tag{4.62}$$

Force equilibrium along the x-direction:

$$BC : B_x - C_x = m_{BC}.a_G \cos 45^\circ \tag{4.63}$$

Torque equilibrium about point D:

$$CD : \frac{C_y}{C_x} = \tan 60^\circ \tag{4.64}$$

B_y and C_y are determined from Eq. 4.61 and 4.62:

$B_y = 803.55$ N, $C_y = 96.46$ N

Substituting C_y in Eq. 4.64 leads to the following:

$C_x = 55.69$ N

Substituting C_x in Eq. 4.63 results in $B_x = 762.79$ N, which can be substituted into Eq. 4.60 to determine the driving torque T: $T = 174.4$ N.

3. Choice (4) is correct.
While torque is changing, we compute the mean torque as follows:

$$\overline{T} = \frac{1}{4\pi}\int_0^{4\pi} T d\theta = \frac{1}{4\pi}\left(-20\pi + \frac{200\pi}{2} - 20\pi\right) = 15 \text{ N.m}$$

$$\text{Power } P = \overline{T}.\omega = 15 \cdot \frac{1000 \cdot 2\pi}{60} = 1.57 \text{ kW}$$

4. Choice (1) is correct.
In the resultant of forces exerted on the mechanism in Choice (1), the support forces at points O_2 and O_4 are ignored. On the other hand, F_{34} and F_{32} are internal forces that must not appear in the equation, although the sum of these forces is zero in this mechanism.
Choice (2) represents the resultant of the torques exerted on Bar 4 about point O_4. Also, Choice (3) is the resultant of forces exerted on this bar, which are true. Also, Choice (4) represents the torque resultant for bar 2 about point O_2, which is true.
5. Choice (2) is correct.
The power transferred from the engine to the differential and from the differential to the wheels is constant. On the other hand, the rpm decreases when the wheel diameter increases, and thus the transferred torque increases. $P = T.\omega$ (Figs. 4.29 and 4.30).

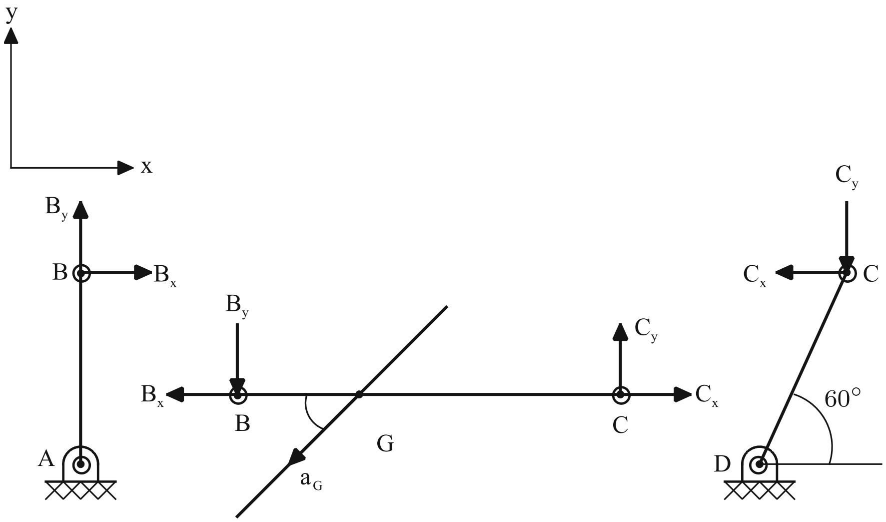


Fig. 4.29 Free-body diagram of each link

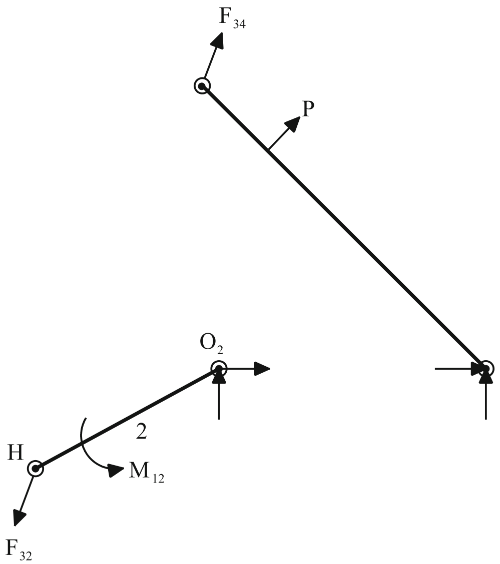


Fig. 4.30 Forces on each link

Chapter 5
Cams

This chapter begins with an introduction, providing an overview of cams and their classification. The chapter further delves into various cam mechanisms, including disk cams with radial flat-faced followers, cams with positive return followers, cylindrical cams, and inverse cams. Each type of cam mechanism is described, highlighting their unique features and applications. Also, the aspect of dynamic loading in cams is addressed.

Overall, this chapter provides a comprehensive exploration of cams, covering their classification, mechanisms, displacement diagrams, and dynamic loading. It offers readers valuable insights into the design, analysis, and application of cams in mechanical systems.

5.1 Introduction

Cams are machine members with an irregular shape that act as a driver by transferring motion to another member named follower. In general, both rolling and sliding exist in cam-follower contact. Cams are among the important mechanisms regarding their capability to create any motion in the follower despite their simplicity. In this mechanism, the motion of the cam (which is usually rotational) is converted to an oscillating or translational motion or a combination of these in the follower. The cam design problem can take two forms: (A) the follower motion is known, and a cam must be designed to create this motion or (B) the cam shape is known, and the problem involves determining the resulting displacement, velocity, and acceleration. Case (A) is discussed in detail in the mechanism design. Here, we will study Case (B). Gruebler's equation can be used to create numerous hybrid mechanisms from cam pairs. However, in practice, the most common application of cam pairs is in simple cam-follower mechanisms, which consist of three links, i.e., a cam pair and

A. Abbasimoshaei, T. A. Kern, *Machine Dynamics*,
https://doi.org/10.1007/978-981-99-6010-1_5

the ground link. This chapter only addresses three-link cam-follower sets, which are mostly called simple cam mechanisms.

5.2 Classification of Cams

Cam mechanisms can be classified in terms of cam-type or follower shape, motion, and position. The most common cam shapes are disk (or plate) cams, translating cams, and cylindrical cams. Figure 5.1A, B, and C displays disk, translating, and cylindrical cams, respectively.

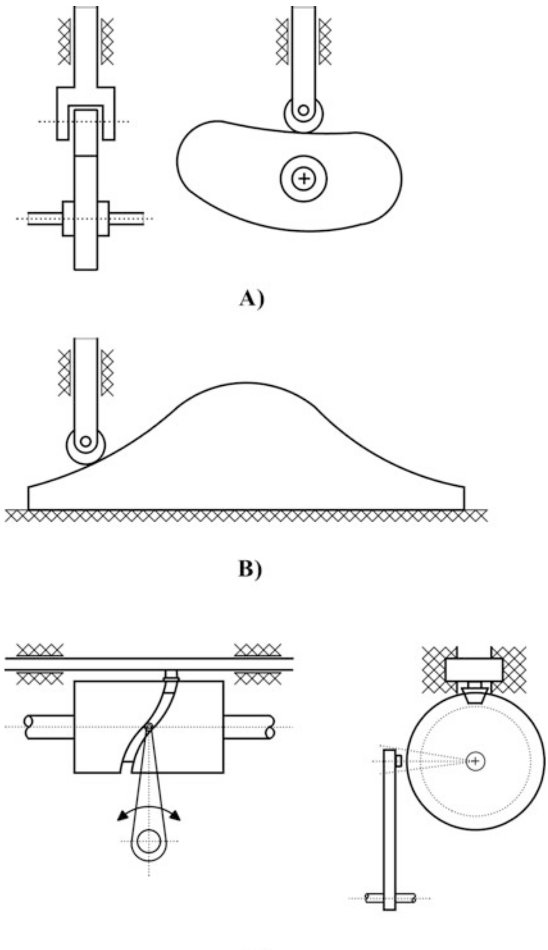


Fig. 5.1 Different types of cams

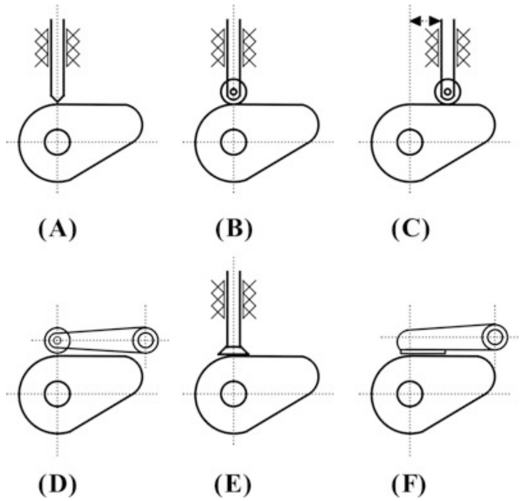


Fig. 5.2 Several categories of followers

Disk cams with reciprocating or oscillating followers are the simplest and most common cam mechanisms. Also, followers are divided into several categories in terms of motion type, cam-follower contact type, etc. Fig. 5.2 displays common followers.

Figure 5.2A shows a disk cam with a radial knife-edge follower. Although this mechanism is theoretically interesting, it is of little practical use due to the contact stresses involved.

Note The follower is called radial when its centerline passes through the center of rotation of the cam.

Figure 5.2B and C depict a disk cam with a radial roller follower and a disk cam with an offset roller follower, respectively. In all the cam mechanisms shown in Fig. 5.2A, B, and D, the cam rotates while the follower reciprocates. Figure 5.2D displays a disk cam with an oscillating roller follower. Also, Fig. 5.2E represents a disk cam with a reciprocating flat-faced follower.

Note The distinction between radial and offset flat-faced followers is unimportant since they are kinematically equivalent. In other words, any follower whose axis is parallel to that shown in the figure undergoes the same motion. However, offset followers may require modifying the length of the follower's face.

Figure 5.2F shows a disk cam with an oscillating flat-faced follower and Fig. 5.3 displays the general terms used for cam mechanisms.

The trace point is a point on the follower that corresponds to the point of contact in knife-edge followers.

Note The trace point of a roller follower is at the center of the roller.

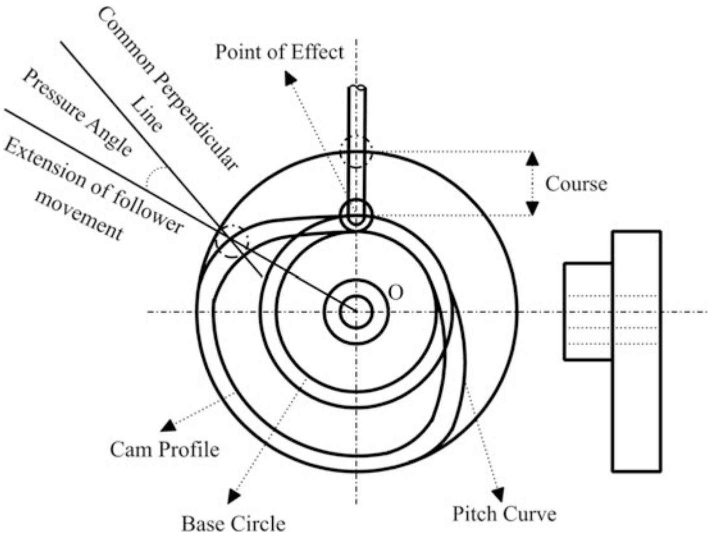


Fig. 5.3 General vocabulary used for cam mechanisms

The pitch curve is a path traveled by the trace point on the cam. The base circle is the smallest circle drawn around the center of rotation of the cam tangent to its surface. The pressure angle is the angle between the direction of motion of the trace point and the common normal of the contact surfaces (line of action). This angle varies during cam rotation. The pitch point is a point on the pitch curve with the maximum pressure angle.

Note The pressure angle is a measure of the instantaneous force transfer and represents the ability of the cam to transfer motion to the follower.

Example The pressure angle of a cam is:

1) The angle between the common tangent of the cam and the follower and the follower's path.
2) The angle between the common normal of the cam and the follower and the follower's path.
3) The angle between the common normal of the cam and the follower and the normal to the follower's path.
4) The angle between the common tangent of the cam and the follower and the normal to the follower's path.

Solution Choice (2) is correct.
The distance between the two extreme positions of the follower is known as the lift or travel.

5.3 Cam Mechanisms

Different cam mechanisms have different characteristics and applications. In the following, some of these mechanisms are introduced in brief.

In circular cams, where the center of rotation is at a distance e from the center of the cam, the position of the follower is determined as follows:

$$y = e(1 - \cos\theta) \tag{5.1}$$

In this equation, $\theta = 0$ corresponds to the state where the cam is at the base height $y = 0$, i.e., its lowest position.

Note In eccentric circular cams, if $\theta = 0$ relates to the lowest position of the cam, the maximum pressure angle occurs at $\theta = 90^\circ$.

Example Calculate the maximum pressure angle in the cam-follower mechanism shown in Fig. 5.4.

1) $\arcsin \frac{e}{R_2+R_3}$ 2) $\arctan \frac{e}{\sqrt{R_2^2+e^2}}$ 3) $\arctan \frac{h}{\sqrt{R_2^2-e^2}}$ 4) $\arctan \frac{d}{\sqrt{R_2^2-e^2}}$

Solution As mentioned previously, the maximum pressure angle occurs at $\theta = 90^\circ$; hence, it suffices to plot the cam geometry corresponding to $\theta = 90^\circ$ (Fig. 5.5).

From the geometry:

$$\sin\phi = \frac{e}{R_2 + R_3} \Longrightarrow \phi = \arcsin(\frac{e}{R_2 + R_3})$$

Choice (1) is correct.

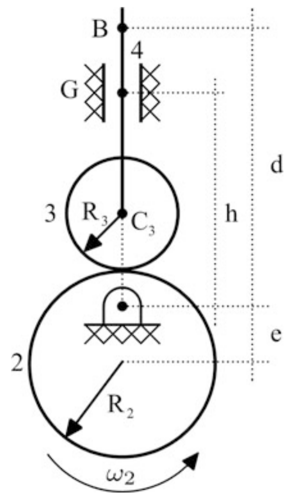


Fig. 5.4 A cam-follower mechanism

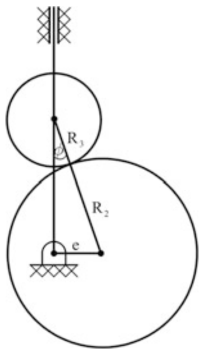


Fig. 5.5 The cam-follower mechanism analysis

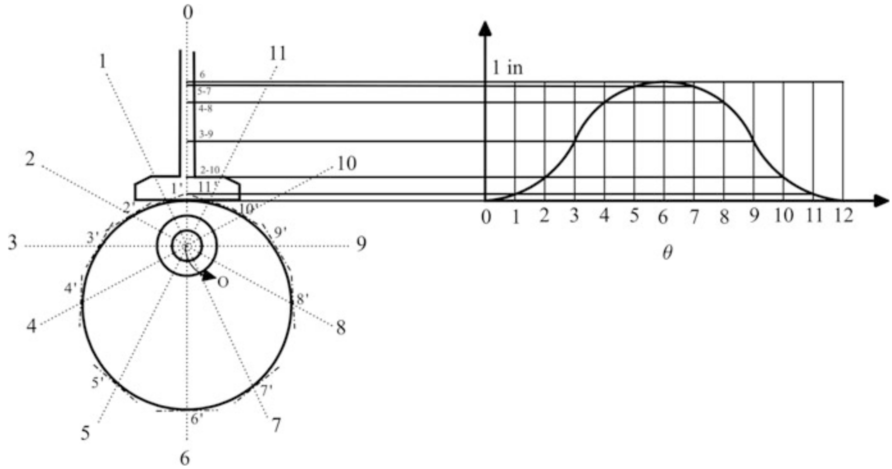


Fig. 5.6 A cam profile

5.3.1 *Disk Cams with Radial Flat-Faced Followers*

Figure 5.6 represents a disk cam with a radial flat-faced follower. When the cam rotates clockwise at a constant angular speed, the follower will lift 1 inch for half a rotation of the cam. The reverse motion occurs in the same manner.

Note To determine the cam profile graphically, the mechanism must be reversed, and the follower must be traveled around the fixed cam. These actions do not affect the relative motion between the cam and the follower.

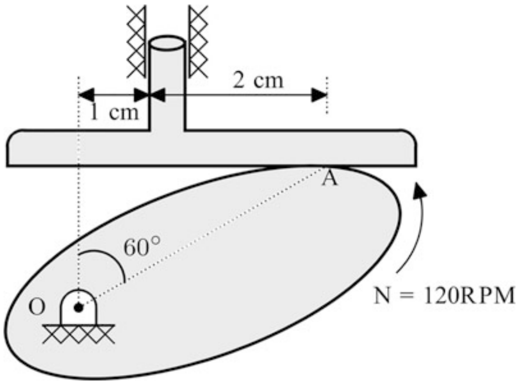


Fig. 5.7 A cam-follower to determine the follower lift speed

A similar procedure is used for disk cams with roller followers, except that the cam profile is obtained by plotting the curve tangent to the different positions of the roller follower.

The pressure angle of the above follower, the base surface of which is normal to its leg, is zero. Also, the lateral force exerted on it is negligible compared to the force exerted on a roller follower.

Example Based on Fig. 5.7, how much is the lift speed of the follower in cm/s at the instant shown?

1) 25.133 2) 29.25 3) 37.7 4) 43.533

Solution We need only to find the vertical component of the cam's velocity at point A. Since the cam and the follower move together, the follower's lift speed will be the same as the obtained speed.

$$\omega = \frac{2\pi N}{60} = \frac{2\pi * 120}{60} = 4\pi \text{ rad/s}$$

On the other hand, based on the trigonometry and geometry of OA (Fig. 5.8):

$$|OA| = \frac{3}{\sin 60} = \sqrt{12}$$

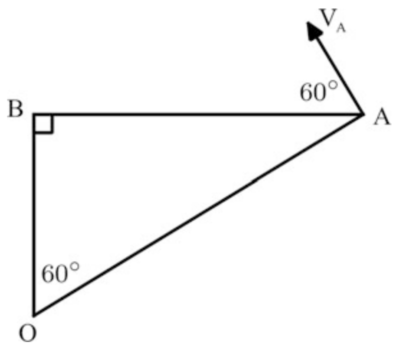
$$V_A = |OA| * \omega = (4\pi)\left(\sqrt{12}\right) \text{ cm/s}$$

$$\text{Follower lift speed} = V_A * \sin 60^\circ = (4\pi)\left(\sqrt{12}\right)\sin 60^\circ = 37.68 \text{ cm/s}$$

Choice (3) is correct.

For a disk cam with a roller follower, a large pressure angle causes a significant lateral force on the follower, which may lead to the bending and gripping of the follower's leg.

Fig. 5.8 Speed analysis



Note The maximum pressure angle, i.e., the angle between the line of action and the centerline of the follower, must be as small as possible.

In most cases, this angle must not exceed 30° to achieve acceptable performance.

Note The pressure angle of all radial flat-faced followers is a fixed value.

Pressure angles can be increased by increasing the minimum radius of the cam. Thus, the follower would travel longer on the cam for a given lift.

Reducing the overall follower lift, changing the follower offset, changing the follower motion (such as constant speed or constant acceleration), increasing the cam rotation for a given follower displacement, and increasing the diameter of the base circle are among the methods to lower pressure angles in cams.

Note In a cam with a roller follower, the curvature radius of the pitch surface must be larger than that of the roller; otherwise, the cam profile will become sharp.

Sometimes, flat-faced or roller followers are of the offset type. These types may be used for structural reasons or, in the case of a roller follower, to reduce the pressure angle during the upward stroke. Nevertheless, although the pressure angle decreases during the upward stroke in this case, it still increases during the downward stroke.

5.3.2 Cams with Positive Return Follower

In disk cams with radial followers, the driver must have more impact than only by spring or gravity. Figure 5.9 shows an example of this cam, where the cam controls the follower motion both in the upward and downward strokes. In this cam, the downward stroke is the same as the upward stroke but in the opposite direction. This cam is also known as the constant-breadth cam.

Such a cam can be designed using two roller followers instead of a single flat-faced follower. If the downward stroke is independent of the upward stroke, two

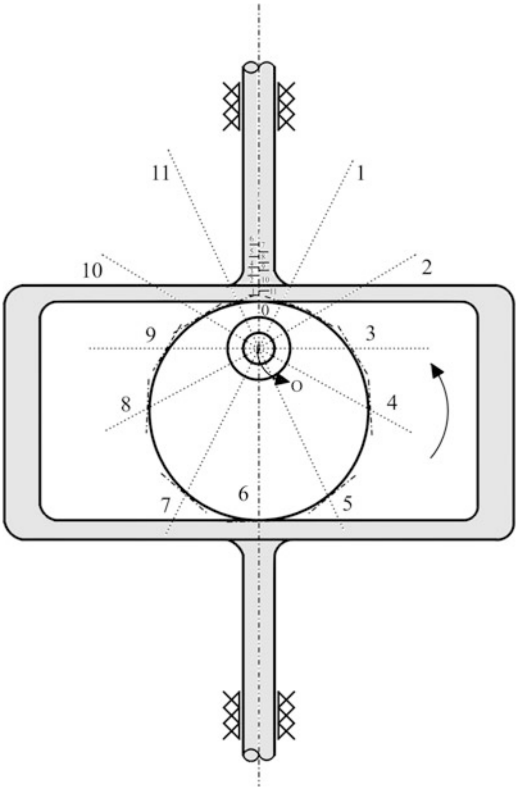


Fig. 5.9 A constant-breadth cam

disks must be used, one for the upward stroke and one for the downward stroke. These dual-disk cams can be used with both roller and flat-faced followers.

5.3.3 Cylindrical Cams

In a machine where the axis of rotation of the cam needs to be parallel to the direction of motion, cylindrical cams can be used to avoid complex gear systems needed when using a disk cam. This type of cam consists of a rotating cylinder, the motion of which is transferred to a follower via a slot embedded in the cylinder.

Note These cams are commonly used in fishing reel mechanisms.

Figure 5.10 illustrates a cylindrical design that transfers its rotation to a follower placed in the slot on the cylinder.

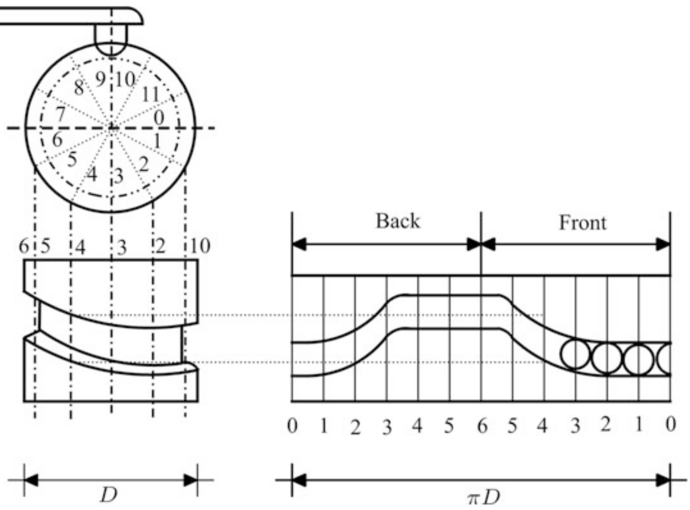


Fig. 5.10 A cylindrical design

5.3.4 Inverse Cams

The roles of the cam and the follower are sometimes reversed, and the follower drives the cam. This reverse mechanism is used in sewing machines and other mechanisms with similar functionality. Figure 5.11 shows a plate cam where the arm oscillations result in the reciprocating motion of a block via a roller in the slot.

5.4 Cam Displacement Diagrams

The displacement diagram is a curve representing the follower's displacement as a function of time. Since this curve ideally shows displacement versus time, the speed and acceleration diagrams can be obtained via its consecutive differentiation. A sample cam displacement diagram is shown in Fig. 5.12.

Degrees or radians instead of seconds are used for selecting the follower motion before considering the angular speed of the cam.

Example Given the displacement diagram of the follower versus the angular displacement of the cam shown in Fig. 5.13, if the angular speed of the cam is ω, which of the following will be the lift speed of the follower at a cam angle of θ_0?

1) $\omega tg\theta$ 2) $\omega \sin\theta$ 3) $\overline{OA}\omega\cos\theta$ 4) $\overline{OA}\omega tg\theta$

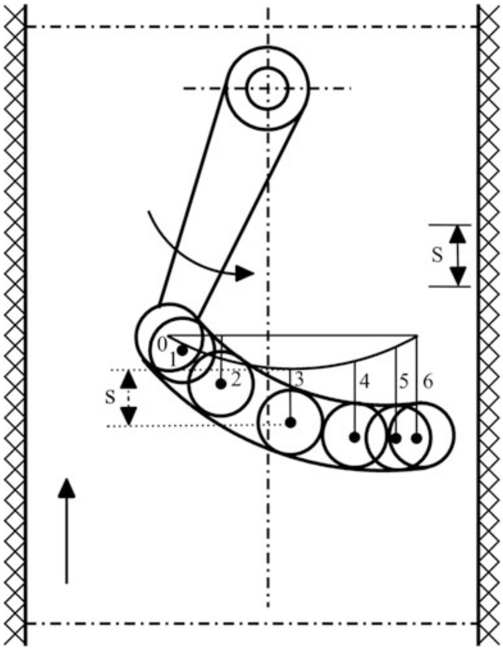


Fig. 5.11 An inverse cam

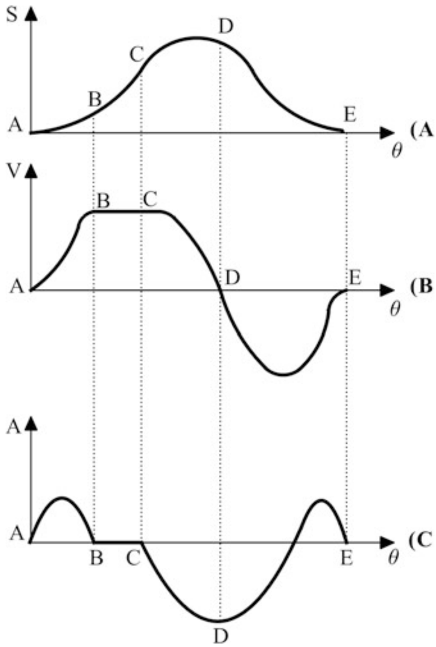
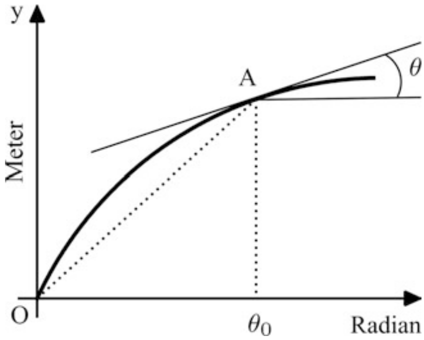


Fig. 5.12 Cam displacement diagrams

Fig. 5.13 Displacement diagram of a follower versus the angular displacement of a cam



Solution The speed of a follower is obtained by differentiating its displacement with respect to time. Therefore:

$$\text{follower speed} = \left.\frac{dy}{dt}\right|_{\theta=\theta_0} = \left.\frac{dy}{d\theta}\right|_{\theta=\theta_0} \cdot \frac{d\theta}{dt} = \left.\frac{dy}{d\theta}\right|_{\theta=\theta_0} \cdot \omega = \omega \tan\theta$$

Choice (1) is correct.

Before determining the diagram of a cam, the motion of the follower must be specified based on the system's needs. If the operational speed is low, the follower motion may be a known type, such as parabolic (with constant acceleration and deceleration), parabolic with constant speed, simple harmonic, or cycloidal.

Note Among these motions, parabolic motion theoretically produces the smallest acceleration for specific amounts of cam displacement and speed; thus, it is rarely used in low-speed applications.

The displacement of a body moving from rest with a constant acceleration is equal to:

$$s = \frac{1}{2}At^2 \tag{5.2}$$

where s is displacement, A is acceleration, and t is time. The graph of this equation is a parabola. Therefore, this motion is often called parabolic motion.

Note In parabolic motion, the distance traveled after time t is proportional to t^2 since the acceleration is constant.

Moreover, the acceleration and deceleration intervals may be unequal depending on the motion conditions. In addition, parabolic motion can be modified in such a way as to create constant-speed motion intervals between the acceleration and deceleration intervals. This speed is often known as the modified constant velocity.

Motion at constant velocity means traveling the same distances at equal time intervals. Therefore, changes in the distance or displacement with respect to time

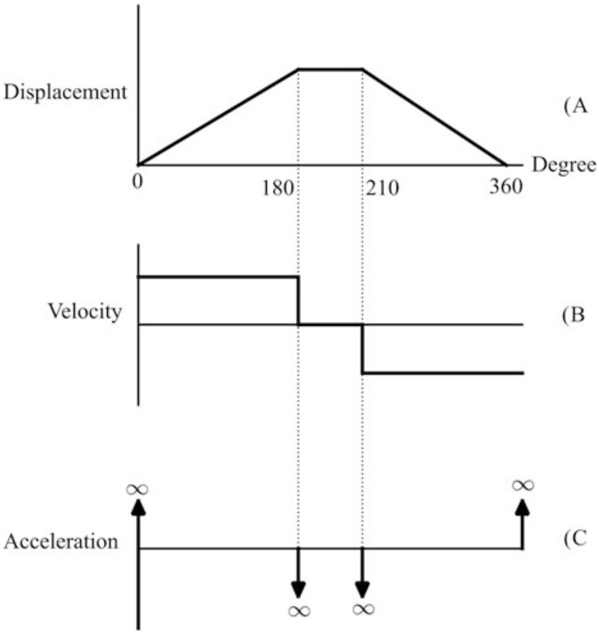


Fig. 5.14 Displacement diagram of a follower traveling upward at constant speed

represent a straight line. Figure 5.14 shows the displacement diagram of a follower traveling upward at constant speed between points B and C, stopping between points C and D, and traveling downward at constant speed between points D and E.

Theoretically, this motion causes infinite acceleration at points B, C, D, and E, leading to impact loads. Hence, such a motion must be avoided. If these diagrams are used for a real cam mechanism, the elastic properties of the cam and the follower will cause the accelerations to be smaller.

Note In the modified constant velocity, intervals of constant acceleration and deceleration are considered before and after the constant speed interval, respectively.

A simple harmonic displacement diagram is displayed in Fig. 5.15. This diagram is plotted using the idea that projecting a point P that moves at a constant speed in a circular path onto the circle's diameter creates a simple harmonic motion.

As shown in Fig. 5.15, with the rotation of the radius OP through the angle ϕ, the cam rotates by θ, and the follower is displaced by s. According to the figure:

$$s = \frac{h}{2} - \frac{h}{2}\cos\phi = \frac{h}{2}(1 - \cos\phi) \tag{5.3}$$

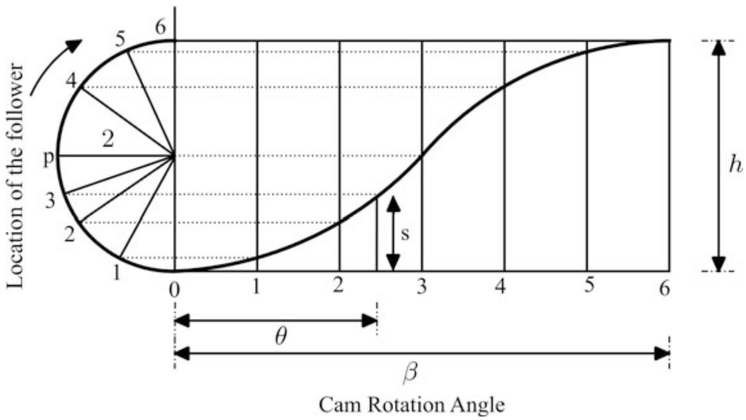


Fig. 5.15 A simple harmonic displacement diagram

Since during a π-radian rotation of the radius OP, the cam rotates by β radians, $\phi = \frac{\pi\theta}{\beta}$. Substituting this into the above equation gives:

$$v = \frac{\pi h\omega}{2\beta}\sin(\frac{\pi\theta}{\beta}) \tag{5.4}$$

A second differentiation with respect to time provides an equation for the acceleration:

$$a = \frac{\pi^2 h\omega^2}{2\beta^2}\cos(\frac{\pi\theta}{\beta}) \tag{5.5}$$

Note In simple harmonic motion, the speed diagram is a sine curve with an amplitude of $\pi h\omega/2\beta$, and the acceleration diagram is a cosine curve with an amplitude of $\pi^2 h\omega^2/2\beta^2$.

Example A follower is lifted L cm harmonically for every β -radian rotation of the cam. If the cam rotates at N rpm, what will be the maximum acceleration of the follower (in cm/s^2)?

1) $\left(2\pi^4 LN^2\right)/\beta^2$
2) $\pi^4 LN^2/\left(1800\beta^2\right)$
3) $\pi^4 L\,(\pi N/30)^2$
4) $2L\left(\frac{\pi}{\beta}\right)^2(\pi N)^2$

Solution In harmonic motion, the equations of motion for a β -radian rotation of the cam, L-cm translation of the follower, and an angular speed of N rpm will be as follows:

$$s = \frac{L}{2}\left[1 - \cos\left(\frac{\pi\theta}{\beta}\right)\right], v = \frac{\pi L\omega}{2\beta}\sin\left(\frac{\pi\theta}{\beta}\right), a = \frac{\pi^2 l\omega^2}{2\beta^2}\cos(\frac{\pi\theta}{\beta})$$

It must be noted that $\theta = \omega t$ and $\omega = \frac{2\pi N}{60}$. On the other hand, the maximum acceleration will be obtained by setting the cosine term equal to 1 in the acceleration equation. Therefore:

$$a_{\max} = \frac{\pi^2 L\omega^2}{2\beta^2} = \frac{\pi^2 L}{2\beta^2}\left(\frac{2\pi N}{60}\right)^2 = \frac{\pi^4 L N^2}{1800\beta^2}$$

Choice (2) is correct.

Figure 5.16 presents the specifications of various states in harmonic motion.

Note The advantage of simple harmonic motion over parabolic and cycloidal motions is that the maximum pressure angle of the radial roller follower is smaller in this motion for equal time intervals. This property reduces the problems involved in the design of follower structures and makes it possible to use less rigid supports. In this case, less power would be required to drive the cam.

While plotting the displacement-time diagram, if the motion is a modified parabolic one, the inflection point must be determined first. In simple harmonic and cycloidal motions, the inflection point is obtained automatically by plotting the motion curve.

Note In parabolic motion, the inflection point will lie in the middle of the displacement and time scale if the time intervals are equal.

The displacement diagram of a cycloidal motion is obtained from a cycloid that is the locus of a point on a circle that rolls on a straight line. In Fig. 5.17, the curve BDE represents a cycloidal displacement diagram, where a total displacement of h occurs for a β -degree rotation of the cam. On the right-hand side, a circle with a perimeter of h is considered that rolls on a straight line FE. A point on the perimeter of this circle traverses the curve FHE, which is known as a cycloid. If the circle rotates by ϕ, the cam will rotate by θ.

Based on Fig. 5.17, the displacement s (which is the coordinate of point P on the diagram) is equal to:

$$s = R\phi - P\sin\phi = R(\phi - \sin\phi) \tag{5.6}$$

Since the circle rotates only once for a total lift of h:

$$\phi = 2\pi\frac{\theta}{\beta} \tag{5.7}$$

and

$$R = \frac{h}{2\pi} \tag{5.8}$$

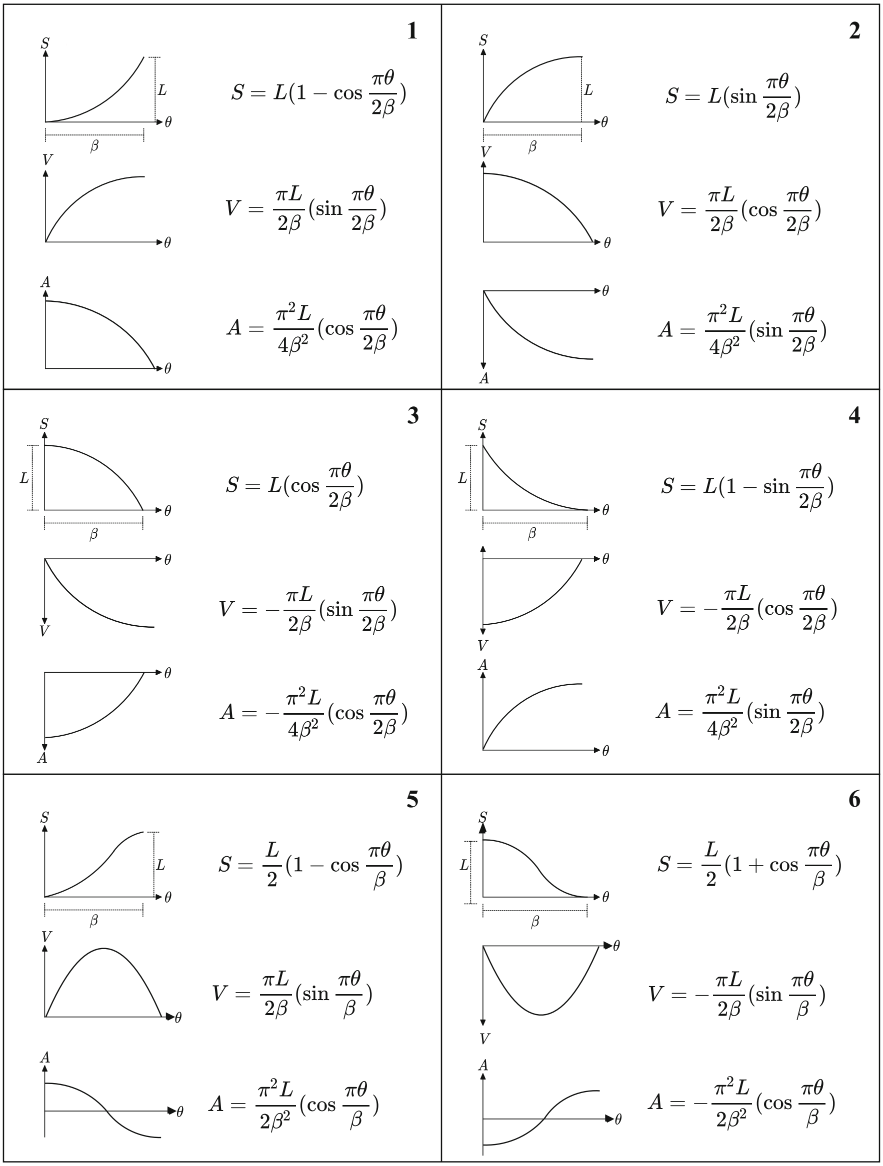


Fig. 5.16 Specifications of various states in a harmonic motion

Substituting ϕ and R from Eqs. (5.8) and (5.9) into Eq. (5.7) gives:

$$s = \frac{h}{2\pi}\left[2\pi\frac{\theta}{\beta} - \sin(2\pi\frac{\theta}{\beta})\right] = h\frac{\theta}{\beta} - \frac{h}{2\pi}\sin(2\pi\frac{\theta}{\beta}) \tag{5.9}$$

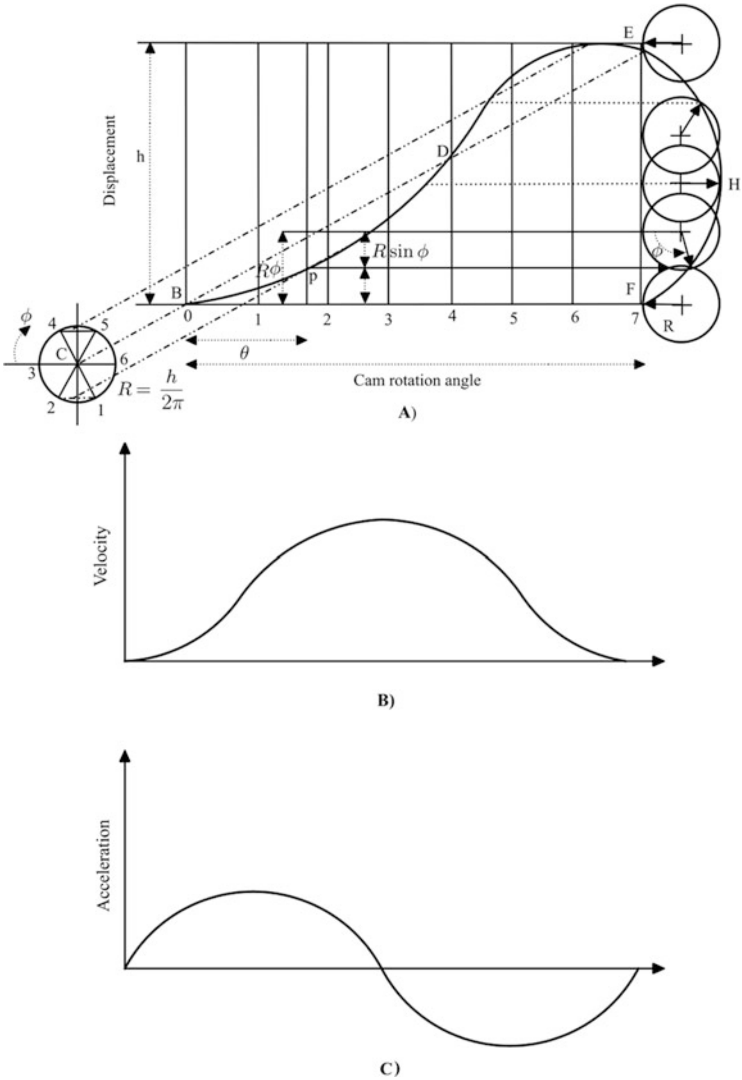


Fig. 5.17 A displacement diagram of a cycloidal motion obtained from a cycloid

Since the angular speed of the cam is considered constant, differentiating Eq. (5.9) provides the speed and acceleration equations:

$$v = \frac{h}{\beta}\omega\left[1 - \cos(\frac{2\pi\theta}{\beta})\right] \tag{5.10}$$

$$a = \frac{2\pi h}{\beta^2}\omega^2 \sin(\frac{2\pi\theta}{\beta}) \tag{5.11}$$

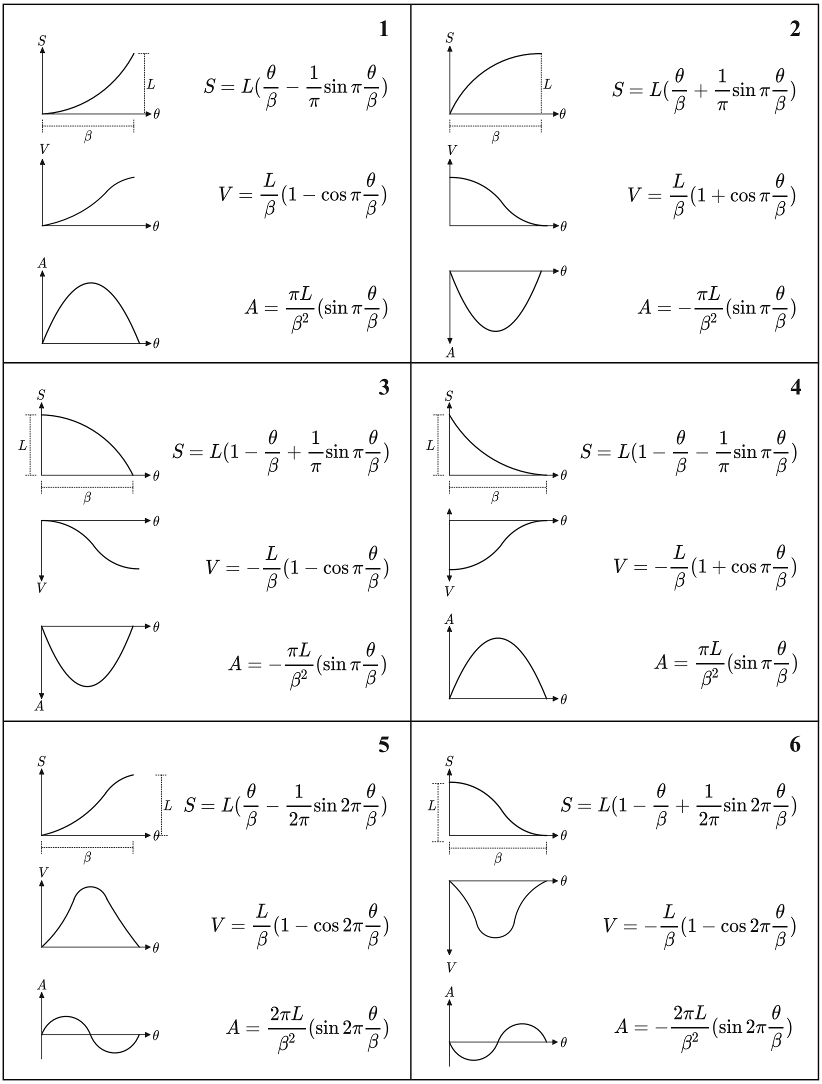
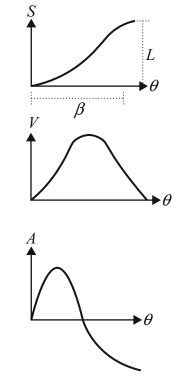


Fig. 5.18 Various states of cycloidal motions

Figure 5.18 presents the various states of cycloidal motion.

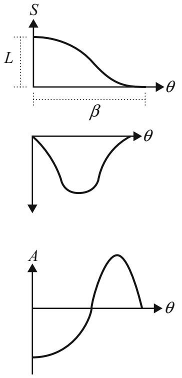
Figure 5.19 displays the characteristics of the 8th-degree polynomial motion for use as the follower motion.



$$S = L[1 - 2.63415(\frac{\theta}{\beta})^2 + 2.78055(\frac{\theta}{\beta})^5 + 3.17060(\frac{\theta}{\beta})^6 - 6.87795(\frac{\theta}{\beta})^7 + 2.56095(\frac{\theta}{\beta})^8]$$

$$V = \frac{L}{\beta}[-5{,}26830(\frac{\theta}{\beta}) - 13{,}90275(\frac{\theta}{\beta})^4 + 19{,}02360(\frac{\theta}{\beta})^5 - 48{,}14565(\frac{\theta}{\beta})^6 + 20{,}48760(\frac{\theta}{\beta})^7]$$

$$A = \frac{L}{\beta^2}[-5{,}26830(\frac{\theta}{\beta}) + 55{,}61100(\frac{\theta}{\beta})^3 + 95{,}11800(\frac{\theta}{\beta})^4 - 288{,}87390(\frac{\theta}{\beta})^5 + 143{,}41320(\frac{\theta}{\beta})^6]$$



$$S = L[6{,}09755(\frac{\theta}{\beta})^3 - 20{,}78040(\frac{\theta}{\beta})^5 + 26{,}73155(\frac{\theta}{\beta})^6 - 13{,}60965(\frac{\theta}{\beta})^7 + 2{,}56095(\frac{\theta}{\beta})^8]$$

$$V = \frac{L}{\beta}[-18{,}29265(\frac{\theta}{\beta})^2 - 103{,}90200(\frac{\theta}{\beta})^4 + 160{,}38930(\frac{\theta}{\beta})^5 - 95{,}26755(\frac{\theta}{\beta})^6 + 20{,}48760(\frac{\theta}{\beta})^7]$$

$$A = \frac{L}{\beta^2}[-36{,}58530(\frac{\theta}{\beta}) - 415{,}60800(\frac{\theta}{\beta})^3 + 801{,}94650(\frac{\theta}{\beta})^4 - 571{,}60530(\frac{\theta}{\beta})^5 + 143{,}41320(\frac{\theta}{\beta})^6]$$

Fig. 5.19 Various states of cycloidal motions with characteristics of 8th-degree polynomial motion

5.5 Dynamic Loading

For cams operating at higher speeds, the follower must be selected not only based on displacement but also based on the forces exerted on the system due to the selected motion. Given the requirement for high-speed machines in the industry, it is necessary to pay attention to the system's dynamic properties and select a cam profile that minimizes the dynamic load and prevents cam-follower separation.

Let us consider a parabolic motion to demonstrate the importance of dynamic loads. This motion is desirable in terms of inertia forces since it produces a small acceleration. However, the increase in acceleration from zero to the constant value occurs almost instantaneously, leading to a high loading rate. The rate of acceleration changes, which is obtained from the third derivative of displacement, is known as the jerk and represents the impact characteristic of the load. The impact can act as a measure of changes in the inertia force with respect to time.

Note The jerk (also known as jolt) of an impact is infinite.

Infinite jerk causes vibration in the follower assembly, thereby affecting the cam's service life. It is noteworthy that the flexibility and clearances in the assembly increase the effect of impact loading.

Note In parabolic motion, where the jerk is infinite, every cycle has two impacts. These result in sudden impacts on the assembly, which may cause undesired vibrations and damage to the mechanism.

The acceleration diagram in simple harmonic motion is continuous only when the ascent and descent times are equivalent to 180°. If these times are not equal or are accompanied by stops before and after, steps will appear in the acceleration diagram, leading to infinite jerk. Cycloidal motion causes the largest maximum acceleration among the studied motions for a given lift. However, it can be related to another cycloidal diagram without causing steps in the acceleration curve and can also be preceded or followed by a stop. Therefore, this motion is the most suitable for high speeds. In this respect, the 8th-degree exponential motion is another type of motion suitable for high speeds.

Example Which of the following displacement equations is appropriate for use in a radial cam-follower system at high speeds? S is the follower displacement, θ is the cam rotation angle, H is the maximum follower displacement, and β is the rotation angle of the cam corresponding to the maximum follower displacement.

1) $S = \frac{H}{2}(1 - \cos\frac{\pi\theta}{\beta})$ 2) $S = \frac{H}{\pi}(\frac{\pi\theta}{\beta} - \frac{1}{2} - \sin\frac{2\pi\theta}{\beta})$
3) $S = \frac{H}{2}\left[(1 - \cos\frac{\pi\theta}{\beta}) - \frac{1}{4}(1 - \cos\frac{2\pi\theta}{\beta})\right]$ 4) $S = H\frac{\theta^2}{\beta^2}(3 - \frac{2\theta}{\beta})$

Solution Choice (2) represents the cycloidal motion and is the most appropriate for high speeds, according to previous discussions.

Choice (2) is correct.

To avoid infinite jerk and its adverse effects on cam mechanisms, profiles satisfying special motion conditions are selected via the following criteria:

1. The cycloidal motion generates zero acceleration at the two endpoints of the motion (start and end points). Therefore, it can be connected to a stop (pause) from both points. No two cycloidal motions must follow each other as it increases the pressure angle and undesirably returns the acceleration to zero.
2. From among cycloidal, harmonic, and 8th-degree polynomial motions, the harmonic motion leads to the minimum peak and pressure angle. Therefore, it is preferable, wherever it is possible, to match its start and end accelerations to those of the endpoints of the previous or subsequent intervals. On the other hand, since the acceleration is zero in the middle interval of the semi-harmonic motion, this motion is used whenever a constant-speed lift occurs after an acceleration motion.

3. The acceleration graph is an asymmetric 8th-degree polynomial, and the peak acceleration and the related pressure angle are between the corresponding values in the harmonic and cycloidal graphs.

Some Examples of “Cams”

1. If the angular speed ω of the cam in Fig. 5.20 is considered constant, the acceleration of the follower will be equal to …

   1) $\frac{\sqrt{2}}{2}\overline{OC}\omega^2$ 2) $\overline{CP}.\omega^2$ 3) Zero   4) $\overline{OP}.\omega^2$

2. In the displayed cam and follower mechanism in Fig. 5.21, the cam rotates at an angular speed of ω. What will be the speed of the follower?

   1) $\frac{r\omega}{2}$ 2) $r\omega\sqrt{\frac{2}{2}}$ 3) $r\omega$ 4) $2r\omega$

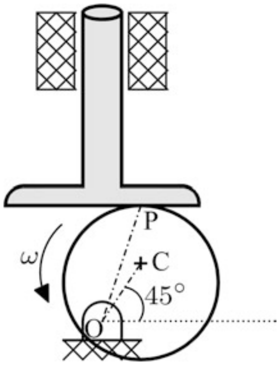


Fig. 5.20 Acceleration of a follower with a cam with constant angular speed

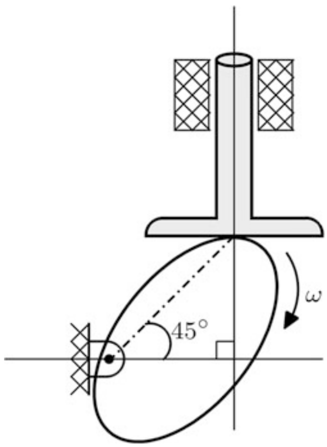


Fig. 5.21 Cam-follower mechanism with a constant speed cam

Fig. 5.22 Determine the lift speed of the follower

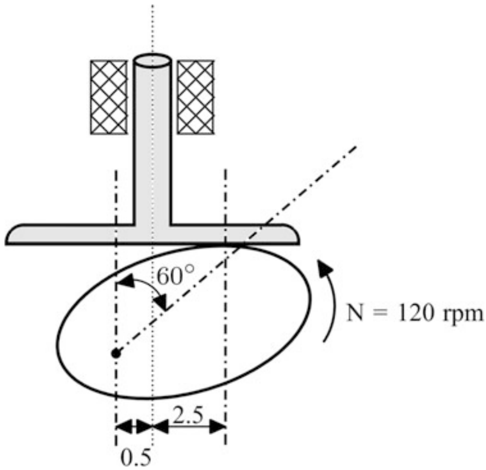
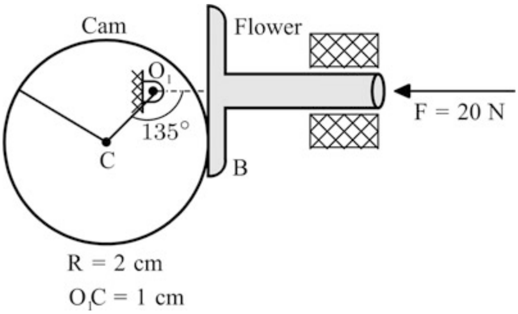


Fig. 5.23 Determine the exerted torque on the cam



3. Based on Fig. 5.22, how much is the lift speed of the follower in cm/s at the instant shown (N = 120 rpm, and the lengths are in cm)?

   1) 31.416 2) 37.25 3) 37.7 4) 43.53

4. The torque T_1 to exert on the camshaft in Fig. 5.23 for the static equilibrium of the cam and follower mechanism is equal to:

   1) 14.1 N.cm (CCW) 2) 14.1 N.cm (CW)

   3) 20 N.cm (CCW) 4) 20 N.cm (CW)

5. In the system shown in Fig. 5.24, the cam has a radius R, the distance O_1O_2 is equal to $\frac{2R}{3}$, and the angular displacement, speed, and acceleration of the cam are θ, ω, and A, respectively. So, what will be the follower travel?

   1) $\frac{R}{2}(\omega^2 \cos\theta + \alpha \sin\theta)$ 2) $\frac{R}{3}(\omega^2 \cos\theta + \alpha \sin\theta)$

   3) $3\frac{R}{2}(\omega^2 \sin\theta + \alpha \cos\theta)$ 4) $2\frac{R}{3}(\omega^2 \cos\theta + \alpha \sin\theta)$

Fig. 5.24 Determine the follower travel

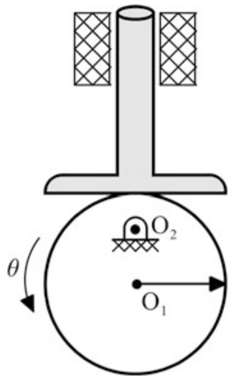
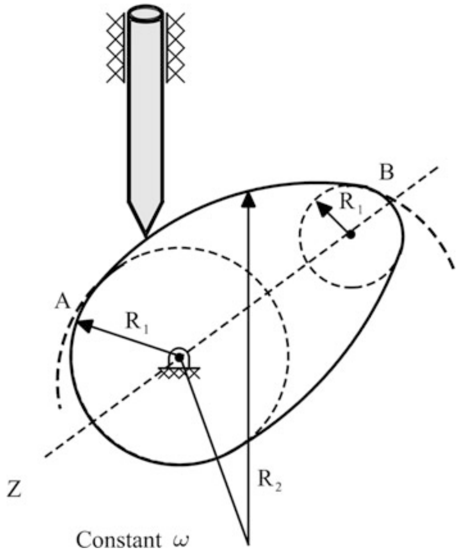


Fig. 5.25 A symmetric cam



6. The plate cam in Fig. 5.25 is symmetric with respect to the line δ. This cam consists of four tangent circular arcs. Which of the following is true about points A and B?

   1) The follower has a continuous and differentiable speed and an infinite jerk.
   2) The speed is continuous, but the acceleration is discontinuous.
   3) The speed and acceleration are both continuous and differentiable.
   4) The speed and acceleration of the follower are both continuous, but its jerk is discontinuous.

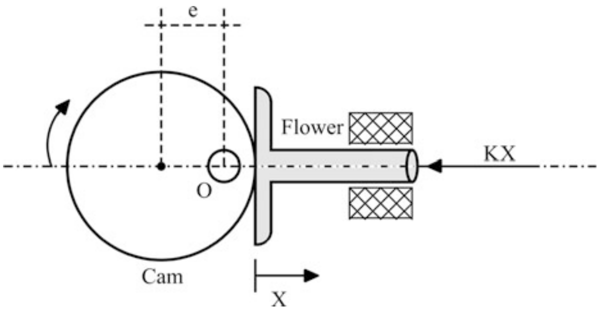


Fig. 5.26 Find the resulting energy

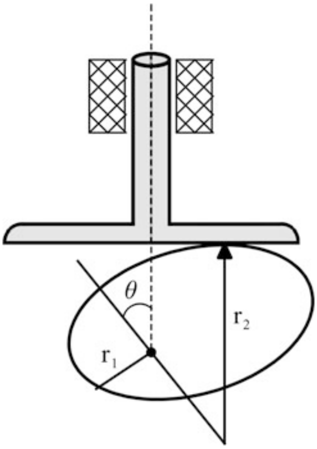


Fig. 5.27 Find the acceleration of the follower

7. When the circular cam rotates 180° from the position shown in Fig. 5.26, which of the following will be equal to the resulting energy? The resistive force is proportional to the displacement, and $x = e(1 - cos\theta)$.

   1) ke^2 2) $\frac{1}{2}ke$ 3) $2ke^2$ 4) $2k$

8. In the cam and follower system in Fig. 5.27, the cam rotates at the constant angular speed ω. At the position where the side circle of radius r_2 is in contact with the follower, the acceleration of the follower will be equal to …

   1) $r_2\omega^2 \sin\theta$ 2) $(r_2 - r_1)\omega^2 \sin\theta$

   3) $r_2\omega^2 \cos\theta$ 4) $(r_2 - r_1)\omega^2 \cos\theta$

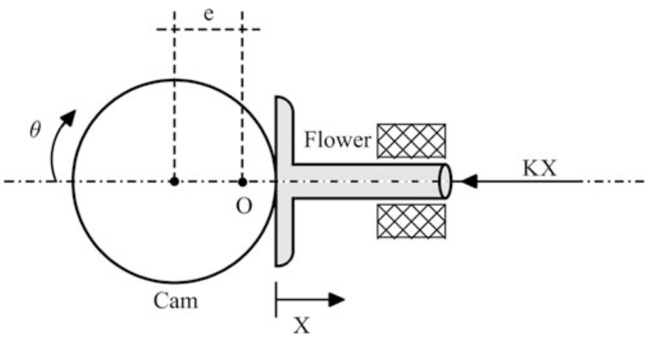


Fig. 5.28 Determine the resulting energy

9. When the circular cam rotates 180° from the position shown in Fig. 5.28, which of the following will be equal to the resulting energy? The resistive force is proportional to the displacement, and $x = e(1 - cos\theta)$.

   1) ke^2 2) $\frac{1}{2}ke$ 3) $2ke^2$ 4) $2ke$

Answers for the Examples of “Cams”

1. Choice (1) is correct.
   The displacement of the follower and, hence, its speed and acceleration are obtained as follows:

$$y = \overline{OC}\sin\theta + PC$$

$$\Rightarrow \dot{y} = \overline{OC}\dot{\theta}\cos\theta$$

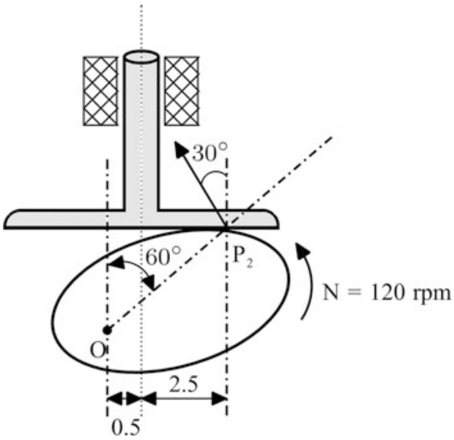
$$\Rightarrow \ddot{y} = -\overline{OC}\dot{\theta}\sin\theta$$

   Substituting $\theta = 45^\circ$ and $\dot{\theta} = \omega$, the acceleration will be equal to: $\frac{\sqrt{2}}{2}\overline{OC}\omega^2$.
2. Choice (2) is correct.
   If ω is the angular speed of the cam, the speed of the follower will be:

$$\left.\begin{array}{l} \frac{dy}{dt} = \frac{dy}{d\theta} * \frac{d\theta}{dt} = \frac{dy}{d\theta} * \omega \\ y = r\cos\theta \Rightarrow \frac{dy}{d\theta} = r\sin\theta \end{array}\right\} \text{follower speed} = \frac{dy}{dt} = r\omega\sin\theta$$

3. Choice (3) is correct.

Fig. 5.29 Speed of the point of contact



Assume P_1 and P_2 in Fig. 5.29 to be the points belonging to the cam and the follower, respectively, coinciding at the point of contact:

$$V_{p1} = |OP_1|\,\omega_1\omega_1 = \frac{120 * 2\pi}{60} = 12.57\left(\frac{rad}{s}\right)|OP_1| = \frac{3}{\sin 60°} = 3.46cm$$

$$V_{p1} = 3.46 * 12.57 = 43.49(\frac{cm}{s})$$

The projections of the speeds of points P_1 and P_2 along the common normal must be equal and in the same direction. Besides, since the direction of V_{p2} is along the common normal:

$$V_{p2} = V_{p1}\cos 30 = 43.49\cos 30° = 37.7(\frac{cm}{s})$$

Shortcut: The solution is the product of the horizontal distance between the point of contact and point O (3 cm) and ω_1.

$$V_{p1} = 3 * 12.57 = 37.7(\frac{cm}{s})$$

4. Choice (1) is correct.
Since the follower is in equilibrium, a force of 20 N is exerted on the cam at point B.

$$T = F.\,|O_1C|\cos 45 = 20 * 1 * \frac{\sqrt{2}}{2} = 14.1 \text{ N.cm (CW)}$$

To neutralize this torque, a torque of equal magnitude but in the opposite direction must be applied to the camshaft (Fig. 5.30).

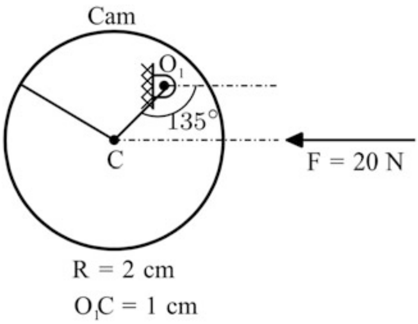


Fig. 5.30 Force exerted on the cam

5. Choice (4) is correct.
   According to the equation mentioned for circular cams:

$$y = e(1 - \cos\theta)$$

   e is the offset, which is equal to $\frac{2R}{3}$ in this problem.

$$y = \frac{2R}{3}(1 - \cos\theta)$$

$$\dot{y} = \frac{2R}{3}\dot{\theta}\sin\theta$$

$$\ddot{y} = \frac{2R}{3}\dot{\theta}\sin\theta + \frac{2R}{3}\dot{\theta}^2\cos\theta = \frac{2R}{3}(\omega^2\cos\theta + \alpha\sin\theta)$$

6. Choice (4) is correct.
   The motion type of the follower is determined by the cam profile. In this profile, the follower lift (characteristic curve) is a continuous curve whose derivatives (speed and acceleration) are also continuous. However, given the change in the radius of curvature of the cam at points A and B, the acceleration derivatives (jerk) can be discontinuous.
7. Choice (3) is correct.
   According to the equation mentioned for circular cams:

$$y = e(1 - \cos\theta)$$

$$y_\theta = 180 = e\,(1 - (-1)) = 2e$$

$$\text{Energy} = E = \frac{1}{2}kx^2 = \frac{1}{2}k(2e)^2 = 2ke^2$$

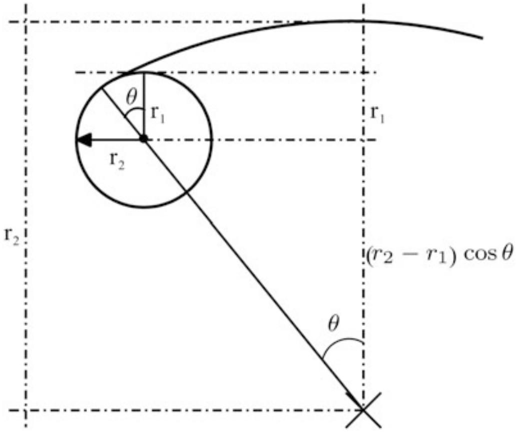


Fig. 5.31 Cam analysis

8. Choice (4) is correct (Fig. 5.31).

$$y = r_2 - r_1 - (r_2 - r_1)\cos\theta$$

$$\dot{y} = (r_2 - r_1)\,\dot{\theta}\,\sin\theta$$

$$\ddot{y} = (r_2 - r_1)\,\dot{\theta}^2\cos\theta = (r_2 - r_1)\,\omega^2\cos\theta$$

9. Choice (4) is correct.
   The displacement due to a 180° rotation is equal to:

$$x = e(1 - \cos\theta) = e\,(1 - cs\,(180)) = 2e$$

$$U = \int F dx = \int (kx)\,dx = \frac{1}{2}kx^2$$

$$U|_{x=2e} = \frac{1}{2}k\,(2e)^2 = 2ke^2$$

Chapter 6
Gear Trains

This chapter provides a comprehensive exploration of gears and their applications in mechanical systems. The chapter begins with an introduction, offering an overview of gear trains and their significance in transmitting motion and power. Then, the vocabulary associated with gears, terminology, and definitions are presented that are essential for understanding gear systems.

Different types of gears are discussed in detail, and their unique characteristics, advantages, and limitations are presented. The chapter further delves into gear trains, which are arrangements of multiple gears that work together to transmit motion and power. Simple and hybrid gear trains are explained, showcasing various configurations and their functionalities.

Overall, this chapter provides a comprehensive overview of gears and gear trains, covering their vocabulary, types, rotation directions, and configurations. It serves as a valuable resource for understanding the principles and applications of gear systems in engineering.

6.1 Introduction

When the load exerted on the mechanism exceeds a specific value, cam systems can no longer meet our needs. Therefore, teeth are devised on the contact surface to create a positive drive. The resulting members are known as gears. Gears are used to transfer rotational motion from one shaft to another or transfer rotational motion from a shaft to a translating member that is assumed to rotate around an axis located at infinity.

Different types of gears, their components, and stress analysis in their teeth are discussed in detail in mechanical design courses. Sometimes, it is necessary to use a combination of several gears, which is called a gear set. In this case, one must find

A. Abbasimoshaei, T. A. Kern, *Machine Dynamics*,
https://doi.org/10.1007/978-981-99-6010-1_6

the angular speed and rotation direction of the output gear given the input angular speed. This topic is discussed in the present chapter.

6.2 Gear Vocabulary

The general geometric principles of all gears are the same. Therefore, we will only state the expressions used for spur gears. These gears are of the simplest form and transfer motion between two parallel shafts. Different parts of this gear are named in Fig. 6.1.

The pitch circle is a hypothetical circle used as the basis for all the calculations. The pitch circles of a pair of engaged gears are tangent. The smaller gear in the pair is named the pinion, while the larger is named the gear.

The circular pitch is the distance measured from a point one tooth to the corresponding point on the next tooth on the pitch circle. Hence, the circular pitch is equal to the sum of the tooth thickness and the distance between two teeth.

The module m is the pitch diameter ratio (in mm) to the number of teeth. The module is the tooth measure index in SI.

Note A pair of gears must have equal modules to engage well.

The diametral pitch is the inverse of the module and is equal to the ratio of the number of teeth to pitch circle diameter (in inches).

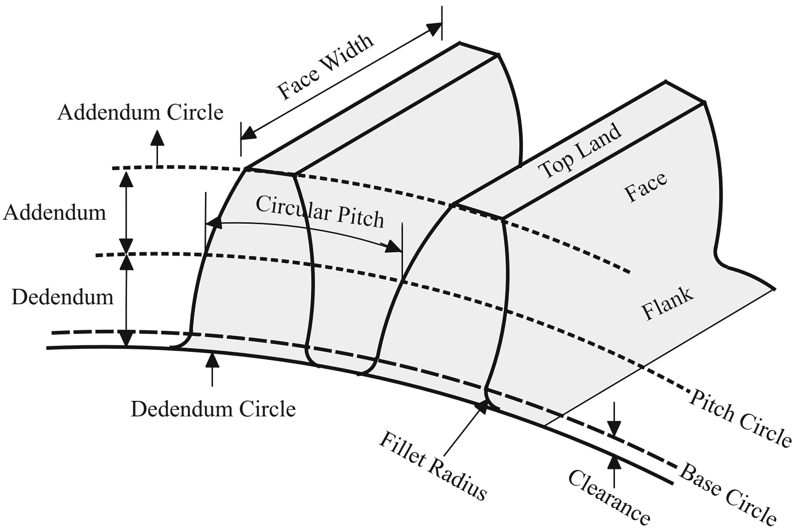


Fig. 6.1 Different parts of a gear

Since the diametral pitch is expressed only in inches, it is also known as the number of teeth per inch. The addendum is the radial distance between the top of the tooth and the pitch circle. In addition, the dedendum is the radial between the bottom of the tooth and the pitch circle. The clearance circle is a circle tangent to the addendum circle of the engaged gear, and the top clearance is the difference between the dedendum of a gear and the addendum of its mating gear.

Note The introduced parameters depend on the characteristics of each gear, not their engagement manner.

When the tooth profiles or cams are designed to maintain a constant angular speed during the engagement, they are said to be conjugate. One such profile is known as the involute profile, which is used for all gears, with a few exceptions. An involute is a curve obtained from the points on a taut thread that is unwound from around a base circle (Fig. 6.2A).

Note Two curves can conjugate: one is the epicyclic curve, which is difficult to construct, and the other is the involute curve.

Note The involute profile minimizes slipping.

An important advantage of the involute shape over others is that it provides easy conjugation when the center distances of the shafts are not accurate.

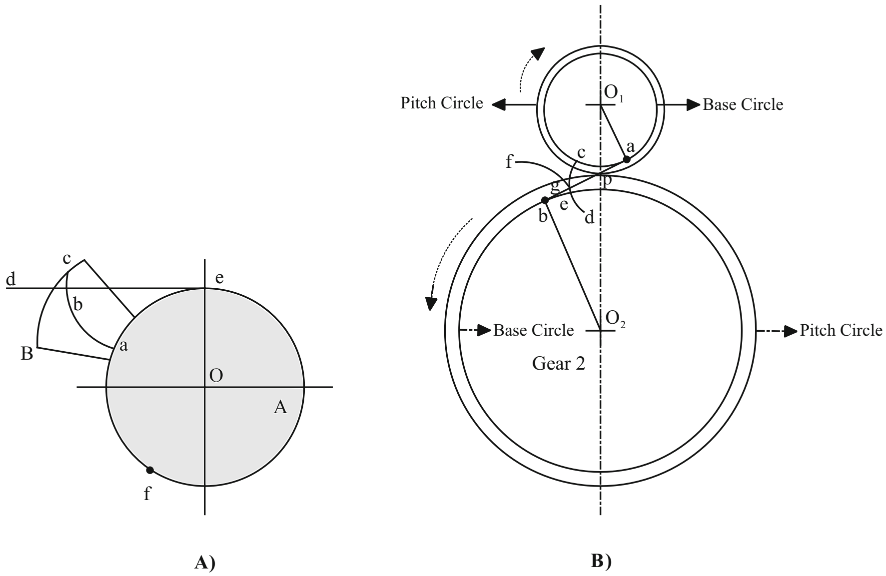


Fig. 6.2 Curves capable of conjugation

6.3 Types of Gears

Based on shaft connection and application, the different types of gears are as follows:

1. Face gears (including spur gears, helical gears, and herringbone gears) to transfer motion between parallel shafts (Fig. 6.3A, B, and C).
2. Bevel gears in different types, such as straight and zerol, are used on orthogonal and skew shafts. An example is shown in Fig. 6.3D.
3. Worm gears and worm shafts are used to transfer motion between orthogonal shafts (Fig. 6.3E).
4. Helical gears have teeth that are angled with respect to the axis of rotation and are used in the same applications as spur gears but produce less noise. When transferring motion between two parallel shafts, these gears are considered a type of face gears. They can also be used to transfer motion between two skew shafts (Fig. 6.3B).

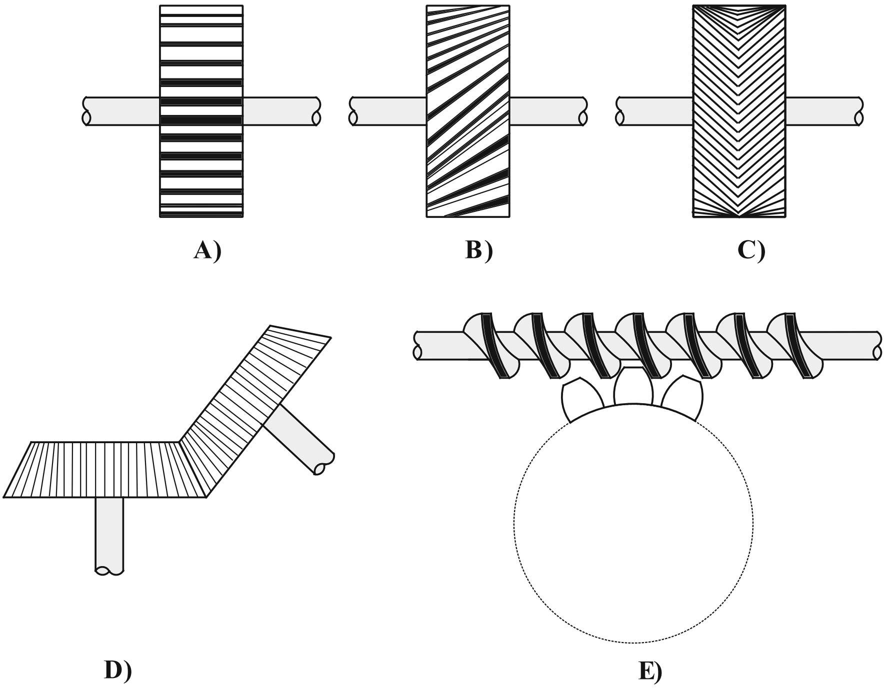


Fig. 6.3 Different types of gears

6.4 Rotation Direction of Engaged Gears

In gear set problems, the aim is usually to find the speed and rotation direction of the output gear from the speed and rotation direction of the input gear. The ratio of the input angular speed to the output angular speed is known as the angular velocity ratio and is expressed by $\frac{\omega_{in}}{\omega_{out}}$.

Figure 6.4 shows pinions driving an external spur gear and an internal spur gear. The angular speed is proportional to the number of teeth in both cases.

The external gear rotates opposite the direction of the pinion rotation, while the internal gear rotates in the same direction as the pinion. Positive and negative signs with respect to speed express the same and opposite directions of rotations. A simple method to identify the rotation direction of bevel gears is to detect whether rolling occurs at the pitch point (the end of the line tangent to the pitch surfaces) and whether the speeds of the two gears are identical at the pitch point. Accordingly, the rotation direction can be obtained from the simple speed equation:

$$\overrightarrow{V}_{p2} = \overrightarrow{V}_{p3} = \overrightarrow{\omega}_2 * \overrightarrow{r}_2 = \overrightarrow{\omega}_3 * \overrightarrow{r}_3 \tag{6.1}$$

where the directions of the pitch radius vectors (r_2 and r_3) for gears 2 and 3 are from the axis of rotation toward the pitch point.

Note For bevel gears, the pitch radius is measured from the larger side of the gear.

Sometimes, it is necessary to reverse the gear's rotation direction without a change in its angular speed. To this end, an idler gear is placed between the driving and driven gears. In this case, the direction of motion changes without changing the angular velocity ratio.

When two spur gears are engaged, their pitch circles roll over each other without slipping. If the pitch radii are represented by r_1 and r_2 and the angular speeds by ω_2 , ω_1:

$$V = |r_2\omega_2| = |r_3\omega_3| \tag{6.2}$$

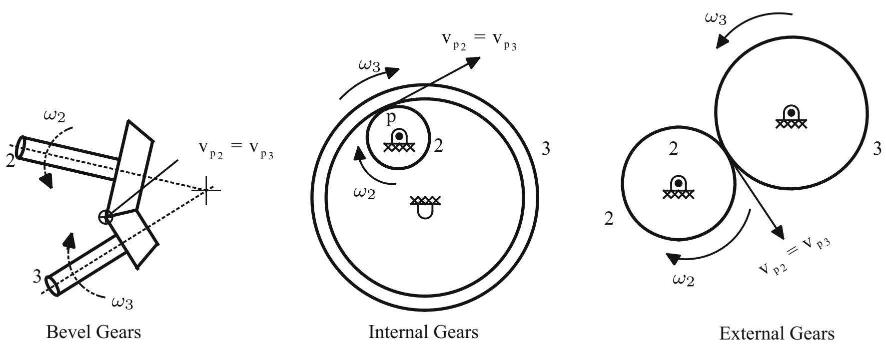


Fig. 6.4 Different engagements of gears

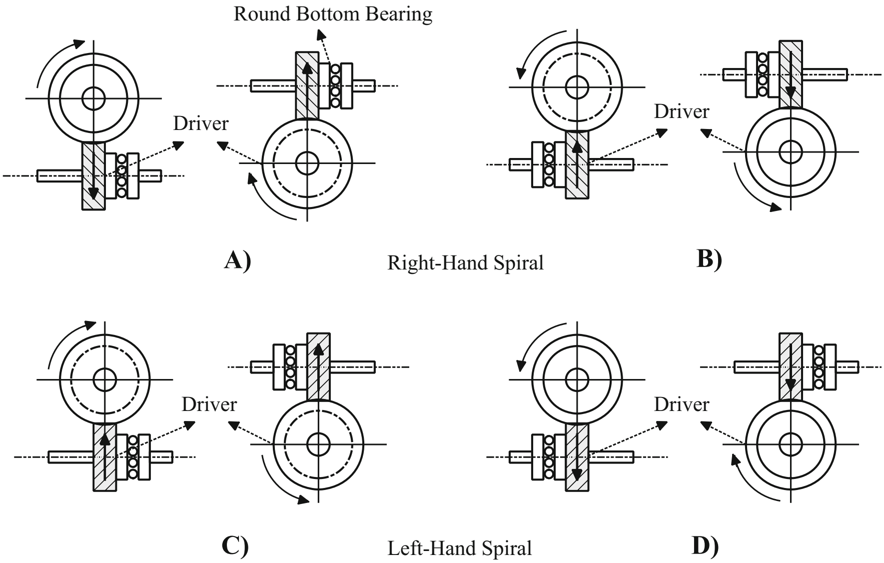


Fig. 6.5 Different arrangements for cross-helical gears with orthogonal shafts

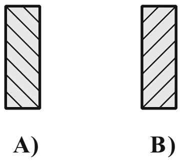


Fig. 6.6 Right- or left-hand helical

where V is the pitch line velocity, the following equation holds between the radius and angular speeds:

$$\left|\frac{\omega_2}{\omega_3}\right| = \frac{r_3}{r_2} \tag{6.3}$$

When dealing with helical gears, the process is slightly more complicated since the direction of the helical teeth (left or right hand) affects the gear's rotation direction. Figure 6.5 shows different arrangements for cross-helical gears with orthogonal shafts.

Note To determine the right- or left-hand helical gear, one needs only to view it from the side. If it looks like Fig. 6.6A, it is right hand, and if it looks like Fig. 6.6B, it is left hand.

Note The tooth direction is the same in two engaged cross-helical gears.

Worm gear systems consist of the worm and the worm gear. The worm acts like a screw and can be single-threaded, double-threaded, or triple thread. On the other hand, the worm gear sets are used in non-intersecting and perpendicular shafts. The helix and teeth directions of a worm and worm gear pair are identical, similar to those of cross-helical gears. However, the two helix angles are different.

Note The worm usually acts as the driver in the worm gear set.

One can considerably reduce the speed and increase the torque using worm and worm gear systems.

Note In a worm and worm gear pair, the speed ratio does not depend on the gear diameters but on the number of threads of the worm and the number of teeth of the worm gear.

Note We perform the same procedure as for the helical gear to determine a worm's right- or left-handedness.

We need only to view it from the side: If it looks like Fig. 6.7A, it is right hand; if it looks like Fig. 6.7B, it is left-handed.

In worm gears, the rotation direction depends on the directions of the threads and the rotation direction of the worm. Figure 6.8 shows the worm gear's rotation direction under different conditions.

There is no need to plot the worm gear teeth in these figures. We should simply note that a worm gear pair (worm and worm gear) is either both left hand or right hand. In other words, a right-hand worm rotates with a right-hand worm gear, and a left-hand worm rotates with a left-hand worm gear.

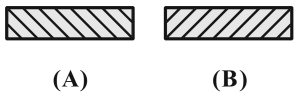


Fig. 6.7 Right- or left-handedness of a worm

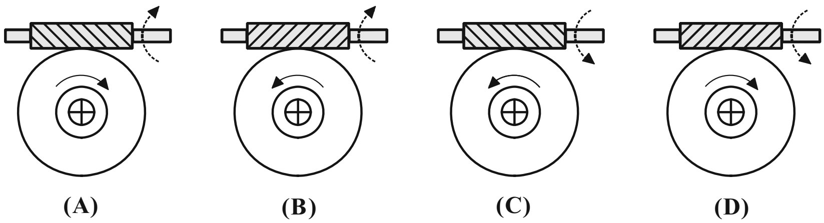


Fig. 6.8 Direction of rotation of a worm gear under different conditions

6.5 Gear Trains

For designers, gears act as a motion transfer or force conversion pair. Gear pairs can be combined in different ways to reach the desired input–output relationship. As discussed in the following, all complex gear trains consist of simple, compound, and epicyclic gear sets.

6.5.1 Simple and Hybrid Gear Trains

Simple gear trains can be divided into two groups depending on the presence of idler gears. A simple gear train has only one gear on every shaft. These shafts rotate on bearings connected to a single frame. The gears may be of any type, such as spur, bevel, hypoid, or worm. Figure 6.9 shows different types of simple gear trains.

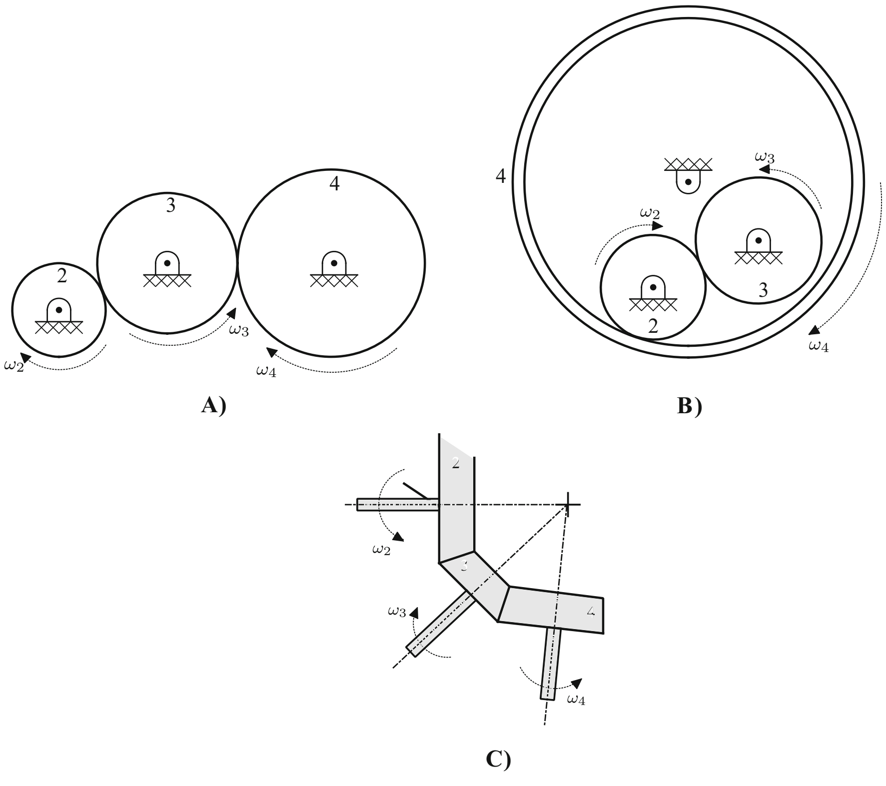


Fig. 6.9 Simple and hybrid gear trains

Idler gears in simple gear trains can have two objectives. One is to change the rotation direction of the output gear, and the other is to fill gaps when two gears cannot be directly engaged due to the locations of their shafts. The second case occurs when there is a size limit, but the shaft locations are constrained for non-kinematic reasons.

Note Increasing the number of idler gears reduces the system efficiency.

Although the simple gear train can contain any number and type of gear, each gear in the system must be able to engage with another gear.

Note If the gears in a simple gear train are engaged correctly, all the gears must have a normal pitch and identical modules.

The speed ratio is constrained for all gear types based on practical considerations for each engagement. For instance, the speed ratio in each engagement must not exceed 5:1 for a simple gear train. For further speed reduction, compound gear trains must be used. A compound gear train is distinguished by the presence of more than one gear on a shaft.

Note Unlike simple gear trains, the gears in compound gear trains are generally of different types.

Example Which of the following is false about gears?

1) In a simple gear train, the gears can rotate about independent shafts, and the system can transfer motion from one gear to another.
2) The intermediate gears whose size does not affect the speed ratio of the gear trains are called idler gears.
3) Idler gears are used when we tend to change the rotation direction of gears or when there is a large distance between the input and output gears.
4) In compound gear trains, one or more gears rotate about different shafts but with equal angular speeds.

Solution Choice (4) is correct.

Let us consider the following gear train. Gear A drives gear B, B drives C, C drives D, and D drives E. We assume that the number of teeth on the gears is N_E, N_D, N_C, N_B, N_A, respectively. Therefore,

$$\frac{\omega_A}{\omega_B} = -\frac{N_B}{N_A}, \quad \frac{\omega_B}{\omega_C} = -\frac{N_C}{N_B}, \quad \frac{\omega_C}{\omega_D} = -\frac{N_D}{N_C}, \quad \frac{\omega_D}{\omega_E} = -\frac{N_E}{N_D} \tag{6.4}$$

The negative sign in the above equations shows that the rotation directions of two engaged gears are opposite (Fig. 6.10).

Note The angular velocity ratio of every mating gear pair is the inverse of the ratio of their number of teeth.

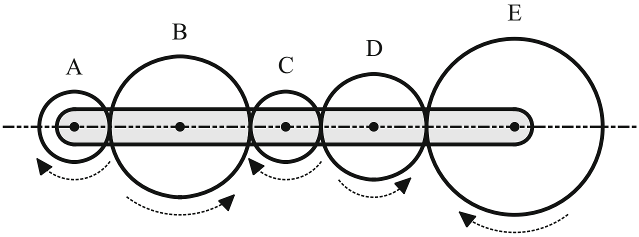


Fig. 6.10 A mating gear pair

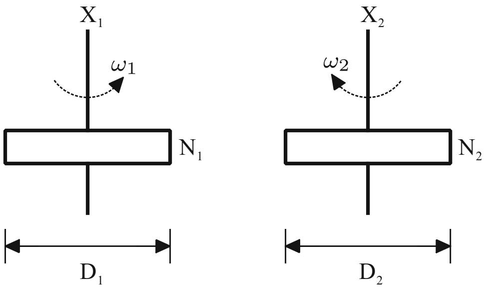


Fig. 6.11 Two spur gears

Example Assume two gears with an angular velocity ratio of 1:1.4 and with shafts at a distance of 6 cm. If the diametral pitch of these two gears is 3, determine the number of teeth of each gear:

1) 21 and 15 2) 18 and 21 3) 24 and 18 4) 15 and 9

Solution In Fig. 6.11, we consider the two spur gears 1 and 2, which rotate at speeds of x_2 , x_1 about fixed parallel shafts $\omega_2$andω_1:

$$\frac{\omega_2}{\omega_1} = -\frac{N_1}{N_2} = -\frac{D_1}{D_2}$$

where D_i , N_i are the number of teeth and diameter of gear number i, respectively. Therefore, since $\frac{\omega_2}{\omega_1} = 1.4$, the ratio $\frac{N_1}{N_2}$ must also be 1.4, which is reflected by choice (1).

Choice (1) is correct.

The velocity ratio (VR) of a gear set is equal to the ratio of the angular velocity of the first gear to that of the last gear in the set. For the system shown,

$$VR = \frac{\omega_A}{\omega_E} = \frac{\omega_A}{\omega_B} * \frac{\omega_B}{\omega_C} * \frac{\omega_C}{\omega_D} * \frac{\omega_D}{\omega_E} \tag{6.5}$$

substituting Eq. 6.4 into this relationship gives

$$VR = \frac{\omega_A}{\omega_E} = \frac{N_B}{N_A} * \frac{N_C}{N_B} * \frac{N_D}{N_C} * \frac{N_E}{N_D} = \frac{N_E}{N_A} \tag{6.6}$$

It can be seen that the VR between the input and output shafts is a function of only the number of teeth of the input and output gears and is independent of the size and number of idler gears.

Note Idler gears merely affect the sign of VR, not its value. In other words, they influence only the direction of the output motion.

In general, one can write for both simple and compound gear trains:

$$\frac{\omega_{in}}{\omega_{out}} = \frac{\omega_{driving}}{\omega_{driven}} = \frac{\text{product of the numbers of teeth of driving gears}}{\text{product of the number of teeth of driven gears}} * (-1)^n \tag{6.7}$$

where n is the number of driven (or driving) gears in the system, provided that all the gears are external.

Note The contact between an internal and an external gear does not change the rotation direction.

Note An idler gear is driven by the gear it receives motion from and drives the gear it transfers motion to. Therefore, it is counted twice among the gears, once as a driving and once as a driven gear.

Note In simple and compound gear trains, the numbers of driving and driven gears are equal.

Example Which of the following is true for a compound gear train, such as a simple gearbox?

1) The sum of the input angular speed and the driving gear's angular speed is equal to the sum of the output angular speed and the driven gear's angular speed.
2) The product of the input angular speed and the driven gear's angular speed is equal to the product of the output angular speed and the driving gear's angular speed.
3) The ratio of the input angular speed to the output angular speed equals the ratio of the driven gear's angular speed to the driving gear's angular speed.
4) The input angular speed is equal to the sum of the angular speeds of the driving and driven gears.

Solution According to Eq. 6.7, the product of the input angular speed and the driven gear's angular speed is equal to the product of the output angular speed and the driving gear's angular speed.

Choice (2) is correct.

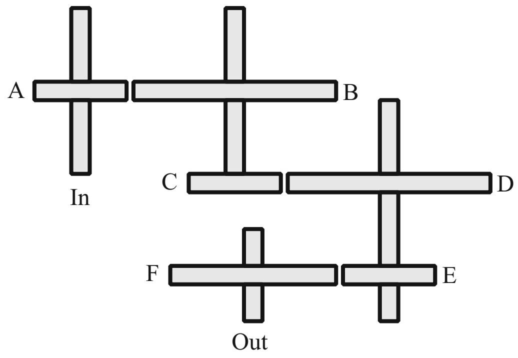


Fig. 6.12 A gear train

Example The output-to-input angular speed ratio in a gear train with the following number of gears is ...

$$N_F = N_A = 30, \quad N_B = 60, \quad N_C = 20, \quad N_D = 40, \quad N_E = 20$$

1) 1/3 2) 1/5

3) 1/6 4) None of the answers

Solution In the system shown, more than one gear is installed on some of the shafts; hence, the gears form a compound gear train (Fig. 6.12). Therefore,

$$\frac{\omega_{in}}{\omega_{out}} = \frac{N_A N_C N_E}{N_B N_D N_F} * (-1)^3 = -\frac{30 * 20 * 20}{60 * 40 * 30} = -\frac{1}{6}$$

In this question, the magnitude of the speed is enough, and there is no need for a negative sign.

Therefore, **choice (3) is correct.**

6.5.2 Epicyclic Gear Train

Both simple and compound gear trains suffer from the constraint that the shafts must rotate within bearings fixed in a certain frame. This constraint prevents a change in the gear train. If one or more shafts can rotate around another shaft in addition to their own axes, the resulting system is called an epicyclic or planetary gear train. These gear trains are widely used in compact gear reduction systems. In these systems, the gear motion causes the driven shaft (or the arm) to rotate along with the rotation of the gears relative to the arm.

Figure 6.13 shows two epicyclic gear trains. Gear 1 is often called the sun gear, and gear 2 is called the planet gear. As can be seen, Arm 3 turns gear 2 around gear

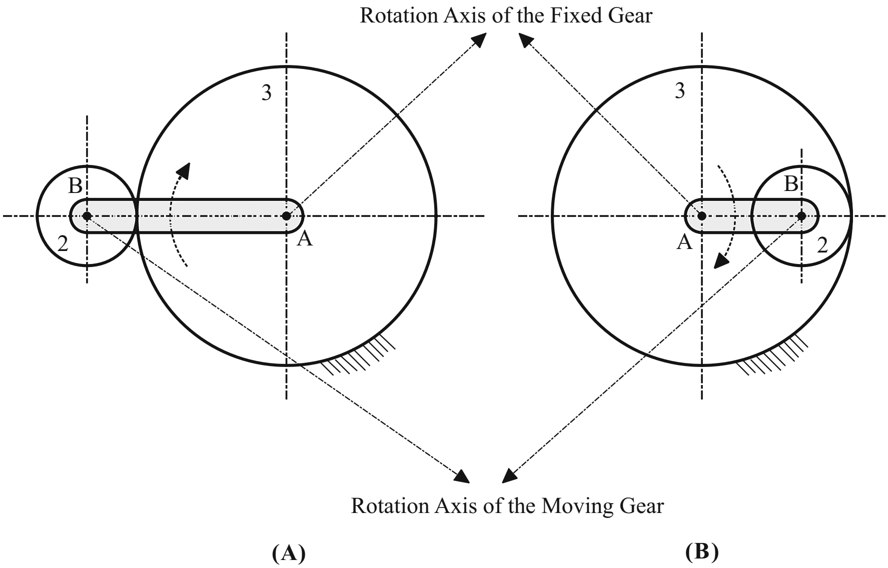


Fig. 6.13 Two epicyclic gear trains

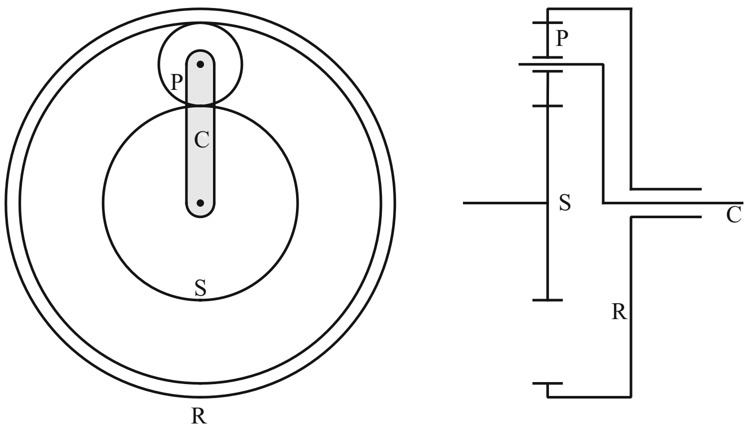


Fig. 6.14 Planetary gears represented graphically

1. In addition, gear 2 rotates about its center, i.e., B, while the center of B turns about the center of A.

Sometimes, planetary gears are graphically represented as in Fig. 6.14. This schematic gives the designer a simple representation of the planetary gear structure.

Two methods can be used to analyze planetary gear trains.

A. Algebraic or tabular method

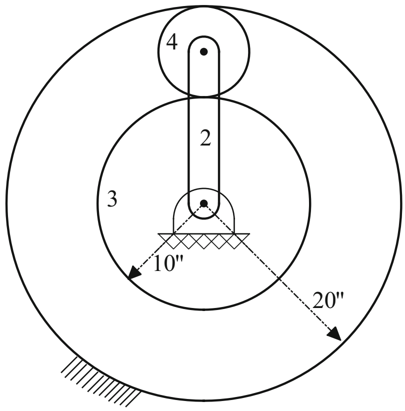


Fig. 6.15 A gear train rotating clockwise

This method begins with a table in which each column corresponds to a member in the gear train, and each row corresponds to each of the following steps:

1) The whole set is locked to the arm, and the arm is rotated one turn clockwise.
2) The arm is held fixed, and the gear that is supposed to be fixed (usually the sun gear) is rotated one turn counterclockwise to return to its initial orientation.
3) The results of Steps 1 and 2 for each member are added to the table.

Note The ratio of the number of teeth of two mating gears is equal to the ratio of their diameters.

Example The gear train shown in Fig. 6.15 rotates clockwise at an angular speed of $\boldsymbol{\omega_2} = \mathbf{10\ rad/s}$ via arm 2. Determine the angular speed of the output shaft connected to gear 3 (solve with tabular method):

1) $\omega_3 = 30$ rad/s 2) $\omega_3 = 20$ rad/s 3) $\omega_3 = 10$ rad/s 4) $\omega_3 = 5$ rad/s

Solution Since mating gears must have the same module, the number of teeth of each gear is proportional to its diameter. Hence, based on the figure, the diameter of gear 3 is half that of the fixed inner gear, and the diameter of gear 4 is half that of gear 3. Therefore, the number of teeth of gear 4 is half that of gear 3, i.e., 5.

The internal large gear is the sun gear, and gear 4 is the planet gear, which rotates around gear 3 via arm 2. Using this information, we draw Table 6.1 and implement the corresponding steps.

Table 6.1 Steps in the tabular method

Members	Arm	Sun	3	4
The set is locked to the arm and revolves once clockwise	+1	+1	+1	+1
The arm is fixed, and the sun gear revolves once counterclockwise	0	−1	+2	−4
Resultant	+1	0	+3	−3

Fig. 6.16 Find shaft A speed

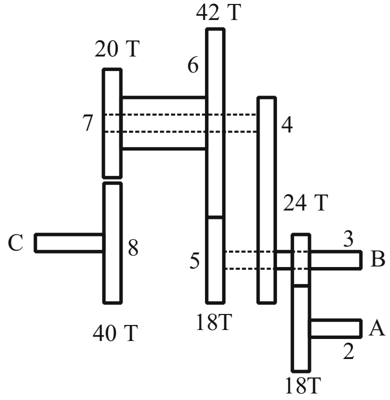


Table 6.2 Table for finding shaft A speed

Members	Arm	5	7 and 6	8	2
The set is locked to the arm and revolves once clockwise.	+1	+1	+1	+1	$-\frac{24}{18}$
The arm is fixed, and the sun gear revolves once counterclockwise.	0	−1	$\frac{24}{18}$	−4	0
Resultant	+1	0	+3	−3	0

It can be seen that the ratio of the rotation of gear 3 to that of the arm is 3. Thus, if $\omega_3 = 10$ rad/s, the angular speed of gear 3 will be $\omega_3 = 30$ rad/s.

Example In Fig. 6.16, shaft B is fixed, and shaft C rotates clockwise at 380 rpm. Which of the following represents the angular speed of shaft A, regardless of direction?

1) 645 rpm 2) 273 rpm 3) 932 rpm 4) 745 rpm

Solution It can be seen that gear 3 forms a single part with the arm and is in contact with gear 2. Moreover, the arm passes through the shafts of gears 6 and 7. Gear 5 acts as a sun gear and is attached to shaft B. Let us now plot the table and implement the steps (Table 6.2).

It is worth noting that gear 2 is independent of the planetary system and can revolve only with the rotation of gear 3.

Gear 2 is attached to shaft A, and gear 8 is attached to shaft C. With the velocity ratio known from the table, an angular speed of 380 rpm for shaft C, and using a simple proportionality, we have

$$\left.\begin{array}{l}\text{Shaft C} \Rightarrow +\dfrac{11}{14} \Rightarrow 380 \\ \\ \text{Shaft A} \Rightarrow -\dfrac{24}{18} \Rightarrow x\end{array}\right\} x = 644.84\ \text{CCW}$$

The following formula can also be used to solve this problem.

Therefore, **choice (1) is correct.**

It can be seen that this method is time-consuming. When the initial and final gears revolve on fixed bearings (fixed shafts) and their axes of rotation are parallel, a simple method called the "relative velocity" can be used.

B. Relative velocity method

If ω_F, ω_L, and ω_A are the angular speeds of the first gear, last gear, and arm relative to the fixed member, respectively, one can write

$$\frac{\omega_{LA}}{\omega_{FA}} = \frac{\omega_L - \omega_A}{\omega_F - \omega_A} \tag{6.8}$$

where ω_{LA} is the angular speed of the last gear relative to the arm, ω_{FA} is the angular speed of the first gear relative to the arm, and $\frac{\omega_{LA}}{\omega_{FA}}$ is the speed ratio of the last gear to the first gear relative to the arm.

Note To find the ratio $\frac{\omega_{LA}}{\omega_{FA}}$, we only need to assume that the arm is fixed and treat the gears as simple and compound systems.

Note Regarding the application of this equation, it must be noted that the first and last gears mate with the planet gear or gears.

Example The gear train shown in Fig. 6.17 rotates clockwise at an angular speed of $\boldsymbol{\omega_2}$= **10 rad/s** via Arm 2. Determine the angular speed of the output shaft connected to gear 3 (solve with formula):

1) $\omega_3 = 30$ rad/s 2) $\omega_3 = 20$ rad/s 3) $\omega_3 = 10$ rad/s 4) $\omega_3 = 5$ rad/s

Solution Assume that the internal gear of gear 1 is considered the first gear, and gear 3 is the last gear. From the above equation, we will have

$$\frac{\omega_{LA}}{\omega_{FA}} = \frac{\omega_L - \omega_A}{\omega_F - \omega_A} \quad \Rightarrow \quad \frac{\omega_3 - \omega_{32}}{\omega_1 - \omega_2} = \frac{\omega_{32}}{\omega_{12}}$$

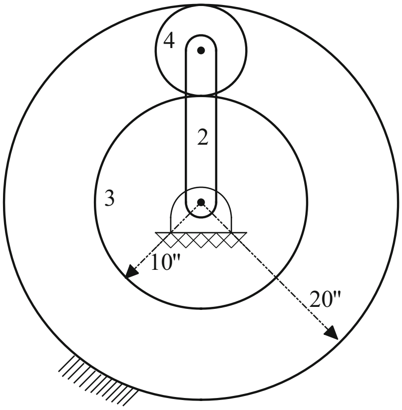


Fig. 6.17 A gear train rotating clockwise

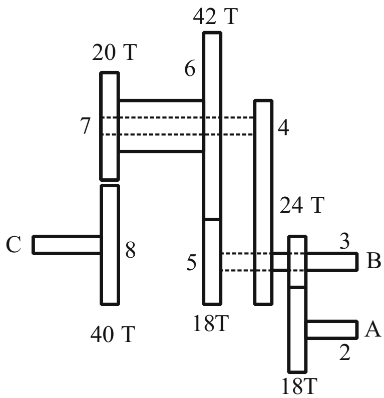


Fig. 6.18 Find the angular speed of shaft A

If the arm is fixed, the ratio $\frac{\omega_{32}}{\omega_{12}} = \frac{N_1}{N_3}$ will be obtained. Substituting this ratio in the above equation gives

$$\frac{\omega_3 - \omega_2}{\omega_1 - \omega_2} = \frac{N_1}{N_3} \quad \Rightarrow \quad \frac{\omega_3 - 10}{0 - 10} = -\frac{2}{1} \quad \Rightarrow \quad \omega_3 = 30$$

Therefore, **choice (1) is correct.**

Example In Fig. 6.18, shaft B is fixed, and shaft C rotates clockwise at 380 rpm. Which of the following represents the angular speed of shaft A, regardless of direction?

1) 645 rpm 2) 273 rpm 3) 932 rpm 4) 745 rpm

Solution Gear 5 attached to shaft B and gear 8 attached to shaft C are engaged via the planetary system. Therefore,

$$\frac{\omega_B - \omega_{arm}}{\omega_C - \omega_{arm}} = \frac{\omega_{B/arm}}{\omega_{C/arm}} = \frac{N_8 N_6}{N_7 N_5}$$

$$\omega_B = 0\,,\ \omega_C = 380\ \text{rpm} \Rightarrow \frac{0 - \omega_{arm}}{380 - \omega_{arm}} = \frac{40 * 42}{20 * 18} \Rightarrow \omega_{arm} = 483.5\ \text{rpm}$$

Gear 3 is engaged with gear 2 and is also attached to the arm; hence,

$$\omega_3 = \omega_{arm}$$

$$\frac{\omega_A}{\omega_3} = \frac{N_3}{N_2} \Rightarrow \frac{\omega_A}{483.5} = \frac{24}{18} \Rightarrow \omega_A = 645\ \text{rpm}$$

Choice (1) is correct.

Some Examples of "Gear Trains"

1. Two planetary gear trains, known as the "differential" and with similar geometries, form a compound differential train with the inputs $\boldsymbol{\omega_1}$ and $\boldsymbol{\omega_9}$ and the outputs $\boldsymbol{\omega_3}$ and $\boldsymbol{\omega_4}$ (Fig. 6.19). Which of the following is true ($\mathbf{\frac{N_1}{N_2}=\frac{N_9}{N_8}}$, $\mathbf{N_3=N_5}$, $\mathbf{N_4=N_7}$)?

   1) $\omega_3 = (\omega_9 - \omega_1)$
   2) $\omega_3 = \frac{N_1}{N_2}(\omega_9 - \omega_1)$
   3) $\omega_3 = \frac{N_1}{N_2}(\omega_9 + \omega_1)$
   4) $\omega_3 = (\omega_9 + \omega_1)$

2. In the planetary gear train shown in Fig. 6.20, $\boldsymbol{\omega_3}$= **600 rpm CCW**. What is the speed of C in rpm?

   1) $\dfrac{600}{1 + \frac{30 \cdot 76}{28 \cdot 18}}$ CCW
   2) $\dfrac{600}{1 - \frac{28 \cdot 18}{30 \cdot 76}}$ CW
   3) $\dfrac{600}{1 + \frac{28 \cdot 18}{30 \cdot 76}}$ CW
   4) $\dfrac{600}{1 - \frac{30 \cdot 76}{28 \cdot 18}}$ CCW

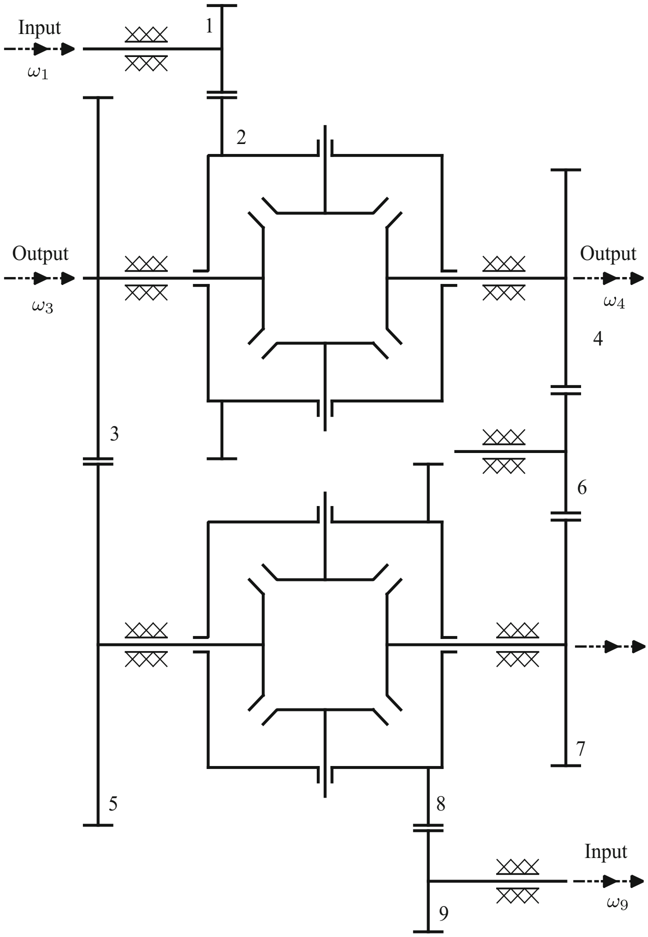


Fig. 6.19 Two planetary gear trains

3. In the epicyclic gear train shown in Fig. 6.21, if the numbers of teeth of A, B, C, D, E, F, and G are identical, which of the following relationships holds between the speed of shaft I and the speed of gear D (A is constant)?

   1) $n_D = \frac{1}{3}n_I$ 2) $n_D = \frac{1}{2}n_I$ 3) $n_D = \frac{2}{3}n_I$ 4) None of them

4. Consider the vehicle differential gear train. The arm and the ring gear are integrated. If $\boldsymbol{\omega_A = 0}$ (the right axle is locked), what gyroscopic torque ($T = I\boldsymbol{\omega_s \omega_p}$) will be exerted on gear B for one revolution of the arm? ($\mathbf{D_A = D_C = 2D_B}$ and I is the moment of inertia of gear B.) (Fig. 6.22)

   1) I 2) 2I 3) 3I 4) 4I

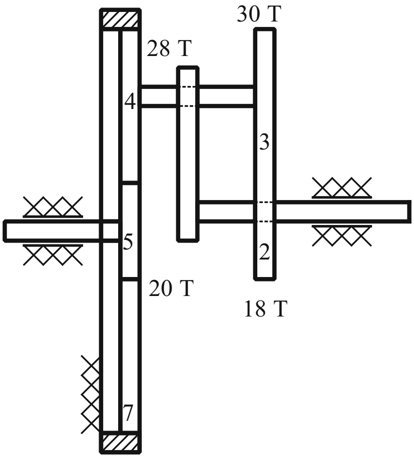


Fig. 6.20 A planetary gear train

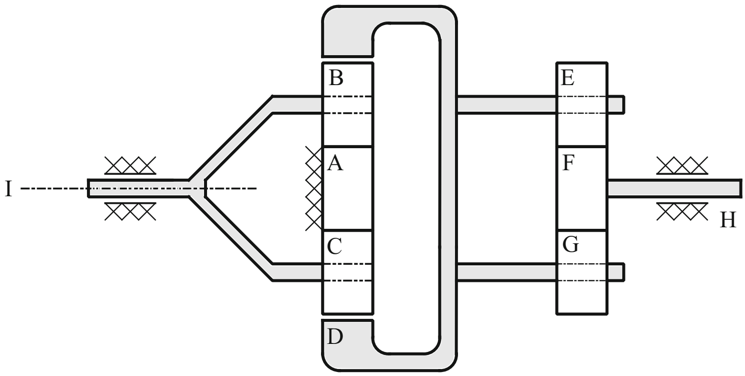


Fig. 6.21 An epicyclic gear train

5. Two planetary gear trains are joined to form a single gear train. The arm of the right set is connected to the left set's sun ring, and the right set's sun gear is connected to the arm of the left set. If all the geometric information is known, what will be the output revolution $\mathbf{n_1}$ for the two clockwise input revolutions $\mathbf{n_3 = n_6 = 1}$ (Fig. 6.23)?

   1) -1 2) The train will lock.

   3) 1 4) The outcome is unpredictable.

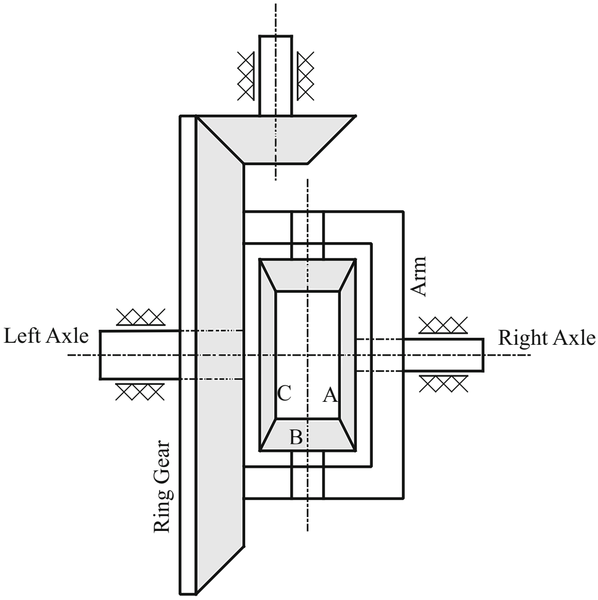


Fig. 6.22 A vehicle differential gear train

6. In the epicyclic gear train shown (Fig. 6.24), gear D rotates at +1000 rpm, and the arm rotates at −1000 rpm. If the number of teeth is $\mathbf{N_{BO}= 57}$, $\mathbf{N_D= 57}$, and $\mathbf{N_A=N_C= N}$, the speed of A will be …

   1) −10,000 rpm 2) −5000 rpm 3) +5000 rpm 4) +10,000 rpm

7. In the gear train shown in Fig. 6.25, if $\mathbf{n_C= 1500 \frac{rev}{min}}$ clockwise, the speed of arm R will be …

   1) 100 rev/min (CCW) 2) 100 rev/min (CW)

   3) 300 rev/min (CCW) 4) 300 rev/min (CW)

8. The gear system shown in Fig. 6.26 has two $\boldsymbol{\omega_2= 500}$ and $\boldsymbol{\omega_5= 300}$ rpm inputs in the same direction. Gear 5 is internal, and the number of teeth of each gear is written between brackets. Find the value and direction of $\boldsymbol{\omega_6}$:

   1) 420 rpm in the same direction as ω_5
   2) $\frac{1900}{3}$ rpm in the same direction as ω_5
   3) 420 rpm opposite to the direction of ω_5
   4) $\frac{1900}{3}$ rpm opposite to the direction of ω_5

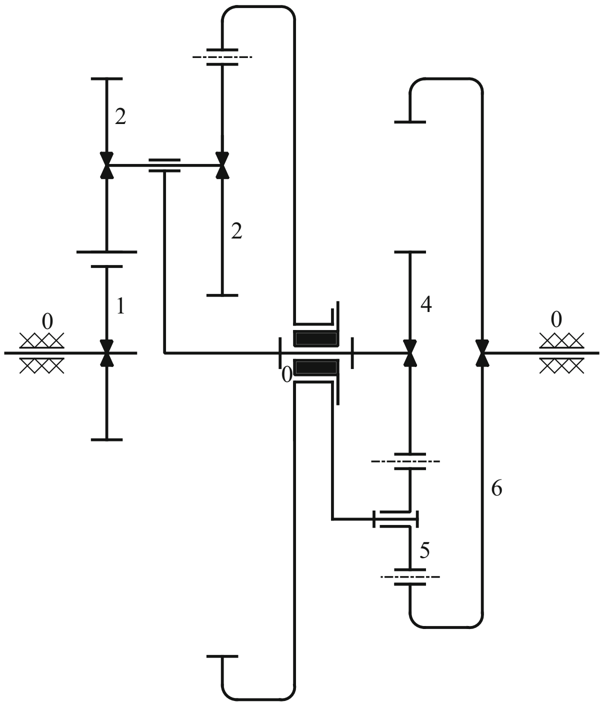


Fig. 6.23 Two planetary gear trains

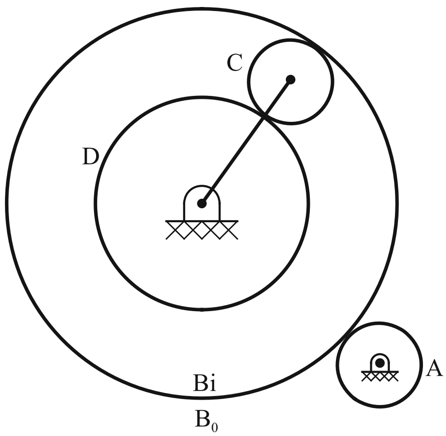


Fig. 6.24 An epicyclic gear train

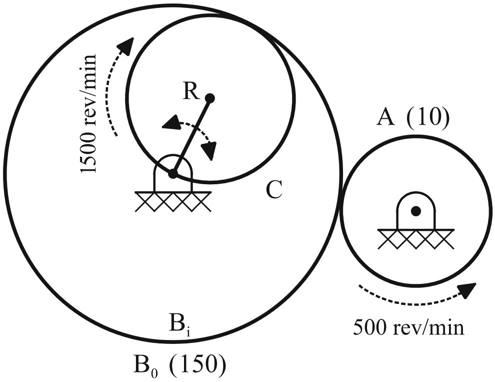


Fig. 6.25 A gear train

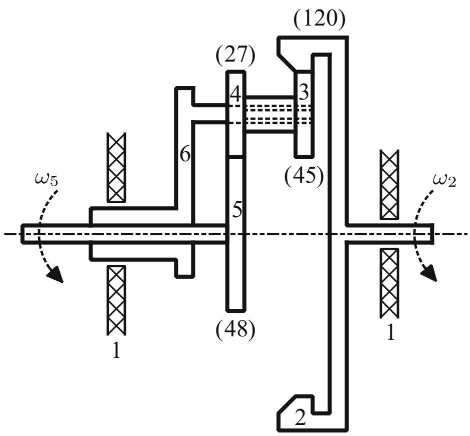


Fig. 6.26 A gear train with two inputs

9. In the gearbox shown in Fig. 6.27, what is the rpm of gear 5 and shaft 6? To determine the direction of motion, view the gearbox from the right side.

   1) $\omega_5 = 3750$ rpm (CW) $\omega_6 = 1200$ rpm (CW)
   2) $\omega_5 = 2840$ rpm (CW) $\omega_6 = 1500$ rpm (CW)
   3) $\omega_5 = 3400$ rpm (CW) $\omega_6 = 1350$ rpm (CW)
   4) $\omega_5 = 1850$ rpm (CW) $\omega_6 = 910$ rpm (CW)

10. In Fig. 6.28, a speed of 60 rpm is input to gear A, and a speed of −60 rpm is input to gear B. The output of gear C in rpm is equal to …

    Number of teeth of A = 25, Number of teeth of B = 35

    1) −10 rpm 2) 60 rpm 3) 10 rpm 4) 0

11. In the gear system in Fig. 6.29, gear A is fixed and has 20 teeth, gear B has 40 teeth, and the internal gear C has 100 teeth and is attached to the output shaft. If

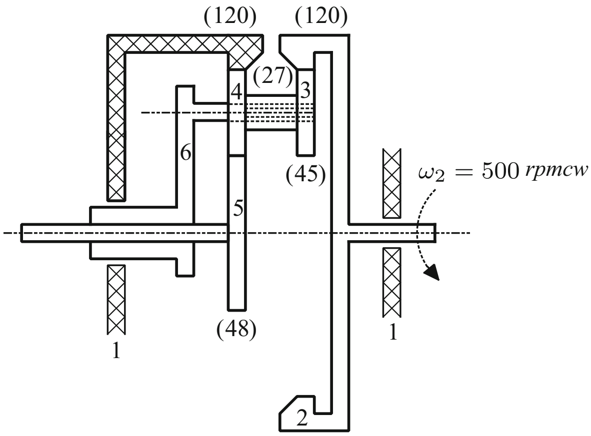


Fig. 6.27 Find gear 5 and shaft 6 speed

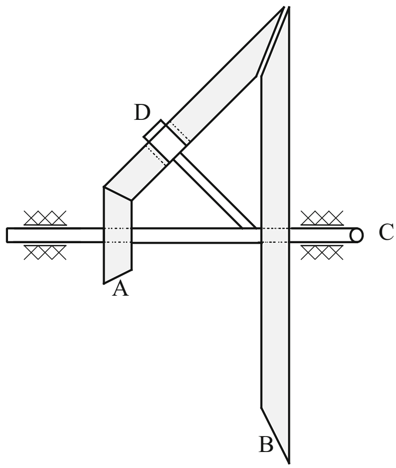


Fig. 6.28 Gear A is input

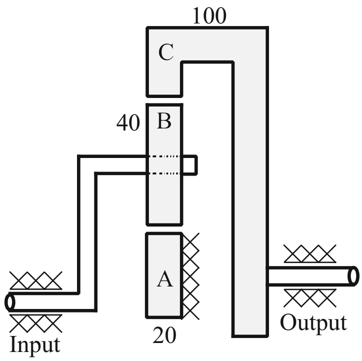


Fig. 6.29 Gear A is fixed

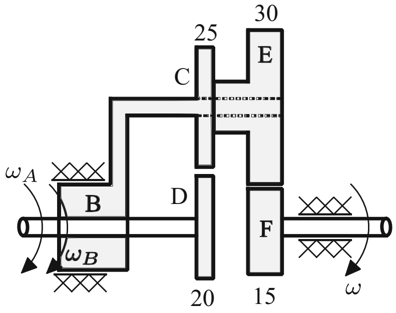


Fig. 6.30 Find the output speed

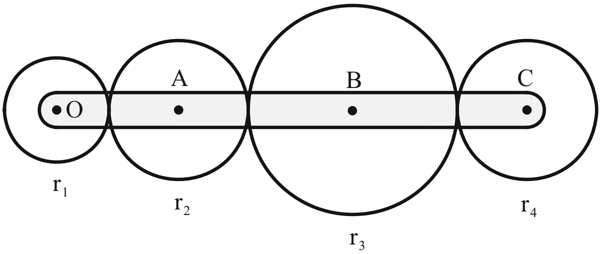


Fig. 6.31 Four gears mating in pairs

the angular speed of the input arm is 1500 rpm, what will be the angular speed of the output shaft?

1) 600 rpm 2) 1250 rpm 3) 1800 rpm 4) 3750 rpm

12. In the planetary gear mechanism shown in Fig. 6.30, if shaft A rotates with a speed of **10 rad/s (CW)** and shaft B rotates with a speed of **20 rad/s (CW)**, what will be the output speed $\boldsymbol{\omega_O}$ (in **rad/s**)?

1) 4 2) 10 3) 28 4) 58

13. Rod OC is pivoted at point O and is connected at points O, B, and C to four gears mating in pairs (Fig. 6.31). The gear with a radius of $\mathbf{r_1}$ is fixed. If the rod rotates at 1 revolution per second counterclockwise, which of the following is the angular speed of the gear with a radius of r_4?

1) $\omega_4 = \frac{r_2 r_1}{r_4 r_3}$ 2) $\omega_4 = \frac{r_2}{r_4}\left(1 - \frac{r_1}{r_2}\right)$

3) $\omega_4 = \frac{r_2}{r_4}\left(1 + \frac{r_1}{r_2}\right)$ 4) $\omega_4 = \frac{\mathbf{r_1}}{\mathbf{r_4}}\left(1 + \frac{\mathbf{r_1}}{\mathbf{r_2}}\right)$

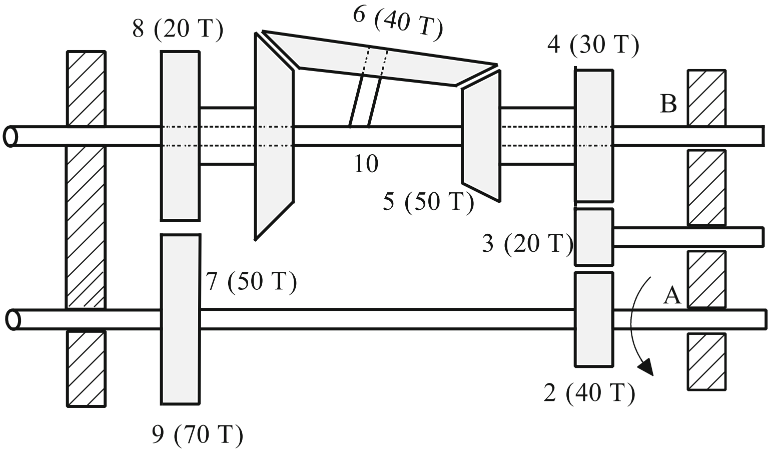


Fig. 6.32 Find the angular velocity of shaft B

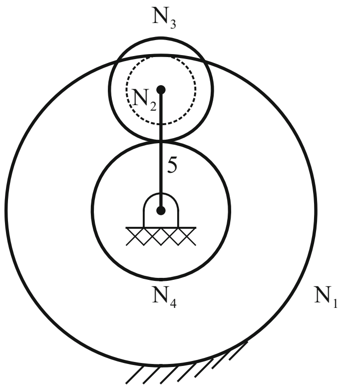


Fig. 6.33 All contacts are rolling type

14. In Fig. 6.32, shaft A rotates at 100 rpm in the direction shown. The angular speed of shaft B is …

    1) In the same direction as shaft A and equal to 269.5 rpm
    2) In the same direction as shaft A and equal to 1333 rpm
    3) Opposite to the direction of shaft A and equal to 269.5 rpm
    4) Opposite to the direction of shaft A and equal to 1333 rpm

15. If the counterclockwise motion is considered constant and all contacts in Fig. 6.33 are of the rolling type, how much will be $\frac{\boldsymbol{\omega_4}}{\boldsymbol{\omega_5}}$?

    1) $\left(1-\frac{N_3}{N_4}\right)\left(1-\frac{N_3}{N_2}\right)$
    2) $\left(1+\frac{N_3}{N_4}\right)\left(1+\frac{N_3}{N_2}\right)$
    3) $-\left(1+\frac{N_3}{N_4}\right)\left(1+\frac{N_3}{N_2}\right)$
    4) $\left(1+\frac{N_4}{N_3}\right)\left(1+\frac{N_2}{N_3}\right)$

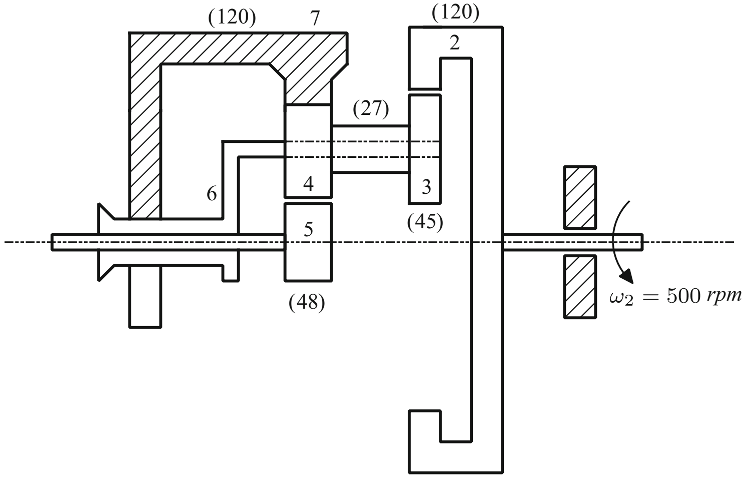


Fig. 6.34 A gear train

16. If $\boldsymbol{\omega_2}$ is known in the gear train shown in Fig. 6.34, which of the following is true about $\boldsymbol{\omega_{51}}$?

1) 600 rpm in the direction of ω_2
2) 1200 rpm opposite to the direction of ω_2
3) 3750 rpm opposite to the direction of ω_2
4) 5250 rpm in the direction of ω_2

17. The magnitude and direction of $\boldsymbol{\omega_2}$, shown in Fig. 6.35, are specified. Which of the following is the magnitude and direction of the angular speed of Member C (Arm 10)?

1) 133.3 rpm in the direction of ω_2
2) 157.3 rpm opposite to the direction of ω_2
3) 200 rpm opposite to the direction of ω_2
4) 400 rpm in the direction of ω_2

18. The gear system shown in Fig. 6.36 is used for speed reduction. The ratio $\frac{\boldsymbol{\omega_F}}{\boldsymbol{\omega_A}}$ must be determined to specify appropriate gears for F and G. Find this ratio.

$$\mathbf{B = 100T \quad c = 20T \quad D = 25T \quad E = 105T}$$

1) $-\frac{4}{21}$ 2) $-\frac{1}{4}$ 3) $-\frac{3}{21}$ 4) $\frac{1}{4}$

19. In Fig. 6.37, if Arm 4 has an angular speed of 2 rad/s clockwise, and gear 2 has an angular speed of 5 rad/s counterclockwise, what will be the angular speed of gear 3 in rad/s?

1) $\omega_3 = 7$ 2) $\omega_3 = 11$ 3) $\omega_3 = 19$ 4) $\omega_3 = 29$

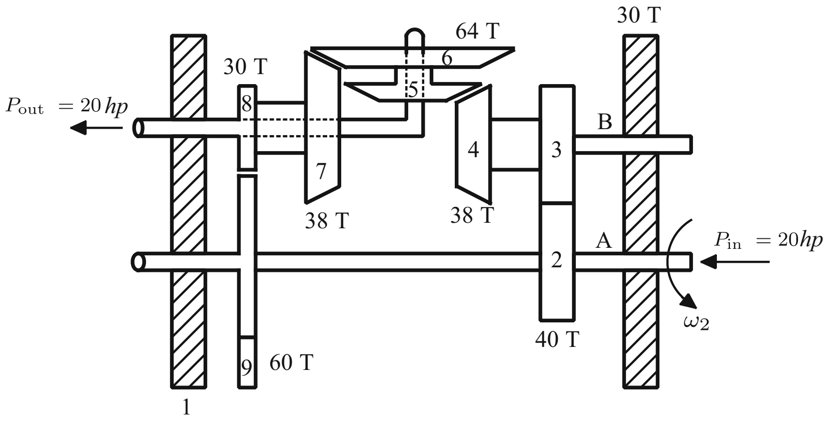


Fig. 6.35 Find the magnitude and direction of the angular speed of Member C

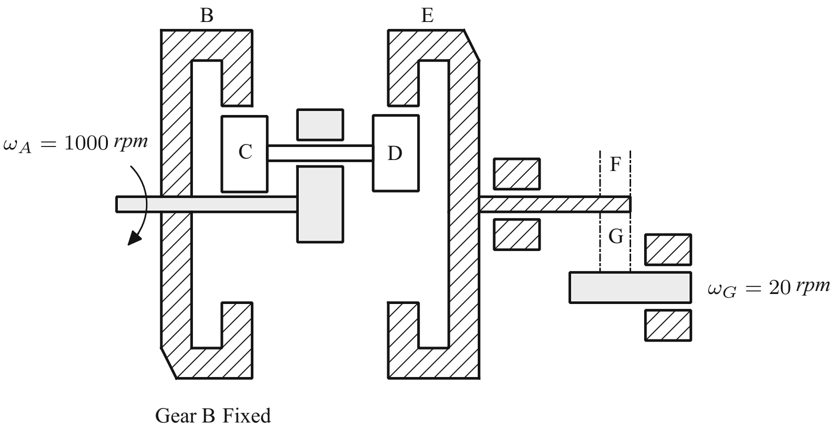


Fig. 6.36 A speed reduction gear system

20. In the epicyclic gear train shown in Fig. 6.38, if Arm 4 rotates at 1 revolution per second clockwise and gear 1 rotates at 5 revolutions per second clockwise, the angular speed of the internal gear 3 will be … revolutions per second … **($R_1 = 2R_2$)**

1) 1- clockwise
2) 1- counterclockwise
3) 5- clockwise
4) 5- counterclockwise

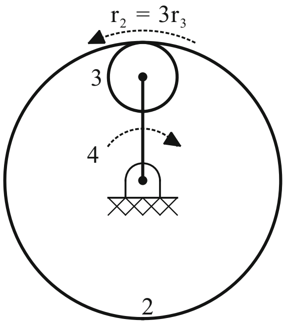


Fig. 6.37 Find angular speed of gear 3

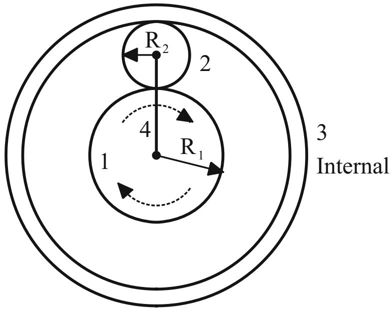


Fig. 6.38 An epicyclic gear train

21. The chain ring A of a bicycle has a diameter of 30 cm and rotates at 60 rpm, while its sprocket B has a diameter of 5 cm. If the diameter of the bicycle's wheel is 70 cm, what is the bicycle's speed in km/h (Fig. 6.39)?

    1) 35.7 2) 40.5 3) 47.5 4) 55.8

22. A hand-operated crane is shown in Fig. 6.40. If the number of teeth $\mathbf{Z_D = 30}$, $\mathbf{Z_C = 80}$, and $\mathbf{Z_B = 20}$, how many teeth must E have for a gear ratio of 25:1 = $\mathbf{n_A{:}n_B}$?

    1) 40 2) 80 3) 120 4) 180

23. In the gear shown in Fig. 6.41, $\boldsymbol{\omega_2 = 12\ (\frac{rad}{s})}$ and $\boldsymbol{\alpha_2 = 48\ (\frac{rad}{s^2})}$. Which is $\boldsymbol{\alpha_3}$?

    1) 48 $\left(\frac{rad}{s^2}\right)$ CCW 2) 144 $\left(\frac{rad}{s^2}\right)$ CW
    3) 152 $\left(\frac{rad}{s^2}\right)$ CW 4) 288 $\left(\frac{rad}{s^2}\right)$ CCW

24. In the planetary gear train shown in Fig. 6.42, a triple-thread left-handed screw E rotates at 1000 rpm clockwise, and shaft I rotates at 100 rpm clockwise when

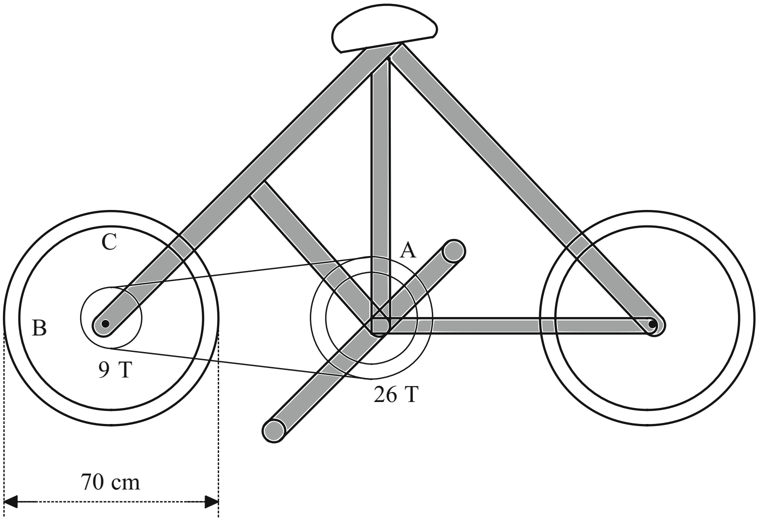


Fig. 6.39 A bicycle system

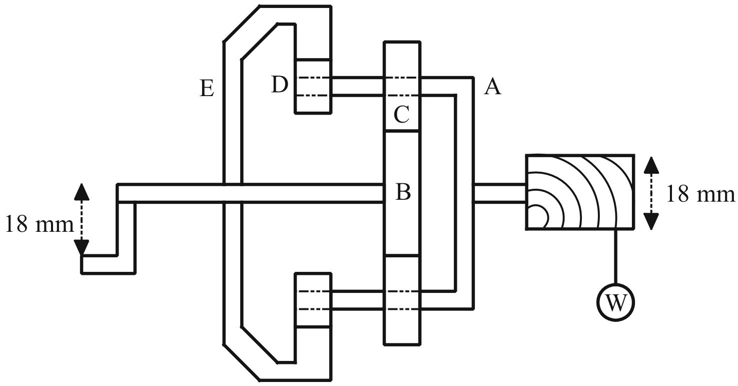


Fig. 6.40 A hand-operated crane

viewed from the right. The speed of the output shaft O, when viewed from the right, is equal to …

1) $\frac{20}{30}$ rpm counterclockwise
2) $\frac{20}{30}$ rpm clockwise
3) $\frac{40}{30}$ rpm counterclockwise
4) $\frac{40}{30}$ rpm clockwise

25. In the gear train shown in Fig. 6.43, the link OBC rotates at 5 revolutions per second clockwise, and the gear centered at O rotates at 5 revolutions per second

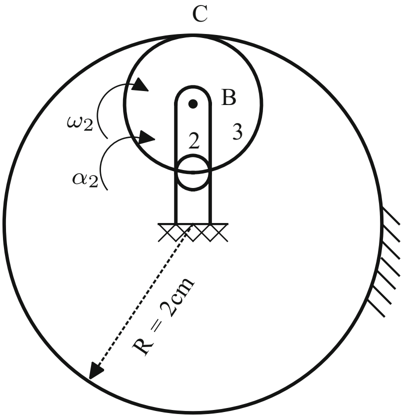


Fig. 6.41 A gear train

$N_{CO} = 100, \quad N_{CI} = 80, \quad N_B = 20, \quad N_A = 40$

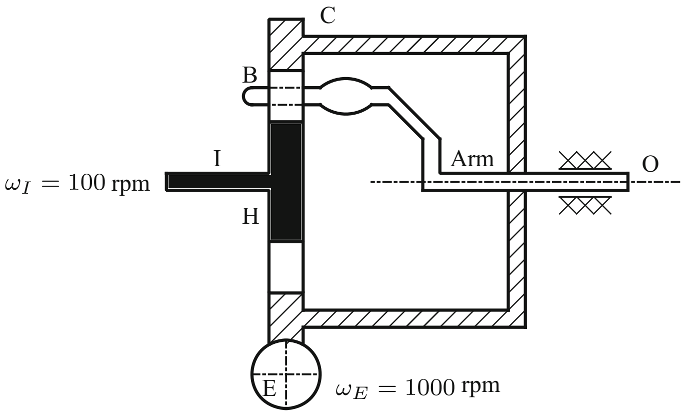


Fig. 6.42 A planetary gear train

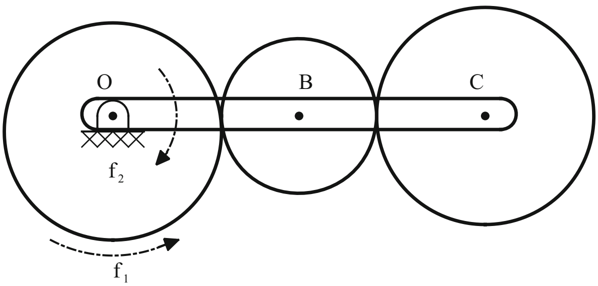


Fig. 6.43 Find the revolutions of the gear centered at C

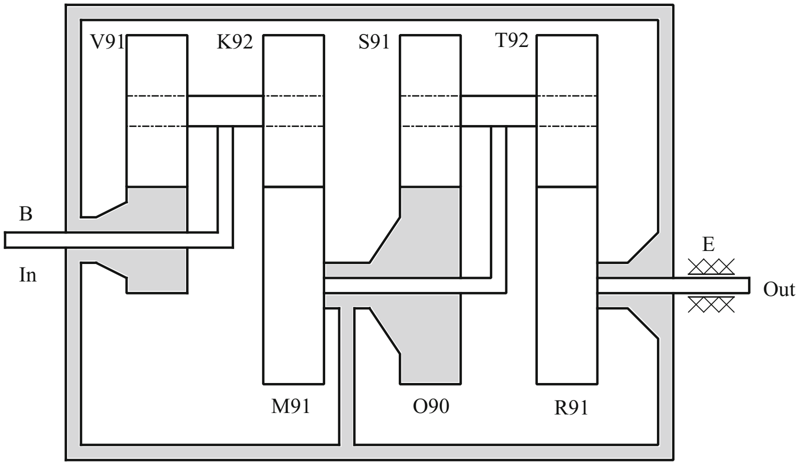


Fig. 6.44 Find the output angular speed

counterclockwise. How many revolutions will the gear centered at C rotate per second?

1) 0
2) 10 revolutions per second clockwise
3) 10 revolutions per second counterclockwise
4) 5 revolutions per second counterclockwise

26. In the gear train shown in Fig. 6.44, find the output angular speed for an input angular speed of 2000 rpm:

1) 200 rpm.
2) 8281 rpm.
3) The output speed is almost equal to the input speed.
4) The output speed decreases to the extent that one revolution will take almost one month.

27. In the mechanism in Fig. 6.45, a gear of radius r moves inside a fixed gear of radius R without slipping, creating a planetary gear train. Link OB of length r is fixed at point O and connected to the center of the gear of radius r via a revolute joint at B. Link CD is constrained to move along the x-direction and is connected to a point on the circumference of the gear of radius r at a point C via a revolute joint. The variable x is obtained as follows:

1) $x = R\sin\theta$ 2) $x = R\tan\theta$

3) $x = R\cos\theta$ 4) $x = R(\sin\theta + \cos\theta)$

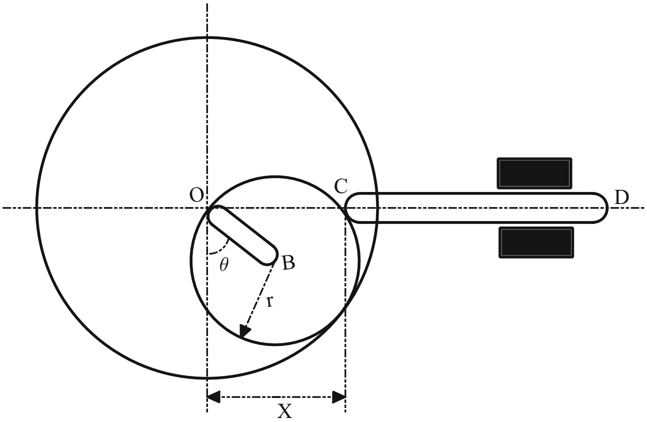


Fig. 6.45 A gear of radius r moves inside a fixed gear of radius R

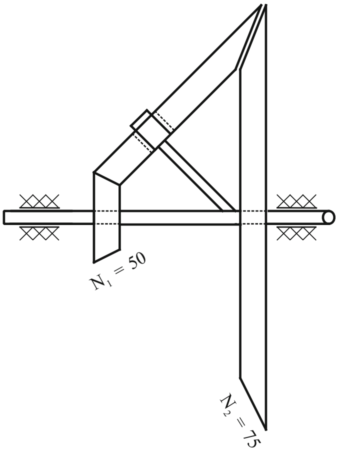


Fig. 6.46 Find the speed of the output shaft connected to the arm

28. In the gear system in Fig. 6.46, the numbers of teeth on the left and right gears are N_1 = 50 and N_2 = 75, respectively. Also, the left and right gears rotate at 2.49 rps clockwise and 1.66 rps counterclockwise, respectively. The speed of the output shaft connected to the arm will be …

    1) 2 rps
    2) 0.5 rps
    3) Zero rps
    4) This design does not work.

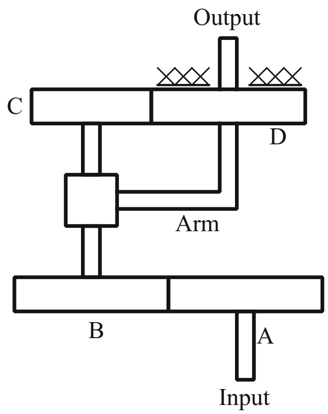


Fig. 6.47 Find the speed of the input shaft

29. In the gear system shown in Fig. 6.47, what should be the speed of the input shaft if the output shaft is to rotate at 1200 rpm (gear D is fixed)?

$$\mathbf{N_A = 51}\ ,\quad \mathbf{N_B = 50}\ ,\quad \mathbf{N_C = 50}\ ,\quad \mathbf{N_D = 50}$$

1) 46.6 rpm
2) 56.5 rpm
3) 82.5 rpm
4) None of the above

30. In the gear system shown in Fig. 6.48, gear 1 rotates at 24 rpm in the indicated direction. Determine the speed of Pinion 9 and the direction of motion of Rack 10.

1) 0.75 rpm; upward
2) 768 rpm; downward
3) 0.75 rpm; downward
4) 768 rpm; upward

31. Which of the following relationships is correct for the speed $\frac{\omega_6}{\omega_1}$ of the compound gear system shown in Fig. 6.49?

1) $\frac{R_6 R_4 R_2}{R_5 R_3 R_1}$
2) $\frac{R_6 R_5 R_3 R_1}{R_4 R_2}$
3) $\frac{R_5 R_3 R_1}{R_6 R_4 R_2}$
4) $\frac{R_4 R_2}{R_6 R_5 R_3 R_1}$

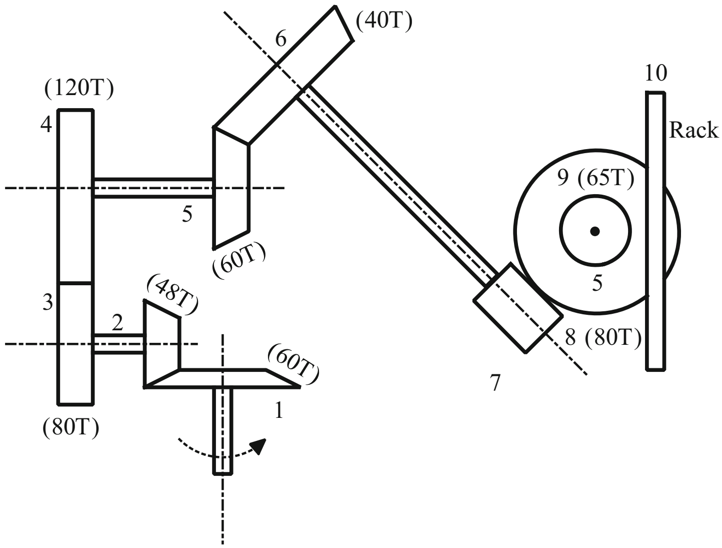


Fig. 6.48 Determine the speed of Pinion 9 and rack 10 motion

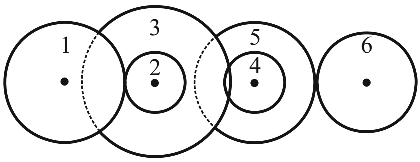


Fig. 6.49 Find the correct relationship

Answers for the Examples of “Gear Trains”

1. Choice (2) is correct.
   We will use the principle of superposition:

$$(1)\ \omega_1 = 0 \quad \Rightarrow \quad \omega_3 = \omega_9 * \left(\frac{N_9}{N_8}\right) * \left(\frac{N_5}{N_3}\right) * (-1)^2 = \omega_9 * \left(\frac{N_1}{N_2}\right)$$

$$(2)\ \omega_9 = 0 \quad \Rightarrow \quad \omega_3 = \omega_1 * \left(\frac{N_1}{N_2}\right) * (-1)^1 = -\omega_1 * \left(\frac{N_1}{N_2}\right)$$

$$\Rightarrow \quad \omega_3 = \omega_{3\ (1)} + \omega_{3\ (2)} = \left(\frac{N_1}{N_2}\right)(\omega_9 - \omega_1)$$

2. Choice (1) is correct.

$\omega_2 = 600$ rpm (CCW)

$$(1)\ \frac{\omega_F - \omega_A}{\omega_L - \omega_A} = \frac{\omega_{FA}}{\omega_{LA}} \quad \Rightarrow \quad \frac{\omega_2 - \omega_C}{\omega_7 - \omega_C} = \frac{\omega_{2C}}{\omega_{7C}}$$

On the other hand, holding the arm fixed relative to $\frac{\omega_{2C}}{\omega_{7C}}$:

$$(2)\ \left(\frac{\omega_{2C}}{\omega_{7C}}\right) = \left(\frac{\omega_2}{\omega_7}\right) = \left(\frac{\omega_2}{\omega_3}\right) * \left(\frac{\omega_4}{\omega_7}\right) = \left(-\frac{N_3}{N_2}\right) * \left(\frac{N_7}{N_4}\right) = -\frac{30}{18} * \frac{76}{28}$$

Substituting Equation (2) into Equation (1) gives

$$\frac{600 - \omega_C}{0 - \omega_C} = -\frac{30 * 76}{28 \div 18} \quad \Rightarrow \quad \omega_C = \frac{600}{1 + \frac{30*76}{28*18}}\ \text{(CCW)}$$

It must be noted that the number of teeth on the internal gear 7 is equal to the sum of the teeth of the external gears inside it, i.e., $N_7 = 28 + 20 + 28 = 76$. The explanation is that the diameter of the outer gear 7 is equal to the sum of the diameters of the inner gears, and the modules of mating gears are identical.

3. Choice (4) is correct.
According to the tabular method:

A	D	I
+1	+1	+1
−1	+1/3	0
0	+4/3	+1

$$\Rightarrow \frac{n_D}{n_I} = \frac{4}{3}$$

According to the relative velocity method:

$$\frac{\omega_A - \omega_I}{\omega_D - \omega_I} = \frac{\omega_{AI}}{\omega_{DI}} = -3 \quad \overbrace{\Rightarrow}^{\omega_A=0} \quad \omega_D = \frac{4}{3}\omega_I$$

None of the choices is correct.

4. Choice (4) is correct.
As discussed in Chap. 4, the gyroscopic couple T is determined by the relationship $T = I\omega_s\omega_p$, where ω_s is the angular speed about the axis of rotation, and ω_p is the angular speed about the spin axis.

We will use the relative velocity method for planetary gear trains:

$$\frac{\omega_L - \omega_A}{\omega_F - \omega_A} = \frac{\omega_{LA}}{\omega_{FA}} \quad \Rightarrow \quad \frac{\omega_C - \omega_{Arm}}{\omega_A - \omega_{Arm}} = -\frac{D_A}{D_C}$$

$$\Rightarrow \quad \frac{\omega_C - 1}{0 - 1} = -1 \quad \Rightarrow \quad \omega_C = 2$$

On the other hand:

$$\frac{\omega_B}{\omega_C} = \frac{D_C}{D_B} \quad \Rightarrow \quad \frac{\omega_B}{2} = \frac{2D_B}{D_B} \quad \Rightarrow \quad \omega_B = 4$$

Therefore, gear B completes 4 revolutions about its rotation axis for every arm revolution. On the other hand, with the rotation of gear A, gear B makes one revolution about the right axle, i.e., $\omega_p = 1$. Therefore,

$$T = I\omega_s\omega_p = 4I$$

5. Choice (2) is correct.
First method:
The rotation of Member 3 causes gear 5 to rotate. Since $n_3 = n_5 = n_6 = 1$, all the members of the right-hand side system move together without relative motion. Hence, they all rotate clockwise at a speed of 1, including gear 4. The rotation of gear 4 causes gear 2 to rotate at a speed of 1. Due to the engagement of gears 2 and 3 and the equal rotation speeds of gear 3 and shaft 2, all the members on the left-hand side also rotate at a speed of 1, including gear 1.
Second method:
In the first case, we set ω_6 equal to 0 and ω_3 equal to 1 in the clockwise direction and determine ω_1 from the following equations:

$$\frac{\omega_4 - \omega_3}{\omega_6 - \omega_3} = \frac{\omega_{43}}{\omega_{63}} \quad , \quad \frac{\omega_1 - \omega_4}{\omega_3 - \omega_4} = \frac{\omega_{14}}{\omega_{34}}$$

In the second case, we set ω_3 equal to 0 and ω_6 equal to 1 in the clockwise direction and calculate ω_1 from the same equations. The output speed ω_1 is obtained from the superposition principle:

$$\omega_{total} = (\omega_1)_{First\ case} + (\omega_1)_{Second\ case} = 1$$

6. Choice (4) is correct.

$$\frac{\omega_B - \omega_{arm}}{\omega_D - \omega_{arm}} = \frac{N_D}{N_{Bi}} = -\frac{2}{4} \quad \Rightarrow \quad \frac{\omega_B - (-1000)}{1000 - (-1000)} = -\frac{2}{4}$$

$$\Rightarrow \quad \omega_B = -2000$$

$$\frac{\omega_A}{\omega_B} = \frac{N_{BO}}{N_A} = -5 \quad \Rightarrow \quad \omega_A = +10{,}000$$

It must be noted that

$$N_{Bi} = 4N,\ i.e.\ \frac{N_{Bi}}{N_D} = \frac{N_{Bi}}{N_D} = \frac{N_D + 2N_C}{N_D} = 2D_{Bi} = D_D + 2D_C$$

7. Choice (3) is correct.
The angular speed of gear B is clockwise and equal to

$$\omega_B = 1500 * \frac{10}{150} = 100$$

$$\frac{\omega_B - \omega_{arm}}{\omega_C - \omega_{arm}} = \frac{1}{2} \quad \Rightarrow \quad \frac{100 - \omega_{arm}}{500 - \omega_{arm}} = +\frac{1}{2} \quad \Rightarrow \quad \omega_{arm} = -300$$

8. Choice (1) is correct.
Member 6 is an arm; hence, Members 5, 6, and 2 form a planetary gear train. Therefore:

$$\frac{\omega_2 - \omega_6}{\omega_5 - \omega_6} = \left(\frac{+N_3}{N_2}\right)\left(\frac{-N_5}{N_4}\right) = \left(+\frac{45}{120}\right)\left(-\frac{48}{27}\right) = -\frac{2}{3}$$

$$\Rightarrow \quad \omega_6 = \frac{3(\omega_2 + \frac{2}{3}\omega_5)}{5} = \frac{3\left(500 + \frac{2}{3} * 300\right)}{5} = 420$$

Since the direction is positive, it has the same direction as ω_5, which is positive. The engagement of gears 2 and 3 is internal for one and external for the other. Thus, a positive sign was used in $\frac{+N_3}{N_2}$. In addition, the engagement of gears 4 and 5 is external for both. Hence, a negative sign was used $\frac{-N_5}{N_4}$.

9. Choice (1) is correct.
In Gears 2 and 7:

$$\frac{\omega_2 - \omega_6}{\omega_7 - \omega_6} = \left(\frac{N_3}{N_2}\right)\left(\frac{N_7}{N_4}\right) = \frac{45}{120} * \frac{102}{17} = 1.417$$

$$\Rightarrow \quad \frac{500 - \omega_6}{0 - \omega_6} = 1.417 \Rightarrow \ \omega_6 = -1200$$

In Gears 5 and 7:

$$\frac{\omega_5 - \omega_6}{\omega_7 - \omega_6} = \left(\frac{-N_4}{N_5}\right) * \left(\frac{N_7}{N_4}\right) = \frac{-N_7}{N_5} = \frac{-102}{48}$$

$$\Rightarrow \frac{\omega_5 - (-1200)}{0 - (-1200)} = \frac{-102}{48} \qquad \Rightarrow \omega_5 = -3750$$

10. Choice (1) is correct.
Gears A and B form a planetary system with Arm C. If we name the middle gear D:

$$\frac{\omega_B - \omega_{arm}}{\omega_A - \omega_{arm}} = \frac{-N_A N_D}{N_D N_B} = \frac{-N_A}{N_B}$$
$$\frac{-60 - \omega_{arm}}{60 - \omega_{arm}} = \frac{-25}{35} \Rightarrow \quad \omega_{arm} = -10 \text{ rpm}$$

The negative sign on the right-hand side is because if the arm is fixed and A rotates, B will rotate in a direction opposite to A.

11. Choice (3) is correct.
Gears A and C form a planetary system. The negative sign on the right-hand side is because if the arm is assumed to be fixed and A rotates, C will rotate in a direction opposite to A.

$$\frac{\omega_C - \omega_{arm}}{\omega_A - \omega_{arm}} = \frac{-N_A N_B}{N_B N_C} = \frac{-N_A}{N_C}$$
$$\frac{\omega_C - 1500}{0 - 1500} = \frac{-20}{100} \qquad \omega_C = 1800 \text{ rpm}$$

12. Choice (1) is correct.
D and F are related through a planetary system. Therefore,

$$\frac{\omega_F - \omega_{arm}}{\omega_D - \omega_{arm}} = \frac{N_D . N_E}{N_C . N_F} \Rightarrow \frac{\omega_o - 20}{10 - 20} = \frac{20 * 30}{25 * 15} \qquad \omega_o = 4 \text{ rad/s}$$

13. Choice (3) is correct.
Gears 1 and 2 are related through a planetary system, and link OC acts as an arm (A). Therefore,

$$(1)\ \frac{\omega_2 - \omega_A}{\omega_1 - \omega_{1A}} \Rightarrow \quad \frac{\omega_2 - 1}{0 - 1} = -\frac{r_1}{r_2} \quad \Rightarrow \quad \omega_2 = 1 + \frac{r_1}{r_2}$$

On the other hand, gears 2 and 4 are related through a simple gear train. Therefore,

$$(2)\ \frac{\omega_4}{\omega_2} = \frac{r_2}{r_4}$$

Combining Equation (1) and (2):

$$\omega_4 = \frac{r_2}{r_4}\ (1 + \frac{r_1}{r_2})$$

14. Choice (3) is correct.
The chains 2-9-8-7 and 2-3-4-5 are simple. Gears 5 and 7 are related through a planetary system. If the rotation direction of shaft A is taken to be positive,

$$\frac{\omega_5}{\omega_2} = \frac{N_2 * N_3}{N_4 * N_3}\ ,\ \frac{\omega_5}{100} = \frac{40}{30}\ ,\ \omega_5 = 133.33\ \text{rpm}$$

The rotation direction of gear 5 is positive.

$$\frac{\omega_7}{\omega_2} = \frac{N_9}{N_8} \qquad \frac{\omega_7}{100} = \frac{70}{20} \qquad \omega_7 = 350\ \text{rpm}$$

The rotation direction of gear 7 is negative.
Now, for the planetary system 7-6-5:

$$\frac{\omega_5 - \omega_{arm}}{\omega_7 - \omega_{arm}} = \frac{-N_7 N_6}{N_6 N_5} \qquad \frac{+133.33 - \omega_B}{-350 - \omega_B} = \frac{-50}{10} \quad \Rightarrow \quad \omega_5 = -269.44\ \text{rpm}$$

The sign is negative; hence, the rotation direction is opposite to that of A.

15. Choice (2) is correct.
Gears 4 and 1 are related through a planetary system, and gear 1 is fixed, and thus:

$$\frac{\omega_1 - \omega_{arm}}{\omega_4 - \omega_{arm}} = \frac{-N_4 N_2}{N_3 N_1}$$

$$\frac{0 - \omega_5}{\omega_4 - \omega_5} = \frac{-N_4 N_2}{N_3 N_1} \quad \Rightarrow \quad \frac{\omega_4 - \omega_5}{\omega_5} = \frac{+N_3 N_1}{N_4 N_2} \quad \Rightarrow \quad \frac{\omega_4}{\omega_5} = 1 + \frac{N_3 N_1}{N_4 N_2}$$

$$r_1 = r_3 + r_2 + r_4 \quad \Rightarrow \quad D_1 = D_3 + D_2 + D_4 \quad \Rightarrow \quad N_1 = N_3 + N_2 + N_4$$

$$\frac{\omega_4}{\omega_5} = 1 + \frac{N_3 N_1}{N_4 N_2}\ (N_3 + N_2 + N_4) = (1 + \frac{N_3}{N_4})(1 + \frac{N_3}{N_2})$$

16. Choice (3) is correct.
    Gears 2 and 7 are related through a planetary system. Also, gear 7 is fixed. Therefore,

$$\frac{\omega_7 - \omega_{arm}}{\omega_2 - \omega_{arm}} = \frac{+N_2 N_4}{N_3 N_7}$$

$$\frac{0 - \omega_{arm}}{500 - \omega_{arm}} = \frac{120 * 27}{45 * 102} \quad \Rightarrow \quad \omega_{arm} = -1200 \text{ rpm}$$

Now, for the planetary system 7-4-5:

$$\frac{\omega_5 - \omega_{arm}}{\omega_7 - \omega_{arm}} = \frac{-N_4 N_7}{N_5 N_4} = \frac{N_7}{N_5} \Rightarrow \frac{\omega_5 + 1200}{0 + 1200} = \frac{-102}{48} \quad \omega_5 = -3750 \text{ rpm}$$

The negative sign on the right-hand side shows that if the arm is assumed to be fixed and gear 7 rotates, gear 5 will rotate in a direction opposite to gear 7.

$$\omega_{51} = \omega_5 - \omega_1 = \omega_5$$

17. Choice (2) is correct.
    Gears 2, 9, 8, and 7 form a simple gear train. If the rotation direction of shaft A is taken to be positive:

$$\frac{\omega_7}{\omega_2} = \frac{N_9}{N_8} \qquad \frac{\omega_7}{100} = \frac{60}{30} \qquad \omega_7 = 200 \text{ rpm}$$

The direction of ω_7 is negative.
Gears 2, 3, and 4 form a simple gear train.

$$\frac{\omega_4}{\omega_2} = \frac{N_2}{N_3} \qquad \frac{\omega_4}{100} = \frac{40}{30} \qquad \omega_4 = 133.3 \text{ rpm}$$

The direction of ω_4 is negative.
Gears 4 and 7 are related through a planetary system.

$$\frac{\omega_7 - \omega_{arm}}{\omega_4 - \omega_{arm}} = \frac{-N_6 N_4}{N_7 N_{45}} \Rightarrow \frac{-200 - \omega_{arm}}{-133.3 - \omega_{arm}} = \frac{-64 * 38}{38 * 36}$$

$$\omega_{arm} = -157.3 \text{ rpm}$$

18. Choice (1) is correct.
Gears E and B are related through a planetary system, and gear B is fixed; hence, assuming the direction of A as positive gives

$$\frac{\omega_B - \omega_{arm}}{\omega_E - \omega_{arm}} = \frac{N_E N_C}{N_D N_B}$$

$$\frac{0 - 1000}{\omega_F - 1000} = \frac{105 + 20}{25 * 100} \qquad \omega_F = -190.47 \text{ rpm}$$

$$\frac{\omega_F}{\omega_A} = \frac{-190.47}{1000} = \frac{-4}{21}$$

19. Choice (3) is correct.
Gears 3 and 2 are related through a planetary system:

$$\frac{\omega_2 - \omega_{arm}}{\omega_3 - \omega_{arm}} = \frac{N_3}{N_2} = \frac{r_3}{r_2} \quad \Rightarrow \quad \frac{-5 - 2}{\omega_3 - 2} = \frac{1}{3} \qquad \omega_3 = -19 \text{ rad/s}$$

20. Choice (2) is correct.
Gears 3 and 1 are related through a planetary system. We take the clockwise direction to be positive.

$$\frac{\omega_3 - \omega_{arm}}{\omega_1 - \omega_{arm}} = \frac{-N_1 N_2}{N_2 N_3} = -\frac{N_1}{N_3} = \frac{-R_1}{R_3} \qquad R_3 = R_1 + 2R_2 = 2R_1$$

$$\frac{\omega_3 - 1}{5 - 1} = \frac{-1}{2} \qquad \omega_3 = -1 \text{ rev/s}$$

The negative sign denotes an opposite direction.

21. Choice (3) is correct.
In the sprockets chain, the ratio of the speeds is the inverse of the ratio of the radii:

$$\omega_A = 60 * \frac{2\pi}{60} = 6.28 \text{ rad/s}$$

$$\frac{\omega_B}{\omega_A} = \frac{R_A}{R_B} \qquad \frac{\omega_B}{6.28} = \frac{15}{2.5} \quad \Rightarrow \quad \omega_B = 37.68$$

r = wheel radius

$$V_B = r.\omega_B = 0.35 * 37.68 = 13.18 \text{ m/s}$$

$$V_B = 47.5 \text{ km/h}$$

22. Choice (4) is correct.
Gear E is fixed, and gears E and B are related through a planetary gear train:

$$\frac{\omega_B - \omega_{arm}}{\omega_E - \omega_{arm}} = \frac{-N_E N_C}{N_B N_D}$$

$$\frac{\omega_B - \omega_A}{0 - \omega_A} = \frac{-80 * N_E}{20 * 30} \quad \Rightarrow \quad \frac{-\omega_B}{\omega_A} + 1 = \frac{-4}{30} N_E$$

$$-25 + 1 = \frac{-4}{30} N_E \qquad N_E = 180$$

The negative sign on the right-hand side shows that if the arm is assumed to be fixed and gear E rotates, gear B will rotate in a direction opposite to gear E.
23. Choice (1) is correct.
Given the stationary nature of point C and the center of rotation,

$$a_B^t = R_2\alpha_2$$

$$a_B^t = R_3\alpha_3 \qquad R_3\alpha_3 = R_2\alpha_2 \qquad \alpha_3 = \alpha_2$$

24. Choice (4) is correct.
The screw E is engaged with the external side of gear C.

$$\frac{\omega_C}{\omega_E} = \frac{\text{Number of threads}}{N_{CO}} \qquad \frac{\omega_C}{1000} = \frac{3}{100}$$

$$\omega_C = 30\text{rpm}$$

The internal side of gear C is connected to gear A through the planetary system. The positive direction is taken to be clockwise. If we assume the arm is fixed and A rotates, E must rotate opposite to A. Thus, a negative sign is added to the right-hand side of the equation.

$$\frac{\omega_{CI} - \omega_{arm}}{\omega_A - \omega_{arm}} = \frac{-N_B * N_A}{N_{CI} * N_B} = \frac{-N_A}{N_{CI}}$$

$$\frac{-30 - \omega_O}{+100 - \omega_O} = \frac{-40}{80} \qquad \omega_O = \frac{40}{3}\text{ rpm}$$

Since ω_O is positive, it rotates clockwise.
25. Choice (4) is correct.
Gears C and O are related through a planetary system:

$$\frac{\omega_C - \omega_{arm}}{\omega_O - \omega_{arm}} = \frac{N_O N_B}{N_B N_C} = 1$$

Based on the figure, the radius or number of teeth of gears O and C are considered identical.

$$\frac{\omega_C - 5}{-5 - 5} = 1 \qquad \omega_C = -5 \text{ rev/s}$$

26. Choice (4) is correct.
Gears L and M are related through a planetary gear train, and gear L is fixed. Therefore,

$$\frac{\omega_L - \omega_{arm}}{\omega_m - \omega_{arm}} = \frac{N_V N_M}{N_L N_K}$$

$$\frac{0 - 2000}{\omega_M - 2000} = \frac{91 * 91}{90 * 92} \qquad \omega_M = 0.24 \text{ rpm}$$

Gears O and R are also connected through a planetary system, and gear O is fixed. Therefore,

$$\frac{\omega_O - \omega_{arm}}{\omega_R - \omega_{arm}} = \frac{N_S N_R}{N_O N_T} \qquad \omega_{arm} = \omega_M$$

$$\frac{0 - 0.24}{\omega_R - 0.24} = \frac{91 * 91}{90 * 92} \qquad \omega_R = 2.89 * 10^{-5} \text{ rpm}$$

Now, we convert the speed:

$$\omega_R = 2.89 * 10^{-5} * 30 * 24 * 60 = \frac{1}{24} \text{ revolutions/month}$$

27. Choice (1) is correct.
According to Fig. 6.50:

$$x = 2r \sin\theta$$

$$2r = R \qquad \Rightarrow \qquad x = R \sin\theta$$

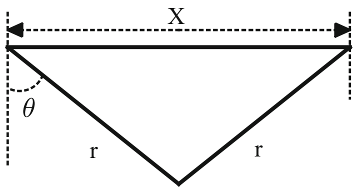


Fig. 6.50 System analyzing

28. Choice (3) is correct.

$$\frac{\omega_2 - \omega_{arm}}{\omega_1 - \omega_{arm}} = \frac{-N_1}{N_2} \quad \Rightarrow \quad \frac{-\frac{1}{66} - \omega_a}{\frac{2}{49} - \omega_a} = \frac{-50}{75} \quad \Rightarrow \quad \omega_a = 0$$

29. Choice (1) is correct.
The system between A and D is a planetary system, and D is fixed. Thus,

$$\frac{\omega_D - \omega_{arm}}{\omega_A - \omega_{arm}} = \frac{N_A N_C}{N_B N_D}$$

$$\frac{0 - 1200}{\omega_A - 1200} = \frac{51 * 51}{50 * 50} \qquad \omega_A = 46.6 \text{ rpm}$$

30. Choice (3) is correct.
Gears 8 and 9 rotate together; hence, $\omega_9 = \omega_8$. On the other hand:

$$\frac{\omega_8}{\omega_1} = \frac{N_1 * N_3 * N_5 * N_7}{N_2 * N_4 * N_6 * N_8} = \frac{60 * 80 * 60 * 2}{48 * 120 * 40 * 80} = 0.75$$

The rotation direction of the gear can be easily found through a standard procedure: It must be noted that in the left-hand worm, according to the right-hand rule, if the fingers of the right hand are curled in the rotation direction of the worm, the thumb represents the direction of motion of the surface on the worm gear tangent to the worm. For the right-hand worm, the thumb represents the opposite of the motion direction of this surface.

31. Choice (3) is correct.

$$\frac{\omega_6}{\omega_1} = \frac{R_5}{R_6} * \frac{R_3}{R_4} * \frac{R_1}{R_2}$$

Chapter 7
Balancing

The focus of this chapter is on the concept of balancing rotating bodies and reciprocating masses. The chapter begins with an introduction, providing an overview of the importance of balancing in mechanical systems to minimize vibration, increase stability, and improve performance.

Then, the effects of unbalanced masses and the resulting vibration are discussed. Various methods of balancing rotating bodies are explained, including balancing masses in a plane, balancing masses in several lateral planes, and the graphical method for balancing. These methods involve the placement of counterweights or the adjustment of mass distribution to achieve balance.

Overall, this chapter provides a comprehensive understanding of balancing techniques for both rotating bodies and reciprocating masses. It emphasizes the importance of balancing in reducing vibration and improving the overall performance of mechanical systems. The methods and concepts presented in this chapter serve as a valuable resource for engineers and designers involved in the development of balanced machinery.

7.1 Introduction

High-speed machinery with rotating or reciprocating masses constitutes a considerable source of vibration. A major issue in machine dynamics and design is the attempt to minimize oscillating forces exerted by such machinery on the environment via their bases. Small imbalances in rapidly rotating masses, such as those used in generators, can produce significant oscillatory forces. A combination of rotating and reciprocating masses is observed in internal combustion engines, pumps, compressors, and other machinery. The above-mentioned are among the main factors causing oscillating forces, which can be partially balanced by using appropriate weights.

A. Abbasimoshaei, T. A. Kern, *Machine Dynamics*,
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In dynamics, the equation of motion of a particle was studied in the form of $\sum \overrightarrow{F} = m\overrightarrow{a}$ and those of rigid body were studied in the form of $\sum \overrightarrow{F} = m_G\overrightarrow{a}_G$ (where m_G and a_G are the mass and acceleration vector of the center of gravity) and $\sum \overrightarrow{M}_G = I_G\overrightarrow{\alpha}$ (where I_G is the moment of inertia tensor about the center of gravity). A vector in the opposite direction of the vector $m\overrightarrow{a}$ can be regarded as an external force and the opposite of the moment of inertia $I_G\overrightarrow{\alpha}$ as an external torque exerted on the system. Then, using D'Alembert's Principle, one can examine the static equilibrium of the mechanism instead of its dynamic equations.

7.2 Balancing of Rotating Bodies

7.2.1 Masses in a Plane

The masses M_1, M_2, etc. are lumped masses all located in a single plane of rotation and connected to point O (center of rotation) via lightweight links. In an arbitrary orientation, the angle of the link connecting mass i to point O with respect to the horizon is named θ_i. The angle between the links is always constant. M_e is the mass that must be added at a radial distance of r_e and an angular position of θ_e to balance the system.

Static Balance

The set of masses M_1, M_2, etc. is said to be in static balance if the system does not rotate when left on its own. This requires that the center of mass of the system be coincident with the center of rotation (point O). Otherwise, the system will tend to rotate toward an orientation where the center of mass is directly below point O so that the torque exerted by the weight about this point becomes zero. In fact, static balance is a balance related to the gravitational effect.

$M_e r_e$ equals the product of the mass of the unbalanced system and the distance from its center of mass to point O, and $\theta_e = \theta_G + \pi$, where θ_G is the angle of the mass center of the unbalanced system relative to the horizon. After adding the balancing mass M_e, the center of mass of the new system coincides with O, and the system reaches static equilibrium. For balancing the system, the net torque of the masses about the vertical axis must equal zero (Fig. 7.1):

$$\sum_{i=1}^{n} M_i r_i \cos\theta_i + M_e r_e \cos\theta_e = 0 \tag{7.1}$$

Note For the static balance of a unique θ_e, an infinite number of M_e and r_e pairs can be found whose product satisfies Eq. 7.1.

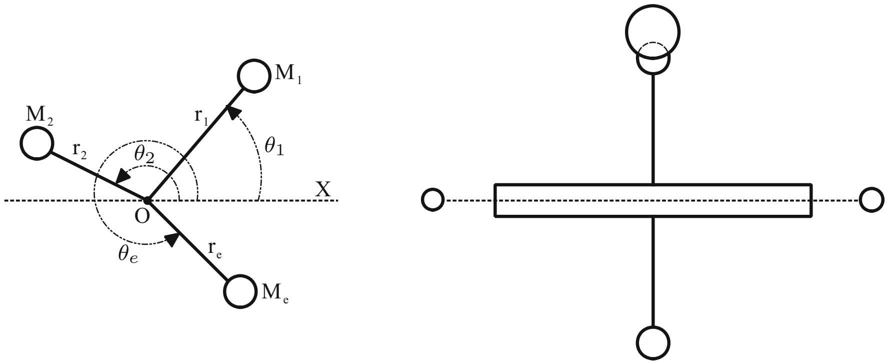


Fig. 7.1 Set of masses

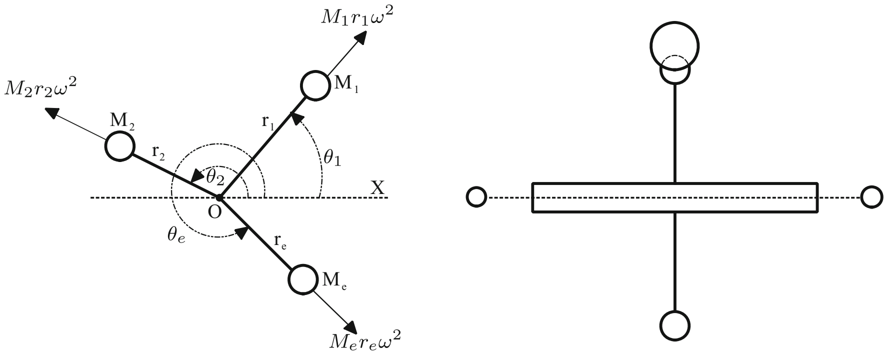


Fig. 7.2 Inputting forces to the set of masses

Rotary systems such as gears, pulleys, wheels, cams, fans, flywheels, and rotor blades that have a thin-disk shape are often balanced statically.

Dynamic Balance

If the system rotates around the center of rotation with an angular speed of ω, the masses will possess centripetal acceleration. Based on D'Alembert's Principle and Fig. 7.2, the inverse of the product of these accelerations and the mass can be considered a static force exerted on the system. When the resultant of these inertia forces is zero, no force is exerted on the support O during rotation. Under these conditions, the system is known to be in dynamic equilibrium.

The system can reach dynamic balance if a mass is added at a suitable location. The condition for the dynamic balance of the system is that net inertia forces exerted on the system in the horizontal and vertical directions should be zero.

$$\sum_{i=1}^{n} M_i r_i \cos\theta_i + M_e r_e \cos\theta_e = 0 \tag{7.2}$$

$$\sum_{i=1}^{n} M_i r_i \sin\theta_i + M_e r_e \sin\theta_e = 0 \tag{7.3}$$

Note If the dynamic balance condition is satisfied, the static balance condition will also be satisfied. However, if a rotor is statically balanced, one cannot directly conclude that it is also dynamically balanced.

Note Similar to static balancing, dynamic balancing is not limited to one state. Although the angle θ_e is unique, an infinite number of M_e and r_e pairs can be used, the product of which equals the required value.

Note Static balance is a reliable criterion for the dynamic balance test only when all the masses are in one lateral plane and the dynamic unbalance of torques is unlikely.

7.2.2 Masses in Several Plane Lateral

Under conditions where the masses are not in one plane, as shown below, the torque of the inertia forces must be considered relative to the case when they are all in one plane.

This case is the most general state of mass distribution on a rigid rotor. To study such systems, we selected two arbitrary planes A and B perpendicular to the axis of rotation. First, we add the mass M_B at a suitable location in plane B to make the torques exerted on the system relative to plane A equal to zero. The condition for the torques becoming zero relative to plane A and about the x- and y-axes is as follows:

$$\sum_{i=1}^{n} M_i r_i a_i \sin\theta_i + M_B r_B a_B \sin\theta_B = 0 \tag{7.4}$$

$$\sum_{i=1}^{n} M_i r_i a_i \cos\theta_i + M_B r_B a_B \cos\theta_B = 0 \tag{7.5}$$

where a_i is the distance between mass i and plane A, and a_B is the distance between plane B and plane A (Fig. 7.3).

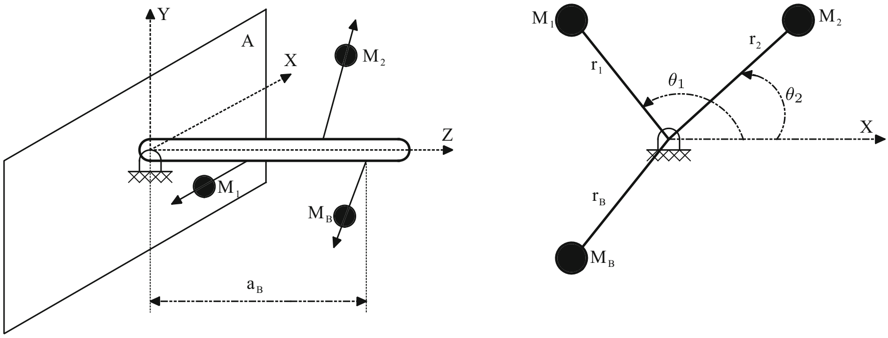


Fig. 7.3 Masses in several plane laterals

Note θ_B is unique; however, there are infinite choices for M_B and R_B. Moreover, there is an infinite number of choices for planes A and B.

As Eqs. 7.4 and 7.5 are established, the z-axis will be the principal axis of the system. As such, the product moments of inertia I_{zy} and I_{zy} become zero, and the system balances in terms of torque during rotation.

Example The necessary and sufficient conditions for the dynamic balance of a system are as follows:

1) The center of mass must be on the axis of rotation.
2) The axis of rotation must be the principal axis.
3) Both conditions (1) and (2) must be satisfied.
4) The angular momentum vector must be normal to the angular velocity vector.

Solution First, a dynamically balanced system is also statically balanced. Hence, the center of mass must be on the axis of rotation. On the other hand, if the masses are not in one plane, the necessary condition for eliminating the torque effects is that the axis of rotation must be the principal axis.

Choice (3) is correct.

After balancing the system in terms of torque by adding the mass M_B in plane B, the mass M_A is added to plane A at a suitable location so that the system can be balanced also in terms of inertia forces. The condition for this is as follows:

$$\sum_{i=1}^{n} M_i r_i \cos\theta_i + M_B r_B \cos\theta_B + M_A r_A \cos\theta_A = 0 \tag{7.6}$$

$$\sum_{i=1}^{n} M_i r_i \sin\theta_i + M_B r_B \sin\theta_B + M_A r_A \sin\theta_A = 0 \tag{7.7}$$

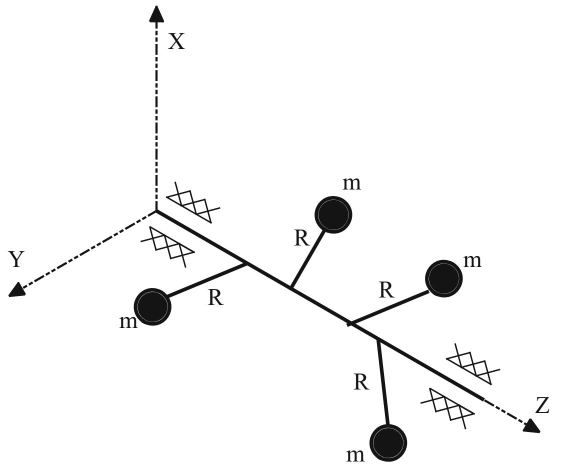


Fig. 7.4 Dynamically balancing set of masses

Example The system can be balanced … for dynamically balancing the set of masses shown on the axis (Fig. 7.4).

1) Only in terms of force
2) Only in terms of torque
3) By adding a mass at a specific point
4) By adding two masses at two specific points

Solution In dynamic balancing, the inertia effects (inertia torques and forces) at the bearings must become zero. Given that the masses are in several planes, two masses are needed at two locations, one for balancing the forces and the other for balancing the torques.

Choice (4) is correct.

7.2.3 Graphical Method for Balancing

Consider a plane state. The mathematical condition for the dynamic balance of masses can be written in the following vectorial form:

$$\sum \vec{F} = \sum (M \vec{r} \omega^2) = \sum (\frac{W}{g} \vec{r} \omega^2) = \frac{\omega^2}{g} \sum (W \vec{r}) \quad \Rightarrow \quad \sum (W \vec{r}) = 0 \tag{7.8}$$

Since ω^2/g is constant for all masses, balance is established by satisfying Eq. 7.8. The $W \vec{r}$ of each mass is vector with the same direction as the inertia force. In

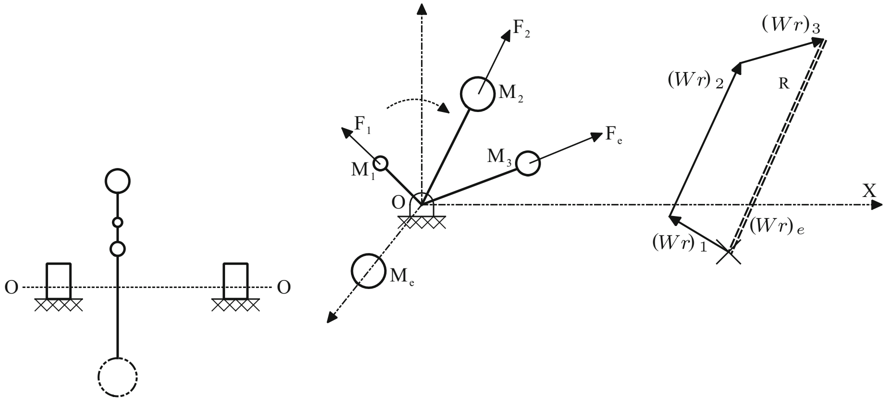


Fig. 7.5 Graphical method for balancing

Fig. 7.5, the $W\vec{r}$ of the three masses are known, and the $W\vec{r}$ of the balancing mass (M_e) must be determined in such a way that Eq. 7.8 is satisfied.

$$R = \sum \left(W\vec{r}\right)$$

$$W_e\vec{r}_e = -\vec{R}$$

In the polygon, the resultant R denotes the unbalance of three masses. Without a balancing mass, the resultant force of the rotating system is $R\omega^2/g$, which causes bending in the shaft and results in forces being developed in the bearings. In Fig. 7.5, the left bearing carries a larger portion of the unbalance load. The shaft bending and bearing forces are minimized by adding a balancing mass (Fig. 7.6).

Note Any number of masses rotating in a common radial plane can be balanced by adding only one mass.

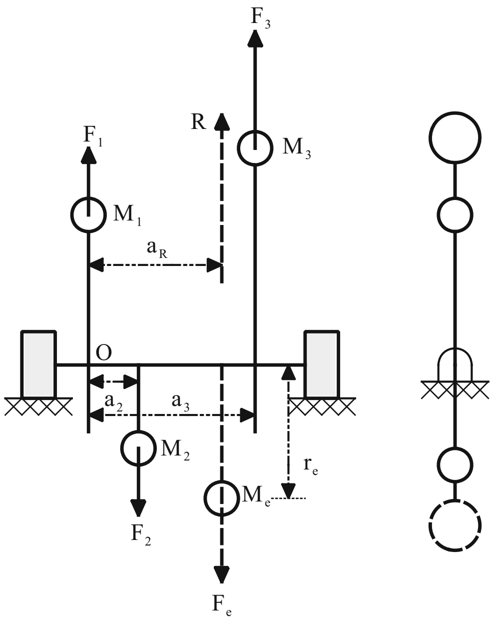
In cases where the masses of a rigid rotor are on the same axial plane, the inertia forces are parallel vectors.

Also here, the inertia forces are balanced by satisfying Eq. 7.8; however, the torque of the inertia forces must also be balanced.

For having balance in the torques, the torques from the inertia about any axis perpendicular to the plane should be zero.

$$\sum (Fa) = \sum (\frac{W}{g} r\omega^2 a) = \frac{\omega^2}{g} \sum \left(W\vec{r}a\right) = 0 \quad \Rightarrow \quad \sum \left(W\vec{r}a\right) = 0 \tag{7.9}$$

Fig. 7.6 Add one mass to make balance



where a is the moment arm of each inertia force. The line of action of R is determined by the principle of moments, where the moments are taken about the center of moment O. In this case, a_R is obtained by dividing $\sum\left(W\vec{r}a\right)$ by $\sum\left(W\vec{r}\right)$.

For the general case below, the two conditions $\sum\left(W\vec{r}\right)=0$ and $\sum\left(W\vec{r}a\right)=0$ must be simultaneously satisfied. In other words, the vector polygons of $W\vec{r}$ and $W\vec{r}a$ must be closed. The procedure is similar to the analytical case.

Figure 7.7C displays the torque polygon about the lateral plane A-A. Although the direction of the torque vectors is determined according to the right-hand rule, these vectors are displayed in the same direction as the inertia forces in Fig. 7.7C. In this figure, first the known torque vectors $(W\vec{r}\ a)_2$ and $(W\vec{r}\ a)_3$ are plotted, and the completing side $(W\vec{r}\ a)_b$ of the polygon expresses the torque vector required for balancing. The direction $(W\vec{r}\ a)_b$ represents an axial plane where M_b must lie. The magnitude of the vector $(W\vec{r}\ a)_b$ is obtained from the relationship $(W\vec{r}\ a)_b/a_b$ and is plotted in the force polygon in Fig. 7.7B. A second mass, such as M_a, is required to close the force polygon with $(W\vec{r})_a$ in order to balance the forces. $(W\vec{r})_a$ and $(W\vec{r})_b$ create the resultant R. By placing M_a in plane A-A, which causes the torque around A-A to become zero, the torque balance polygon (Fig. 7.7C) remains unchanged. As a result, both Eqs. 7.8 and 7.9 are satisfied. Figure 7.7D shows the torque polygon, in which the torques are considered about

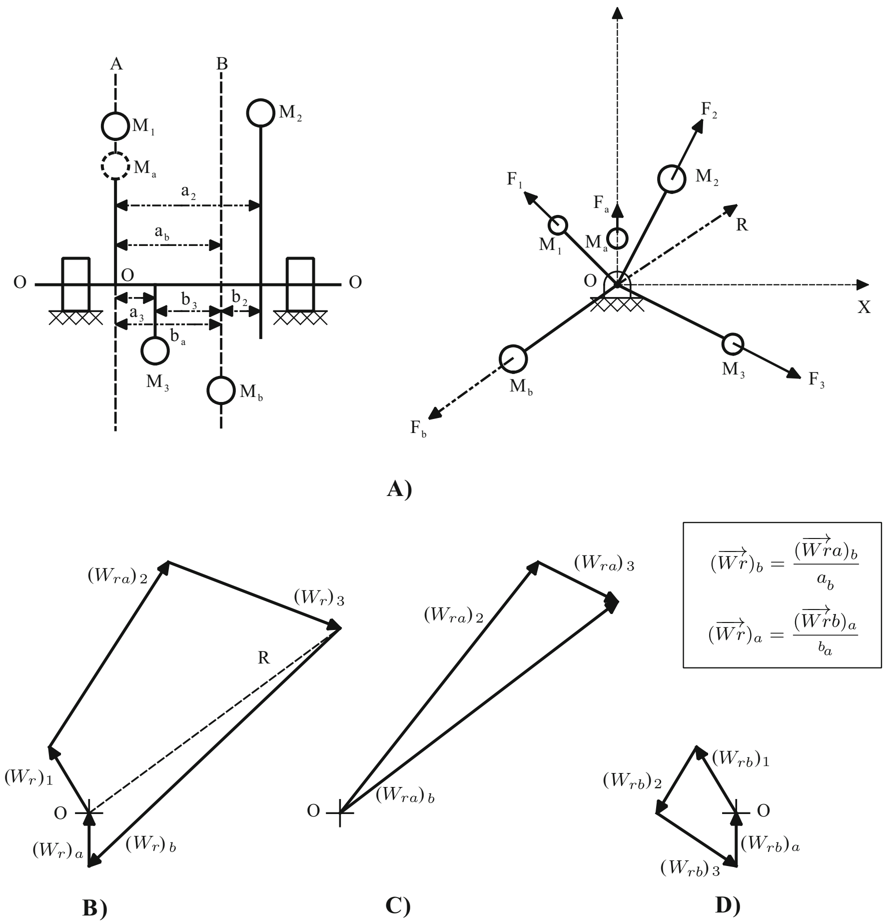


Fig. 7.7 Torque polygons

plane B-B to determine the torque vector $(W\overrightarrow{r}\ b)_a$ relative to M_a in the plane A-A. The vector $(W\overrightarrow{r})_a$ resulting from this polygon is similar to the previous solution.

7.3 Balancing of Reciprocating Masses

Balancing reciprocating mechanisms, such as the slider-crank mechanism, is widely used in machines such as internal combustion engines and compressors. Hence, extensive research has been conducted on balancing these mechanisms.

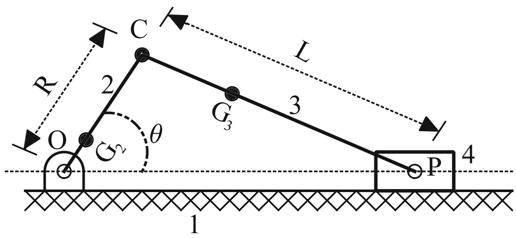


Fig. 7.8 A piston-crank system

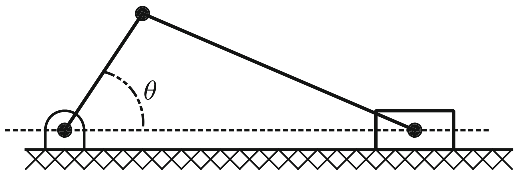


Fig. 7.9 A piston-crank system

In the slider-crank mechanism shown below, the velocity and acceleration of point P are simply obtained by differentiating its position with respect to time, as follows:

$$V_P = -R\omega(\sin\theta + \frac{R}{2L}\sin 2\theta) \tag{7.10}$$

$$A_P = -R\omega^2(\cos\theta + \frac{R}{L}\cos 2\theta) \tag{7.11}$$

In this equation, ω is the angular velocity of the crank. The other variables are shown below. The positive value of A_P in this equation shows that the piston accelerates away from O, and a negative value shows that the piston accelerates toward O (Fig. 7.8).

Example Which of the following is true about the piston-crank mechanism (Fig. 7.9)?

1) If the length of the piston rod is equal to that of the crank, the mechanism cannot work.
2) The maximum acceleration of the piston occurs at $\theta = 180^\circ$, i.e., the bottom dead center.
3) The longer the piston rod is, the closer the piston's motion to simple harmonic motion will be.
4) When the crank angle is 90°, the piston speed will be maximum.

Solution If we differentiate the position vector of the piston with respect to time, we obtain the velocity, and if we differentiate the velocity vector with respect to time, we obtain the acceleration of the piston. These relationships are as follows:

$$V_P = -R\omega(\sin\theta + \frac{R}{2L}\sin 2\theta)$$
$$A_P = -R\omega^2(\cos\theta + \frac{R}{L}\cos 2\theta)$$

According to the equations, the longer the piston rod (L) is, the smaller the second term will be, and the closer we will be to harmonic motion. Hence, Choice (3) is correct (Fig. 7.10).

Choice (1) is incorrect since, based on the equations, if R=L, the mechanism will have a velocity. Choice (2) is incorrect since the maximum acceleration occurs at $\theta = 0$. Choice (4) is also incorrect since $\theta = 90°$ does not make the derivative of velocity with respect to θ zero:

$$\frac{dV_P}{d\theta} = -R\omega^2\left(\cos\theta + \frac{R}{L}\cos 2\theta\right) \qquad if\ \theta = 90 \qquad \frac{dV_P}{d\theta} = \frac{R^2\,\omega}{L} \neq 0$$

Choice (3) is correct.

This mechanism can be dynamically made equivalent to the following figure. M''_C is the equivalent mass of link OC at point C, which is obtained by equating the inertia forces of the two states (Fig. 7.11).

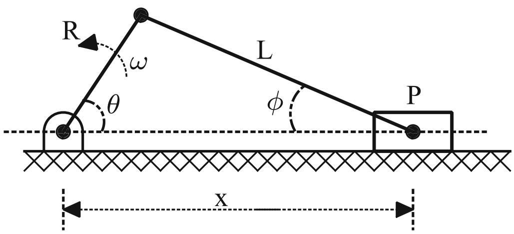


Fig. 7.10 Piston-crank mechanism analyze

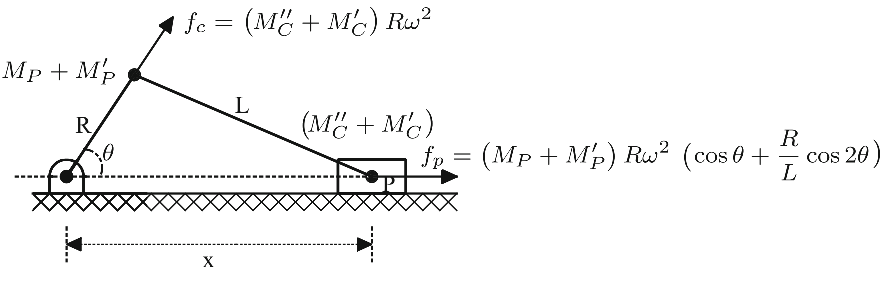


Fig. 7.11 Equivalent mechanism

We can write

$$M''_C R\omega^2 = M_C R_2 \omega^2 \quad \Rightarrow \quad M''_C = \frac{R_2}{R} M_C \tag{7.12}$$

where M_C is the mass of link OC, R_2 is the distance of its mass center from point O, and R is the length of this link. Moreover, M'_C and M'_P are the equivalent masses of link PC at points C and P, which can be determined as follows:

$$\begin{aligned} M'_C + M'_P &= M \\ M'_C h_C &= M'_P h_P \end{aligned} \tag{7.13}$$

where M_P is the slider mass, M is the mass of link PC, and h_P and h_C are the distances of the mass center of link PC from points P and C, respectively. Equation 7.13 considers the fact that the mass and mass center position are the same in the original and equivalent dynamic systems.

The inertia force in the piston pin is always opposite to the acceleration and equals

$$f_P = \underbrace{\left(M_P + M'_P\right) R\omega^2 \cos\theta}_{\text{Primary}} + \underbrace{\left(M_P + M'_P\right) R\omega^2 \frac{R}{L} \cos 2\theta}_{\text{Secondary}} \tag{7.14}$$

In this equation, a positive value obtained for f_P indicates that the inertia force is positive away from point O, and a negative value means the opposite. The first part of the equation is a function of the angle θ and is known as the initial inertia force. Since the second part is a function of 2θ, it is known as the secondary inertia force.

As such

$$f_C = (M''_P + M'_C) R\omega^2 \tag{7.15}$$

The forces θ and f_C affect point O for all values of f_P. Therefore, the vibratory force F_s is exerted at the crankshaft bearings for all crank conditions, and

$$f_O = f_C + f_P \tag{7.16}$$

Placing a balancing mass against link OC on the crank can balance the centrifugal force f_C and also partly balance the inertia force f_P. The optimal force of the balancing weight equals the centrifugal force f_C and a fraction of the inertia force f_P so that the maximum horizontal and vertical forces exerted on the bearing O are equal.

The vibration of a piston engine is basically due to the inertia forces of reciprocating masses at the piston's pin. The masses rotating with the crankshaft are usually balanced and do not transfer a vibratory force to the crankcase. Based on the free-body diagram of the slider-crank shown above (Fig. 7.12), the effect of the inertia force F of the reciprocating masses is the transfer of a force to the

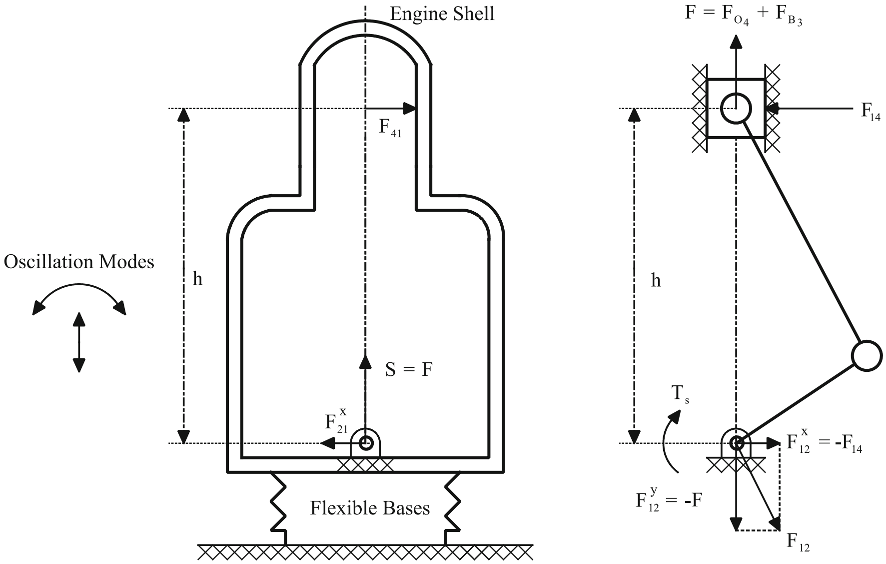


Fig. 7.12 Free-body diagram of a slider-crank

crankcase at the cylinder wall and the main bearings. The vertical component of the main bearing force F_{12}^{x} and the cylinder wall force F_{14} are equal in magnitude but opposite in direction. Since they are not codirectional, they generate the couple $F_{14}h$.

As shown by the free-body diagram of the crankcase (Fig. 7.12), the effect of the reciprocating masses on the crankcase is in the form of the vibration force S = F and the vibration couple $F_{14}h$. Since the magnitude and direction of both vibration force and couple vary over the operating cycle of the engine, forced vibrations are applied to the crankcase. If the crankcase is installed on flexible mounts, its vibration modes are the upper and lower modes resulting from the force S and the lateral vibration due to the couple.

Note One can reduce the resultant vibration force to zero by combining several slider-crank mechanisms and constructing a multi-cylinder engine. In this case, the individual vibration forces cancel each other out although the resultant vibration couple does not reduce to zero.

Figure 7.13 shows an n-cylinder inline engine. The angle made by crank 1 with the vertical (cylinder axis) is θ_1. Since the crankshaft is rigid, the angles between the cranks are constant. The angle between cranks 2 and 1 is ϕ_2, that between cranks 3 and 1 is ϕ_3, and so on.

Attention ϕ_1 is equal to zero.

Equation 7.14 determines the inertia forces exerted on various pistons, named $f_n, \ldots, f_3, f_2, f_1$.

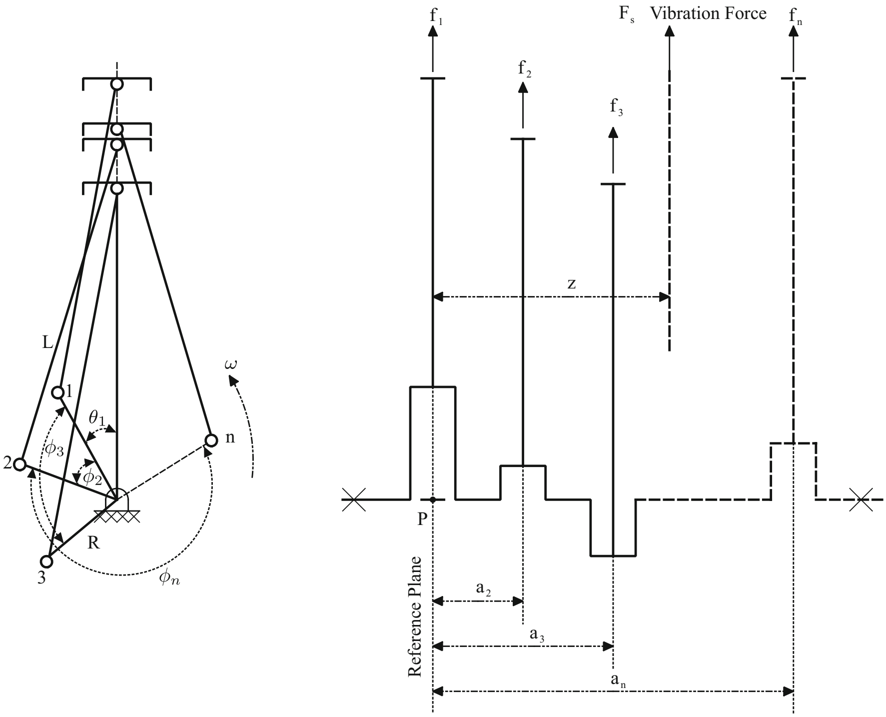


Fig. 7.13 An n-cylinder inline engine

Hence

$$f_1 = \left(M_P + M'_P\right) R\omega^2 \left[\cos\theta\,(\theta_1 + \phi_1) + \frac{R}{L}\cos 2(\theta_1 + \phi_1)\right]$$

$$f_2 = \left(M_P + M'_P\right) R\omega^2 \left[\cos\theta\,(\theta_1 + \phi_2) + \frac{R}{L}\cos 2(\theta_1 + \phi_2)\right]$$

.

. (7.17)

.

$$f_n = \left(M_P + M'_P\right) R\omega^2 \left[\cos\theta\,(\theta_1 + \phi_n) + \frac{R}{L}\cos 2(\theta_1 + \phi_n)\right]$$

The algebraic sum of the inertia forces equals the vibration force F_s. Therefore

$$F_s = \left(M_P + M'_P\right) R\omega^2 \left[\sum_{i=1}^{n} \cos\theta\,(\theta_1 + \phi_i) + \frac{R}{L}\sum_{i=1}^{n} \cos 2(\theta_1 + \phi_i)\right] \tag{7.18}$$

Using trigonometric relationships, this equation can be expressed as follows:

$$\begin{aligned} F_s = (M_P + M'_P)R\omega^2 \Bigg[& \cos\theta_1 \sum_{i=1}^{n} \cos\phi_i - \sin\theta_1 \sum_{i=1}^{n} \sin\phi_i + \\ & \frac{R}{L}\cos 2\theta_1 \sum_{i=1}^{n} \cos 2\phi_i - \frac{R}{L}\sin 2\theta \sum_{i=1}^{n} \sin 2\phi_i \Bigg] \end{aligned} \tag{7.19}$$

For the balancing of the inertia forces, their resultant must be zero for all crank states θ_1, i.e., $F_s = 0$. Therefore

$$\left.\begin{aligned} \textstyle\sum_{i=1}^{n} \cos\phi_i = 0 \\ \textstyle\sum_{i=1}^{n} \sin\phi_i = 0 \end{aligned}\right\} \text{(Necessary condition for the balance of the primary forces)} \tag{7.20}$$

$$\left.\begin{aligned} \textstyle\sum_{i=1}^{n} \cos 2\phi_i = 0 \\ \textstyle\sum_{i=1}^{n} \sin 2\phi_i = 0 \end{aligned}\right\} \text{(Necessary condition for the balance of the secondary forces)} \tag{7.21}$$

The torque of the vibration forces can be determined by obtaining the torques of the inertia forces relative to a point in their plane. For instance, if we consider the torques relative to point P and assume the moment of the vibration force be equal to M,

$$M = f_1 a_1 + f_2 a_2 + f_3 a_3 + \ldots + f_n a_n \tag{7.22}$$

This equation is equivalent to the product of each term in Eq. 7.19 and the relevant moment arm. Therefore

$$\begin{aligned} M = \left(M_P + M'_P\right) R\omega^2 \Bigg[& \cos\theta_1 \sum_{i=1}^{n} a_i \cos\phi_i - \sin\theta_1 \sum_{i=1}^{n} a_i \sin\phi_i + \\ & \frac{R}{L}\cos 2\theta_1 \sum_{i=1}^{n} a_i \cos 2\phi_i - \frac{R}{L}\sin 2\theta \sum_{i=1}^{n} a_i \sin 2\phi_i \Bigg] \end{aligned} \tag{7.23}$$

For the torque of the vibration force to be zero, M must equal zero for all angular positions θ_1. Hence

$$\left.\begin{array}{l}\sum_{i=1}^{n}\cos\phi_i=0\\ \sum_{i=1}^{n}\sin\phi_i=0\end{array}\right\}\text{(Necessary condition for the balance of the primary forces)}\tag{7.24}$$

$$\left.\begin{array}{l}\sum_{i=1}^{n}\cos 2\phi_i=0\\ \sum_{i=1}^{n}\sin 2\phi_i=0\end{array}\right\}\text{(Necessary condition for the balance of the secondary forces)}\tag{7.25}$$

The primary torques refer to the resultant of the torques of the primary forces of all the pistons about the location of piston 1. Similarly, the secondary torques refer to the resultant of the torques of the secondary forces of all the pistons about the location piston 1 (Fig. 7.14).

Example Consider the cylinder engine shown in the following figure with a crankshaft containing cranks with equal relative angles and longitudinal positions. The masses of the reciprocating parts of the different cylinders are equal. The causes of unbalance in this engine are as follows:

1) The primary force and primary torque
2) The secondary force and secondary torque

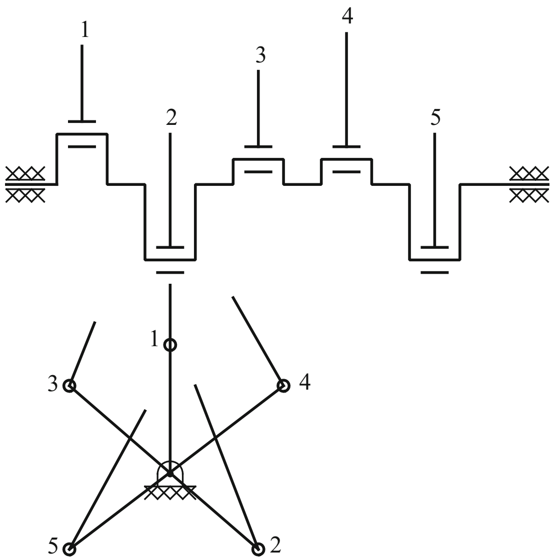


Fig. 7.14 A cylinder engine mechanism

3) Primary torque and secondary torque
4) Primary and secondary forces and torques

Solution

$$\phi_1 = 0 \qquad \phi_2 = 3\left(\frac{2\pi}{5}\right) \qquad \phi_3 = \frac{2\pi}{5} \qquad \phi_4 = 4\left(\frac{2\pi}{5}\right) \qquad \phi_5 = 2\left(\frac{2\pi}{5}\right)$$

$$a_1 = 0 \quad a_2 = a \quad a_3 = 2a \quad a_4 = 3a \quad a_5 = 4a$$

Primary force balance:

$$\sum \cos\phi_i = \cos(0) + \cos(\frac{6\pi}{5}) + \cos(\frac{2\pi}{5}) + \cos(\frac{8\pi}{5}) + \cos(\frac{4\pi}{5}) = 0$$

$$\sum \sin\phi_i = \sin(0) + \sin(\frac{6\pi}{5}) + \sin(\frac{2\pi}{5}) + \sin(\frac{8\pi}{5}) + \sin(\frac{4\pi}{5}) = 0$$

Secondary force balance:

$$\sum \cos 2\phi_i = \cos(0) + \cos(\frac{12\pi}{5}) + \cos(\frac{4\pi}{5}) + \cos(\frac{16\pi}{5}) + \cos(\frac{8\pi}{5}) = 0$$

$$\sum \sin 2\phi_i = \sin(0) + \sin(\frac{12\pi}{5}) + \sin(\frac{4\pi}{5}) + \sin(\frac{16\pi}{5}) + \sin(\frac{8\pi}{5}) = 0$$

Primary and secondary forces are not the causes of unbalance. Therefore, choice (3) is correct since the rest of the choices contain primary or secondary forces.

Choice (3) is correct.

Some Examples of "Balancing"

1. Three unbalanced masses $\mathbf{m_1 = m}$, $\mathbf{m_2 = m}$, **and** $\mathbf{m_3 = 2m}$ are arranged on the shaft so that static balance is established (Fig. 7.15). Which of the following is true?

$$\mathbf{r_1 = r_2 = r_3 = r}$$

1) $2\sin\alpha = \sin(\alpha + \beta)$
2) $\cos\alpha = 2cos(\alpha + \beta)$
3) $2\cos\alpha = \cos(\alpha + \beta)$
4) $\sin\alpha = 2sin(\alpha + \beta)$

2. Three disks A, B, and C are installed on a rigid and homogeneous shaft, as shown in Fig. 7.16. If the equivalent unbalance of each disk equals

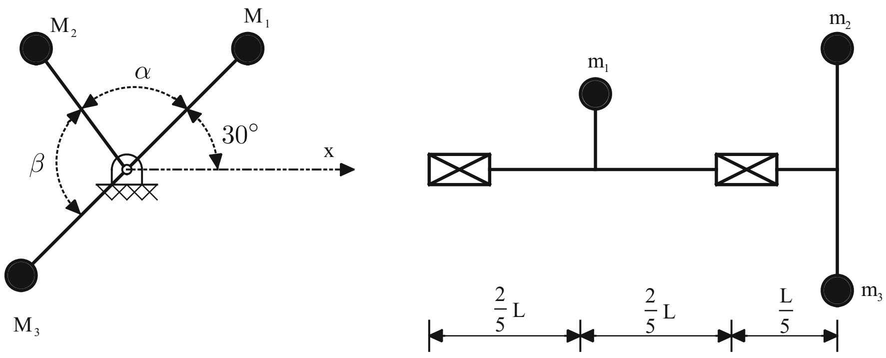


Fig. 7.15 Set of masses on a shaft

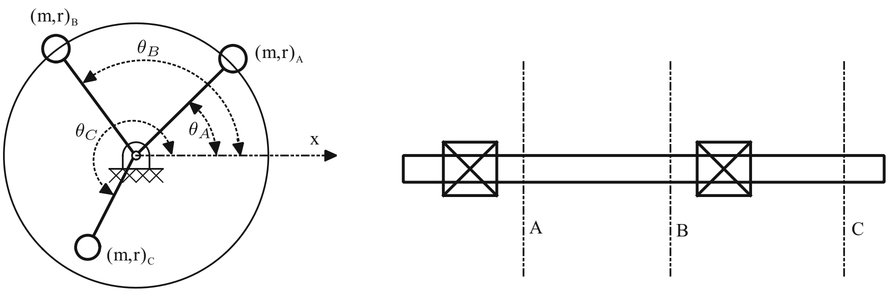


Fig. 7.16 Disks on a shaft

$(mr)_C$, $(mr)_B$, $(mr)_A$ and the system is statically balanced, which of the following equations is true?

1) $(mr)_A \cos\theta_A - (mr)_B \cos\theta_B - (mr)_C \cos\theta_C = 0$
2) $(mr)_A \cos\theta_A - (mr)_B \sin\theta_B - (mr)_C \cos\theta_C = 0$
3) $(mr)_A + (mr)_B \cos(\theta_B + \theta_A) + (mr)_C \cos(\theta_C + \theta_A) = 0$
4) $(mr)_A + (mr)_B \sin(\theta_B - \theta_A) + (mr)_C \cos(\theta_C - \theta_A) = 0$

3. Which of the following is true about the engine vibratory forces in the mechanism shown?

   1) The vibratory forces are balanced in the x- and y-directions.
   2) The vibratory forces are balanced only in the x-direction.
   3) The vibratory forces are balanced only in y-direction.
   4) The vibratory forces are unbalanced in the x- and y-directions (Fig. 7.17).

4. In the following rotating system, $a = c = 300$ mm, $R_1 = R_2 = 60$ mm, $b = 600$ mm, $m_1 = 1$ kg, and $m_2 = 3$ kg. Determine the reaction force at support A if the speed of shaft AB is 100 rpm (Fig. 7.18).
   1) 8.8 kN 2) 23.5 kN 3) 13.15 kN 4) 10.15 kN

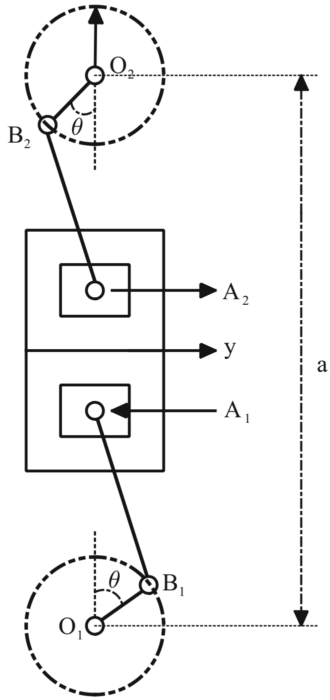


Fig. 7.17 Engine vibratory force

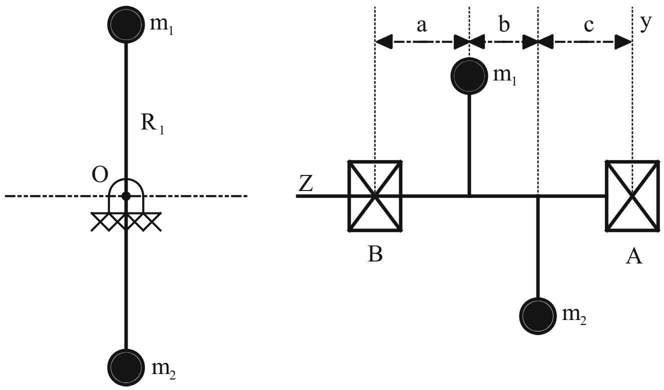


Fig. 7.18 Determine the reaction force at support A

5. Which of the following is true about the two-cylinder engine shown below (Fig. 7.19)?

   1) The primary vibration moment is balanced.
   2) The secondary vibration moment is balanced.

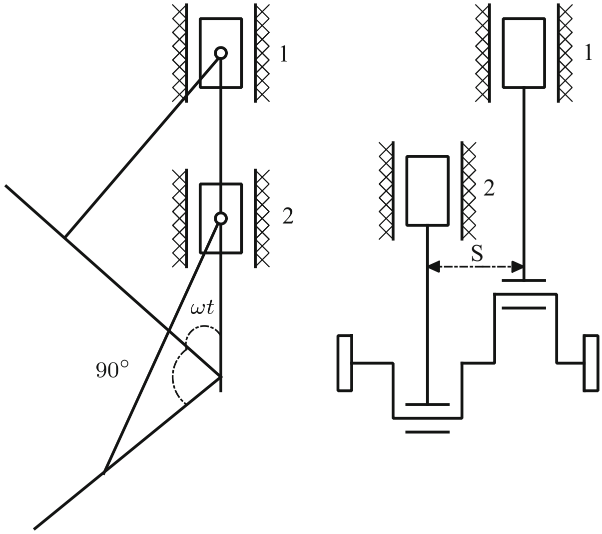


Fig. 7.19 A two-cylinder engine

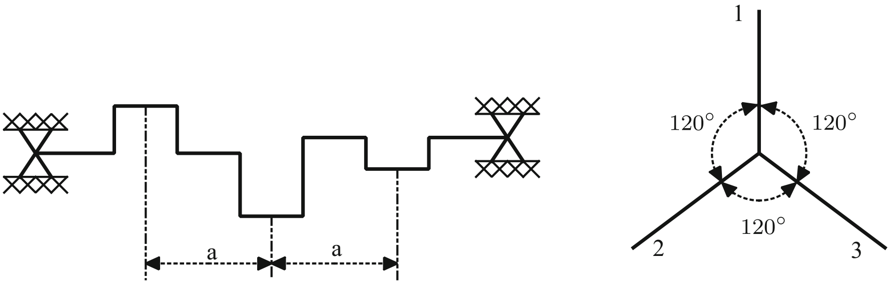


Fig. 7.20 A two-stroke three-cylinder engine

3) The primary vibration force is balanced.
4) The secondary vibration force is balanced.

6. A two-stroke three-cylinder engine has the following piston arrangement (Fig. 7.20). Which of the following is true?

1) Only the primary forces are balanced.
2) The primary and secondary forces are balanced.
3) Only the primary and secondary torques are balanced.
4) None of the forces or torques is balanced.

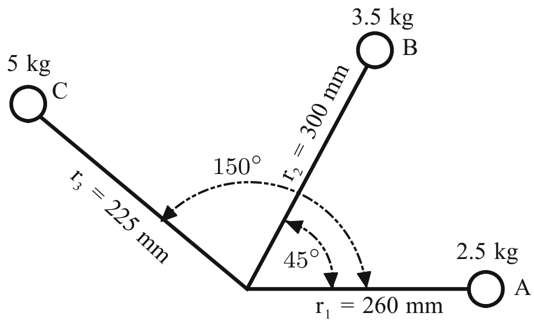


Fig. 7.21 Find the most suitable angle and balancing mass

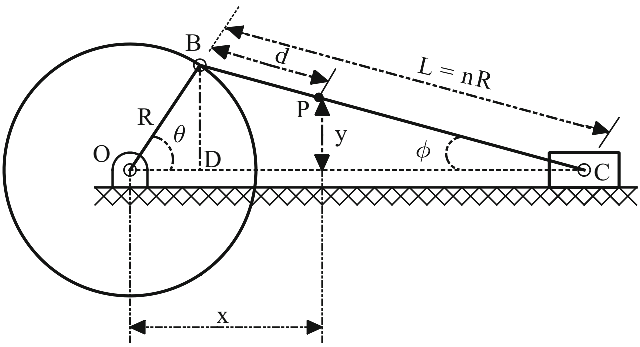


Fig. 7.22 A slider-crank mechanism

7. For the system to be balanced, which of the following angles and balancing masses seem more suitable (Fig. 7.21)?
   1) $\theta = \tan^{-1}(7.33) + 180$
      $m = 4.95\,\text{kg}$
   2) $\theta = \tan^{-1}(7.33)$
      $m = 6.6\,\text{kg}$
   3) $\theta = \tan^{-1}(7.33) + 180$
      $m = 6.6\,\text{kg}$
   4) None of them
8. The necessary and sufficient conditions for the system to be dynamically balanced are as follows:
   1) The center of mass must be on the axis of rotation.
   2) The axis of rotation must be the principal axis of the system.
   3) Both conditions (1) and (2) must be satisfied.
   4) The angular momentum vector must be normal to the angular velocity vector.
9. Which of the following pair of equations represent the velocity and acceleration of point C in the slider-crank mechanism shown (Fig. 7.22)?

1) $V = -R\omega_2(\sin\theta + \frac{1}{2n}\sin 2\theta)$

$A = -R{\omega_2}^2(\cos\theta + \frac{1}{n}\cos 2\theta)$

2) $V = R\omega_2(\cos\theta + \frac{1}{2n}\sin 2\theta)$

$A = R{\omega_2}^2(\sin\theta + \frac{1}{n}\cos 2\theta)$

3) $V = -R\omega_2(\sin\theta + \frac{1}{2n}\cos 2\theta)$

$A = R{\omega_2}^2(\cos\theta + \frac{1}{n}\sin 2\theta)$

4) $V = -R\omega_2(\cos\theta + \frac{1}{2n}\cos 2\theta)$

$A = -R{\omega_2}^2(\sin\theta + \frac{1}{n}\sin 2\theta)$

Answers for the Examples of "Balancing"

1. Choice (4) is correct.

   In static balance, the system does not tend to rotate regardless of its orientation. Hence, by rotating the system 60° counterclockwise and applying static balance conditions,

$$\sum m_i r_i \cos\theta_i = 0$$

$$m_1 r_1 \cos\left(\frac{\pi}{2}\right) + m_2 r_2 \cos\left(\frac{\pi}{2} + \alpha\right) + m_3 r_3 \cos\left(\frac{\pi}{2} + \alpha + \beta\right) = 0$$

$$r_1 = r_2 = r_3 = r \qquad m_1 = m_2 = m \qquad m_3 = 2m$$

$$\sin\alpha - 2\sin(\alpha + \beta) = 0 \quad \Rightarrow \quad \sin\alpha = 2\sin(\alpha + \beta)$$

2. Choice (4) is correct.

   In static balance, the system does not tend to rotate regardless of its orientation (Fig. 7.23). By rotating the system by θ_A clockwise and applying static balance conditions,

$$\sum m_i r_i \cos\theta_i = 0$$

   In this state, the angles made by masses A, B, and C with the horizon are zero, $(\theta_B - \theta_A)$, and $(\theta_C - \theta_A)$, respectively. Therefore

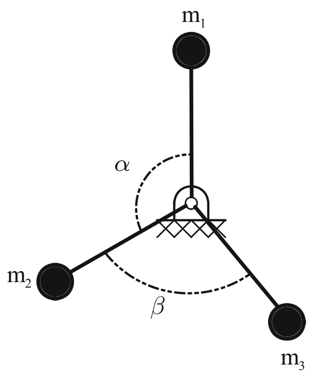
$$(mr)_A + (mr)_B \cos(\theta_B + \theta_A) + (mr)_C \cos(\theta_C + \theta_A) = 0$$

   Hence, choice (4) is correct.

3. Choice (1) is correct.

   In the reciprocating mechanism, the forces exerted on the bearing include the inertia force of the crank O_1 and the inertia force O_2 of the slider. These forces

Fig. 7.23 Set of masses



are opposite to each other. Therefore, the forces are balanced in both the x- and y-directions.

4. Choice (3) is correct.

   f_2 and f_1 are inertia forces and do not have components in the x-direction; thus, R_A does not have a component in the x-direction.

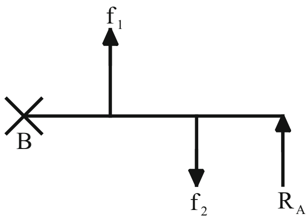
$$\omega = \frac{2\pi N}{60} = \frac{2\pi * 100}{60} = 10.47 \text{ rad/s}$$

$$\sum M_B = 0 \qquad R_A * 1200 - f_2 * 900 + f_1 * 300 = 0 \qquad (1)$$

$$f_1 = m_1 R_1 \omega^2 = 1 * 0.06 * (10.47)^2 = 6.57 \text{ N}$$

$$f_2 = m_2 R_2 \omega^2 = 3 * 0.06 * (10.47)^2 = 19.73 \text{ N}$$

Substituting f_2 *and* f into Equation (1) results in $R_A = 13.15$ N. Its positive value validates the chosen direction.



5. Choice (4) is correct.

ϕ is the angle between the cranks; therefore

$$\phi_1 = 0 \qquad \phi_2 = 90$$
$$a_1 = 0 \qquad a_2 = s$$

$$\text{Primary force balance} \begin{cases} \sum \cos\phi_i = \cos 0 + \cos 90 = 0 \\ \sum \sin\phi_i = \sin 0 + \sin 90 = 1 \neq 0 \end{cases}$$

$$\text{Secondary force balance} \begin{cases} \sum \cos 2\phi_i = \cos 0 + \cos 180 = 0 \\ \sum \sin 2\phi_i = \sin 0 + \sin 180 = 0 \end{cases}$$

We can see that the secondary forces are balanced. Therefore, choice (4) is correct.

The primary and secondary torques can be examined in the forms mentioned in the text.

6. Choice (2) is correct.

$$\phi_1 = 0 \qquad \phi_2 = \frac{2\pi}{3} \qquad \phi_3 = 4\frac{\pi}{3}$$
$$a_1 = 0 \qquad a_2 = a \qquad a_3 = 2a$$

$$\text{Primary force balance} \begin{cases} \sum \cos\phi_i = \cos 0 + \cos\frac{2\pi}{3} + \cos\frac{4\pi}{3} = 0 \\ \sum \sin\phi_i = \sin 0 + \sin\frac{2\pi}{3} + \sin\frac{4\pi}{3} = 0 \end{cases}$$

$$\text{Secondary force balance} \begin{cases} \sum \cos 2\phi_i = \cos 0 + \cos\frac{4\pi}{3} + \cos\frac{8\pi}{3} = 0 \\ \sum \sin 2\phi_i = \sin 0 + \sin\frac{4\pi}{3} + \sin\frac{8\pi}{3} = 0 \end{cases}$$

It can be seen that the primary and secondary forces are balanced.

7. Choice (4) is correct.

Based on the equations required for static and dynamic balance,

(1)

$$\sum_{i=1}^{n} M_i R_i \cos\theta_i + M_A R_A \cos\theta_A = 0$$

$$\Rightarrow \quad 2.5 + 260 * \cos 0 + 3.5 * 300 * \cos 60 + 5 * 225 * \cos 150$$

$$+ M_A R_A \cos\theta_A = 0$$

(2)

$$\sum_{i=1}^{n} M_i R_i \cos\theta_i + M_A R_A \sin\theta_A = 0$$

$$\Rightarrow \quad 2.5 + 260 * \sin 0 + 3.5 * 300 * \sin 60 + 5 * 225 * \sin 150$$

$$+ M_A R_A \sin\theta_A = 0$$

Taking the unknowns to one side and dividing the two equations, $\tan\theta_A =$ 7.33 $\Rightarrow \begin{cases} \theta_A = \tan^{-1}(7.33) \\ \theta_A = \tan^{-1}(7.33) + 180 \end{cases}$

However, based on Equation (2) and the fact that all the terms in this equation are positive, $M_A R_A \sin\theta_A$ must be negative to satisfy this equation; therefore, $\theta_A = \tan^{-1}(7.33) + 180$. However, according to the existing data and the absence of the radius of the balancing mass, it is impossible to calculate this mass. Therefore, none of the choices is correct.

8. Choice (3) is correct.
9. Choice (1) is correct.

Based on the previous discussions for the slider-crank mechanism,

$$V = -R\omega(\sin\theta + \frac{R}{2L}\sin 2\theta)$$

$$A = -R\omega^2(\cos\theta + \frac{R}{L}\cos 2\theta)$$

By substituting L=nR in the above equation, we obtain choice (1).

This problem could also be solved without the above equations. This is because if the velocities in the choices are differentiated, only choice (1) provides the correct acceleration.