An Optimal Fourth-Order Iterative Method for Multiple Roots of Nonlinear Equations

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1 Introduction

The problems of real world natural phenomena are usually express by nonlinear equation which exact roots are infeasible due to inherent complexities. Analytical methods for solving equations are not applicable for such type of equations. Due to this inconvenience, we use iterative methods for solving nonlinear equation. Most of researchers in Numerical Analysis are trying to construct iterative methods for solving nonlinear equations.

Newton's method is one popular iterative method for solving nonlinear equations, which quadraticaly converges for simple roots but linear for multiple roots. For a nonlinear equation $\zeta(\upsilon) = 0$ having multiple roots with multiplicity $\omega \geq 1$ modified Newton's methods is given as:

$$\upsilon_{n+1} = \upsilon_n - \omega \frac{\zeta(\upsilon_n)}{\zeta'(\upsilon_n)}, \tag{1}$$

This iterative method (1) is quadraticaly convergence [1, 2].

Chebyshev's Method for multiple is given as follows:

$$\upsilon_{n+1} = \upsilon_n - \frac{\omega(3-\omega)}{2} \frac{\zeta(\upsilon_n)}{\zeta'(\upsilon_n)} - \frac{\omega^2}{2} \frac{\left(\zeta(\upsilon_n)^2\right)\zeta'(\upsilon_n)}{\zeta'(\upsilon_n)^3}, \tag{2}$$

which is third order of convergence [1]. However, these are one point iteration functions. The one point iteration functions required higher-order derivative to increase

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the order of convergence (see in Traub [1] and Ostrowski [2]). To reduce this difficulties, several research scholar whose are in the field of Numerical Analysis, are trying to get multipoint iteration function of higher-order [3].

In the year 2009, Shengguo et al. [4] construct an iterative scheme for multiple root which is fourth-order of convergence.

$$y_n = \upsilon_n - \frac{2\omega}{\omega+1}\frac{\zeta(\upsilon_n)}{\zeta'(\upsilon_n)}$$

$$\upsilon_{n+1} = \upsilon_n - \frac{\frac{1}{2}\omega(\omega-2)\left(\frac{\omega}{\omega+2}\right)^{\omega}\zeta'(y_n) - \frac{\omega^2}{2}\zeta'(\upsilon_n)}{\zeta'(\upsilon_n) - \left(\frac{\omega}{\omega+2}\right)^{\omega}\zeta'(y_n)} \tag{3}$$

Li et al. [5] introduced a fourth-order scheme for multiple roots, in 2010.

$$y_n = \upsilon_n - \frac{2\omega}{\omega+2}\frac{\zeta(\upsilon_n)}{\zeta'(\upsilon_n)}$$

$$\upsilon_{n+1} = \upsilon_n - a_3\frac{\zeta(\upsilon_n)}{\zeta'(y_n)} - \frac{\zeta(\upsilon_n)}{b_1\zeta'(\upsilon_n) + b_2\zeta'(y_n)} \tag{4}$$

where

$$a_3 = -\frac{(\omega-2)\omega\left(\frac{\omega}{\omega+2}\right)^{\omega}(\omega+2)^3}{2\left(\omega^3-4\omega+8\right)}$$

$$b_1 = -\frac{\left(\omega^3-4\omega+8\right)^2}{\omega\left(\omega^2+2\omega-4\right)\left(\omega^4+4\omega^3-4\omega^2-16\omega+16\right)}$$

$$b_2 = \frac{\omega^2\left(\frac{\omega}{\omega+2}\right)^{-\omega}\left(\omega^3-4\omega+8\right)}{\left(\omega^2+2\omega-4\right)\left(\omega^4+4\omega^3-4\omega^2-16\omega+16\right)}$$

In 2019, Bhel and Al-Hamadan [5] presented a fourth-order method for multiple roots which is optimal:

$$y_n = \upsilon_n - \omega\frac{\zeta(\upsilon_n)}{\zeta'(\upsilon_n)}$$

$$z_n = \upsilon_n - \omega\frac{\zeta(y_n)}{\zeta'(\upsilon_n)}\left(\frac{1-\mu}{1-2\mu}\right)Q(\mu). \tag{5}$$

where $\mu = \left(\frac{\zeta(y_n)}{\zeta(\upsilon_n)}\right)^{\frac{1}{\omega}}$ and $Q(\mu)$ is weight function.

In this article, we present an optimal fourth-order iterative scheme for multiple roots using weight function. We check the behaviour of the developed scheme a using numbers nonlinear examples. Form the result, it is notice that the developed method perform better as compare to other standard iterative schemes available in the literature. The remaining part of this article is sort out as follows. In the second section, we are presenting two fourth-order iterative schemes function using weight function and the proof is also provided. In the third section, we present the comparison result of existing methods with the new methods. Conclusion is presented in the last section.

2 Development of the Method

In the year 2019, Francisco et al. [6] present an iterative scheme which is written as follows:

$$y_n = \upsilon_n - \frac{\zeta(\upsilon_n)}{\zeta(\upsilon_n)}$$

$$\upsilon_{n+1} = \upsilon_n - \frac{\zeta^{(2)}(\upsilon_n) + \zeta(\upsilon_n)\zeta(y_n) + 2\zeta^{(2)}(y_n)}{\{\zeta(\upsilon_n)\zeta'(\upsilon_n)\}} \tag{6}$$

The order of convergence of the scheme defined in Eq. (6) is four for the nonlinear functions having simple roots. We are trying to improve the method in Eq. (6) to an iterative scheme for solving nonlinear equations having multiple roots. The scheme is given as follows:

$$y_n = \upsilon_n - \omega\frac{\zeta(\upsilon_n)}{\zeta'(\upsilon_n)}$$

$$\upsilon_{n+1} = \upsilon_n - \omega\Gamma(t_n)\frac{\zeta^2(\upsilon_n) + \zeta(\upsilon_n)\zeta(y_n) + 2\zeta^2(y_n)}{\zeta(\upsilon_n)\zeta'(\upsilon_n)} \tag{7}$$

where $t_n = \left(\frac{\zeta(y_n)}{\zeta(\upsilon_n)}\right)^{\frac{1}{\omega}}$

Theorem 1 If $\zeta : R \to R$ has a multiple zero β with multiplicity $\omega = 2$ and is sufficiently differentiable function in the neighbourhood of the roots β. . Then, the order of convergence of iterative methods defined in Eq. (7) is four, if $\Gamma(t_n)$ satisfies $\Gamma(0) = 1$, $\Gamma'(0) = 1$, and $\Gamma''(0) = 2$, then (7) has the following error equation

$$\upsilon_{n+1} - \beta = \left(-\frac{1}{48}k_1^3\big(\Gamma^{(3)}(0) - 27\big) - \frac{k_2k_1}{4}\right)e_n^4 + O\big(e_n^5\big) \tag{8}$$

Proof
Since β is a root of $\zeta(\upsilon)$ with multiplicity $\omega = 2$, let $e_n = \upsilon_n - \beta$ be the error at the nth iteration. Using Taylor's series expansion on $\zeta(\upsilon_n)$ and $\zeta'(\upsilon_n)$ we get:

$$\zeta(\upsilon_n) = \frac{\zeta''(\beta)}{2!} e_n^2 \big(1 + e_n k_1 + e_n^2 k_2 + e_n^3 k_3 + e_n^4 k_4 + O(e_n)^5\big) \tag{9}$$

And

$$\zeta'(\upsilon_n) = \frac{\zeta^{(2)}(\beta)}{2!} e_n^2 (2 + 3k_1 e_n + 4k_2 e_n^2 + 5k_3 e_n^3 + 6k_4 e_n^4 + O(e_n)^5 \tag{10}$$

where $k_i = \frac{2!}{(2+i)!} \frac{\zeta^{(2+i)}(\beta)}{\zeta^{(2)}(\beta)}$, $i = 1, 2, 3, \ldots$
Using (9) and (10) in the second step of (7)

$$y_n - \beta = \frac{k_1 e_n^2}{2} + \left(k_2 - \frac{3k_1^2}{4}\right) e_n^3 + \left(\frac{9k_1^3}{8} - \frac{5k_2 k_1}{2} + \frac{3k_3}{2}\right) e_n^4 + O\big(e^5\big) \tag{11}$$

Taylor's expansion of $\zeta(y_n)$ about β, we have

$$\zeta(y_n) = \frac{\zeta^{(2)}(\beta)}{2!} e_n^2 \left(\frac{1}{4} k_1^2 e_n^2 + k_1 \left(k_2 - \frac{3k_1^2}{4}\right) e_n^3 + \left(\frac{29k_1^4}{16} - 4k_2 k_1^2 + \frac{3}{2} k_3 k_1 + k_2^2\right) e_n^4\right) + O\big(e_n^5\big) \tag{12}$$

Using (9) and (12), we get

$$\begin{aligned} \upsilon_{n+1} - \beta = {} & (1 - \Gamma(0)) e_n + \frac{1}{2}\, k_1 \big(\Gamma(0) - \Gamma'(0)\big) e_n^2 \\ & + \left(k_2 \big(\Gamma(0) - \Gamma'(0)\big) - \frac{1}{8}\, k_1^2 \,\big(-10\, \Gamma'(0) + \Gamma''(0) + 8\Gamma(0)\big)\right) e_n^3 \\ & + \frac{1}{48} \left(\begin{array}{l} k_1^3 \,\big(27 \,\big(-5\, \Gamma'(0) + \Gamma''(0) + 4\, \Gamma(0)\big) - \Gamma^{(3)}(0)\big) \\ \quad - 12\, k_2\, k_1 \,\big(-17\, \Gamma'(0) + 2\, \Gamma''(0) + 14\, \Gamma(0)\big) + 72\, k_3 \,\big(\Gamma(0) - \Gamma'(0)\big) \end{array}\right) e_n^4 \\ & + O\big(e_n^5\big) \end{aligned} \tag{13}$$

Using (9) (12) and (13) in the last step of (7).

$$\begin{aligned} \upsilon_{n+1} - \beta = {} & (1 - \Gamma(0)) e_n + \frac{1}{2} k_1 \big(\Gamma(0) - \Gamma'(0)\big) e_n^2 \\ & + \left(k_2 \big(\Gamma(0) - \Gamma'(0)\big) - \frac{1}{8} k_1^2 \big(-10\Gamma'(0) + \Gamma''(0) + 8\Gamma(0)\big)\right) e_n^3 \\ & + \frac{1}{48} \left(\begin{array}{l} k_1^3 \big(27\big(-5\Gamma'(0) + \Gamma''(0) + 4\Gamma(0)\big) - \Gamma^{(3)}(0)\big) \\ \quad - 12 k_2 k_1 \big(-17\Gamma'(0) + 2\Gamma''(0) + 14\Gamma(0)\big) + 72 k_3 \big(\Gamma(0) - \Gamma'(0)\big) \end{array}\right) e_n^4 \\ & + O\big(e_n^5\big) \end{aligned} \tag{14}$$

If we are putting $\Gamma(0) = 1$, $\Gamma'(0) = 1$ and $\Gamma''(0) = 2$, error equation become

$$\upsilon_{n+1} - \beta = \left(-\frac{1}{48}k_1^3\big(\Gamma^{(3)}(0) - 27\big) - \frac{k_2 k_1}{4}\right)e_n^4 + O\big(e_n^5\big) \tag{15}$$

Thus the proof is completed.

Theorem 2 If $\zeta : R \to R$ has a multiple zero β with multiplicity $\omega \geq 3$ and is sufficiently differentiable function in the neighbourhood of the roots β. Then, the order of convergence of iterative methods defined in Eq. (7) is four, if $\Gamma(t_n)$ satisfies $\Gamma(0) = 1$, $\Gamma'(0) = 1$, and $\Gamma''(0) = 4$, then (7) has the following error equation

$$\upsilon_{n+1} - \beta = \left(\frac{k_1^3\big(\Gamma^{(3)}(0) + 3\omega + 27\big) - 6k_2 k_1 \omega}{6\omega^3}\right)e_n^4 + O\big(e_n^5\big) \tag{16}$$

Proof
The proof is same as the proof of theorem 1.

3 Numerical Results

In this section, we deal with computational aspects of the proposed scheme with other existing methods such as fourth-order methods given in Eq. (3) denoted as (LM), method in Eq. (4) denoted as (LCM) and Behl Method (BM) given in Eq. (5) by applying on various nonlinear examples. In Tables 1, 2, 3 and 4 we have presented $|\zeta(\upsilon_n)|$, absolute of difference between the successive iterations $|\upsilon_n - \upsilon_{n-1}|$. Approximate roots. (υ_n). obtained after completion of 4 iterations and the computational order of convergence (COC) for each example are also presented. The COC is obtained by using the following formula [1]:

$$\rho \approx \frac{Log\left|\frac{\upsilon_{n+1}-\beta}{(\upsilon_n-\beta)}\right|}{Log\left|\frac{\upsilon_n-\beta}{\upsilon_{n-1}-\beta}\right|}$$

Table 1 Convergence behaviour for $\zeta_1(\upsilon)$

Method	υ_4	$\|\upsilon_n - \upsilon_{n-1}\|$	$\|\zeta(\upsilon)\|$	COC
LM	1.2917332924436028	5.5927×10^{-40}	1.1200×10^{-312}	4.0000
LCM	1.2917332924436028	6.1797×10^{-36}	1.1200×10^{-312}	4.0000
BM	1.2917332924436028	1.1567×10^{-38}	2.0891×10^{-301}	4.0000
NMM	1.2917332924436028	7.0179×10^{-41}	7.664×10^{-319}	4.0000

Table 2 Convergence behaviour for $\zeta_2(\upsilon)$

Method	υ_4	$\lvert\upsilon_n - \upsilon_{n-1}\rvert$	$\lvert\zeta(\upsilon)\rvert$	COC
LM	2.0000000000000000	6.0530×10^{-57}	$3.5493 \times 10^{-11259}$	4.0000
LCM	2.0000000000000000	2.9024×10^{-57}	$3.3971 \times 10^{-11259}$	4.0000
BM	2.0000000000000000	4.8282×10^{-59}	$8.1432 \times 10^{-11619}$	4.0000
NMM	2.0000000000000000	7.0179×10^{-61}	$7.6647 \times 10^{-11719}$	4.0000

Table 3 Convergence behaviour for $\zeta_3(\upsilon)$

Method	υ_4	. $\lvert\upsilon_n - \upsilon_{n-1}\rvert$	$\lvert\zeta(\upsilon)\rvert$	COC
LM	2.8500000000000000	6.3883×10^{-84}	7.5412×10^{-370}	4.0000
LCM	2.8500000000000000	6.3883×10^{-84}	7.5412×10^{-370}	4.0000
BM	2.8500000000000000	1.3949×10^{-49}	9.8347×10^{-396}	4.0000
NMM	2.8500000000000000	1.4067×10^{-49}	1.0569×10^{-397}	4.0000

Table 4 Convergence behaviour for $\zeta_4(\upsilon)$

Method	υ_4	$\lvert\upsilon_n - \upsilon_{n-1}\rvert$	$\lvert\zeta(\upsilon)\rvert$	COC
LM	2.4905398276083051	1.0312×10^{-19}	2.3983×10^{-1505}	4.0000
LCM	2.4905398276083051	4.0539×10^{-20}	8.8944×10^{-1538}	4.0000
BM	2.4905398276083051	1.9627×10^{-19}	1.7767×10^{-1477}	4.0000
NMM	2.4905398276083051	1.9627×10^{-19}	2.1011×10^{-1667}	4.0000

The numerical result has been carried out with Mathematica \$12\$ software, * denotes for divergence.

Example 1
$\zeta_1(\upsilon) = \left(9 - 2\upsilon - 2\upsilon^4 + cos2\upsilon\right)\left(5 - \upsilon^4 - sin^2\upsilon\right)$, $\upsilon_0 = 1.5, \omega = 2.$

Example 2
$\zeta_2(\upsilon) = \left((\upsilon - 1)^3 - 1\right)^{50}$, $\upsilon_0 = 2.1, \omega = 50$

Example 3
$\zeta_3(\upsilon) = \upsilon^4 + 11.50\upsilon^3 + 47.49\upsilon^2 + 83.06325\upsilon + 51.23266875$, $\upsilon_0 = -2.7$ *and* ω = 2.

Example 4
$\zeta_4(\upsilon) = \left(e^{-\upsilon^2+\upsilon+3} - \upsilon + 2\right)^{20}$, $\upsilon_0 = 2.1$ and $\omega = 20$,

4 Conclusion

We construct a new fourth-order iterative method base on Newton's and Francisco's methods using weight functional approach. It attends its optimal order. We test, by comparing the newly developed methods with other methods having the same convergence are order using several nonlinear equations having multiple roots. The results obtained from the comparison tables illustrate the superiority of the method over the existing methods, despite choosing the same test problem and the same initial guess. Tables confirm that our iterative method has smaller value of $|\zeta(\upsilon)|$ and $|\upsilon_n - \upsilon_{n-1}|$.

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