Improved Polar Extensions of an Inequality for a Complex Polynomial with All Zeros on a Circle

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1 Introduction

Let $f(w) = \sum_{j=0}^{m} c_j w^j$ be a polynomial of degree m and let

$$\text{Max}(f, r) = \max_{|w|=r} |f(w)|.$$

If $f(w)$ has no zero in $|w| < \rho$, $\rho \geq 1$, Malik [8] proved that

$$\text{Max}\big(f', 1\big) \leq \frac{m}{1+\rho} \text{Max}(f, 1), \tag{1}$$

for which the equality holds for the polynomial $(w + \rho)^m$.

A natural question that arise is whether there exists an analogous inequality of (1) for $f(w)$ having no zero in $|w| < \rho$, $\rho \leq 1$. In this regard, Govil [5, 6] proved the following two results.

Theorem 1 [5] *If $f(w)$ is a polynomial of degree m having no zero in $|w| < \rho$, $\rho \leq 1$, then*

$$\text{Max}\big(f', 1\big) \leq \frac{m}{1+\rho^m} \text{Max}(f, 1), \tag{2}$$

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provided that $\left|f'(w)\right|$ *and* $\left|F'(w)\right|$ *attain their maxima at the same point on the circle* $|w| = 1$*, where*

$$F(w) = w^m \overline{f\left(\frac{1}{\overline{w}}\right)}.$$

Theorem 2 [6] *If* $f(w)$ *is a polynomial of degree m having all its zeros on* $|w| = \rho$*,* $\rho \leq 1$*,*

$$\text{Max}\big(f', 1\big) \leq \frac{m}{\rho^m + \rho^{m-1}} \text{Max}(f, 1). \tag{3}$$

In literature we find refinements, generalizations and extensions of Theorems 1 and 2 by involving some coefficients of the polynomial $f(w)$ (see [2–4]).

Definition 1 *Let* $f(w)$ *be a polynomial of degree m and let* β *be any complex number. The polar derivative of* $f(w)$ *with respect to the point* β*, denoted by* $D_\beta f(w)$*, is defined as*

$$D_\beta f(w) = mf(w) + (\beta - w) f'(w).$$

$D_\beta f(w)$ is a polynomial of degree at most $m - 1$. It can be considered as a generalized form of the ordinary derivative of $f(w)$ with respect to w due to the fact that

$$\lim_{\beta \to \infty} \frac{D_\beta f(w)}{\beta} = f'(w).$$

Polar derivative extension of (1) was proved by Aziz [1], who under the same hypothesis on $f(w)$ proved that

$$\text{Max}\big(D_\beta f(w), 1\big) \leq m\left(\frac{\rho + |\beta|}{1 + \rho}\right)\text{Max}(f, 1), \quad \text{where } |\beta| \geq 1. \tag{4}$$

2 Lemmas

We need the following results to prove our results.

The following lemma is due to Govil and Rahman [7].

Lemma 1 [7] *If* $f(w)$ *is a polynomial of degree m having all its zeros on* $|w| = \rho$*,* $\rho \leq 1$*, then on* $|w| = 1$.

$$\left|f'(w)\right| + \left|F'(w)\right| \leq m \text{ Max}(f, 1). \tag{5}$$

The next two lemmas are due to Barchand et al. [3].

Lemma 2 [3] *If* $f(w) = \sum_{j=0}^{n} c_j w^j$ *is a polynomial of degree* m *having all its zeros in* $|w| \leq \rho$, $\rho \leq 1$, *then*

$$\frac{1}{\rho m}\left(\frac{c_{m-j}}{c_m}\right) \leq 1. \tag{6}$$

Lemma 3 [3] *If* $f(w) = \sum_{j=0}^{n} c_j w^j$ *is a polynomial of degree* m *having all its zeros on* $|w| = \rho$, $\rho \leq 1$, *then on* $|w| = 1$.

$$\text{Max}\big(f', 1\big) \leq \frac{m}{\rho^m + \rho^{m-1}} E_\rho \ \text{Max}(f, 1), \tag{7}$$

where

$$E_\rho = \frac{(1+|t|)\big(\rho^2+|t|\big)+\rho(m-1)\big|s-t^2\big|}{(1-|t|)\big(1-\rho+\rho^2+\rho|t|\big)+\rho(m-1)\big|s-t^2\big|},$$

$$t = \frac{1}{\rho m}\left(\frac{\overline{c}_{m-1}}{\overline{c}_m}\right)$$

$$s = \frac{2}{\rho^2 m(m-1)}\left(\frac{\overline{c}_{m-2}}{\overline{c}_m}\right).$$

Lemma 3 is, in fact, a refinement of Theorem 2 due to Govil [6] (see Remark 3).

Remark 1
Under the hypothesis of Lemma 3, we have (see [3, Lemma 2.6]).

$$|t| = \frac{1}{\rho m}\left|\frac{\overline{c}_{m-1}}{\overline{c}_m}\right| \leq 1.$$

The following lemma was proved by Malik [8].

Lemma 4 [8] *If* $f(w)$ *is a polynomial of degree* m *having no zero in* $|w| < \rho$, $\rho \geq 1$, *then*

$$\rho\big|f'(w)\big| \leq \big|F'(w)\big|, \tag{8}$$

where $F(w) = w^m \overline{f\left(\frac{1}{\overline{w}}\right)}$.

Lemma 5 *If* $f(w)$ *is a polynomial of degree* m *having all its zeros on* $|w| = \rho$, $\rho \leq 1$, *then on* $|w| = 1$.

$$\big|F'(w)\big| \leq \rho\big|f'(w)\big|. \tag{9}$$

Proof
Since $f(w)$ has all its zeros on $|w| = \rho$, $\rho \leq 1$, then $F(w)$ has all its zeros on $|w| = \frac{1}{\rho}$, $\frac{1}{\rho} \geq 1$. This implies that $F(w)$ has no zeros in $|w| < \frac{1}{\rho}$, $\frac{1}{\rho} \geq 1$. Thus, applying Lemma 1 to the polynomial $F(w)$ we get, on $|w| = 1$.

$$\frac{1}{\rho}\left|F'(w)\right| \leq \left|f'(w)\right|.$$

$$\therefore \left|F'(w)\right| \leq \rho\left|f'(w)\right|.$$

3 Main Results

In this paper, we prove polar extensions of Theorem 2. Precisely, we prove the following result.

Theorem 3 *If* $f(w) = \sum_{j=0}^{n} c_j w^j$ *is a polynomial of degree m having all its zeros on* $|w| = \rho$, $\rho \leq 1$, *and* $\beta \in \mathbb{C}$ *with* $|\beta| \geq 1$, *then on* $|w| = 1$.

(a)

$$\left|D_\beta f(w)\right| \leq m\left(1 + \frac{|\beta| - 1}{\rho^m + \rho^{m-1}} E_\rho\right)\text{Max}(f, 1). \tag{10}$$

(b)

$$\left|D_\beta f(w)\right| \leq m\left(\frac{|\beta| + \rho}{\rho^m + \rho^{m-1}} E_\rho\right)\text{Max}(f, 1). \tag{11}$$

where

$$E_\rho = \frac{(1 - |t|)\left(\rho^2 + |t|\right) + \rho(m-1)\left|s - t^2\right|}{(1 - |t|)\left(1 - \rho + \rho^2 + \rho|t|\right) + \rho(m-1)\left|s - t^2\right|}, \tag{12}$$

$$t = \frac{1}{\rho m}\left(\frac{\overline{c}_{m-1}}{\overline{c}_m}\right) \tag{13}$$

$$s = \frac{2}{\rho^2 m(m-1)}\left(\frac{\overline{c}_{m-2}}{\overline{c}_m}\right). \tag{14}$$

Proof
Let $F(w) = w^m \overline{f\left(\frac{1}{\overline{w}}\right)}$. Then on $|w| = 1$.

$$\left|F'(w)\right| = \left|mf(w) - wf'(w)\right|. \tag{15}$$

For $\beta \in \mathbb{C}$, polar derivative of $f(w)$ with respect to β is

$$D_\beta f(w) = mf(w) + (\beta - w)f'(w).$$

Therefore,

$$\left|D_\beta f(w)\right| = \left|mf(w) + (\beta - w)f'(w)\right|$$

$$\leq \left|mf(w) - wf'(w)\right| + |\beta|\left|f'(w)\right| \tag{16}$$

$$= \left|F'(w)\right| + |\beta|\left|f'(w)\right| \tag{17}$$

$$= \left|F'(w)\right| + \left|f'(w)\right| + (|\beta| - 1)\left|f'(w)\right|. \tag{18}$$

Using (5) of Lemma 1 in (18) we have on $|w| = 1$

$$\left|D_\beta f(w)\right| \leq m\text{Max}(f, 1) + (|\beta| - 1)\text{Max}\big(f', 1\big). \tag{19}$$

Now using (7) of Lemma 3 in (19) we have for $|w| = 1$

$$\left|D_\beta f(w)\right| \leq m\left(1 + \frac{|\beta| - 1}{\rho^m + \rho^{m-1}} E_\rho\right)\text{Max}(f, 1),$$

which proves (a).

We now prove (b).

Using (9) of Lemma 5 in (17), we get for $|w| = 1$

$$\left|D_\beta f(w)\right|$$

$$\leq \rho\left|f'(w)\right| + |\beta|\left|f'(w)\right| \tag{20}$$

$$\leq (|\beta| + \rho)\text{Max}\big(f', 1\big). \tag{21}$$

Using (7) of Lemma 3, we obtain for $|w| = 1$.

$$\left|D_\beta f(w)\right| \leq m\left(\frac{|\beta| + \rho}{\rho^m + \rho^{m-1}} E_\rho\right)\text{Max}(f, 1),$$

which proves *(b)*.

Remark 2
Dividing inequalities (10) and (11) by $|\beta|$ and taking $|\beta| \to \infty$, both reduce to

$$\operatorname{Max}\big(f', 1\big) \leq \frac{m}{\rho^m + \rho^{m-1}} E_\rho \operatorname{Max}(f, 1),$$

which is the conclusion of Lemma 3, where E_ρ is given by (12).

Remark 3
Inequalities (10) and (11) are improved extensions of inequality (3) to polar derivative. In other words, the ordinary form of (10) and (11) obtained in Remark 2 is an improvement of (3). To see this, it is sufficient to show that

$$E_\rho \leq 1.$$

That is,

$$\frac{(1-|t|)\big(\rho^2+|t|\big)+\rho(m-1)\big|s-t^2\big|}{(1-|t|)\big(1-\rho+\rho^2+\rho|t|\big)+\rho(m-1)\big|s-t^2\big|} \leq 1.$$

i.e., $\rho^2 + |t| \leq 1 - \rho + \rho^2 + \rho|t|$
which holds as $\rho \leq 1$ and $|t| \leq 1$ (by Remark 1).

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