Chapter 6
Self-duality, Pieri Formula and Cauchy Formulas

Abstract Self-duality (evaluation symmetry), which we are going to discuss below, is one of the most characteristic properties of Macdonald polynomials. In this chapter, we explain how the Pieri formulas (multiplication formula by e_r) are obtained from the action of Macdonald–Ruijsenaars operators $D_x^{(r)}$ through the self-duality. We also investigate the Cauchy formula and the dual Cauchy formula for Macdonald polynomials and the relevant kernel identities.

6.1 Self-duality and Pieri Formula

We have seen in the previous chapter that, for generic $q, t \in \mathbb{C}^*$ the Macdonald polynomials $P_\lambda(x) = P_\lambda(x; q, t)$ $(\lambda \in \mathcal{P}_n)$ are joint eigenfunctions of the commuting family of Macdonald–Ruijsenaars q-difference operators $D_x^{(r)}$ $(r = 1, \ldots, n)$, and that they form a $\mathbb{C}$ basis of the ring of symmetric polynomials $\mathbb{C}[x]^{\mathfrak{S}_n}$:

$$\mathbb{C}[x]^{\mathfrak{S}_n} = \bigoplus_{\lambda \in \mathcal{P}_n} \mathbb{C}\, P_\lambda(x), \quad D_x(u) P_\lambda(x) = d_\lambda(u) P_\lambda(x). \tag{6.1}$$

Note that, under our assumption $|q| < 1$, the genericity condition for t is fulfilled if $t^k \notin q^{\mathbb{Z}_{<0}}$ for $k = 1, \ldots, n-1$, and in particular, if $|t| < 1$. Also, if we regard q, t as variables (indeterminates), the (monic) Macdonald polynomials $P_\lambda(x)$ are determined uniquely as symmetric polynomials with coefficients in the field $\mathbb{Q}(q, t)$ of rational functions in (q, t); their coefficients are regular in the domain $|q| < 1$, $|t| < 1$.

6.1.1 Principal Specialization

As to the values of Schur functions $s_\lambda(x)$ at the base point $x = t^\delta$, we gave two explicit formulas in Propositions 3.1 and 3.2. Those evaluation formulas are generalized to the case of Macdonald polynomials as follows.

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Theorem 6.1 (Principal specialization) *For any* $\lambda \in \mathcal{P}_n$*, the value of* $P_\lambda(x)$ *at* $x = t^\delta$ *is given explicitly by*

$$P_\lambda(t^\delta) = t^{n(\lambda)} \prod_{s\in\lambda} \frac{1 - t^{n-l'_\lambda(s)} q^{a'_\lambda(s)}}{1 - t^{l_\lambda(s)+1} q^{a_\lambda(s)}} = \frac{t^{n(\lambda)} \prod_{i=1}^{n} (t^{n-i+1}; q)_{\lambda_i}}{\prod_{1\le i\le j\le n} (t^{j-i+1} q^{\lambda_i - \lambda_j}; q)_{\lambda_j - \lambda_{j+1}}}, \tag{6.2}$$

where $n(\lambda) = \sum_{i=1}^{n} (i-1)\lambda_i$ *and, for each* $s = (i, j) \in \lambda$, $l'_\lambda(s) = i - 1$ *and* $a'_\lambda(s) = j - 1$ *denote the* co-leg length *and the* co-arm length*, respectively.*

The proof of this evaluation formula at $x = t^\delta$ will be given in Sect. 6.3, under the assumption that Theorem 6.2 (below) of self-duality holds.

6.1.2 Self-duality

At this moment, we know at least that $P_\lambda(t^\delta) \neq 0$ as a rational function of (q, t), since the Schur functions are the special case of Macdonald polynomials where $t = q$, i.e. $P_\lambda(x; q, q) = s_\lambda(x)$. Keeping this in mind, we normalize the Macdonald polynomials as

$$\widetilde{P}_\lambda(x) = \frac{P_\lambda(x)}{P_\lambda(t^\delta)} \qquad (\lambda \in \mathcal{P}_n) \tag{6.3}$$

so that $\widetilde{P}_\lambda(t^\delta) = 1$. Then we have the following self-duality (evaluation symmetry).

Theorem 6.2 (Self-duality) *The normalized Macdonald polynomials* $\widetilde{P}_\lambda(x) = P_\lambda(x)/P_\lambda(t^\delta)$ *satisfy*

$$\widetilde{P}_\lambda(t^\delta q^\mu) = \widetilde{P}_\mu(t^\delta q^\lambda) \tag{6.4}$$

for all pairs $(\lambda, \mu) \in \mathcal{P}_n \times \mathcal{P}_n$.

We regard $x = t^\delta q^\lambda$ as the *position variables* and $\xi = t^\delta q^\mu$ as the *spectral variables*. Then (6.4) means that the *normalized* Macdonald polynomial $\widetilde{P}_\lambda(t^\delta q^\mu)$, regarded as a function of $(\lambda, \mu) \in \mathcal{P}_n \times \mathcal{P}_n$, is invariant under the exchange of position and spectral variables on the discrete set.

We include a proof of Theorem 6.2 due to Koornwinder [14, 20] in Sect. 6.4.

6.1.3 Pieri Formula

For each $\lambda \in \mathcal{P}_n$, the Macdonald polynomial $P_\lambda(x)$ multiplied by an elementary symmetric function $e_r(x)$ $(r = 1, \dots, n)$ can be expanded into a linear combination of Macdonald polynomials:

$$e_r(x)P_\mu(x) = \sum_{\substack{\lambda\in\mathcal{P}_n \\ \lambda\le\mu+(1^r)}} \psi'_{\lambda/\mu}\, P_\lambda(x), \tag{6.5}$$

with some coefficients $\psi'_{\lambda/\mu} \in \mathbb{Q}(q,t)$. This type of expansion formula is called the *Pieri formula*. In order to describe the expansion coefficients in the Pieri formula, we introduce certain rational functions in (q,t).

For each pair $\lambda, \mu \in \mathcal{P}$ of partitions with $\mu \subseteq \lambda$ (i.e. $\mu_i \le \lambda_i$ for all $i \ge 1$), we define a rational function $\psi_{\lambda/\mu}(q,t) \in \mathbb{Q}(q,t)$ by

$$\psi_{\lambda/\mu}(q,t) = \prod_{1\le i\le j\le \ell(\mu)} \frac{(t^{j-i+1}q^{\mu_i-\mu_j};q)_{\lambda_i-\mu_i}}{(t^{j-i}q^{\mu_i-\mu_j+1};q)_{\lambda_i-\mu_i}} \frac{(t^{j-i}q^{\mu_i-\lambda_{j+1}+1};q)_{\lambda_i-\mu_i}}{(t^{j-i+1}q^{\mu_i-\lambda_{j+1}};q)_{\lambda_i-\mu_i}}, \tag{6.6}$$

and set

$$\psi'_{\lambda/\mu}(q,t) = \psi_{\lambda'/\mu'}(t,q). \tag{6.7}$$

Recall that a skew diagram λ/μ is called a *horizontal strip* ("h-strip" for short) if the complement $\lambda\backslash\mu$ contains at most one square in each column. Similarly, we say that a skew diagram λ/μ is a *vertical strip* ("v-strip" for short) if the complement $\lambda\backslash\mu$ contains at most one square in each row. Note that $\psi_{\lambda/\mu}(q,t) = 0$ unless λ/μ is a horizontal strip, and that $\psi'_{\lambda/\mu}(q,t) = 0$ unless λ/μ is a vertical strip.

Theorem 6.3 (Pieri formula) *For each $\mu \in \mathcal{P}_n$ and $r = 1,\ldots,n$, $P_\mu(x)$ multiplied by $e_r(x)$ is expanded in terms of Macdonald polynomials as*

$$e_r(x)P_\mu(x) = \sum_{\substack{\lambda/\mu:\,\text{v-strip} \\ |\lambda/\mu|=r}} \psi'_{\lambda/\mu}\, P_\lambda(x) \tag{6.8}$$

with coefficients $\psi'_{\lambda/\mu} = \psi'_{\lambda/\mu}(q,t)$ in (6.6)–(6.7), where the sum is over all partitions $\lambda \in \mathcal{P}_n$ with $\mu \subseteq \lambda$, $|\lambda| = |\mu| + r$, such that the skew diagram λ/μ is a vertical strip.

Theorem 6.3 will be proved in Sects. 6.2 and 6.3 before Sect. 6.4, assuming that Theorem 6.2 holds.

6.2 Self-duality Implies the Pieri Formula

Note that the fact that $P_\lambda(t^\delta) \ne 0$ (as a rational function of (q,t)) follows from the principal specialization of the special case $t = q$, where $P_\lambda(x|q,q) = s_\lambda(x)$. Assuming that the self-duality (6.4) has been established, we explain here how one can obtain the Pieri formula (6.8) and the evaluation formula (6.2) from the q-difference equations for $P_\lambda(x)$, by way of the self-duality.

For each $r = 1, \dots, n$, the eigenfunction equation

$$D_x^{(r)} \widetilde{P}_\lambda(x) = e_r(t^\delta q^\lambda) \widetilde{P}_\lambda(x) \tag{6.9}$$

implies

$$\sum_{|I|=r} A_I(x) \widetilde{P}_\lambda(q^{\varepsilon_I} x) = e_r(t^\delta q^\lambda) \widetilde{P}_\lambda(x), \tag{6.10}$$

where $\varepsilon_I = \sum_{i \in I} \varepsilon_i$. Evaluating this formula at $x = t^\delta q^\mu$ ($\mu \in \mathcal{P}_n$), we obtain

$$\sum_{|I|=r} A_I(t^\delta q^\mu) \widetilde{P}_\lambda(t^\delta q^{\mu+\varepsilon_I}) = e_r(t^\delta q^\lambda) \widetilde{P}_\lambda(t^\delta q^\mu). \tag{6.11}$$

Suppose that $\nu = \mu + \varepsilon_I$ is *not* a partition, i.e. $\mu_{i-1} = \mu_i$ for some $i \in \{2, \dots, n\}$ with $i \in I$ and $i - 1 \notin I$. In such a case, we have

$$A_I(t^\delta q^\mu) = t^{\binom{|I|}{2}} \prod_{i \in I,\, j \notin J} \frac{t^{n-i+1} q^{\mu_i} - t^{n-j} q^{\mu_j}}{t^{n-i} q^{\mu_i} - t^{n-j} q^{\mu_j}} = 0 \tag{6.12}$$

since $t x_i - x_j = t^{n-i+1} q^{\mu_i} - t^{n-i+1} q^{\mu_{i-1}} = 0$ ($i \in I$, $j = i - 1 \notin I$). This means that the sum in the left-hand side of (6.10) is over all $I \subseteq \{1, \dots, n\}$ with $|I| = r$ such that $\nu = \mu + \varepsilon_I$ is a partition. A skew partition ν/μ is a *vertical strip* if and only if $\nu = \mu + \varepsilon_I$ for some $I \subseteq \{1, \dots, n\}$. In the following, for each pair $\mu, \nu \in \mathcal{P}_n$ with $\mu \subseteq \nu$, we set $A_{\nu/\mu} = A_I(t^\delta q^\mu)$ if ν/μ is a vertical strip with $\nu = \mu + \varepsilon_I$ and $A_{\nu/\mu} = 0$ otherwise. Then we have

$$\sum_{\nu/\mu:\ \text{v-strip},\ |\nu/\mu|=r} A_{\nu/\mu} \widetilde{P}_\lambda(t^\delta q^\nu) = e_r(t^\delta q^\lambda) \widetilde{P}_\lambda(t^\delta q^\mu). \tag{6.13}$$

We now apply the self-duality (6.4) to obtain

$$\sum_{\nu/\mu:\ \text{v-strip},\ |\nu/\mu|=r} A_{\nu/\mu} \widetilde{P}_\nu(t^\delta q^\lambda) = e_r(t^\delta q^\lambda) \widetilde{P}_\mu(t^\delta q^\lambda). \tag{6.14}$$

This means that equality

$$e_r(x) \widetilde{P}_\mu(x) = \sum_{\nu/\mu:\ \text{v-strip},\ |\nu/\mu|=r} A_{\nu/\mu} \widetilde{P}_\nu(x) \tag{6.15}$$

holds for $x = t^\delta q^\lambda$ ($\lambda \in \mathcal{P}_n$). It also implies that (6.15) is an identity in the ring $\mathbb{C}[x]^{\mathfrak{S}_n}$ of symmetric polynomials, since a polynomial $f(x) \in \mathbb{C}[x]$ such that $f(t^\delta q^\lambda) = 0$ for all $\lambda \in \mathcal{P}_n$ must be zero as a polynomial in x. Namely, if the self-duality (6.4) has been established, the q-difference equations (6.9) for $\lambda \in \mathcal{P}_n$ implies the *Pieri formulas* (6.15) for the normalized Macdonald polynomials $\widetilde{P}_\mu(x)$ with coefficients $A_{\nu/\mu}$.

Exercise 6.1 Prove: if a polynomial $f(x) \in \mathbb{C}[x]$ vanishes at all points $x = t^{\delta} q^{\lambda}$ ($\lambda \in \mathcal{P}_n$), then $f(x) = 0$ as a polynomial in x.

Supposing that λ/μ is a vertical strip, we express λ as $\lambda = \mu + \varepsilon_I$ with a subset $I \subseteq \{1, \ldots, n\}$ with $|I| = |\lambda/\mu| = r$. In this setting, we derive an explicit formula for the Pieri coefficient

$$A_{\lambda/\mu} = A_I(t^{\delta} q^{\mu}) \tag{6.16}$$

for $\widetilde{P}_{\mu}(x)$. Since $\Delta(x) = \prod_{b=1}^{n} x^{b-1} \prod_{1 \le a < b \le n} (1 - x_a/x_b)$, we have

$$\begin{aligned} A_{\lambda/\mu} = A_I(t^{\delta} q^{\mu}) &= \frac{\Delta(t^{\delta+\varepsilon_I} q^{\mu})}{\Delta(t^{\delta} q^{\mu})} \\ &= t^{n(\varepsilon_I)} \prod_{\substack{1 \le a < b \le n \\ a \in I, b \notin I}} \frac{1 - t^{b-a+1} q^{\mu_a - \mu_b}}{1 - t^{b-a} q^{\mu_a - \mu_b}} \prod_{\substack{1 \le a < b \le n \\ a \notin I, b \in I}} \frac{1 - t^{b-a-1} q^{\mu_a - \mu_b}}{1 - t^{b-a} q^{\mu_a - \mu_b}}. \end{aligned} \tag{6.17}$$

We use the conjugate partitions $\lambda', \mu' \in \mathcal{P}$, noting that they satisfy the interlacing property

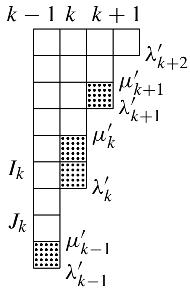
$$n \ge \lambda'_1 \ge \mu'_1 \ge \lambda'_2 \ge \mu'_2 \ge \lambda'_3 \ge \ldots. \tag{6.18}$$

Then, the subset $I \subseteq \{1, \ldots, n\}$ and its complement $J = \{1, \ldots, n\} \backslash I$ are parametrized as

$$I = \bigsqcup_{k \ge 1} I_k, \quad I_k = (\mu'_k, \lambda'_k],$$

$$J = \bigsqcup_{k \ge 1} J_k, \quad J_k = (\lambda'_k, \mu'_{k-1}]$$

$$(\lambda'_0 = \mu'_0 = n) \tag{6.19}$$



in the notation of an interval $(a, b] = \{k \in \mathbb{Z} \mid a < k \le b\}$ of integers. Note that, $\mu_i = k - 1, \lambda_i = k$ if $i \in I_k$ and $\mu_j = \lambda_j = k - 1$ if $j \in J_k$). Then we have

$$\begin{aligned} A_{\lambda/\mu} &= t^{n(\varepsilon_I)} \prod_{\substack{j \le i \\ a \in I_i \\ b \in J_j}} \frac{1 - t^{b-a+1} q^{i-j}}{1 - t^{b-a} q^{i-j}} \prod_{\substack{i < j \\ a \in J_j \\ b \in I_i}} \frac{1 - t^{b-a-1} q^{j-i}}{1 - t^{b-a} q^{j-i}} \\ &= t^{n(\varepsilon_I)} \prod_{j \le i} \frac{(t^{\mu'_{j-1} - \lambda'_i + 1} q^{i-j}; t)_{\lambda'_i - \mu'_i}}{(t^{\lambda'_j - \lambda'_i + 1} q^{i-j}; t)_{\lambda'_i - \mu'_i}} \prod_{i < j} \frac{(t^{\mu'_i - \mu'_{j-1}} q^{j-i}; t)_{\mu'_{j-1} - \lambda'_j}}{(t^{\lambda'_i - \mu'_{j-1}} q^{j-i}; t)_{\mu'_{j-1} - \lambda'_j}} \end{aligned} \tag{6.20}$$

and finally

$$A_{\lambda/\mu} == t^{n(\varepsilon_I)} \prod_{j\geq 1} (t^{n-\lambda'_j+1} q^{j-1}; t)_{\lambda'_j-\mu'_j}$$
$$\cdot \frac{\prod_{i<j} (t^{\mu'_i-\lambda'_j+1} q^{j-i-1}; t)_{\lambda'_j-\mu'_j}}{\prod_{i\leq j} (t^{\lambda'_i-\lambda'_j+1} q^{j-i}; t)_{\lambda'_j-\mu'_j}} \frac{\prod_{i\leq j} (t^{\mu'_i-\mu'_j} q^{j-i+1}; t)_{\mu'_j-\lambda'_{j+1}}}{\prod_{i\leq j} (t^{\lambda'_i-\mu'_j} q^{j-i+1}; t)_{\mu'_j-\lambda'_{j+1}}}. \quad (6.21)$$

In combinatorial terms of Young diagrams, this can be written alternatively as

$$A_{\lambda/\mu} = \frac{t^{n(\lambda)} \prod_{i=1}^{n} (t^{n-i+1}; q)_{\lambda_i}}{t^{n(\mu)} \prod_{i=1}^{n} (t^{n-i+1}; q)_{\mu_i}}$$
$$\cdot \frac{\prod_{s\in\mu\cap R_{\lambda/\mu}} (1 - t^{l_\mu(s)+1} q^{a_\mu(s)})}{\prod_{s\in\lambda\cap R_{\lambda/\mu}} (1 - t^{l_\lambda(s)+1} q^{a_\lambda(s)})} \frac{\prod_{s\in\mu\setminus R_{\lambda/\mu}} (1 - t^{l_\mu(s)} q^{a_\mu(s)+1})}{\prod_{s\in\lambda\setminus R_{\lambda/\mu}} (1 - t^{l_\lambda(s)} q^{a_\lambda(s)+1})}, \quad (6.22)$$

where $R_{\lambda/\mu} = I \times \mathbb{Z}_{>0}$ denotes the union of rows intersecting with the vertical strip λ/μ.

6.3 Principal Specialization: Evaluation at $x = t^\delta$

The normalized Macdonald polynomials $\widetilde{P}_\lambda(x)$ can be written as

$$\widetilde{P}_\lambda(x) = \frac{1}{a_\lambda} P_\lambda(x) = \frac{1}{a_\lambda} m_\lambda(x) + (\text{lower-order terms}), \quad a_\lambda = P_\lambda(t^\delta). \quad (6.23)$$

We compare the coefficients of $m_\lambda(x)$ of the both sides of (6.15) for $\lambda = \mu + \varpi_r$, $\varpi_r = \varepsilon_1 + \cdots + \varepsilon_r = (1^r)$. Then we obtain

$$\frac{1}{a_\mu} = A_{\lambda/\mu} \frac{1}{a_\lambda}, \quad \text{i.e.} \quad a_\lambda = a_\mu A_{\lambda/\mu} \quad (6.24)$$

for $\lambda = \mu + \varpi_r$.

We make use of this recurrence formula for the case where $\ell(\mu) \leq r$ and $\lambda = \mu + \varpi_r$. Since

$$A_{\lambda/\mu} = t^{\binom{r}{2}} \prod_{i=1}^{r} \frac{1 - t^{n-i+1} q^{\mu_i}}{1 - t^{r-i+1} q^{\mu_i}} = t^{n(\varpi_r)} \prod_{s\in\lambda\setminus\mu} \frac{1 - t^{n-l'_\lambda(s)} q^{a_\lambda(s)}}{1 - t^{l_\lambda(s)+1} q^{a_\lambda(s)}}, \quad (6.25)$$

by $a_\lambda = a_\mu A_{\lambda/\mu}$, we obtain

$$a_\lambda = a_\mu \cdot t^{n(\varpi_r)} \prod_{s\in\lambda\setminus\mu} \frac{1 - t^{n-l'_\lambda(s)} q^{a_\lambda(s)}}{1 - t^{l_\lambda(s)+1} q^{a_\lambda(s)}}, \quad (6.26)$$

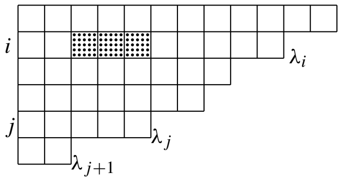
if $\ell(\mu) \leq r$ and $\lambda = \mu + \varpi_r$. Noting that any $\lambda \in \mathcal{P}_n$ is expressed as $\lambda = \varpi_{\lambda'_1} + \cdots + \varpi_{\lambda'_l}$, $l = \lambda_1$, we can apply the recurrence formula (6.26) to obtain

$$a_\lambda = t^{n(\lambda)} \prod_{s \in \lambda} \frac{1 - t^{n - l'_\lambda(s)} q^{a_\lambda(s)}}{1 - t^{l_\lambda(s)+1} q^{a_\lambda(s)}} = \prod_{s \in \lambda} \frac{t^{l'_\lambda(s)} - t^n q^{a'_\lambda(s)}}{1 - t^{l_\lambda(s)+1} q^{a_\lambda(s)}} \tag{6.27}$$

for any $\lambda \in \mathcal{P}_n$. In terms of the components of λ, a_λ is expressed alternatively as

$$a_\lambda = \frac{t^{n(\lambda)} \prod_{i=1}^{n} (t^{n-i+1}; q)_{\lambda_i}}{\prod_{1 \leq i \leq j \leq n} (t^{j-i+1} q^{\lambda_i - \lambda_j}; q)_{\lambda_j - \lambda_{j+1}}}. \tag{6.28}$$

Note that the pair (i, j) of indices with $1 \leq i \leq j \leq n$ in the denominator covers the sequence of squares $s = (i, k)$ with $k \in (\lambda_{j+1}, \lambda_j]$, for which $l_\lambda(s) = j - i$ and $a_\lambda(s) = \lambda_i - k$.



(6.29)

Formulas (6.26)–(6.27) are the explicit formulas for $P_\lambda(t^\delta) = a_\lambda$ in Theorem 6.1.

Also, the Pieri coefficients $\psi'_{\lambda/\mu}$ in (6.8) for $\lambda = \mu + \varepsilon_I$, $I \subseteq \{1, \ldots, n\}$ are obtained from (6.15) by

$$\psi'_{\lambda/\mu} = \frac{a_\mu}{a_\lambda} A_{\lambda/\mu}, \quad A_{\lambda/\mu} = A_I(t^\delta q^\mu). \tag{6.30}$$

Writing down this formula in terms of $\lambda, \mu \in \mathcal{P}_n$, we obtain the explicit formula for $\psi'_{\lambda/\mu} = \psi'_{\lambda/\mu}(q, t)$ as in (6.6). By (6.22) and (6.27) we obtain

$$\begin{aligned} \psi'_{\lambda/\mu} &= \frac{a_\mu}{a_\lambda} A_{\lambda/\mu} \\ &= \frac{\prod_{s \in \lambda} (1 - t^{l_\lambda(s)+1} q^{a_\lambda(s)})}{\prod_{s \in \mu} (1 - t^{l_\mu(s)+1} q^{a_\mu(s)})} \\ &\quad \cdot \frac{\prod_{s \in \mu \cap R_{\lambda/\mu}} (1 - t^{l_\mu(s)+1} q^{a_\mu(s)})}{\prod_{s \in \lambda \cap R_{\lambda/\mu}} (1 - t^{l_\lambda(s)+1} q^{a_\lambda(s)})} \frac{\prod_{s \in \mu \setminus R_{\lambda/\mu}} (1 - t^{l_\mu(s)} q^{a_\mu(s)+1})}{\prod_{s \in \lambda \setminus R_{\lambda/\mu}} (1 - t^{l_\lambda(s)} q^{a_\lambda(s)+1})} \\ &= \prod_{s \in \lambda \setminus R_{\lambda/\mu}} \frac{1 - t^{l_\lambda(s)+1} q^{a_\lambda(s)}}{1 - t^{l_\lambda(s)} q^{a_\lambda(s)+1}} \prod_{s \in \mu \setminus R_{\lambda/\mu}} \frac{1 - t^{l_\mu(s)} q^{a_\mu(s)+1}}{1 - t^{l_\mu(s)+1} q^{a_\mu(s)}}. \end{aligned} \tag{6.31}$$

In terms of the components of λ, μ, this formula can be rewritten as

$$\psi'_{\lambda/\mu} = \prod_{i \le j} \frac{(t^{\lambda'_i-\mu'_j+1}q^{j-i};t)_{\mu'_j-\lambda'_{j+1}}}{(t^{\lambda'_i-\mu'_j}q^{j-i+1};t)_{\mu'_j-\lambda'_{j+1}}} \frac{(t^{\mu'_i-\mu'_j}q^{j-i+1};t)_{\mu'_j-\lambda'_{j+1}}}{(t^{\mu'_i-\mu'_j+1}q^{j-i};t)_{\mu'_j-\lambda'_{j+1}}}$$
$$= \prod_{i \le j} \frac{(t^{\mu'_i-\lambda'_{j+1}+1}q^{j-i};t)_{\lambda'_i-\mu'_i}}{(t^{\mu'_i-\lambda'_{j+1}}q^{j-i+1};t)_{\lambda'_i-\mu'_i}} \frac{(t^{\mu'_i-\mu'_j}q^{j-i+1};t)_{\lambda'_i-\mu'_i}}{(t^{\mu'_i-\mu'_j+1}q^{j-i};t)_{\lambda'_i-\mu'_i}}. \tag{6.32}$$

This gives a proof of Theorem 6.3 (under the assumption that Theorem 6.2 holds). Note that the two expressions in (6.32) are transformed into each other through the formula

$$\frac{(q^l a;q)_k}{(a;q)_k} = \frac{(a;q)_{k+l}}{(a;q)_k(a;q)_l} = \frac{(q^k a;q)_l}{(a;q)_l} \qquad (k,l \in \mathbb{N}) \tag{6.33}$$

for q-shifted factorials.

6.4 Koornwinder's Proof of Self-duality

In this section, we present Koornwinder's inductive argument which proves the self-duality and the Pieri formula for $\widetilde{P}_\lambda(x)$ simultaneously (see Macdonald [20] and Koornwinder [14]).

For $\mu \in \mathcal{P}_n$ and $r = 0, 1, \ldots, n$, we consider the expansion of $e_r(x)\widetilde{P}_\mu(x)$ in terms of $\widetilde{P}_\lambda(x)$ $(\lambda \in \mathcal{P}_n)$:

$$e_r(x)\widetilde{P}_\mu(x) = \sum_{\lambda \in \mathcal{P}_n,\, \lambda \le \mu+\varpi_r} B_{\lambda/\mu}\widetilde{P}_\lambda(x). \tag{6.34}$$

The coefficients $B_{\lambda/\mu}$ are defined for all $\lambda \in \mathcal{P}_n$ such that $\lambda \le \mu + \varpi_r$; we set $B_{\lambda/\mu} = 0$ otherwise. For each pair $\lambda, \mu \in \mathcal{P}_n$ with $\mu \subseteq \lambda$, we set $A_{\lambda/\mu} = A_I(t^\delta q^\mu)$ if λ/μ is a vertical strip with $\lambda = \mu + \varepsilon_I$, $I \subseteq \{1, \ldots, n\}$, and $A_{\lambda/\mu} = 0$ otherwise.

We prove the following two statements for $\lambda \in \mathcal{P}_n$ simultaneously by the induction on $|\lambda|$ combined with the dominance order of partitions:

(a)$_\lambda$ $\widetilde{P}_\lambda(t^\delta q^\mu) = \widetilde{P}_\mu(t^\delta q^\lambda)$ for all $\mu \in \mathcal{P}_n$.

(b)$_\lambda$ Suppose that $r \in \{1, \ldots, n\}$ and $\lambda - \varpi_r \in \mathcal{P}_n$, and set $\kappa = \lambda - \varpi_r$. Then, $B_{\nu/\kappa} = A_{\nu/\kappa}$ for any $\nu \in \mathcal{P}_n$ with $\nu \le \lambda = \kappa + \varpi_r$.

For the induction, we use the partial order $\nu \preceq_{\text{d-dom}} \mu$ for $\nu, \mu \in \mathcal{P}_n$ defined by

$$\nu \preceq_{\text{d-dom}} \mu \iff |\nu| < |\mu| \;\text{ or }\; (|\nu| = |\mu| \text{ and } \nu \le \mu). \tag{6.35}$$

Statement (a)$_\mu$ holds for $\mu = 0$ since $\widetilde{P}_\lambda(t^\delta) = 1$ for all $\lambda \in \mathcal{P}_n$, while (b)$_\mu$ is empty for $\mu = 0$.

Assuming that $\lambda \in \mathcal{P}_n$ and $|\lambda| > 0$, we first prove $(\mathrm{b})_\lambda$. Suppose that $\kappa \in \mathcal{P}_n$, $r \in \{1, \dots, n\}$ and $\lambda = \kappa + \varpi_r$. By the argument of Sect. 6.2, (6.13), we know

$$e_r(t^\delta q^\mu)\widetilde{P}_\mu(t^\delta q^\kappa) = \sum_{\substack{\nu/\kappa:\,\text{v-strip}\\ |\nu/\kappa|=r}} A_{\nu/\kappa}\widetilde{P}_\mu(t^\delta q^\nu) \qquad (\mu \in \mathcal{P}_n). \tag{6.36}$$

Note that we have $\nu \le \kappa + \varpi_r = \lambda$ if ν/μ is a vertical strip with $|\nu/\kappa| = r$. On the other hand, we have

$$e_r(t^\delta q^\mu)\widetilde{P}_\kappa(t^\delta q^\mu) = \sum_{\nu \le \lambda} B_{\nu/\kappa}\widetilde{P}_\nu(t^\delta q^\mu) \qquad (\mu \in \mathcal{P}_n) \tag{6.37}$$

by (6.34). Since $|\kappa| < |\lambda|$, we have $\widetilde{P}_\kappa(t^\delta q^\mu) = \widetilde{P}_\mu(t^\delta q^\kappa)$ by the induction hypothesis. Also, we have $\widetilde{P}_\nu(t^\delta q^\mu) = \widetilde{P}_\mu(t^\delta q^\mu)$ for all pair $\mu, \nu \le \lambda$ by the induction hypothesis; in fact we have $\mu < \lambda$ or $\nu < \lambda$ if $\mu \ne \nu$. Hence we have

$$e_r(t^\delta q^\mu)\widetilde{P}_\mu(t^\delta q^\kappa) = \sum_{\nu \le \lambda} B_{\nu/\kappa}\widetilde{P}_\mu(t^\delta q^\nu) \qquad (\mu \in \mathcal{P}_n, \mu \le \lambda). \tag{6.38}$$

From (6.36) and (6.38), we obtain

$$\sum_{\substack{\nu/\kappa:\text{v-strip}\\ |\nu/\kappa|=r}} A_{\nu/\kappa}\widetilde{P}_\mu(t^\delta q^\nu) = \sum_{\nu \le \lambda} B_{\nu/\kappa}\widetilde{P}_\mu(t^\delta q^\nu) \qquad (\mu \in \mathcal{P}_n, \mu \le \lambda). \tag{6.39}$$

Then, statement $(\mathrm{b})_\lambda$ follows if we confirm that $\det\left(\widetilde{P}_\mu(t^\delta q^\nu)\right)_{\mu,\nu \le \lambda} \ne 0$, which will be proved below in Lemma 6.1.

Knowing that $(\mathrm{b})_\lambda$ holds, we can rewrite (6.37) as

$$e_r(t^\delta q^\mu)\widetilde{P}_\kappa(t^\delta q^\mu) = \sum_{\substack{\nu/\mu:\,\text{v-strip}\\ |\nu/\mu|=r}} A_{\nu/\kappa}\widetilde{P}_\nu(t^\delta q^\mu) \qquad (\mu \in \mathcal{P}_n). \tag{6.40}$$

We now compare (6.36) and (6.40) for arbitrary $\mu \in \mathcal{P}_n$. Since $\widetilde{P}_\mu(t^\delta q^\kappa) = \widetilde{P}_\kappa(t^\delta q^\mu)$ and $\widetilde{P}_\mu(t^\delta q^\nu) = \widetilde{P}_\nu(t^\delta q^\mu)$ for any $\nu < \lambda = \kappa + \varpi_r$, we obtain

$$A_{\lambda/\kappa}\widetilde{P}_\mu(t^\delta q^\lambda) = A_{\lambda/\kappa}\widetilde{P}_\lambda(t^\delta q^\mu). \tag{6.41}$$

Since $A_{\lambda/\kappa} \ne 0$, we obtain $\widetilde{P}_\mu(t^\delta q^\lambda) = \widetilde{P}_\lambda(t^\delta q^\mu)$ for all $\mu \in \mathcal{P}_n$, as desired.

Lemma 6.1 *For any* $\lambda \in \mathcal{P}_n$, $\det\left(\widetilde{P}_\mu(t^\delta q^\nu)\right)_{\mu,\nu \le \lambda} \ne 0$.

Proof This statement is equivalent to $\det\left(P_\mu(t^\delta q^\nu)\right)_{\mu,\nu \le \lambda} \ne 0$ since $\widetilde{P}_\mu(x) = P_\mu(x)/a_\mu$, and further to $\det\left(m_\mu(t^\delta q^\nu)\right)_{\mu,\nu \le \lambda} \ne 0$ since $P_\mu(x) = m_\mu(x) +$ (lower order terms with respect to $\le$). Note that

$$m_\mu(t^\delta q^\nu) = t^{\langle \mu, \delta \rangle} q^{\langle \mu, \nu \rangle} + (\text{lower degree terms in } t), \tag{6.42}$$

and hence

$$\begin{aligned} &\det \left(m_\mu(t^\delta q^\nu)\right)_{\mu,\nu \le \lambda} \\ &= \det \left(t^{\langle \mu, \delta \rangle} q^{\langle \mu, \nu \rangle}\right)_{\mu,\nu \le \lambda} + (\text{lower degree terms in } t) \\ &= t^{\sum_{\mu \le \lambda} \langle \mu, \delta \rangle} \det \left(q^{\langle \mu, \nu \rangle}\right)_{\mu,\nu \le \lambda} + (\text{lower degree terms in } t). \end{aligned} \tag{6.43}$$

Setting $N = \#\{\mu \in \mathcal{P}_n \mid \mu \le \lambda\}$, parametrize all $\mu \in \mathcal{P}_n$ with $\mu \le \lambda$ as $\mu^{(1)}, \ldots, \mu^{(N)}$. Then we have

$$\begin{aligned} \det \left(q^{\langle \mu, \nu \rangle}\right)_{\mu,\nu \le \lambda} &= \det \left(q^{\langle \mu^{(i)}, \mu^{(j)} \rangle}\right)_{i,j=1}^N = \sum_{\sigma \in \mathfrak{S}_N} \operatorname{sgn}(\sigma) q^{\sum_{i=1}^N \langle \mu^{(i)}, \mu^{(\sigma(i))} \rangle} \\ &= q^{\sum_{i=1}^N \langle \mu^{(i)}, \mu^{(i)} \rangle} + (\text{lower degree terms in } q). \end{aligned} \tag{6.44}$$

In fact, for $\sigma \ne 1$, inequality $\sum_{i=1}^N \langle \mu^{(i)} - \mu^{(\sigma(i))}, \mu^{(i)} - \mu^{(\sigma(i))} \rangle > 0$ implies $\sum_{i=1}^N \langle \mu^{(i)}, \mu^{(i)} \rangle > \sum_{i=1}^N \langle \mu^{(i)}, \mu^{(\sigma(i))} \rangle$. Hence we have $\det \left(q^{\langle \mu, \nu \rangle}\right)_{\mu,\nu \le \lambda} \ne 0$. □

We remark that the self-duality of Theorem 6.2 can also be proved by means of the *Cherednik involution* of the double affine Hecke algebra (see Sect. 8.5).

6.5 Cauchy Formula and Dual Cauchy Formula

The Cauchy formula of Theorem 3.2 and the dual Cauchy formula of Theorem 3.4 for Schur functions can be generalized to the case of Macdonald polynomials.

Theorem 6.4 (Cauchy formula) *For two sets of variables* $x = (x_1, \ldots, x_m)$ *and* $y = (y_1, \ldots, y_n)$*, we have*

$$\prod_{i=1}^m \prod_{j=1}^n \frac{(t x_i y_j; q)_\infty}{(x_i y_j; q)_\infty} = \sum_{\ell(\lambda) \le \min\{m,n\}} b_\lambda P_\lambda(x) P_\lambda(y), \tag{6.45}$$

where λ *runs over all partitions with* $\ell(\lambda) \le \min\{m, n\}$*, and the coefficients* b_λ *are given by*

$$b_\lambda = \prod_{s \in \lambda} \frac{1 - t^{l_\lambda(s)+1} q^{a_\lambda(s)}}{1 - t^{l_\lambda(s)} q^{a_\lambda(s)+1}} = \prod_{1 \le i \le j \le \ell(\lambda)} \frac{(t^{j-i+1} q^{\lambda_i - \lambda_j}; q)_{\lambda_j - \lambda_{j+1}}}{(t^{j-i} q^{\lambda_i - \lambda_j + 1}; q)_{\lambda_j - \lambda_{j+1}}}. \tag{6.46}$$

We remark that, when $q = t$, formula (6.45) reduces to the Cauchy formula

$$\prod_{i=1}^{m}\prod_{j=1}^{n}\frac{1}{1-x_iy_j} = \sum_{\ell(\lambda)\le\min\{m,n\}} s_\lambda(x)s_\lambda(y), \tag{6.47}$$

with coefficients $b_\lambda = 1$. In Sect. 6.6, we give a proof of the fact that the left-hand side of (6.45) has an expansion formula of the form (6.45) for some constants b_λ ($\lambda \in \mathcal{P}_n$); a derivation of the explicit formula (6.46) for b_λ will be given in Sect. 7.3. In Macdonald's monograph [20], the notation $Q_\lambda(y) = b_\lambda P_\lambda(y)$ for the "dual" Macdonald polynomials is consistently used in view of their roles in duality arguments.

Theorem 6.5 (Dual Cauchy formula) *For two sets of variables $x = (x_1, \ldots, x_m)$ and $y = (y_1, \ldots, y_n)$, we have*

$$\prod_{i=1}^{m}\prod_{j=1}^{n}(1 + x_iy_j) = \sum_{\lambda\subseteq(n^m)} P_\lambda(x; q, t)P_{\lambda'}(y; t, q), \tag{6.48}$$

$$\prod_{i=1}^{m}\prod_{j=1}^{n}(x_i + y_j) = \sum_{\lambda\subseteq(n^m)} P_\lambda(x; q, t)P_{\lambda^c}(y; t, q), \tag{6.49}$$

where the sum is over all partitions λ contained in the $m \times n$ rectangle (n^m); $\lambda' = (\lambda'_1, \ldots, \lambda'_n)$ and $\lambda^c = (m - \lambda'_n, \ldots, m - \lambda'_1)$ denote the conjugate partition of λ and the complementary partion of λ in (n^m) respectively.

In what follows, we set

$$\Pi_{m,n}(x; y) = \prod_{i=1}^{m}\prod_{j=1}^{n}\frac{(tx_iy_j; q)_\infty}{(x_iy_j; q)_\infty} \tag{6.50}$$

and regard $\Pi_{m,n}(x; y)$ as a formal power series in $\mathbb{C}[[x, y]]^{\mathfrak{S}_m\times\mathfrak{S}_n}$.[1] We also set

$$\Pi^\vee_{m,n}(x; y) = \prod_{i=1}^{m}\prod_{j=1}^{n}(1 + x_iy_j) \in \mathbb{C}[x, y]^{\mathfrak{S}_m\times\mathfrak{S}_n}. \tag{6.51}$$

It is sometimes more convenient to use the generating function

$$\Psi_{m,n}(x; y) = \prod_{i=1}^{m}\prod_{j=1}^{n}(x_i + y_j) \in \mathbb{C}[x, y]^{\mathfrak{S}_m\times\mathfrak{S}_n}. \tag{6.52}$$

[1] In fact, $\Pi_{m,n}(x; y)$ is a meromorphic function on $\mathbb{C}^m \times \mathbb{C}^n$ under our assumption $|q| < 1$. It is also holomorphic in the domain $|x_iy_j| < 1$ for $i \in \{1, \ldots, m\}$, $j \in \{1, \ldots, n\}$.

Since

$$\Psi_{m,n}(x;y) = (y_1 \cdots y_n)^m \prod_{i=1}^{m}\prod_{j=1}^{n}(1 + x_i y_j^{-1}) = (y_1 \ldots y_n)^m \Pi^{\vee}_{m,n}(x;y^{-1}), \quad (6.53)$$

formula (6.48) is equivalent to

$$\prod_{i=1}^{m}\prod_{j=1}^{n}(x_i + y_j) = \sum_{\lambda\in(n^m)} P_\lambda(x;q,t)(y_1\cdots y_n)^m P_{\lambda'}(y^{-1};t,q). \quad (6.54)$$

By Proposition 5.1, for each partition $\lambda \subseteq (n^m)$ we have

$$(y_1\cdots y_n)^m P_{\lambda'}(y^{-1};t,q) = P_{\lambda^{c}}(y;t,q), \quad (6.55)$$

where

$$\lambda^{c} = ((m^n) - \lambda')^{\vee} = (m - \lambda'_n, \ldots, m - \lambda'_1) \quad (6.56)$$

denotes the *complementary partition* of λ in the $m \times n$ rectangle. Hence formula (6.48) is equivalent to (6.49). We give a proof of the dual Cauchy formula (6.49) in the second half of Sect. 6.6.

6.6 Kernel Identities

6.6.1 Kernel Identity for the Cauchy Formula

We consider the case where $m = n$. We first remark that there exists an expansion formula as (6.45) with *some* constants b_λ, if and only if $\Pi(x;y) = \Pi_{n,n}(x;y)$ satisfies the *kernel identity*

$$D_x(u)\Pi(x;y) = D_y(u)\Pi(x;y). \quad (6.57)$$

Expand $\Pi(x;y)$ in terms of Macdonald polynomials $P_\lambda(x)$ ($\lambda \in \mathcal{P}_n$) as

$$\Pi(x;y) = \sum_{\lambda\in\mathcal{P}_n} P_\lambda(x) Q_\lambda(y), \qquad Q_\lambda(y) \in \mathbb{C}[y]^{\mathfrak{S}_n} \quad (\lambda \in \mathcal{P}_n). \quad (6.58)$$

Since

$$D_x(u)\Pi(x;y) = \sum_{\lambda\in\mathcal{P}_n} P_\lambda(x) Q_\lambda(y) \prod_{i=1}^{n}(1 - ut^{n-j}q^{\lambda_j}),$$
$$D_y(u)\Pi(x;y) = \sum_{\lambda\in\mathcal{P}_n} P_\lambda(x) D_y(u) Q_\lambda(y), \quad (6.59)$$

identity (6.57) implies $D_y(u)Q_\lambda(y) = Q_\lambda(y)\prod_{i=1}^n(1-ut^{n-j}q^{\lambda_j})$ and hence, $Q_\lambda(x) = b_\lambda P_\lambda(x)$ for some $b_\lambda \in \mathbb{C}$.

Proposition 6.1 *For two sets of variables* $x = (x_1, \ldots, x_n)$, $y = (y_1, \ldots, y_n)$, *the formal power series*

$$\Pi(x; y) = \prod_{i=1}^n \prod_{j=1}^n \frac{(tx_iy_j; q)_\infty}{(x_iy_j; q)_\infty} \in \mathbb{C}[\![x, y]\!]^{\mathfrak{S}_n \times \mathfrak{S}_n} \tag{6.60}$$

satisfies the kernel identity

$$D_x(u)\Pi(x; y) = D_y(u)\Pi(x; y). \tag{6.61}$$

Proof Recall that

$$\begin{aligned} D_x(u) &= \sum_{I\subseteq\{1,\ldots,n\}} (-u)^{|I|} t^{\binom{|I|}{2}} \prod_{i\in I,\, j\notin I} \frac{tx_i - x_j}{x_i - x_j} \prod_{i\in I} T_{q,x_i}, \\ D_y(u) &= \sum_{K\subseteq\{1,\ldots,n\}} (-u)^{|K|} t^{\binom{|K|}{2}} \prod_{k\in K,\, l\notin K} \frac{ty_k - y_l}{y_k - y_l} \prod_{k\in K} T_{q,y_k}. \end{aligned} \tag{6.62}$$

Since

$$\begin{aligned} \prod_{i\in I} T_{q,x_i}\Pi(x; y) &= \prod_{i\in I}\prod_{l=1}^n \frac{1 - x_iy_l}{1 - tx_iy_l} \cdot \Pi(x; y), \\ \prod_{k\in K} T_{q,y_k}\Pi(x; y) &= \prod_{k\in K}\prod_{j=1}^n \frac{1 - x_jy_k}{1 - tx_jy_k} \cdot \Pi(x; y), \end{aligned} \tag{6.63}$$

Equation (6.61) is equivalent to the *source identity*

$$\begin{aligned} &\sum_{I\subseteq\{1,\ldots,n\}} (-u)^{|I|} t^{\binom{|I|}{2}} \prod_{i\in I;\, j\notin I} \frac{tx_i - x_j}{x_i - x_j} \prod_{i\in I}\prod_{l=1}^n \frac{1 - x_iy_l}{1 - tx_iy_l} \\ &= \sum_{K\subseteq\{1,\ldots,n\}} (-u)^{|K|} t^{\binom{|K|}{2}} \prod_{k\in K;\, l\notin K} \frac{ty_k - y_l}{y_k - y_l} \prod_{k\in K}\prod_{j=1}^n \frac{1 - x_jy_k}{1 - tx_jy_k}. \end{aligned} \tag{6.64}$$

An important observation is that this identity does *not* involve q. This means that, in order to prove (6.64), it is sufficient to prove (6.61) for $q = t$. However, we already know that (6.61) holds when $q = t$ by the Cauchy formula for Schur functions. □

The existence of an expansion formula of the form (6.45) for difference number of variables m, n follows from the stability of Macdonald polynomials as in Exercise 4.2. Also, for a given partition $\lambda \in \mathcal{P}$, the coefficient b_λ of $P_\lambda(x)P_\lambda(y)$ in

(6.45) is determined independently of the choice of m, n such that $m \geq \ell(\lambda), n \geq \ell(\lambda)$.

It should be noted that we need some other arguments to obtain the explicit formula (6.46) for b_λ; a proof of (6.46) will be given in Sect. 7.3, on the basis of compatibility of the Cauchy and the dual Cauchy formula for Macdonald polynomials.

In the setting of Theorem 6.4, suppose that $m \geq n$. Then for each $\lambda \in \mathcal{P}_n$, we have

$$D_x(u)P_\lambda(x) = P_\lambda(x)\prod_{i=1}^{n}(1 - ut^{m-i}q^{\lambda_i})\prod_{i=n+1}^{m}(1 - ut^{m-i})$$
$$D_y(v)P_\lambda(y) = P_\lambda(y)\prod_{i=1}^{n}(1 - vt^{n-i}q^{\lambda_i}). \tag{6.65}$$

We also remark that (6.45) for the case where $m \geq n$ corresponds to the kernel identity

$$D_x(u)\Pi_{m,n}(x;y) = (u;t)_{m-n}D_y(ut^{m-n})\Pi_{m,n}(x;y). \tag{6.66}$$

Remark 6.1 We have used here the kernel identity for $\Pi_{m,n}(x;y)$ to prove the Cauchy formula for Macdonald polynomials. Another important application of the kernel identity is the integral transform of the form

$$\varphi(x) = \frac{1}{(2\pi\sqrt{-1})^n}\int_{\mathbb{T}^n}\Pi_{m,n}(x;y)\psi(y)w(y)\frac{dy_1\cdots dy_n}{y_1\cdots y_n}. \tag{6.67}$$

It transforms joint eigenfunctions $\psi(y)$ of the Macdonald–Ruijsenaars operators $D_y(v)$ in y variables to joint eigenfunctions $\varphi(x)$ of $D_x(u)$ in x variables. This property is a consequence of the kernel identity for $\Pi_{m,n}(x;y)$ combined with the self-adjointness of $D_y(v)$ with respect to the weight function $w(y)$.

6.6.2 Kernel Identity for the Dual Cauchy Formula

Here we give a proof of formula (6.49) which is equivalent to (6.48), on the basis of a relevant kernel identity.

For two sets of variables $x = (x_1, \ldots, x_m)$ and $y = (y_1, \ldots, y_n)$, we set

$$P_\mu(x) = P_\mu(x;q,t) \ \ (\mu \in \mathcal{P}_m), \quad P_\nu^\circ(y) = P_\nu(y;t,q) \ \ (\nu \in \mathcal{P}_n). \tag{6.68}$$

We also denote by

$$D_y^\circ = \sum_{k=1}^{n}\prod_{1\leq l\leq n;\ l\neq k}\frac{qy_k - y_l}{y_k - y_l}T_{t,y_k} \tag{6.69}$$

the t-difference operator obtained from D_y by exchanging q and t.

Note that the polynomial

$$\Psi_{m,n}(x;y) = \prod_{i=1}^{m}\prod_{j=1}^{n}(x_i + y_j) \in \mathbb{C}[x,y]^{\mathfrak{S}_m\times\mathfrak{S}_n} \tag{6.70}$$

is of degree mn in (x, y), and symmetric both in x and in y. Since $\Psi_{m,n}(x; y)$ is of degree $\le n$ in each x_i and of degree $\le m$ in each y_j, it can be expressed as

$$\Psi_{m,n}(x;y) = \sum_{\mu\subseteq(n^m);\ \nu\subseteq(m^n)} c_{\mu,\nu} P_\mu(x) P^\circ_\nu(y) \tag{6.71}$$

with some constants $c_{\mu,\nu}$. For each partition $\mu \subseteq (n^m)$, we defined the *complementary partition* μ^{c} in the $m \times n$ rectangle by $\mu^{\mathrm{c}} = (m - \mu'_n, n - \mu'_{n-1}, \ldots, m - \mu'_1)$ (see the figure in (3.82)). In this setting, we show that $c_{\mu,\nu} = 0$ unless $\nu = \mu^{\mathrm{c}}$, namely

$$\Psi_{m,n}(x;y) = \prod_{i=1}^{m}\prod_{j=1}^{n}(x_i + y_j) = \sum_{\lambda\subseteq(n^m)} c_\lambda P_\lambda(x) P^\circ_{\lambda^{\mathrm{c}}}(y) \tag{6.72}$$

for some constants $c_\lambda \in \mathbb{C}$.

In the eigenfunction equation

$$D_x P_\lambda(x) = d_\lambda P_\lambda(x), \quad d_\lambda = \sum_{i=1}^{m} t^{m-i} q^{\lambda_i}, \tag{6.73}$$

the eigenvalue d_λ has the following combinatorial meaning:

$$\begin{aligned}\frac{1}{q-1}\Big(d_\lambda - \frac{t^m-1}{t-1}\Big) &= \frac{1}{q-1}\sum_{i=1}^{m} t^{m-i}(q^{\lambda_i} - 1)\\ &= \sum_{i=1}^{m}\sum_{j=1}^{\lambda_i} t^{m-i} q^{j-1} = \sum_{s=(i,j)\in D(\lambda)} t^{m-i} q^{j-1}.\end{aligned} \tag{6.74}$$

Similarly, as for the eigenvalue $d^\circ_{\lambda^{\mathrm{c}}}$ in the equation

$$D^\circ_y P^\circ_{\lambda^{\mathrm{c}}}(y) = d^\circ_{\lambda^{\mathrm{c}}} P^\circ_{\lambda^{\mathrm{c}}}(y), \quad d^\circ_{\lambda^{\mathrm{c}}} = \sum_{j=1}^{n} q^{n-j} t^{\lambda^{\mathrm{c}}_j}, \tag{6.75}$$

we have

$$\frac{1}{t-1}\Big(d^\circ_{\lambda^{\mathrm{c}}} - \frac{q^n-1}{q-1}\Big) = \sum_{j=1}^{n}\sum_{i=1}^{\lambda^{\mathrm{c}}_i} t^{i-1} q^{n-j} = \sum_{s=(i,j)\in D(\lambda)^{\mathrm{c}}} t^{m-i} q^{j-1}, \tag{6.76}$$

where $D(\lambda)^{\mathrm{c}}$ stands for the complement of $D(\lambda)$ in the rectangle $\{1,\ldots,m\}\times\{1,\ldots,n\}$. Since

$$\sum_{s=(i,j)\in(n^m)} t^{m-i}q^{j-1}=\frac{t^m-1}{t-1}\frac{q^n-1}{q-1}, \tag{6.77}$$

the existence of a formula in the form (6.72) is equivalent to

$$\begin{aligned}\Big(\frac{1}{q-1}\Big(D_x-\frac{t^m-1}{t-1}\Big)+\frac{1}{t-1}\Big(D_y^{\circ}-\frac{q^n-1}{q-1}\Big)\Big)\Psi_{m,n}(x;y)\\ =\frac{t^m-1}{t-1}\frac{q^n-1}{q-1}\Psi_{m,n}(x;y),\end{aligned} \tag{6.78}$$

namely,

$$\Big(\frac{1}{q-1}D_x+\frac{1}{t-1}D_y^{\circ}\Big)\Psi_{m,n}(x;y)=\frac{t^mq^n-1}{(t-1)(q-1)}\Psi_{m,n}(x;y). \tag{6.79}$$

Proposition 6.2 *For two sets of variables $x=(x_1,\ldots,x_m)$ and $y=(y_1,\ldots,y_n)$, the polynomial*

$$\Psi_{m,n}(x;y)=\prod_{i=1}^{m}\prod_{j=1}^{n}(x_i+y_j)\in\mathbb{C}[x,y]^{\mathfrak{S}_m\times\mathfrak{S}_n} \tag{6.80}$$

satisfies the kernel identity

$$\Big(\frac{1}{q-1}D_x+\frac{1}{t-1}D_y^{\circ}\Big)\Psi_{m,n}(x;y)=\frac{t^mq^n-1}{(t-1)(q-1)}\Psi_{m,n}(x;y). \tag{6.81}$$

Proof This kernel identity is equivalent to the following identity of rational functions:

$$\begin{aligned}&\frac{1}{q-1}\sum_{i=1}^{m}\prod_{j\neq i}\frac{tx_i-x_j}{x_i-x_j}\prod_{l=1}^{n}\frac{qx_i+y_l}{x_i+y_l}\\ &+\frac{1}{t-1}\sum_{k=1}^{n}\prod_{l\neq k}\frac{qy_k-y_l}{y_k-y_l}\prod_{j=1}^{m}\frac{x_j+ty_k}{x_j+y_k}=\frac{t^mq^n-1}{(t-1)(q-1)},\end{aligned} \tag{6.82}$$

which can be verified directly by the residue calculus combined with induction on the number of variables. In fact, equality (6.82) for $n=0$ is the same as (4.2). When $n>0$, we regard the left-hand side as a rational function of y_n. Then, we see that the residues at $y_n=y_k$ ($k=1,\ldots,n-1$) and at $y_n=-x_i$ ($i=1,\ldots,m$) are all zero. We can also verify that the limit as $y_n\to 0$ gives the value of the right-hand side, by using the induction hypothesis of the case $(m,n-1)$. □

Finally, we show that $c_\lambda = 1$ for all $\lambda \subseteq (n^m)$. We denote by $\mathcal{A}_{m,n}$ the set of all $m \times n$ integer matrices $A = (a_{ij})_{1 \le i \le m;\ 1 \le j \le n}$ such that $a_{ij} \in \{0, 1\}$ for all i, j. Also, for a pair of multi-indices $(\mu, \nu) \in \mathbb{N}^m \times \mathbb{N}^n$, we denote by $\mathcal{A}_{\mu,\nu}$ the set of all $A = (a_{ij}) \in \mathcal{A}_{m,n}$ such that

$$\sum_{j=1}^{n} a_{ij} = \mu_i \quad (i = 1, \ldots, m), \quad \sum_{i=1}^{m} a_{ij} = \nu_j \quad (j = 1, \ldots, n). \tag{6.83}$$

Then $\Psi_{m,n}(x; y)$ can be expanded as follows:

$$\begin{aligned}\Psi_{m,n}(x; y) &= \sum_{A=(a_{ij}) \in \mathcal{A}_{m,n}} \prod_{i=1}^{m} \prod_{j=1}^{n} (x_i^{a_{ij}} y_j^{1-a_{ij}}) \\ &= \sum_{\mu \in \mathbb{N}^m, \nu \in \mathbb{N}^n} (\#\mathcal{A}_{\mu,\nu})\, x^{\mu} y^{(m^n)-\nu} = \sum_{\mu \in \mathbb{N}^m, \nu \in \mathbb{N}^n} (\#\mathcal{A}_{\mu,(m^n)-\nu})\, x^{\mu} y^{\nu} \\ &= \sum_{\mu, \nu \subseteq (n^m)} (\#\mathcal{A}_{\mu,(m^n)-\nu^{\mathrm{c}}})\, m_\mu(x)\, m_{\nu^{\mathrm{c}}}(y). \end{aligned} \tag{6.84}$$

Since $(m^n) - \nu^{\mathrm{c}} = (\nu'_n, \ldots, \nu'_1)$ is the reversal of $\nu' = (\nu'_1, \ldots, \nu'_n)$, we obtain

$$\Psi_{m,n}(x; y) = \sum_{\mu, \nu \subseteq (n^m)} (\#\mathcal{A}_{\mu,\nu'})\, m_\mu(x)\, m_{\nu^{\mathrm{c}}}(y). \tag{6.85}$$

We now look at the coefficients of $m_\mu(x) m_{\mu^{\mathrm{c}}}(y)$ for partitions $\mu \subseteq (n^m)$.

Lemma 6.2 *For each partition $\mu \subseteq (n^m)$, $\#\mathcal{A}_{\mu,\mu'} = 1$.*

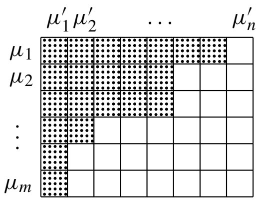
Proof Define $A = (a_{ij})_{1 \le i \le m, 1 \le j \le n}$ by

$$a_{ij} = 1 \quad (1 \le j \le \mu_i), \qquad a_{ij} = 0 \quad (\mu_i < j \le n) \tag{6.86}$$

for all $i \in \{1, \ldots, m\}$, so that

$$\left\{ s = (i, j) \in \{1, \ldots, m\} \times \{1, \ldots, n\} \mid a_{ij} = 1 \right\} = D(\mu). \tag{6.87}$$

Then, one can verify that this matrix A it is the only element of $\mathcal{A}_{\mu,\mu'}$. □



(6.88)

This lemma implies that, for each partition $\mu \subseteq (n^m)$, the coefficient of $x^\mu y^{\mu^c}$ in the expansion of $\Psi_{m,n}(x; y)$ is precisely 1. In the right-hand side of (6.72), the monomial $x^\mu y^{\mu^c}$ arises only if there exists a partition $\lambda \in (n^m)$ such that $\mu \le \lambda$ and $\mu^c \le \lambda^c$. One can directly verify that the condition $\mu^c \le \lambda^c$ implies $\mu' \le \lambda'$, and hence $\mu \ge \lambda$. Together with $\mu \le \lambda$, we obtain $\lambda = \mu$. This implies that the monomial $x^\mu y^{\mu^c}$ arises only from the term $P_\mu(x) P^\circ_{\mu^c}(y)$. This also means that the coefficient of $x^\mu y^{\mu^c}$ on the right-hand side is given by c_μ. Hence we have $c_\mu = 1$ for all partitions $\mu \subseteq (n^m)$. This completes the proof of the dual Cauchy formula (6.49) of Theorem 6.5, and also the proof of (6.48).