Chapter 5
Orthogonality and Higher-Order q-Difference Operators

Abstract We show that the Macdonald polynomials satisfy the orthogonality relation with respect to a certain scalar product on the ring of symmetric polynomials. We also explain how this orthogonality is related with the existence of commuting family of higher-order q-difference operators for which Macdonald polynomials are joint eigenfunctions.

5.1 Scalar Product and Orthogonality

As always, we fix the parameters $q, t \in \mathbb{C}^*$ with $|q| < 1$. Also, keeping the convention of the previous chapter, we suppose that the parameters q, t satisfy the genericity condition (4.10).

5.1.1 Weight Function and Scalar Product

We define a meromorphic function $w(x) = w(x; q, t)$ on $(\mathbb{C}^*)^n$ by

$$w(x) = \prod_{1\le i<j\le n} \frac{(x_i/x_j; q)_\infty}{(tx_i/x_j; q)_\infty} \frac{(x_j/x_i; q)_\infty}{(tx_j/x_i; q)_\infty}. \tag{5.1}$$

Note that $w(x)$ is $\mathfrak{S}_n$-invariant and also $w(x^{-1}) = w(x)$. We assume $|t| < 1$ so that $w(x)$ is holomorphic in a neighborhood of the n-dimensional torus

$$\mathbb{T}^n = \left\{x = (x_1, \dots, x_n) \in (\mathbb{C}^*)^n \mid |x_i| = 1\ (i = 1, \dots, n)\right\} \subset (\mathbb{C}^*)^n. \tag{5.2}$$

For a pair of holomorphic functions $f(x), g(x)$ in a neighborhood of $\mathbb{T}^n$, we define the scalar product (symmetric bilinear form) $\langle f, g\rangle$ as

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$$\langle f, g\rangle = \frac{1}{n!}\frac{1}{(2\pi\sqrt{-1})^n}\int_{\mathbb{T}^n} f(x^{-1})g(x)w(x)\frac{dx_1\cdots dx_n}{x_1\cdots x_n} \tag{5.3}$$

by the integral over $\mathbb{T}^n$ with orientation such that

$$\frac{1}{(2\pi\sqrt{-1})^n}\int_{\mathbb{T}^n}\frac{dx_1\cdots dx_n}{x_1\cdots x_n} = 1. \tag{5.4}$$

The scalar product is alternatively expressed as

$$\langle f, g\rangle = \frac{1}{n!}\mathrm{CT}\Big[f(x^{-1})g(x)w(x)\Big], \tag{5.5}$$

in terms of the *constant term* CT (coefficient of 1) of the Laurent expansion of a holomorphic function around $\mathbb{T}^n$.

Theorem 5.1 *Suppose that* $|t| < 1$. *Then, the Macdonald polynomials are orthogonal with respect to the scalar product defined by* (5.3):

$$\langle P_\lambda, P_\mu\rangle = \delta_{\lambda,\mu}\, N_\lambda \quad (\lambda, \mu \in \mathcal{P}_n) \tag{5.6}$$

for some constants $N_\lambda \in \mathbb{C}$ $(\lambda \in \mathcal{P}_n)$.

We remark that, if $q, t \in \mathbb{R}$ and $|q| < 1$, $|t| < 1$, the Macdonald polynomials have real coefficients, and $\langle\ ,\ \rangle$ defines a positive definite scalar product on $\mathbb{R}[x]^{\mathfrak{S}_n}$.

Remark 5.1 In Macdonald's monograph [20, Sect. VI.9], the scalar product $\langle f, g\rangle$ of (5.3) is called *another scalar product* and denoted by $\langle f, g\rangle'_n$. It should be noted that our scalar product is different from Macdonald's $\langle f, g\rangle_n$ defined by [20, Chap. VI, (2.20)].

5.1.2 Constant Term and Scalar Products

It is known [20, Sect. VI.9] that the constant term and the scalar products are determined explicitly as follows.

Theorem 5.2 *For each* $\lambda \in \mathcal{P}_n$, *the scalar product* $N_\lambda = \langle P_\lambda, P_\lambda\rangle$ *is explicitly evaluated as*

$$N_\lambda = \prod_{1\le i<j\le n}\frac{(q^{\lambda_i-\lambda_j}t^{j-i};q)_\infty(q^{\lambda_i-\lambda_j+1}t^{j-i};q)_\infty}{(q^{\lambda_i-\lambda_j}t^{j-i+1};q)_\infty(q^{\lambda_i-\lambda_j+1}t^{j-i-1};q)_\infty}. \tag{5.7}$$

In particular, the constant term of the weight function $\mathrm{CT}\Big[w(x)\Big] = n!N_\phi = n!\langle 1, 1\rangle$ *is given by*

$$\mathrm{CT}\Big[w(x)\Big] = n!\left(\frac{(t;q)_\infty}{(q;q)_\infty}\right)^n \prod_{i=1}^{n} \frac{(t^{i-1}q;q)_\infty}{(t^i;q)_\infty}. \tag{5.8}$$

In this book, we will *not* go into the proof of these explicit formulas. For proofs of this theorem, we refer the reader to Macdonald [20, Sect. VI.9], and Mimachi [21] (see also Macdonald [22]).

5.2 Proof of Orthogonality

The orthogonality of Macdonald polynomials is a consequence of the facts that:

(1) The q-difference operator D_x is (formally) *self-adjoint* with respect to the weight function $w(x)$.
(2) The partitions $\lambda \in \mathcal{P}_n$ are separated by the eigenvalues of D_x, namely $d_\lambda \neq d_\mu$ for any distinct pair $\lambda, \mu \in \mathcal{P}_n$.

Along this idea, we explain step by step how the orthogonality of Theorem 5.1 can be established.

5.2.1 Cauchy's Theorem as a Basis of q-Difference de Rham Theory

Let $\varphi(z)$ be a holomorphic function in an neighborhood of a closed curve C in $\mathbb{C}^*$. We suppose that the contour C can be deformed continuously to qC in a domain where $\varphi(z)$ is holomorphic. Note that this condition is satisfied either if the domain of holomorphy of $\varphi(z)$ is sufficiently large, or if q is sufficiently close to 1. Then, by Cauchy's theorem, we have

$$\int_C \varphi(qz)\frac{dz}{z} = \int_{qC} \varphi(z)\frac{dz}{z} = \int_C \varphi(z)\frac{dz}{z}, \tag{5.9}$$

namely

$$\int_C T_{q,z}\big(\varphi(z)\big)\frac{dz}{z} = \int_C \varphi(z)\frac{dz}{z}, \quad \text{i.e.} \quad \int_C (T_{q,z}-1)(\varphi(z))\frac{dz}{z} = 0. \tag{5.10}$$

In particular, we have

$$\int_C T_{q,z}\big(\varphi(z)\big)\psi(z)\frac{dz}{z} = \int_C \varphi(z) T_{q,z}^{-1}\big(\psi(z)\big)\frac{dz}{z}. \tag{5.11}$$

This formula plays the role of integration by parts.

5.2.2 Formal Adjoint of a q-Difference Operator

Let $L_x \in \mathcal{D}_{q,x} = \mathbb{C}(x)[T_{q,x}^{\pm 1}]$ be a q-difference operator in $x = (x_1, \ldots, x_n)$ with rational coefficients:

$$L_x = \sum_{\mu \in \mathbb{Z}^n} a_\mu(x) T_{q,x}^\mu \quad \text{(finite sum)}, \quad a_\mu(x) \in \mathbb{C}(x) \ \ (\mu \in \mathbb{Z}^n), \tag{5.12}$$

where $T_{q,x}^\mu = T_{q,x_1}^{\mu_1} \cdots T_{q,x_n}^{\mu_n}$. We define the *formal adjoint* L_x^* of L_x by

$$L_x^* = \sum_{\mu \in \mathbb{Z}^n} T_{q,x}^{-\mu} a_\mu(x), \tag{5.13}$$

so that $(L_x M_x)^* = M_x^* L_x^*$. Then, we have

$$\begin{aligned}
&\int_{\mathbb{T}^n} (L_x f)(x^{-1}) g(x) w(x) \frac{dx}{x} \\
&= \int_{\mathbb{T}^n} (L_{x^{-1}} f(x^{-1})) g(x) w(x) \frac{dx}{x} \\
&= \int_{\mathbb{T}^n} f(x^{-1}) \big(L_{x^{-1}}^* g(x) w(x) \big) \frac{dx}{x} \\
&= \int_{\mathbb{T}^n} f(x^{-1}) \big(w(x)^{-1} L_{x^{-1}}^* w(x) g(x) \big) w(x) \frac{dx}{x},
\end{aligned} \tag{5.14}$$

and hence

$$\langle Lf, g \rangle = \langle f, L^\dagger g \rangle, \quad L^\dagger = w(x)^{-1} L_{x^{-1}}^* w(x), \tag{5.15}$$

provided that q is sufficiently close to 1 and that Cauchy's theorem can be applied to L_x. We say that L_x is *formally self-adjoint* with respect to $w(x)$ if $L_x^\dagger = L_x$, namely $w(x) L_x w(x)^{-1} = L_{x^{-1}}^*$.

5.2.3 D_x Is Self-Adjoint with Respect to $w(x)$

Note that

$$\frac{T_{q,x_i} w(x)}{w(x)} = \prod_{j \neq i} \frac{1 - t x_i / x_j}{1 - x_i / x_j} \prod_{j \neq i} \frac{1 - x_j / q x_i}{1 - t x_j / q x_i} = \frac{A_i(x)}{T_{q,x_i} A_i(x^{-1})} \quad (i = 1, \ldots, n). \tag{5.16}$$

This implies that

$$\begin{aligned} w(x) D_x w(x)^{-1} &= \sum_{i=1}^{n} A_i(x) \frac{w(x)}{T_{q,x_i} w(x)} T_{q,x_i} = \sum_{i=1}^{n} (T_{q,x_i} A_i(x^{-1})) T_{q,x_i} \\ &= \sum_{i=1}^{n} T_{q,x_i} A_i(x^{-1}) = D^{*}_{x^{-1}}. \end{aligned} \tag{5.17}$$

It can be verified directly that $\langle D_x f, g\rangle = \langle f, D_x g\rangle$ if $|t| < |q| < 1$. Note that the poles of $A_i(x)$ along $\Delta(x) = 0$ are canceled by the zeros of $w(x)$.

5.2.4 Orthogonality

Since D_x is self-adjoint with respect to the scalar product, for any $\lambda, \mu \in \mathcal{P}_n$ we have the equality

$$\langle D_x P_\lambda(x), P_\mu(x)\rangle = \langle P_\lambda(x), D_x P_\mu(x)\rangle, \tag{5.18}$$

and hence

$$d_\lambda \langle P_\lambda, P_\mu\rangle = d_\mu \langle P_\lambda, P_\mu\rangle. \tag{5.19}$$

Under our assumption that $d_\lambda \neq d_\mu$ $(\lambda \neq \mu)$, we obtain $\langle P_\lambda, P_\mu\rangle = 0$ $(\lambda \neq \mu)$.

5.3 Commuting Family of q-Difference Operators

5.3.1 Macdonald–Ruijsenaars Operator of rth Order

For each $r = 0, 1, \ldots, n$, we define the *Macdonald–Ruijsenaars q-difference operator $D_x^{(r)}$* of rth order by

$$D_x^{(r)} = \sum_{I \subseteq \{1,\ldots,n\};\, |I| = r} A_I(x) T^I_{q,x}, \quad A_I(x) = t^{\binom{|I|}{2}} \prod_{i \in I,\, j \notin I} \frac{t x_i - x_j}{x_i - x_j}, \tag{5.20}$$

where $T^I_{q,x} = \prod_{i\in I} T_{q,x_i}$, so that $D_x^{(0)} = 1$, $D_x^{(1)} = D_x$ and $D_n^{(n)} = t^{\binom{n}{2}} T_{q,x_1} \cdots T_{q,x_n}$.

Example: $D_x^{(r)}$ $(n = 3,\ r = 1, 2, 3)$

$$\begin{aligned}
D_x^{(1)} &= \frac{(tx_1 - x_2)(tx_1 - x_3)}{(x_1 - x_2)(x_1 - x_3)} T_{q,x_1} + \frac{(tx_2 - x_1)(tx_2 - x_3)}{(x_2 - x_1)(x_2 - x_3)} T_{q,x_2} \\
&\quad + \frac{(tx_3 - x_1)(tx_3 - x_2)}{(x_3 - x_1)(x_3 - x_2)} T_{q,x_3} \\
D_x^{(2)} &= t\frac{(tx_1 - x_3)(tx_2 - x_3)}{(x_1 - x_3)(x_2 - x_3)} T_{q,x_1} T_{q,x_2} + t\frac{(tx_1 - x_2)(tx_3 - x_2)}{(x_1 - x_2)(x_3 - x_2)} T_{q,x_1} T_{q,x_3} \\
&\quad + t\frac{(tx_2 - x_1)(tx_3 - x_1)}{(x_2 - x_1)(x_3 - x_1)} T_{q,x_2} T_{q,x_3} \\
D_x^{(3)} &= t^3 T_{q,x_1} T_{q,x_2} T_{q,x_3}
\end{aligned} \tag{5.21}$$

Exercise 5.1 Show that the coefficients $A_I(x)$ can be expressed as

$$A_I(x) = \frac{T_{t,x}^I \Delta(x)}{\Delta(x)} \qquad (I \subseteq \{1, \dots, n\}) \tag{5.22}$$

in terms of the difference product $\Delta(x) = \prod_{1 \le i < j \le n} (x_i - x_j)$.

As we will see below, the q-difference operators $D_x^{(r)}$ $(r = 1, \dots, n)$ commute with each other, and are simultaneously diagonalized on $\mathbb{C}[x]^{\mathfrak{S}_n}$ by the Macdonald polynomials.

5.3.2 *Fundamental Properties of $D_x^{(r)}$*

By the same method as we applied to D_x, one can directly verify:

(1) The q-difference operators $D_x^{(r)}$ $(r = 1, \dots, n)$ are invariant under the action of $\mathfrak{S}_n$.
(2) The linear operators $D_x^{(r)} : \mathbb{C}(x) \to \mathbb{C}(x)$ stabilize $\mathbb{C}[x]^{\mathfrak{S}_n}$, i.e. $D_x^{(r)}(\mathbb{C}[x]^{\mathfrak{S}_n}) \subseteq \mathbb{C}[x]^{\mathfrak{S}_n}$.

As to the triangularity of $D_x^{(r)}$, we have:

Lemma 5.1 *The linear operators $D_x^{(r)} : \mathbb{C}[x]^{\mathfrak{S}_n} \to \mathbb{C}[x]^{\mathfrak{S}_n}$ $(r = 0, 1, \dots, n)$ are triangular with respect to the dominance order of $m_\lambda(x)$: For each $\lambda \in \mathcal{P}_n$,*

$$D_x^{(r)} m_\lambda(x) = \sum_{\mu \le \lambda} d_{\lambda,\mu}^{(r)} m_\mu(x) = d_\lambda^{(r)} m_\lambda(x) + \sum_{\mu < \lambda} d_{\lambda,\mu}^{(r)} m_\mu(x), \tag{5.23}$$

where $d_\lambda^{(r)} = d_{\lambda,\lambda}^{(r)} = e_r(t^\delta q^\lambda)$ are the elementary symmetric functions of degree r in $t^\delta q^\lambda = (t^{n-1} q^{\lambda_1}, t^{n-2} q^{\lambda_2}, \dots, q^{\lambda_n})$.

Proof We follow the same approach as in the case of $D_x = D_x^{(1)}$ (Lemma 4.1). For each $I \subseteq \{1, \ldots, n\}$ with $|I| = r$, we have

$$A_I(x) = t^{\binom{r}{2}} \prod_{\substack{i<j \\ i\in I,\, j\notin I}} t\, \frac{1 - x_j/tx_i}{1 - x_j/x_i} \prod_{\substack{i<j \\ i\notin I,\, j\in I}} \frac{1 - tx_j/x_i}{1 - x_j/x_i}$$
$$= t^{\sum_{i\in I}(n-i)} + \text{(lower-order terms)}, \tag{5.24}$$

where, for $I = \{i_1 < \cdots < i_r\}$, the exponent of t is computed as

$$\binom{r}{2} + \#\{(i, j) \mid i < j,\ i \in I, j \notin I\}$$
$$= \binom{r}{2} + \sum_{k=1}^{r} ((n - i_k) + (r - k)) = \sum_{i\in I} (n - i). \tag{5.25}$$

Hence, we have

$$D_x^{(r)} x^\mu = \sum_{|I|=r} A_I(x) q^{\sum_{i\in I} \mu_i} x^\mu$$
$$= \Big(\sum_{|I|=r} t^{\sum_{i\in I}(n-i)} q^{\sum_{i\in I} \mu_i} \Big) x^\mu + \text{lower-order terms} \tag{5.26}$$
$$= e_r(t^\delta q^\mu) x^\mu + \text{(lower-order terms)}. \tag{5.27}$$

This implies

$$D_x^{(r)} m_\lambda(x) = e_r(t^\delta q^\lambda) m_\lambda(x) + \text{(lower-order terms)} \qquad (\lambda \in \mathcal{P}_n), \tag{5.28}$$

as desired. □

It is convenient to introduce the generating function for $D_x^{(r)}$ $(r = 0, 1, \ldots, n)$ with an extra parameter u:

$$D_x(u) = \sum_{r=0}^{n} (-u)^r D_x^{(r)} = \sum_{I\subseteq\{1,\ldots,n\}} (-u)^{|I|} A_I(x) T_{q,x}^I. \tag{5.29}$$

Then, by Lemma 5.1, we have

$$D_x(u) m_\lambda(x) = d_\lambda(u) m_\lambda(x) + \sum_{\mu<\lambda} d_\mu^\lambda(u)\, m_\mu(x),$$
$$d_\lambda(u) = \sum_{r=0}^{n} (-u)^r e_r(t^\delta q^\lambda) = \prod_{i=1}^{n} (1 - ut^{n-i} q^{\lambda_i}). \tag{5.30}$$

5.3.3 Macdonald Polynomials as Joint Eigenfunctions

We prove the following two theorems in the subsequent sections.

Theorem 5.3 *The q-difference operators $D_x^{(r)}$ $(r = 1, \ldots, n)$ commute with each other*:

$$D_x^{(r)} D_x^{(s)} = D_x^{(s)} D_x^{(r)} \qquad (r, s = 1, \ldots, n), \tag{5.31}$$

Theorem 5.4 *For each $\lambda \in \mathcal{P}_n$, the Macdonald polynomial $P_\lambda(x)$ satisfies the joint eigenfunction equations*

$$D_x^{(r)} P_\lambda(x) = d_\lambda^{(r)} P_\lambda(x), \quad d_\lambda^{(r)} = e_r(t^\delta q^\lambda) \quad (r = 1, \ldots, n). \tag{5.32}$$

We have assumed the genericity condition (4.10) of parameters for the existence of Macdonald polynomials, as well as $|q| < 1$. In this setting, Theorems 5.3 and 5.4 are equivalent. In fact:

Theorem 5.3 implies Theorem 5.4: By the commutativity of $D_x^{(r)}$ with $D_x = D_x^{(1)}$, we have

$$D_x D_x^{(r)} P_\lambda(x) = D_x^{(r)} D_x P_\lambda(x) = d_\lambda D_x^{(r)} P_\lambda(x), \tag{5.33}$$

namely $D_x^{(r)} P_\lambda(x)$ is an eigenfunction of D_x with eigenvalue d_λ. Since the eigenspace of D_x in $\mathbb{C}[x]^{\mathfrak{S}_n}$ with d_λ is one-dimensional, we have $D_x^{(r)} P_\lambda(x) = \varepsilon P_\lambda(x)$ for some constant $\varepsilon \in \mathbb{C}$. Since $P_\lambda(x) = m_\lambda(x) +$ (lower-order terms) and also $D_x^{(r)} m_\lambda(x) = d_\lambda^{(r)} m_\lambda(x) +$ (lower-order terms), we conclude $\varepsilon = d_\lambda^{(r)}$ as desired. Conversely:

Theorem 5.4 implies Theorem 5.3. Since $D_x^{(r)}$ $(r = 1, \ldots, n)$ are simultaneously diagonalized by $P_\lambda(x)$ $(\lambda \in \mathcal{P}_n)$, for any pair $r, s \in \{1, \ldots, n\}$ the commutator $[D_x^{(r)}, D_x^{(s)}] = D_x^{(r)} D_x^{(s)} - D_x^{(s)} D_x^{(r)}$ is 0 as a linear operator on $\mathbb{C}[x]^{\mathfrak{S}_n}$. From this, it follows that $[D_x^{(r)}, D_x^{(s)}] = 0$ as a q-difference operator thanks to the following lemma.

Lemma 5.2 *Let $L_x \in \mathcal{D}_{q,x} = \mathbb{C}(x)[T_{q,x}^{\pm 1}]$ be a q-difference operator with rational function coefficients, and suppose that $L_x f(x) = 0$ for all $f(x) \in \mathbb{C}[x]^{\mathfrak{S}_n}$. Then $L_x = 0$ as a q-difference operator.*

Proof Without losing generality, we may assume that L_x has the form

$$L_x = \sum_{\mu \in \mathbb{N}^n : |\mu| \le d} a_\mu(x) T_{q,x}^\mu, \quad d \in \mathbb{N}, \tag{5.34}$$

namely, $L_x \in \mathbb{C}(x)[T_{q,x}]$ and $\operatorname{ord} L_x \le d$. Supposing that $L_x|_{\mathbb{C}[x]^{\mathfrak{S}_n}} = 0$, we prove $L_x = 0$ by the induction on d. Since this statement is obvious for $d = 0$, we assume $d > 0$. Introducing variables $y = (y_1, \ldots, y_d)$, we consider the polynomial

$$F(x; y) = \prod_{i=1}^n \prod_{k=1}^d (1 - x_i y_k) \in \mathbb{C}[x]^{\mathfrak{S}_n}[y] \tag{5.35}$$

in (x, y). Then we have $L_x F(x; y) = 0$, namely

$$\sum_{|\mu| \leq d} a_\mu(x) F(q^\mu x; y) = \sum_{|\mu| \leq d} a_\mu(x) \prod_{i=1}^{n} \prod_{k=1}^{d} (1 - q^{\mu_i} x_i y_k) = 0. \tag{5.36}$$

For each $\alpha \in \mathbb{N}^n$ with $|\alpha| = d$, we define the reference point $\eta_\alpha(x) \in (\mathbb{C}^*)^d$ by

$$\eta_\alpha(x) = (1/x_1, 1/qx_1, \ldots, 1/q^{\alpha_1 - 1}x_1; \ldots; 1/x_n, 1/qx_n, \ldots, 1/q^{\alpha_n - 1}x_n). \tag{5.37}$$

Then we have

$$\begin{aligned} F(q^\mu x, \eta_\alpha(x)) &= \prod_{i=1}^{n} \prod_{j=1}^{n} \prod_{\nu=0}^{\alpha_j - 1} (1 - q^{\mu_i} x_i / q^\nu x_j) \\ &= \prod_{i=1}^{n} \prod_{j=1}^{n} (q^{\mu_i - \alpha_j + 1} x_i / x_j; q)_{\alpha_j} \end{aligned} \tag{5.38}$$

Note that $F(q^\mu x; \eta_\alpha(x))$ contains $\prod_{i=1}^{n} (q^{\mu_i - \alpha_i + 1}; q)_{\alpha_i}$ as diagonal factors. If $|\mu| \leq d$ and $\mu \neq \alpha$, there exists an index $i \in \{1, \ldots, n\}$ such that $\mu_i < \alpha_i$, and hence $(q^{\mu_i - \alpha_i + 1}; q)_{\alpha_i} = 0$. This means that, if $|\mu| \leq d$, $F(q^\mu x; \eta_\alpha(x)) = 0$ unless $\mu = \alpha$. Also, we have $F(q^\alpha x; \eta_\alpha(x)) = \prod_{i,j=1}^{n} (q^{\alpha_i - \alpha_j + 1} x_i / x_j; q)_{\alpha_j} \neq 0$. Hence, evaluating (5.36) at $y = \eta_\alpha(x)$, we obtain

$$L_x F(x, y)\big|_{y = \eta_\alpha(x)} = a_\alpha(x) F(q^\alpha x; \eta_\alpha(x)) = 0. \tag{5.39}$$

This implies that $a_\alpha(x) = 0$ for all $\alpha \in \mathbb{N}^n$ with $|\alpha| = 0$, namely ord $L_x < d$. Hence, by the induction on d we conclude that $L_x = 0$. □

5.4 Commutativity of the Operators $D_x^{(r)}$

In this section, we give two proofs of Theorem 5.3 of commutativity of the q-difference operators $D_x^{(r)}$ $(r = 1, \ldots, n)$. One proof, due to Macdonald [20], is based on the orthogonality of Macdonald polynomials, and the other is a direct proof due to Ruijsenaars [30]. Theorem 5.4 follows from Theorem 5.3 as we already explained in the previous section.

5.4.1 Orthogonality Implies Commutativity

One can show that, for each $r = 1, \dots, n$, $D_x^{(r)}$ is formally self-adjoint with respect to the scalar product defined by $w(x)$, by a method similar to the one we used in the case of $D_x = D_x^{(1)}$. Since $D_x^{(r)} : \mathbb{C}[x]^{\mathfrak{S}_n} \to \mathbb{C}[x]^{\mathfrak{S}_n}$ is lower triangular with respect to the dominance order, we have

$$D_x^{(r)} P_\lambda(x) = \sum_{\mu \le \lambda} a_{\lambda,\mu}^{(r)} P_\mu(x), \tag{5.40}$$

for some $a_{\lambda,\mu}^{(r)} \in \mathbb{C}$, with leading coefficient $a_{\lambda,\lambda}^{(r)} = d_\lambda^{(r)}$. Since

$$\left\langle D_x^{(r)} P_\lambda, P_\mu \right\rangle = a_{\lambda,\mu}^{(r)} \left\langle P_\mu, P_\mu \right\rangle, \quad \left\langle P_\lambda, D_x^{(r)} P_\mu \right\rangle = 0 \quad (\mu < \lambda), \tag{5.41}$$

and $\left\langle P_\mu, P_\mu \right\rangle \ne 0$, we have $a_{\lambda,\mu}^{(r)} = 0$ for $\mu < \lambda$. This means that $D_x^{(r)} P_\lambda(x) = d_\lambda^{(r)} P_\lambda(x)$. In this way, the linear operators $D_x^{(r)} : \mathbb{C}[x]^{\mathfrak{S}_n} \to \mathbb{C}[x]^{\mathfrak{S}_n}$ $(r = 1, \dots, n)$ are simultaneously diagonalized by the Macdonald basis. This gives a proof of Theorem 5.4, as well as Theorem 5.3 by the argument we already explained in the previous section.

5.4.2 A Direct Proof of Commutativity

Here we explain a direct proof of Theorem 5.3 of commutativity, following the idea of Ruijsenaars [30].

The composition $D_x^{(r)} D_x^{(s)}$ is computed as

$$D_x^{(r)} D_x^{(s)} = \sum_{|I|=r, |J|=s} A_I(x) A_J(q^{\varepsilon_I} x) T_{q,x}^{\varepsilon_I + \varepsilon_J}, \tag{5.42}$$

where $\varepsilon_I = \sum_{i \in I} \varepsilon_i$, $\varepsilon_i = (\delta_{i,j})_{1 \le j \le n} \in \mathbb{Z}^n$. Setting $K = I \cap J$, $L = (I \cup J) \backslash K$, $P = I \backslash K$, $Q = J \backslash K$, we rewrite (5.42) as

$$D_x^{(r)} D_x^{(s)} = \sum_{\substack{K \cap L = \phi \\ |K| \le \min\{r,s\}}} \Bigg(\sum_{\substack{P \sqcup Q = L \\ |K|+|P|=r, |K|+|Q|=s}} A_{K \sqcup P}(x) A_{K \sqcup Q}(q^{\varepsilon_K + \varepsilon_P} x) \Bigg) T_{q,x}^{2\varepsilon_K + \varepsilon_L}. \tag{5.43}$$

Then the commutativity $D_x^{(r)} D_x^{(s)} = D_x^{(s)} D_x^{(r)}$ is equivalent to the following statement: For each $K, L \subseteq \{1, \dots, n\}$ with $K \cap L = \phi$, and for any $p, q \in \mathbb{Z}_{\ge 0}$ such that $p + q = |L|$,

$$\sum_{\substack{P\sqcup Q=L\\|P|=p,|Q|=q}} A_{K\sqcup P}(x)A_{K\sqcup Q}(q^{\varepsilon_K+\varepsilon_P}x) = \sum_{\substack{P\sqcup Q=L\\|P|=p,|Q|=q}} A_{K\sqcup Q}(x)A_{K\sqcup P}(q^{\varepsilon_K+\varepsilon_Q}x). \tag{5.44}$$

Analyzing this equality carefully, we show that the statement (5.44) is reduced to an identity of rational functions, which we call the *Ruijsenaars identity*.

For each pair (I, J) of subsets of $\{1, \dots, n\}$ such that $I \cap J = \phi$, we set

$$A_{I,J}(x) = \prod_{i\in I;\ j\in J} \frac{1 - tx_i/x_j}{1 - x_i/x_j} \tag{5.45}$$

so that

$$A_I(x) = t^{\binom{|I|}{2}} A_{I,I^c}(x), \quad I^c = \{1, \dots, n\}\setminus I. \tag{5.46}$$

We use below the properties that $A_{I,J}(x)$ is *distributive* in I and J in the sense

$$A_{I_1\sqcup I_2,J}(x) = A_{I_1,J}(x)A_{I_2,J}(x), \quad A_{I,J_1\sqcup J_2}(x) = A_{I,J_1}(x)A_{I,J_2}(x), \tag{5.47}$$

and that $A_{I,J}(x)$ depends on the ratios x_i/x_j $(i \in I, j \in J)$ only.

We set $M = \{1, \dots, n\}\setminus(K \sqcup L)$, so that $K \sqcup P \sqcup Q \sqcup M = \{1, \dots, n\}$, to obtain

$$\begin{aligned}
&t^{-\binom{|K\sqcup P|}{2}-\binom{|K\sqcup Q|}{2}} A_{K\sqcup P}(x)A_{K\sqcup Q}(q^{\varepsilon_K+\varepsilon_P}x)\\
&= A_{K\sqcup P,M\sqcup Q}(x)A_{K\sqcup Q,M\sqcup P}(q^{\varepsilon_K+\varepsilon_P}x)\\
&= A_{K,M}(x)A_{K,Q}(x)A_{P,M}(x)A_{P,Q}(x)\\
&\quad\cdot A_{K,M}(q^{\varepsilon_K}x)A_{K,P}(q^{\varepsilon_K+\varepsilon_P}x)A_{Q,M}(x)A_{Q,P}(q^{\varepsilon_P}x)\\
&= A_{K,M}(x)A_{K,P}(x)A_{K,Q}(x)A_{P,M}(x)A_{Q,M}(x)A_{K,M}(q^{\varepsilon_K}x)\\
&\quad\cdot A_{P,Q}(x)A_{Q,P}(q^{\varepsilon_P}x)\\
&= A_{K,M}(x)A_{K,L}(x)A_{L,M}(x)A_{K,M}(q^{\varepsilon_K}x)\cdot A_{P,Q}(x)A_{Q,P}(q^{\varepsilon_P}x).
\end{aligned} \tag{5.48}$$

Exchanging the roles of P and Q, we have

$$\begin{aligned}
&t^{-\binom{|K\sqcup Q|}{2}-\binom{|K\sqcup P|}{2}} A_{K\sqcup Q}(x)A_{K\sqcup P}(q^{\varepsilon_K+\varepsilon_Q}x)\\
&= A_{K,M}(x)A_{K,L}(x)A_{L,M}(x)A_{K,M}(q^{\varepsilon_K}x)\cdot A_{Q,P}(x)A_{P,Q}(q^{\varepsilon_Q}x).
\end{aligned} \tag{5.49}$$

Hence, equality (5.44) is equivalent to:

$$\sum_{\substack{P\sqcup Q=L\\|P|=p,|Q|=q}} A_{P,Q}(x)A_{Q,P}(q^{\varepsilon_P}x) = \sum_{\substack{P\sqcup Q=L\\|P|=p,|Q|=q}} A_{Q,P}(x)A_{P,Q}(q^{\varepsilon_Q}x) \tag{5.50}$$

for any $L \subseteq \{1, \ldots, n\}$ and p, q with $p + q = |L|$.

Changing the notation, we see that the commutativity of the Macdonald–Ruijsenaars operators is reduced to proving the identity

$$\sum_{\substack{I \sqcup J = \{1,\ldots,n\} \\ |I|=r, |J|=s}} A_{I,J}(x) A_{J,I}(q^{\varepsilon_I} x) = \sum_{\substack{I \sqcup J = \{1,\ldots,n\} \\ |I|=r, |J|=s}} A_{J,I}(x) A_{I,J}(q^{\varepsilon_J}) \tag{5.51}$$

for any r,s such that $r + s = n$. To be explicit,

Lemma 5.3 (Ruijsenaars identity) *For any $r, s \in \mathbb{Z}_{\geq 0}$ with $r + s = n$,*

$$\begin{aligned} &\sum_{\substack{I \sqcup J = \{1,\ldots,n\} \\ |I|=r,\, |J|=s}} \prod_{\substack{i \in I \\ j \in J}} \frac{(1 - t x_i/x_j)(1 - t x_j/q x_i)}{(1 - x_i/x_j)(1 - x_j/q x_i)} \\ &= \sum_{\substack{I \sqcup J = \{1,\ldots,n\} \\ |I|=r,\, |J|=s}} \prod_{\substack{i \in I \\ j \in J}} \frac{(1 - t x_j/x_i)(1 - t x_i/q x_j)}{(1 - x_j/x_i)(1 - x_i/q x_j)}. \end{aligned} \tag{5.52}$$

Proof We denote by $F_{r,s}(x)$ the left-hand side of (5.52):

$$F_{r,s}(x) = \sum_{\substack{I \sqcup J = [n] \\ |I|=r,\, |J|=s}} \prod_{\substack{i \in I \\ j \in J}} F_{I,J}(x), \quad F_{I,J}(x) = \prod_{\substack{i \in I \\ j \in J}} \frac{(t x_i - x_j)(q x_i - t x_j)}{(x_i - x_j)(q x_i - x_j)}, \tag{5.53}$$

where $[n] = \{1, \ldots, n\}$. Then the right-hand side is given by $F_{r,s}(x^{-1}) = F_{s,r}(x)$. We remark that $F_{r,s}(x)$ is a symmetric function and $\Delta(x) F_{r,s}(x)$ is regular along the divisors $x_i - x_j = 0$ $(1 \leq i < j \leq n)$. From this fact it follows that $F_{r,s}(x)$ itself is regular along these divisors.

We prove by induction on n that $G_{r,s}(x) = F_{r,s}(x) - F_{r,s}(x^{-1}) = 0$ for any pair (r, s) such that $r + s = n$. We first remark that $G_{r,s}(x) = 0$ if $r = 0$ or $s = 0$, and that $G_{r,s}(x) = 0$ for $n = r + s = 0, 1$. Assuming that $r, s \geq 1$, we regard $F_{r,s}(x)$ as rational functions of x_n:

$$\begin{aligned} F_{r,s}(x) = &\sum_{\substack{I \sqcup J = [n] \\ |I|=r, |J|=s,\ n \in I}} \prod_{j \in J} \frac{(t x_n - x_j)(q x_n - t x_j)}{(x_n - x_j)(q x_n - x_j)} F_{I \setminus \{n\}, J}(x_{\widehat{n}}) \\ &+ \sum_{\substack{I \sqcup J = [n] \\ |I|=r, |J|=s,\ n \in J}} \prod_{i \in I} \frac{(x_n - t x_i)(t x_n - q x_i)}{(x_n - x_i)(x_n - q x_i)} F_{I \setminus \{n\}, J}(x_{\widehat{n}}), \end{aligned} \tag{5.54}$$

where $x_{\widehat{n}} = (x_1, \dots, x_{n-1})$. Note that $F_{r,s}(x)$ has at most simple poles at $x_n = qx_k, q^{-1}x_k$ for $k = 1, \dots, n-1$; it is regular at $x_n = x_k$ as mentioned above.[1] We look at the residues at $x_n = qx_k$:

$$
\begin{aligned}
&\mathrm{Res}(F_{r,s}(x)dx_n|x_n = qx_k)\\
&= \frac{(1-t)(t-q)}{q-1} \sum_{\substack{I\sqcup J=[n]\\ |I|=r,|J|=s;\ k\in I,n\in J}} \prod_{i\in I\setminus\{k\}} \frac{(qx_k - tx_i)(tx_k - x_i)}{(qx_k - x_i)(x_k - x_i)} F_{I,J\setminus\{n\}}(x_{\widehat{n}})\\
&= \frac{(1-t)(t-q)}{q-1} \sum_{\substack{I\sqcup J=[n]\\ |I|=r,|J|=s;\ k\in I,n\in J}} \prod_{i\in I\setminus\{k\}} \frac{(qx_k - tx_i)(tx_k - x_i)}{(qx_k - x_i)(x_k - x_i)}\\
&\qquad \cdot \prod_{j\in J\setminus\{n\}} \frac{(tx_k - x_j)(qx_k - tx_j)}{(x_k - x_j)(qx_k - x_j)} \cdot F_{I\setminus\{k\},J\setminus\{n\}}(x_{\widehat{n}})\\
&= \frac{(1-t)(t-q)}{q-1} \prod_{l\neq k,n} \frac{(tx_k - x_l)(qx_k - tx_l)}{(x_k - x_l)(qx_k - x_l)} \sum_{\substack{I'\sqcup J'=[n]\setminus\{k,n\}\\ |I'|=r-1,|J'|=s-1}} F_{I',J'}(x_{\widehat{k},\widehat{n}}),
\end{aligned}
\tag{5.55}
$$

where $x_{\widehat{n}} = (x_1, \dots, x_{n-1})$ and $x_{\widehat{k},\widehat{n}} = (x_1, \dots, \widehat{k}, \dots, x_{n-1})$. Similarly, we compute

$$
\begin{aligned}
&\mathrm{Res}(F_{r,s}(x^{-1})dx_n|x_n = qx_k)\\
&= \frac{(1-t)(t-q)}{q-1} \prod_{l\neq k,n} \frac{(tx_k - x_l)(qx_k - tx_l)}{(x_k - x_l)(qx_k - x_l)} \sum_{\substack{I'\sqcup J'=[n]\setminus\{k,n\}\\ |I'|=r-1,|J'|=s-1}} F_{J',I'}(x_{\widehat{x_k},\widehat{n}}).
\end{aligned}
\tag{5.56}
$$

Hence, for $G_{r,s}(x) = F_{r,s}(x) - F_{r,s}(x^{-1})$ we have

$$
\begin{aligned}
&\mathrm{Res}(G_{r,s}(x)dx_n|x_n = qx_k)\\
&= \frac{(1-t)(t-q)}{q-1} \prod_{l\neq k,n} \frac{(tx_k - x_l)(qx_k - tx_l)}{(x_k - x_l)(qx_k - x_l)}\\
&\quad \cdot \sum_{\substack{I'\sqcup J'=[n]\setminus\{k,n\}\\ |I'|=r-1,|J'|=s-1}} \big(F_{I',J'}(x_{\widehat{x_k},\widehat{n}}) - F_{J',I'}(x_{\widehat{x_k},\widehat{n}})\big) = 0
\end{aligned}
\tag{5.57}
$$

for $k = 1, \dots, n-1$, by the induction hypothesis of the case of $n-2$ variables. By the same argument we obtain $\mathrm{Res}(G_{r,s}(x)dx_n|x_n = q^{-1}x_k) = 0$ for $k = 1, \dots, n-1$. This implies that $G_{r,s}(x)$ is constant with respect to x_n. Since $G_{r,s}(x)$ is symmetric with respect to $x = (x_1, \dots, x_n)$, we conclude that $G_{r,s}(x)$ is a constant, i.e. does not depend on x_i $(i = 1, \dots, n)$. However, $G_{r,s}(x) = F_{r,s}(x) - F_{r,s}(x^{-1})$ satisfies $G_{r,s}(x^{-1}) = -G_{r,s}(x)$, and hence we obtain $G_{r,s}(x) = 0$. □

[1] One can also show directly that $\mathrm{Res}(F_{r,s}(x^{\pm 1})dx_n|x_n = x_k) = 0$ $(k = 1, \dots, n-1)$, by a computation similar to the one presented below.

We remark that Ruijsenaars [30] proved the commutativity of the elliptic version of $D_x^{(r)}$ $(r = 1, \ldots, n)$ along the same line as above, on the basis of the corresponding identity for the Weierstrass sigma functions.

Remark 5.2 In Chap. 8, we will explain a construction of the q-difference operators $D_x^{(r)}$ as well as their commutativity, following the idea of Cherednik based on a representation of the affine Hecke algebra.

5.5 Refinement of the Existence Theorem

Once commutativity of the Macdonald–Ruijsenaars operators $D_x^{(r)}$ $(r = 1, \ldots, n)$ has been established, the existence theorem of Macdonald polynomials can be refined as we formulate below. Here we fix the parameters $q, t \in \mathbb{C}^*$ with $|q| < 1$, and suppose that the parameter $t \in \mathbb{C}^*$ satisfies the condition $t^k \notin q^{\mathbb{Z}_{<0}}$ for $k = 1, \ldots, n-1$. In this setting we give a proof of existence of the Macdonald polynomials, independently of the previous existence theorem (Theorem 4.1).

Theorem 5.5 *Suppose that the parameter t satisfies the condition that $t^k \notin q^{\mathbb{Z}_{<0}}$ $(k = 1, \ldots, n-1)$. Then, for each partition $\lambda \in \mathcal{P}_n$ there exists a unique symmetric polynomial $P_\lambda(x) \in \mathbb{C}[x]^{\mathfrak{S}_n}$ such that*

$$(1)\quad D_x^{(r)} P_\lambda(x) = d_\lambda^{(r)} P_\lambda(x) \quad (r = 1, \ldots, n), \tag{5.58}$$

$$(2)\quad P_\lambda(x) = m_\lambda(x) + \sum_{\mu<\lambda} u_\mu^\lambda m_\mu(x) \quad (u_\mu^\lambda \in \mathbb{C}). \tag{5.59}$$

We remark that, in terms of the generating function $D_x(u) = \sum_{r=0}^{n} (-u)^r D_x^{(r)}$, the joint eigenfunction equations for $P_\lambda(x)$ are unified in the form

$$D_x(u) P_\lambda(x) = d_\lambda(u) P_\lambda(x), \quad d_\lambda(u) = \prod_{i=1}^{n} (1 - u t^{n-i} q^{\lambda_i}). \tag{5.60}$$

Note that, for a pair $\lambda, \mu \in \mathcal{P}_n$, $d_\lambda(u) = d_\mu(u)$ as polynomials in u if and only if there exists a permutation $\sigma \in \mathfrak{S}_n$ such that

$$t^{n-i} q^{\mu_i} = t^{n-\sigma(i)} q^{\lambda_{\sigma(i)}} \quad (i = 1, \ldots, n). \tag{5.61}$$

Under our assumption $|q| < 1$, we have:

Lemma 5.4 *Suppose that $t^k \notin q^{\mathbb{Z}_{<0}}$ $(k = 1, \ldots, n-1)$. Then, $d_\lambda(u) \neq d_\mu(u)$ for any distinct pair $\lambda, \mu \in \mathcal{P}_n$ as polynomials in u, and also for generic $u \in \mathbb{C}$.*

Proof We first show that, if $|t| \leq 1$, then $d_\lambda(u) \neq d_\mu(u)$ as polynomials in u for any distinct pair $\lambda, \mu \in \mathcal{P}_n$. Under the assumption $|t| \leq 1$, the sequence $|t^{n-i} q^{\lambda_i}|$

$(i = 1, \ldots, n)$ is weakly increasing for any $\lambda \in \mathcal{P}_n$. From this it follows that, if $d_\lambda(u) = d_\mu(u)$ for $\lambda, \mu \in \mathcal{P}_n$, then we have $|t^{n-i}q^{\lambda_i}| = |t^{n-i}q^{\mu_i}|$ $(i = 1, \ldots, n)$. Hence, for $i = 1, \ldots, n$, we have $|q|^{\lambda_i} = |q|^{\mu_i}$ and $\lambda_i = \mu_i$ since $|q| < 1$. Namely, if $|t| \le 1$, then $d_\lambda(u) = d_\mu(u)$ implies $\lambda = \mu$.

We now consider the case $|t| > 1$. Suppose that $d_\lambda(u) = d_\mu(u)$ as polynomials in u for some distinct pair $\lambda, \mu \in \mathcal{P}_n$. Then, there exists a permutation $\sigma \in \mathfrak{S}_n$ such that

$$t^{n-i}q^{\mu_i} = t^{n-\sigma(i)}q^{\lambda_{\sigma(i)}} \quad (i = 1, \ldots, n). \tag{5.62}$$

Since $\lambda \ne \mu$, we have $\sigma \ne 1$, and hence there exists an index $\sigma(i) > i$. Then we have $t^{\sigma(i)-i} = q^{\lambda_{\sigma(i)}-\mu_i} \in q^{\mathbb{Z}}$, which means $t^k \in q^{\mathbb{Z}}$ for $k = \sigma(i) - i \in \{1, \ldots, n-1\}$. Since $|t| > 1$, $t^k \in q^{\mathbb{Z}_{<0}}$ for some $k \in \{1, \ldots, n-1\}$.

Suppose that $d_\lambda(u) \ne d_\mu(u)$ for any distinct pair $\lambda, \mu \in \mathcal{P}_n$. Since the set

$$S = \left\{a \in \mathbb{C}^* \mid d_\lambda(a) = d_\mu(a) \text{ for some distinct pair } \lambda, \mu \in \mathcal{P}_n\right\} \tag{5.63}$$

is countable, the complement $\mathbb{C}^* \backslash S$ is non-empty. Then, for any $c \in \mathbb{C}^* \backslash S$, we have $d_\lambda(c) \ne d_\mu(c)$ for any distinct pair $\lambda, \mu \in \mathcal{P}_n$. □

Proof (***of Theorem*** 5.5) Under the assumption that $t^k \notin q^{\mathbb{Z}_{<0}}$ for $k = 1, \ldots, n-1$, by Lemma 5.4 we can find a constant $c \in \mathbb{C}$ such that $d_\lambda(c) \ne d_\mu(c)$ for any distinct pair $\lambda, \mu \in \mathcal{P}_n$. From the facts that $D_x(c) : \mathbb{C}[x]^{\mathfrak{S}_n} \to \mathbb{C}[x]^{\mathfrak{S}_n}$ is triangular with respect to the dominance order and that the eigenvalues $d_\lambda(c)$ separate $\mathcal{P}_n$, it follows that for each $\lambda \in \mathcal{P}_n$ there exists a unique symmetric polynomial $P_\lambda(x) \in \mathbb{C}[x]^{\mathfrak{S}_n}$ such that $P_\lambda(x) = m_\lambda(x) + (\text{lower-order terms}) \in \mathbb{C}[x]^{\mathfrak{S}_n}$ and $D_x(c)P_\lambda(x) = d_\lambda(c)P_\lambda(x)$. Note that $P_\lambda(x)$ $(\lambda \in \mathcal{P}_n)$ form a $\mathbb{C}$-basis of $\mathbb{C}[x]^{\mathfrak{S}_n}$, and have mutually distinct eigenvalues $d_\lambda(c)$ with respect to the linear operators $D_x(c)$. We remark that these $P_\lambda(x)$ do not depend on the choice of c, as we will see below.

Since $D_x^{(r)}$ commutes with $D_x(c)$ for $r = 1, \ldots, n$, we have $D_x(c)D_x^{(r)}P_\lambda(x) = D_x^{(r)}D_x(c)P_\lambda(c) = d_\lambda(c)D_x^{(r)}P_\lambda(x)$. This means that $D_x^{(r)}P_\lambda(x)$ is an eigenfunction of $D_x(c)$ with eigenvalue $d_\lambda(c)$, and hence $D_x^{(r)}P_\lambda(x)$ is a contant multiple of $P_\lambda(x)$ by the fact that the eigenspace of $D_x(c)$ with eigenvalue $d_\lambda(c)$ is one-dimensional. Since $D_x^{(r)}P_\lambda(x) = d_\lambda^{(r)}m_\lambda(x) + (\text{lower-order terms})$, we have $D_x^{(r)}P_\lambda(x) = d_\lambda^{(r)}P_\lambda(x)$. Namely, we obtain

$$\mathbb{C}[x]^{\mathfrak{S}_n} = \bigoplus_{\lambda \in \mathcal{P}_n} \mathbb{C}\, P_\lambda(x), \quad D_x(u)P_\lambda(x) = d_\lambda(u)P_\lambda(x). \tag{5.64}$$

This also implies that the polynomials $P_\lambda(x)$ do not depend on the choice of $c \in \mathbb{C}^*$ with which we started. □

5.6 Some Remarks Related to $D_x(u)$

5.6.1 Macdonald Polynomials in x^{-1}

Consider the q-difference operators $D^{(r)}_{x^{-1}}$ $(r = 0, 1, \ldots, n)$ in the variables $x^{-1} = (x_1^{-1}, \ldots, x_n^{-1})$ such that

$$D^{(r)}_{x^{-1}} f(x^{-1}) = D^{(r)}_x f(x)\Big|_{x\to x^{-1}}. \tag{5.65}$$

These operators are then explicitly given by

$$D^{(r)}_{x^{-1}} = \sum_{|I|=r} t^{\binom{r}{2}} \prod_{i\in I,\, j\notin I} \frac{tx_j - x_i}{x_j - x_i} \prod_{i\in I} T^{-1}_{q,x_i}. \tag{5.66}$$

Lemma 5.5 *For each $r = 0, 1, \ldots, n$,*

$$D^{(r)}_x = t^{(n-1)r-\binom{n}{2}} D^{(n-r)}_{x^{-1}} T_{q,x_1} \cdots T_{q,x_n}. \tag{5.67}$$

In terms of the generating function, we have

$$D_x(u) = (-u)^n t^{\binom{n}{2}} D_{x^{-1}}(u^{-1}t^{-n+1}) T_{q,x_1} \cdots T_{q,x_n}. \tag{5.68}$$

We leave the proof of this lemma as an exercise.

Let $\lambda \in \mathcal{P}_n$ be a partition and suppose that λ is contained in the $n \times l$ rectangle $(\lambda_1 \le l)$. Then we have

$$(x_1 \cdots x_n)^l P_\lambda(x^{-1}) = m_{(l^n)-\lambda^\vee}(x) + (\text{lower-order terms}) \in \mathbb{C}[x]^{\mathfrak{S}_n}, \tag{5.69}$$

where $\lambda^\vee = (\lambda_n, \ldots, \lambda_1)$ denotes the *reversal* of $\lambda = (\lambda_1, \ldots, \lambda_n)$.

Proposition 5.1 *For each partition $\lambda \in \mathcal{P}_n$ with $\lambda_1 \le l$, $l \in \mathbb{N}$, we have*

$$(x_1 \cdots x_n)^l P_\lambda(x^{-1}) = P_{(l^n)-\lambda^\vee}(x), \quad \lambda^\vee = (\lambda_n, \ldots, \lambda_1). \tag{5.70}$$

One can verify the eigenfunction equation

$$D_x(u)(x_1 \cdots x_n)^l P_\lambda(x^{-1}) = \prod_{i=1}^n (1 - ut^{n-i} q^{l-\lambda_{n+1-i}}) \cdot (x_1 \cdots x_n)^l P_\lambda(x^{-1}) \tag{5.71}$$

by using Lemma 5.5.

5.6.2 Determinant Representation of $D_x(u)$

The generating function $D_x(u)$ of the Macdonald–Ruijsenaars q-difference operators can also be expressed in terms of the determinant of a matrix of q-difference operators.

For an $n \times n$ matrix $L = (L_{ij})_{i,j=1}^n$ with entires in a ring, possibly non-commutative, we use the notation $\det(L)$ for the *column determinant*

$$\det(L) = \sum_{\sigma \in \mathfrak{S}_n} \operatorname{sgn}(\sigma) L_{\sigma(1)1} \cdots L_{\sigma(n)n}. \tag{5.72}$$

Theorem 5.6 *The generating function $D_x(u) = \sum_{r=0}^n (-u)^r D_x^{(r)}$ of the Macdonald–Ruijsenaars operators is represented by the column determinant*

$$\begin{aligned} D_x(u) &= \frac{1}{\Delta(x)} \det\Big(x_i^{n-j}\big(1 - ut^{n-j} T_{q,x_i}\big)\Big)_{i,j=1}^n \\ &= \frac{1}{\Delta(x)} \sum_{\sigma \in \mathfrak{S}_n} \operatorname{sgn}(\sigma) \prod_{j=1}^n x_{\sigma(j)}^{n-j}\big(1 - ut^{n-j} T_{q,x_{\sigma(j)}}\big). \end{aligned} \tag{5.73}$$

We remark that the q-difference operators $L_{ij} = x_i^{n-j}(1 - ut^{n-j} T_{q,x_i})$ satisfy the commutativity $L_{ij} L_{kl} = L_{kl} L_{ij}$ $(i \neq k)$. This implies that the product $\prod_{j=1}^n$ above does not depend on the ordering.

For a q-difference operator $L_x = \sum_{\mu \in \mathbb{Z}^n} a_\mu(x) T_{q,x}^\mu \in \mathbb{C}(x)[T_{q,x}^{\pm 1}]$, we define its *symbol* by

$$\operatorname{symb}(L_x) = \sum_{\mu \in \mathbb{Z}^n} a_\mu(x) \xi^\mu \in \mathbb{C}(x)[\xi^{\pm 1}], \quad \xi = (\xi_1, \ldots, \xi_n). \tag{5.74}$$

Note that two q-difference operators L_x, M_x coincide if $\operatorname{symb}(L_x) = \operatorname{symb}(M_x)$. We compute the symbol of $D_x(u)$ as follows:

$$\begin{aligned} &\operatorname{symb}(D_x(u)) \\ &= \sum_{I \subseteq \{1,\ldots,n\}} (-u)^{|I|} \frac{T_{t,x}^{\varepsilon_I} \Delta(x)}{\Delta(x)} \xi^{\varepsilon_I} = \frac{1}{\Delta(x)} \Big(\sum_{I \subseteq \{1,\ldots,n\}} (-u)^{|I|} \xi^{\varepsilon_I} T_{t,x}^{\varepsilon_I} \Big) \Delta(x) \\ &= \frac{1}{\Delta(x)} \prod_{i=1}^n (1 - u\, \xi_i\, T_{t,x_i}) \Delta(x) = \frac{1}{\Delta(x)} \det\big(x_i^{n-j}(1 - u\, t^{n-j} \xi_i)\big)_{i,j=1}^n \\ &= \frac{1}{\Delta(x)} \sum_{\sigma \in \mathfrak{S}_n} \operatorname{sgn}(\sigma) \prod_{j=1}^n x_{\sigma(j)}^{n-j} (1 - u\, t^{n-j} \xi_{\sigma(j)}), \end{aligned} \tag{5.75}$$

which coincides with the symbol of the right-hand side of (5.73).

5.6.3 *Limit to the Differential (Jack) Case*

If we set $q = e^{\varepsilon}$ with a small parameter ε, we have

$$\begin{aligned} T_{q,x_i} x^{\mu} &= q^{\mu_i} x^{\mu} = \sum_{k=0}^{\infty} \frac{(\mu_i \varepsilon)^k}{k!} x^{\mu} \\ &= \sum_{k=0}^{\infty} \frac{(\varepsilon x_i \partial_{x_i})^k}{k!} x^{\mu} = e^{\varepsilon x_i \partial_{x_i}} x^{\mu} = q^{x_i \partial_{x_i}} x^{\mu}. \end{aligned} \tag{5.76}$$

In view of this fact, we rewrite the q-shift operators as $T_{q,x_i} = q^{x_i \partial_{x_i}}$ by the Euler operators $x_i \partial_{x_i} = x_i \partial / \partial x_i$ $(i = 1, \ldots, n)$. Then we take the scaling limit of $D_x(u)/(1-q)^n$ as $q \to 1$ with $t = q^{\beta}$, $u = q^{v}$:

$$\begin{aligned} S_x(v) &= \lim_{q \to 1} \frac{1}{(1-q)^n} \left(D_x(q^v) \big|_{t=q^{\beta}} \right) \\ &= \frac{1}{\Delta(x)} \lim_{q \to 1} \det \left(x_i^{n-j} \frac{1 - q^{v + (n-j)\beta + x_i \partial_{x_i}}}{1-q} \right)_{i,j=1}^{n} \\ &= \frac{1}{\Delta(x)} \det \left(x_i^{n-j} (v + x_i \partial_{x_i} + (n-j)\beta) \right)_{i,j=1}^{n}. \end{aligned} \tag{5.77}$$

The resulting operator $S_x(v)$ satisfies

$$S_x(v) P_{\lambda}^{(\beta)}(x) = P_{\lambda}^{(\beta)}(x) \prod_{i=1}^{n} (v + \lambda_i + (n-i)\beta) \qquad (\lambda \in \mathcal{P}_n), \tag{5.78}$$

where $P_{\lambda}^{(\beta)}(x) = \lim_{q \to 1} P_{\lambda}(x; q, q^{\beta})$ are the *Jack polynomials*. Denoting by $S_x^{(r)}$ the coefficients of v^{n-r} of $S_x(v)$, we obtain a commuting family of differential operators $S_x^{(r)}$, called the *Sekiguchi–Debiard operators*, such that

$$S_x^{(r)} P_{\lambda}^{(\beta)}(x) = e_r(\lambda + \beta\delta) P_{\lambda}^{(\beta)}(x) \qquad (r = 0, 1, \ldots, n), \tag{5.79}$$

where $\delta = (n-1, n-2, \ldots, 0)$. The eigenvalues $e_r(\lambda + \beta\delta)$ are the rth elementary symmetric functions of $\lambda_i + (n-i)\beta$ $(i = 1, \ldots, n)$.

From the determinant representation (5.77), by a computation similar to that of (5.75) we obtain the following expression for the Sekiguchi–Debiard operators:

$$S_x^{(r)} = \sum_{|K|=r} \sum_{J \subseteq K} \beta^{|K \setminus J|} \frac{(x\partial_x)^{K \setminus J} (\Delta(x))}{\Delta(x)} (x\partial_x)^J \quad (r = 0, 1, \ldots, n), \tag{5.80}$$

where the sum is over all pairs (J, K) of subsets of $\{1, \ldots, n\}$ such that $|K| = r$ and $J \subseteq K$.[2] In particular, we have

$$S_x^{(1)} = \sum_{i=1}^{n} x_i \partial_{x_i} + \beta e_1(\delta),$$

$$S_x^{(2)} = \sum_{1 \le i < j \le n} x_i \partial_{x_i} x_j \partial_{x_j} + \beta \sum_{i=1}^{n} \left(e_1(\delta) - \sum_{j \ne i} \frac{x_i}{x_i - x_j} \right) x_i \partial_{x_i} + \beta^2 e_2(\delta), \quad (5.81)$$

where $e_1(\delta) = \frac{1}{2} n(n-1)$ and $e_2(\delta) = \frac{1}{24} n(n-1)(n-2)(3n-1)$. Recall that power sums are represented as

$$p_1 = e_1, \quad p_2 = e_1^2 - 2e_2, \quad p_3 = e_1^3 - 3e_1 e_2 + 3e_3, \quad \ldots, \quad (5.82)$$

by elementary symmetric functions. In view of these formulas, we introduce the differential operators $L_x^{(k)}$ $(k = 1, 2, \ldots)$ by

$$L_x^{(1)} = S_x^{(1)}, \; L_x^{(2)} = \left(S_x^{(1)}\right)^2 - 2S_x^{(2)}, \; L_x^{(3)} = \left(S_x^{(1)}\right)^3 - 3S_x^{(1)} S_x^{(2)} + 3S_x^{(3)}, \; \ldots . \quad (5.83)$$

Then we have

$$L_x^{(k)} P_\lambda^{(\beta)}(x) = p_k(\lambda + \beta\delta) P_\lambda^{(\beta)}(x) \quad (k = 1, 2, \ldots), \quad (5.84)$$

with eigenvalues $p_k(\lambda + \beta\delta) = \sum_{i=1}^{n} (\lambda_i + (n-i)\beta)^k$ expressed by power sums. Explicitly, $L_x^{(1)}$ and $L_x^{(2)}$ are given by

$$L_x^{(1)} = \sum_{i=1}^{n} x_i \partial_{x_i} + \beta p_1(\delta),$$

$$L_x^{(2)} = \sum_{i=1}^{n} \left(x_i \partial_{x_i}\right)^2 + 2\beta \sum_{i=1}^{n} \left(\sum_{j \ne i} \frac{x_i}{x_i - x_j} \right) x_i \partial_{x_i} + \beta^2 p_2(\delta), \quad (5.85)$$

where $p_1(\delta) = \frac{1}{2} n(n-1)$ and $p_2(\delta) = \frac{1}{6} n(n-1)(2n-1)$.[3] We now conjugate these operators by the power $\Delta(x)^\beta$ of the difference product:

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[2] For a differential operator $L_x = \sum_{\mu \in \mathbb{N}^n} a_\mu(x)(x\partial_x)^\mu$ (finite sum), consider the symbol $\mathrm{symb}(L_x) = \sum_{\mu \in \mathbb{N}^n} a_\mu(x)\lambda^\mu$ with $\lambda = (\lambda_1, \ldots, \lambda)$ regarded as variables. Note also that $L_x(x^\lambda) = \mathrm{symb}(L_x)x^\lambda$.

[3] Set $U_i(x) = \frac{x_i \partial_{x_i}(\Delta(x))}{\Delta(x)} = \sum_{j \ne i} \frac{x_i}{x_i - x_j}$ for each i, and $U_{ij}(x) = x_i \partial_{x_i}(U_j(x)) = \frac{x_i x_j}{(x_i - x_j)^2}$ for distinct pair i, j, so that $\frac{x_i \partial_{x_i} x_j \partial_{x_j}(\Delta(x))}{\Delta(x)} = U_i(x)U_j(x) + U_{ij}(x)$. Then we have $\sum_{i=1}^{n} U_i(x) = p_1(\delta)$ and $\sum_{i=1}^{n} U_i^2 - 2\sum_{1 \le i < j \le n} U_{ij}(x) = p_2(\delta)$. Use these formulas to derive (5.81) and (5.85).

$$P = \Delta(x)^{\beta} L_x^{(1)} \Delta(x)^{-\beta} = \sum_{i=1}^{n} x_i \partial_{x_i}, \tag{5.86}$$

$$H = \Delta(x)^{\beta} L_x^{(2)} \Delta(x)^{-\beta} = \sum_{i=1}^{n} \left(x_i \partial_{x_i}\right)^2 - 2\beta(\beta-1) \sum_{1 \le i < j \le n} \frac{x_i x_j}{(x_i - x_j)^2}. \tag{5.87}$$

Then the functions $\psi_\lambda(x) = P_\lambda^{(\beta)}(x)\Delta(x)^\beta$ ($\lambda \in \mathcal{P}_n$) satisfy

$$P\psi_\lambda(x) = p_1(\lambda + \beta\delta)\psi_\lambda(x), \quad H\psi_\lambda(x) = p_2(\lambda + \beta\delta)\psi_\lambda(x). \tag{5.88}$$

The operators P and H are the momentum operator and the Hamiltonian for the *Calogero–Sutherland model* with coupling constant β. Note that, in terms of the angular coordinates θ_i ($i = 1, \ldots, n$) such that $x_i = e^{\sqrt{-1}\theta_i}$, the operators P and H are expressed as

$$P = \frac{1}{\sqrt{-1}} \sum_{i=1}^{n} \partial_{\theta_i} \quad H = -\sum_{i=1}^{n} \partial_{\theta_i}^2 + \frac{\beta(\beta-1)}{2} \sum_{1 \le i < j \le n} \frac{1}{\sin^2 \frac{\theta_i - \theta_j}{2}}. \tag{5.89}$$