Chapter 2
Preliminaries on Symmetric Functions

Abstract In this section, we recall some basic material on symmetric functions as preliminaries to the theory of Schur functions and Macdonald polynomials.

2.1 Symmetric Functions $e_k(x)$, $h_k(x)$ and $p_k(x)$

We first introduce three sequences of symmetric polynomials in n variables $x = (x_1, \ldots, x_n)$ which are constantly used in the theory of symmetric functions. They are the *elementary symmetric functions* $e_k(x)$ $(k \geq 0)$, the *complete homogeneous symmetric functions* $h_k(x)$ $(k \geq 0)$, and the *power sum symmetric functions* $p_k(x)$ $(k \geq 1)$:

$$e_k(x) = \sum_{1 \leq i_1 < i_2 < \cdots < i_k \leq n} x_{i_1} x_{i_2} \cdots x_{i_k} \quad (k \geq 0), \qquad e_k(x) = 0 \quad (k > n), \tag{2.1}$$

$$h_k(x) = \sum_{\mu_1 + \mu_2 + \cdots + \mu_n = k} x_1^{\mu_1} x_2^{\mu_2} \cdots x_n^{\mu_n} = \sum_{1 \leq j_1 \leq \cdots \leq j_k \leq n} x_{j_1} x_{j_2} \cdots x_{j_k} \quad (k \geq 0), \tag{2.2}$$

$$p_k(x) = x_1^k + x_2^k + \cdots + x_n^k \quad (k \geq 1). \tag{2.3}$$

Conventionally, we define $e_k(x) = 0$, $h_k(x) = 0$ for $k < 0$. As for the power sums, we leave $p_0(x)$ undefined since it depends on the dimension n.

We introduce the generating functions

$$E(x; u) = \sum_{k=0}^{n} e_k(x) u^k = \prod_{i=1}^{n} (1 + x_i u), \tag{2.4}$$

$$H(x; u) = \sum_{k=0}^{\infty} h_k(x) u^k = \prod_{i=1}^{n} \frac{1}{1 - x_i u} \tag{2.5}$$

for the elementary and the complete homogenous symmetric functions, regarding them as formal power series in u: $E(x; u), H(x; u) \in \mathbb{C}[x][\![u]\!]$. The first equality (2.4) essentially represents the relationship between the coefficients of a general

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polynomial of degree n and its roots. The second equality (2.5) is verified as

$$
\begin{aligned}
\prod_{i=1}^{n} \frac{1}{1-x_i u} &= \frac{1}{1-x_1 u} \cdots \frac{1}{1-x_n u} \\
&= (1 + x_1 u + x_1^2 u^2 + \cdots) \cdots (1 + x_n u + x_n^2 u^2 + \cdots) \\
&= \sum_{\mu_1,\ldots,\mu_n \geq 0} x_i^{\mu_1} \cdots x_n^{\mu_n} u^{\mu_1+\cdots+\mu_n} \\
&= \sum_{k=0}^{\infty} \left(\sum_{\mu_1+\cdots+\mu_n=k} x_i^{\mu_1} \cdots x_n^{\mu_n} \right) u^k = \sum_{k=0}^{\infty} h_k(x) u^k. \qquad (2.6)
\end{aligned}
$$

Note also that the two generating functions $E(x; u)$, $H(x; u)$ are related through the formula

$$E(x; -u) H(x; u) = 1. \qquad (2.7)$$

The following identities of formal power series in z are frequently used in the theory of symmetric functions:

$$\log(1+z) = \sum_{k=1}^{\infty} (-1)^{k-1} \frac{z^k}{k}, \quad -\log(1-z) = \sum_{k=1}^{\infty} \frac{z^k}{k}, \qquad (2.8)$$

and

$$\exp\left(\sum_{k=1}^{\infty} (-1)^{k-1} \frac{z^k}{k} \right) = 1 + z, \quad \exp\left(\sum_{k=1}^{\infty} \frac{z^k}{k} \right) = \frac{1}{1-z}. \qquad (2.9)$$

We apply (2.9) to the generating function

$$P(x; u) = \sum_{k=1}^{\infty} p_k(x) \frac{u^k}{k} \qquad (2.10)$$

of the power sum symmetric functions. Then we obtain

$$
\begin{aligned}
\exp(P(x; u)) &= \exp\left(\sum_{k=1}^{\infty} p_k(x) \frac{u^k}{k} \right) = \exp\left(\sum_{k=1}^{\infty} \sum_{i=1}^{n} x_i^k \frac{u^k}{k} \right) \\
&= \prod_{i=1}^{n} \exp\left(\sum_{k=1}^{\infty} \frac{x_i^k u^k}{k} \right) = \prod_{i=1}^{n} \frac{1}{1-x_i u}. \qquad (2.11)
\end{aligned}
$$

Hence we have

$$\exp(P(x; u)) = H(x; u), \quad \exp(-P(x; u)) = E(x; -u). \qquad (2.12)$$

As we will see later, the identities (2.7), (2.12) of generating functions provide powerful tools for analyzing the relationship among the three sequences of symmetric functions $e_k(x)$, $h_k(x)$ and $p_k(x)$.

2.2 Fundamental Theorem of Symmetric Polynomials

We denote the ring of symmetric polynomials in x by

$$\mathbb{C}[x]^{\mathfrak{S}_n} = \{f \in \mathbb{C}[x] \mid \sigma(f) = f \quad (\sigma \in \mathfrak{S}_n)\} \subseteq \mathbb{C}[x]. \tag{2.13}$$

Then the *fundamental theorem of symmetric polynomials* is formulated as follows:

Theorem 2.1 *The ring $\mathbb{C}[x]^{\mathfrak{S}_n}$ of symmetric polynomials in n variables is generated as a $\mathbb{C}$-algebra by the elementary symmetric functions $e_1(x), \dots, e_n(x)$. Furthermore, $e_1(x), \dots, e_n(x)$ are algebraically independent over $\mathbb{C}$.*

This means that for any symmetric polynomial $f(x) \in \mathbb{C}[x]^{\mathfrak{S}_n}$ in $x = (x_1, \dots, x_n)$ there exists a unique polynomial $F(y) \in \mathbb{C}[y]$ in n variables $y = (y_1, \dots, y_n)$ such that

$$f(x) = F(e_1(x), \dots, e_n(x)). \tag{2.14}$$

We denote the $\mathbb{C}$-vector space of alternating polynomials in x by

$$\mathbb{C}[x]^{\mathfrak{S}_n,\mathrm{sgn}} = \{f \in \mathbb{C}[x] \mid \sigma(f) = \mathrm{sgn}(\sigma) f \quad (\sigma \in \mathfrak{S}_n)\} \subseteq \mathbb{C}[x], \tag{2.15}$$

where $\mathrm{sgn}(\sigma)$ denotes the sign of a permutation $\sigma \in \mathfrak{S}_n$. Among all nonzero alternating polynomials, the difference product (Vandermonde determinant)

$$\Delta(x) = \prod_{1 \le i < j \le n} (x_i - x_j) = \det(x_i^{n-j})_{i,j=1}^n, \quad \deg_x \Delta(x) = \binom{n}{2} \tag{2.16}$$

has the smallest degree. In fact we have:

Theorem 2.2 *Any alternating polynomial $f(x)$ in $x = (x_1, \dots, x_n)$ is expressed as the product $f(x) = \Delta(x) g(x)$ of the difference product $\Delta(x)$ and a symmetric polynomial $g(x) \in \mathbb{C}[x]^{\mathfrak{S}_n}$. Namely, $\mathbb{C}[x]^{\mathfrak{S}_n,\mathrm{sgn}} = \Delta(x)\mathbb{C}[x]^{\mathfrak{S}_n}$.*

Note that

$$\sigma(\Delta(x)) = \mathrm{sgn}(\sigma)\Delta(x) \qquad (\sigma \in \mathfrak{S}_n), \tag{2.17}$$

and that $\mathrm{sgn}(\sigma)$ is expressed as $\mathrm{sgn}(\sigma) = (-1)^{\ell(\sigma)}$ in terms of the *number of inversions* $\ell(\sigma)$ of σ defined by

$$\ell(\sigma) = \#\left\{(i,j) \in \{1,\ldots,n\}^2 \mid i < j \text{ and } \sigma(i) > \sigma(j)\right\}. \tag{2.18}$$

We will give proofs of Theorems 2.1 and 2.2 later in Sect. 2.5.

2.3 Wronski Relations and Newton Relations

Knowing that any symmetric polynomial can be expressed by elementary symmetric functions, it would be natural to ask how the complete homogenous functions $h_k(x)$ and the power sums $p_k(x)$ are expressed explicitly in terms of $e_k(x)$. Here are some examples: Suppressing the dependence on the x variables, we have

$$\begin{aligned} h_1 = e_1,\ h_2 = e_1^2 - e_2,\ h_3 = e_1^3 - 2e_1e_2 + e_3,\\ h_4 = e_1^4 - 3e_1^2e_2 + 2e_1e_3 + e_2^2 - e_4,\ \ldots, \end{aligned} \tag{2.19}$$

$$\begin{aligned} p_1 = e_1,\ p_2 = e_1^2 - 2e_2,\ p_3 = e_1^3 - 3e_1e_2 + 3e_3,\\ p_4 = e_1^4 - 4e_1^2e_2 + 4e_1e_3 + 2e_2^2 - 4e_4,\ \ldots. \end{aligned} \tag{2.20}$$

Formulas of this kind can be generated by means of the formulas (2.7) and (2.12) for the generating functions.

Formula (2.7) relating $E(x;u)$ and $H(x;u)$ is equivalent to the infinite number of relations

$$\sum_{i+j=k} (-1)^i e_i h_j = 0 \qquad (k = 1, 2, 3, \ldots), \tag{2.21}$$

called *Wronski's relations*. To be explicit,

$$h_1 - e_1 = 0,\ \ h_2 - e_1h_1 + e_2 = 0,,\ \ h_3 - e_1h_2 + e_2h_3 - e_3 = 0,\ \ \ldots. \tag{2.22}$$

Using these formulas recursively, we see that all h_k are expressed in terms of $e_1, \ldots, e_k$, and *vice versa*. Wronski's relations can also be formulated as the system of linear equations

$$\begin{bmatrix} 1 & & & & \\ e_1 & 1 & & 0 & \\ e_2 & e_1 & 1 & & \\ \vdots & \ddots & \ddots & \ddots & \\ e_{k-1} & \ldots & e_2 & e_1 & 1 \end{bmatrix} \begin{bmatrix} h_1 \\ -h_2 \\ h_3 \\ \vdots \\ (-1)^{k-1}h_k \end{bmatrix} = \begin{bmatrix} e_1 \\ e_2 \\ e_3 \\ \vdots \\ e_k \end{bmatrix}. \tag{2.23}$$

Then, by Cramer's formula we obtain

$$h_k = (-1)^{k-1} \det \begin{bmatrix} 1 & & & & e_1 \\ e_1 & 1 & & & e_2 \\ e_2 & e_1 & 1 & & e_3 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ e_{k-1} & \dots & e_2 & e_1 & e_k \end{bmatrix} = \det \begin{bmatrix} e_1 & 1 & & & \\ e_2 & e_1 & 1 & & \\ \vdots & \vdots & \ddots & \ddots & \\ e_{k-1} & e_{k-2} & \dots & e_1 & 1 \\ e_k & e_{k-1} & \dots & e_2 & e_1 \end{bmatrix}. \tag{2.24}$$

Formula (2.7) also implies

$$\begin{aligned} H(x;-u) &= E(x;u)^{-1} = (1 + e_1 u + e_2 u^2 + \cdots)^{-1} \\ &= \sum_{d=0}^{\infty} (-1)^d (e_1 u + e_2 u^2 + \cdots)^d \\ &= \sum_{d=0}^{\infty} (-1)^d \sum_{\mu_1+\mu_2+\cdots=d} \frac{d!}{\mu_1!\mu_2!\cdots} e_1^{\mu_1} e_2^{\mu_2} \cdots u^{\mu_1+2\mu_2+\cdots} \\ &= \sum_{k=0}^{\infty} \left(\sum_{\mu_1+2\mu_2+\cdots=k} \frac{(-1)^{|\mu|}|\mu|!}{\mu_1!\mu_2!\cdots} e_1^{\mu_1} e_2^{\mu_2} \cdots \right) u^k. \end{aligned} \tag{2.25}$$

Hence we obtain the explicit formula

$$h_k = \sum_{\|\mu\|=k} \frac{(-1)^{k-|\mu|}|\mu|!}{\mu_1!\mu_2!\cdots} e_1^{\mu_1} e_2^{\mu_2} \cdots, \quad \|\mu\| = \sum_{i\geq 1} i\mu_i, \tag{2.26}$$

expressing h_k in terms of $e_1, e_2, \dots, e_k$. Since the roles of e_i and h_j are interchangeable in (2.22), we also obtain

$$e_k = \sum_{\|\mu\|=k} \frac{(-1)^{k-|\mu|}|\mu|!}{\mu_1!\mu_2!\cdots} h_1^{\mu_1} h_2^{\mu_2} \cdots . \tag{2.27}$$

This implies that $\mathbb{C}[x]^{\mathfrak{S}_n} = \mathbb{C}[h_1, \dots, h_n]$ and that $h_1, \dots, h_n$ are algebraically independent as well. Since $e_k = 0$ $(k > n)$, each formula (2.27) for $k > n$ represents an explicit algebraic dependence among $h_1, h_2, \dots, h_k$.

Similar computations can be performed for the relationship between e_k and p_k. We apply the differential operator $u\partial_u$, $\partial_u = d/du$, to the second formula of (2.12) to obtain

$$-(u\partial_u P(x;u))E(x;-u) = u\partial_u E(x;-u). \tag{2.28}$$

This means that

$$-(p_1 u + p_2 u^2 + \cdots)(1 - e_1 u + e_2 u^2 - \cdots) = -e_1 u + 2e_2 u^2 - 3e_3 u^3 - \cdots, \tag{2.29}$$

and hence we obtain

$$p_k - e_1 p_{k-1} + \cdots + (-1)^{k-1} e_{k-1} p_1 + (-1)^k k e_k = 0 \quad (k = 1, 2, \ldots). \tag{2.30}$$

These recurrence relations between the elementary symmetric functions and the power sums are called *Newton's relations*. Rewriting these as a system of linear equations for $p_1, p_2, \ldots$, and then solving it by Cramer's formula, we obtain the determinant formula for p_k:

$$p_k = \det \begin{bmatrix} e_1 & 1 & & & \\ 2e_2 & e_1 & 1 & & \\ \vdots & \vdots & \ddots & \ddots & \\ (k-1)e_{k-1} & e_{k-2} & \ldots & e_1 & 1 \\ ke_k & e_{k-1} & \ldots & e_2 & e_1 \end{bmatrix}. \tag{2.31}$$

The second formula of (2.12) also implies

$$\begin{aligned} -P(x; -u) &= \log E(x; u) = \log(1 + e_1 u + e_2 u^2 + \cdots) \\ &= \sum_{d=1}^{\infty} (-1)^{d-1} \frac{1}{d} (e_1 u + e_2 u^2 + \cdots)^d \\ &= \sum_{d=1}^{\infty} (-1)^{d-1} \frac{1}{d} \sum_{\mu_1 + \mu_2 + \cdots = d} \frac{d!}{\mu_1! \mu_2! \cdots} e_1^{\mu_1} e_2^{\mu_2} \cdots u^{\mu_1 + 2\mu_2 + \cdots} \\ &= \sum_{k=1}^{\infty} \left(\sum_{\|\mu\| = k} \frac{(-1)^{|\mu|-1} (|\mu| - 1)!}{\mu_1! \mu_2! \cdots} e_1^{\mu_1} e_2^{\mu_2} \cdots \right) u^k. \end{aligned} \tag{2.32}$$

Hence we obtain

$$\frac{p_k}{k} = \sum_{\|\mu\| = k} \frac{(-1)^{k-|\mu|} (|\mu| - 1)!}{\mu_1! \mu_2! \cdots} e_1^{\mu_1} e_2^{\mu_2} \cdots \qquad (k = 1, 2, \ldots) \tag{2.33}$$

This also implies that $\mathbb{C}[x]^{\mathfrak{S}_n} = \mathbb{C}[p_1, \ldots, p_n]$ and that $p_1, \ldots, p_n$ are algebraically independent.

The method explained here can be applied to derive other formulas (recurrence formulas, determinant formulas and explicit formulas) representing e_k, h_k and p_k by each other.

2.4 Monomial Symmetric Functions

We have seen so far that the first n members (up to degree n) of any of the three sequences e_k, h_k, p_k can be taken as a generator system of the $\mathbb{C}$-algebra $\mathbb{C}[x]^{\mathfrak{S}_n}$. The monomial symmetric functions $m_\lambda(x)$, as well as the Schur functions $s_\lambda(x)$ which we will discuss later, appear as bases of $\mathbb{C}[x]^{\mathfrak{S}_n}$ regarded as a $\mathbb{C}$-vector space.

Let $f(x) \in \mathbb{C}[x]$ be an arbitrary polynomial in $x = (x_1, \dots, x_n)$, and express it as a finite sum of the form

$$f = f(x_1, \dots, x_n) = \sum_{\mu_1, \dots, \mu_n \geq 0} a_{\mu_1, \dots, \mu_n} x_1^{\mu_1} \cdots x_n^{\mu_n}. \tag{2.34}$$

Then the action of a permutation $\sigma \in \mathfrak{S}_n$ on f is defined by

$$\sigma(f) = f(x_{\sigma(1)}, \dots, x_{\sigma(n)}) = \sum_{\mu_1, \dots, \mu_n \geq 0} a_{\mu_1, \dots, \mu_n} x_{\sigma(1)}^{\mu_1} \cdots x_{\sigma(n)}^{\mu_n}. \tag{2.35}$$

We are using the same symbol σ of permutation for the $\mathbb{C}$-algebra automorphism of $\mathbb{C}[x]$ that maps x_i to $x_{\sigma(i)}$ $(i = 1, \dots, n)$. In what follows, we will freely use the *multi-index notation* for monomials in $x = (x_1, \dots, x_n)$: For each multi-index (or *composition* in combinatorial terminology) $\mu = (\mu_1, \dots, \mu_n) \in \mathbb{N}^n$, we set

$$x^\mu = x_1^{\mu_1} \cdots x_n^{\mu_n}, \quad \deg_x x^\mu = |\mu| = \mu_1 + \cdots + \mu_n. \tag{2.36}$$

Noting that the action of $\sigma \in \mathfrak{S}_n$ on x^μ is given by

$$\sigma(x^\mu) = x_{\sigma(1)}^{\mu_1} \cdots x_{\sigma(i)}^{\mu_i} \cdots x_{\sigma(n)}^{\mu_n} = x_1^{\mu_{\sigma^{-1}(1)}} \cdots x_j^{\mu_{\sigma^{-1}(j)}} \cdots x_n^{\mu_{\sigma^{-1}(n)}}, \tag{2.37}$$

we specify the (left) action of $\mathfrak{S}_n$ on $\mu \in \mathbb{N}^n$ as

$$\sigma.\mu = (\mu_{\sigma^{-1}(1)}, \dots, \mu_{\sigma^{-1}(n)}) \tag{2.38}$$

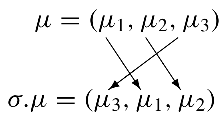
by permuting the positions (rather than the components). Then we have

$$\sigma(x^\mu) = x^{\sigma.\mu} \qquad (\mu \in \mathbb{N}^n, \sigma \in \mathfrak{S}_n). \tag{2.39}$$

Let us illustrate this definition with an example:

Action of a permutation on multi-indices

$n = 3, \quad \sigma = (123) = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}$ (cyclic permutation):



We now express an arbitrary polynomial $f(x) \in \mathbb{C}[x]$ as

$$f(x) = \sum_{\mu \in \mathbb{N}^n} a_\mu \, x^\mu \quad \text{(finite sum)} \tag{2.40}$$

in the multi-index notation, and rewrite the action of $\sigma \in \mathfrak{S}_n$ on f as

$$\sigma(f(x)) = \sum_{\mu \in \mathbb{N}^n} a_\mu \sigma(x^\mu) = \sum_{\mu \in \mathbb{N}^n} a_\mu x^{\sigma.\mu} = \sum_{\mu \in \mathbb{N}^n} a_{\sigma^{-1}.\mu} x^\mu, \tag{2.41}$$

where we have replaced μ by $\sigma^{-1}.\mu$ in the last step. Hence we have $\sigma(f(x)) = f(x)$ if and only if

$$a_\mu = a_{\sigma^{-1}.\mu} \quad \text{for all} \quad \mu \in \mathbb{N}^n. \tag{2.42}$$

This implies that $f(x)$ is a symmetric polynomial if and only if the coefficients a_μ, regarded as a function of $\mu \in \mathbb{N}^n$, are constant on each $\mathfrak{S}_n$-orbit in $\mathbb{N}^n$.

Note that, for any $\mu = (\mu_1, \ldots, \mu_n) \in \mathbb{N}^n$, the $\mathfrak{S}_n$-orbit $\mathfrak{S}_n.\mu \subseteq \mathbb{N}^n$ contains a unique partition $\lambda \in \mathcal{P}_n$ obtained by rearranging the components of μ. This means that the set $\mathcal{P}_n$ of partitions is a *transversal* (fundamental domain) of the $\mathfrak{S}_n$-set $\mathbb{N}^n$ (a complete set of representatives of the $\mathfrak{S}_n$-orbits in $\mathbb{N}^n$). For each $\lambda \in \mathcal{P}_n$, we denote by

$$m_\lambda(x) = \sum_{\mu \in \mathfrak{S}_n.\lambda} x^\mu = x^\lambda + \cdots \tag{2.43}$$

the sum of all monomials attached to the elements in $\mathfrak{S}_n.\lambda$. This $m_\lambda(x)$ is called the *monomial symmetric function* of monomial type λ; each monomial obtained from x^λ by permutation appears precisely once (with coefficient 1). An alternative definition of $m_\lambda(x)$ can be given as

$$m_\lambda(x) = \frac{1}{|\mathfrak{S}_{n,\lambda}|} \sum_{\sigma \in \mathfrak{S}_n} \sigma.x^\lambda = \frac{1}{|\mathfrak{S}_{n,\lambda}|} \sum_{\sigma \in \mathfrak{S}_n} x^{\sigma.\mu} \tag{2.44}$$

by symmetrizing the monomial x^λ, where $\mathfrak{S}_{n,\lambda} = \{\sigma \in \mathfrak{S}_n \mid \sigma.\lambda = \lambda\}$ denotes the stabilizer subgroup of λ. (See the examples given below.)

If $f(x) \in \mathbb{C}[x]$ is a symmetric polynomial, we have

$$\begin{aligned} f(x) &= \sum_{\mu \in \mathbb{N}^n} a_\mu x^\mu = \sum_{\lambda \in \mathcal{P}_n} \sum_{\mu \in \mathfrak{S}_n.\lambda} a_\mu x^\mu \\ &= \sum_{\lambda \in \mathcal{P}_n} a_\lambda \sum_{\mu \in \mathfrak{S}_n.\lambda} x^\mu = \sum_{\lambda \in \mathcal{P}_n} a_\lambda m_\lambda(x). \end{aligned} \tag{2.45}$$

This means that a polynomial $f(x) \in \mathbb{C}[x]$ is symmetric if and only if it is expressed as a finite linear combination of monomial symmetric functions

$$f(x) = \sum_{\lambda \in \mathcal{P}_n} a_\lambda \, m_\lambda(x) \quad \text{(finite sum)}. \tag{2.46}$$

Since $m_\lambda(x)$ $(\lambda \in \mathcal{P}_n)$ are linearly independent over $\mathbb{C}$, we conclude that they form a $\mathbb{C}$-basis of $\mathbb{C}[x]^{\mathfrak{S}_n}$.

Theorem 2.3 *The monomial symmetric functions $m_\lambda(x)$ $(\lambda \in \mathcal{P}_n)$ form a $\mathbb{C}$-basis of the ring $\mathbb{C}[x]^{\mathfrak{S}_n}$ of symmetric polynomials*:

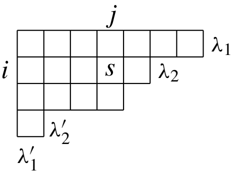
$$\mathbb{C}[x]^{\mathfrak{S}_n} = \bigoplus_{\lambda \in \mathcal{P}_n} \mathbb{C}\, m_\lambda(x). \tag{2.47}$$

□

In order to visualize a partition $\lambda = (\lambda_1, \lambda_2, \ldots) \in \mathcal{P}$, we frequently identify λ with the *diagram* of λ,

$$D(\lambda) = \{s = (i, j) \in \mathbb{Z} \times \mathbb{Z} \mid 1 \le i \le \ell(\lambda),\ \ 1 \le j \le \lambda_i\}, \tag{2.48}$$

or the *Young diagram*

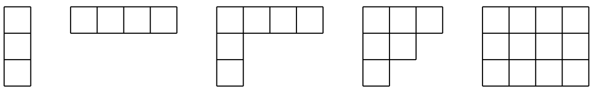
$D(\lambda) =$ 

(2.49)

of squares $s = (i, j)$ with rows and columns labeled by $i = 1, 2, \ldots$ and $j = 1, 2, \ldots$ respectively. By abuse of notation, we also write $s \in \lambda$ instead of $s \in D(\lambda)$. We define the *conjugate partition* (transpose) $\lambda' = (\lambda'_1, \lambda'_2, \ldots) \in \mathcal{P}$ of λ, denoting by $\lambda'_j = \#\{i \ge 1 \mid \lambda_i \ge j\}$ the number of squares in the jth column of $D(\lambda)$ for each $j = 1, 2, \ldots$.

Given a partition $\lambda = (\lambda_1, \lambda_2, \ldots)$, let $m_j \in \mathbb{N}$ be the number (multiplicity) of j's appearing in λ for $j = 1, 2, \ldots$. We often express a partition as $\lambda = (1^{m_1} 2^{m_2} \ldots)$, or $\lambda = (\ldots 2^{m_2} 1^{m_1})$, specifying the multiplicities of parts of λ.

Young diagrams of special shapes



(1^3) column — (4) row — (41^2) hook — (321) staircase — (4^3) rectangle

Monomial symmetric functions

(1) Single column $\lambda = (1^r) = (1, \ldots, 1, 0, \ldots, 0)$ with r 1's $(0 \leq r \leq n)$:

$$m_{(1^r)}(x) = x_1 \cdots x_r + \cdots = \sum_{1 \leq i_1 < \cdots < i_r \leq n} x_{i_1} \cdots x_{i_r} = e_r(x) \tag{2.50}$$

(2) Single row $\lambda = (l) = (l, 0, \ldots, 0)$ $(l = 1, 2, \ldots)$:

$$m_{(l)}(x) = x_1^l + \cdots = \sum_{i=1}^{n} x_i^l = p_l(x). \tag{2.51}$$

(3) When $n \geq 3$, there are three partitions $\lambda \in \mathcal{P}_n$ with $|\lambda| = 3$:

$$(3) = (3, 0, \ldots), \quad (21) = (2, 1, 0, \ldots), \quad (1^3) = (1, 1, 1, 0, \ldots). \tag{2.52}$$

Any homogeneous symmetric polynomial of degree 3 is a linear combination of the monomial symmetric functions $m_{(3)}(x)$, $m_{(21)}(x)$ and $m_{(1^3)}(x)$. When $n = 3$, they are given explicitly by

$$\begin{aligned} m_{(3)}(x) &= x_1^3 + x_2^3 + x_3^3, \\ m_{(21)}(x) &= x_1^2 x_2 + x_1^2 x_3 + x_1 x_2^2 + x_1 x_3^2 + x_2^2 x_3 + x_2 x_3^2, \\ m_{(1^3)}(x) &= x_1 x_2 x_3. \end{aligned} \tag{2.53}$$

Note that, if we symmetrize x_1^3, $x_1^2 x_2$, $x_1 x_2 x_3$ by $\mathfrak{S}_3$, we obtain $2m_{(3)}(x)$, $m_{(2,1)}(x)$, $6m_{(1^3)}(x)$, respectively, where 2, 1, 6 are the orders of the stabilizer subgroups of (3), (2, 1), (1^3).

Among all monomials x^μ appearing in $m_\lambda(x)$, x^λ is the leading (maximal) term with respect to the partial order $\leq$, called the *dominance order*. For $\mu, \nu \in \mathbb{N}^n$, the dominance order $\mu \leq \nu$ is defined by the condition

$$\mu_1 + \cdots + \mu_i \leq \nu_1 + \cdots + \nu_i \quad (i = 1, \ldots, n-1) \quad \text{and} \quad |\mu| = |\nu|. \tag{2.54}$$

Exercise 2.1 Prove the following:
(1) If $\lambda \in \mathcal{P}_n$ is a partition, then any $\mu \in \mathfrak{S}_n.\lambda$ satisfies $\mu \leq \lambda$.
(2) If $\mu, \nu \in \mathbb{N}^n$ and $\mu \leq \nu$, then $\mu \preceq_{\text{lex}} \nu$ under the lexicographic order of $\mathbb{N}^n$.[1]

Remark 2.1 We denote by $P = \mathbb{Z}^n = \mathbb{Z}\varepsilon_1 \oplus \cdots \oplus \mathbb{Z}\varepsilon_n$ the set of all multi-indices of integers, where ε_i $(i = 1, \ldots, n)$ are the unit vectors. In the language of rep-

[1] For $\mu, \nu \in \mathbb{N}^n$, $\mu \preceq_{\text{lex}} \nu$ means that, either $\mu = \nu$, or if $\mu \neq \nu$, then $\mu_k < \nu_k$ for the smallest index $k \in \{1, \ldots, n\}$ such that $\mu_k \neq \nu_k$.

resentation theory, P is the *weight lattice* of the general linear group GL_n. We extend the definition of the dominance order to P by the same condition (2.54). We remark that the dominance order $\mu \leq \nu$ for $\mu, \nu \in P$ is equivalent to $\nu - \mu \in Q_+ = \mathbb{N}\alpha_1 \oplus \cdots \oplus \mathbb{N}\alpha_{n-1}$, where $\alpha_i = \varepsilon_i - \varepsilon_{i+1} \in \mathbb{Z}^n$ $(i = 1, \ldots, n-1)$ are the *simple roots* of the root system of type A_{n-1}. This fact can be seen by the fact that the simple roots $\alpha_1, \ldots, \alpha_{n-1}$ together with $\alpha_n = \varepsilon_n$ form the dual basis of the *fundamental weights* $\varpi_i = \varepsilon_1 + \cdots + \varepsilon_i$ $(i = 1, \ldots, n)$, with respect to the standard scalar product on $P = \mathbb{Z}^n$ such that $\langle \varepsilon_i, \varepsilon_j \rangle = \delta_{i,j}$ $(i, j \in \{1, \ldots, n\})$.

2.5 Comments on Fundamental Theorems

In this section, we outline the proofs of Theorems 2.1 and 2.2.

For two sets of variables $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$ we consider the $\mathbb{C}$-algebra homomorphism $\phi : \mathbb{C}[y] \to \mathbb{C}[x]^{\mathfrak{S}_n}$ defined by

$$\phi(F(y)) = F(e_1(x), \ldots, e_n(x)) \qquad (F(y) \in \mathbb{C}[y]). \tag{2.55}$$

Note that this algebra homomorphism ϕ is uniquely determined by the condition $\phi(y_r) = e_r(x)$ $(r = 1, \ldots, n)$. Then, Theorem 2.1 is equivalent to saying that $\phi : \mathbb{C}[y] \to \mathbb{C}[x]^{\mathfrak{S}_n}$ is an isomorphism of $\mathbb{C}$-algebras.

We define the grading of $\mathbb{C}[y]$ by

$$\mathbb{C}[y] = \bigoplus_{d=0}^{\infty} \mathbb{C}[y]_d, \quad \mathbb{C}[y]_d = \bigoplus_{\nu \in \mathbb{N}^n;\, \|\nu\| = d} \mathbb{C}\, y^\nu \quad (d \in \mathbb{N}), \tag{2.56}$$

where $\|\nu\| = \nu_1 + 2\nu_2 + \cdots + n\nu_n$, assigning the degree $\deg_y y_r = r$ to each y_r $(r = 1, \ldots, n)$. Then $\phi : \mathbb{C}[y] \to \mathbb{C}[x]^{\mathfrak{S}_n}$ preserves the grading, with $\mathbb{C}[x]^{\mathfrak{S}_n}$ regarded as a graded algebra with $\deg_x x_i = 1$ $(i = 1, \ldots, n)$. Then we show that

$$\phi : \mathbb{C}[y]_d = \bigoplus_{\nu \in \mathbb{N}^n;\, \|\nu\| = d} \mathbb{C}\, y^\nu \to \mathbb{C}[x]_d^{\mathfrak{S}_n} = \bigoplus_{\lambda \in \mathcal{P}_n;\, |\lambda| = d} \mathbb{C}\, m_\lambda(x) \tag{2.57}$$

defines a $\mathbb{C}$-isomorphism for all $d = 0, 1, 2, \ldots$. In fact, for each $\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathcal{P}_n$ with $|\lambda| = \lambda_1 + \cdots + \lambda_n = d$, we express the conjugate partition $\lambda' \in \mathcal{P}$ as

$$\lambda = (\lambda'_1, \lambda'_2, \ldots, \lambda'_l) = (1^{\nu_1} 2^{\nu_2} \cdots n^{\nu_n}) \qquad (l = \lambda_1), \tag{2.58}$$

in terms of the multiplicities ν_i of i in λ'. Then the multi-index $\nu = (\nu_1, \ldots, \nu_n) \in \mathbb{N}^n$ satisfies

$$\|\nu\| = \nu_1 + 2\nu_2 + \cdots + n\nu_n = \lambda'_1 + \cdots + \lambda'_l = |\lambda'| = |\lambda| = d. \tag{2.59}$$

This correspondence $\lambda \to \nu$ defines a bijection

$$\{\lambda \in \mathcal{P}_n \mid |\lambda| = d\} \overset{\sim}{\to} \left\{\nu \in \mathbb{N}^n \mid \|\nu\| = d\right\} \tag{2.60}$$

between the two indexing sets. Note also that λ is determined from ν by $\lambda_i = \nu_i + \cdots + \nu_n$ $(i = 1, \dots, n)$. Under this correspondence, the image of y^ν by ϕ is computed as

$$\begin{aligned}
\phi(y^\nu) &= e_1(x)^{\nu_1} e_2(x)^{\nu_2} \cdots e_n(x)^{\nu_n} \\
&= (x_1 + \cdots)^{\nu_1} (x_1 x_2 + \cdots)^{\nu_2} \cdots (x_1 \cdots x_n)^{\nu_n} \\
&= x_1^{\nu_1} (x_1 x_2)^{\nu_2} \cdots (x_1 \cdots x_n)^{\nu_n} + (\text{lower-order terms}), \\
&= x^\lambda + (\text{lower-order terms}) \\
&= m_\lambda(x) + (\text{lower-order terms}),
\end{aligned} \tag{2.61}$$

with respect to the lexicographic order (as well as the dominance order) of $\mathbb{N}^n$. This triangularity of ϕ implies that $\phi : \mathbb{C}[y]_d \to \mathbb{C}[x]_d^{\mathfrak{S}_n}$ is an isomorphism of $\mathbb{C}$-vector space.

Example: $n = 5$, $\lambda = (7, 5, 4, 1, 0)$, $\lambda' = (4, 3, 3, 3, 2, 1, 1)$, $\nu = (2, 1, 3, 1, 0)$

To each column of length r, attach the elementary symmetric function e_r.

$$\begin{array}{cc|c|c|c|c|c|c|c|}
 & \lambda' & 4 & 3 & 3 & 3 & 2 & 1 & 1 \\
\cline{3-9}
 & 7 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
\cline{3-9}
\lambda & 5 & 2 & 2 & 2 & 2 & 2 \\
\cline{3-7}
 & 4 & 3 & 3 & 3 & 3 \\
\cline{3-6}
 & 1 & 4 \\
\cline{3-3}
 & 0
\end{array}
\qquad
\begin{aligned}
& e_4 e_3 e_3 e_3 e_2 e_1 e_1 \\
&= e_1^2 e_2 e_3^3 e_4 \\
&= x_1^2 (x_1 x_2)(x_1 x_2 x_3)^3 (x_1 x_2 x_3 x_4) + \cdots \\
&= x_1^7 x_2^5 x_3^4 x_4 + \cdots \\
&= m_{(7541)}(x) + \cdots
\end{aligned} \tag{2.62}$$

Example: Symmetric polynomials of degree 3

Note that $(3)' = (111)$, $(21)' = (21)$, $(1^3)' = 3$.

$$\begin{aligned}
e_1^3 &= m_{(3)} + 3m_{(21)} + 6m_{(1^3)}, & m_{(3)} &= e_1^3 - 3e_2 e_1 + 3e_3, \\
e_2 e_1 &= m_{(21)} + 3m_{(1^3)}, & m_{(21)} &= e_2 e_1 - 3e_3, \\
e_3 &= m_{(1^3)}, & m_{(1^3)} &= e_3.
\end{aligned} \tag{2.63}$$

Theorem 2.2 can be proved by using the factor theorem for polynomials in one variable. We prove that any alternating polynomial $f(x)$ in $x = (x_1, \ldots, x_n)$ is divisible by $\Delta(x)$ by the induction on the number of variables. Regard $f(x)$ as a polynomial $p(u) = f(u, x_2, \ldots, x_n) \in \mathbb{C}[x_2, \ldots, x_n][u]$ of the first variable. Since $f(x)$ is alternating, one has $p(x_j) = f(x_j, x_2, \ldots, x_n) = 0$ for $j = 2, \ldots, n$, and hence $p(u)$ is expressed as $p(u) = q(u)(u - x_2) \cdots (u - x_n)$ for some $q(u) \in C[x_2, \ldots, x_n][u]$, namely

$$f(x_1, \ldots, x_n) = g(x_1, x_2, \ldots, x_n) \prod_{j=2}^{n} (x_1 - x_j) \tag{2.64}$$

for some $g(x_1, \ldots, x_n) \in \mathbb{C}[x]$. Since $g(x)$ is alternating in $(x_2, \ldots, x_n)$, it is expressed as

$$g(x_1, \ldots, x_n) = h(x_1, \ldots, x_n) \Delta(x_2, \ldots, x_n) \tag{2.65}$$

with some $h(x) \in \mathbb{C}[x]$ by the induction hypothesis. Hence we obtain

$$\begin{aligned} f(x_1, \ldots, x_n) &= h(x_1, \ldots, x_n) \prod_{j=2}^{n} (x_1 - x_j)\, \Delta(x_2, \ldots, x_n) \\ &= h(x_1, \ldots, x_n) \Delta(x_1, \ldots, x_n). \end{aligned} \tag{2.66}$$

From $f(x), \Delta(x) \in \mathbb{C}[x]^{\mathfrak{S}_n, \mathrm{sgn}}$, it also follows that $h(x) \in \mathbb{C}[x]^{\mathfrak{S}_n}$.

Remark 2.2 The statements of Theorems 2.1 and 2.2 hold in a slightly more general setting, including the case of symmetric and alternating polynomials over $\mathbb{Z}$. In fact, we have the isomorphism

$$\phi: \quad R[y] \xrightarrow{\sim} R[x]^{\mathfrak{S}_n}, \quad \phi(y_i) = e_i(x) \quad (i = 1, \ldots, n), \tag{2.67}$$

of commutative rings, for any *integral domain*[2] R. We also have

$$R[x]^{\mathfrak{S}_n, \mathrm{sgn}} = \Delta(x) R[x]^{\mathfrak{S}_n} \tag{2.68}$$

provided that $1 \neq -1$ in the integral domain R. The proofs given above apply to this general setting without any essential change.

[2] A commutative ring with 1 satisfying the property that $f, g \in R$, $fg = 0 \implies (f = 0 \text{ or } g = 0)$.