SpringerBriefs in Mathematical Physics 50

Masatoshi Noumi

# Macdonald Polynomials

Commuting Family of $q$-Difference Operators and Their Joint Eigenfunctions

Springer

# SpringerBriefs in Mathematical Physics

Volume 50

SpringerBriefs are characterized in general by their size (50–125 pages) and fast production time (2–3 months compared to 6 months for a monograph).

Briefs are available in print but are intended as a primarily electronic publication to be included in Springer's e-book package.

**Typical works might include:**

- An extended survey of a field
- A link between new research papers published in journal articles
- A presentation of core concepts that doctoral students must understand in order to make independent contributions
- Lecture notes making a specialist topic accessible for non-specialist readers.

**SpringerBriefs in Mathematical Physics** showcase, in a compact format, topics of current relevance in the field of mathematical physics. Published titles will encompass all areas of theoretical and mathematical physics. This series is intended for mathematicians, physicists, and other scientists, as well as doctoral students in related areas.


- 50 – 125 published pages, including all tables, figures, and references
- Softcover binding
- Copyright to remain in author's name
- Versions in print, eBook, and MyCopy

Masatoshi Noumi

# Macdonald Polynomials

## Commuting Family of $q$-Difference Operators and Their Joint Eigenfunctions

Springer

Masatoshi Noumi
Department of Mathematics
Rikkyo University
Tokyo, Japan

ISSN 2197-1757 ISSN 2197-1765 (electronic)
SpringerBriefs in Mathematical Physics
ISBN 978-981-99-4586-3 ISBN 978-981-99-4587-0 (eBook)
https://doi.org/10.1007/978-981-99-4587-0



This Springer imprint is published by the registered company Springer Nature Singapore Pte Ltd.
The registered company address is: 152 Beach Road, #21-01/04 Gateway East, Singapore 189721, Singapore

# Preface

This book is intended to provide an introduction to the theory of Macdonald polynomials from the viewpoint of commuting $q$-difference operators and their joint eigenfunctions. It is an extended version of lecture notes for a series of online lectures "Introduction to Macdonald Polynomials," which I gave at KTH Royal Institute of Technology, Stockholm, during the period of February and March 2021.

Macdonald polynomials refer to a class of symmetric $q$-orthogonal polynomials in many variables. They include important classes of special functions such as Schur functions and Hall–Littlewood polynomials, and play important roles in various situations of mathematics and physics. After an overview of Schur functions, I will introduce Macdonald polynomials (of type $A$, in the $GL_{\mathrm{n}}$ version) as eigenfunctions of a $q$-difference operator, called the Macdonald–Ruijsenaars operator, in the ring of symmetric polynomials. Starting from this definition, I explain various remarkable properties of Macdonald polynomials such as orthogonality, evaluation formulas and self-duality, with emphasis on the roles of commuting $q$-difference operators.

The main reference for this theory is in Macdonald's book

> *Symmetric Functions and Hall Polynomials*. Second Edition. Oxford University Press, 1995, x+475 pp.
>
> Chapter VI: Symmetric functions with two parameters.

A characteristic feature of Macdonald's approach in his monograph is the use of symmetric functions in an *infinite number of variables*. In view of the introductory nature of this book, I decided to avoid the approach using infinite variables here, and to put more emphasis instead on the roles of the commuting family of $q$-difference operators for which Macdonald polynomials are joint eigenfunctions. I tried to make this book self-contained, and to give proofs to fundamental formulas in Macdonald theory within the framework of finite variables, as much as possible. I hope that this exposition will be helpful to a wider class of readers with various backgrounds.

In this book, I adopted the *classical* approach to Macdonald polynomials which does *not* rely on the theory of (double) affine Hecke algebras. For the Macdonald–Cherednik theory based on affine Hecke algebras, I refer the reader to Macdonald

[22], Cherednik [5] and other textbooks. In this direction, I only added a chapter on affine Hecke algebras and $q$-Dunkl operators, to provide an idea (without getting into the detail of proofs) about how the commuting family of $q$-difference operators arises in the framework of affine Hecke algebras.

I also included some materials which I could not deal with in the online lectures I gave at KTH. I really enjoyed meeting regularly online with many friends from various parts of the world, with whom I shared scientific interests and discussions. My thanks go to all the participants of the online lectures. I am grateful to the Knut and Alice Wallenberg Foundation for funding my guest professorship of the year 2020/2021 at KTH, which provided me with an invaluable opportunity of giving lectures and writing lecture notes on this subject of great concern to myself. Also, I would like to express my thanks to colleagues at KTH, especially Edwin Langmann and Jonatan Lenells, for their kind hospitality and friendship during my stay in Stockholm.

Tokyo, Japan Masatoshi Noumi

# Contents

# Chapter 1
# Overview of Macdonald Polynomials

**Abstract** The Macdonald polynomials are a family of symmetric polynomials in $n$ variables indexed by partitions. They are characterized as joint eigenfunctions of a commuting family of $q$-difference operators acting on the ring of symmetric polynomials. This chapter is a summary of the material which is developed in the rest of this book. Some of the notations used throughout this book is also introduced.

## 1.1 Macdonald Polynomials

We begin with an overview of the *Macdonald polynomials*[1]

$$P_\lambda(x) = P_\lambda(x; q, t) \in \mathbb{C}[x]^{\mathfrak{S}_n} \tag{1.1}$$

which we are going to discuss throughout this book. They are symmetric polynomials in $n$ variables $x = (x_1, \ldots, x_n)$ with parameters $q, t \in \mathbb{C}$, indexed by the partitions $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ with $\ell(\lambda) \le n$. By a *partition*, we mean a weakly decreasing sequence of nonnegative integers

$$\lambda = (\lambda_1, \lambda_2, \ldots); \quad \lambda_i \in \mathbb{N} = \mathbb{Z}_{\ge 0} \ \ (i = 1, 2, \ldots), \quad \lambda_1 \ge \lambda_2 \ge \cdots \ge 0, \tag{1.2}$$

with a finite number of *parts* (nonzero components); the 0's in the tail are frequently omitted. We denote by $\ell(\lambda) \in \mathbb{N}$ the number of parts of $\lambda$, and by $|\lambda| = \lambda_1 + \lambda_2 + \cdots$ the degree (sum of all parts) of $\lambda$. We denote by $\mathcal{P}$ the set of all partitions, and by $\mathcal{P}_n$ the set of all $\lambda \in \mathcal{P}$ with $\ell(\lambda) \le n$. We identify $\mathcal{P}_n$ with the indexing set of Macdonald polynomials:

---

[1] In this book, we use the term "Macdonald polynomials" in the narrow sense, meaning Macdonald polynomials of type $A_{n-1}$ (in the $\mathrm{GL}_n$ version). They are called the "symmetric functions with two parameters" in Macdonald's monograph [20, Chap. VI]. They are a special case of Macdonald polynomials associated with root systems, which are Weyl group invariant Laurent polynomials with parameters $q$ and $t = (t_\alpha)_\alpha$. The Macdonald polynomials associated with non-reduced root systems (of type $C^\vee C$ in the terminology of [22]) are called the *Koornwinder polynomials*.

 
M. Noumi, *Macdonald Polynomials*, SpringerBriefs in Mathematical Physics,
https://doi.org/10.1007/978-981-99-4587-0_1

$$\mathcal{P}_n = \left\{ \lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{N}^n \mid \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \geq 0 \right\}. \tag{1.3}$$

We denote by $\mathfrak{S}_n$ the *symmetric group* of degree $n$ (the set of all bijections $\sigma : \{1, \ldots, n\} \to \{1, \ldots, n\}$). It acts on the ring $\mathbb{C}[x] = \mathbb{C}[x_1, \ldots, x_n]$ of polynomials in $x = (x_1, \ldots, x_n)$ by permuting the indices of the variables $x_i$. We denote by $\mathbb{C}[x]^{\mathfrak{S}_n}$ the ring of symmetric ($\mathfrak{S}_n$-invariant) polynomials in $x$.

As a $\mathbb{C}$-vector space, $\mathbb{C}[x]^{\mathfrak{S}_n}$ has two fundamental bases,

$$\mathbb{C}[x]^{\mathfrak{S}_n} = \bigoplus_{\lambda \in \mathcal{P}_n} \mathbb{C}\, m_\lambda(x) = \bigoplus_{\lambda \in \mathcal{P}_n} \mathbb{C}\, s_\lambda(x), \tag{1.4}$$

both of which are indexed by $\mathcal{P}_n$. These symmetric polynomials

$$m_\lambda(x) = \sum_{\mu \in \mathfrak{S}_n.\lambda} x^\mu = x^\lambda + \cdots, \quad s_\lambda(x) = \frac{\det \left( x_i^{\lambda_j + n - j} \right)_{i,j=1}^n}{\det \left( x_i^{n-j} \right)_{i,j=1}^n} = x^\lambda + \cdots \tag{1.5}$$

are called the *monomial symmetric functions* (orbit sums) and the *Schur functions*,[2] respectively. Both $m_\lambda(x)$ and $s_\lambda(x)$ have the leading term $x^\lambda = x_1^{\lambda_1} \cdots x_n^{\lambda_n}$ with respect to a partial order $\leq$ of partitions, called the *dominance order* (see (2.54) for the definition). The Macdonald polynomials provide a family of $\mathbb{C}$-bases of $\mathbb{C}[x]^{\mathfrak{S}_n}$ with two parameters $(q, t)$, including $m_\lambda(x)$ and $s_\lambda(x)$ as special cases.

The Macdonald polynomials $P_\lambda(x; q, t)$ are defined (or characterized) as the eigenfunctions of the *Macdonald–Ruijsenaars $q$-difference operator*

$$D_x = \sum_{i=1}^n \prod_{\substack{1 \leq j \leq n \\ j \neq i}} \frac{t x_i - x_j}{x_i - x_j} T_{q,x_i} = \prod_{j=2}^n \frac{t x_1 - x_j}{x_1 - x_j} T_{q,x_1} + \cdots \tag{1.6}$$

acting on $\mathbb{C}[x]^{\mathfrak{S}_n}$. Here, $T_{q,x_i}$ stands for the *$q$-shift operator* with respect to the variable $x_i$: $T_{q,x_i} f(x_1, \ldots, x_i, \ldots, x_n) = f(x_1, \ldots, q x_i, \ldots, x_n)$.

**Theorem 1.1** (Macdonald) *For each partition $\lambda \in \mathcal{P}_n$ with $\ell(\lambda) \leq n$, there exists a unique symmetric polynomial $P_\lambda(x) = P_\lambda(x; q, t) \in \mathbb{C}[x]^{\mathfrak{S}_n}$ in $x$, homogeneous of degree $|\lambda|$ and depending rationally on $(q, t)$, such that*

$$\text{(1)} \quad D_x P_\lambda(x) = d_\lambda P_\lambda(x), \quad d_\lambda = q^{\lambda_1} t^{n-1} + q^{\lambda_2} t^{n-2} + \cdots + q^{\lambda_n}, \tag{1.7}$$

$$\text{(2)} \quad P_\lambda(x) = m_\lambda(x) + (\textit{lower-order terms with respect to } \leq). \tag{1.8}$$

This theorem will be proved in Sect. 4.1 (Theorem 4.1).

[2] Some people would restrict the usage of the term "symmetric functions" to the case of symmetric formal power series in an infinite number of variables $x = (x_1, x_2, \ldots)$. We will not strictly follow this rule, since polynomials are functions, whereas formal power series are *not* functions in general.

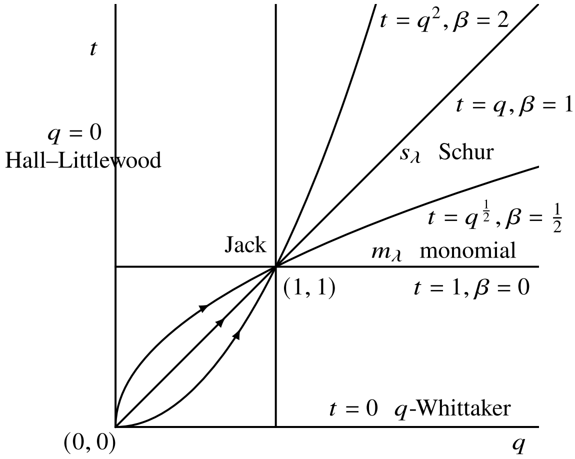



**Fig. 1.1** Space of parameters $(q, t)$

For generic $(q, t)$, the Macdonald polynomials $P_\lambda(x; q, t)$ $(\lambda \in \mathcal{P}_n)$ form a $\mathbb{C}$-basis of $\mathbb{C}[x]^{\mathfrak{S}_n}$:

$$\mathbb{C}[x]^{\mathfrak{S}_n} = \bigoplus_{\lambda \in \mathcal{P}_n} \mathbb{C}\, P_\lambda(x; q, t). \tag{1.9}$$

They specialize to $m_\lambda(x)$ when $t = 1$, and to $s_\lambda(x)$ when $t = q$. Also, in the limit as $q \to 1$ with scaling $t = q^\beta$, they recover the *Jack polynomials* $P_\lambda^{(\beta)}(x) = \lim_{q \to 1} P_\lambda(x; q, q^\beta)$.[3] Two other important special cases are the *Hall–Littlewood polynomials* $P_\lambda(x; t) = P_\lambda(x; 0, t)$ with $q = 0$, and the *q-Whittaker functions* $P_\lambda(x; q, 0)$ with $t = 0$ (Fig. 1.1).

We remark that the Jack polynomials $P_\lambda^{(\beta)}(x)$ are orthogonal polynomials associated with the Heckman–Opdam system (or Calogero–Sutherland system) of type $A_{n-1}$; we refer the reader to [15, Chap. 8] and Sect. 5.6 of this book for Heckman–Opdam and Calogero–Sutherland systems. They are the polynomial joint eigenfunctions of a commuting family of differential operators, called the *Sekiguchi–Debiard operators*. The Macdonald polynomials are also the orthogonal polynomials (polynomial joint eigenfunctions) associated with the commuting family of Macdonald–Ruijsenaars $q$-difference operators, which define a difference version of the differential system of Heckman–Opdam (relativistic version of the non-relativistic system of Calogero–Sutherland).

**Remark 1.1** In the parameterization $t = q^\beta$, the three values $\beta = \frac{1}{2}, 1, 2$ are special in this case of type $A_{n-1}$. The Jack polynomials $P_\lambda^{(\beta)}(x)$ for $\beta = \frac{1}{2}, 1, 2$ arise as the zonal spherical functions associated with finite-dimensional representations of the symmetric pairs $(\mathfrak{g}, \mathfrak{k}) = (\mathfrak{gl}_n, \mathfrak{so}_n)$, $(\mathfrak{gl}_n \times \mathfrak{gl}_n, \mathfrak{gl}_n)$, $(\mathfrak{gl}_{2n}, \mathfrak{sp}_{2n})$, respectively (see Gangolli–Varadarajan [7] or Heckman–Schlichtkrull [11]). In particu-

[3] In Macdonald's monograph [20, Sect. VI.10], the notation $P_\lambda^{(\alpha)}$ is used for Jack polynomials with the convention $\alpha = 1/\beta$.

lar, $P_\lambda^{(\beta)}(x)$ with $\beta = \frac{1}{2}$ are called the *zonal polynomials*, and play crucial roles in statistics. The Macdonald polynomials $P_\lambda(x; q, t)$ with $t = q^{\frac{1}{2}}, q, q^2$ are similarly interpreted as the zonal spherical functions of the corresponding quantum symmetric pairs $(U_q(\mathfrak{g}), U_q^{\mathrm{tw}}(\mathfrak{k}))$ (see Noumi [24], Noumi–Sugitani [28]). Here, $U_q(\mathfrak{g})$ denotes the standard quantized universal enveloping algebra of Drinfeld and Jimbo, whereas $U_q^{\mathrm{tw}}(\mathfrak{k})$ is a coideal subalgebra of $U_q(\mathfrak{g})$ corresponding to the subalgebra $U(\mathfrak{k}) \subseteq U(\mathfrak{g})$.

## 1.2 Fundamental Properties of Macdonald Polynomials

The Macdonald polynomials have various remarkable properties. We highlight below some of the fundamental properties of Macdonald polynomials, which are in fact intimately related with each other.

**(a) Specializations**: As we already mentioned above, from the Macdonald polynomials $P_\lambda(x; q, t)$, one can obtain the Schur, Jack, Hall–Littlewood and $q$-Whittaker functions by specializations or limiting procedures with respect to the parameters $(q, t)$.

**(b) Orthogonality**: When $q, t \in \mathbb{R}$ and $|q| < 1, |t| < 1$, the Macdonald polynomials $P_\lambda(x) = P_\lambda(x; q, t)$ are orthogonal polynomials on the torus $\mathbb{T}^n = \{|x_1| = \cdots = |x_n| = 1\}$ with respect to the scalar product defined by a certain weight function. Explicit formulas are also known for the square norms of $P_\lambda(x)$.

**(c) Commuting family of $q$-difference operators**: There exists a commuting family of higher-order $q$-difference operators $D_x^{(1)}, \ldots, D_x^{(n)}$ with $D_x^{(1)} = D_x$, acting on the ring $\mathbb{C}[x]^{\mathfrak{S}_n}$ of symmetric polynomials. The operators $D_x^{(r)}$ $(r = 1, \ldots, n)$ are algebraically independent, and the Macdonald polynomials $P_\lambda(x)$ are joint eigenfunctions of them. See Sect. 5.3 for the explicit formulas of these operators.

**(d) Principal specialization and self-duality**: The value of $P_\lambda(x)$ at the base point $x = t^\delta = (t^{n-1}, t^{n-2}, \ldots, 1)$ can be evaluated explicitly as a product of simple factors. Also, the normalized Macdonald polynomials $\widetilde{P}_\lambda(x) = P_\lambda(x)/P_\lambda(t^\delta)$ are self-dual in the sense $\widetilde{P}_\lambda(t^\delta q^\mu) = \widetilde{P}_\mu(t^\delta q^\lambda)$ with respect to discrete sets of the position variables $x = q^\mu t^\delta$ and the spectral variables $\xi = q^\lambda t^\delta$.

**(e) Pieri formula**: The Macdonald polynomial $P_\mu(x)$ of degree $d$ multiplied by the elementary symmetric function $e_r(x)$ of degree $r$ $(r = 0, 1, 2, \ldots, n)$ can be expanded into a linear combination of Macdonald polynomials of degree $d + r$ with explicitly determined coefficients. This Pieri formula is obtained from the eigenfunction equations for the higher-order $q$-difference operator $D_x^{(r)}$ via the self-duality of Macdonald polynomials.

**(f) Recurrence formula and tableau representation**: The Macdonald polynomials of $n$ variables $x = (x_1, \ldots, x_n)$ admit a recurrence formula regarding the number of variables with explicitly determined coefficients. A repeated application of this recurrence formula leads to an explicit formula for $P_\lambda(x)$, called the *tableau representation*, as a sum of certain weights over all semi-standard tableaux.

These materials will be discussed in subsequent chapters, after preliminaries on symmetric functions and the Schur functions. There are several topics which we will not cover, such as the integral forms and combinatorial formulas for Macdonald polynomials, for which we refer the reader to some other textbooks and individual papers (see Haglund [10] and Ram–Yip [29] for example).

## 1.3 Outlook

The Macdonald polynomials can be regarded as a class of classical orthogonal polynomials and special functions of hypergeometric type in many variables. From that viewpoint, the *Encyclopedia of Special Functions* [15], published recently, would be a helpful guide for learning various backgrounds and recent developments around the subjects of Macdonald polynomials and Koornwinder polynomials.

It is one of the main problems of special functions in many variables to find *good* commuting families of linear differential/difference operators and to understand their joint eigenfunctions in appropriate function spaces. Typically, we consider linear operators which are invariant under the action of a Weyl group. In physics terminology, such problems could be equivalently formulated as (one-dimensional) integrable quantum many-body problems of Calogero type, where the existence of a sufficiently large family of commuting linear operators is interpreted as quantum integrability. Systems of commuting differential and difference operators are called *non-relativistic* and *relativistic* respectively, according to Ruijsenaars. Also, depending on the nature of functions appearing as coefficients of the linear operators, we distinguish three hierarchies: *rational*, *trigonometric* and *elliptic*. (See Remark 3.2 for the three variations of Cauchy's lemma.) For these problems, it would be important to pursue systematic approaches which cover quantum integrable systems, both differential and difference, and of all three cases with rational, trigonometric and elliptic coefficients.

In terms of the angular coordinates $\theta_i$ $(i = 1, \ldots, n)$ such that $x_i = e^{\sqrt{-1}\theta_i}$, the Jack polynomials and the Macdonald polynomials are concerned with differential and difference systems of trigonometric type, respectively. To be more precise, the self-duality (Sect. 6.1) implies that the Macdonald polynomials are considered as trigonometric both in position variables and in spectral variables (with respect to the Pieri formulas); this property is one of the characteristic features of Macdonald polynomials. On the other hand, the Jack polynomials are concerned with trigonometric differential systems in position variables and with rational difference systems in spectral variables. We also remark that, as a variant of the Macdonald–Ruijsenaars system of $A$ type, a coupled system of two groups of particles, called the *deformed Macdonald–Ruijsenaars system*, is introduced by Sergeev–Veselov [32], for which the eigenfunctions are described by the *super Macdonald polynomials*.

Finally, we give some comments on the difference systems with elliptic coefficients. The elliptic counterpart of the Macdonald–Ruijsenaars system (of $A$ type) is called the *elliptic Ruijsenaars system*, for which integrability was already established in the pioneering work of Ruijsenaars [30]. The $BC$ version of the elliptic Ruijsenaars system is called the *elliptic van Diejen system* (see van Diejen [33] and Komori–Hikami [13]). As for the eigenfunctions (eigenstates) of these systems with elliptic coefficients, however, our knowledge is rather restricted in comparison with the theory of Macdonald polynomials. For recent topics on eigenfunctions of the elliptic Ruijsenaars system, we refer to [17, 18] and references therein. For recent works on eigenfunctions of the elliptic van Diejen system, see [2, 34] for example. To completely understand eigenfunctions of these systems with elliptic coefficients would be one of the important, challenging problems in the theory of special functions in many variables.

# Chapter 2
# Preliminaries on Symmetric Functions

**Abstract** In this section, we recall some basic material on symmetric functions as preliminaries to the theory of Schur functions and Macdonald polynomials.

## 2.1 Symmetric Functions $e_k(x)$, $h_k(x)$ and $p_k(x)$

We first introduce three sequences of symmetric polynomials in $n$ variables $x = (x_1, \ldots, x_n)$ which are constantly used in the theory of symmetric functions. They are the *elementary symmetric functions* $e_k(x)$ $(k \geq 0)$, the *complete homogeneous symmetric functions* $h_k(x)$ $(k \geq 0)$, and the *power sum symmetric functions* $p_k(x)$ $(k \geq 1)$:

$$e_k(x) = \sum_{1 \leq i_1 < i_2 < \cdots < i_k \leq n} x_{i_1} x_{i_2} \cdots x_{i_k} \quad (k \geq 0), \qquad e_k(x) = 0 \quad (k > n), \tag{2.1}$$

$$h_k(x) = \sum_{\mu_1 + \mu_2 + \cdots + \mu_n = k} x_1^{\mu_1} x_2^{\mu_2} \cdots x_n^{\mu_n} = \sum_{1 \leq j_1 \leq \cdots \leq j_k \leq n} x_{j_1} x_{j_2} \cdots x_{j_k} \quad (k \geq 0), \tag{2.2}$$

$$p_k(x) = x_1^k + x_2^k + \cdots + x_n^k \quad (k \geq 1). \tag{2.3}$$

Conventionally, we define $e_k(x) = 0$, $h_k(x) = 0$ for $k < 0$. As for the power sums, we leave $p_0(x)$ undefined since it depends on the dimension $n$.

We introduce the generating functions

$$E(x; u) = \sum_{k=0}^{n} e_k(x) u^k = \prod_{i=1}^{n} (1 + x_i u), \tag{2.4}$$

$$H(x; u) = \sum_{k=0}^{\infty} h_k(x) u^k = \prod_{i=1}^{n} \frac{1}{1 - x_i u} \tag{2.5}$$

for the elementary and the complete homogenous symmetric functions, regarding them as formal power series in $u$: $E(x; u), H(x; u) \in \mathbb{C}[x][\![u]\!]$. The first equality (2.4) essentially represents the relationship between the coefficients of a general



M. Noumi, *Macdonald Polynomials*, SpringerBriefs in Mathematical Physics,
https://doi.org/10.1007/978-981-99-4587-0_2

polynomial of degree $n$ and its roots. The second equality (2.5) is verified as

$$\begin{aligned}\prod_{i=1}^{n}\frac{1}{1-x_iu} &= \frac{1}{1-x_1u}\cdots\frac{1}{1-x_nu}\\ &= (1+x_1u+x_1^2u^2+\cdots)\cdots(1+x_nu+x_n^2u^2+\cdots)\\ &= \sum_{\mu_1,\ldots,\mu_n\geq 0} x_i^{\mu_1}\cdots x_n^{\mu_n}u^{\mu_1+\cdots+\mu_n}\\ &= \sum_{k=0}^{\infty}\left(\sum_{\mu_1+\cdots+\mu_n=k} x_i^{\mu_1}\cdots x_n^{\mu_n}\right)u^k = \sum_{k=0}^{\infty}h_k(x)u^k. \end{aligned} \tag{2.6}$$

Note also that the two generating functions $E(x;u)$, $H(x;u)$ are related through the formula

$$E(x;-u)H(x;u) = 1. \tag{2.7}$$

The following identities of formal power series in $z$ are frequently used in the theory of symmetric functions:

$$\log(1+z) = \sum_{k=1}^{\infty}(-1)^{k-1}\frac{z^k}{k}, \quad -\log(1-z) = \sum_{k=1}^{\infty}\frac{z^k}{k}, \tag{2.8}$$

and

$$\exp\left(\sum_{k=1}^{\infty}(-1)^{k-1}\frac{z^k}{k}\right) = 1+z, \quad \exp\left(\sum_{k=1}^{\infty}\frac{z^k}{k}\right) = \frac{1}{1-z}. \tag{2.9}$$

We apply (2.9) to the generating function

$$P(x;u) = \sum_{k=1}^{\infty}p_k(x)\frac{u^k}{k} \tag{2.10}$$

of the power sum symmetric functions. Then we obtain

$$\begin{aligned}\exp(P(x;u)) &= \exp\left(\sum_{k=1}^{\infty}p_k(x)\frac{u^k}{k}\right) = \exp\left(\sum_{k=1}^{\infty}\sum_{i=1}^{n}x_i^k\frac{u^k}{k}\right)\\ &= \prod_{i=1}^{n}\exp\left(\sum_{k=1}^{\infty}\frac{x_i^ku^k}{k}\right) = \prod_{i=1}^{n}\frac{1}{1-x_iu}. \end{aligned} \tag{2.11}$$

Hence we have

$$\exp(P(x;u)) = H(x;u), \quad \exp(-P(x;u)) = E(x;-u). \tag{2.12}$$

As we will see later, the identities (2.7), (2.12) of generating functions provide powerful tools for analyzing the relationship among the three sequences of symmetric functions $e_k(x)$, $h_k(x)$ and $p_k(x)$.

## 2.2 Fundamental Theorem of Symmetric Polynomials

We denote the ring of symmetric polynomials in $x$ by

$$\mathbb{C}[x]^{\mathfrak{S}_n} = \{f \in \mathbb{C}[x] \mid \sigma(f) = f \quad (\sigma \in \mathfrak{S}_n)\} \subseteq \mathbb{C}[x]. \tag{2.13}$$

Then the *fundamental theorem of symmetric polynomials* is formulated as follows:

**Theorem 2.1** *The ring $\mathbb{C}[x]^{\mathfrak{S}_n}$ of symmetric polynomials in $n$ variables is generated as a $\mathbb{C}$-algebra by the elementary symmetric functions $e_1(x), \ldots, e_n(x)$. Furthermore, $e_1(x), \ldots, e_n(x)$ are algebraically independent over $\mathbb{C}$.*

This means that for any symmetric polynomial $f(x) \in \mathbb{C}[x]^{\mathfrak{S}_n}$ in $x = (x_1, \ldots, x_n)$ there exists a unique polynomial $F(y) \in \mathbb{C}[y]$ in $n$ variables $y = (y_1, \ldots, y_n)$ such that

$$f(x) = F(e_1(x), \ldots, e_n(x)). \tag{2.14}$$

We denote the $\mathbb{C}$-vector space of alternating polynomials in $x$ by

$$\mathbb{C}[x]^{\mathfrak{S}_n, \mathrm{sgn}} = \{f \in \mathbb{C}[x] \mid \sigma(f) = \mathrm{sgn}(\sigma) f \quad (\sigma \in \mathfrak{S}_n)\} \subseteq \mathbb{C}[x], \tag{2.15}$$

where $\mathrm{sgn}(\sigma)$ denotes the sign of a permutation $\sigma \in \mathfrak{S}_n$. Among all nonzero alternating polynomials, the difference product (Vandermonde determinant)

$$\Delta(x) = \prod_{1 \le i < j \le n} (x_i - x_j) = \det(x_i^{n-j})_{i,j=1}^n, \quad \deg_x \Delta(x) = \binom{n}{2} \tag{2.16}$$

has the smallest degree. In fact we have:

**Theorem 2.2** *Any alternating polynomial $f(x)$ in $x = (x_1, \ldots, x_n)$ is expressed as the product $f(x) = \Delta(x) g(x)$ of the difference product $\Delta(x)$ and a symmetric polynomial $g(x) \in \mathbb{C}[x]^{\mathfrak{S}_n}$. Namely, $\mathbb{C}[x]^{\mathfrak{S}_n, \mathrm{sgn}} = \Delta(x)\mathbb{C}[x]^{\mathfrak{S}_n}$.*

Note that

$$\sigma(\Delta(x)) = \mathrm{sgn}(\sigma)\Delta(x) \qquad (\sigma \in \mathfrak{S}_n), \tag{2.17}$$

and that $\mathrm{sgn}(\sigma)$ is expressed as $\mathrm{sgn}(\sigma) = (-1)^{\ell(\sigma)}$ in terms of the *number of inversions* $\ell(\sigma)$ of $\sigma$ defined by

$$\ell(\sigma) = \#\left\{(i, j) \in \{1, \ldots, n\}^2 \mid i < j \text{ and } \sigma(i) > \sigma(j)\right\}. \tag{2.18}$$

We will give proofs of Theorems 2.1 and 2.2 later in Sect. 2.5.

## 2.3 Wronski Relations and Newton Relations

Knowing that any symmetric polynomial can be expressed by elementary symmetric functions, it would be natural to ask how the complete homogenous functions $h_k(x)$ and the power sums $p_k(x)$ are expressed explicitly in terms of $e_k(x)$. Here are some examples: Suppressing the dependence on the $x$ variables, we have

$$\begin{aligned} h_1 = e_1,\ h_2 = e_1^2 - e_2,\ h_3 = e_1^3 - 2e_1e_2 + e_3,\\ h_4 = e_1^4 - 3e_1^2e_2 + 2e_1e_3 + e_2^2 - e_4,\ \ldots, \end{aligned} \tag{2.19}$$

$$\begin{aligned} p_1 = e_1,\ p_2 = e_1^2 - 2e_2,\ p_3 = e_1^3 - 3e_1e_2 + 3e_3,\\ p_4 = e_1^4 - 4e_1^2e_2 + 4e_1e_3 + 2e_2^2 - 4e_4,\ \ldots. \end{aligned} \tag{2.20}$$

Formulas of this kind can be generated by means of the formulas (2.7) and (2.12) for the generating functions.

Formula (2.7) relating $E(x; u)$ and $H(x; u)$ is equivalent to the infinite number of relations

$$\sum_{i+j=k} (-1)^i e_i h_j = 0 \qquad (k = 1, 2, 3, \ldots), \tag{2.21}$$

called *Wronski's relations*. To be explicit,

$$h_1 - e_1 = 0,\ \ h_2 - e_1h_1 + e_2 = 0,,\ \ h_3 - e_1h_2 + e_2h_3 - e_3 = 0,\ \ \ldots. \tag{2.22}$$

Using these formulas recursively, we see that all $h_k$ are expressed in terms of $e_1, \ldots, e_k$, and *vice versa*. Wronski's relations can also be formulated as the system of linear equations

$$\begin{bmatrix} 1 & & & & \\ e_1 & 1 & & 0 & \\ e_2 & e_1 & 1 & & \\ \vdots & \ddots & \ddots & \ddots & \\ e_{k-1} & \ldots & e_2 & e_1 & 1 \end{bmatrix} \begin{bmatrix} h_1 \\ -h_2 \\ h_3 \\ \vdots \\ (-1)^{k-1}h_k \end{bmatrix} = \begin{bmatrix} e_1 \\ e_2 \\ e_3 \\ \vdots \\ e_k \end{bmatrix}. \tag{2.23}$$

Then, by Cramer's formula we obtain

$$h_k = (-1)^{k-1} \det \begin{bmatrix} 1 & & & & e_1 \\ e_1 & 1 & & & e_2 \\ e_2 & e_1 & 1 & & e_3 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ e_{k-1} & \dots & e_2 & e_1 & e_k \end{bmatrix} = \det \begin{bmatrix} e_1 & 1 & & & \\ e_2 & e_1 & 1 & & \\ \vdots & \vdots & \ddots & \ddots & \\ e_{k-1} & e_{k-2} & \dots & e_1 & 1 \\ e_k & e_{k-1} & \dots & e_2 & e_1 \end{bmatrix}. \tag{2.24}$$

Formula (2.7) also implies

$$\begin{aligned} H(x; -u) &= E(x; u)^{-1} = (1 + e_1 u + e_2 u^2 + \cdots)^{-1} \\ &= \sum_{d=0}^{\infty} (-1)^d (e_1 u + e_2 u^2 + \cdots)^d \\ &= \sum_{d=0}^{\infty} (-1)^d \sum_{\mu_1+\mu_2+\cdots=d} \frac{d!}{\mu_1!\mu_2!\cdots} e_1^{\mu_1} e_2^{\mu_2} \cdots u^{\mu_1+2\mu_2+\cdots} \\ &= \sum_{k=0}^{\infty} \left( \sum_{\mu_1+2\mu_2+\cdots=k} \frac{(-1)^{|\mu|}|\mu|!}{\mu_1!\mu_2!\cdots} e_1^{\mu_1} e_2^{\mu_2} \cdots \right) u^k. \end{aligned} \tag{2.25}$$

Hence we obtain the explicit formula

$$h_k = \sum_{\|\mu\|=k} \frac{(-1)^{k-|\mu|}|\mu|!}{\mu_1!\mu_2!\cdots} e_1^{\mu_1} e_2^{\mu_2} \cdots, \quad \|\mu\| = \sum_{i \geq 1} i\mu_i, \tag{2.26}$$

expressing $h_k$ in terms of $e_1, e_2, \dots, e_k$. Since the roles of $e_i$ and $h_j$ are interchangeable in (2.22), we also obtain

$$e_k = \sum_{\|\mu\|=k} \frac{(-1)^{k-|\mu|}|\mu|!}{\mu_1!\mu_2!\cdots} h_1^{\mu_1} h_2^{\mu_2} \cdots . \tag{2.27}$$

This implies that $\mathbb{C}[x]^{\mathfrak{S}_n} = \mathbb{C}[h_1, \dots, h_n]$ and that $h_1, \dots, h_n$ are algebraically independent as well. Since $e_k = 0$ $(k > n)$, each formula (2.27) for $k > n$ represents an explicit algebraic dependence among $h_1, h_2, \dots, h_k$.

Similar computations can be performed for the relationship between $e_k$ and $p_k$. We apply the differential operator $u\partial_u$, $\partial_u = d/du$, to the second formula of (2.12) to obtain

$$-(u\partial_u P(x; u))E(x; -u) = u\partial_u E(x; -u). \tag{2.28}$$

This means that

$$-(p_1 u + p_2 u^2 + \cdots)(1 - e_1 u + e_2 u^2 - \cdots) = -e_1 u + 2e_2 u^2 - 3e_3 u^3 - \cdots, \tag{2.29}$$

and hence we obtain

$$p_k - e_1 p_{k-1} + \cdots + (-1)^{k-1} e_{k-1} p_1 + (-1)^k k e_k = 0 \quad (k = 1, 2, \ldots). \tag{2.30}$$

These recurrence relations between the elementary symmetric functions and the power sums are called *Newton's relations*. Rewriting these as a system of linear equations for $p_1, p_2, \ldots$, and then solving it by Cramer's formula, we obtain the determinant formula for $p_k$:

$$p_k = \det \begin{bmatrix} e_1 & 1 & & & \\ 2e_2 & e_1 & 1 & & \\ \vdots & \vdots & \ddots & \ddots & \\ (k-1)e_{k-1} & e_{k-2} & \ldots & e_1 & 1 \\ ke_k & e_{k-1} & \ldots & e_2 & e_1 \end{bmatrix}. \tag{2.31}$$

The second formula of (2.12) also implies

$$\begin{aligned} -P(x; -u) &= \log E(x; u) = \log(1 + e_1 u + e_2 u^2 + \cdots) \\ &= \sum_{d=1}^{\infty} (-1)^{d-1} \frac{1}{d} (e_1 u + e_2 u^2 + \cdots)^d \\ &= \sum_{d=1}^{\infty} (-1)^{d-1} \frac{1}{d} \sum_{\mu_1 + \mu_2 + \cdots = d} \frac{d!}{\mu_1! \mu_2! \cdots} e_1^{\mu_1} e_2^{\mu_2} \cdots u^{\mu_1 + 2\mu_2 + \cdots} \\ &= \sum_{k=1}^{\infty} \left( \sum_{\|\mu\| = k} \frac{(-1)^{|\mu|-1} (|\mu| - 1)!}{\mu_1! \mu_2! \cdots} e_1^{\mu_1} e_2^{\mu_2} \cdots \right) u^k. \end{aligned} \tag{2.32}$$

Hence we obtain

$$\frac{p_k}{k} = \sum_{\|\mu\| = k} \frac{(-1)^{k-|\mu|} (|\mu| - 1)!}{\mu_1! \mu_2! \cdots} e_1^{\mu_1} e_2^{\mu_2} \cdots \qquad (k = 1, 2, \ldots) \tag{2.33}$$

This also implies that $\mathbb{C}[x]^{\mathfrak{S}_n} = \mathbb{C}[p_1, \ldots, p_n]$ and that $p_1, \ldots, p_n$ are algebraically independent.

The method explained here can be applied to derive other formulas (recurrence formulas, determinant formulas and explicit formulas) representing $e_k$, $h_k$ and $p_k$ by each other.

## 2.4 Monomial Symmetric Functions

We have seen so far that the first $n$ members (up to degree $n$) of any of the three sequences $e_k, h_k, p_k$ can be taken as a generator system of the $\mathbb{C}$-algebra $\mathbb{C}[x]^{\mathfrak{S}_n}$. The monomial symmetric functions $m_\lambda(x)$, as well as the Schur functions $s_\lambda(x)$ which we will discuss later, appear as bases of $\mathbb{C}[x]^{\mathfrak{S}_n}$ regarded as a $\mathbb{C}$-vector space.

Let $f(x) \in \mathbb{C}[x]$ be an arbitrary polynomial in $x = (x_1, \ldots, x_n)$, and express it as a finite sum of the form

$$f = f(x_1, \ldots, x_n) = \sum_{\mu_1, \ldots, \mu_n \geq 0} a_{\mu_1, \ldots, \mu_n} x_1^{\mu_1} \cdots x_n^{\mu_n}. \tag{2.34}$$

Then the action of a permutation $\sigma \in \mathfrak{S}_n$ on $f$ is defined by

$$\sigma(f) = f(x_{\sigma(1)}, \ldots, x_{\sigma(n)}) = \sum_{\mu_1, \ldots, \mu_n \geq 0} a_{\mu_1, \ldots, \mu_n} x_{\sigma(1)}^{\mu_1} \cdots x_{\sigma(n)}^{\mu_n}. \tag{2.35}$$

We are using the same symbol $\sigma$ of permutation for the $\mathbb{C}$-algebra automorphism of $\mathbb{C}[x]$ that maps $x_i$ to $x_{\sigma(i)}$ $(i = 1, \ldots, n)$. In what follows, we will freely use the *multi-index notation* for monomials in $x = (x_1, \ldots, x_n)$: For each multi-index (or *composition* in combinatorial terminology) $\mu = (\mu_1, \ldots, \mu_n) \in \mathbb{N}^n$ , we set

$$x^\mu = x_1^{\mu_1} \cdots x_n^{\mu_n}, \quad \deg_x x^\mu = |\mu| = \mu_1 + \cdots + \mu_n. \tag{2.36}$$

Noting that the action of $\sigma \in \mathfrak{S}_n$ on $x^\mu$ is given by

$$\sigma(x^\mu) = x_{\sigma(1)}^{\mu_1} \cdots x_{\sigma(i)}^{\mu_i} \cdots x_{\sigma(n)}^{\mu_n} = x_1^{\mu_{\sigma^{-1}(1)}} \cdots x_j^{\mu_{\sigma^{-1}(j)}} \cdots x_n^{\mu_{\sigma^{-1}(n)}}, \tag{2.37}$$

we specify the (left) action of $\mathfrak{S}_n$ on $\mu \in \mathbb{N}^n$ as

$$\sigma.\mu = (\mu_{\sigma^{-1}(1)}, \ldots, \mu_{\sigma^{-1}(n)}) \tag{2.38}$$

by permuting the positions (rather than the components). Then we have

$$\sigma(x^\mu) = x^{\sigma.\mu} \qquad (\mu \in \mathbb{N}^n, \sigma \in \mathfrak{S}_n). \tag{2.39}$$

Let us illustrate this definition with an example:

**Action of a permutation on multi-indices**

$n = 3, \quad \sigma = (123) = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}$ (cyclic permutation):

$$\mu = (\mu_1, \mu_2, \mu_3)$$

$$\sigma.\mu = (\mu_3, \mu_1, \mu_2)$$

We now express an arbitrary polynomial $f(x) \in \mathbb{C}[x]$ as

$$f(x) = \sum_{\mu \in \mathbb{N}^n} a_\mu \, x^\mu \quad \text{(finite sum)} \tag{2.40}$$

in the multi-index notation, and rewrite the action of $\sigma \in \mathfrak{S}_n$ on $f$ as

$$\sigma(f(x)) = \sum_{\mu \in \mathbb{N}^n} a_\mu \sigma(x^\mu) = \sum_{\mu \in \mathbb{N}^n} a_\mu x^{\sigma.\mu} = \sum_{\mu \in \mathbb{N}^n} a_{\sigma^{-1}.\mu} x^\mu, \tag{2.41}$$

where we have replaced $\mu$ by $\sigma^{-1}.\mu$ in the last step. Hence we have $\sigma(f(x)) = f(x)$ if and only if

$$a_\mu = a_{\sigma^{-1}.\mu} \quad \text{for all} \quad \mu \in \mathbb{N}^n. \tag{2.42}$$

This implies that $f(x)$ is a symmetric polynomial if and only if the coefficients $a_\mu$, regarded as a function of $\mu \in \mathbb{N}^n$, are constant on each $\mathfrak{S}_n$-orbit in $\mathbb{N}^n$.

Note that, for any $\mu = (\mu_1, \ldots, \mu_n) \in \mathbb{N}^n$, the $\mathfrak{S}_n$-orbit $\mathfrak{S}_n.\mu \subseteq \mathbb{N}^n$ contains a unique partition $\lambda \in \mathcal{P}_n$ obtained by rearranging the components of $\mu$. This means that the set $\mathcal{P}_n$ of partitions is a *transversal* (fundamental domain) of the $\mathfrak{S}_n$-set $\mathbb{N}^n$ (a complete set of representatives of the $\mathfrak{S}_n$-orbits in $\mathbb{N}^n$). For each $\lambda \in \mathcal{P}_n$, we denote by

$$m_\lambda(x) = \sum_{\mu \in \mathfrak{S}_n.\lambda} x^\mu = x^\lambda + \cdots \tag{2.43}$$

the sum of all monomials attached to the elements in $\mathfrak{S}_n.\lambda$. This $m_\lambda(x)$ is called the *monomial symmetric function* of monomial type $\lambda$; each monomial obtained from $x^\lambda$ by permutation appears precisely once (with coefficient 1). An alternative definition of $m_\lambda(x)$ can be given as

$$m_\lambda(x) = \frac{1}{|\mathfrak{S}_{n,\lambda}|} \sum_{\sigma \in \mathfrak{S}_n} \sigma.x^\lambda = \frac{1}{|\mathfrak{S}_{n,\lambda}|} \sum_{\sigma \in \mathfrak{S}_n} x^{\sigma.\mu} \tag{2.44}$$

by symmetrizing the monomial $x^\lambda$, where $\mathfrak{S}_{n,\lambda} = \{\sigma \in \mathfrak{S}_n \mid \sigma.\lambda = \lambda\}$ denotes the stabilizer subgroup of $\lambda$. (See the examples given below.)

If $f(x) \in \mathbb{C}[x]$ is a symmetric polynomial, we have

$$\begin{aligned} f(x) &= \sum_{\mu \in \mathbb{N}^n} a_\mu x^\mu = \sum_{\lambda \in \mathcal{P}_n} \sum_{\mu \in \mathfrak{S}_n.\lambda} a_\mu x^\mu \\ &= \sum_{\lambda \in \mathcal{P}_n} a_\lambda \sum_{\mu \in \mathfrak{S}_n.\lambda} x^\mu = \sum_{\lambda \in \mathcal{P}_n} a_\lambda m_\lambda(x). \end{aligned} \tag{2.45}$$

This means that a polynomial $f(x) \in \mathbb{C}[x]$ is symmetric if and only if it is expressed as a finite linear combination of monomial symmetric functions

$$f(x) = \sum_{\lambda \in \mathcal{P}_n} a_\lambda \, m_\lambda(x) \quad \text{(finite sum)}. \tag{2.46}$$

Since $m_\lambda(x)$ $(\lambda \in \mathcal{P}_n)$ are linearly independent over $\mathbb{C}$, we conclude that they form a $\mathbb{C}$-basis of $\mathbb{C}[x]^{\mathfrak{S}_n}$.

**Theorem 2.3** *The monomial symmetric functions $m_\lambda(x)$ $(\lambda \in \mathcal{P}_n)$ form a $\mathbb{C}$-basis of the ring $\mathbb{C}[x]^{\mathfrak{S}_n}$ of symmetric polynomials*:

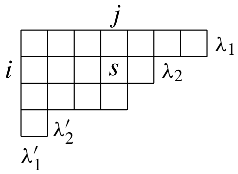
$$\mathbb{C}[x]^{\mathfrak{S}_n} = \bigoplus_{\lambda \in \mathcal{P}_n} \mathbb{C}\, m_\lambda(x). \tag{2.47}$$

□

In order to visualize a partition $\lambda = (\lambda_1, \lambda_2, \ldots) \in \mathcal{P}$, we frequently identify $\lambda$ with the *diagram* of $\lambda$,

$$D(\lambda) = \{s = (i, j) \in \mathbb{Z} \times \mathbb{Z} \mid 1 \le i \le \ell(\lambda),\ \ 1 \le j \le \lambda_i\}, \tag{2.48}$$

or the *Young diagram*

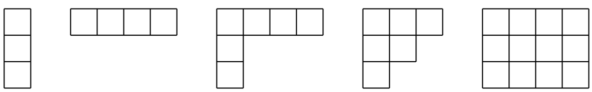
$D(\lambda) =$ 


(2.49)

of squares $s = (i, j)$ with rows and columns labeled by $i = 1, 2, \ldots$ and $j = 1, 2, \ldots$ respectively. By abuse of notation, we also write $s \in \lambda$ instead of $s \in D(\lambda)$. We define the *conjugate partition* (transpose) $\lambda' = (\lambda'_1, \lambda'_2, \ldots) \in \mathcal{P}$ of $\lambda$, denoting by $\lambda'_j = \#\{i \ge 1 \mid \lambda_i \ge j\}$ the number of squares in the $j$th column of $D(\lambda)$ for each $j = 1, 2, \ldots$.

Given a partition $\lambda = (\lambda_1, \lambda_2, \ldots)$, let $m_j \in \mathbb{N}$ be the number (multiplicity) of $j$'s appearing in $\lambda$ for $j = 1, 2, \ldots$. We often express a partition as $\lambda = (1^{m_1} 2^{m_2} \ldots)$, or $\lambda = (\ldots 2^{m_2} 1^{m_1})$, specifying the multiplicities of parts of $\lambda$.

**Young diagrams of special shapes**



$(1^3)$ column | $(4)$ row | $(41^2)$ hook | $(321)$ staircase | $(4^3)$ rectangle

**Monomial symmetric functions**

(1) Single column $\lambda = (1^r) = (1, \ldots, 1, 0, \ldots, 0)$ with $r$ 1's $(0 \le r \le n)$:

$$m_{(1^r)}(x) = x_1 \cdots x_r + \cdots = \sum_{1 \le i_1 < \cdots < i_r \le n} x_{i_1} \cdots x_{i_r} = e_r(x) \tag{2.50}$$

(2) Single row $\lambda = (l) = (l, 0, \ldots, 0)$ $(l = 1, 2, \ldots)$:

$$m_{(l)}(x) = x_1^l + \cdots = \sum_{i=1}^{n} x_i^l = p_l(x). \tag{2.51}$$

(3) When $n \ge 3$, there are three partitions $\lambda \in \mathcal{P}_n$ with $|\lambda| = 3$:

$$(3) = (3, 0, \ldots), \quad (21) = (2, 1, 0, \ldots), \quad (1^3) = (1, 1, 1, 0, \ldots). \tag{2.52}$$

Any homogeneous symmetric polynomial of degree 3 is a linear combination of the monomial symmetric functions $m_{(3)}(x)$, $m_{(21)}(x)$ and $m_{(1^3)}(x)$. When $n = 3$, they are given explicitly by

$$\begin{aligned} m_{(3)}(x) &= x_1^3 + x_2^3 + x_3^3, \\ m_{(21)}(x) &= x_1^2 x_2 + x_1^2 x_3 + x_1 x_2^2 + x_1 x_3^2 + x_2^2 x_3 + x_2 x_3^2, \\ m_{(1^3)}(x) &= x_1 x_2 x_3. \end{aligned} \tag{2.53}$$

Note that, if we symmetrize $x_1^3$, $x_1^2 x_2$, $x_1 x_2 x_3$ by $\mathfrak{S}_3$, we obtain $2m_{(3)}(x)$, $m_{(2,1)}(x)$, $6m_{(1^3)}(x)$, respectively, where 2, 1, 6 are the orders of the stabilizer subgroups of (3), (2, 1), $(1^3)$.

---

Among all monomials $x^\mu$ appearing in $m_\lambda(x)$, $x^\lambda$ is the leading (maximal) term with respect to the partial order $\le$, called the *dominance order*. For $\mu, \nu \in \mathbb{N}^n$, the dominance order $\mu \le \nu$ is defined by the condition

$$\mu_1 + \cdots + \mu_i \le \nu_1 + \cdots + \nu_i \quad (i = 1, \ldots, n-1) \quad \text{and} \quad |\mu| = |\nu|. \tag{2.54}$$

**Exercise 2.1** Prove the following:
(1) If $\lambda \in \mathcal{P}_n$ is a partition, then any $\mu \in \mathfrak{S}_n.\lambda$ satisfies $\mu \le \lambda$.
(2) If $\mu, \nu \in \mathbb{N}^n$ and $\mu \le \nu$, then $\mu \preceq_{\text{lex}} \nu$ under the lexicographic order of $\mathbb{N}^n$.[1]

**Remark 2.1** We denote by $P = \mathbb{Z}^n = \mathbb{Z}\varepsilon_1 \oplus \cdots \oplus \mathbb{Z}\varepsilon_n$ the set of all multi-indices of integers, where $\varepsilon_i$ $(i = 1, \ldots, n)$ are the unit vectors. In the language of rep-

[1] For $\mu, \nu \in \mathbb{N}^n$, $\mu \preceq_{\text{lex}} \nu$ means that, either $\mu = \nu$, or if $\mu \ne \nu$, then $\mu_k < \nu_k$ for the smallest index $k \in \{1, \ldots, n\}$ such that $\mu_k \ne \nu_k$.

resentation theory, $P$ is the *weight lattice* of the general linear group $\mathrm{GL}_n$. We extend the definition of the dominance order to $P$ by the same condition (2.54). We remark that the dominance order $\mu \le \nu$ for $\mu, \nu \in P$ is equivalent to $\nu - \mu \in Q_+ = \mathbb{N}\alpha_1 \oplus \cdots \oplus \mathbb{N}\alpha_{n-1}$, where $\alpha_i = \varepsilon_i - \varepsilon_{i+1} \in \mathbb{Z}^n$ $(i = 1, \dots, n-1)$ are the *simple roots* of the root system of type $A_{n-1}$. This fact can be seen by the fact that the simple roots $\alpha_1, \dots, \alpha_{n-1}$ together with $\alpha_n = \varepsilon_n$ form the dual basis of the *fundamental weights* $\varpi_i = \varepsilon_1 + \cdots + \varepsilon_i$ $(i = 1, \dots, n)$, with respect to the standard scalar product on $P = \mathbb{Z}^n$ such that $\langle \varepsilon_i, \varepsilon_j \rangle = \delta_{i,j}$ $(i, j \in \{1, \dots, n\})$.

## 2.5 Comments on Fundamental Theorems

In this section, we outline the proofs of Theorems 2.1 and 2.2.

For two sets of variables $x = (x_1, \dots, x_n)$ and $y = (y_1, \dots, y_n)$ we consider the $\mathbb{C}$-algebra homomorphism $\phi : \mathbb{C}[y] \to \mathbb{C}[x]^{\mathfrak{S}_n}$ defined by

$$\phi(F(y)) = F(e_1(x), \dots, e_n(x)) \qquad (F(y) \in \mathbb{C}[y]). \tag{2.55}$$

Note that this algebra homomorphism $\phi$ is uniquely determined by the condition $\phi(y_r) = e_r(x)$ $(r = 1, \dots, n)$. Then, Theorem 2.1 is equivalent to saying that $\phi : \mathbb{C}[y] \to \mathbb{C}[x]^{\mathfrak{S}_n}$ is an isomorphism of $\mathbb{C}$-algebras.

We define the grading of $\mathbb{C}[y]$ by

$$\mathbb{C}[y] = \bigoplus_{d=0}^{\infty} \mathbb{C}[y]_d, \quad \mathbb{C}[y]_d = \bigoplus_{\nu \in \mathbb{N}^n;\, \|\nu\| = d} \mathbb{C}\, y^\nu \quad (d \in \mathbb{N}), \tag{2.56}$$

where $\|\nu\| = \nu_1 + 2\nu_2 + \cdots + n\nu_n$, assigning the degree $\deg_y y_r = r$ to each $y_r$ $(r = 1, \dots, n)$. Then $\phi : \mathbb{C}[y] \to \mathbb{C}[x]^{\mathfrak{S}_n}$ preserves the grading, with $\mathbb{C}[x]^{\mathfrak{S}_n}$ regarded as a graded algebra with $\deg_x x_i = 1$ $(i = 1, \dots, n)$. Then we show that

$$\phi : \mathbb{C}[y]_d = \bigoplus_{\nu \in \mathbb{N}^n;\, \|\nu\| = d} \mathbb{C}\, y^\nu \to \mathbb{C}[x]_d^{\mathfrak{S}_n} = \bigoplus_{\lambda \in \mathcal{P}_n;\, |\lambda| = d} \mathbb{C}\, m_\lambda(x) \tag{2.57}$$

defines a $\mathbb{C}$-isomorphism for all $d = 0, 1, 2, \dots$. In fact, for each $\lambda = (\lambda_1, \dots, \lambda_n) \in \mathcal{P}_n$ with $|\lambda| = \lambda_1 + \cdots + \lambda_n = d$, we express the conjugate partition $\lambda' \in \mathcal{P}$ as

$$\lambda = (\lambda'_1, \lambda'_2, \dots, \lambda'_l) = (1^{\nu_1} 2^{\nu_2} \cdots n^{\nu_n}) \qquad (l = \lambda_1), \tag{2.58}$$

in terms of the multiplicities $\nu_i$ of $i$ in $\lambda'$. Then the multi-index $\nu = (\nu_1, \dots, \nu_n) \in \mathbb{N}^n$ satisfies

$$\|\nu\| = \nu_1 + 2\nu_2 + \cdots + n\nu_n = \lambda'_1 + \cdots + \lambda'_l = |\lambda'| = |\lambda| = d. \tag{2.59}$$

This correspondence $\lambda \to \nu$ defines a bijection

$$\{\lambda \in \mathcal{P}_n \mid |\lambda| = d\} \xrightarrow{\sim} \left\{\nu \in \mathbb{N}^n \mid \|\nu\| = d\right\} \tag{2.60}$$

between the two indexing sets. Note also that $\lambda$ is determined from $\nu$ by $\lambda_i = \nu_i + \cdots + \nu_n$ $(i = 1, \ldots, n)$. Under this correspondence, the image of $y^\nu$ by $\phi$ is computed as

$$\begin{aligned}
\phi(y^\nu) &= e_1(x)^{\nu_1} e_2(x)^{\nu_2} \cdots e_n(x)^{\nu_n} \\
&= (x_1 + \cdots)^{\nu_1} (x_1x_2 + \cdots)^{\nu_2} \cdots (x_1 \cdots x_n)^{\nu_n} \\
&= x_1^{\nu_1} (x_1x_2)^{\nu_2} \cdots (x_1 \cdots x_n)^{\nu_n} + \text{(lower-order terms)}, \\
&= x^\lambda + \text{(lower-order terms)} \\
&= m_\lambda(x) + \text{(lower-order terms)},
\end{aligned} \tag{2.61}$$

with respect to the lexicographic order (as well as the dominance order) of $\mathbb{N}^n$. This triangularity of $\phi$ implies that $\phi : \mathbb{C}[y]_d \to \mathbb{C}[x]_d^{\mathfrak{S}_n}$ is an isomorphism of $\mathbb{C}$-vector space.

**Example: $n = 5$, $\lambda = (7, 5, 4, 1, 0)$, $\lambda' = (4, 3, 3, 3, 2, 1, 1)$, $\nu = (2, 1, 3, 1, 0)$**

To each column of length $r$, attach the elementary symmetric function $e_r$.

$$\begin{array}{cc|ccccccc}
 & & & & & \lambda' & & & \\
 & & 4 & 3 & 3 & 3 & 2 & 1 & 1 \\
\hline
 & 7 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
\lambda & 5 & 2 & 2 & 2 & 2 & 2 & & \\
 & 4 & 3 & 3 & 3 & 3 & & & \\
 & 1 & 4 & & & & & & \\
 & 0 & & & & & & &
\end{array}
\qquad
\begin{aligned}
& e_4e_3e_3e_3e_2e_1e_1 \\
&= e_1^2 e_2 e_3^3 e_4 \\
&= x_1^2 (x_1x_2)(x_1x_2x_3)^3 (x_1x_2x_3x_4) + \cdots \\
&= x_1^7 x_2^5 x_3^4 x_4 + \cdots \\
&= m_{(7541)}(x) + \cdots
\end{aligned} \tag{2.62}$$

**Example: Symmetric polynomials of degree** 3

Note that $(3)' = (111)$, $(21)' = (21)$, $(1^3)' = 3$.

$$\begin{aligned}
e_1^3 &= m_{(3)} + 3m_{(21)} + 6m_{(1^3)}, & m_{(3)} &= e_1^3 - 3e_2e_1 + 3e_3, \\
e_2e_1 &= m_{(21)} + 3m_{(1^3)}, & m_{(21)} &= e_2e_1 - 3e_3, \\
e_3 &= m_{(1^3)}, & m_{(1^3)} &= e_3.
\end{aligned} \tag{2.63}$$

Theorem 2.2 can be proved by using the factor theorem for polynomials in one variable. We prove that any alternating polynomial $f(x)$ in $x=(x_1,\ldots,x_n)$ is divisible by $\Delta(x)$ by the induction on the number of variables. Regard $f(x)$ as a polynomial $p(u)=f(u,x_2,\ldots,x_n)\in\mathbb{C}[x_2,\ldots,x_n][u]$ of the first variable. Since $f(x)$ is alternating, one has $p(x_j)=f(x_j,x_2,\ldots,x_n)=0$ for $j=2,\ldots,n$, and hence $p(u)$ is expressed as $p(u)=q(u)(u-x_2)\cdots(u-x_n)$ for some $q(u)\in C[x_2,\ldots,x_n][u]$, namely

$$f(x_1,\ldots,x_n)=g(x_1,x_2,\ldots,x_n)\prod_{j=2}^{n}(x_1-x_j) \tag{2.64}$$

for some $g(x_1,\ldots,x_n)\in\mathbb{C}[x]$. Since $g(x)$ is alternating in $(x_2,\ldots,x_n)$, it is expressed as

$$g(x_1,\ldots,x_n)=h(x_1,\ldots,x_n)\Delta(x_2,\ldots,x_n) \tag{2.65}$$

with some $h(x)\in\mathbb{C}[x]$ by the induction hypothesis. Hence we obtain

$$\begin{aligned} f(x_1,\ldots,x_n)&=h(x_1,\ldots,x_n)\prod_{j=2}^{n}(x_1-x_j)\,\Delta(x_2,\ldots,x_n)\\ &=h(x_1,\ldots,x_n)\Delta(x_1,\ldots,x_n). \end{aligned} \tag{2.66}$$

From $f(x),\Delta(x)\in\mathbb{C}[x]^{\mathfrak{S}_n,\mathrm{sgn}}$, it also follows that $h(x)\in\mathbb{C}[x]^{\mathfrak{S}_n}$.

**Remark 2.2** The statements of Theorems 2.1 and 2.2 hold in a slightly more general setting, including the case of symmetric and alternating polynomials over $\mathbb{Z}$. In fact, we have the isomorphism

$$\phi:\quad R[y]\xrightarrow{\sim}R[x]^{\mathfrak{S}_n},\quad \phi(y_i)=e_i(x)\quad (i=1,\ldots,n), \tag{2.67}$$

of commutative rings, for any *integral domain*[2] $R$. We also have

$$R[x]^{\mathfrak{S}_n,\mathrm{sgn}}=\Delta(x)R[x]^{\mathfrak{S}_n} \tag{2.68}$$

provided that $1\neq -1$ in the integral domain $R$. The proofs given above apply to this general setting without any essential change.

[2] A commutative ring with 1 satisfying the property that $f,g\in R$, $fg=0\implies(f=0$ or $g=0)$.

# Chapter 3
# Schur Functions

**Abstract** As a warmup for our discussion of Macdonald polynomials, we review fundamental properties of Schur functions. We start here with two definitions of the Schur functions, one by combinatorics of semi-standard tableaux, and the other in terms of ratios of Vandermonde-type determinants. Then we establish the equivalence of the two definitions by means of the Cauchy formula. It should be noted that the theory of Macdonald polynomials is modeled in many respects on that of Schur functions.

## 3.1 Definitions of the Schur Functions

### *3.1.1 Two Definitions*

We now move on to the *Schur functions* $s_\lambda(x)$ $(\lambda \in \mathcal{P}_n)$; they are a family of symmetric polynomials indexed by the same set $\mathcal{P}_n$ of partitions $\lambda$ with $\ell(\lambda) \le n$ as in the case of $m_\lambda(x)$. Each $s_\lambda(x)$ is homogeneous of degree $|\lambda|$ and has the leading term $x^\lambda$ with respect to the dominance order:

$$s_\lambda(x) = x^\lambda + \cdots = m_\lambda(x) + \cdots . \tag{3.1}$$

With this property, they also form a $\mathbb{C}$-basis of the ring of symmetric polynomials:

$$\mathbb{C}[x]^{\mathfrak{S}_n} = \bigoplus_{\lambda \in \mathcal{P}_n} \mathbb{C}\, s_\lambda(x). \tag{3.2}$$

As we will see below, $s_\lambda(x)$ are in fact symmetric polynomials with nonnegative integer coefficients, i.e. $s_\lambda(x) \in \mathbb{N}[x]^{\mathfrak{S}_n}$.

We give two definitions of the Schur functions here, denoting them by $s_\lambda^{\mathrm{comb}}(x)$ and $s_\lambda^{\det}(x)$ respectively, and show later that they in fact coincide.

 
M. Noumi, *Macdonald Polynomials*, SpringerBriefs in Mathematical Physics,
https://doi.org/10.1007/978-981-99-4587-0_3

**Definition 3.1** (*combinatorial*) For each $\lambda \in \mathcal{P}_n$, we define the Schur function $s_\lambda^{\mathrm{comb}}(x)$ as the sum

$$s_\lambda^{\mathrm{comb}}(x) = \sum_{T \in \mathrm{SSTab}_n(\lambda)} x^{\mathrm{wt}(T)} \tag{3.3}$$

of monomials $x^{\mathrm{wt}(T)}$ over the set $\mathrm{SSTab_n}(\lambda)$ of all *semi-standard tableaux* $T$ of shape $\lambda$ in letters $\{1, \ldots, n\}$.

We explain below the precise meaning of a semi-standard tableau $T$ and its weight $\mathrm{wt}(T)$. By definition we have $s_\lambda^{\mathrm{comb}}(x) \in \mathbb{N}[x]$, but it is not obvious why it should be symmetric since this definition depends strongly on the ordering of the indexing set $\{1, \ldots, n\}$.

**Definition 3.2** (*determinantal*) For each $\lambda \in \mathcal{P}_n$, we define the Schur function $s_\lambda^{\mathrm{det}}(x)$ as the ratio of two determinants of Vandermonde type:

$$s_\lambda^{\mathrm{det}}(x) = \frac{\det\left(x_i^{\lambda_j+n-j}\right)_{i,j=1}^n}{\det\left(x_i^{n-j}\right)_{i,j=1}^n} = \frac{\det\left(x_i^{\lambda_j+n-j}\right)_{i,j=1}^n}{\Delta(x)}, \tag{3.4}$$

where $\Delta(x) = \prod_{1 \le i < j \le n}(x_i - x_j)$ stands for the difference product.

Since the numerator $\det\left(x_i^{\lambda_j+n-j}\right)_{i,j=1}^n \in \mathbb{Z}[x]$ is an alternating polynomial in $\mathbb{Z}[x]^{\mathfrak{S}_n,\mathrm{sgn}}$, it is divisible by $\Delta(x)$ in the polynomial ring $\mathbb{Z}[x]$ with integer coefficients. Hence the resulting $s_\lambda^{\mathrm{det}}(x)$ is a symmetric polynomial with coefficients in $\mathbb{Z}$, i.e. $s_\lambda^{\mathrm{det}}(x) \in \mathbb{Z}[x]^{\mathfrak{S}_n}$ (see Remark 2.2). It is not obvious, however, why they should have coefficients in $\mathbb{N} = \mathbb{Z}_{\ge 0}$.

**Theorem 3.1** *For any* $\lambda \in \mathcal{P}_n$, *we have* $s_\lambda^{\mathrm{comb}}(x) = s_\lambda^{\mathrm{det}}(x)$.

Namely, the two definitions of the Schur functions give the same polynomials, which we denote by $s_\lambda(x)$. An immediate consequence of this theorem is that the Schur functions are symmetric polynomials with coefficients in $\mathbb{N} = \mathbb{Z}_{\ge 0}$, i.e. $s_\lambda(x) \in \mathbb{N}[x]^{\mathfrak{S}_n}$. The equivalence of the two definitions will be established later in Sect. 3.5 on the basis of Cauchy's formula.

### *3.1.2 Combinatorial Definition*

By a *semi-standard tableau* $T$ of shape $\lambda$ in letters $\{1, \ldots, n\}$, we mean a mapping $T : D(\lambda) \to \{1, \ldots, n\}$ such that the numbers $T(s)$ $(s \in D(\lambda))$ are weakly increasing along the rows and strictly increasing along the columns.[1] For example,

[1] $T$ is called a *column strict tableau* in the terminology of Macdonald [20].

$$T = \begin{array}{|c|c|c|c|c|}\hline 1 & 1 & 2 & 3 & 4 \\ \hline 2 & 3 & 3 \\ \cline{1-3} 4 \\ \cline{1-1}\end{array} \qquad\qquad \begin{array}{c} a \leq b \\ \wedge \\ c \end{array} \tag{3.5}$$

Namely, $T$ should satisfy

$$\begin{aligned} T(i,j) &\leq T(i,j+1) \quad (1 \leq i \leq \ell(\lambda),\, 1 \leq j < \lambda_i), \\ T(i,j) &< T(i+1,j) \quad (1 \leq j \leq \lambda_1,\ 1 \leq i < \lambda'_j). \end{aligned} \tag{3.6}$$

We denote by $\mathrm{SSTab}_n(\lambda)$ the set of all semi-standard tableaux of shape $\lambda$ in letters $\{1, \ldots, n\}$. For each semi-standard tableau $T$, we denote by $\mathrm{wt}(T)$ the composition (multi-index)

$$\mu = (\mu_1, \ldots, \mu_n) \in \mathbb{N}^n, \quad \mu_i = \#\{s \in D(\lambda) \mid T(s) = i\} \quad (i = 1, \ldots, n) \tag{3.7}$$

obtained by counting the number of $i$'s in the tableau $T$ for each $i$; $\mathrm{wt}(T)$ is called the *weight* of $T$. In the example of $T$ in (3.5), we have

$$\mathrm{wt}(T) = (2, 2, 3, 2), \quad x^{\mathrm{wt}(T)} = x_1^2 x_2^2 x_3^3 x_4^2. \tag{3.8}$$

## $s_\lambda^{\mathrm{comb}}(x)$ attached to columns and rows

(1) Single column $\lambda = (1^r)$: $(r = 0, 1, 2, \ldots)$

$$s_{(1^r)}^{\mathrm{comb}}(x) = \sum_{1 \leq i_1 < \cdots < i_r \leq n} x_{i_1} \cdots x_{i_r} = e_r(x). \tag{3.9}$$

(2) Single row $\lambda = (l)$: $(l = 0, 1, 2, \ldots)$

$$s_{(l)}^{\mathrm{comb}}(x) = \sum_{1 \leq j_1 \leq \cdots \leq j_l \leq n} x_{j_1} \cdots x_{j_r} = h_l(x). \tag{3.10}$$

---

## Example of $s_\lambda^{\mathrm{comb}}(x)$: $n = 3$, $\lambda = (2, 1, 0)$

When $n = 3$ and $\lambda = (2, 1, 0)$, there are 8 semi-standard tableaux of shape $\lambda$.

$$\begin{array}{cccccccc} \begin{array}{|c|c|}\hline 1 & 1 \\ \hline 2 \\ \cline{1-1}\end{array} & \begin{array}{|c|c|}\hline 1 & 1 \\ \hline 3 \\ \cline{1-1}\end{array} & \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 2 \\ \cline{1-1}\end{array} & \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1}\end{array} & \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1}\end{array} & \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 3 \\ \cline{1-1}\end{array} & \begin{array}{|c|c|}\hline 2 & 2 \\ \hline 3 \\ \cline{1-1}\end{array} & \begin{array}{|c|c|}\hline 2 & 3 \\ \hline 3 \\ \cline{1-1}\end{array} \\ x_1^2 x_2 & x_1^2 x_3 & x_1 x_2^2 & x_1 x_2 x_3 & x_1 x_2 x_3 & x_1 x_3^2 & x_2^2 x_3 & x_2 x_3^2 \end{array} \tag{3.11}$$

Hence we have

$$\begin{aligned} s^{\text{comb}}_{(21)}(x) &= x_1^2x_2 + x_1^2x_3 + x_1x_2^2 + x_1x_3^2 + x_2^2x_3 + x_2x_3^2 + 2x_1x_2x_3 \\ &= m_{(21)}(x) + 2m_{(1^3)}(x). \end{aligned} \tag{3.12}$$

---

Note that the definition of $s^{\text{comb}}_\lambda(x)$ strongly depends on the ordering of $1, 2, \dots, n$. By definition we have $s^{\text{comb}}_\lambda(x) \in \mathbb{N}[x]$, but why are they symmetric?

For each $\mu \in \mathbb{N}^n$ with $|\mu| = |\lambda|$, we set

$$\text{SSTab}_n(\lambda)_\mu = \{T \in \text{SSTab}_n(\lambda) \mid \text{wt}(T) = \mu\}. \tag{3.13}$$

The number

$$K_{\lambda,\mu} = \#\text{SSTab}_n(\lambda)_\mu \in \mathbb{N} \tag{3.14}$$

of semi-standard tableaux of shape $\lambda$ with weight $\mu$ is called the *Kostka number*. Then we have

$$s^{\text{comb}}_\lambda(x) = \sum_{\mu\in\mathbb{N}^n} \left(\#\text{SSTab}_n(\lambda)_\mu\right) x^\mu = \sum_{\mu\in\mathbb{N}^n} K_{\lambda,\mu}\, x^\mu. \tag{3.15}$$

In fact we have

$$s^{\text{comb}}_\lambda(x) = x^\lambda + \sum_{\mu<\lambda} K_{\lambda,\mu}\, x^\mu, \tag{3.16}$$

namely, $s^{\text{comb}}_\lambda(x)$ has the leading term $x^\lambda$ with respect to the dominance order.

**Exercise 3.1** Let $\lambda \in \mathcal{P}_n$. Prove the following:
(1) If $T \in \text{SSTab}_n(\lambda)$ and $\text{wt}(T) = \mu$, then $\mu \le \lambda$.
(2) $K_{\lambda,\lambda} = 1$, and $K_{\lambda,\mu} > 0$ if and only if $\mu \le \lambda$.

**Remark 3.1** As we mentioned already, each $s^{\text{comb}}_\lambda(x)$ is in fact a symmetric polynomial. This statement is equivalent to $K_{\lambda,\mu} = K_{\lambda,\sigma.\mu}$ ($\mu \in \mathbb{N}^n$) for any permutation $\sigma \in \mathfrak{S}_n$. We remark that, for each adjacent transposition $s_i = (i, i+1)$ ($i = 1, \dots, n-1$), there is a bijection

$$\text{SSTab}_n(\lambda)_\mu \xrightarrow{\sim} \text{SSTab}_n(\lambda)_{s_i.\mu} \tag{3.17}$$

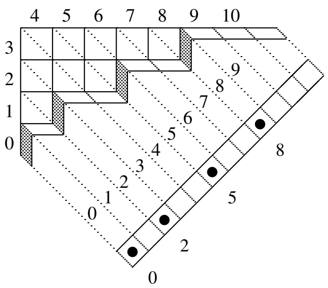
called the *Bender–Knuth involution*. It implies that $K_{\lambda,\mu} = K_{\lambda,s_i.\mu}$ ($\mu \in \mathbb{N}^n$) for $i = 1, \dots, n-1$, and hence $K_{\lambda,\mu} = K_{\lambda,\sigma.\mu}$ ($\mu \in \mathbb{N}^n$) for any $\sigma \in \mathfrak{S}_n$. For a combinatorial proof of $\mathfrak{S}_n$-invariance of this sort, see Sagan's textbook [31, Proposition 4.4.2] for example.

### *3.1.3 Determinantal Definition*

For each $\lambda \in \mathcal{P}_n$, we defined $s_\lambda^{\det}(x)$ as the ratio of two determinants in Definition 3.2. We denote by $\delta = (n-1, n-2, \ldots, 0)$ the *staircase* partition of $n-1$ parts so that $\delta_i = n - i$ $(i = 1, \ldots, n)$. Then the definition of $s_\lambda^{\det}(x)$ can be rewritten as

$$s_\lambda^{\det}(x) = \frac{\det\left(x_i^{(\lambda+\delta)_j}\right)_{i,j=1}^n}{\det\left(x_i^{\delta_j}\right)_{i,j=1}^n}. \tag{3.18}$$

We give here a remark on the strict partition $l = \lambda + \delta$ appearing in the numerator.[2] The sequence $l = (l_1, \ldots, l_n)$, $l_j = \lambda_j + n - j$ $(j = 1, \ldots, n)$, can be read off from the boundary of the Young diagram as shown below.




$$\begin{aligned} n &= 4 \\ \lambda &= (5, 3, 1, 0) \\ \delta &= (3, 2, 1, 0) \\ \lambda + \delta &= (8, 5, 2, 0) \end{aligned} \tag{3.19}$$

The subset $M = \{l_1, \ldots, l_n\} \subseteq \mathbb{N}$ is often called the *Maya diagram* attached to $\lambda$.

**Example of $s_\lambda^{\det}(x)$: $n = 3$, $\lambda = (2, 1, 0)$**

Since $\lambda + \delta = (4, 2, 0)$, we have

$$s_{(21)}^{\det}(x) = \det\begin{bmatrix} x_1^4 & x_1^2 & 1 \\ x_2^4 & x_2^2 & 1 \\ x_3^4 & x_3^2 & 1 \end{bmatrix} \Big/ \det\begin{bmatrix} x_1^2 & x_1 & 1 \\ x_2^2 & x_2 & 1 \\ x_3^2 & x_3 & 1 \end{bmatrix} = \frac{\Delta(x_1^2, x_2^2, x_3^2)}{\Delta(x_1, x_2, x_3)}. \tag{3.20}$$

Hence

$$\begin{aligned} s_{(21)}^{\det}(x) &= \frac{(x_1^2 - x_2^2)(x_1^2 - x_3^2)(x_2^2 - x_3^2)}{(x_1 - x_2)(x_1 - x_3)(x_2 - x_3)} = (x_1 + x_2)(x_1 + x_3)(x_2 + x_3) \\ &= x_1^2 x_2 + x_1^2 x_3 + x_1 x_2^2 + x_1 x_3^2 + x_2^2 x_3 + x_2 x_3^2 + 2x_1 x_2 x_3 \\ &= m_{(21)}(x) + 2m_{(1^3)}(x). \end{aligned} \tag{3.21}$$

---

[2] A partition $\lambda = (\lambda_1, \lambda_2, \ldots)$ with $\ell(\lambda) = l$ is called *strict* if $\lambda_1 > \lambda_2 > \cdots > \lambda_l > 0$.

**Exercise 3.2** Show that $s^{\det}_{(1^k)}(x) = e_k(x)$, $s^{\det}_{(k)}(x) = h_k(x)$ $(k = 0, 1, 2, \ldots)$.

A possible approach would be to use the following identities:

$$\prod_{j=1}^{n}(u - x_j) \cdot \Delta(x_1, \ldots, x_n) = \Delta(u, x_1, \ldots, x_n),$$
$$\frac{\Delta(x_1, \ldots, x_n)}{\prod_{i=1}^{n}(1 - x_i u)} = \det\left(\frac{x_i^{n-j}}{1 - x_i u}\right)_{i,j=1}^{n}. \tag{3.22}$$

**Exercise 3.3** Prove that both $s^{\mathrm{comb}}_\lambda(x)$ and $s^{\det}_\lambda(x)$ carry the following properties.
(1) For any $\lambda \in \mathcal{P}_n$ and $k \in \mathbb{N}$, $s_{\lambda+(k^n)}(x) = (x_1 \ldots x_n)^k s_\lambda(x)$, where $(k^n) = (k, \ldots, k)$ denotes the $n \times k$ rectangle.
(2) Let $\lambda \in \mathcal{P}_n$ and $m < n$. Then we have

$$s_\lambda(x_1, \ldots, x_m, 0, \ldots, 0) = \begin{cases} s_\lambda(x_1, \ldots, x_m) & (\ell(\lambda) \le m), \\ 0 & (\ell(\lambda) > m). \end{cases} \tag{3.23}$$

## 3.2 Principal Specialization and Self-duality

Before giving a proof of Theorem 3.1, we explain some consequences of the equivalence of the two definitions of Schur functions. From this section on, we set $s_\lambda(x) = s^{\det}_\lambda(x)$.

### *3.2.1 Principal Specialization: Evaluation at $x = t^\delta$*

According to the combinatorial definition, the Schur function $s^{\mathrm{comb}}_\lambda(x)$ counts the semi-standard tableaux $T$ of shape $\lambda$ with weights $x^{\mathrm{wt}(T)}$. In particular, we have

$$s_\lambda(1, \ldots, 1) = s^{\mathrm{comb}}_\lambda(1, \ldots, 1) = \sum_{T \in \mathrm{SSTab}_n(\lambda)} 1 = \#\mathrm{SSTab}_n(\lambda). \tag{3.24}$$

In terms of the determinantal definition, the evaluation of $s_\lambda(x)$ at $x = (1, \ldots, 1)$ is a subtle question since the denominator $\Delta(x)$ vanishes at this point. In order to avoid this singularity, we first evaluate $s_\lambda(x)$ at $t^\delta = (t^{n-1}, t^{n-2}, \ldots, 1)$ and then take the limit as $t \to 1$.

**Proposition 3.1** (Principal specialization) *For each $\lambda \in \mathcal{P}_n$, we have*

$$s_\lambda(t^\delta) = \frac{\Delta(t^{\lambda+\delta})}{\Delta(t^\delta)} = t^{n(\lambda)} \prod_{1 \le i < j \le n} \frac{1 - t^{\lambda_i - \lambda_j + j - i}}{1 - t^{j-i}}, \tag{3.25}$$

*where $n(\lambda) = \sum_{i=1}^{n}(i - 1)\lambda_i$.*

***Proof*** In fact, we have

$$
\begin{aligned}
s_\lambda(t^\delta) &= \frac{\det\left(t^{\delta_i(\lambda+\delta)_j}\right)_{i,j=1}^n}{\Delta(t^\delta)} = \frac{\Delta(t^{\lambda+\delta})}{\Delta(t^\delta)} \\
&= \prod_{1\le i<j\le n} \frac{t^{\lambda_i+n-i} - t^{\lambda_j+n-j}}{t^{n-i} - t^{n-j}} = \prod_{1\le i<j\le n} t^{\lambda_j} \frac{1 - t^{\lambda_i-\lambda_j+j-i}}{1 - t^{j-i}}.
\end{aligned}
\tag{3.26}
$$

□

We are now allowed to take the limit as $t \to 1$ in (3.25), to obtain an explicit formula

$$
\#\mathrm{SSTab}_n(\lambda) = s_\lambda(1, \ldots, 1) = \frac{\Delta(\lambda+\delta)}{\Delta(\delta)} = \prod_{1\le i<j\le n} \frac{\lambda_i - \lambda_j + j - i}{j - i} \tag{3.27}
$$

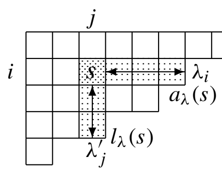
for the number of semi-standard tableaux of shape $\lambda$.

### 3.2.2 *Hook Length Formula*

Formulas (3.25) and (3.27) can be rewritten into a combinatorial expression of the Young diagram. For each square $s = (i, j) \in D(\lambda)$, we define the *content* $c_\lambda(s)$ and the *hook length* $h_\lambda(s)$ by

$$
c_\lambda(s) = j - i, \quad h_\lambda(s) = \lambda_i + \lambda'_j - i - j + 1. \tag{3.28}
$$

Note that, in terms of the *arm length* $a_\lambda(s) = \lambda_i - j$ and the *leg length* $l_\lambda(s) = \lambda'_j - i$, the hook length is expressed as $h_\lambda(s) = a_\lambda(s) + l_\lambda(s) + 1$.




$$
a_\lambda(s) = \lambda_i - j \qquad l_\lambda(s) = \lambda'_j - i \qquad h_\lambda(s) = a_\lambda(s) + l_\lambda(s) + 1 \tag{3.29}
$$

In terms of the Maya diagram $M = \{l_1, \ldots, l_n\}$, a square $s \in \lambda$ is in one-to-one correspondence with a pair $(k, l)$ of nonnegative integers such that $k < l$, $k \notin M$, $l \in M$; the hook length is then interpreted as $h_\lambda(s) = l - k$.

**Proposition 3.2** (Hook length formula) *For each $\lambda \in \mathcal{P}_n$, we have*

$$
s_\lambda(t^\delta) = t^{n(\lambda)} \prod_{s\in\lambda} \frac{1 - t^{n+c_\lambda(s)}}{1 - t^{h_\lambda(s)}}, \tag{3.30}
$$

*and*

$$s_\lambda(1,\ldots,1) = \#\mathrm{SSTab}_n(\lambda) = \prod_{s\in\lambda} \frac{n+c_\lambda(s)}{h_\lambda(s)}. \tag{3.31}$$

***Proof*** We show

$$\prod_{1\le i<j\le n} \frac{\lambda_i-\lambda_j+j-i}{j-i} = \prod_{s\in\lambda} \frac{n+c_\lambda(s)}{h_\lambda(s)}. \tag{3.32}$$

Setting $l_i = \lambda_i + n - i$ $(i = 1,\ldots,n)$, consider the Maya digram $M = \{l_1,\ldots,l_n\}$ attached to $\lambda$. In terms of $M$, we see

$$\prod_{1\le i<j\le n} \frac{\lambda_i-\lambda_j+j-i}{j-i} = \prod_{1\le i<j\le n} \frac{l_i-l_j}{j-i} = \frac{\prod_{0\le k<l;k,l\in M}(l-k)}{\prod_{0\le k<l<n}(l-k)} \tag{3.33}$$

and

$$\prod_{s\in\lambda} h_\lambda(s) = \prod_{\substack{0\le k<l\\ k\notin M,l\in M}} (l-k). \tag{3.34}$$

Since

$$\prod_{\substack{0\le k<l\\ l\in M}} (l-k) = \prod_{\substack{0\le k<l\\ k,l\in M}} (l-k) \prod_{\substack{0\le k<l\\ k\notin M,l\in M}} (l-k), \tag{3.35}$$

we have

$$\begin{aligned}
&\prod_{1\le i<j\le n} \frac{\lambda_i-\lambda_j+j-i}{j-i} \cdot \prod_{s\in\lambda} h_\lambda(s) & (3.36)\\
&= \frac{\prod_{0\le k<l;k,l\in M}(l-k)\prod_{0\le k<l;\ k\notin M,l\in M}(l-k)}{\prod_{0\le k<l<n}(l-k)}\\
&= \frac{\prod_{0\le k<l;l\in M}(l-k)}{\prod_{0\le k<l<n}(l-k)} = \frac{\prod_{i=1}^n(\lambda_i+n-i)!}{\prod_{i=1}^n(n-i)!}\\
&= \prod_{i=1}^n (n-i+1)_{\lambda_i} = \prod_{s\in\lambda}(n+c_\lambda(s)), & (3.37)
\end{aligned}$$

where we have used the notation of shifted factorials $(a)_k = a(a+1)\cdots(a+k-1)$ $(k=0,1,\ldots)$. The same proof applies to the formula for $s_\lambda(t^\delta)$ as well. □

**Hook length formula**

(1) $n = 3, \lambda = (2, 1, 0)$.

$$\prod_{s\in\lambda} \frac{n + c_\lambda(s)}{h_\lambda(s)} = \frac{\begin{array}{|c|c|}\hline 3 & 4 \\ \hline 2 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}}{\begin{array}{|c|c|}\hline 3 & 1 \\ \hline 1 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}} = 2 \cdot 4 = 8. \tag{3.38}$$

(2) $n = 4, \lambda = (5, 3, 1, 0)$.

$$\prod_{s\in\lambda} \frac{n + c_\lambda(s)}{h_\lambda(s)} = \frac{\begin{array}{|c|c|c|c|c|}\hline 4 & 5 & 6 & 7 & 8 \\ \hline 3 & 4 & 5 & \multicolumn{2}{c}{} \\ \cline{1-3} 2 & \multicolumn{4}{c}{} \\ \cline{1-1}\end{array}}{\begin{array}{|c|c|c|c|c|}\hline 7 & 5 & 4 & 2 & 1 \\ \hline 4 & 2 & 1 & \multicolumn{2}{c}{} \\ \cline{1-3} 1 & \multicolumn{4}{c}{} \\ \cline{1-1}\end{array}} = 360. \tag{3.39}$$

**Exercise 3.4** Confirm that the hook length formula implies the following:
(1) Single column $\lambda = (1^r)$: $s_{(1^r)}(1, \ldots, 1) = \binom{n}{r}$ $(r \geq 0)$.
(2) Single row $\lambda = (l)$: $s_{(l)}(1, \ldots, 1) = \binom{n+l-1}{l}$ $(l \geq 0)$.

### 3.2.3 Self-duality

The values of $s_\lambda(x)$ at the discrete set $x = t^{\mu+\delta}$ $(\mu \in \mathcal{P}_n)$ have a remarkable duality property (evaluation symmetry).

**Proposition 3.3** (Self-duality) *For any pair of partitions $\lambda, \mu \in \mathcal{P}_n$, we have*

$$\frac{s_\lambda(t^{\mu+\delta})}{s_\lambda(t^\delta)} = \frac{s_\mu(t^{\lambda+\delta})}{s_\mu(t^\delta)}. \tag{3.40}$$

***Proof*** Since $s_\lambda(t^\delta) = \Delta(t^{\lambda+\delta})/\Delta(t^\delta)$, we have

$$\frac{s_\lambda(t^{\mu+\delta})}{s_\lambda(t^\delta)} = \frac{\Delta(t^\delta)\det\left(t^{(\mu+\delta)_i(\lambda+\delta)_j}\right)_{i,j=1}^n}{\Delta(t^{\lambda+\delta})\Delta(t^{\mu+\delta})}. \tag{3.41}$$

This formula is symmetric with respect to exchanging $\lambda$ and $\mu$. □

Regarding $x = t^\delta$ as a base point, we set

$$\widetilde{s}_\lambda(x) = \frac{s_\lambda(x)}{s_\lambda(t^\delta)} \tag{3.42}$$

so that $\widetilde{s}_\lambda(t^\delta) = 1$. Then Proposition 3.3 implies that $\widetilde{s}_\lambda(t^{\mu+\delta}) = \widetilde{s}_\mu(t^{\lambda+\delta})$ for any pair of partitions $\lambda, \mu \in \mathcal{P}_n$. Namely, regarded as a function of $(\lambda, \mu) \in \mathcal{P}_n \times \mathcal{P}_n$, $\widetilde{s}_\lambda(t^{\mu+\delta})$ is invariant under the exchange of the arguments $\lambda$ and $\mu$.

## 3.3 Cauchy Formula

In this section, we give a proof of the *Cauchy formula* for Schur functions; it will be used in Sect. 3.5 to establish the equivalence of two definitions of the Schur functions.

### *3.3.1 Cauchy Determinant*

**Lemma 3.1** (Cauchy) *For two sets of variables $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$, we have*

$$\det\left(\frac{1}{x_i + y_j}\right)_{i,j=1}^n = \frac{\Delta(x)\Delta(y)}{\prod_{i,j=1}^n (x_i + y_j)}, \tag{3.43}$$

$$\det\left(\frac{1}{1 - x_i y_j}\right)_{i,j=1}^n = \frac{\Delta(x)\Delta(y)}{\prod_{i,j=1}^n (1 - x_i y_j)}. \tag{3.44}$$

□

The two formulas of Lemma 3.1 are equivalent to each other; the second formula is obtained from the first by change of variables $x_i \to -x_i^{-1}$ $(i = 1, \ldots, n)$ and *vice versa*.

**Exercise 3.5** Prove Cauchy's lemma (3.43) by means of the property of alternating polynomials.

**Exercise 3.6** (1) For any $n \times n$ matrix $\left(a_{ij}\right)_{i,j=1}^n$ with $a_{nn} \neq 0$, its determinant is expressed as follows by a determinant of $2 \times 2$ minors (a variant of the *Dodgson condensation*):

$$\det\left(a_{ij}\right)_{i,j=1}^n = a_{nn}^{-n+2} \det\left(a_{ij}a_{nn} - a_{in}a_{nj}\right)_{i,j=1}^n. \tag{3.45}$$

(2) Use (3.45) to give an inductive proof of Cauchy's lemma.

**Remark 3.2** Lemma 3.1 can be extended to the following family of determinant formulas involving an extra parameter $u$:

$$\det\left(\frac{u+x_i+y_j}{u(x_i+y_j)}\right)_{i,j=1}^{n} = \frac{u+\sum_{i=1}^{n}x_i+\sum_{j=1}^{n}y_j}{u}\frac{\Delta(x)\Delta(y)}{\prod_{i,j=1}^{n}(x_i+y_j)}, \tag{3.46}$$

$$\det\left(\frac{1-ux_iy_j}{(1-u)(1-x_iy_j)}\right)_{i,j=1}^{n} = \frac{1-ux_1\cdots x_ny_1\cdots y_n}{1-u}\frac{\Delta(x)\Delta(y)}{\prod_{i,j=1}^{n}(1-x_iy_j)}, \tag{3.47}$$

$$\det\left(\frac{\sigma(u+x_i+y_j)}{\sigma(u)\sigma(x_i+y_j)}\right)_{i,j=1}^{n}$$
$$= \frac{\sigma(u+\sum_{i=1}^{n}x_i+\sum_{j=1}^{n}y_j)}{\sigma(u)}\frac{\prod_{1\le i<j\le n}\sigma(x_i-x_j)\sigma(y_i-y_j)}{\prod_{i,j=1}^{n}\sigma(x_i+y_j)}, \tag{3.48}$$

where $\sigma(z)=\sigma(z|\Omega)$ stands for the *Weierstrass sigma function* attached to a period lattice $\Omega=\mathbb{Z}\omega_1\oplus\mathbb{Z}\omega_2\subset\mathbb{C}$ of rank two ($\mathrm{Im}(\omega_2/\omega_1)>0$), defined by

$$\sigma(z|\Omega) = z\prod_{\omega\in\Omega,\ \omega\neq 0}\left(1-\frac{z}{\omega}\right)e^{z/\omega+z^2/2\omega^2} \qquad (z\in\mathbb{C}). \tag{3.49}$$

These three variations (rational, trigonometric and elliptic) play crucial roles in various situations of integrable systems. Here, formula (3.47) is called trigonometric in the sense of additive variables $\theta_i$ such that $x_i=e^{\sqrt{-1}\theta_i}$.

## 3.3.2 Cauchy Formula for Schur Functions

In what follows, we use the notation of Schur functions $s_\lambda(x)$ for $s_\lambda^{\det}(x)$.

**Theorem 3.2** (Cauchy formula) *For two sets of variables $x=(x_1,\ldots,x_n)$ and $y=(y_1,\ldots,y_n)$, the following identity holds in the ring $\mathbb{C}[[x,y]]$ of formal power series in $x$ and $y$:*

$$\prod_{i,j=1}^{n}\frac{1}{1-x_iy_j} = \sum_{\lambda\in\mathcal{P}_n}s_\lambda(x)s_\lambda(y). \tag{3.50}$$

***Proof*** We make use of the multiplicative version (3.44) of Cauchy's lemma.

$$\frac{\Delta(x)\Delta(y)}{\prod_{i,j=1}^{n}(1-x_iy_j)} = \det\left(\frac{1}{1-x_iy_j}\right)_{i,j=1}^{\infty}$$
$$= \sum_{\sigma\in\mathfrak{S}_n}\mathrm{sgn}(\sigma)\frac{1}{(1-x_{\sigma(1)}y_1)\cdots(1-x_{\sigma(n)}y_n)}$$
$$= \sum_{\sigma\in\mathfrak{S}_n}\mathrm{sgn}(\sigma)\sum_{k_1,\ldots,k_n\ge 0}(x_{\sigma(1)}y_1)^{k_1}\cdots(x_{\sigma(n)}y_n)^{k_n}$$

$$= \sum_{k_1,\ldots,k_n \geq 0} \left( \sum_{\sigma \in \mathfrak{S}_n} \operatorname{sgn}(\sigma) x_{\sigma(1)}^{k_1} \cdots x_{\sigma(n)}^{k_n} \right) y_1^{k_1} \cdots y_n^{k_n}$$

$$= \sum_{k_1,\ldots,k_n \geq 0} \Delta_{k_1,\ldots,k_n}(x)\, y_1^{k_1} \cdots y_n^{k_n}, \tag{3.51}$$

where we have used the notation of determinants

$$\Delta_{k_1,\ldots,k_n}(x) = \det\left(x_i^{k_j}\right)_{i,j=1}^{n} \tag{3.52}$$

of Vandermonde type (alternating polynomials of monomial type $(k_1, \ldots, k_n)$). Note that $\Delta_{n-1,n-2,\ldots,0}(x) = \Delta(x)$. Since $\Delta_{k_1,\ldots,k_n}(x)$ is alternating in $(k_1, \ldots, k_n)$, we have only to consider the cases where $k_1, \ldots, k_n$ are mutually distinct. In such a case, there exists a unique sequence $(l_1, \ldots, l_n) \in \mathbb{N}^n$ and a permutation $\sigma \in \mathfrak{S}_n$ such that

$$l_1 > \cdots > l_n \geq 0, \quad (k_1, \ldots, k_n) = (l_{\sigma(1)}, \ldots, l_{\sigma(n)}). \tag{3.53}$$

Then we have

$$\Delta_{k_1,\ldots,k_n}(x) = \Delta_{l_{\sigma(1)},\ldots,l_{\sigma(n)}}(x) = \operatorname{sgn}(\sigma)\Delta_{l_1,\ldots,l_n}(x). \tag{3.54}$$

Hence,

$$\frac{\Delta(x)\Delta(y)}{\prod_{i,j=1}^{n}(1 - x_i y_j)} = \sum_{l_1>\ldots>l_n\geq 0} \sum_{\sigma \in \mathfrak{S}_n} \operatorname{sgn}(\sigma)\Delta_{l_1,\ldots,l_n}(x)\, y_1^{l_{\sigma(1)}} \cdots y_n^{l_{\sigma(n)}}$$

$$= \sum_{l_1>\ldots>l_n\geq 0} \Delta_{l_1,\ldots,l_n}(x) \sum_{\sigma \in \mathfrak{S}_n} \operatorname{sgn}(\sigma) y_1^{l_{\sigma(1)}} \cdots y_n^{l_{\sigma(n)}}$$

$$= \sum_{l_1>\ldots>l_n\geq 0} \Delta_{l_1,\ldots,l_n}(x)\Delta_{l_1,\ldots,l_n}(y). \tag{3.55}$$

Each $l = (l_1, \ldots, l_n) \in \mathbb{N}^n$ with $l_1 > \cdots > l_n \geq 0$ is uniquely expressed in the form $l = \lambda + \delta$ with $\lambda \in \mathcal{P}_n$, and we have $\Delta_l(x) = \Delta_{\lambda+\delta}(x) = \Delta(x)s_\lambda(x)$ by the definition of $s_\lambda(x) = s_\lambda^{\det}(x)$. Hence we obtain

$$\frac{\Delta(x)\Delta(y)}{\prod_{i,j=1}^{n}(1 - x_i y_j)} = \sum_{\lambda \in \mathcal{P}_n} \Delta_{\lambda+\delta}(x)\Delta_{\lambda+\delta}(y)$$

$$= \Delta(x)\Delta(y) \sum_{\lambda \in \mathcal{P}_n} s_\lambda(x)s_\lambda(y), \tag{3.56}$$

as desired. □

It is convenient to introduce the *signed* version

$$f_{k_1,\dots,k_n}(x) = \frac{\Delta_{k_1,\dots,k_n}(x)}{\Delta(x)} \quad (k_1,\dots,k_n \in \mathbb{N}) \tag{3.57}$$

of $s_\lambda(x)$ with alternating indices $(k_1,\dots,k_n)$. Note that, if $l = (l_1,\dots,l_n) \in \mathbb{N}^n$ is *strict* in the sense $l_1 > \dots > l_n \geq 0$, then we have $f_{l_1,\dots,l_n}(x) = s_\lambda(x)$ for the partition $\lambda \in \mathcal{P}_n$ such that $l = \lambda + \delta$. In terms of these functions, Cauchy's formula is written as

$$\begin{aligned}\frac{\Delta(y)}{\prod_{i,j=1}^n (1 - x_i y_j)} &= \sum_{\lambda \in \mathcal{P}_n} s_\lambda(x) \Delta_{\lambda+\delta}(y) \\ &= \sum_{k_1,\dots,k_n \geq 0} f_{k_1,\dots,k_n}(x)\, y_1^{k_1} \cdots y_n^{k_n}.\end{aligned} \tag{3.58}$$

This formula will be used in Sect. 3.5 to establish equivalence of the two definitions of Schur functions.

We also remark that Cauchy's formula can be generalized to the case of two sets of variables with unequal dimensions: For two sets of variables $x = (x_1,\dots,x_m)$ and $y = (y_1,\dots,y_n)$,

$$\prod_{i=1}^m \prod_{j=1}^n \frac{1}{1 - x_i y_j} = \sum_{\ell(\lambda) \leq \min\{m,n\}} s_\lambda(x_1,\dots,x_m)\, s_\lambda(y_1,\dots,y_n). \tag{3.59}$$

This formula is obtained from the $m = n$ case by setting unnecessary variables to zero, thanks to the stability property of Exercise 3.3 (2).

## 3.4 Recurrence on the Number of Variables

We recall the combinatorial definition of Schur functions:

$$s_\lambda^{\mathrm{comb}}(x) = \sum_{T \in \mathrm{SSTab}_n(\lambda)} x^{\mathrm{wt}(T)}. \tag{3.60}$$

Given a semi-standard tableau $T \in \mathrm{SSTab}_n(\lambda)$ of shape $\lambda$ in letters $\{1,\dots,n\}$, let $T'$ be the sub-tableau of $T$ consisting of letters in $\{1,\dots,n-1\}$. Then by the condition of a semi-standard tableau, the shape $\mu = (\mu_1,\mu_2,\dots)$ of $T'$ is a partition satisfying the *interlacing property*

$$\lambda_1 \geq \mu_1 \geq \lambda_2 \geq \mu_2 \geq \lambda_3 \geq \cdots. \tag{3.61}$$

A pair $(\lambda, \mu)$ of partitions in $\mathcal{P}$ with $\mu \subseteq \lambda$ (i.e. $\mu_i \leq \lambda_i$ for all $i \geq 1$) is referred to as a *skew diagram* $\lambda/\mu$; we also use the notation $\lambda\backslash\mu$ for the set-theoretic difference $D(\lambda)\backslash D(\mu)$ of diagrams.

We say that a skew diagram $\lambda/\mu$ is a *horizontal strip* ("h-strip" for short) if the pair $(\lambda, \mu)$ satisfies the interlacing property (3.61). In terms of the Young diagrams, this condition is equivalent to saying that the difference $\lambda\backslash\mu$ has at most one square in each column. In this terminology, $s_\lambda^{\text{comb}}(x)$ can be expanded in the form

$$s_\lambda^{\text{comb}}(x) = \sum_{\lambda/\mu:\ \text{h-strip}} \ \sum_{T' \in \text{SSTab}_{n-1}(\mu)} (x')^{\text{wt}(T')}\, x_n^{|\lambda|-|\mu|} \tag{3.62}$$

$$= \sum_{\lambda/\mu:\ \text{h-strip}} s_\mu^{\text{comb}}(x')\, x_n^{|\lambda|-|\mu|}, \tag{3.63}$$

where $x' = (x_1, \ldots, x_{n-1})$. Namely,

$$s_\lambda^{\text{comb}}(x_1, \ldots, x_n) = \sum_{\lambda/\mu:\ \text{h-strip}} s_\mu^{\text{comb}}(x_1, \ldots, x_{n-1})\, x_n^{|\lambda/\mu|}, \tag{3.64}$$

where $|\lambda/\mu| = |\lambda| - |\mu|$. The combinatorial Schur functions $s_\lambda^{\text{comb}}(x)$ are completely determined by this recurrence formula with respect to the number of variables.

In order to establish the equivalence of the two definitions of Schur functions, we prove that $s_\lambda(x) = s_\lambda^{\det}(x)$ satisfy the same recurrence formula. Since

$$s_\lambda(x) = (x_1 \cdots x_n)^{\lambda_n} s_{\lambda-(\lambda_n^n)}(x), \quad s_\lambda^{\text{comb}}(x) = (x_1 \cdots x_n)^{\lambda_n} s_{\lambda-(\lambda_n^n)}^{\text{comb}}(x), \tag{3.65}$$

we have only to consider the case where $\lambda_n = 0$.

**Theorem 3.3** *The Schur functions $s_\lambda(x)$ satisfy the following recurrence formula with respect to the number of variables $n$: For any $\lambda \in \mathcal{P}_n$,*

$$s_\lambda(x_1, \ldots, x_n) = \sum_{\substack{\mu \subseteq \lambda \\ \lambda/\mu:\ \textit{h-strip}}} s_\mu(x_1, \ldots, x_{n-1})\, x_n^{|\lambda/\mu|}, \tag{3.66}$$

*where the sum is over all partitions $\mu \subseteq \lambda$ such that $\lambda/\mu$ is a horizontal strip.*

Recurrence formulas of this kind are called *branching formulas* as well. We give a proof of this theorem in Sect. 3.5.

Applying this recurrence formula repeatedly, we obtain an alternative expression of the tableau representation of $s_\lambda(x)$:

$$s_\lambda(x) = \sum_{\substack{\emptyset=\lambda^{(0)}\subseteq\lambda^{(1)}\subseteq\cdots\subseteq\lambda^{(n)}=\lambda \\ |\lambda^{(i)}/\lambda^{(i-1)}|:\ \text{h-strip}}} \prod_{i=1}^{n} x_i^{|\lambda^{(i)}/\lambda^{(i-1)}|} \tag{3.67}$$

where the sum is taken over all weakly increasing sequences of partitions $\emptyset = \lambda^{(0)} \subseteq \lambda^{(1)} \subseteq \cdots \subseteq \lambda^{(n)} = \lambda$ connecting $\emptyset$ (empty diagram) and $\lambda$ by $n$ steps such that the successive skew diagrams $\lambda^{(i)}/\lambda^{(i-1)}$ are all horizontal strips. It is also convenient to display such a sequence of partitions $\lambda^{(1)}, \ldots, \lambda^{(n)}$ as a table

$$\begin{bmatrix} \lambda_1^{(n)} \ \lambda_2^{(n)} \ \lambda_3^{(n)} \ \ldots \ \lambda_n^{(n)} \\ \lambda_1^{(n-1)} \lambda_2^{(n-1)} \ldots \lambda_{n-1}^{(n-1)} \\ \cdots \\ \lambda_1^{(2)} \ \lambda_2^{(2)} \\ \lambda_1^{(1)} \end{bmatrix} \tag{3.68}$$

with interlacing property $\lambda_j^{(i)} \geq \lambda_j^{(i-1)} \geq \lambda_{j+1}^{(i)}$ for $1 \leq j < i \leq n$, called a *Gelfand–Tsetlin pattern*.

## 3.5 Equivalence of the Two Definitions

In this section, we give a proof of Theorem 3.3, thereby establishing the equivalence of two definitions of Schur functions.

The recurrence formula (3.66) for $s_\lambda(x)$ (with $\lambda_n = 0$) can be proved by means of Cauchy's formula (3.58) for $f_{l_1,\ldots,l_n}(x) = \Delta_{l_1,\ldots,l_n}(x)/\Delta(x)$:

$$\sum_{l_1,\ldots,l_n \geq 0} f_{l_1,\ldots,l_n}(x_1, \ldots, x_n)\, y_1^{l_1} \cdots y_n^{l_n} = \frac{\Delta(y_1, \ldots, y_n)}{\prod_{i,j=1}^n (1 - x_i y_j)}. \tag{3.69}$$

In this formula, we set $y_n = 0$ to obtain

$$\begin{aligned} &\sum_{l_1,\ldots,l_{n-1} \geq 0} f_{l_1,\ldots,l_{n-1},0}(x_1, \ldots, x_n)\, y_1^{l_1} \cdots y_{n-1}^{l_{n-1}} \\ &= \frac{\Delta(y_1, \ldots, y_{n-1})}{\prod_{i,j=1}^{n-1} (1 - x_i y_j)} \frac{y_1 \cdots y_{n-1}}{\prod_{j=1}^{n-1} (1 - x_n y_j)} \\ &= \left( \sum_{k_1,\ldots,k_{n-1} \geq 0} f_{k_1,\ldots,k_{n-1}}(x_1, \ldots, x_{n-1}) y_1^{k_1} \cdots y_{n-1}^{k_{n-1}} \right) \\ &\quad \cdot \left( \sum_{r_1,\ldots,r_{n-1} \geq 0} x_n^{\sum_j r_j} y_1^{r_1+1} \cdots y_{n-1}^{r_{n-1}+1} \right). \end{aligned} \tag{3.70}$$

We now look at the coefficient of $y_1^{l_1} \cdots y_{n-1}^{l_{n-1}}$ assuming that $l_1 > l_2 > \cdots > l_{n-1} \geq 0$:

$$f_{l_1,\dots,l_{n-1},0}(x_1,\dots,x_n) = \sum_{\substack{k_1,\dots,k_{n-1}\geq 0\\ 0\leq k_i<l_i}} f_{k_1,\dots,k_{n-1}}(x_1,\dots,x_{n-1})\, x_n^{\sum_j l_j-\sum_j k_j-(n-1)} \tag{3.71}$$

where the sum is taken over all $(k_1,\dots,k_{n-1}) \in \mathbb{N}^{n-1}$ such that $0 \leq k_i < l_i$ $(i = 1,\dots,n-1)$, namely

$$(k_1,\dots,k_{n-1}) \in [0,l_1) \times [0,l_2) \times \cdots \times [0,l_{n-1}), \tag{3.72}$$

where we have used the symbol $[a,b) = \{k \in \mathbb{Z} \mid a \leq k < b\}$ for an interval of integers. Notice that, in the expression

$$f_{l_1,\dots,l_{n-1},0}(x_1,\dots,x_n) = \sum_{\substack{(k_1,k_2,\dots,k_{n-1})\\ \in[0,l_1)\times[0,l_2)\times\cdots\times[0,l_{n-1}),}} f_{k_1,\dots,k_{n-1}}(x_1,\dots,x_{n-1})\, x_n^{\sum_j l_j-\sum_j k_j-(n-1)}, \tag{3.73}$$

the summand $f_{k_1,\dots,k_{n-1}}(x_1,\dots,x_{n-1})$ is alternating with respect to the permutation of $k_1,\dots,k_{n-1}$. Thanks to this alternating property, the sum over the first two indices $k_1, k_2$ reduces as

$$\sum_{(k_1,k_2)\in[0,l_1)\times[0,l_2)} f_{k_1,\dots,k_{n-1}}(x_1,\dots,x_{n-1})\, x_n^{\sum_j l_j-\sum_j k_j-(n-1)} = \sum_{(k_1,k_2)\in[l_2,l_1)\times[0,l_2)} f_{k_1,\dots,k_{n-1}}(x_1,\dots,x_{n-1})\, x_n^{\sum_j l_j-\sum_j k_j-(n-1)}, \tag{3.74}$$

since the sum of an alternating function over a symmetric region gives zero (Fig. 3.1).

Repeating this procedure with $(k_2,k_3) \in [0,l_2) \times [0,l_3)$ and so on, we finally obtain

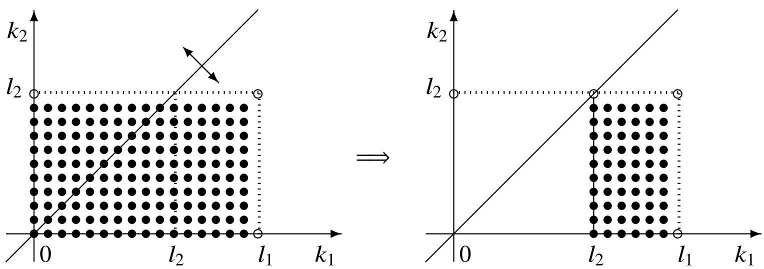



**Fig. 3.1** Reducing the region of summation indices

$$f_{l_1,\ldots,l_{n-1},0}(x_1,\ldots,x_n) \\ = \sum_{\substack{(k_1,k_2,\ldots,k_{n-1}) \\ \in[l_2,l_1)\times[l_3,l_2)\times\cdots\times[0,l_{n-1}),}} f_{k_1,\ldots,k_{n-1}}(x_1,\ldots,x_{n-1})\, x_n^{\sum_j l_j - \sum_j k_j - (n-1)}, \tag{3.75}$$

where the sum is taken over all $(k_1,\ldots,k_{n-1})$ such that

$$l_1 > k_1 \geq l_2 > k_2 \geq l_3 > \cdots \geq l_{n-1} > k_{n-1} \geq 0. \tag{3.76}$$

Then passing to the expressions by partitions $\lambda = (\lambda_1,\ldots,\lambda_{n-1},0) \in \mathcal{P}_n$ and $\mu = (\mu_1,\ldots,\mu_{n-1}) \in \mathcal{P}_{n-1}$ such that

$$l_i = \lambda_i + n - i, \quad k_i = \mu_i + n - i - 1 \qquad (i = 1,\ldots,n-1), \tag{3.77}$$

we obtain

$$\lambda_1 \geq \mu_1 \geq \lambda_2 \geq \mu_2 \geq \cdots \geq \lambda_{n-1} \geq \mu_{n-1} \geq 0, \tag{3.78}$$

and hence

$$s_\lambda(x_1,\ldots,x_n) = \sum_{\substack{\mu\subseteq\lambda \\ \lambda/\mu:\ \text{h-strip}}} s_\mu(x_1,\ldots,x_{n-1})\, x_n^{|\lambda|-|\mu|}, \tag{3.79}$$

as desired.

## 3.6 Dual Cauchy Formula

We propose two versions of the dual Cauchy formula for Schur functions.

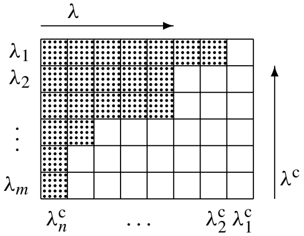
**Theorem 3.4** (Dual Cauchy formulas) *For two sets of variables $x = (x_1,\ldots,x_m)$ and $y = (y_1,\ldots,y_n)$, we have*

$$\prod_{i=1}^{m}\prod_{j=1}^{n}(1 + x_i y_j) = \sum_{\lambda\subseteq(n^m)} s_\lambda(x) s_{\lambda'}(y), \tag{3.80}$$

$$\prod_{i=1}^{m}\prod_{j=1}^{n}(x_i + y_j) = \sum_{\lambda\subseteq(n^m)} s_\lambda(x) s_{\lambda^{\mathrm{c}}}(y), \tag{3.81}$$

*where the sum is over all partitions $\lambda$ contained in the $m \times n$ rectangle $(n^m) = (n,\ldots,n)$; $\lambda'$ denotes the conjugate partition of $\lambda$ (see Sect. 2.4), and $\lambda^{\mathrm{c}} = (m - \lambda'_n, m - \lambda'_{n-1},\ldots,m-\lambda'_1)$.*

We call $\lambda^{\mathrm{c}}$ the *complementary partition* of $\lambda$ in the $m \times n$ rectangle.




$$\lambda^c = (\lambda_1^c, \ldots, \lambda_n^c) = (m - \lambda'_n, \ldots, m - \lambda'_1) \tag{3.82}$$

For the proof of these formulas, we use a lemma on determinants. For an $N \times N$ matrix $Z = (z_{i,j})_{i,j=1}^N$, we denote by

$$\det Z^{i_1,\ldots,i_r}_{j_1,\ldots,j_r} = \det \left(z_{i_a,j_b}\right)_{a,b=1}^r \tag{3.83}$$

the $r \times r$ minor determinant of $Z$ with row indices $i_1, \ldots, i_r$ and column indices $j_1, \ldots, j_r$. When $(i_1, \ldots, i_r) = (1, \ldots, r)$, we simply write $\det Z_{j_1,\ldots,j_r}$ for $\det Z^{1,\ldots,r}_{j_1,\ldots,j_r}$. Also, for two subsets $I, J \subseteq \{1, \ldots, N\}$ of indices with $|I| = |J| = r$, we use the notation $\det Z^I_J = \det Z^{i_1,\ldots,i_r}_{j_1,\ldots,j_r}$ and $\det Z_J = \det Z_{j_1,\ldots,j_r}$ taking the increasing sequences $i_1 < \ldots < i_r$ and $j_1 < \ldots < j_r$ such that $I = \{i_1, \ldots, i_r\}$ and $J = \{j_1, \ldots, j_r\}$.

**Lemma 3.2** *Setting $N = m + n$, let $X = (x_{i,j})_{1\le i\le m, 1\le j\le N}$ be an $m \times N$ matrix, and $Y = (y_{i,j})_{1\le i\le n, 1\le j\le N}$ an $n \times N$ matrix. Define the $N \times N$ matrix $Z = (z_{i,j})_{1\le i,j\le N}$ by*

$$z_{ij} = x_{i,j} \quad (1 \le i \le n), \qquad z_{m+i,j} = y_{i,j} \quad (1 \le i \le n) \tag{3.84}$$

*for all $j = 1, \ldots, N$. Then the determinant of $Z$ is expressed as*

$$\det Z = \sum_{\substack{K \sqcup L = \{1,\ldots,N\} \\ |K|=m,\ |L|=n}} \epsilon(K; L) \det X_K \det Y_L \tag{3.85}$$

*in terms of minor determinants of $X$ and $Y$, where the sum is over all pairs of subsets $K, L \subseteq \{1, \ldots, N\}$ such that $|K| = m$, $|L| = n$ and $K \sqcup L = \{1, \ldots, N\}$, and $\epsilon(K; L)$ denotes the sign defined by*

$$\epsilon(K; L) = (-1)^{\ell(K;L)}, \quad \ell(K; L) = \#\{(k, l) \in K \times L \mid k > l\}. \tag{3.86}$$

For the proof of this lemma, we refer the reader to [25], for example. □

***Proof*** **(of Theorem 3.4)** Taking the variables $x = (x_1, \ldots, x_m)$ and $y = (y_1, \ldots, y_n)$ as in Theorem 3.4, we apply this lemma to the matrices

$$X = \left(x_i^{j-1}\right)_{1\le i\le m; 1\le j\le N}, \quad Y = \left(y_i^{j-1}\right)_{1\le i\le n; 1\le j\le N}, \quad N = m + n. \tag{3.87}$$

Then we have

$$(-1)^{\binom{N}{2}} \det Z = \Delta(x, y) = \Delta(x)\Delta(y) \prod_{i=1}^{m} \prod_{j=1}^{n} (x_i - y_j). \tag{3.88}$$

On the other hand, by Lemma 3.2 we have

$$\det Z = \sum_{\substack{K \sqcup L = \{1,\dots,N\} \\ |K|=m, |L|=n}} \epsilon(K; L) \det X_K \det Y_L. \tag{3.89}$$

Hence we obtain

$$(-1)^{\binom{N}{2}} \prod_{i=1}^{m} \prod_{j=1}^{n} (x_i - y_j) = \sum_{\substack{K \sqcup L = \{1,\dots,N\} \\ |K|=m, |L|=n}} \frac{\det X_K}{\Delta(x)} \frac{\det Y_L}{\Delta(y)}. \tag{3.90}$$

Taking two sequences $k_1 > k_2 > \dots > k_m \geq 0$ and $l_1 > l_2 > \dots > l_n \geq 0$ such that $K = \{k_m + 1, \dots, k_1 + 1\}$ and $L = \{l_n + 1, \dots, l_1 + 1\}$. Then we have

$$\begin{aligned} \det X_K &= (-1)^{\binom{m}{2}} \det \left(x_i^{k_j}\right)_{i,j=1}^{m} = (-1)^{\binom{m}{2}} \Delta_{k_1,\dots,k_m}(x) \\ \det Y_L &= (-1)^{\binom{n}{2}} \det \left(y_i^{l_j}\right)_{i,j=1}^{n} = (-1)^{\binom{n}{2}} \Delta_{l_1,\dots,l_n}(x). \end{aligned} \tag{3.91}$$

For each pair $(K, L)$, we take two partitions $\mu \in \mathcal{P}_m$ and $\nu \in \mathcal{P}_n$ such that $k_i = \mu_i + m - i$ $(i = 1, \dots, m)$ and $l_i = \nu_i + n - i$ $(i = 1, \dots, n)$. Then one can show that $\nu = (m - \mu'_n, \dots, m - \mu'_1) = \mu^{\mathrm{c}}$ and $\epsilon(K; L) = (-1)^{|\mu|}$. Hence, we can rewrite (3.90) as

$$(-1)^{mn} \prod_{i=1}^{m} \prod_{j=1}^{n} (x_i - y_j) = \sum_{\mu \in (n^m)} (-1)^{|\mu|} s_\mu(x) s_{\mu^{\mathrm{c}}}(y). \tag{3.92}$$

Replacing $y_j$ by $-y_j$, we obtain the dual Cauchy formula (3.81). Formula (3.80) is obtained from (3.81) by the relation

$$(y_1 \cdots y_n)^m s_{\lambda'}(y^{-1}) = s_{\lambda^{\mathrm{c}}}(y), \tag{3.93}$$

which can be verified directly from the determinantal definition of the Schur function. □

## 3.7 Jacobi–Trudi Formula

From the Cauchy and the dual Cauchy formulas, one can read off various properties of Schur functions. For example, one can derive a determinant formula, called the *Jacobi–Trudi formula*, which represents a general Schur function $s_\lambda(x)$ in terms of complete homogeneous symmetric functions $h_k(x)$ or elementary symmetric functions $e_k(x)$

**Theorem 3.5** (Jacobi–Trudi formula) *Let $\lambda \in \mathcal{P}_n$ and $\ell(\lambda') \le m$. Then we have*

$$\text{(1)} \quad s_\lambda(x) = \det\left(h_{\lambda_i+j-i}(x)\right)_{i,j=1}^{n}. \tag{3.94}$$

$$\text{(2)} \quad s_\lambda(x) = \det\left(e_{\lambda'_i+j-i}(x)\right)_{i,j=1}^{m}. \tag{3.95}$$

In these formulas, we understand $h_k(x) = 0$, $e_k(x)$ for $k < 0$. Explicitly,

$$s_\lambda = \det\begin{bmatrix} h_{\lambda_1} & h_{\lambda_1+1} & \dots & h_{\lambda_1+n-1} \\ h_{\lambda_2-1} & h_{\lambda_2} & \dots & h_{\lambda_2+n-2} \\ \vdots & & \ddots & \vdots \\ h_{\lambda_n-n+1} & h_{\lambda_n-n+2} & \dots & h_{\lambda_n} \end{bmatrix} = \begin{bmatrix} e_{\lambda'_1} & e_{\lambda'_1+1} & \dots & e_{\lambda'_1+n-1} \\ e_{\lambda'_2-1} & e_{\lambda'_2} & \dots & e_{\lambda'_2+n-2} \\ \vdots & & \ddots & \vdots \\ e_{\lambda'_n-n+1} & e_{\lambda'_n-n+2} & \dots & e_{\lambda'_n} \end{bmatrix}. \tag{3.96}$$

Note that the size of the determinant can be reduced as

$$s_\lambda(x) = \det\left(h_{\lambda_i+j-i}(x)\right)_{i,j=1}^{\ell(\lambda)}, \quad s_\lambda(x) = \det\left(e_{\lambda'_i+j-i}(x)\right)_{i,j=1}^{\ell(\lambda')}, \tag{3.97}$$

since the $(i, j)$ entries of the matrix vanish for $i > \ell(\lambda)$ (or $i > \ell(\lambda')$) and $j < i$.

***Proof*** (1) We rewrite the Cauchy formula (3.50) as

$$\Delta(x) \prod_{i,j=1}^{n} \frac{1}{1 - x_i y_j} = \sum_{\lambda \in \mathcal{P}_n} \Delta_{\lambda+\delta}(x) s_\lambda(y). \tag{3.98}$$

Then $s_\lambda(y)$ is the coefficient of $x^{\lambda+\delta}$ in the right-hand side. On the other hand,

$$\Delta(x) \prod_{i,j=1}^{n} \frac{1}{1 - x_i y_j} = \Delta(x) \prod_{i=1}^{n} (1 + x_i h_1(y) + x_i^2 h_2(y) + \cdots) \tag{3.99}$$

$$= \sum_{\sigma \in \mathfrak{S}_n} \operatorname{sgn}(\sigma) x^{\sigma.\delta} \sum_{\mu \in \mathbb{N}^n} x^\mu h_\mu(y), \tag{3.100}$$

where $h_\mu(y) = h_{\mu_1}(y) \cdots h_{\mu_n}(y)$. Taking the coefficient of $x^{\lambda+\delta}$, we obtain

$$s_\lambda(y) = \sum_{\sigma \in \mathfrak{S}_n} \operatorname{sgn}(\sigma) h_{\lambda+\delta-\sigma.\delta}(y)$$
$$= \det \left(h_{\lambda_i+\delta_i-\delta_j}\right)_{i,j=1}^n = \det \left(h_{\lambda_i+j-i}(y)\right)_{i,j=1}^n, \tag{3.101}$$

which proves (3.94).

(2) We rewrite the dual Cauchy formula (3.80) as

$$\Delta(x) \prod_{i=1}^m \prod_{j=1}^n (1 + x_i y_j) = \sum_{\lambda \subseteq (n^m)} \Delta_{\lambda+\delta}(x)\, s_{\lambda'}(y). \tag{3.102}$$

Then $s_{\lambda'}(y)$ is the coefficient of $x^{\lambda+\delta}$ in the right-hand side. On the other hand,

$$\Delta(x) \prod_{i=1}^m \prod_{j=1}^n (1 + x_i y_j) = \Delta(x) \prod_{i=1}^m (1 + x_i e_1(y) + \cdots + x_i^n e_n(y))$$
$$= \sum_{\sigma \in \mathfrak{S}_m} \operatorname{sgn}(\sigma) x^{\sigma.\delta} \sum_{\mu \in \mathbb{N}^m} x^\mu e_\mu(y), \tag{3.103}$$

where $e_\mu(y) = e_{\mu_1}(y) \cdots e_{\mu_n}(y)$. Taking the coefficient of $x^{\lambda+\delta}$ in this formula, we obtain

$$s_{\lambda'}(y) = \sum_{\sigma \in \mathfrak{S}_m} \operatorname{sgn}(\sigma) e_{\lambda+\delta-\delta.\sigma}(y)$$
$$= \det \left(e_{\lambda_i+\delta_i-\delta_j}\right)_{i,j=1}^n = \det \left(e_{\lambda_i+j-i}\right)_{i,j=1}^n, \tag{3.104}$$

as desired. □

## 3.8 $q$-Difference and Differential Equations

For each $i = 1, \ldots, n$, we define the *$q$-shift operator* $T_{q,x_i}$ in $x_i$ by

$$T_{q,x_i} \varphi(x_1, \ldots, x_i, \ldots, x_n) = \varphi(x_1, \ldots, q x_i, \ldots, x_n) \quad (i = 1, \ldots, n) \tag{3.105}$$

leaving $x_j$ for $j \neq i$ unchanged. For $r = 0, 1, \ldots, n$, we define the $q$-difference operators $D_x^{(r)}$ by

$$\begin{aligned} D_x^{(r)} &= \sum_{\substack{I\subseteq\{1,\ldots,n\}\\|I|=r}} \frac{T_{q,x}^I(\Delta(x))}{\Delta(x)} T_{q,x}^I \\ &= \sum_{\substack{I\subseteq\{1,\ldots,n\}\\|I|=r}} q^{\binom{r}{2}} \prod_{i\in I;\, j\notin J} \frac{qx_i - x_j}{x_i - x_j} \prod_{i\in I} T_{q,x_i}, \end{aligned} \tag{3.106}$$

where $T_{q,x}^I = \prod_{i\in I} T_{q,x_i}$. As we will see below, the $q$-difference operators $D_x^{(r)}$ ($r = 1,\ldots,n$) commutes with each other. We remark that these $q$-difference operators $D_x^{(r)}$ are the special case of *Macdonald–Ruijsenaars operators* with $q = t$ to be discussed from the next chapter on.

**Theorem 3.6** *For each $\lambda \in \mathcal{P}_n$, the Schur function $s_\lambda(x)$ satisfies the system of $q$-difference equations*

$$D_x^{(r)} s_\lambda(x) = e_r(q^{\lambda+\delta}) s_\lambda(x) \quad (r = 0, 1, \ldots, n), \tag{3.107}$$

*where the eigenvalues $e_r(q^{\lambda+\delta})$ are the elementary symmetric functions of $q^{\lambda_i+n-i}$ ($i = 1,\ldots,n$).*

In fact, the $q$-shift operator $T_{q,x_i}$ acts on monomials in $x = (x_1,\ldots,x_n)$ by

$$T_{q,x_i}(x^\mu) = q^{\mu_i} x^\mu, \quad \mu = (\mu_1,\ldots,\mu_n) \in \mathbb{N}^n \tag{3.108}$$

for $i = 1,\ldots,n$. Hence, for each polynomials $f(\xi) \in \mathbb{C}[\xi]$ in $\xi = (\xi_1,\ldots,\xi_n)$, the $q$-difference operator $f(T_{q,x}) = f(T_{q,x_1},\ldots,T_{q,x_n})$ acts on monomials by

$$f(T_{q,x})x^\mu = f(q^\mu)x^\mu \quad (\mu \in \mathbb{N}^n). \tag{3.109}$$

If $f(\xi)$ is $\mathfrak{S}_n$-invarinat, then $f(T_{q,x})$ acts on monomial symmetric functions $m_\lambda(x) = \sum_{\mu\in\mathfrak{S}_n.\lambda} x^\mu$ by

$$f(T_{q,x})m_\lambda(x) = f(q^\lambda)m_\lambda(x) \qquad (\lambda \in \mathcal{P}_n), \tag{3.110}$$

since $f(q^\mu) = f(q^{\sigma.\lambda}) = f(q^\lambda)$ for $\mu = \sigma.\lambda$, $\sigma \in \mathfrak{S}_n$. Taking elementary symmetric functions $e_r(\xi)$ for $f(\xi)$, we obtain

$$e_r(T_{q,x})m_\lambda(x) = e_r(q^\lambda)m_\lambda(x) \quad (\lambda \in \mathcal{P}_n, r = 0, 1, \ldots, n). \tag{3.111}$$

Similarly, the alternating polynomials $\Delta_{\lambda+\delta}(x) = \sum_{\mu\in\mathfrak{S}_n.\lambda} \operatorname{sgn}(\sigma) x^{\sigma.(\lambda+\delta)}$ ($\lambda \in \mathcal{P}_n$) satisfy

$$f(T_{q,x})\Delta_{\lambda+\delta}(x) = f(q^{\lambda+\delta})\Delta_{\lambda+\delta}(x) \qquad (\lambda \in \mathcal{P}_n) \tag{3.112}$$

for all $f(\xi) \in \mathbb{C}[\xi]^{\mathfrak{S}_n}$. By conjugation, we introduce the $q-$difference operators

$$D_x^f = \Delta(x) f(T_{q,x}) \Delta(x)^{-1}. \tag{3.113}$$

Then, we see that the Schur functions $s_\lambda(x) = \Delta_{\lambda+\delta}(x)/\Delta(x)$ satisfy the $q$-difference equations

$$D_x^f s_\lambda(x) = f(q^{\lambda+\delta}) s_\lambda(x) \qquad (\lambda \in \mathcal{P}_n) \tag{3.114}$$

for all symmetric polynomials $f(\xi) = \mathbb{C}[\xi]^{\mathfrak{S}_n}$. The $q$-difference operators $D_x^{(r)}$ of (3.106) are the special cases of $D_x^f$, where $f = e_r$ $(r = 0, 1, \dots, n)$. We also remark that the $q$-difference operators $D_x^f$ for all $f(\xi) \in \mathbb{C}[\xi]^{\mathfrak{S}_n}$ pairwise commute since they are conjugations of $f(T_{q,x})$ by $\Delta(x)$.

The differential operators $x_i \partial_{x_i} = x_i \partial/\partial x_i$ acts on monomials in $x = (x_1, \dots, x_n)$ by

$$x_i \partial_{x_i} x^\mu = \mu_i x^\mu, \quad \mu = (\mu_1, \dots, \mu_n) \in \mathbb{N}^n \tag{3.115}$$

for $i = 1, \dots, n$. Hence for any polynomial $f(\xi) \in \mathbb{C}[\xi]$ in $\xi = (\xi_1, \dots, \xi_n)$, we have

$$f(x\partial_x) x^\mu = f(\mu) x^\mu \quad (\mu \in \mathbb{N}^n). \tag{3.116}$$

Hence for all $f(\xi) \in \mathbb{C}[\xi]^{\mathfrak{S}_n}$, we have

$$f(x\partial_x) m_\lambda(x) = f(\lambda) m_\lambda(x), \quad f(x\partial_x) \Delta_{\lambda+\delta}(x) = f(\lambda+\delta) \Delta_{\lambda+\delta}(x). \tag{3.117}$$

By conjugation, we introduce the differential operator

$$L_x^f = \Delta(x) f(x\partial_x) \Delta(x)^{-1}. \tag{3.118}$$

Then, we see that the Schur functions satisfy the differential equations

$$L_x^f s_\lambda(x) = f(\lambda+\delta) s_\lambda(x) \qquad (\lambda \in \mathcal{P}_n) \tag{3.119}$$

for all $f(\xi) = \mathbb{C}[\xi]^{\mathfrak{S}_n}$. In particular, for $L_x^{(r)} = L_x^{e_r}$ we have

$$L_x^{(r)} s_\lambda(x) = e_r(\lambda+\delta) s_\lambda(x) \quad (\lambda \in \mathcal{P}_n, r = 0, 1, \dots, n), \tag{3.120}$$

where

$$L_x^{(r)} = \sum_{|K|=r} \frac{1}{\Delta(x)} (x\partial_x)^K \Delta(x) = \sum_{|K|=r} \sum_{I \sqcup J = K} \frac{(x\partial_x)^I (\Delta(x))}{\Delta(x)} (x\partial_x)^J \tag{3.121}$$

with the notation $(x\partial_x)^I = \prod_{i \in I} x_i \partial_{x_i}$.

## 3.9 Link to the Representation Theory of $\mathrm{GL}_n$ (Overview)

In this section, we outline how the Schur functions arise, and how their properties are interpreted, in the context of representation theory of general linear groups. For the detail, see Goodman–Wallach [9] for example.

By a *representation* of a group $G$, we mean a $\mathbb{C}$-vector space $M$ endowed with a group homomorphism $\pi_M : G \to \mathrm{GL}_{\mathbb{C}}(M)$, where $\mathrm{GL}_{\mathbb{C}}(M)$ denotes the group of invertible $\mathbb{C}$-linear transformations of $M$. In this situation, we also say that $M$ is a *G-module*, and use the notation of the left action $g.v = \pi_M(g)(v)$ of $g \in G$ on $v \in M$. Suppose that $M$ is finite-dimensional, and fix a $\mathbb{C}$-basis $v_1, \ldots, v_N$ of $M$. For each $g \in G$, we take the matrix representation $\Phi(g) = \big(\varphi_{ij}(g)\big)_{i,j=1}^{N}$ of $\pi_M(g) : M \to M$ with respect to the basis $(v_1, \ldots, v_N)$:

$$g.v_j = \pi_M(g)(v_j) = \sum_{i=1}^{N} v_i \varphi_{ij}(g) \quad (i = 1, \ldots, N). \tag{3.122}$$

Then we obtain an $N \times N$ matrix $\Phi_M(g) = \Phi(g)$ whose entries are functions on $G$ satisfying the condition

$$\Phi(1_G) = I_N, \quad \Phi(g_1 g_2) = \Phi(g_1)\Phi(g_2), \quad \Phi(g^{-1}) = \Phi(g)^{-1}. \tag{3.123}$$

### 3.9.1 *Polynomial Representations of* $\mathrm{GL}_n$

We consider the case of the general linear group $\mathrm{GL}_n = \mathrm{GL}_n(\mathbb{C})$ of degree $n$. Expressing a general element of $\mathrm{GL}_n$ as $g = (g_{ij})_{i,j=1}^{n}$, we regard $g_{ij}$ $(1 \le i, j \le n)$ as the canonical coordinates of $\mathrm{GL}_n$. A representation $M$ of $\mathrm{GL}_n$ is called a *polynomial representation* if the matrix elements $\varphi_{ij}(g)$ are all polynomials of the coordinates $g_{ij}$ $(1 \le i, j \le n)$. It is known that any polynomial representation is completely reducible, and the isomorphism classes of irreducible representations are parametrized by the partitions $\lambda \in \mathcal{P}$ with $\ell(\lambda) \le n$. Namely, for each $\lambda \in \mathcal{P}_n$, there exists an irreducible polynomial representation $V(\lambda) = V_n(\lambda)$ (with highest weight $\lambda$), uniquely determined up to isomorphism, such that $V(\lambda) \not\simeq V(\mu)$ if $\lambda \ne \mu$, and that any polynomial representation $M$ is decomposed into a direct sum of the form

$$M \simeq \bigoplus_{\lambda \in \mathcal{P}_n} V(\lambda)^{\oplus m_\lambda} \tag{3.124}$$

with some multiplicities $m_\lambda \in \mathbb{N}$. We remark that $V(1^r) = V(\varpi_r)$, $\varpi_r = \varepsilon_1 + \cdots + \varepsilon_r$ (fundamental weights) $(r = 0, 1, \ldots, n)$ attached to single columns are the alternating tensor representation $\Lambda^r(V)$ of the vector space $V = \mathbb{C}$ on which $\mathrm{GL}_n$ is defined, and $V((l)) = V(l\varepsilon_1)$ ($l = 0, 1, 2, \ldots$) attached to single rows are the symmetric tensor representation $S^l(V)$.

We denote by $H_n \subseteq \mathrm{GL}_n$ the diagonal subgroup of $\mathrm{GL}_n$. Expressing a general element of $H_n$ as $g_x = \mathrm{diag}(x_1, \ldots, x_n)$, we regard $x = (x_1, \ldots, x_n)$ as coordinates of $H_n$, and identify $H_n$ with $(\mathbb{C}^*)^n$, $\mathbb{C}^* = \mathbb{C}\setminus\{0\}$. For each polynomial representation $M$ of $\mathrm{GL}_n$, we define the function $\mathrm{ch}_M(x)$ of $x = (x_1, \ldots, x_n) \in H_n = (\mathbb{C}^*)^n$ by

$$\mathrm{ch}_M(x) = \mathrm{tr}(\pi_M(g_x) : M \to M) = \mathrm{tr}\,\Phi_M(g_x), \tag{3.125}$$

and call it the *character* of the representation $M$. For each $\mu \in \mathbb{N}^n$, we denote by

$$M_\mu = \{v \in M \mid g_x.v = x^\mu v \ \ (x \in H_n)\} \subseteq M \tag{3.126}$$

the subspace of *weight* $\mu$. Since $M$ decomposes into the direct sum $M = \bigoplus_{\mu\in\mathbb{N}^n} M_\mu$ of weight subspaces, we have

$$\mathrm{ch}_M(x) = \sum_{\mu\in\mathbb{N}^n} (\dim_{\mathbb{C}} M_\mu) x^\mu \in \mathbb{C}[x]. \tag{3.127}$$

In this sense, the character $\mathrm{ch}_M(x)$ provides the generating function for counting the weight multiplicities in $M$. Note that $\mathrm{ch}_M(1) = \dim_{\mathbb{C}} M$. Also, for two polynomial representations $M, N$, the character of the tensor product representation $M \otimes N$ is given by the multiplication of the two characters as functions on $H_n$, namely $\mathrm{ch}_{M\otimes N}(x) = \mathrm{ch}_M(x)\mathrm{ch}_N(x)$.

A fundamental fact in the representation theory of $\mathrm{GL}_n$ is that the Schur function $s_\lambda(x)$ attached to each $\lambda \in \mathcal{P}_n$ appears as the character of the irreducible polynomial representation $V(\lambda)$, namely, $\mathrm{ch}_{V(\lambda)}(x) = s_\lambda(x)$.

### *3.9.2 Weyl Character Formula and Branching Rules*

In the context of representation theory, the determinant representation

$$s_\lambda(x) = \frac{\det(x_i^{\lambda_j+n-j})_{i,j=1}^n}{\det(x_i^{n-j})_{i,j=1}^n} = \frac{\Delta_{\lambda+\delta}(x)}{\Delta(x)} \tag{3.128}$$

is called the *Weyl character formula*. The combinatorial description of $s_\lambda(x)$ in terms of semi-standard tableaux arises from the *branching rule* for the restriction of $V(\lambda)$ along the tower of subgroups

$$\mathrm{GL}_n \supset \mathrm{GL}_{n-1} \supset \cdots \supset \mathrm{GL}_1. \tag{3.129}$$

In fact, if we restrict the representation $V(\lambda) = V_n(\lambda)$ ($\lambda \in \mathcal{P}_n$) from $\mathrm{GL}_n$ to $\mathrm{GL}_{n-1}$, it decomposes into the direct sum

$$V_n(\lambda) \simeq \bigoplus_{\mu \in \mathcal{P}_{n-1},\ \lambda/\mu:\ \text{h-strip}} V_{n-1}(\mu) \qquad (\lambda \in \mathcal{P}_n) \tag{3.130}$$

of irreducible $\mathrm{GL}_{n-1}$-modules. Passing to the level of characters, this multiplicity-free decomposition of $V_n(\lambda)$ gives rise to the recurrence formula for Schur functions of Theorem 3.3 with respect to the number of variables. Repeating this restriction procedure, we find that $V(\lambda) = V_n(\lambda)$ for each $\lambda \in \mathcal{P}_n$ has a $\mathbb{C}$-basis $v_T$ ($T \in \mathrm{SSTab}_n(\lambda)$) parameterized by the semi-standard tableaux of shape $\lambda$ such that $g_x.v_T = x^{\mathrm{wt}(T)} v_T$:

$$V(\lambda) = \bigoplus_{\mu \in \mathbb{N}^n} V(\lambda)_\mu, \quad V(\lambda)_\mu = \bigoplus_{T \in \mathrm{SSTab}_n(\lambda)_\mu} \mathbb{C}\, v_T. \tag{3.131}$$

This gives rise to the tableau representation

$$s_\lambda(x) = \sum_{T \in \mathrm{SSTab}_n(\lambda)} x^{\mathrm{wt}(T)} = \sum_{\mu \in \mathbb{N}^n} K_{\lambda,\mu}\, x^\mu \tag{3.132}$$

of the character $s_\lambda(x)$; in particular, the Kostka numbers count the multiplicities of weights in $V(\lambda)$, i.e. $K_{\lambda,\mu} = \dim_{\mathbb{C}} V(\lambda)_\mu$. In the language of representation theory, we have shown in this chapter that, from the Weyl character formula, one can derive the combinatorial description of the weight subspaces of irreducible representations $V(\lambda)$ ($\lambda \in \mathcal{P}_n$)

### *3.9.3* $(\mathbf{GL}_m, \mathbf{GL}_n)$ *Duality*

We also give a remark on the Cauchy formula for Schur functions. We consider the space $\mathrm{Mat}_{m,n} = \mathrm{Mat}_{m,n}(\mathbb{C})$ of all $m \times n$ matrices. Denoting a general element of $\mathrm{Mat}_{m,n}$ as $T = \big(t_{ij}\big)_{1 \le i \le m;\, 1 \le j \le n}$, we regard $t_{ij}$ as the canonical coordinates of $\mathrm{Mat}_{m,n}$. Then the coordinate ring of $\mathrm{Mat}_{m,n}$ is identified with the ring of polynomials in $t_{ij}$, i.e. $\mathcal{A}(\mathrm{Mat}_{m,n}) = \mathbb{C}[t_{ij}\ (1 \le i \le m,\ 1 \le j \le n)]$. We regard $\mathcal{A}(\mathrm{Mat}_{m,n})$ as a representation of the product group $\mathrm{GL}_m \times \mathrm{GL}_n$ through the action of $(g, h) \in \mathrm{GL}_m \times \mathrm{GL}_n$ defined by

$$((g, h).\varphi)(T) = \varphi(g^{\mathrm{t}}\, T\, h) \qquad (\varphi \in \mathcal{A}(\mathrm{Mat}_{m,n}),\ T \in \mathrm{Mat}_{m,n}). \tag{3.133}$$

Then it turns out that $\mathcal{A}(\mathrm{Mat}_{m,n})$ has the irreducible decomposition

$$\mathcal{A}(\mathrm{Mat}_{m,n}) \simeq \bigoplus_{\ell(\lambda) \le \min\{m,n\}} V_m(\lambda) \otimes V_n(\lambda), \tag{3.134}$$

where the sum is over all partitions $\lambda$ with $\ell(\lambda) \leq \min\{m, n\}$. From this $(GL_m, GL_n)$ *duality*, we obtain the identity

$$\mathrm{ch}_{\mathcal{A}(\mathrm{Mat}_{m,n})}(x, y) = \sum_{\ell(\lambda)\leq\min\{m,n\}} \mathrm{ch}_{V_m(\lambda)}(x)\, \mathrm{ch}_{V_n(\lambda)}(y) \tag{3.135}$$

for the (formal) character of the $GL_m \times GL_n$-module $\mathcal{A}(\mathrm{Mat}_{m,n})$, which is precisely the Cauchy formula (3.59) for Schur functions. In fact, for each $(x, y) \in H_m \times H_n$, the action of $(g_x, g_y) \in GL_m \times GL_n$ on the coordinates $t_{ij}$ is given by

$$(g_x, g_y).t_{ij} = x_i t_{ij} y_j \quad (1 \leq i \leq m,\ 1 \leq j \leq n). \tag{3.136}$$

Hence, $(g_x, g_y)$ acts on the monomials $t^A = \prod_{i=1}^m \prod_{j=1}^n t_{ij}^{a_{ij}}$ attached to $A = (a_{ij})_{ij} \in \mathrm{Mat}_{m,n}(\mathbb{N})$ by

$$(g_x, g_y).t^A = \prod_{i=1}^m \prod_{j=1}^n (x_i t_{ij} y_j)^{a_{ij}} = x^{\mu(A)} t^A y^{\nu(A)}, \tag{3.137}$$

where the weights $\mu(A) \in \mathbb{N}^m$ and $\nu(A) \in \mathbb{N}^n$ are the row sum and the column sum of $A$ respectively, i.e. $\mu(A)_i = \sum_{j=1}^n a_{ij}$, $\nu(A)_j = \sum_{i=1}^m a_{ij}$. Noting that

$$\mathcal{A}(\mathrm{Mat}_{m,n}) = \bigoplus_{A\in\mathrm{Mat}_{m,n}(\mathbb{N})} \mathbb{C}\, t^A, \tag{3.138}$$

we obtain

$$\begin{aligned} \mathrm{ch}_{\mathcal{A}(\mathrm{Mat}_{m,n})}(x, y) &= \sum_{A\in\mathrm{Mat}_{m,n}(\mathbb{N})} x^{\mu(A)} y^{\nu(A)} \\ &= \sum_{A=(a_{ij})} \prod_{i=1}^m \prod_{j=1}^n (x_i y_j)^{a_{ij}} = \prod_{i=1}^m \prod_{j=1}^n \frac{1}{1 - x_i y_j}. \end{aligned} \tag{3.139}$$

Since $\mathrm{ch}_{V_m(\lambda)}(x) = s_\lambda(x)$ and $\mathrm{ch}_{V_n(\lambda)}(y) = s_\lambda(y)$, formula (3.134) implies the Cauchy formula

$$\prod_{i=1}^m \prod_{j=1}^n \frac{1}{1 - x_i y_j} = \sum_{\ell(\lambda)\leq\min\{m,n\}} s_\lambda(x)\, s_\lambda(y). \tag{3.140}$$

# Chapter 4
# Macdonald Polynomials: Definition and Examples

**Abstract** The Macdonald polynomials are defined as eigenfunctions of the Macdonald–Ruijsenaars $q$-difference operator acting on the ring of symmetric polynomials. We also investigate some special cases where Macdonald polynomials can be explicitly described, including the case of single rows.

## 4.1 Macdonald–Ruijsenaars $q$-Difference Operator

### *4.1.1 Macdonald–Ruijsenaars Operator $D_x$*

We regard the variables $x = (x_1, \ldots, x_n)$ as the canonical coordinates of the $n$-dimensional algebraic torus $(\mathbb{C}^*)^n$. We fix parameters $q, t \in \mathbb{C}^* = \mathbb{C}\setminus \{0\}$ with $|q| < 1$.

The *Macdonald–Ruijsenaars $q$-difference operator* of first order with parameter $t$ is defined by

$$D_x = \sum_{i=1}^{n} A_i(x) T_{q,x_i} = \sum_{i=1}^{n} \prod_{1\le j\le n;\ j\ne i} \frac{tx_i - x_j}{x_i - x_j} T_{q,x_i}, \tag{4.1}$$

where $T_{q,x_i}$ stands for the $q$-shift operator in the variable $x_i$:

$$T_{q,x_i} f(x_1, \ldots, x_i, \ldots, x_n) = f(x_1, \ldots, qx_i, \ldots, x_n) \quad (i = 1, \ldots, n). \tag{4.2}$$

We remark that the coefficients of $D_x$ are expressed as

$$A_i(x) = \prod_{j\ne i} \frac{tx_i - x_j}{x_i - x_j} = \frac{T_{t,x_i}\Delta(x)}{\Delta(x)}, \qquad \Delta(x) = \prod_{1\le i<j\le n} (x_i - x_j) \tag{4.3}$$

in terms of the difference product $\Delta(x)$ of $x$.

In the following, we denote by $\mathcal{D}_{q,x} = \mathbb{C}(x)[T_{q,x}^{\pm 1}]$ the ring of *$q$-difference operators* with rational function coefficients

 
M. Noumi, *Macdonald Polynomials*, SpringerBriefs in Mathematical Physics,
https://doi.org/10.1007/978-981-99-4587-0_4

$$L_x = \sum_{\mu \in \mathbb{Z}^n} a_\mu(x)\, T_{q,x}^{\mu} \quad \text{(finite sum)}, \quad a_\mu(x) \in \mathbb{C}(x) \quad (\mu \in \mathbb{Z}^n), \tag{4.4}$$

where $T_{q,x}^{\mu} = T_{q,x_1}^{\mu_1} \cdots T_{q,x_n}^{\mu_n}$. Note that the symmetric group $\mathfrak{S}_n$ acts on $\mathcal{D}_{q,x}$ through the $\mathbb{C}$-algebra automorphisms $\sigma : \mathcal{D}_{q,x} \to \mathcal{D}_{q,x}$ $(\sigma \in \mathfrak{S}_n)$ such that $\sigma(x_i) = x_{\sigma(i)}$, $\sigma(T_{q,x_i}) = T_{q,x_{\sigma(i)}}$ $(i = 1, \ldots, n)$. Note also that $\mathcal{D}_{q,x}$ acts naturally on the field $\mathbb{C}(x)$ of rational functions in $x$.

### 4.1.2 Fundamental Properties of $D_x$

We first give remarks on fundamental properties of the Macdonald–Ruijsenaars operator $D_x$.

(1) The $q$-difference operator $D_x$ is $\mathfrak{S}_n$-invariant, and hence the linear operator $D_x : \mathbb{C}(x) \to \mathbb{C}(x)$ stabilizes the field $\mathbb{C}(x)^{\mathfrak{S}_n}$ of symmetric rational functions, i.e. $D_x(\mathbb{C}(x)^{\mathfrak{S}_n}) \subseteq \mathbb{C}(x)^{\mathfrak{S}_n}$.

In fact, the definition of $D_x$ does not depend on the ordering of $\{1, 2, \ldots, n\}$.

(2) The linear operator $D_x : \mathbb{C}(x) \to \mathbb{C}(x)$ stabilizes the ring $\mathbb{C}[x]^{\mathfrak{S}_n}$ of symmetric polynomials, i.e. $D_x(\mathbb{C}[x]^{\mathfrak{S}_n}) \subseteq \mathbb{C}[x]^{\mathfrak{S}_n}$.

If $f(x) \in \mathbb{C}[x]^{\mathfrak{S}_n}$, then $g(x) = D_x(f(x))$ is a symmetric rational function by the $\mathfrak{S}_n$-invariance of $D_x$. Since $\Delta(x)D_x$ has polynomial coefficients, $\Delta(x)g(x) \in \mathbb{C}[x]$ is an alternating polynomial, and hence divisible by $\Delta(x)$. This means that $g(x) = D_x(f(x)) \in \mathbb{C}[x]^{\mathfrak{S}_n}$. Warning: $D_x : \mathbb{C}(x) \to \mathbb{C}(x)$ does *not* stabilize the polynomial ring $\mathbb{C}[x]$.

**Lemma 4.1** *The linear operator $D_x : \mathbb{C}[x]^{\mathfrak{S}_n} \to \mathbb{C}[x]^{\mathfrak{S}_n}$ is triangular with respect to the dominance order of $m_\lambda(x)$. Namely, for each $\lambda \in \mathcal{P}_n$,*

$$D_x m_\lambda(x) = \sum_{\mu \le \lambda} d_\mu^\lambda\, m_\mu(x) = d_\lambda\, m_\lambda(x) + \sum_{\mu < \lambda} d_\mu^\lambda\, m_\mu(x) \tag{4.5}$$

*for some constants $d_\mu^\lambda \in \mathbb{C}$, where $d_\lambda = d_\lambda^\lambda = \sum_{i=1}^n t^{n-i} q^{\lambda_i}$.*

***Proof*** As for the dominance order, by Remark 2.1 we know that $\nu \le \mu$ if and only if

$$x^\nu = x^\mu (x_2/x_1)^{k_1} \cdots (x_n/x_{n-1})^{k_{n-1}} \tag{4.6}$$

for some $k_1, \ldots, k_{n-1} \in \mathbb{N}$. In view of this fact, for each $\mu \in \mathbb{N}^n$ we consider the asymptotic expansion of $D_x x^\mu$ in the region $|x_1| \gg |x_2| \gg \cdots \gg |x_n|$, assuming that $|x_2/x_1|, \ldots, |x_n/x_{n-1}|$ are very small. For each $i = 1, \ldots, n$, the coefficient of $T_{q,x_i}$ can be expanded into formal power series of $x_2/x_1, \ldots, x_n/x_{n-1}$ as

$$\begin{aligned}\prod_{j\neq i}\frac{tx_i-x_j}{x_i-x_j}&=\prod_{1\le j<i}\frac{tx_i-x_j}{x_i-x_j}\prod_{i<j\le n}\frac{tx_i-x_j}{x_i-x_j}\\&=\prod_{1\le j<i}\frac{1-tx_i/x_j}{1-x_i/x_j}\prod_{i<j\le n}t\frac{1-x_j/tx_i}{1-x_j/x_i}\\&=\big(1+(\text{lower-order terms})\big)\big(t^{n-i}+(\text{lower-order terms})\big)\\&=t^{n-i}+(\text{lower-order terms}).\end{aligned}\tag{4.7}$$

Hence we have

$$\begin{aligned}D_x x^\mu&=\sum_{i=1}^n\prod_{j\neq i}\frac{tx_i-x_j}{x_i-x_j}q^{\mu_i}x^\mu\\&=x^\mu\left(\sum_{i=1}^n t^{n-i}q^{\mu_i}+(\text{lower-order terms})\right)\\&\in x^\mu\,\mathbb{C}[\![x_2/x_1,\dots,x_n/x_{n-1}]\!]\end{aligned}\tag{4.8}$$

Hereafter, we set $d_\mu=\sum_{i=1}^n t^{n-i}q^{\mu_i}$ for each $\mu\in\mathbb{N}^n$. Then, for each $\lambda\in\mathcal{P}_n$ we have

$$\begin{aligned}D_x m_\lambda(x)&=\sum_{\mu\in\mathfrak{S}_n.\lambda}D_x x^\mu\\&=\sum_{\mu\in\mathfrak{S}_n.\lambda}\big(d_\mu x^\mu+(\text{lower-order terms})\big)\\&=d_\lambda x^\lambda+(\text{lower-order terms})\\&=d_\lambda m_\lambda(x)+\sum_{\mu<\lambda}d_\mu^\lambda\, m_\mu(x),\end{aligned}\tag{4.9}$$

since we know that $D_x m_\lambda(x)$ is a symmetric polynomial. This implies the triangularity of $D_x$ with respect to $\le$ as mentioned above. □

### 4.1.3 Diagonalization of $D_x$

With these preparatory remarks, we prove that $D_x$ is diagonalizable on the ring $\mathbb{C}[x]^{\mathfrak{S}_n}$ of symmetric polynomials if the parameters $q,t$ are *generic*. In what follows, we suppose that the parameters $q,t$ are generic in the sense that

$$\lambda,\mu\in\mathcal{P}_n,\quad \lambda\neq\mu\quad\Longrightarrow\quad d_\lambda\neq d_\mu.\tag{4.10}$$

Since $d_\lambda-d_\mu=\sum_{i=1}^n t^{n-i}(q^{\lambda_i}-q^{\mu_i})$, this condition is fulfilled if $1,t,\dots,t^{n-1}$ are linearly independent over $\mathbb{Q}(q)$.

**Theorem 4.1** (Macdonald) *Suppose that the parameters $q, t \in \mathbb{C}^*$ satisfy the genericity condition (4.10). Then, for each partition $\lambda \in \mathcal{P}_n$ there exists a unique symmetric polynomial $P_\lambda(x) = P_\lambda(x; q, t) \in \mathbb{C}[x]^{\mathfrak{S}_n}$, homogenous of degree $|\lambda|$, such that*

$$(1)\ \ D_x P_\lambda(x) = d_\lambda P_\lambda(x), \qquad d_\lambda = \sum_{i=1}^{n} t^{n-i} q^{\lambda_i}, \tag{4.11}$$

$$(2)\ \ P_\lambda(x) = m_\lambda(x) + \sum_{\mu<\lambda} u_\mu^\lambda m_\mu(x) \quad (u_\mu^\lambda \in \mathbb{C}). \tag{4.12}$$

This eigenfunction $P_\lambda(x) \in \mathbb{C}[x]^{\mathfrak{S}_n}$ is called the *Macdonald polynomial* attached to the partition $\lambda \in \mathcal{P}_n$. We remark that, when we regard $q, t$ as indeterminates, $P_\lambda(x; q, t)$ is determined as a unique symmetric polynomial in $\mathbb{Q}(q, t)[x]^{\mathfrak{S}_n}$, where $\mathbb{Q}(q, t)$ denotes the field of rational functions in $(q, t)$.

***Proof*** (of Theorem 4.1) We assume that $P_\lambda(x)$ is expressed as

$$P_\lambda(x) = \sum_{\nu\le\lambda} u_\nu^\lambda m_\nu(x), \quad u_\lambda^\lambda = 1. \tag{4.13}$$

With the coefficients $d_\mu^\lambda$ as in (4.5), we have

$$D_x P_\lambda(x) = \sum_{\nu\le\lambda} u_\nu^\lambda D_x m_\nu(x) = \sum_{\mu\le\lambda} \left( \sum_{\mu\le\nu\le\lambda} u_\nu^\lambda d_\mu^\nu \right) m_\mu(x). \tag{4.14}$$

Hence the eigenfunction equation $D_x P_\lambda(x) = \varepsilon P_\lambda(x)$, $(\varepsilon \in \mathbb{C})$ is equivalent to the system of equations

$$\varepsilon\, u_\mu^\lambda = \sum_{\mu\le\nu\le\lambda} u_\nu^\lambda d_\mu^\nu, \quad \text{i.e.} \quad (\varepsilon - d_\mu) u_\mu^\lambda = \sum_{\mu<\nu\le\lambda} u_\nu^\lambda d_\mu^\nu \tag{4.15}$$

for $\mu \in \mathcal{P}_n$ with $\mu \le \lambda$. From the case where $\mu = \lambda$, $(\varepsilon - d_\lambda) u_\lambda^\lambda = 0$, $u_\lambda^\lambda = 1$, we obtain $\varepsilon = d_\lambda$. With this eigenvalue, the equations

$$(d_\lambda - d_\mu) u_\mu^\lambda = \sum_{\mu<\nu\le\lambda} u_\nu^\lambda d_\mu^\nu \qquad (\mu \in \mathcal{P}_n,\ \mu < \lambda) \tag{4.16}$$

for the coefficients $u_\mu^\lambda$ $(\mu < \lambda)$ can be solved in a unique way by the descending induction with respect to $\le$, provided that $d_\mu \ne d_\lambda$ for all $\mu < \lambda$. □

Note that, from the triangularity (2) of $P_\lambda(x)$, it also follows that the Macdonald polynomials $P_\lambda(x)$ $(\lambda \in \mathcal{P}_n)$ form a $\mathbb{C}$-basis of the ring $\mathbb{C}[x]^{\mathfrak{S}_n}$ of symmetric polynomials, namely

$$\mathbb{C}[x]^{\mathfrak{S}_n} = \bigoplus_{\lambda\in\mathcal{P}_n} \mathbb{C}\, P_\lambda(x). \tag{4.17}$$

**Remark 4.1** The construction of $P_\lambda(x)$ mentioned above can be explained in a different way in terms of linear algebra. For a partition $\lambda \in \mathcal{P}_n$ given, consider the finite dimensional $\mathbb{C}$-vector space $V_{\leq\lambda} = \bigoplus_{\mu\in\mathcal{P}_n;\mu\leq\lambda} \mathbb{C}\, m_\mu(x)$. The linear operator $D_x$ stabilizes $V_{\leq\lambda}$, and is represented by an upper triangular matrix with respect to $\leq$ under the basis $m_\mu = m_\mu(x)$ $(\mu \leq \lambda)$:

$$D_x(\ldots, m_\mu, \ldots, m_\lambda) = (\ldots, m_\mu, \ldots, m_\lambda) \begin{pmatrix} \ddots & & * \\ & d_\mu & \\ & & \ddots \\ 0 & & & d_\lambda \end{pmatrix}. \tag{4.18}$$

Then, by the Cayley–Hamilton theorem we obtain $\prod_{\mu\leq\lambda}(D_x - d_\mu)\big|_{V_{\leq\lambda}} = 0$. From this, we obtain

$$(D_x - d_\lambda) \prod_{\mu<\lambda} (D_x - d_\mu)\, m_\lambda(x) = 0. \tag{4.19}$$

Since

$$\prod_{\mu<\lambda}(D_x - d_\mu) m_\lambda(x) = \prod_{\mu<\lambda}(d_\lambda - d_\mu) m_\lambda(x) + \text{(lower-order terms)}, \tag{4.20}$$

we see that

$$P_\lambda(x) = \prod_{\mu<\lambda} \frac{D_x - d_\mu}{d_\lambda - d_\mu}\big(m_\lambda(x)\big) = m_\lambda(x) + \text{(lower-order terms)} \tag{4.21}$$

gives the eigenfunction of $D_x$ with leading term $m_\lambda(x)$, under the condition that $d_\mu \neq d_\lambda$ for all $\mu < \lambda$.

**Remark 4.2** The Macdonald–Ruijsenaars operator $D_x$ also stabilizes the ring $\mathbb{C}[x^{\pm 1}]^{\mathfrak{S}_n}$ of symmetric Laurent polynomials. In this setting, for generic $q, t$, the linear operator $D_x : \mathbb{C}[x^{\pm 1}]^{\mathfrak{S}_n} \to \mathbb{C}[x^{\pm 1}]^{\mathfrak{S}_n}$ is diagonalized by the Macdonald polynomials $P_\lambda(x)$ parameterized by $n$-tuples of integers $\lambda = (\lambda_1, \ldots, \lambda_n) \in P = \mathbb{Z}^n$ such that $\lambda_1 \geq \ldots \geq \lambda_n$. In the language of representation theory, the set of such $\lambda$ is denoted by $P_+$, and called the cone of *dominant integral weights* of $\mathrm{GL}_n$:

$$P_+ = \big\{\lambda \in P \mid \langle\alpha_i, \lambda\rangle \geq 0 \ \ (i = 1, \ldots, n)\big\}. \tag{4.22}$$

For each $\lambda \in P_+$, we have $\mu = \lambda + (l^n) \in \mathcal{P}_n$ for a sufficiently large $l \in \mathbb{Z}_{\geq 0}$. Then, the symmetric Laurent polynomial

$$P_\lambda(x) = P_{\mu-(l^n)}(x) = (x_1 \cdots x_n)^{-l} P_\mu(x), \tag{4.23}$$

defined by the Macdonald polynomial $P_\mu(x)$ attached to the partition $\mu \in \mathcal{P}_n$, satisfies the eigenfunction equation

$$D_x P_\lambda(x) = d_\lambda P_\lambda(x), \quad d_\lambda = \sum_{i=1}^{n} t^{n-i} q^{\lambda_i}. \tag{4.24}$$

Furthermore, these $P_\lambda(x)$ $(\lambda \in P_+)$ form a $\mathbb{C}$-basis of the ring $\mathbb{C}[x^{\pm 1}]^{\mathfrak{S}_n}$ of symmetric Laurent polynomials, namely, $\mathbb{C}[x^{\pm 1}]^{\mathfrak{S}_n} = \bigoplus_{\lambda \in P_+} \mathbb{C} P_\lambda(x)$.

## 4.2 Some Examples

**Single columns**

$P_{(1^r)}(x) = e_r(x) \quad (r = 0, 1, \ldots, n)$. In particular, $P_{(1^n)}(x) = x_1 \cdots x_n$.

If $\lambda$ is a single column $(1^r)$ $(r = 0, 1, \ldots, n)$, $P_{(1^r)}(x)$ is the elementary symmetric function $e_r(x)$ of degree $r$, since $(1^r)$ is minimal with respect to the dominance order. The equation $D_x e_r(x) = d_{(1^r)} e_r(x)$ already implies a nontrivial identity

$$\sum_{i=1}^{n} \prod_{j \neq i} \frac{t x_i - x_j}{x_i - x_j} e_r(x_1, \ldots, q x_i, \ldots, x_n) = d_{(1^r)} e_r(x),$$
$$d_{(1^r)} = t^{n-1} q + \cdots + t^{n-r} q + t^{n-r-1} + \cdots + 1 = q t^{n-r} \frac{1 - t^r}{1 - t} + \frac{1 - t^{n-r}}{1 - t}. \tag{4.25}$$

In particular, from $D_x(1) = d_0 1$ we obtain

$$\sum_{i=1}^{n} \prod_{j \neq i} \frac{t x_i - x_j}{x_i - x_j} = \frac{1 - t^n}{1 - t}. \tag{4.26}$$

**Exercise 4.1** Derive (4.26) from the partial fraction expansion in $u$,

$$\prod_{j=1}^{n} \frac{t u - x_j}{u - x_j} = \sum_{i=1}^{n} \frac{(t-1) x_i}{u - x_i} \prod_{j \neq i} \frac{t x_i - x_j}{x_i - x_j} + t^n. \tag{4.27}$$

**Adding columns of length $n$**

$P_{\lambda+(k^n)}(x) = (x_1 \cdots x_n)^k P_\lambda(x) \qquad (\lambda \in \mathcal{P}_n;\ k = 0, 1, 2, \ldots).$

Similarly to the case of Schur functions, by adding a column of length $n$ the Macdonald polynomial $P_\lambda(x)$ is multiplied by $x_1 \cdots x_n$. This follows from the commutation relation $D_x\, x_1 \cdots x_n = q x_1 \ldots x_n D_x$ between $D_x$ and the multiplication by $x_1 \cdots x_n$.

**Cases where $t = 1$ and $t = q$**

$$t = 1:\quad P_\lambda(x) = m_\lambda(x), \qquad t = q:\quad P_\lambda(x) = s_\lambda(x).$$

When $t = 1$, we have $D_x = \sum_{i=1}^n T_{q,x_i}$ and $d_\lambda = \sum_{i=1}^n q^{\lambda_i} = \sum_{j\geq 0} m_j q^j$ for each $\lambda \in \mathcal{P}_n$ with $\lambda = (1^{m_1} 2^{m_2} \ldots)$. Note that $d_\mu \neq d_\lambda$ for $\mu \neq \lambda$ if $q$ is transcendental over $\mathbb{Q}$. In this case $P_\lambda(x)$ coincides with the monomial symmetric function $m_\lambda(x)$.

When $t = q$, we have

$$D_x = \sum_{i=1}^n \frac{T_{q,x_i}(\Delta(x))}{\Delta(x)} T_{q,x_i} = \frac{1}{\Delta(x)}\Big(\sum_{i=1}^n T_{q,x_i}\Big)\Delta(x), \quad d_\lambda = \sum_{i=1}^n q^{\lambda_i+n-i}. \tag{4.28}$$

In this case, for each $\lambda \in \mathcal{P}_n$, we have $\sum_{i=1}^n T_{q,x_i}(x^{\sigma.(\lambda+\delta)}) = (\sum_{i=1}^n q_i^{(\lambda+\delta)_i}) x^{\sigma.(\lambda+\delta)}$ $= d_\lambda x^{\sigma.(\lambda+\delta)}$ for all $\sigma \in \mathfrak{S}_n$. From this we obtain $\sum_{i=1}^n T_{q,x_i} \Delta_{\lambda+\delta}(x) = d_\lambda \Delta_{\lambda+\delta}(x)$ and hence $D_x s_\lambda(x) = d_\lambda s_\lambda(x)$.

**Case where $n = 1$**

$P_{(l)}(x_1) = x_1^l \quad (l = 0, 1, 2, \ldots).$

**Case where $n = 2$**

For any $\lambda = (\lambda_1, \lambda_2) \in \mathcal{P}_2$, $\lambda_1 \geq \lambda_2 \geq 0$, we have

$$P_{(\lambda_1,\lambda_2)}(x_1, x_2) = (x_1 x_2)^{\lambda_2} P_{(l,0)}(x_1, x_2), \quad l = \lambda_1 - \lambda_2, \tag{4.29}$$

$$P_{(l.0)}(x_1, x_2) = \frac{(q;q)_l}{(t;q)_l} \sum_{\mu_1+\mu_2=l} \frac{(t;q)_{\mu_1}(t;q)_{\mu_2}}{(q;q)_{\mu_1}(q;q)_{\mu_2}} x_1^{\mu_1} x_2^{\mu_2}, \tag{4.30}$$

where $(t;q)_k = (1-t)(1-qt)\cdots(1-q^{k-1}t) \quad (k = 0, 1, 2, \ldots)$ denotes the *$q$-shifted factorial.*

In the context of $q$-orthogonal polynomials [12], $P_{(l,0)}(x_1, x_2)$ correspond to the *$q$-ultraspherical polynomials*. As to the eigenfunctions in the case of two variables, we discuss some detail in the next section.

**Exercise 4.2** (*Stability*) Let $\lambda \in \mathcal{P}_n$ and $m < n$. Then we have

$$P_\lambda(x_1, \ldots, x_m, 0, \ldots, 0) = \begin{cases} P_\lambda(x_1, \ldots, x_m) & (\ell(\lambda) \le m), \\ 0 & (\ell(\lambda) > m). \end{cases} \tag{4.31}$$

## 4.3 Eigenfunctions in Two Variables

Restricting ourselves to the case of two variables, we investigate below a class of eigenfunctions of the $q$-difference operator $D_x$ which are expressed as formal power series.

### *4.3.1 Eigenfunctions in Power Series*

In the case of two variables, the eigenfunction equation for $D_x$

$$\frac{tx_1 - x_2}{x_1 - x_2}\varphi(qx_1, x_2) + \frac{x_1 - tx_2}{x_1 - x_2}\varphi(x_1, qx_2) = \varepsilon\, \varphi(x_1, x_2) \tag{4.32}$$

can be solved in a larger class of power series.

Note that, for $\mu = (\mu_1, \mu_2)$, $\lambda = (\lambda_1, \lambda_2) \in \mathbb{Z}^n$,

$$\begin{aligned} \mu \le \lambda \quad &\Longleftrightarrow \quad (\mu_1 \le \lambda_1,\ \mu_1 + \mu_2 = \lambda_1 + \lambda_2) \\ &\Longleftrightarrow \quad (\mu_1, \mu_2) = (\lambda_1 - k, \lambda_2 + k) \quad \text{for some } k \in \mathbb{N}. \end{aligned} \tag{4.33}$$

Extending this relation to multi-indices of complex numbers, we consider a formal power series of the form

$$\varphi(x_1, x_2) = \sum_{k \ge 0} c_k x_1^{\lambda_1 - k} x_2^{\lambda_2 + k} = x_1^{\lambda_1} x_2^{\lambda_2} \sum_{k \ge 0} c_k (x_2/x_1)^k, \quad c_0 = 1, \tag{4.34}$$

for arbitrary $\lambda_1, \lambda_2 \in \mathbb{C}$, and solve the eigenfunction equation (4.32).

**Proposition 4.1** *Let $\lambda = (\lambda_1, \lambda_2) \in \mathbb{C}^2$ be generic. Then, the eigenfunction equation (4.32) has a unique solution of the form (4.34) with eigenvalue $\varepsilon = tq^{\lambda_1} + q^{\lambda_2}$. It is determined explicitly as*

$$\varphi(x_1, x_2) = x_1^{\lambda_1} x_2^{\lambda_2} \sum_{k=0}^{\infty} \frac{(t; q)_k (q^{\lambda_2 - \lambda_1}; q)_k}{(q; q)_k (q^{\lambda_2 - \lambda_1 + 1}/t; q)_k} (qx_2/tx_1)^k, \tag{4.35}$$

*where $(a; q)_k = (1 - a)(1 - qa) \cdots (1 - q^{k-1}a)$ $(k = 0, 1, 2, \ldots)$.*

***Proof*** Setting $z = x_2/x_1$, we rewrite this equation by means of $f(z) = \sum_{k\geq 0} c_k z^k$. Since $\varphi(x_1, x_2) = x_1^{\lambda_1} x_2^{\lambda_2} f(x_2/x_1)$, we obtain the equation

$$\frac{t - z}{1 - z} q^{\lambda_1} f(q^{-1}z) + \frac{1 - tz}{1 - z} q^{\lambda_2} f(qz) = \varepsilon f(z) \tag{4.36}$$

to be satisfied by $f(z)$, namely,

$$(t - z)q^{\lambda_1} f(q^{-1}z) + (1 - tz)q^{\lambda_2} f(qz) = \varepsilon(1 - z) f(z). \tag{4.37}$$

This equation gives rise to the recurrence formulas for the coefficients

$$(tq^{\lambda_1-k} + q^{\lambda_2+k} - \varepsilon)c_k = (q^{\lambda_1-k+1} + tq^{\lambda_2+k-1} - \varepsilon)c_{k-1} \quad (k \in \mathbb{Z}), \tag{4.38}$$

with $c_k = 0$ for $k < 0$. This formula for $k = 0$ determines the eigenvalue as $\varepsilon = tq^{\lambda_1} + q^{\lambda_2}$. Then the resulting recurrence formulas

$$(1 - q^k)(1 - q^{\lambda_2-\lambda_1+k}/t)c_k = (q/t)(1 - tq^{k-1})(1 - q^{\lambda_2-\lambda_1+k-1})c_{k-1} \tag{4.39}$$

for $k = 1, 2, \ldots$ are solved as

$$c_k = \frac{(t; q)_k (q^{\lambda_2-\lambda_1}; q)_k}{(q; q)_k (q^{\lambda_2-\lambda_1+1}/t; q)_k} (q/t)^k \quad (k = 0, 1, 2, \ldots), \tag{4.40}$$

by the notation of $q$-shifted factorials $(a; q)_k = (1 - a)(1 - qa) \cdots (1 - q^{k-1}a)$ $(k = 0, 1, 2, \ldots)$. □

We remark that, under our assumption $|q| < 1$, this formal solution $\varphi(x_1, x_2)$ is absolutely convergent in the domain $|qx_2/tx_1| < 1$.

### 4.3.2 Macdonald Polynomials in Two Variables

When $\lambda = (\lambda_1, \lambda_2)$ is a partition and $l = \lambda_1 - \lambda_2 \in \mathbb{N}$, the power series solution $\varphi(x_1, x_2)$ constructed above reduces to a polynomial in $(x_1, x_2)$. In fact, we have

$$\begin{aligned}\varphi(x_1, x_2) &= x_1^{\lambda_1} x_2^{\lambda_2} \sum_{k=0}^{l} \frac{(t; q)_k (q^{-l}; q)_k}{(q; q)_k (q^{-l+1}/t; q)_k} (qx_2/tx_1)^k \\ &= (x_1 x_2)^{\lambda_2} \sum_{k=0}^{l} \frac{(t; q)_k (q^{-l}; q)_k}{(q; q)_k (q^{-l+1}/t; q)_k} (q/t)^k x_1^{l-k} x_2^k\end{aligned} \tag{4.41}$$

since $(q^{-l};q)_k = 0$ for $k > l$. Also, by

$$\frac{(q^{-l};q)_k}{(q^{-l+1}/t;q)_k} = \frac{(q^{l-k+1};q)_k}{(q^{l-k}t;q)_l}(t/q)^k = \frac{(q;q)_l}{(t;q)_l}\frac{(t;q)_{l-k}}{(q;q)_{l-k}}(t/q)^k \tag{4.42}$$

we obtain

$$\begin{aligned}\varphi(x_1,x_2) &= (x_1x_2)^{\lambda_2}\frac{(q;q)_l}{(t;q)_l}\sum_{k=0}^{l}\frac{(t;q)_{l-k}}{(q;q)_{l-k}}\frac{(t;q)_k}{(q;q)_k}x_1^{l-k}x_2^k \\ &= (x_1x_2)^{\lambda_2}\frac{(q;q)_l}{(t;q)_l}\sum_{\mu_1+\mu_2=l}\frac{(t;q)_{\mu_1}(t;q)_{\mu_2}}{(q;q)_{\mu_1}(q;q)_{\mu_2}}x_1^{\mu_1}x_2^{\mu_2},\end{aligned} \tag{4.43}$$

which is manifestly a symmetric polynomial in $(x_1, x_2)$. In this way we obtain the expression of general Macdonald polynomials in two variables.

**Proposition 4.2** *The Macdonald polynomials in two variables are explicitly given as follows: For each partition* $\lambda = (\lambda_1, \lambda_2) \in \mathcal{P}_2$,

$$P_{(\lambda_1,\lambda_2)}(x_1,x_2) = (x_1x_2)^{\lambda_2}P_{(l,0)}(x_1,x_2), \quad l = \lambda_1 - \lambda_2, \tag{4.44}$$

$$\begin{aligned}P_{(l,0)}(x_1,x_2) &= \frac{(q;q)_l}{(t;q)_l}\sum_{\mu_1+\mu_2=l}\frac{(t;q)_{\mu_1}(t;q)_{\mu_2}}{(q;q)_{\mu_1}(q;q)_{\mu_2}}x_1^{\mu_1}x_2^{\mu_2} \\ &= x_1^l\sum_{k=0}^{l}\frac{(t;q)_k(q^{-l};q)_k}{(q;q)_k(q^{-l+1}/t;q)_k}(qx_2/tx_1)^k.\end{aligned} \tag{4.45}$$

□

## 4.4 *q*-Binomial Theorem and *q*-Hypergeometric Series

In this section, we quickly review some basic facts about $q$-hypergeometric series. They will be used to see how Macdonald polynomials attached to single rows are related to $q$-hypergeometric series.

### *4.4.1 q-Binomial Theorem*

Under our assumption $|q| < 1$, the infinite product

$$(z;q)_\infty = \prod_{i=0}^{\infty}(1 - q^i z) \qquad (z \in \mathbb{C}) \tag{4.46}$$

is absolutely convergent, and defines a holomorphic function in $z \in \mathbb{C}$. Note that the $q$-shifted factorial $(z; q)_k$ for $k = 0, 1, 2, \ldots$ is expressed as the ratio $(z; q)_\infty/(q^k z; q)_\infty$.

**Proposition 4.3** ($q$-Binomial theorem) *For any $a \in \mathbb{C}$, one has*

$$\frac{(az; q)_\infty}{(z; q)_\infty} = \sum_{k=0}^{\infty} \frac{(a; q)_k}{(q; q)_k} z^k \quad (|z| < 1). \tag{4.47}$$

***Proof*** Note that the left-hand side $f(z) = (az; q)_\infty/(z; q)_\infty$ is a meromorphic function on $\mathbb{C}$ at most with simple poles at $z = 1, q^{-1}, q^{-2}, \ldots$. Since

$$f(qz) = \frac{(qaz; q)_\infty}{(qz; q)_\infty} = \frac{1 - z}{1 - az} \frac{(az; q)_\infty}{(z; q)_\infty} = \frac{1 - z}{1 - az} f(z), \tag{4.48}$$

$f(z)$ satisfies the $q$-difference equation

$$(1 - az) f(qz) = (1 - z) f(z) \tag{4.49}$$

with initial condition $f(0) = 1$. In terms of the Taylor expansion $f(z) = \sum_{k=0}^{\infty} c_k z^k$ of $f(z)$ around $z = 0$, this equation gives rise to the recurrence formulas

$$(1 - q^k) c_k = (1 - q^{k-1} a) c_{k-1} \qquad (k = 0, 1, 2, \ldots) \tag{4.50}$$

with $c_{-1} = 0$, $c_0 = 1$. Hence we obtain

$$\begin{aligned} c_k &= \frac{1 - q^{k-1} a}{1 - q^k} c_{k-1} = \frac{(1 - q^{k-1} a)(1 - q^{k-2} a)}{(1 - q^k)(1 - q^{k-1})} c_{k-2} \\ &= \cdots = \frac{(a; q)_k}{(q; q)_k} \qquad (k = 0, 1, 2, \ldots). \end{aligned} \tag{4.51}$$

□

We remark that this $q$-binomial theorem contains two formulas

$$\frac{1}{(z; q)_\infty} = \sum_{k=0}^{\infty} \frac{1}{(q; q)_k} z^k, \quad (z; q)_\infty = \sum_{k=0}^{\infty} \frac{(-1)^k q^{\binom{k}{2}}}{(q; q)_k} z^k. \tag{4.52}$$

The first one is the special case where $a = 0$, while the second is obtained by replacing $z$ with $z/a$, and then by taking the limit as $a \to \infty$. These two formulas can be found, with proofs similar to the one mentioned above, in Euler's monograph *Introductio in Analysin Infinitorum* (1748) [6, Caput XVI]. According to [8], the $q$-binomial theorem in the form of (4.47) appeared in the middle of 19th century in works of Cauchy, Heine and others.

### 4.4.2 *q-Hypergeometric Series*

We introduce the notation of $q$-hypergeometric series ${}_{r+1}\phi_r$:

$$ {}_{r+1}\phi_r\left[\begin{matrix} a_0,\ a_1,\ \dots,\ a_r \\ b_1,\ \dots,\ b_r \end{matrix};\ q,\ z\right] = \sum_{k=0}^{\infty} \frac{(a_0;q)_k(a_1;q)_k\cdots(a_r;q)_k}{(q;q)_k(b_1;q)_k\cdots(b_r;q)_k} z^k. \tag{4.53} $$

We remark that this series is absolutely convergent for $|z| < 1$. Note that ${}_{r+1}\phi_r$ series (4.53) is a $q$-version of the generalized hypergeometric series

$$ {}_{r+1}F_r\left[\begin{matrix} \alpha_0,\ \alpha_1,\ \dots,\ \alpha_r \\ \beta_1,\ \dots,\beta_r \end{matrix};\ z\right] = \sum_{k=0}^{\infty} \frac{(\alpha_0)_k(\alpha_1)_k\cdots(\alpha_r;q)_k}{(1)_k(\beta_1)_k\cdots(\beta_r)_k} z^k \tag{4.54} $$

defined with shifted factorials $(\alpha)_k = \alpha(\alpha+1)\cdots(\alpha+k-1)$ $(k = 0, 1, 2, \dots)$. The ${}_{r+1}F_r$ series of (4.54) is obtained, at least in the sense of formal power series in $z$, from the ${}_{r+1}\phi_r$ series of (4.53) with $a_i = q^{\alpha_i}$, $b_i = q^{\beta_i}$, $c_i = q^{\gamma_i}$ by the limiting procedure as $q \to 1$.

The $q$-binomial theorem can be interpreted as the summation formula for general ${}_1\phi_0$ series:

$$ {}_1\phi_0\left[\begin{matrix} a \\ \cdot \end{matrix};\ q,\ z\right] = \sum_{k=0}^{\infty} \frac{(a;q)_k}{(q;q)_k} z^k = \frac{(az;q)_\infty}{(z;q)_\infty}. \tag{4.55} $$

This is a $q$-version of Newton's binomial theorem:

$$ {}_1F_0\left[\begin{matrix} \alpha \\ \cdot \end{matrix};\ z\right] = \sum_{k=0}^{\infty} \frac{(\alpha)_k}{k!} z^k = \sum_{k=0}^{\infty} \binom{-\alpha}{k} (-z)^k = (1-z)^{-\alpha}. \tag{4.56} $$

The ${}_2\phi_1$ series

$$ {}_2\phi_1\left[\begin{matrix} a,\ b, \\ c \end{matrix};\ q,\ z\right] = \sum_{k=0}^{\infty} \frac{(a;q)_k(b;q)_k}{(q;q)_k(c;q)_k} z^k \tag{4.57} $$

is a $q$-version of the Gauss hypergeometric series

$$ {}_2F_1\left[\begin{matrix} \alpha,\ \beta, \\ \gamma \end{matrix};\ q,\ z\right] = \sum_{k=0}^{\infty} \frac{(\alpha)_k(\beta)_k}{(1)_k(\gamma)_k} z^k. \tag{4.58} $$

The eigenfunction (4.35) of $D_x$ in the case of two variables is expressed by ${}_2\phi_1$ series as

$$ \varphi(x_1, x_2) = x_1^{\lambda_1} x_2^{\lambda_2}\ {}_2\phi_1\left[\begin{matrix} t,\ q^{\lambda_2-\lambda_1} \\ q^{\lambda_2-\lambda_1+1}/t \end{matrix};\ q,\ qx_2/tx_1\right]. \tag{4.59} $$

This class of eigenfunctions includes Macdonald polynomials as special cases where $\lambda = (\lambda_1, \lambda_2) \in \mathcal{P}_2$ are partitions.

### 4.4.3 Generating Function of Two Variable Macdonald Polynomials

We also know that the Macdonald polynomials $P_{(l)}(x_1, x_2)$ attached to single rows in two variables are expressed as

$$P_{(l)}(x_1, x_2) = \frac{(q;q)_l}{(t;q)_l} \sum_{\mu_1+\mu_2=l} \frac{(t;q)_{\mu_1}(t;q)_{\mu_2}}{(q;q)_{\mu_1}(q;q)_{\mu_2}} x_1^{\mu_1} x_2^{\mu_2} \tag{4.60}$$

for $l = 0, 1, 2, \ldots$. This expression implies that $P_{(l)}(x_1, x_2)$ arise as the expansion coefficients of a generating function as specified below.

**Proposition 4.4** *With an extra variable u, we have*

$$\frac{(tx_1u;q)_\infty (tx_2u;q)_\infty}{(x_1u;q)_\infty (x_2u;q)_\infty} = \sum_{l=0}^{\infty} \frac{(t;q)_l}{(q;q)_l} P_{(l)}(x_1, x_2)\, u^l. \tag{4.61}$$

***Proof*** We expand the left-hand side by the $q$-binomial theorem:

$$\begin{aligned}
\frac{(tx_1u;q)_\infty (tx_2u;q)_\infty}{(x_1u;q)_\infty (x_2u;q)_\infty} &= \left(\sum_{\mu_1=0}^{\infty} \frac{(t;q)_{\mu_1}}{(q;q)_{\mu_1}} (x_1u)^{\mu_1}\right)\left(\sum_{\mu_2=0}^{\infty} \frac{(t;q)_{\mu_2}}{(q;q)_{\mu_1}} (x_2u)^{\mu_2}\right) \\
&= \sum_{l=0}^{\infty} \left(\sum_{\mu_1+\mu_2=l} \frac{(t;q)_{\mu_1}(t;q)_{\mu_2}}{(q;q)_{\mu_1}(q;q)_{\mu_2}} x_1^{\mu_1} x_2^{\mu_2}\right) u^l \\
&= \sum_{l=0}^{\infty} \frac{(t;q)_l}{(q;q)_l} P_{(l)}(x_1, x_2)\, u^l.
\end{aligned} \tag{4.62}$$

□

## 4.5 Macdonald Polynomials Attached to Single Rows

### 4.5.1 Macdonald Polynomials $P_{(l)}(x)$ and $g_l(x)$

Returning to the general setting of $n$ variables $x = (x_1, \ldots, x_n)$, we define a sequence of polynomials $g_l(x) = g_l(x; q, t)$ $(l = 0, 1, 2, \ldots)$ by means of the generating function

$$G(x;u) = \prod_{i=1}^{n} \frac{(tx_iu;q)_\infty}{(x_iu;q)_\infty} = \sum_{l=0}^{\infty} g_l(x)\, u^l. \tag{4.63}$$

By the $q$-binomial theorem, this function is expanded as

$$G(x;u) = \sum_{\mu_1,\ldots,\mu_n\geq 0} \frac{(t;q)_{\mu_1}\cdots(t;q)_{\mu_n}}{(q;q)_{\mu_1}\cdots(q;q)_{\mu_n}} x_1^{\mu_1}\cdots x_n^{\mu_n} y^{\mu_1+\cdots+\mu_n}$$
$$= \sum_{l=0}^{\infty}\left(\sum_{\mu_1+\ldots+\mu_n=l} \frac{(t;q)_{\mu_1}\cdots(t;q)_{\mu_n}}{(q;q)_{\mu_1}\cdots(q;q)_{\mu_n}} x_1^{\mu_1}\cdots x_n^{\mu_n}\right) y^l. \quad (4.64)$$

Hence we have

$$g_l(x) = \sum_{\mu_1+\cdots+\mu_n=l} \frac{(t;q)_{\mu_1}\cdots(t;q)_{\mu_n}}{(q;q)_{\mu_1}\cdots(q;q)_{\mu_n}} x_1^{\mu_1}\cdots x_n^{\mu_n}$$
$$= \frac{(t;q)_l}{(q;q)_l} x_1^l + \text{(lower-order terms)} \quad (l = 0, 1, 2, \ldots). \quad (4.65)$$

**Theorem 4.2** *For each $l = 0, 1, 2, \ldots$, $g_l(x)$ satisfies the eigenfunction equation*

$$D_x g_l(x) = d_{(l)} g_l(x),$$
$$d_{(l)} = t^{n-1}q^l + t^{n-2} + \cdots + 1 = t^{n-1}q^l + \frac{1-t^{n-1}}{1-t}, \quad (4.66)$$

*and hence*

$$g_l(x) = \frac{(t;q)_l}{(q;q)_l} P_{(l)}(x). \quad (4.67)$$

***Proof*** Since $G(x;u) = \sum_{l=0}^{\infty} g_l(x)\, u^l$, we have

$$D_x G(x;u) = \sum_{l=0}^{\infty} D_x g_l(x)\, u^l, \quad T_{q,u} G(x;u) = \sum_{l=0}^{\infty} q^l\, g_l(x)\, u^l. \quad (4.68)$$

Hence, the eigenfunction equations

$$D_x g_l(x) = \left(t^{n-1}q^l + \frac{1-t^{n-1}}{1-t}\right) g_l(x) \quad (4.69)$$

for $g_l(x)$ $(l = 0, 1, 2, \ldots)$ are equivalent to the identity

$$D_x G(x;u) = \left(t^{n-1}T_{q,u} + \frac{1-t^{n-1}}{1-t}\right) G(x;u) \quad (4.70)$$

for the generating function. Noting that

$$T_{q,x_i}G(x;u) = \frac{1-x_iu}{1-tx_iu}G(x;u), \quad T_{q,u}G(x;u) = \prod_{j=1}^{n}\frac{1-x_ju}{1-tx_ju}G(x;u), \tag{4.71}$$

we see that identity (4.70) is equivalent to the identity

$$\sum_{i=1}^{n}\prod_{j\neq i}\frac{tx_i-x_j}{x_i-x_j}\frac{1-x_iu}{1-tx_iu} = t^{n-1}\prod_{j=1}^{n}\frac{1-x_ju}{1-tx_ju} + \frac{1-t^{n-1}}{1-t} \tag{4.72}$$

of rational functions. As rational functions of $u$, both sides are of the form

$$\frac{p(u)}{q(u)}, \quad p(u), q(u) \in \mathbb{C}[u], \quad \deg_u p(u) \leq n, \quad \deg_u q(u) = n. \tag{4.73}$$

Then, one can verify identity (4.72) directly by comparing the residues at $n$ points $u = 1/t_ix_i$ $(i = 1, \ldots, n)$ and the value at $u = 0$ which reduces to (4.26). □

### *4.5.2 Expression in Terms of $\phi_D$*

Similarly to the case of two variables, the Macdonald polynomials

$$g_l(x) = \sum_{\mu_1+\cdots+\mu_n=l}\frac{(t;q)_{\mu_1}\cdots(t;q)_{\mu_n}}{(q;q)_{\mu_1}\cdots(q;q)_{\mu_n}}x_1^{\mu_1}\cdots x_n^{\mu_n} \tag{4.74}$$

attached to single rows are expressed in terms of certain $q$-hypergeometric seires in $n-1$ variables. Using $\mu_1 = l - \mu_2 - \cdots - \mu_n$, we rewrite the factor containing $\mu_1$ as follows:

$$\begin{aligned}\frac{(t;q)_{\mu_1}}{(q;q)_{\mu_1}} &= \frac{(t;q)_{l-\mu_2-\cdots-\mu_n}}{(q;q)_{l-\mu_2-\cdots-\mu_n}} \\ &= \frac{(t;q)_l}{(q;q)_l}\frac{(q^{l-\mu_2-\cdots-\mu_n+1};q)_{\mu_2+\cdots+\mu_n}}{(q^{l-\mu_2-\cdots-\mu_n}t;q)_{\mu_2+\cdots+\mu_n}} \\ &= \frac{(t;q)_l}{(q;q)_l}\frac{(q^{-l};q)_{\mu_2+\cdots+\mu_n}}{(q^{-l+1}/t;q)_{\mu_2+\cdots+\mu_n}}(q/t)^{\mu_2+\cdots+\mu_n}.\end{aligned} \tag{4.75}$$

Hence we have

$$\begin{aligned}g_l(x) = \frac{(t;q)_l}{(q;q)_l}x_1^l\sum_{\mu_2,\ldots,\mu_n\geq 0}&\frac{(q^{-l};q)_{\mu_2+\cdots+\mu_n}}{(q^{-l+1}/t;q)_{\mu_2+\cdots+\mu_n}}\frac{(t;q)_{\mu_2}\cdots(t;q)_{\mu_n}}{(q;q)_{\mu_2}\cdots(q;q)_{\mu_n}} \\ &\cdot(qx_2/tx_1)^{\mu_2}\cdots(qx_n/tx_1)^{\mu_n}.\end{aligned} \tag{4.76}$$

We now introduce the $q$-hypergeometric series

$$\phi_D\left[\begin{matrix}a;\ b_1,\dots,b_m\\ c\end{matrix};\ q;\ z_1,\dots,z_m\right] = \sum_{k_1,\dots,k_m\geq 0} \frac{(a;q)_{k_1+\cdots+k_m}}{(c;q)_{k_1+\cdots+k_m}} \frac{(b_1;q)_{k_1}\cdots(b_m;q)_{k_m}}{(q;q)_{k_1}\cdots(q;q)_{k_m}} z_1^{k_1}\cdots z_m^{k_m} \tag{4.77}$$

in $m$ variables, which is a $q$-analogue of Lauricella's $F_D$ (Appell's $F_1$ when $m=2$).[1] Then the Macdonald polynomials attached to single rows are expressed in terms of $\phi_D$ in $n-1$ variables as

$$P_{(l)}(x_1,\dots,x_n) = x_1^l\,\phi_D\left[\begin{matrix}q^{-l};\ t,\dots,t\\ q^{-l+1}/t\end{matrix};\ q;\ qx_2/tx_1,\dots,qx_n/tx_1\right] \tag{4.78}$$

for $l = 0, 1, 2, \dots$.

### 4.5.3 Wronski Relations

We compare this generating function

$$G(x;u) = \prod_{i=1}^n \frac{(tx_i y;q)_\infty}{(x_i u;q)_\infty} = \sum_{l=0}^\infty g_l(x)u^l. \tag{4.79}$$

with that of elementary symmetric functions

$$E(x;u) = \prod_{i=1}^n (1+x_i u) = \sum_{r=0}^n e_r(x)u^r. \tag{4.80}$$

Note that $G(x;u)$ satisfies the $q$-difference equation

$$G(x;qu) = \prod_{i=1}^n \frac{(qtx_i u;q)_\infty}{(qx_i u;q)_\infty} = \prod_{i=1}^n \frac{1-x_i u}{1-tx_i u} G(x;u) \tag{4.81}$$

with respect to $u$. This means that

$$\prod_{i=1}^n (1-tx_i u)\cdot G(x;qu) = \prod_{i=1}^n (1-x_i u)\cdot G(x;u), \tag{4.82}$$

[1] See [15, Chap. 6] for Appell's and Lauricella's hypergeometric series in many variables.

namely,

$$E(x; -tu)G(x; qu) = E(x; -u)G(x; u). \tag{4.83}$$

Comparing the coefficients of $u^k$ in this equality, we obtain

$$\sum_{i+j=k} (-1)^i t^i q^j e_i(x) g_j(x) = \sum_{i+j=k} (-1)^i e_i(x) g_j(x) \quad (k = 0, 1, 2, \ldots), \tag{4.84}$$

and hence

$$\sum_{i+j=k} (-1)^i (t^i q^j - 1) e_i(x) g_j(x) = 0 \quad (k = 1, 2, \ldots). \tag{4.85}$$

This formula is the counterpart of Wronski's relations in the case of Macdonald polynomials between those attached to single columns and single rows.

# Chapter 5
# Orthogonality and Higher-Order $q$-Difference Operators

**Abstract** We show that the Macdonald polynomials satisfy the orthogonality relation with respect to a certain scalar product on the ring of symmetric polynomials. We also explain how this orthogonality is related with the existence of commuting family of higher-order $q$-difference operators for which Macdonald polynomials are joint eigenfunctions.

## 5.1 Scalar Product and Orthogonality

As always, we fix the parameters $q, t \in \mathbb{C}^*$ with $|q| < 1$. Also, keeping the convention of the previous chapter, we suppose that the parameters $q, t$ satisfy the genericity condition (4.10).

### *5.1.1 Weight Function and Scalar Product*

We define a meromorphic function $w(x) = w(x; q, t)$ on $(\mathbb{C}^*)^n$ by

$$w(x) = \prod_{1\le i<j\le n} \frac{(x_i/x_j; q)_\infty}{(tx_i/x_j; q)_\infty} \frac{(x_j/x_i; q)_\infty}{(tx_j/x_i; q)_\infty}. \tag{5.1}$$

Note that $w(x)$ is $\mathfrak{S}_n$-invariant and also $w(x^{-1}) = w(x)$. We assume $|t| < 1$ so that $w(x)$ is holomorphic in a neighborhood of the $n$-dimensional torus

$$\mathbb{T}^n = \left\{x = (x_1, \dots, x_n) \in (\mathbb{C}^*)^n \mid |x_i| = 1 \ (i = 1, \dots, n)\right\} \subset (\mathbb{C}^*)^n. \tag{5.2}$$

For a pair of holomorphic functions $f(x)$, $g(x)$ in a neighborhood of $\mathbb{T}^n$, we define the scalar product (symmetric bilinear form) $\langle f, g\rangle$ as



M. Noumi, *Macdonald Polynomials*, SpringerBriefs in Mathematical Physics,
https://doi.org/10.1007/978-981-99-4587-0_5

$$\langle f, g\rangle = \frac{1}{n!}\frac{1}{(2\pi\sqrt{-1})^n}\int_{\mathbb{T}^n} f(x^{-1})g(x)w(x)\frac{dx_1\cdots dx_n}{x_1\cdots x_n} \tag{5.3}$$

by the integral over $\mathbb{T}^n$ with orientation such that

$$\frac{1}{(2\pi\sqrt{-1})^n}\int_{\mathbb{T}^n}\frac{dx_1\cdots dx_n}{x_1\cdots x_n} = 1. \tag{5.4}$$

The scalar product is alternatively expressed as

$$\langle f, g\rangle = \frac{1}{n!}\mathrm{CT}\Big[f(x^{-1})g(x)w(x)\Big], \tag{5.5}$$

in terms of the *constant term* CT (coefficient of 1) of the Laurent expansion of a holomorphic function around $\mathbb{T}^n$.

**Theorem 5.1** *Suppose that $|t| < 1$. Then, the Macdonald polynomials are orthogonal with respect to the scalar product defined by* (5.3):

$$\langle P_\lambda, P_\mu\rangle = \delta_{\lambda,\mu}\,N_\lambda \quad (\lambda, \mu \in \mathcal{P}_n) \tag{5.6}$$

*for some constants $N_\lambda \in \mathbb{C}$ ($\lambda \in \mathcal{P}_n$).*

We remark that, if $q, t \in \mathbb{R}$ and $|q| < 1$, $|t| < 1$, the Macdonald polynomials have real coefficients, and $\langle\ ,\ \rangle$ defines a positive definite scalar product on $\mathbb{R}[x]^{\mathfrak{S}_n}$.

**Remark 5.1** In Macdonald's monograph [20, Sect. VI.9], the scalar product $\langle f, g\rangle$ of (5.3) is called *another scalar product* and denoted by $\langle f, g\rangle'_n$. It should be noted that our scalar product is different from Macdonald's $\langle f, g\rangle_n$ defined by [20, Chap. VI, (2.20)].

### 5.1.2 Constant Term and Scalar Products

It is known [20, Sect. VI.9] that the constant term and the scalar products are determined explicitly as follows.

**Theorem 5.2** *For each $\lambda \in \mathcal{P}_n$, the scalar product $N_\lambda = \langle P_\lambda, P_\lambda\rangle$ is explicitly evaluated as*

$$N_\lambda = \prod_{1\le i<j\le n}\frac{(q^{\lambda_i-\lambda_j}t^{j-i};q)_\infty(q^{\lambda_i-\lambda_j+1}t^{j-i};q)_\infty}{(q^{\lambda_i-\lambda_j}t^{j-i+1};q)_\infty(q^{\lambda_i-\lambda_j+1}t^{j-i-1};q)_\infty}. \tag{5.7}$$

*In particular, the constant term of the weight function* $\mathrm{CT}\Big[w(x)\Big] = n!N_\phi = n!\langle 1, 1\rangle$ *is given by*

$$\mathrm{CT}\Big[w(x)\Big] = n!\left(\frac{(t;q)_\infty}{(q;q)_\infty}\right)^n \prod_{i=1}^{n} \frac{(t^{i-1}q;q)_\infty}{(t^i;q)_\infty}. \tag{5.8}$$

In this book, we will *not* go into the proof of these explicit formulas. For proofs of this theorem, we refer the reader to Macdonald [20, Sect. VI.9], and Mimachi [21] (see also Macdonald [22]).

## 5.2 Proof of Orthogonality

The orthogonality of Macdonald polynomials is a consequence of the facts that:

(1) The $q$-difference operator $D_x$ is (formally) *self-adjoint* with respect to the weight function $w(x)$.

(2) The partitions $\lambda \in \mathcal{P}_n$ are separated by the eigenvalues of $D_x$, namely $d_\lambda \neq d_\mu$ for any distinct pair $\lambda, \mu \in \mathcal{P}_n$.

Along this idea, we explain step by step how the orthogonality of Theorem 5.1 can be established.

### *5.2.1 Cauchy's Theorem as a Basis of q-Difference de Rham Theory*

Let $\varphi(z)$ be a holomorphic function in an neighborhood of a closed curve $C$ in $\mathbb{C}^*$. We suppose that the contour $C$ can be deformed continuously to $qC$ in a domain where $\varphi(z)$ is holomorphic. Note that this condition is satisfied either if the domain of holomorphy of $\varphi(z)$ is sufficiently large, or if $q$ is sufficiently close to 1. Then, by Cauchy's theorem, we have

$$\int_C \varphi(qz)\frac{dz}{z} = \int_{qC} \varphi(z)\frac{dz}{z} = \int_C \varphi(z)\frac{dz}{z}, \tag{5.9}$$

namely

$$\int_C T_{q,z}\big(\varphi(z)\big)\frac{dz}{z} = \int_C \varphi(z)\frac{dz}{z}, \quad \text{i.e.} \quad \int_C (T_{q,z}-1)(\varphi(z))\frac{dz}{z} = 0. \tag{5.10}$$

In particular, we have

$$\int_C T_{q,z}\big(\varphi(z)\big)\psi(z)\frac{dz}{z} = \int_C \varphi(z)T_{q,z}^{-1}\big(\psi(z)\big)\frac{dz}{z}. \tag{5.11}$$

This formula plays the role of integration by parts.

### 5.2.2 Formal Adjoint of a q-Difference Operator

Let $L_x \in \mathcal{D}_{q,x} = \mathbb{C}(x)[T_{q,x}^{\pm 1}]$ be a $q$-difference operator in $x = (x_1, \dots, x_n)$ with rational coefficients:

$$L_x = \sum_{\mu \in \mathbb{Z}^n} a_\mu(x) T_{q,x}^{\mu} \quad \text{(finite sum)}, \quad a_\mu(x) \in \mathbb{C}(x) \ \ (\mu \in \mathbb{Z}^n), \tag{5.12}$$

where $T_{q,x}^{\mu} = T_{q,x_1}^{\mu_1} \cdots T_{q,x_n}^{\mu_n}$. We define the *formal adjoint* $L_x^*$ of $L_x$ by

$$L_x^* = \sum_{\mu \in \mathbb{Z}^n} T_{q,x}^{-\mu} a_\mu(x), \tag{5.13}$$

so that $(L_x M_x)^* = M_x^* L_x^*$. Then, we have

$$\begin{aligned}
&\int_{\mathbb{T}^n} (L_x f)(x^{-1}) g(x) w(x) \frac{dx}{x} \\
&= \int_{\mathbb{T}^n} (L_{x^{-1}} f(x^{-1})) g(x) w(x) \frac{dx}{x} \\
&= \int_{\mathbb{T}^n} f(x^{-1}) \big( L_{x^{-1}}^* g(x) w(x) \big) \frac{dx}{x} \\
&= \int_{\mathbb{T}^n} f(x^{-1}) \big( w(x)^{-1} L_{x^{-1}}^* w(x) g(x) \big) w(x) \frac{dx}{x},
\end{aligned} \tag{5.14}$$

and hence

$$\langle Lf, g \rangle = \langle f, L^\dagger g \rangle, \quad L^\dagger = w(x)^{-1} L_{x^{-1}}^* w(x), \tag{5.15}$$

provided that $q$ is sufficiently close to 1 and that Cauchy's theorem can be applied to $L_x$. We say that $L_x$ is *formally self-adjoint* with respect to $w(x)$ if $L_x^\dagger = L_x$, namely $w(x) L_x w(x)^{-1} = L_{x^{-1}}^*$.

### 5.2.3 $D_x$ Is Self-Adjoint with Respect to $w(x)$

Note that

$$\frac{T_{q,x_i} w(x)}{w(x)} = \prod_{j \neq i} \frac{1 - t x_i / x_j}{1 - x_i / x_j} \prod_{j \neq i} \frac{1 - x_j / q x_i}{1 - t x_j / q x_i} = \frac{A_i(x)}{T_{q,x_i} A_i(x^{-1})} \quad (i = 1, \dots, n). \tag{5.16}$$

This implies that

$$
\begin{aligned}
w(x) D_x w(x)^{-1} &= \sum_{i=1}^{n} A_i(x) \frac{w(x)}{T_{q,x_i} w(x)} T_{q,x_i} = \sum_{i=1}^{n} (T_{q,x_i} A_i(x^{-1})) T_{q,x_i} \\
&= \sum_{i=1}^{n} T_{q,x_i} A_i(x^{-1}) = D^{*}_{x^{-1}}.
\end{aligned}
\tag{5.17}
$$

It can be verified directly that $\langle D_x f, g\rangle = \langle f, D_x g\rangle$ if $|t| < |q| < 1$. Note that the poles of $A_i(x)$ along $\Delta(x) = 0$ are canceled by the zeros of $w(x)$.

### 5.2.4 *Orthogonality*

Since $D_x$ is self-adjoint with respect to the scalar product, for any $\lambda, \mu \in \mathcal{P}_n$ we have the equality

$$\langle D_x P_\lambda(x), P_\mu(x)\rangle = \langle P_\lambda(x), D_x P_\mu(x)\rangle, \tag{5.18}$$

and hence

$$d_\lambda \langle P_\lambda, P_\mu\rangle = d_\mu \langle P_\lambda, P_\mu\rangle. \tag{5.19}$$

Under our assumption that $d_\lambda \neq d_\mu$ $(\lambda \neq \mu)$, we obtain $\langle P_\lambda, P_\mu\rangle = 0$ $(\lambda \neq \mu)$.

## 5.3 Commuting Family of $q$-Difference Operators

### 5.3.1 *Macdonald–Ruijsenaars Operator of rth Order*

For each $r = 0, 1, \ldots, n$, we define the *Macdonald–Ruijsenaars $q$-difference operator $D^{(r)}_x$* of $r$th order by

$$D^{(r)}_x = \sum_{I \subseteq \{1,\ldots,n\};\, |I|=r} A_I(x) T^I_{q,x}, \quad A_I(x) = t^{\binom{|I|}{2}} \prod_{i\in I,\, j\notin I} \frac{t x_i - x_j}{x_i - x_j}, \tag{5.20}$$

where $T^I_{q,x} = \prod_{i\in I} T_{q,x_i}$, so that $D^{(0)}_x = 1$, $D^{(1)}_x = D_x$ and $D^{(n)}_n = t^{\binom{n}{2}} T_{q,x_1} \cdots T_{q,x_n}$.

**Example: $D_x^{(r)}$ ($n = 3,\ r = 1, 2, 3$)**

$$\begin{aligned}
D_x^{(1)} &= \frac{(tx_1 - x_2)(tx_1 - x_3)}{(x_1 - x_2)(x_1 - x_3)} T_{q,x_1} + \frac{(tx_2 - x_1)(tx_2 - x_3)}{(x_2 - x_1)(x_2 - x_3)} T_{q,x_2} \\
&\quad + \frac{(tx_3 - x_1)(tx_3 - x_2)}{(x_3 - x_1)(x_3 - x_2)} T_{q,x_3} \\
D_x^{(2)} &= t\frac{(tx_1 - x_3)(tx_2 - x_3)}{(x_1 - x_3)(x_2 - x_3)} T_{q,x_1} T_{q,x_2} + t\frac{(tx_1 - x_2)(tx_3 - x_2)}{(x_1 - x_2)(x_3 - x_2)} T_{q,x_1} T_{q,x_3} \\
&\quad + t\frac{(tx_2 - x_1)(tx_3 - x_1)}{(x_2 - x_1)(x_3 - x_1)} T_{q,x_2} T_{q,x_3} \\
D_x^{(3)} &= t^3 T_{q,x_1} T_{q,x_2} T_{q,x_3}
\end{aligned} \tag{5.21}$$

**Exercise 5.1** Show that the coefficients $A_I(x)$ can be expressed as

$$A_I(x) = \frac{T_{t,x}^I \Delta(x)}{\Delta(x)} \qquad (I \subseteq \{1, \ldots, n\}) \tag{5.22}$$

in terms of the difference product $\Delta(x) = \prod_{1 \le i < j \le n} (x_i - x_j)$.

As we will see below, the $q$-difference operators $D_x^{(r)}$ ($r = 1, \ldots, n$) commute with each other, and are simultaneously diagonalized on $\mathbb{C}[x]^{\mathfrak{S}_n}$ by the Macdonald polynomials.

### 5.3.2 *Fundamental Properties of $D_x^{(r)}$*

By the same method as we applied to $D_x$, one can directly verify:

(1) The $q$-difference operators $D_x^{(r)}$ ($r = 1, \ldots, n$) are invariant under the action of $\mathfrak{S}_n$.
(2) The linear operators $D_x^{(r)} : \mathbb{C}(x) \to \mathbb{C}(x)$ stabilize $\mathbb{C}[x]^{\mathfrak{S}_n}$, i.e. $D_x^{(r)}(\mathbb{C}[x]^{\mathfrak{S}_n}) \subseteq \mathbb{C}[x]^{\mathfrak{S}_n}$.

As to the triangularity of $D_x^{(r)}$, we have:

**Lemma 5.1** *The linear operators $D_x^{(r)} : \mathbb{C}[x]^{\mathfrak{S}_n} \to \mathbb{C}[x]^{\mathfrak{S}_n}$ ($r = 0, 1, \ldots, n$) are triangular with respect to the dominance order of $m_\lambda(x)$: For each $\lambda \in \mathcal{P}_n$,*

$$D_x^{(r)} m_\lambda(x) = \sum_{\mu \le \lambda} d_{\lambda,\mu}^{(r)} m_\mu(x) = d_\lambda^{(r)} m_\lambda(x) + \sum_{\mu < \lambda} d_{\lambda,\mu}^{(r)} m_\mu(x), \tag{5.23}$$

*where $d_\lambda^{(r)} = d_{\lambda,\lambda}^{(r)} = e_r(t^\delta q^\lambda)$ are the elementary symmetric functions of degree $r$ in $t^\delta q^\lambda = (t^{n-1} q^{\lambda_1}, t^{n-2} q^{\lambda_2}, \ldots, q^{\lambda_n})$.*

***Proof*** We follow the same approach as in the case of $D_x = D_x^{(1)}$ (Lemma 4.1). For each $I \subseteq \{1, \ldots, n\}$ with $|I| = r$, we have

$$A_I(x) = t^{\binom{r}{2}} \prod_{\substack{i<j \\ i\in I,\, j\notin I}} t\,\frac{1 - x_j/tx_i}{1 - x_j/x_i} \prod_{\substack{i<j \\ i\notin I,\, j\in I}} \frac{1 - tx_j/x_i}{1 - x_j/x_i}$$
$$= t^{\sum_{i\in I}(n-i)} + \text{(lower-order terms)}, \tag{5.24}$$

where, for $I = \{i_1 < \cdots < i_r\}$, the exponent of $t$ is computed as

$$\binom{r}{2} + \#\{(i, j) \mid i < j,\ i \in I, j \notin I\}$$
$$= \binom{r}{2} + \sum_{k=1}^{r}((n - i_k) + (r - k)) = \sum_{i\in I}(n - i). \tag{5.25}$$

Hence, we have

$$D_x^{(r)} x^\mu = \sum_{|I|=r} A_I(x) q^{\sum_{i\in I}\mu_i} x^\mu$$
$$= \Big(\sum_{|I|=r} t^{\sum_{i\in I}(n-i)} q^{\sum_{i\in I}\mu_i}\Big) x^\mu + \text{lower-order terms} \tag{5.26}$$
$$= e_r(t^\delta q^\mu) x^\mu + \text{(lower-order terms)}. \tag{5.27}$$

This implies

$$D_x^{(r)} m_\lambda(x) = e_r(t^\delta q^\lambda) m_\lambda(x) + \text{(lower-order terms)} \qquad (\lambda \in \mathcal{P}_n), \tag{5.28}$$

as desired. □

It is convenient to introduce the generating function for $D_x^{(r)}$ $(r = 0, 1, \ldots, n)$ with an extra parameter $u$:

$$D_x(u) = \sum_{r=0}^{n} (-u)^r D_x^{(r)} = \sum_{I\subseteq\{1,\ldots,n\}} (-u)^{|I|} A_I(x) T_{q,x}^I. \tag{5.29}$$

Then, by Lemma 5.1, we have

$$D_x(u) m_\lambda(x) = d_\lambda(u) m_\lambda(x) + \sum_{\mu<\lambda} d_\mu^\lambda(u)\, m_\mu(x),$$
$$d_\lambda(u) = \sum_{r=0}^{n} (-u)^r e_r(t^\delta q^\lambda) = \prod_{i=1}^{n} (1 - ut^{n-i} q^{\lambda_i}). \tag{5.30}$$

### *5.3.3 Macdonald Polynomials as Joint Eigenfunctions*

We prove the following two theorems in the subsequent sections.

**Theorem 5.3** *The $q$-difference operators $D_x^{(r)}$ $(r = 1, \ldots, n)$ commute with each other*:

$$D_x^{(r)} D_x^{(s)} = D_x^{(s)} D_x^{(r)} \qquad (r, s = 1, \ldots, n), \tag{5.31}$$

**Theorem 5.4** *For each $\lambda \in \mathcal{P}_n$, the Macdonald polynomial $P_\lambda(x)$ satisfies the joint eigenfunction equations*

$$D_x^{(r)} P_\lambda(x) = d_\lambda^{(r)} P_\lambda(x), \quad d_\lambda^{(r)} = e_r(t^\delta q^\lambda) \quad (r = 1, \ldots, n). \tag{5.32}$$

We have assumed the genericity condition (4.10) of parameters for the existence of Macdonald polynomials, as well as $|q| < 1$. In this setting, Theorems 5.3 and 5.4 are equivalent. In fact:

**Theorem 5.3 implies Theorem 5.4**: By the commutativity of $D_x^{(r)}$ with $D_x = D_x^{(1)}$, we have

$$D_x D_x^{(r)} P_\lambda(x) = D_x^{(r)} D_x P_\lambda(x) = d_\lambda D_x^{(r)} P_\lambda(x), \tag{5.33}$$

namely $D_x^{(r)} P_\lambda(x)$ is an eigenfunction of $D_x$ with eigenvalue $d_\lambda$. Since the eigenspace of $D_x$ in $\mathbb{C}[x]^{\mathfrak{S}_n}$ with $d_\lambda$ is one-dimensional, we have $D_x^{(r)} P_\lambda(x) = \varepsilon P_\lambda(x)$ for some constant $\varepsilon \in \mathbb{C}$. Since $P_\lambda(x) = m_\lambda(x) +$ (lower-order terms) and also $D_x^{(r)} m_\lambda(x) = d_\lambda^{(r)} m_\lambda(x) +$ (lower-order terms), we conclude $\varepsilon = d_\lambda^{(r)}$ as desired. Conversely:

**Theorem 5.4 implies Theorem 5.3**. Since $D_x^{(r)}$ $(r = 1, \ldots, n)$ are simultaneously diagonalized by $P_\lambda(x)$ $(\lambda \in \mathcal{P}_n)$, for any pair $r, s \in \{1, \ldots, n\}$ the commutator $[D_x^{(r)}, D_x^{(s)}] = D_x^{(r)} D_x^{(s)} - D_x^{(s)} D_x^{(r)}$ is 0 as a linear operator on $\mathbb{C}[x]^{\mathfrak{S}_n}$. From this, it follows that $[D_x^{(r)}, D_x^{(s)}] = 0$ as a $q$-difference operator thanks to the following lemma.

**Lemma 5.2** *Let $L_x \in \mathcal{D}_{q,x} = \mathbb{C}(x)[T_{q,x}^{\pm 1}]$ be a $q$-difference operator with rational function coefficients, and suppose that $L_x f(x) = 0$ for all $f(x) \in \mathbb{C}[x]^{\mathfrak{S}_n}$. Then $L_x = 0$ as a $q$-difference operator.*

***Proof*** Without losing generality, we may assume that $L_x$ has the form

$$L_x = \sum_{\mu \in \mathbb{N}^n : |\mu| \le d} a_\mu(x) T_{q,x}^\mu, \quad d \in \mathbb{N}, \tag{5.34}$$

namely, $L_x \in \mathbb{C}(x)[T_{q,x}]$ and ord $L_x \le d$. Supposing that $L_x|_{\mathbb{C}[x]^{\mathfrak{S}_n}} = 0$, we prove $L_x = 0$ by the induction on $d$. Since this statement is obvious for $d = 0$, we assume $d > 0$. Introducing variables $y = (y_1, \ldots, y_d)$, we consider the polynomial

$$F(x; y) = \prod_{i=1}^n \prod_{k=1}^d (1 - x_i y_k) \in \mathbb{C}[x]^{\mathfrak{S}_n}[y] \tag{5.35}$$

in $(x, y)$. Then we have $L_x F(x; y) = 0$, namely

$$\sum_{|\mu| \le d} a_\mu(x) F(q^\mu x; y) = \sum_{|\mu| \le d} a_\mu(x) \prod_{i=1}^{n} \prod_{k=1}^{d} (1 - q^{\mu_i} x_i y_k) = 0. \tag{5.36}$$

For each $\alpha \in \mathbb{N}^n$ with $|\alpha| = d$, we define the reference point $\eta_\alpha(x) \in (\mathbb{C}^*)^d$ by

$$\eta_\alpha(x) = (1/x_1, 1/qx_1, \dots, 1/q^{\alpha_1 - 1} x_1; \dots; 1/x_n, 1/qx_n, \dots, 1/q^{\alpha_n - 1} x_n). \tag{5.37}$$

Then we have

$$\begin{aligned} F(q^\mu x, \eta_\alpha(x)) &= \prod_{i=1}^{n} \prod_{j=1}^{n} \prod_{\nu=0}^{\alpha_j - 1} (1 - q^{\mu_i} x_i / q^\nu x_j) \\ &= \prod_{i=1}^{n} \prod_{j=1}^{n} (q^{\mu_i - \alpha_j + 1} x_i / x_j; q)_{\alpha_j} \end{aligned} \tag{5.38}$$

Note that $F(q^\mu x; \eta_\alpha(x))$ contains $\prod_{i=1}^{n} (q^{\mu_i - \alpha_i + 1}; q)_{\alpha_i}$ as diagonal factors. If $|\mu| \le d$ and $\mu \ne \alpha$, there exists an index $i \in \{1, \dots, n\}$ such that $\mu_i < \alpha_i$, and hence $(q^{\mu_i - \alpha_i + 1}; q)_{\alpha_i} = 0$. This means that, if $|\mu| \le d$, $F(q^\mu x; \eta_\alpha(x)) = 0$ unless $\mu = \alpha$. Also, we have $F(q^\alpha x; \eta_\alpha(x)) = \prod_{i,j=1}^{n} (q^{\alpha_i - \alpha_j + 1} x_i / x_j; q)_{\alpha_j} \ne 0$. Hence, evaluating (5.36) at $y = \eta_\alpha(x)$, we obtain

$$L_x F(x, y)\big|_{y = \eta_\alpha(x)} = a_\alpha(x) F(q^\alpha x; \eta_\alpha(x)) = 0. \tag{5.39}$$

This implies that $a_\alpha(x) = 0$ for all $\alpha \in \mathbb{N}^n$ with $|\alpha| = 0$, namely ord $L_x < d$. Hence, by the induction on $d$ we conclude that $L_x = 0$. □

## 5.4 Commutativity of the Operators $D_x^{(r)}$

In this section, we give two proofs of Theorem 5.3 of commutativity of the $q$-difference operators $D_x^{(r)}$ $(r = 1, \dots, n)$. One proof, due to Macdonald [20], is based on the orthogonality of Macdonald polynomials, and the other is a direct proof due to Ruijsenaars [30]. Theorem 5.4 follows from Theorem 5.3 as we already explained in the previous section.

### 5.4.1 Orthogonality Implies Commutativity

One can show that, for each $r = 1, \dots, n$, $D_x^{(r)}$ is formally self-adjoint with respect to the scalar product defined by $w(x)$, by a method similar to the one we used in the case of $D_x = D_x^{(1)}$. Since $D_x^{(r)} : \mathbb{C}[x]^{\mathfrak{S}_n} \to \mathbb{C}[x]^{\mathfrak{S}_n}$ is lower triangular with respect to the dominance order, we have

$$D_x^{(r)} P_\lambda(x) = \sum_{\mu \le \lambda} a_{\lambda,\mu}^{(r)} P_\mu(x), \tag{5.40}$$

for some $a_{\lambda,\mu}^{(r)} \in \mathbb{C}$, with leading coefficient $a_{\lambda,\lambda}^{(r)} = d_\lambda^{(r)}$. Since

$$\langle D_x^{(r)} P_\lambda, P_\mu \rangle = a_{\lambda,\mu}^{(r)} \langle P_\mu, P_\mu \rangle, \quad \langle P_\lambda, D_x^{(r)} P_\mu \rangle = 0 \quad (\mu < \lambda), \tag{5.41}$$

and $\langle P_\mu, P_\mu \rangle \neq 0$, we have $a_{\lambda,\mu}^{(r)} = 0$ for $\mu < \lambda$. This means that $D_x^{(r)} P_\lambda(x) = d_\lambda^{(r)} P_\lambda(x)$. In this way, the linear operators $D_x^{(r)} : \mathbb{C}[x]^{\mathfrak{S}_n} \to \mathbb{C}[x]^{\mathfrak{S}_n}$ $(r = 1, \dots, n)$ are simultaneously diagonalized by the Macdonald basis. This gives a proof of Theorem 5.4, as well as Theorem 5.3 by the argument we already explained in the previous section.

### 5.4.2 A Direct Proof of Commutativity

Here we explain a direct proof of Theorem 5.3 of commutativity, following the idea of Ruijsenaars [30].

The composition $D_x^{(r)} D_x^{(s)}$ is computed as

$$D_x^{(r)} D_x^{(s)} = \sum_{|I|=r, |J|=s} A_I(x) A_J(q^{\varepsilon_I} x) T_{q,x}^{\varepsilon_I + \varepsilon_J}, \tag{5.42}$$

where $\varepsilon_I = \sum_{i \in I} \varepsilon_i$, $\varepsilon_i = (\delta_{i,j})_{1 \le j \le n} \in \mathbb{Z}^n$. Setting $K = I \cap J$, $L = (I \cup J) \backslash K$, $P = I \backslash K$, $Q = J \backslash K$, we rewrite (5.42) as

$$\begin{aligned} &D_x^{(r)} D_x^{(s)} \\ &= \sum_{\substack{K \cap L = \phi \\ |K| \le \min\{r,s\}}} \Bigg( \sum_{\substack{P \sqcup Q = L \\ |K|+|P|=r, |K|+|Q|=s}} A_{K \sqcup P}(x) A_{K \sqcup Q}(q^{\varepsilon_K + \varepsilon_P} x) \Bigg) T_{q,x}^{2\varepsilon_K + \varepsilon_L}. \end{aligned} \tag{5.43}$$

Then the commutativity $D_x^{(r)} D_x^{(s)} = D_x^{(s)} D_x^{(r)}$ is equivalent to the following statement: For each $K, L \subseteq \{1, \dots, n\}$ with $K \cap L = \phi$, and for any $p, q \in \mathbb{Z}_{\ge 0}$ such that $p + q = |L|$,

$$\sum_{\substack{P \sqcup Q = L \\ |P|=p, |Q|=q}} A_{K \sqcup P}(x) A_{K \sqcup Q}(q^{\varepsilon_K + \varepsilon_P} x) = \sum_{\substack{P \sqcup Q = L \\ |P|=p, |Q|=q}} A_{K \sqcup Q}(x) A_{K \sqcup P}(q^{\varepsilon_K + \varepsilon_Q} x). \tag{5.44}$$

Analyzing this equality carefully, we show that the statement (5.44) is reduced to an identity of rational functions, which we call the *Ruijsenaars identity*.

For each pair $(I, J)$ of subsets of $\{1, \ldots, n\}$ such that $I \cap J = \phi$, we set

$$A_{I,J}(x) = \prod_{i \in I;\, j \in J} \frac{1 - t x_i / x_j}{1 - x_i / x_j} \tag{5.45}$$

so that

$$A_I(x) = t^{\binom{|I|}{2}} A_{I, I^c}(x), \quad I^c = \{1, \ldots, n\} \setminus I. \tag{5.46}$$

We use below the properties that $A_{I,J}(x)$ is *distributive* in $I$ and $J$ in the sense

$$A_{I_1 \sqcup I_2, J}(x) = A_{I_1, J}(x) A_{I_2, J}(x), \quad A_{I, J_1 \sqcup J_2}(x) = A_{I, J_1}(x) A_{I, J_2}(x), \tag{5.47}$$

and that $A_{I,J}(x)$ depends on the ratios $x_i / x_j$ $(i \in I, j \in J)$ only.

We set $M = \{1, \ldots, n\} \setminus (K \sqcup L)$, so that $K \sqcup P \sqcup Q \sqcup M = \{1, \ldots, n\}$, to obtain

$$\begin{aligned}
& t^{-\binom{|K \sqcup P|}{2} - \binom{|K \sqcup Q|}{2}} A_{K \sqcup P}(x) A_{K \sqcup Q}(q^{\varepsilon_K + \varepsilon_P} x) \\
&= A_{K \sqcup P, M \sqcup Q}(x) A_{K \sqcup Q, M \sqcup P}(q^{\varepsilon_K + \varepsilon_P} x) \\
&= A_{K,M}(x) A_{K,Q}(x) A_{P,M}(x) A_{P,Q}(x) \\
&\quad \cdot A_{K,M}(q^{\varepsilon_K} x) A_{K,P}(q^{\varepsilon_K + \varepsilon_P} x) A_{Q,M}(x) A_{Q,P}(q^{\varepsilon_P} x) \\
&= A_{K,M}(x) A_{K,P}(x) A_{K,Q}(x) A_{P,M}(x) A_{Q,M}(x) A_{K,M}(q^{\varepsilon_K} x) \\
&\quad \cdot A_{P,Q}(x) A_{Q,P}(q^{\varepsilon_P} x) \\
&= A_{K,M}(x) A_{K,L}(x) A_{L,M}(x) A_{K,M}(q^{\varepsilon_K} x) \cdot A_{P,Q}(x) A_{Q,P}(q^{\varepsilon_P} x).
\end{aligned} \tag{5.48}$$

Exchanging the roles of $P$ and $Q$, we have

$$\begin{aligned}
& t^{-\binom{|K \sqcup Q|}{2} - \binom{|K \sqcup P|}{2}} A_{K \sqcup Q}(x) A_{K \sqcup P}(q^{\varepsilon_K + \varepsilon_Q} x) \\
&= A_{K,M}(x) A_{K,L}(x) A_{L,M}(x) A_{K,M}(q^{\varepsilon_K} x) \cdot A_{Q,P}(x) A_{P,Q}(q^{\varepsilon_Q} x).
\end{aligned} \tag{5.49}$$

Hence, equality (5.44) is equivalent to:

$$\sum_{\substack{P \sqcup Q = L \\ |P|=p, |Q|=q}} A_{P,Q}(x) A_{Q,P}(q^{\varepsilon_P} x) = \sum_{\substack{P \sqcup Q = L \\ |P|=p, |Q|=q}} A_{Q,P}(x) A_{P,Q}(q^{\varepsilon_Q} x) \tag{5.50}$$

for any $L \subseteq \{1, \ldots, n\}$ and $p, q$ with $p + q = |L|$.

Changing the notation, we see that the commutativity of the Macdonald–Ruijsenaars operators is reduced to proving the identity

$$\sum_{\substack{I \sqcup J = \{1,\ldots,n\} \\ |I|=r, |J|=s}} A_{I,J}(x) A_{J,I}(q^{\varepsilon_I} x) = \sum_{\substack{I \sqcup J = \{1,\ldots,n\} \\ |I|=r, |J|=s}} A_{J,I}(x) A_{I,J}(q^{\varepsilon_J}) \tag{5.51}$$

for any $r$,$s$ such that $r + s = n$. To be explicit,

**Lemma 5.3** (Ruijsenaars identity) *For any $r, s \in \mathbb{Z}_{\geq 0}$ with $r + s = n$,*

$$\sum_{\substack{I \sqcup J = \{1,\ldots,n\} \\ |I|=r,\, |J|=s}} \prod_{\substack{i \in I \\ j \in J}} \frac{(1 - t x_i / x_j)(1 - t x_j / q x_i)}{(1 - x_i / x_j)(1 - x_j / q x_i)}$$
$$= \sum_{\substack{I \sqcup J = \{1,\ldots,n\} \\ |I|=r,\, |J|=s}} \prod_{\substack{i \in I \\ j \in J}} \frac{(1 - t x_j / x_i)(1 - t x_i / q x_j)}{(1 - x_j / x_i)(1 - x_i / q x_j)}. \tag{5.52}$$

***Proof*** We denote by $F_{r,s}(x)$ the left-hand side of (5.52):

$$F_{r,s}(x) = \sum_{\substack{I \sqcup J = [n] \\ |I|=r,\, |J|=s}} \prod_{\substack{i \in I \\ j \in J}} F_{I,J}(x), \quad F_{I,J}(x) = \prod_{\substack{i \in I \\ j \in J}} \frac{(t x_i - x_j)(q x_i - t x_j)}{(x_i - x_j)(q x_i - x_j)}, \tag{5.53}$$

where $[n] = \{1, \ldots, n\}$. Then the right-hand side is given by $F_{r,s}(x^{-1}) = F_{s,r}(x)$. We remark that $F_{r,s}(x)$ is a symmetric function and $\Delta(x) F_{r,s}(x)$ is regular along the divisors $x_i - x_j = 0$ $(1 \leq i < j \leq n)$. From this fact it follows that $F_{r,s}(x)$ itself is regular along these divisors.

We prove by induction on $n$ that $G_{r,s}(x) = F_{r,s}(x) - F_{r,s}(x^{-1}) = 0$ for any pair $(r, s)$ such that $r + s = n$. We first remark that $G_{r,s}(x) = 0$ if $r = 0$ or $s = 0$, and that $G_{r,s}(x) = 0$ for $n = r + s = 0, 1$. Assuming that $r, s \geq 1$, we regard $F_{r,s}(x)$ as rational functions of $x_n$:

$$F_{r,s}(x) = \sum_{\substack{I \sqcup J = [n] \\ |I|=r, |J|=s,\ n \in I}} \prod_{j \in J} \frac{(t x_n - x_j)(q x_n - t x_j)}{(x_n - x_j)(q x_n - x_j)} F_{I \setminus \{n\}, J}(x_{\widehat{n}})$$
$$+ \sum_{\substack{I \sqcup J = [n] \\ |I|=r, |J|=s,\ n \in J}} \prod_{i \in I} \frac{(x_n - t x_i)(t x_n - q x_i)}{(x_n - x_i)(x_n - q x_i)} F_{I \setminus \{n\}, J}(x_{\widehat{n}}), \tag{5.54}$$

where $x_{\widehat{n}} = (x_1, \ldots, x_{n-1})$. Note that $F_{r,s}(x)$ has at most simple poles at $x_n = qx_k, q^{-1}x_k$ for $k = 1, \ldots, n-1$; it is regular at $x_n = x_k$ as mentioned above.[1] We look at the residues at $x_n = qx_k$:

$$
\begin{aligned}
&\operatorname{Res}(F_{r,s}(x)dx_n|x_n = qx_k)\\
&= \frac{(1-t)(t-q)}{q-1} \sum_{\substack{I \sqcup J = [n] \\ |I|=r, |J|=s;\ k \in I, n \in J}} \prod_{i \in I \setminus \{k\}} \frac{(qx_k - tx_i)(tx_k - x_i)}{(qx_k - x_i)(x_k - x_i)} F_{I, J\setminus\{n\}}(x_{\widehat{n}})\\
&= \frac{(1-t)(t-q)}{q-1} \sum_{\substack{I \sqcup J = [n] \\ |I|=r, |J|=s;\ k \in I, n \in J}} \prod_{i \in I \setminus \{k\}} \frac{(qx_k - tx_i)(tx_k - x_i)}{(qx_k - x_i)(x_k - x_i)}\\
&\qquad \cdot \prod_{j \in J \setminus \{n\}} \frac{(tx_k - x_j)(qx_k - tx_j)}{(x_k - x_j)(qx_k - x_j)} \cdot F_{I\setminus\{k\}, J\setminus\{n\}}(x_{\widehat{n}})\\
&= \frac{(1-t)(t-q)}{q-1} \prod_{l \neq k,n} \frac{(tx_k - x_l)(qx_k - tx_l)}{(x_k - x_l)(qx_k - x_l)} \sum_{\substack{I' \sqcup J' = [n]\setminus\{k,n\} \\ |I'|=r-1, |J'|=s-1}} F_{I',J'}(x_{\widehat{k},\widehat{n}}),
\end{aligned}
\tag{5.55}
$$

where $x_{\widehat{n}} = (x_1, \ldots, x_{n-1})$ and $x_{\widehat{k},\widehat{n}} = (x_1, \ldots, \widehat{k}, \ldots, x_{n-1})$. Similarly, we compute

$$
\begin{aligned}
&\operatorname{Res}(F_{r,s}(x^{-1})dx_n|x_n = qx_k)\\
&= \frac{(1-t)(t-q)}{q-1} \prod_{l \neq k,n} \frac{(tx_k - x_l)(qx_k - tx_l)}{(x_k - x_l)(qx_k - x_l)} \sum_{\substack{I' \sqcup J' = [n]\setminus\{k,n\} \\ |I'|=r-1, |J'|=s-1}} F_{J',I'}(x_{\widehat{x_k},\widehat{n}}).
\end{aligned}
\tag{5.56}
$$

Hence, for $G_{r,s}(x) = F_{r,s}(x) - F_{r,s}(x^{-1})$ we have

$$
\begin{aligned}
&\operatorname{Res}(G_{r,s}(x)dx_n|x_n = qx_k)\\
&= \frac{(1-t)(t-q)}{q-1} \prod_{l \neq k,n} \frac{(tx_k - x_l)(qx_k - tx_l)}{(x_k - x_l)(qx_k - x_l)}\\
&\quad \cdot \sum_{\substack{I' \sqcup J' = [n]\setminus\{k,n\} \\ |I'|=r-1, |J'|=s-1}} \left(F_{I',J'}(x_{\widehat{x_k},\widehat{n}}) - F_{J',I'}(x_{\widehat{x_k},\widehat{n}})\right) = 0
\end{aligned}
\tag{5.57}
$$

for $k = 1, \ldots, n-1$, by the induction hypothesis of the case of $n-2$ variables. By the same argument we obtain $\operatorname{Res}(G_{r,s}(x)dx_n|x_n = q^{-1}x_k) = 0$ for $k = 1, \ldots, n-1$. This implies that $G_{r,s}(x)$ is constant with respect to $x_n$. Since $G_{r,s}(x)$ is symmetric with respect to $x = (x_1, \ldots, x_n)$, we conclude that $G_{r,s}(x)$ is a constant, i.e. does not depend on $x_i$ $(i = 1, \ldots, n)$. However, $G_{r,s}(x) = F_{r,s}(x) - F_{r,s}(x^{-1})$ satisfies $G_{r,s}(x^{-1}) = -G_{r,s}(x)$, and hence we obtain $G_{r,s}(x) = 0$. □

[1] One can also show directly that $\operatorname{Res}(F_{r,s}(x^{\pm 1})dx_n|x_n = x_k) = 0$ $(k = 1, \ldots, n-1)$, by a computation similar to the one presented below.

We remark that Ruijsenaars [30] proved the commutativity of the elliptic version of $D_x^{(r)}$ $(r = 1, \dots, n)$ along the same line as above, on the basis of the corresponding identity for the Weierstrass sigma functions.

**Remark 5.2** In Chap. 8, we will explain a construction of the $q$-difference operators $D_x^{(r)}$ as well as their commutativity, following the idea of Cherednik based on a representation of the affine Hecke algebra.

## 5.5 Refinement of the Existence Theorem

Once commutativity of the Macdonald–Ruijsenaars operators $D_x^{(r)}$ $(r = 1, \dots, n)$ has been established, the existence theorem of Macdonald polynomials can be refined as we formulate below. Here we fix the parameters $q, t \in \mathbb{C}^*$ with $|q| < 1$, and suppose that the parameter $t \in \mathbb{C}^*$ satisfies the condition $t^k \notin q^{\mathbb{Z}_{<0}}$ for $k = 1, \dots, n-1$. In this setting we give a proof of existence of the Macdonald polynomials, independently of the previous existence theorem (Theorem 4.1).

**Theorem 5.5** *Suppose that the parameter $t$ satisfies the condition that $t^k \notin q^{\mathbb{Z}_{<0}}$ $(k = 1, \dots, n-1)$. Then, for each partition $\lambda \in \mathcal{P}_n$ there exists a unique symmetric polynomial $P_\lambda(x) \in \mathbb{C}[x]^{\mathfrak{S}_n}$ such that*

$$(1)\ \ D_x^{(r)} P_\lambda(x) = d_\lambda^{(r)} P_\lambda(x) \ \ (r = 1, \dots, n), \tag{5.58}$$

$$(2)\ \ P_\lambda(x) = m_\lambda(x) + \sum_{\mu<\lambda} u_\mu^\lambda m_\mu(x) \ \ \ (u_\mu^\lambda \in \mathbb{C}). \tag{5.59}$$

We remark that, in terms of the generating function $D_x(u) = \sum_{r=0}^n (-u)^r D_x^{(r)}$, the joint eigenfunction equations for $P_\lambda(x)$ are unified in the form

$$D_x(u) P_\lambda(x) = d_\lambda(u) P_\lambda(x), \quad d_\lambda(u) = \prod_{i=1}^n (1 - u t^{n-i} q^{\lambda_i}). \tag{5.60}$$

Note that, for a pair $\lambda, \mu \in \mathcal{P}_n$, $d_\lambda(u) = d_\mu(u)$ as polynomials in $u$ if and only if there exists a permutation $\sigma \in \mathfrak{S}_n$ such that

$$t^{n-i} q^{\mu_i} = t^{n-\sigma(i)} q^{\lambda_{\sigma(i)}} \quad (i = 1, \dots, n). \tag{5.61}$$

Under our assumption $|q| < 1$, we have:

**Lemma 5.4** *Suppose that $t^k \notin q^{\mathbb{Z}_{<0}}$ $(k = 1, \dots, n-1)$. Then, $d_\lambda(u) \neq d_\mu(u)$ for any distinct pair $\lambda, \mu \in \mathcal{P}_n$ as polynomials in $u$, and also for generic $u \in \mathbb{C}$.*

***Proof*** We first show that, if $|t| \leq 1$, then $d_\lambda(u) \neq d_\mu(u)$ as polynomials in $u$ for any distinct pair $\lambda, \mu \in \mathcal{P}_n$. Under the assumption $|t| \leq 1$, the sequence $|t^{n-i} q^{\lambda_i}|$

$(i = 1, \ldots, n)$ is weakly increasing for any $\lambda \in \mathcal{P}_n$. From this it follows that, if $d_\lambda(u) = d_\mu(u)$ for $\lambda, \mu \in \mathcal{P}_n$, then we have $|t^{n-i}q^{\lambda_i}| = |t^{n-i}q^{\mu_i}|$ $(i = 1, \ldots, n)$. Hence, for $i = 1, \ldots, n$, we have $|q|^{\lambda_i} = |q|^{\mu_i}$ and $\lambda_i = \mu_i$ since $|q| < 1$. Namely, if $|t| \le 1$, then $d_\lambda(u) = d_\mu(u)$ implies $\lambda = \mu$.

We now consider the case $|t| > 1$. Suppose that $d_\lambda(u) = d_\mu(u)$ as polynomials in $u$ for some distinct pair $\lambda, \mu \in \mathcal{P}_n$. Then, there exists a permutation $\sigma \in \mathfrak{S}_n$ such that

$$t^{n-i}q^{\mu_i} = t^{n-\sigma(i)}q^{\lambda_{\sigma(i)}} \quad (i = 1, \ldots, n). \tag{5.62}$$

Since $\lambda \ne \mu$, we have $\sigma \ne 1$, and hence there exists an index $\sigma(i) > i$. Then we have $t^{\sigma(i)-i} = q^{\lambda_{\sigma(i)}-\mu_i} \in q^{\mathbb{Z}}$, which means $t^k \in q^{\mathbb{Z}}$ for $k = \sigma(i) - i \in \{1, \ldots, n-1\}$. Since $|t| > 1$, $t^k \in q^{\mathbb{Z}_{<0}}$ for some $k \in \{1, \ldots, n-1\}$.

Suppose that $d_\lambda(u) \ne d_\mu(u)$ for any distinct pair $\lambda, \mu \in \mathcal{P}_n$. Since the set

$$S = \left\{a \in \mathbb{C}^* \mid d_\lambda(a) = d_\mu(a) \text{ for some distinct pair } \lambda, \mu \in \mathcal{P}_n\right\} \tag{5.63}$$

is countable, the complement $\mathbb{C}^* \backslash S$ is non-empty. Then, for any $c \in \mathbb{C}^* \backslash S$, we have $d_\lambda(c) \ne d_\mu(c)$ for any distinct pair $\lambda, \mu \in \mathcal{P}_n$. □

***Proof*** (***of Theorem*** 5.5) Under the assumption that $t^k \notin q^{\mathbb{Z}_{<0}}$ for $k = 1, \ldots, n-1$, by Lemma 5.4 we can find a constant $c \in \mathbb{C}$ such that $d_\lambda(c) \ne d_\mu(c)$ for any distinct pair $\lambda, \mu \in \mathcal{P}_n$. From the facts that $D_x(c) : \mathbb{C}[x]^{\mathfrak{S}_n} \to \mathbb{C}[x]^{\mathfrak{S}_n}$ is triangular with respect to the dominance order and that the eigenvalues $d_\lambda(c)$ separate $\mathcal{P}_n$, it follows that for each $\lambda \in \mathcal{P}_n$ there exists a unique symmetric polynomial $P_\lambda(x) \in \mathbb{C}[x]^{\mathfrak{S}_n}$ such that $P_\lambda(x) = m_\lambda(x) + (\text{lower-order terms}) \in \mathbb{C}[x]^{\mathfrak{S}_n}$ and $D_x(c)P_\lambda(x) = d_\lambda(c)P_\lambda(x)$. Note that $P_\lambda(x)$ $(\lambda \in \mathcal{P}_n)$ form a $\mathbb{C}$-basis of $\mathbb{C}[x]^{\mathfrak{S}_n}$, and have mutually distinct eigenvalues $d_\lambda(c)$ with respect to the linear operators $D_x(c)$. We remark that these $P_\lambda(x)$ do not depend on the choice of $c$, as we will see below.

Since $D_x^{(r)}$ commutes with $D_x(c)$ for $r = 1, \ldots, n$, we have $D_x(c)D_x^{(r)}P_\lambda(x) = D_x^{(r)}D_x(c)P_\lambda(c) = d_\lambda(c)D_x^{(r)}P_\lambda(x)$. This means that $D_x^{(r)}P_\lambda(x)$ is an eigenfunction of $D_x(c)$ with eigenvalue $d_\lambda(c)$, and hence $D_x^{(r)}P_\lambda(x)$ is a contant multiple of $P_\lambda(x)$ by the fact that the eigenspace of $D_x(c)$ with eigenvalue $d_\lambda(c)$ is one-dimensional. Since $D_x^{(r)}P_\lambda(x) = d_\lambda^{(r)}m_\lambda(x) + (\text{lower-order terms})$, we have $D_x^{(r)}P_\lambda(x) = d_\lambda^{(r)}P_\lambda(x)$. Namely, we obtain

$$\mathbb{C}[x]^{\mathfrak{S}_n} = \bigoplus_{\lambda \in \mathcal{P}_n} \mathbb{C}\, P_\lambda(x), \quad D_x(u)P_\lambda(x) = d_\lambda(u)P_\lambda(x). \tag{5.64}$$

This also implies that the polynomials $P_\lambda(x)$ do not depend on the choice of $c \in \mathbb{C}^*$ with which we started. □

## 5.6 Some Remarks Related to $D_x(u)$

### 5.6.1 Macdonald Polynomials in $x^{-1}$

Consider the $q$-difference operators $D^{(r)}_{x^{-1}}$ $(r = 0, 1, \ldots, n)$ in the variables $x^{-1} = (x_1^{-1}, \ldots, x_n^{-1})$ such that

$$D^{(r)}_{x^{-1}} f(x^{-1}) = D^{(r)}_x f(x)\Big|_{x \to x^{-1}}. \tag{5.65}$$

These operators are then explicitly given by

$$D^{(r)}_{x^{-1}} = \sum_{|I|=r} t^{\binom{r}{2}} \prod_{i \in I,\, j \notin I} \frac{t x_j - x_i}{x_j - x_i} \prod_{i \in I} T^{-1}_{q,x_i}. \tag{5.66}$$

**Lemma 5.5** *For each $r = 0, 1, \ldots, n$,*

$$D^{(r)}_x = t^{(n-1)r - \binom{n}{2}} D^{(n-r)}_{x^{-1}} T_{q,x_1} \cdots T_{q,x_n}. \tag{5.67}$$

*In terms of the generating function, we have*

$$D_x(u) = (-u)^n t^{\binom{n}{2}} D_{x^{-1}}(u^{-1} t^{-n+1}) T_{q,x_1} \cdots T_{q,x_n}. \tag{5.68}$$

We leave the proof of this lemma as an exercise.

Let $\lambda \in \mathcal{P}_n$ be a partition and suppose that $\lambda$ is contained in the $n \times l$ rectangle $(\lambda_1 \le l)$. Then we have

$$(x_1 \cdots x_n)^l P_\lambda(x^{-1}) = m_{(l^n)-\lambda^\vee}(x) + (\text{lower-order terms}) \in \mathbb{C}[x]^{\mathfrak{S}_n}, \tag{5.69}$$

where $\lambda^\vee = (\lambda_n, \ldots, \lambda_1)$ denotes the *reversal* of $\lambda = (\lambda_1, \ldots, \lambda_n)$.

**Proposition 5.1** *For each partition $\lambda \in \mathcal{P}_n$ with $\lambda_1 \le l$, $l \in \mathbb{N}$, we have*

$$(x_1 \cdots x_n)^l P_\lambda(x^{-1}) = P_{(l^n)-\lambda^\vee}(x), \quad \lambda^\vee = (\lambda_n, \ldots, \lambda_1). \tag{5.70}$$

One can verify the eigenfunction equation

$$D_x(u)(x_1 \cdots x_n)^l P_\lambda(x^{-1}) = \prod_{i=1}^{n} (1 - u t^{n-i} q^{l-\lambda_{n+1-i}}) \cdot (x_1 \cdots x_n)^l P_\lambda(x^{-1}) \tag{5.71}$$

by using Lemma 5.5.

### 5.6.2 Determinant Representation of $D_x(u)$

The generating function $D_x(u)$ of the Macdonald–Ruijsenaars $q$-difference operators can also be expressed in terms of the determinant of a matrix of $q$-difference operators.

For an $n \times n$ matrix $L = (L_{ij})_{i,j=1}^{n}$ with entires in a ring, possibly non-commutative, we use the notation $\det(L)$ for the *column determinant*

$$\det(L) = \sum_{\sigma\in\mathfrak{S}_n} \operatorname{sgn}(\sigma) L_{\sigma(1)1} \cdots L_{\sigma(n)n}. \tag{5.72}$$

**Theorem 5.6** *The generating function $D_x(u) = \sum_{r=0}^{n}(-u)^r D_x^{(r)}$ of the Macdonald–Ruijsenaars operators is represented by the column determinant*

$$\begin{aligned} D_x(u) &= \frac{1}{\Delta(x)} \det\Big(x_i^{n-j}\big(1 - ut^{n-j}T_{q,x_i}\big)\Big)_{i,j=1}^{n} \\ &= \frac{1}{\Delta(x)} \sum_{\sigma\in\mathfrak{S}_n} \operatorname{sgn}(\sigma) \prod_{j=1}^{n} x_{\sigma(j)}^{n-j}\big(1 - ut^{n-j}T_{q,x_{\sigma(j)}}\big). \end{aligned} \tag{5.73}$$

We remark that the $q$-difference operators $L_{ij} = x_i^{n-j}(1 - ut^{n-j}T_{q,x_i})$ satisfy the commutativity $L_{ij}L_{kl} = L_{kl}L_{ij}$ $(i \neq k)$. This implies that the product $\prod_{j=1}^{n}$ above does not depend on the ordering.

For a $q$-difference operator $L_x = \sum_{\mu\in\mathbb{Z}^n} a_\mu(x)T_{q,x}^{\mu} \in \mathbb{C}(x)[T_{q,x}^{\pm1}]$, we define its *symbol* by

$$\operatorname{symb}(L_x) = \sum_{\mu\in\mathbb{Z}^n} a_\mu(x)\xi^\mu \in \mathbb{C}(x)[\xi^{\pm1}], \quad \xi = (\xi_1, \ldots, \xi_n). \tag{5.74}$$

Note that two $q$-difference operators $L_x$, $M_x$ coincide if $\operatorname{symb}(L_x) = \operatorname{symb}(M_x)$. We compute the symbol of $D_x(u)$ as follows:

$$\begin{aligned} &\operatorname{symb}(D_x(u)) \\ &= \sum_{I\subseteq\{1,\ldots,n\}} (-u)^{|I|} \frac{T_{t,x}^{\varepsilon_I}\Delta(x)}{\Delta(x)} \xi^{\varepsilon_I} = \frac{1}{\Delta(x)}\Big(\sum_{I\subseteq\{1,\ldots,n\}} (-u)^{|I|}\xi^{\varepsilon_I}T_{t,x}^{\varepsilon_I}\Big)\Delta(x) \\ &= \frac{1}{\Delta(x)} \prod_{i=1}^{n} (1 - u\,\xi_i\,T_{t,x_i})\Delta(x) = \frac{1}{\Delta(x)} \det\big(x_i^{n-j}(1 - u\,t^{n-j}\xi_i)\big)_{i,j=1}^{n} \\ &= \frac{1}{\Delta(x)} \sum_{\sigma\in\mathfrak{S}_n} \operatorname{sgn}(\sigma) \prod_{j=1}^{n} x_{\sigma(j)}^{n-j}(1 - u\,t^{n-j}\xi_{\sigma(j)}), \end{aligned} \tag{5.75}$$

which coincides with the symbol of the right-hand side of (5.73).

### *5.6.3 Limit to the Differential (Jack) Case*

If we set $q = e^{\varepsilon}$ with a small parameter $\varepsilon$, we have

$$\begin{aligned} T_{q,x_i} x^{\mu} &= q^{\mu_i} x^{\mu} = \sum_{k=0}^{\infty} \frac{(\mu_i \varepsilon)^k}{k!} x^{\mu} \\ &= \sum_{k=0}^{\infty} \frac{(\varepsilon x_i \partial_{x_i})^k}{k!} x^{\mu} = e^{\varepsilon x_i \partial_{x_i}} x^{\mu} = q^{x_i \partial_{x_i}} x^{\mu}. \end{aligned} \tag{5.76}$$

In view of this fact, we rewrite the $q$-shift operators as $T_{q,x_i} = q^{x_i \partial_{x_i}}$ by the Euler operators $x_i \partial_{x_i} = x_i \partial / \partial x_i$ $(i = 1, \dots, n)$. Then we take the scaling limit of $D_x(u)/(1-q)^n$ as $q \to 1$ with $t = q^{\beta}$, $u = q^{v}$:

$$\begin{aligned} S_x(v) &= \lim_{q \to 1} \frac{1}{(1-q)^n} \big( D_x(q^v)\big|_{t=q^{\beta}} \big) \\ &= \frac{1}{\Delta(x)} \lim_{q \to 1} \det\Big( x_i^{n-j} \frac{1 - q^{v+(n-j)\beta + x_i \partial_{x_i}}}{1-q} \Big)_{i,j=1}^{n} . \\ &= \frac{1}{\Delta(x)} \det\Big( x_i^{n-j} (v + x_i \partial_{x_i} + (n-j)\beta) \Big)_{i,j=1}^{n} . \end{aligned} \tag{5.77}$$

The resulting operator $S_x(v)$ satisfies

$$S_x(v) P_{\lambda}^{(\beta)}(x) = P_{\lambda}^{(\beta)}(x) \prod_{i=1}^{n} (v + \lambda_i + (n-i)\beta) \qquad (\lambda \in \mathcal{P}_n), \tag{5.78}$$

where $P_{\lambda}^{(\beta)}(x) = \lim_{q \to 1} P_{\lambda}(x; q, q^{\beta})$ are the *Jack polynomials*. Denoting by $S_x^{(r)}$ the coefficients of $v^{n-r}$ of $S_x(v)$, we obtain a commuting family of differential operators $S_x^{(r)}$, called the *Sekiguchi–Debiard operators*, such that

$$S_x^{(r)} P_{\lambda}^{(\beta)}(x) = e_r(\lambda + \beta\delta) P_{\lambda}^{(\beta)}(x) \qquad (r = 0, 1, \dots, n), \tag{5.79}$$

where $\delta = (n-1, n-2, \dots, 0)$. The eigenvalues $e_r(\lambda + \beta\delta)$ are the $r$th elementary symmetric functions of $\lambda_i + (n-i)\beta$ $(i = 1, \dots, n)$.

From the determinant representation (5.77), by a computation similar to that of (5.75) we obtain the following expression for the Sekiguchi–Debiard operators:

$$S_x^{(r)} = \sum_{|K|=r} \sum_{J \subseteq K} \beta^{|K \setminus J|} \frac{(x\partial_x)^{K \setminus J} (\Delta(x))}{\Delta(x)} (x\partial_x)^J \quad (r = 0, 1, \dots, n), \tag{5.80}$$

where the sum is over all pairs $(J, K)$ of subsets of $\{1, \ldots, n\}$ such that $|K| = r$ and $J \subseteq K$.[2] In particular, we have

$$S_x^{(1)} = \sum_{i=1}^{n} x_i \partial_{x_i} + \beta e_1(\delta),$$

$$S_x^{(2)} = \sum_{1 \le i < j \le n} x_i \partial_{x_i} x_j \partial_{x_j} + \beta \sum_{i=1}^{n} \left( e_1(\delta) - \sum_{j \ne i} \frac{x_i}{x_i - x_j} \right) x_i \partial_{x_i} + \beta^2 e_2(\delta), \quad (5.81)$$

where $e_1(\delta) = \frac{1}{2} n(n-1)$ and $e_2(\delta) = \frac{1}{24} n(n-1)(n-2)(3n-1)$. Recall that power sums are represented as

$$p_1 = e_1, \quad p_2 = e_1^2 - 2e_2, \quad p_3 = e_1^3 - 3e_1 e_2 + 3e_3, \quad \ldots, \quad (5.82)$$

by elementary symmetric functions. In view of these formulas, we introduce the differential operators $L_x^{(k)}$ $(k = 1, 2, \ldots)$ by

$$L_x^{(1)} = S_x^{(1)}, \; L_x^{(2)} = \left(S_x^{(1)}\right)^2 - 2S_x^{(2)}, \; L_x^{(3)} = \left(S_x^{(1)}\right)^3 - 3S_x^{(1)} S_x^{(2)} + 3S_x^{(3)}, \; \ldots. \quad (5.83)$$

Then we have

$$L_x^{(k)} P_\lambda^{(\beta)}(x) = p_k(\lambda + \beta\delta) P_\lambda^{(\beta)}(x) \quad (k = 1, 2, \ldots), \quad (5.84)$$

with eigenvalues $p_k(\lambda + \beta\delta) = \sum_{i=1}^{n} (\lambda_i + (n-i)\beta)^k$ expressed by power sums. Explicitly, $L_x^{(1)}$ and $L_x^{(2)}$ are given by

$$L_x^{(1)} = \sum_{i=1}^{n} x_i \partial_{x_i} + \beta p_1(\delta),$$

$$L_x^{(2)} = \sum_{i=1}^{n} \left(x_i \partial_{x_i}\right)^2 + 2\beta \sum_{i=1}^{n} \left( \sum_{j \ne i} \frac{x_i}{x_i - x_j} \right) x_i \partial_{x_i} + \beta^2 p_2(\delta), \quad (5.85)$$

where $p_1(\delta) = \frac{1}{2} n(n-1)$ and $p_2(\delta) = \frac{1}{6} n(n-1)(2n-1)$.[3] We now conjugate these operators by the power $\Delta(x)^\beta$ of the difference product:

---

[2] For a differential operator $L_x = \sum_{\mu \in \mathbb{N}^n} a_\mu(x)(x\partial_x)^\mu$ (finite sum), consider the symbol $\mathrm{symb}(L_x) = \sum_{\mu \in \mathbb{N}^n} a_\mu(x) \lambda^\mu$ with $\lambda = (\lambda_1, \ldots, \lambda)$ regarded as variables. Note also that $L_x(x^\lambda) = \mathrm{symb}(L_x) x^\lambda$.

[3] Set $U_i(x) = \frac{x_i \partial_{x_i}(\Delta(x))}{\Delta(x)} = \sum_{j \ne i} \frac{x_i}{x_i - x_j}$ for each $i$, and $U_{ij}(x) = x_i \partial_{x_i}(U_j(x)) = \frac{x_i x_j}{(x_i - x_j)^2}$ for distinct pair $i, j$, so that $\frac{x_i \partial_{x_i} x_j \partial_{x_j}(\Delta(x))}{\Delta(x)} = U_i(x) U_j(x) + U_{ij}(x)$. Then we have $\sum_{i=1}^{n} U_i(x) = p_1(\delta)$ and $\sum_{i=1}^{n} U_i^2 - 2 \sum_{1 \le i < j \le n} U_{ij}(x) = p_2(\delta)$. Use these formulas to derive (5.81) and (5.85).

$$P = \Delta(x)^{\beta} L_x^{(1)} \Delta(x)^{-\beta} = \sum_{i=1}^{n} x_i \partial_{x_i}, \tag{5.86}$$

$$H = \Delta(x)^{\beta} L_x^{(2)} \Delta(x)^{-\beta} = \sum_{i=1}^{n} \big(x_i \partial_{x_i}\big)^2 - 2\beta(\beta-1) \sum_{1 \le i < j \le n} \frac{x_i x_j}{(x_i - x_j)^2}. \tag{5.87}$$

Then the functions $\psi_\lambda(x) = P_\lambda^{(\beta)}(x)\Delta(x)^{\beta}$ ($\lambda \in \mathcal{P}_n$) satisfy

$$P\psi_\lambda(x) = p_1(\lambda + \beta\delta)\psi_\lambda(x), \quad H\psi_\lambda(x) = p_2(\lambda + \beta\delta)\psi_\lambda(x). \tag{5.88}$$

The operators $P$ and $H$ are the momentum operator and the Hamiltonian for the *Calogero–Sutherland model* with coupling constant $\beta$. Note that, in terms of the angular coordinates $\theta_i$ ($i = 1, \dots, n$) such that $x_i = e^{\sqrt{-1}\theta_i}$, the operators $P$ and $H$ are expressed as

$$P = \frac{1}{\sqrt{-1}} \sum_{i=1}^{n} \partial_{\theta_i} \quad H = -\sum_{i=1}^{n} \partial_{\theta_i}^2 + \frac{\beta(\beta-1)}{2} \sum_{1 \le i < j \le n} \frac{1}{\sin^2 \frac{\theta_i - \theta_j}{2}}. \tag{5.89}$$

# Chapter 6
# Self-duality, Pieri Formula and Cauchy Formulas

**Abstract** Self-duality (evaluation symmetry), which we are going to discuss below, is one of the most characteristic properties of Macdonald polynomials. In this chapter, we explain how the Pieri formulas (multiplication formula by $e_r$) are obtained from the action of Macdonald–Ruijsenaars operators $D_x^{(r)}$ through the self-duality. We also investigate the Cauchy formula and the dual Cauchy formula for Macdonald polynomials and the relevant kernel identities.

## 6.1 Self-duality and Pieri Formula

We have seen in the previous chapter that, for generic $q, t \in \mathbb{C}^*$ the Macdonald polynomials $P_\lambda(x) = P_\lambda(x; q, t)$ $(\lambda \in \mathcal{P}_n)$ are joint eigenfunctions of the commuting family of Macdonald–Ruijsenaars $q$-difference operators $D_x^{(r)}$ $(r = 1, \ldots, n)$, and that they form a $\mathbb{C}$ basis of the ring of symmetric polynomials $\mathbb{C}[x]^{\mathfrak{S}_n}$:

$$\mathbb{C}[x]^{\mathfrak{S}_n} = \bigoplus_{\lambda \in \mathcal{P}_n} \mathbb{C}\, P_\lambda(x), \quad D_x(u) P_\lambda(x) = d_\lambda(u) P_\lambda(x). \tag{6.1}$$

Note that, under our assumption $|q| < 1$, the genericity condition for $t$ is fulfilled if $t^k \notin q^{\mathbb{Z}_{<0}}$ for $k = 1, \ldots, n-1$, and in particular, if $|t| < 1$. Also, if we regard $q, t$ as variables (indeterminates), the (monic) Macdonald polynomials $P_\lambda(x)$ are determined uniquely as symmetric polynomials with coefficients in the field $\mathbb{Q}(q, t)$ of rational functions in $(q, t)$; their coefficients are regular in the domain $|q| < 1$, $|t| < 1$.

### *6.1.1 Principal Specialization*

As to the values of Schur functions $s_\lambda(x)$ at the base point $x = t^\delta$, we gave two explicit formulas in Propositions 3.1 and 3.2. Those evaluation formulas are generalized to the case of Macdonald polynomials as follows.

 
M. Noumi, *Macdonald Polynomials*, SpringerBriefs in Mathematical Physics,
https://doi.org/10.1007/978-981-99-4587-0_6

**Theorem 6.1** (Principal specialization) *For any $\lambda \in \mathcal{P}_n$, the value of $P_\lambda(x)$ at $x = t^\delta$ is given explicitly by*

$$P_\lambda(t^\delta) = t^{n(\lambda)} \prod_{s\in\lambda} \frac{1 - t^{n-l'_\lambda(s)} q^{a'_\lambda(s)}}{1 - t^{l_\lambda(s)+1} q^{a_\lambda(s)}} = \frac{t^{n(\lambda)} \prod_{i=1}^{n} (t^{n-i+1}; q)_{\lambda_i}}{\prod_{1\le i\le j\le n} (t^{j-i+1} q^{\lambda_i - \lambda_j}; q)_{\lambda_j - \lambda_{j+1}}}, \tag{6.2}$$

*where $n(\lambda) = \sum_{i=1}^{n} (i-1)\lambda_i$ and, for each $s = (i, j) \in \lambda$, $l'_\lambda(s) = i - 1$ and $a'_\lambda(s) = j - 1$ denote the* co-leg length *and the* co-arm length*, respectively.*

The proof of this evaluation formula at $x = t^\delta$ will be given in Sect. 6.3, under the assumption that Theorem 6.2 (below) of self-duality holds.

### 6.1.2 Self-duality

At this moment, we know at least that $P_\lambda(t^\delta) \neq 0$ as a rational function of $(q, t)$, since the Schur functions are the special case of Macdonald polynomials where $t = q$, i.e. $P_\lambda(x; q, q) = s_\lambda(x)$. Keeping this in mind, we normalize the Macdonald polynomials as

$$\widetilde{P}_\lambda(x) = \frac{P_\lambda(x)}{P_\lambda(t^\delta)} \qquad (\lambda \in \mathcal{P}_n) \tag{6.3}$$

so that $\widetilde{P}_\lambda(t^\delta) = 1$. Then we have the following self-duality (evaluation symmetry).

**Theorem 6.2** (Self-duality) *The normalized Macdonald polynomials $\widetilde{P}_\lambda(x) = P_\lambda(x)/P_\lambda(t^\delta)$ satisfy*

$$\widetilde{P}_\lambda(t^\delta q^\mu) = \widetilde{P}_\mu(t^\delta q^\lambda) \tag{6.4}$$

*for all pairs $(\lambda, \mu) \in \mathcal{P}_n \times \mathcal{P}_n$.*

We regard $x = t^\delta q^\lambda$ as the *position variables* and $\xi = t^\delta q^\mu$ as the *spectral variables*. Then (6.4) means that the *normalized* Macdonald polynomial $\widetilde{P}_\lambda(t^\delta q^\mu)$, regarded as a function of $(\lambda, \mu) \in \mathcal{P}_n \times \mathcal{P}_n$, is invariant under the exchange of position and spectral variables on the discrete set.

We include a proof of Theorem 6.2 due to Koornwinder [14, 20] in Sect. 6.4.

### 6.1.3 Pieri Formula

For each $\lambda \in \mathcal{P}_n$, the Macdonald polynomial $P_\lambda(x)$ multiplied by an elementary symmetric function $e_r(x)$ $(r = 1, \ldots, n)$ can be expanded into a linear combination of Macdonald polynomials:

$$e_r(x)P_\mu(x) = \sum_{\substack{\lambda\in\mathcal{P}_n \\ \lambda\le\mu+(1^r)}} \psi'_{\lambda/\mu}\, P_\lambda(x), \tag{6.5}$$

with some coefficients $\psi'_{\lambda/\mu} \in \mathbb{Q}(q,t)$. This type of expansion formula is called the *Pieri formula*. In order to describe the expansion coefficients in the Pieri formula, we introduce certain rational functions in $(q,t)$.

For each pair $\lambda, \mu \in \mathcal{P}$ of partitions with $\mu \subseteq \lambda$ (i.e. $\mu_i \le \lambda_i$ for all $i \ge 1$), we define a rational function $\psi_{\lambda/\mu}(q,t) \in \mathbb{Q}(q,t)$ by

$$\psi_{\lambda/\mu}(q,t) = \prod_{1\le i\le j\le \ell(\mu)} \frac{(t^{j-i+1}q^{\mu_i-\mu_j};q)_{\lambda_i-\mu_i}}{(t^{j-i}q^{\mu_i-\mu_j+1};q)_{\lambda_i-\mu_i}} \frac{(t^{j-i}q^{\mu_i-\lambda_{j+1}+1};q)_{\lambda_i-\mu_i}}{(t^{j-i+1}q^{\mu_i-\lambda_{j+1}};q)_{\lambda_i-\mu_i}}, \tag{6.6}$$

and set

$$\psi'_{\lambda/\mu}(q,t) = \psi_{\lambda'/\mu'}(t,q). \tag{6.7}$$

Recall that a skew diagram $\lambda/\mu$ is called a *horizontal strip* ("h-strip" for short) if the complement $\lambda\backslash\mu$ contains at most one square in each column. Similarly, we say that a skew diagram $\lambda/\mu$ is a *vertical strip* ("v-strip" for short) if the complement $\lambda\backslash\mu$ contains at most one square in each row. Note that $\psi_{\lambda/\mu}(q,t) = 0$ unless $\lambda/\mu$ is a horizontal strip, and that $\psi'_{\lambda/\mu}(q,t) = 0$ unless $\lambda/\mu$ is a vertical strip.

**Theorem 6.3** (Pieri formula) *For each $\mu \in \mathcal{P}_n$ and $r = 1, \ldots, n$, $P_\mu(x)$ multiplied by $e_r(x)$ is expanded in terms of Macdonald polynomials as*

$$e_r(x)P_\mu(x) = \sum_{\substack{\lambda/\mu:\, v\text{-}strip \\ |\lambda/\mu|=r}} \psi'_{\lambda/\mu}\, P_\lambda(x) \tag{6.8}$$

*with coefficients $\psi'_{\lambda/\mu} = \psi'_{\lambda/\mu}(q,t)$ in (6.6)–(6.7), where the sum is over all partitions $\lambda \in \mathcal{P}_n$ with $\mu \subseteq \lambda$, $|\lambda| = |\mu| + r$, such that the skew diagram $\lambda/\mu$ is a vertical strip.*

Theorem 6.3 will be proved in Sects. 6.2 and 6.3 before Sect. 6.4, assuming that Theorem 6.2 holds.

## 6.2 Self-duality Implies the Pieri Formula

Note that the fact that $P_\lambda(t^\delta) \neq 0$ (as a rational function of $(q,t)$) follows from the principal specialization of the special case $t = q$, where $P_\lambda(x|q,q) = s_\lambda(x)$. Assuming that the self-duality (6.4) has been established, we explain here how one can obtain the Pieri formula (6.8) and the evaluation formula (6.2) from the $q$-difference equations for $P_\lambda(x)$, by way of the self-duality.

For each $r=1,\dots,n$, the eigenfunction equation

$$D_x^{(r)}\widetilde{P}_\lambda(x)=e_r(t^\delta q^\lambda)\widetilde{P}_\lambda(x) \tag{6.9}$$

implies

$$\sum_{|I|=r}A_I(x)\widetilde{P}_\lambda(q^{\varepsilon_I}x)=e_r(t^\delta q^\lambda)\widetilde{P}_\lambda(x), \tag{6.10}$$

where $\varepsilon_I=\sum_{i\in I}\varepsilon_i$. Evaluating this formula at $x=t^\delta q^\mu$ ($\mu\in\mathcal{P}_n$), we obtain

$$\sum_{|I|=r}A_I(t^\delta q^\mu)\widetilde{P}_\lambda(t^\delta q^{\mu+\varepsilon_I})=e_r(t^\delta q^\lambda)\widetilde{P}_\lambda(t^\delta q^\mu). \tag{6.11}$$

Suppose that $\nu=\mu+\varepsilon_I$ is *not* a partition, i.e. $\mu_{i-1}=\mu_i$ for some $i\in\{2,\dots,n\}$ with $i\in I$ and $i-1\notin I$. In such a case, we have

$$A_I(t^\delta q^\mu)=t^{\binom{|I|}{2}}\prod_{i\in I,\,j\notin J}\frac{t^{n-i+1}q^{\mu_i}-t^{n-j}q^{\mu_j}}{t^{n-i}q^{\mu_i}-t^{n-j}q^{\mu_j}}=0 \tag{6.12}$$

since $tx_i-x_j=t^{n-i+1}q^{\mu_i}-t^{n-i+1}q^{\mu_{i-1}}=0$ ($i\in I,\ j=i-1\notin I$). This means that the sum in the left-hand side of (6.10) is over all $I\subseteq\{1,\dots,n\}$ with $|I|=r$ such that $\nu=\mu+\varepsilon_I$ is a partition. A skew partition $\nu/\mu$ is a *vertical strip* if and only if $\nu=\mu+\varepsilon_I$ for some $I\subseteq\{1,\dots,n\}$. In the following, for each pair $\mu,\nu\in\mathcal{P}_n$ with $\mu\subseteq\nu$, we set $A_{\nu/\mu}=A_I(t^\delta q^\mu)$ if $\nu/\mu$ is a vertical strip with $\nu=\mu+\varepsilon_I$ and $A_{\nu/\mu}=0$ otherwise. Then we have

$$\sum_{\nu/\mu:\text{ v-strip},\ |\nu/\mu|=r}A_{\nu/\mu}\widetilde{P}_\lambda(t^\delta q^\nu)=e_r(t^\delta q^\lambda)\widetilde{P}_\lambda(t^\delta q^\mu). \tag{6.13}$$

We now apply the self-duality (6.4) to obtain

$$\sum_{\nu/\mu:\text{ v-strip},\ |\nu/\mu|=r}A_{\nu/\mu}\widetilde{P}_\nu(t^\delta q^\lambda)=e_r(t^\delta q^\lambda)\widetilde{P}_\mu(t^\delta q^\lambda). \tag{6.14}$$

This means that equality

$$e_r(x)\widetilde{P}_\mu(x)=\sum_{\nu/\mu:\text{ v-strip},\ |\nu/\mu|=r}A_{\nu/\mu}\widetilde{P}_\nu(x) \tag{6.15}$$

holds for $x=t^\delta q^\lambda$ ($\lambda\in\mathcal{P}_n$). It also implies that (6.15) is an identity in the ring $\mathbb{C}[x]^{\mathfrak{S}_n}$ of symmetric polynomials, since a polynomial $f(x)\in\mathbb{C}[x]$ such that $f(t^\delta q^\lambda)=0$ for all $\lambda\in\mathcal{P}_n$ must be zero as a polynomial in $x$. Namely, if the self-duality (6.4) has been established, the $q$-difference equations (6.9) for $\lambda\in\mathcal{P}_n$ implies the *Pieri formulas* (6.15) for the normalized Macdonald polynomials $\widetilde{P}_\mu(x)$ with coefficients $A_{\nu/\mu}$.

**Exercise 6.1** Prove: if a polynomial $f(x) \in \mathbb{C}[x]$ vanishes at all points $x = t^{\delta} q^{\lambda}$ ($\lambda \in \mathcal{P}_n$), then $f(x) = 0$ as a polynomial in $x$.

Supposing that $\lambda/\mu$ is a vertical strip, we express $\lambda$ as $\lambda = \mu + \varepsilon_I$ with a subset $I \subseteq \{1, \ldots, n\}$ with $|I| = |\lambda/\mu| = r$. In this setting, we derive an explicit formula for the Pieri coefficient

$$A_{\lambda/\mu} = A_I(t^{\delta} q^{\mu}) \tag{6.16}$$

for $\widetilde{P}_{\mu}(x)$. Since $\Delta(x) = \prod_{b=1}^{n} x^{b-1} \prod_{1 \le a < b \le n}(1 - x_a/x_b)$, we have

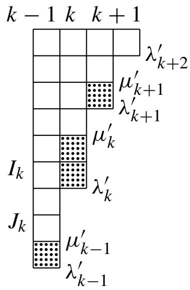
$$\begin{aligned} A_{\lambda/\mu} = A_I(t^{\delta} q^{\mu}) &= \frac{\Delta(t^{\delta+\varepsilon_I} q^{\mu})}{\Delta(t^{\delta} q^{\mu})} \\ &= t^{n(\varepsilon_I)} \prod_{\substack{1 \le a < b \le n \\ a \in I, b \notin I}} \frac{1 - t^{b-a+1} q^{\mu_a - \mu_b}}{1 - t^{b-a} q^{\mu_a - \mu_b}} \prod_{\substack{1 \le a < b \le n \\ a \notin I, b \in I}} \frac{1 - t^{b-a-1} q^{\mu_a - \mu_b}}{1 - t^{b-a} q^{\mu_a - \mu_b}}. \end{aligned} \tag{6.17}$$

We use the conjugate partitions $\lambda', \mu' \in \mathcal{P}$, noting that they satisfy the interlacing property

$$n \ge \lambda'_1 \ge \mu'_1 \ge \lambda'_2 \ge \mu'_2 \ge \lambda'_3 \ge \ldots. \tag{6.18}$$

Then, the subset $I \subseteq \{1, \ldots, n\}$ and its complement $J = \{1, \ldots, n\} \backslash I$ are parametrized as

$$\begin{aligned} I &= \bigsqcup_{k \ge 1} I_k, \quad I_k = (\mu'_k, \lambda'_k], \\ J &= \bigsqcup_{k \ge 1} J_k, \quad J_k = (\lambda'_k, \mu'_{k-1}] \\ &(\lambda'_0 = \mu'_0 = n) \end{aligned} \tag{6.19}$$




in the notation of an interval $(a, b] = \{k \in \mathbb{Z} \mid a < k \le b\}$ of integers. Note that, $\mu_i = k - 1, \lambda_i = k$ if $i \in I_k$ and $\mu_j = \lambda_j = k - 1$ if $j \in J_k$). Then we have

$$\begin{aligned} A_{\lambda/\mu} &= t^{n(\varepsilon_I)} \prod_{\substack{j \le i \\ a \in I_i \\ b \in J_j}} \frac{1 - t^{b-a+1} q^{i-j}}{1 - t^{b-a} q^{i-j}} \prod_{\substack{i < j \\ a \in J_j \\ b \in I_i}} \frac{1 - t^{b-a-1} q^{j-i}}{1 - t^{b-a} q^{j-i}} \\ &= t^{n(\varepsilon_I)} \prod_{j \le i} \frac{(t^{\mu'_{j-1} - \lambda'_i + 1} q^{i-j}; t)_{\lambda'_i - \mu'_i}}{(t^{\lambda'_j - \lambda'_i + 1} q^{i-j}; t)_{\lambda'_i - \mu'_i}} \prod_{i < j} \frac{(t^{\mu'_i - \mu'_{j-1}} q^{j-i}; t)_{\mu'_{j-1} - \lambda'_j}}{(t^{\lambda'_i - \mu'_{j-1}} q^{j-i}; t)_{\mu'_{j-1} - \lambda'_j}} \end{aligned} \tag{6.20}$$

and finally

$$A_{\lambda/\mu} = = t^{n(\varepsilon_I)} \prod_{j\geq 1} (t^{n-\lambda'_j+1} q^{j-1}; t)_{\lambda'_j-\mu'_j}$$
$$\cdot \frac{\prod_{i<j} (t^{\mu'_i-\lambda'_j+1} q^{j-i-1}; t)_{\lambda'_j-\mu'_j}}{\prod_{i\leq j} (t^{\lambda'_i-\lambda'_j+1} q^{j-i}; t)_{\lambda'_j-\mu'_j}} \frac{\prod_{i\leq j} (t^{\mu'_i-\mu'_j} q^{j-i+1}; t)_{\mu'_j-\lambda'_{j+1}}}{\prod_{i\leq j} (t^{\lambda'_i-\mu'_j} q^{j-i+1}; t)_{\mu'_j-\lambda'_{j+1}}}. \quad (6.21)$$

In combinatorial terms of Young diagrams, this can be written alternatively as

$$A_{\lambda/\mu} = \frac{t^{n(\lambda)} \prod_{i=1}^n (t^{n-i+1}; q)_{\lambda_i}}{t^{n(\mu)} \prod_{i=1}^n (t^{n-i+1}; q)_{\mu_i}}$$
$$\cdot \frac{\prod_{s\in\mu\cap R_{\lambda/\mu}} (1 - t^{l_\mu(s)+1} q^{a_\mu(s)})}{\prod_{s\in\lambda\cap R_{\lambda/\mu}} (1 - t^{l_\lambda(s)+1} q^{a_\lambda(s)})} \frac{\prod_{s\in\mu\setminus R_{\lambda/\mu}} (1 - t^{l_\mu(s)} q^{a_\mu(s)+1})}{\prod_{s\in\lambda\setminus R_{\lambda/\mu}} (1 - t^{l_\lambda(s)} q^{a_\lambda(s)+1})}, \quad (6.22)$$

where $R_{\lambda/\mu} = I \times \mathbb{Z}_{>0}$ denotes the union of rows intersecting with the vertical strip $\lambda/\mu$.

## 6.3 Principal Specialization: Evaluation at $x = t^\delta$

The normalized Macdonald polynomials $\widetilde{P}_\lambda(x)$ can be written as

$$\widetilde{P}_\lambda(x) = \frac{1}{a_\lambda} P_\lambda(x) = \frac{1}{a_\lambda} m_\lambda(x) + (\text{lower-order terms}), \quad a_\lambda = P_\lambda(t^\delta). \quad (6.23)$$

We compare the coefficients of $m_\lambda(x)$ of the both sides of (6.15) for $\lambda = \mu + \varpi_r$, $\varpi_r = \varepsilon_1 + \cdots + \varepsilon_r = (1^r)$. Then we obtain

$$\frac{1}{a_\mu} = A_{\lambda/\mu} \frac{1}{a_\lambda}, \quad \text{i.e.} \quad a_\lambda = a_\mu A_{\lambda/\mu} \quad (6.24)$$

for $\lambda = \mu + \varpi_r$.

We make use of this recurrence formula for the case where $\ell(\mu) \leq r$ and $\lambda = \mu + \varpi_r$. Since

$$A_{\lambda/\mu} = t^{\binom{r}{2}} \prod_{i=1}^r \frac{1 - t^{n-i+1} q^{\mu_i}}{1 - t^{r-i+1} q^{\mu_i}} = t^{n(\varpi_r)} \prod_{s\in\lambda\setminus\mu} \frac{1 - t^{n-l'_\lambda(s)} q^{a_\lambda(s)}}{1 - t^{l_\lambda(s)+1} q^{a_\lambda(s)}}, \quad (6.25)$$

by $a_\lambda = a_\mu A_{\lambda/\mu}$, we obtain

$$a_\lambda = a_\mu \cdot t^{n(\varpi_r)} \prod_{s\in\lambda\setminus\mu} \frac{1 - t^{n-l'_\lambda(s)} q^{a_\lambda(s)}}{1 - t^{l_\lambda(s)+1} q^{a_\lambda(s)}}, \quad (6.26)$$

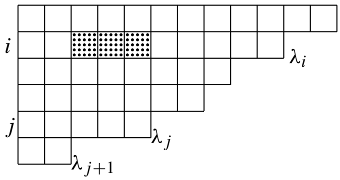
if $\ell(\mu) \le r$ and $\lambda = \mu + \varpi_r$. Noting that any $\lambda \in \mathcal{P}_n$ is expressed as $\lambda = \varpi_{\lambda'_1} + \cdots + \varpi_{\lambda'_l}$, $l = \lambda_1$, we can apply the recurrence formula (6.26) to obtain

$$a_\lambda = t^{n(\lambda)} \prod_{s\in\lambda} \frac{1 - t^{n-l'_\lambda(s)} q^{a_\lambda(s)}}{1 - t^{l_\lambda(s)+1} q^{a_\lambda(s)}} = \prod_{s\in\lambda} \frac{t^{l'_\lambda(s)} - t^n q^{a'_\lambda(s)}}{1 - t^{l_\lambda(s)+1} q^{a_\lambda(s)}} \tag{6.27}$$

for any $\lambda \in \mathcal{P}_n$. In terms of the components of $\lambda$, $a_\lambda$ is expressed alternatively as

$$a_\lambda = \frac{t^{n(\lambda)} \prod_{i=1}^n (t^{n-i+1}; q)_{\lambda_i}}{\prod_{1\le i\le j\le n} (t^{j-i+1} q^{\lambda_i - \lambda_j}; q)_{\lambda_j - \lambda_{j+1}}}. \tag{6.28}$$

Note that the pair $(i, j)$ of indices with $1 \le i \le j \le n$ in the denominator covers the sequence of squares $s = (i, k)$ with $k \in (\lambda_{j+1}, \lambda_j]$, for which $l_\lambda(s) = j - i$ and $a_\lambda(s) = \lambda_i - k$.




(6.29)

Formulas (6.26)–(6.27) are the explicit formulas for $P_\lambda(t^\delta) = a_\lambda$ in Theorem 6.1.

Also, the Pieri coefficients $\psi'_{\lambda/\mu}$ in (6.8) for $\lambda = \mu + \varepsilon_I$, $I \subseteq \{1, \dots, n\}$ are obtained from (6.15) by

$$\psi'_{\lambda/\mu} = \frac{a_\mu}{a_\lambda} A_{\lambda/\mu}, \quad A_{\lambda/\mu} = A_I(t^\delta q^\mu). \tag{6.30}$$

Writing down this formula in terms of $\lambda, \mu \in \mathcal{P}_n$, we obtain the explicit formula for $\psi'_{\lambda/\mu} = \psi'_{\lambda/\mu}(q, t)$ as in (6.6). By (6.22) and (6.27) we obtain

$$\begin{aligned}
\psi'_{\lambda/\mu} &= \frac{a_\mu}{a_\lambda} A_{\lambda/\mu} \\
&= \frac{\prod_{s\in\lambda}(1 - t^{l_\lambda(s)+1} q^{a_\lambda(s)})}{\prod_{s\in\mu}(1 - t^{l_\mu(s)+1} q^{a_\mu(s)})} \\
&\quad \cdot \frac{\prod_{s\in\mu\cap R_{\lambda/\mu}}(1 - t^{l_\mu(s)+1} q^{a_\mu(s)})}{\prod_{s\in\lambda\cap R_{\lambda/\mu}}(1 - t^{l_\lambda(s)+1} q^{a_\lambda(s)})} \frac{\prod_{s\in\mu\setminus R_{\lambda/\mu}}(1 - t^{l_\mu(s)} q^{a_\mu(s)+1})}{\prod_{s\in\lambda\setminus R_{\lambda/\mu}}(1 - t^{l_\lambda(s)} q^{a_\lambda(s)+1})} \\
&= \prod_{s\in\lambda\setminus R_{\lambda/\mu}} \frac{1 - t^{l_\lambda(s)+1} q^{a_\lambda(s)}}{1 - t^{l_\lambda(s)} q^{a_\lambda(s)+1}} \prod_{s\in\mu\setminus R_{\lambda/\mu}} \frac{1 - t^{l_\mu(s)} q^{a_\mu(s)+1}}{1 - t^{l_\mu(s)+1} q^{a_\mu(s)}}.
\end{aligned} \tag{6.31}$$

In terms of the components of $\lambda$, $\mu$, this formula can be rewritten as

$$\psi'_{\lambda/\mu} = \prod_{i\le j} \frac{(t^{\lambda'_i-\mu'_j+1}q^{j-i};t)_{\mu'_j-\lambda'_{j+1}}}{(t^{\lambda'_i-\mu'_j}q^{j-i+1};t)_{\mu'_j-\lambda'_{j+1}}} \frac{(t^{\mu'_i-\mu'_j}q^{j-i+1};t)_{\mu'_j-\lambda'_{j+1}}}{(t^{\mu'_i-\mu'_j+1}q^{j-i};t)_{\mu'_j-\lambda'_{j+1}}}$$
$$= \prod_{i\le j} \frac{(t^{\mu'_i-\lambda'_{j+1}+1}q^{j-i};t)_{\lambda'_i-\mu'_i}}{(t^{\mu'_i-\lambda'_{j+1}}q^{j-i+1};t)_{\lambda'_i-\mu'_i}} \frac{(t^{\mu'_i-\mu'_j}q^{j-i+1};t)_{\lambda'_i-\mu'_i}}{(t^{\mu'_i-\mu'_j+1}q^{j-i};t)_{\lambda'_i-\mu'_i}}. \tag{6.32}$$

This gives a proof of Theorem 6.3 (under the assumption that Theorem 6.2 holds). Note that the two expressions in (6.32) are transformed into each other through the formula

$$\frac{(q^l a;q)_k}{(a;q)_k} = \frac{(a;q)_{k+l}}{(a;q)_k(a;q)_l} = \frac{(q^k a;q)_l}{(a;q)_l} \qquad (k,l\in\mathbb{N}) \tag{6.33}$$

for $q$-shifted factorials.

## 6.4 Koornwinder's Proof of Self-duality

In this section, we present Koornwinder's inductive argument which proves the self-duality and the Pieri formula for $\widetilde{P}_\lambda(x)$ simultaneously (see Macdonald [20] and Koornwinder [14]).

For $\mu \in \mathcal{P}_n$ and $r = 0, 1, \ldots, n$, we consider the expansion of $e_r(x)\widetilde{P}_\mu(x)$ in terms of $\widetilde{P}_\lambda(x)$ ($\lambda \in \mathcal{P}_n$):

$$e_r(x)\widetilde{P}_\mu(x) = \sum_{\lambda\in\mathcal{P}_n,\,\lambda\le\mu+\varpi_r} B_{\lambda/\mu}\widetilde{P}_\lambda(x). \tag{6.34}$$

The coefficients $B_{\lambda/\mu}$ are defined for all $\lambda \in \mathcal{P}_n$ such that $\lambda \le \mu + \varpi_r$; we set $B_{\lambda/\mu} = 0$ otherwise. For each pair $\lambda, \mu \in \mathcal{P}_n$ with $\mu \subseteq \lambda$, we set $A_{\lambda/\mu} = A_I(t^\delta q^\mu)$ if $\lambda/\mu$ is a vertical strip with $\lambda = \mu + \varepsilon_I$, $I \subseteq \{1, \ldots, n\}$, and $A_{\lambda/\mu} = 0$ otherwise.

We prove the following two statements for $\lambda \in \mathcal{P}_n$ simultaneously by the induction on $|\lambda|$ combined with the dominance order of partitions:

(a)$_\lambda$ $\widetilde{P}_\lambda(t^\delta q^\mu) = \widetilde{P}_\mu(t^\delta q^\lambda)$ for all $\mu \in \mathcal{P}_n$.

(b)$_\lambda$ Suppose that $r \in \{1, \ldots, n\}$ and $\lambda - \varpi_r \in \mathcal{P}_n$, and set $\kappa = \lambda - \varpi_r$. Then, $B_{\nu/\kappa} = A_{\nu/\kappa}$ for any $\nu \in \mathcal{P}_n$ with $\nu \le \lambda = \kappa + \varpi_r$.

For the induction, we use the partial order $\nu \preceq_{\text{d-dom}} \mu$ for $\nu, \mu \in \mathcal{P}_n$ defined by

$$\nu \preceq_{\text{d-dom}} \mu \quad\Longleftrightarrow\quad |\nu| < |\mu| \ \text{ or } \ (|\nu| = |\mu| \text{ and } \nu \le \mu). \tag{6.35}$$

Statement (a)$_\mu$ holds for $\mu = 0$ since $\widetilde{P}_\lambda(t^\delta) = 1$ for all $\lambda \in \mathcal{P}_n$, while (b)$_\mu$ is empty for $\mu = 0$.

Assuming that $\lambda \in \mathcal{P}_n$ and $|\lambda| > 0$, we first prove (b)$_\lambda$. Suppose that $\kappa \in \mathcal{P}_n$, $r \in \{1, \dots, n\}$ and $\lambda = \kappa + \varpi_r$. By the argument of Sect. 6.2, (6.13), we know

$$e_r(t^\delta q^\mu)\widetilde{P}_\mu(t^\delta q^\kappa) = \sum_{\substack{\nu/\kappa:\text{ v-strip}\\ |\nu/\kappa|=r}} A_{\nu/\kappa}\widetilde{P}_\mu(t^\delta q^\nu) \qquad (\mu \in \mathcal{P}_n). \tag{6.36}$$

Note that we have $\nu \le \kappa + \varpi_r = \lambda$ if $\nu/\mu$ is a vertical strip with $|\nu/\kappa| = r$. On the other hand, we have

$$e_r(t^\delta q^\mu)\widetilde{P}_\kappa(t^\delta q^\mu) = \sum_{\nu\le\lambda} B_{\nu/\kappa}\widetilde{P}_\nu(t^\delta q^\mu) \qquad (\mu \in \mathcal{P}_n) \tag{6.37}$$

by (6.34). Since $|\kappa| < |\lambda|$, we have $\widetilde{P}_\kappa(t^\delta q^\mu) = \widetilde{P}_\mu(t^\delta q^\kappa)$ by the induction hypothesis. Also, we have $\widetilde{P}_\nu(t^\delta q^\mu) = \widetilde{P}_\mu(t^\delta q^\mu)$ for all pair $\mu, \nu \le \lambda$ by the induction hypothesis; in fact we have $\mu < \lambda$ or $\nu < \lambda$ if $\mu \ne \nu$. Hence we have

$$e_r(t^\delta q^\mu)\widetilde{P}_\mu(t^\delta q^\kappa) = \sum_{\nu\le\lambda} B_{\nu/\kappa}\widetilde{P}_\mu(t^\delta q^\nu) \qquad (\mu \in \mathcal{P}_n, \mu \le \lambda). \tag{6.38}$$

From (6.36) and (6.38), we obtain

$$\sum_{\substack{\nu/\kappa:\text{v-strip}\\ |\nu/\kappa|=r}} A_{\nu/\kappa}\widetilde{P}_\mu(t^\delta q^\nu) = \sum_{\nu\le\lambda} B_{\nu/\kappa}\widetilde{P}_\mu(t^\delta q^\nu) \qquad (\mu \in \mathcal{P}_n, \mu \le \lambda). \tag{6.39}$$

Then, statement (b)$_\lambda$ follows if we confirm that $\det\big(\widetilde{P}_\mu(t^\delta q^\nu)\big)_{\mu,\nu\le\lambda} \ne 0$, which will be proved below in Lemma 6.1.

Knowing that (b)$_\lambda$ holds, we can rewrite (6.37) as

$$e_r(t^\delta q^\mu)\widetilde{P}_\kappa(t^\delta q^\mu) = \sum_{\substack{\nu/\mu:\text{ v-strip}\\ |\nu/\mu|=r}} A_{\nu/\kappa}\widetilde{P}_\nu(t^\delta q^\mu) \qquad (\mu \in \mathcal{P}_n). \tag{6.40}$$

We now compare (6.36) and (6.40) for arbitrary $\mu \in \mathcal{P}_n$. Since $\widetilde{P}_\mu(t^\delta q^\kappa) = \widetilde{P}_\kappa(t^\delta q^\mu)$ and $\widetilde{P}_\mu(t^\delta q^\nu) = \widetilde{P}_\nu(t^\delta q^\mu)$ for any $\nu < \lambda = \kappa + \varpi_r$, we obtain

$$A_{\lambda/\kappa}\widetilde{P}_\mu(t^\delta q^\lambda) = A_{\lambda/\kappa}\widetilde{P}_\lambda(t^\delta q^\mu). \tag{6.41}$$

Since $A_{\lambda/\kappa} \ne 0$, we obtain $\widetilde{P}_\mu(t^\delta q^\lambda) = \widetilde{P}_\lambda(t^\delta q^\mu)$ for all $\mu \in \mathcal{P}_n$, as desired.

**Lemma 6.1** *For any* $\lambda \in \mathcal{P}_n$, $\det\big(\widetilde{P}_\mu(t^\delta q^\nu)\big)_{\mu,\nu\le\lambda} \ne 0$.

***Proof*** This statement is equivalent to $\det\big(P_\mu(t^\delta q^\nu)\big)_{\mu,\nu\le\lambda} \ne 0$ since $\widetilde{P}_\mu(x) = P_\mu(x)/a_\mu$, and further to $\det\big(m_\mu(t^\delta q^\nu)\big)_{\mu,\nu\le\lambda} \ne 0$ since $P_\mu(x) = m_\mu(x) +$ (lower order terms with respect to $\le$). Note that

$$m_\mu(t^\delta q^\nu) = t^{\langle \mu,\delta\rangle} q^{\langle \mu,\nu\rangle} + (\text{lower degree terms in}\, t), \tag{6.42}$$

and hence

$$\begin{aligned} &\det\left(m_\mu(t^\delta q^\nu)\right)_{\mu,\nu\le\lambda} \\ &= \det\left(t^{\langle\mu,\delta\rangle} q^{\langle\mu,\nu\rangle}\right)_{\mu,\nu\le\lambda} + (\text{lower degree terms in}\, t) \\ &= t^{\sum_{\mu\le\lambda}\langle\mu,\delta\rangle} \det\left(q^{\langle\mu,\nu\rangle}\right)_{\mu,\nu\le\lambda} + (\text{lower degree terms in}\, t). \end{aligned} \tag{6.43}$$

Setting $N = \#\{\mu \in \mathcal{P}_n \mid \mu \le \lambda\}$, parametrize all $\mu \in \mathcal{P}_n$ with $\mu \le \lambda$ as $\mu^{(1)}, \dots, \mu^{(N)}$. Then we have

$$\begin{aligned} \det\left(q^{\langle\mu,\nu\rangle}\right)_{\mu,\nu\le\lambda} &= \det\left(q^{\langle\mu^{(i)},\mu^{(j)}\rangle}\right)_{i,j=1}^N = \sum_{\sigma\in\mathfrak{S}_N} \operatorname{sgn}(\sigma) q^{\sum_{i=1}^N \langle\mu^{(i)},\mu^{(\sigma(i))}\rangle} \\ &= q^{\sum_{i=1}^N \langle\mu^{(i)},\mu^{(i)}\rangle} + (\text{lower degree terms in}\, q). \end{aligned} \tag{6.44}$$

In fact, for $\sigma \ne 1$, inequality $\sum_{i=1}^N \langle \mu^{(i)} - \mu^{(\sigma(i))}, \mu^{(i)} - \mu^{(\sigma(i))}\rangle > 0$ implies $\sum_{i=1}^N \langle\mu^{(i)}, \mu^{(i)}\rangle > \sum_{i=1}^N \langle\mu^{(i)}, \mu^{(\sigma(i))}\rangle$. Hence we have $\det\left(q^{\langle\mu,\nu\rangle}\right)_{\mu,\nu\le\lambda} \ne 0$. □

We remark that the self-duality of Theorem 6.2 can also be proved by means of the *Cherednik involution* of the double affine Hecke algebra (see Sect. 8.5).

## 6.5 Cauchy Formula and Dual Cauchy Formula

The Cauchy formula of Theorem 3.2 and the dual Cauchy formula of Theorem 3.4 for Schur functions can be generalized to the case of Macdonald polynomials.

**Theorem 6.4** (Cauchy formula) *For two sets of variables $x = (x_1, \dots, x_m)$ and $y = (y_1, \dots, y_n)$, we have*

$$\prod_{i=1}^m \prod_{j=1}^n \frac{(t x_i y_j; q)_\infty}{(x_i y_j; q)_\infty} = \sum_{\ell(\lambda)\le\min\{m,n\}} b_\lambda P_\lambda(x) P_\lambda(y), \tag{6.45}$$

*where $\lambda$ runs over all partitions with $\ell(\lambda) \le \min\{m,n\}$, and the coefficients $b_\lambda$ are given by*

$$b_\lambda = \prod_{s\in\lambda} \frac{1 - t^{l_\lambda(s)+1} q^{a_\lambda(s)}}{1 - t^{l_\lambda(s)} q^{a_\lambda(s)+1}} = \prod_{1\le i\le j\le\ell(\lambda)} \frac{(t^{j-i+1} q^{\lambda_i-\lambda_j}; q)_{\lambda_j-\lambda_{j+1}}}{(t^{j-i} q^{\lambda_i-\lambda_j+1}; q)_{\lambda_j-\lambda_{j+1}}}. \tag{6.46}$$

We remark that, when $q = t$, formula (6.45) reduces to the Cauchy formula

$$\prod_{i=1}^{m}\prod_{j=1}^{n}\frac{1}{1 - x_i y_j} = \sum_{\ell(\lambda)\le\min\{m,n\}} s_\lambda(x)s_\lambda(y), \tag{6.47}$$

with coefficients $b_\lambda = 1$. In Sect. 6.6, we give a proof of the fact that the left-hand side of (6.45) has an expansion formula of the form (6.45) for some constants $b_\lambda$ ($\lambda \in \mathcal{P}_n$); a derivation of the explicit formula (6.46) for $b_\lambda$ will be given in Sect. 7.3. In Macdonald's monograph [20], the notation $Q_\lambda(y) = b_\lambda P_\lambda(y)$ for the "dual" Macdonald polynomials is consistently used in view of their roles in duality arguments.

**Theorem 6.5** (Dual Cauchy formula) *For two sets of variables $x = (x_1, \ldots, x_m)$ and $y = (y_1, \ldots, y_n)$, we have*

$$\prod_{i=1}^{m}\prod_{j=1}^{n}(1 + x_i y_j) = \sum_{\lambda\subseteq(n^m)} P_\lambda(x; q, t)P_{\lambda'}(y; t, q), \tag{6.48}$$

$$\prod_{i=1}^{m}\prod_{j=1}^{n}(x_i + y_j) = \sum_{\lambda\subseteq(n^m)} P_\lambda(x; q, t)P_{\lambda^c}(y; t, q), \tag{6.49}$$

*where the sum is over all partitions $\lambda$ contained in the $m \times n$ rectangle $(n^m)$; $\lambda' = (\lambda'_1, \ldots, \lambda'_n)$ and $\lambda^c = (m - \lambda'_n, \ldots, m - \lambda'_1)$ denote the conjugate partition of $\lambda$ and the complementary partion of $\lambda$ in $(n^m)$ respectively.*

In what follows, we set

$$\Pi_{m,n}(x; y) = \prod_{i=1}^{m}\prod_{j=1}^{n}\frac{(tx_i y_j; q)_\infty}{(x_i y_j; q)_\infty} \tag{6.50}$$

and regard $\Pi_{m,n}(x; y)$ as a formal power series in $\mathbb{C}[\![x, y]\!]^{\mathfrak{S}_m\times\mathfrak{S}_n}$.[1] We also set

$$\Pi^\vee_{m,n}(x; y) = \prod_{i=1}^{m}\prod_{j=1}^{n}(1 + x_i y_j) \in \mathbb{C}[x, y]^{\mathfrak{S}_m\times\mathfrak{S}_n}. \tag{6.51}$$

It is sometimes more convenient to use the generating function

$$\Psi_{m,n}(x; y) = \prod_{i=1}^{m}\prod_{j=1}^{n}(x_i + y_j) \in \mathbb{C}[x, y]^{\mathfrak{S}_m\times\mathfrak{S}_n}. \tag{6.52}$$

---

[1] In fact, $\Pi_{m,n}(x; y)$ is a meromorphic function on $\mathbb{C}^m \times \mathbb{C}^n$ under our assumption $|q| < 1$. It is also holomorphic in the domain $|x_i y_j| < 1$ for $i \in \{1, \ldots, m\}$, $j \in \{1, \ldots, n\}$.

Since

$$\Psi_{m,n}(x;y) = (y_1 \cdots y_n)^m \prod_{i=1}^{m}\prod_{j=1}^{n}(1 + x_i y_j^{-1}) = (y_1 \dots y_n)^m \Pi^{\vee}_{m,n}(x; y^{-1}), \quad (6.53)$$

formula (6.48) is equivalent to

$$\prod_{i=1}^{m}\prod_{j=1}^{n}(x_i + y_j) = \sum_{\lambda \in (n^m)} P_\lambda(x; q, t)(y_1 \cdots y_n)^m P_{\lambda'}(y^{-1}; t, q). \quad (6.54)$$

By Proposition 5.1, for each partition $\lambda \subseteq (n^m)$ we have

$$(y_1 \cdots y_n)^m P_{\lambda'}(y^{-1}; t, q) = P_{\lambda^c}(y; t, q), \quad (6.55)$$

where

$$\lambda^c = ((m^n) - \lambda')^{\vee} = (m - \lambda'_n, \dots, m - \lambda'_1) \quad (6.56)$$

denotes the *complementary partition* of $\lambda$ in the $m \times n$ rectangle. Hence formula (6.48) is equivalent to (6.49). We give a proof of the dual Cauchy formula (6.49) in the second half of Sect. 6.6.

## 6.6 Kernel Identities

### *6.6.1 Kernel Identity for the Cauchy Formula*

We consider the case where $m = n$. We first remark that there exists an expansion formula as (6.45) with *some* constants $b_\lambda$, if and only if $\Pi(x; y) = \Pi_{n,n}(x; y)$ satisfies the *kernel identity*

$$D_x(u)\Pi(x; y) = D_y(u)\Pi(x; y). \quad (6.57)$$

Expand $\Pi(x; y)$ in terms of Macdonald polynomials $P_\lambda(x)$ ($\lambda \in \mathcal{P}_n$) as

$$\Pi(x; y) = \sum_{\lambda \in \mathcal{P}_n} P_\lambda(x) Q_\lambda(y), \qquad Q_\lambda(y) \in \mathbb{C}[y]^{\mathfrak{S}_n} \quad (\lambda \in \mathcal{P}_n). \quad (6.58)$$

Since

$$\begin{aligned} D_x(u)\Pi(x; y) &= \sum_{\lambda \in \mathcal{P}_n} P_\lambda(x) Q_\lambda(y) \prod_{i=1}^{n}(1 - u t^{n-j} q^{\lambda_j}), \\ D_y(u)\Pi(x; y) &= \sum_{\lambda \in \mathcal{P}_n} P_\lambda(x) D_y(u) Q_\lambda(y), \end{aligned} \quad (6.59)$$

identity (6.57) implies $D_y(u)Q_\lambda(y) = Q_\lambda(y)\prod_{i=1}^{n}(1-ut^{n-j}q^{\lambda_j})$ and hence, $Q_\lambda(x) = b_\lambda P_\lambda(x)$ for some $b_\lambda \in \mathbb{C}$.

**Proposition 6.1** *For two sets of variables* $x = (x_1, \ldots, x_n)$, $y = (y_1, \ldots, y_n)$, *the formal power series*

$$\Pi(x; y) = \prod_{i=1}^{n}\prod_{j=1}^{n} \frac{(tx_iy_j; q)_\infty}{(x_iy_j; q)_\infty} \in \mathbb{C}[\![x, y]\!]^{\mathfrak{S}_n \times \mathfrak{S}_n} \tag{6.60}$$

*satisfies the kernel identity*

$$D_x(u)\Pi(x; y) = D_y(u)\Pi(x; y). \tag{6.61}$$

***Proof*** Recall that

$$D_x(u) = \sum_{I\subseteq\{1,\ldots,n\}} (-u)^{|I|} t^{\binom{|I|}{2}} \prod_{i\in I,\, j\notin I} \frac{tx_i - x_j}{x_i - x_j} \prod_{i\in I} T_{q,x_i},$$
$$D_y(u) = \sum_{K\subseteq\{1,\ldots,n\}} (-u)^{|K|} t^{\binom{|K|}{2}} \prod_{k\in K,\, l\notin K} \frac{ty_k - y_l}{y_k - y_l} \prod_{k\in K} T_{q,y_k}. \tag{6.62}$$

Since

$$\prod_{i\in I} T_{q,x_i}\Pi(x; y) = \prod_{i\in I}\prod_{l=1}^{n} \frac{1 - x_iy_l}{1 - tx_iy_l} \cdot \Pi(x; y),$$
$$\prod_{k\in K} T_{q,y_k}\Pi(x; y) = \prod_{k\in K}\prod_{j=1}^{n} \frac{1 - x_jy_k}{1 - tx_jy_k} \cdot \Pi(x; y), \tag{6.63}$$

Equation (6.61) is equivalent to the *source identity*

$$\sum_{I\subseteq\{1,\ldots,n\}} (-u)^{|I|} t^{\binom{|I|}{2}} \prod_{i\in I;\, j\notin I} \frac{tx_i - x_j}{x_i - x_j} \prod_{i\in I}\prod_{l=1}^{n} \frac{1 - x_iy_l}{1 - tx_iy_l}$$
$$= \sum_{K\subseteq\{1,\ldots,n\}} (-u)^{|K|} t^{\binom{|K|}{2}} \prod_{k\in K;\, l\notin K} \frac{ty_k - y_l}{y_k - y_l} \prod_{k\in K}\prod_{j=1}^{n} \frac{1 - x_jy_k}{1 - tx_jy_k}. \tag{6.64}$$

An important observation is that this identity does *not* involve $q$. This means that, in order to prove (6.64), it is sufficient to prove (6.61) for $q = t$. However, we already know that (6.61) holds when $q = t$ by the Cauchy formula for Schur functions. □

The existence of an expansion formula of the form (6.45) for difference number of variables $m$, $n$ follows from the stability of Macdonald polynomials as in Exercise 4.2. Also, for a given partition $\lambda \in \mathcal{P}$, the coefficient $b_\lambda$ of $P_\lambda(x)P_\lambda(y)$ in

(6.45) is determined independently of the choice of $m, n$ such that $m \geq \ell(\lambda)$, $n \geq \ell(\lambda)$.

It should be noted that we need some other arguments to obtain the explicit formula (6.46) for $b_\lambda$; a proof of (6.46) will be given in Sect. 7.3, on the basis of compatibility of the Cauchy and the dual Cauchy formula for Macdonald polynomials.

In the setting of Theorem 6.4, suppose that $m \geq n$. Then for each $\lambda \in \mathcal{P}_n$, we have

$$D_x(u)P_\lambda(x) = P_\lambda(x)\prod_{i=1}^{n}(1-ut^{m-i}q^{\lambda_i})\prod_{i=n+1}^{m}(1-ut^{m-i})$$
$$D_y(v)P_\lambda(y) = P_\lambda(y)\prod_{i=1}^{n}(1-vt^{n-i}q^{\lambda_i}). \tag{6.65}$$

We also remark that (6.45) for the case where $m \geq n$ corresponds to the kernel identity

$$D_x(u)\Pi_{m,n}(x;y) = (u;t)_{m-n}D_y(ut^{m-n})\Pi_{m,n}(x;y). \tag{6.66}$$

**Remark 6.1** We have used here the kernel identity for $\Pi_{m,n}(x;y)$ to prove the Cauchy formula for Macdonald polynomials. Another important application of the kernel identity is the integral transform of the form

$$\varphi(x) = \frac{1}{(2\pi\sqrt{-1})^n}\int_{\mathbb{T}^n}\Pi_{m,n}(x;y)\psi(y)w(y)\frac{dy_1\cdots dy_n}{y_1\cdots y_n}. \tag{6.67}$$

It transforms joint eigenfunctions $\psi(y)$ of the Macdonald–Ruijsenaars operators $D_y(v)$ in $y$ variables to joint eigenfunctions $\varphi(x)$ of $D_x(u)$ in $x$ variables. This property is a consequence of the kernel identity for $\Pi_{m,n}(x;y)$ combined with the self-adjointness of $D_y(v)$ with respect to the weight function $w(y)$.

### *6.6.2 Kernel Identity for the Dual Cauchy Formula*

Here we give a proof of formula (6.49) which is equivalent to (6.48), on the basis of a relevant kernel identity.

For two sets of variables $x = (x_1, \ldots, x_m)$ and $y = (y_1, \ldots, y_n)$, we set

$$P_\mu(x) = P_\mu(x;q,t) \ \ (\mu \in \mathcal{P}_m), \quad P^\circ_\nu(y) = P_\nu(y;t,q) \ \ (\nu \in \mathcal{P}_n). \tag{6.68}$$

We also denote by

$$D^\circ_y = \sum_{k=1}^{n}\prod_{1\leq l\leq n;\ l\neq k}\frac{qy_k - y_l}{y_k - y_l}T_{t,y_k} \tag{6.69}$$

the $t$-difference operator obtained from $D_y$ by exchanging $q$ and $t$.

Note that the polynomial

$$\Psi_{m,n}(x;y) = \prod_{i=1}^{m}\prod_{j=1}^{n}(x_i + y_j) \in \mathbb{C}[x,y]^{\mathfrak{S}_m\times\mathfrak{S}_n} \tag{6.70}$$

is of degree $mn$ in $(x, y)$, and symmetric both in $x$ and in $y$. Since $\Psi_{m,n}(x; y)$ is of degree $\le n$ in each $x_i$ and of degree $\le m$ in each $y_j$, it can be expressed as

$$\Psi_{m,n}(x;y) = \sum_{\mu\subseteq(n^m);\ \nu\subseteq(m^n)} c_{\mu,\nu} P_\mu(x) P^\circ_\nu(y) \tag{6.71}$$

with some constants $c_{\mu,\nu}$. For each partition $\mu \subseteq (n^m)$, we defined the *complementary partition* $\mu^c$ in the $m \times n$ rectangle by $\mu^c = (m - \mu'_n, n - \mu'_{n-1}, \ldots, m - \mu'_1)$ (see the figure in (3.82)). In this setting, we show that $c_{\mu,\nu} = 0$ unless $\nu = \mu^c$, namely

$$\Psi_{m,n}(x;y) = \prod_{i=1}^{m}\prod_{j=1}^{n}(x_i + y_j) = \sum_{\lambda\subseteq(n^m)} c_\lambda P_\lambda(x) P^\circ_{\lambda^c}(y) \tag{6.72}$$

for some constants $c_\lambda \in \mathbb{C}$.

In the eigenfunction equation

$$D_x P_\lambda(x) = d_\lambda P_\lambda(x), \quad d_\lambda = \sum_{i=1}^{m} t^{m-i} q^{\lambda_i}, \tag{6.73}$$

the eigenvalue $d_\lambda$ has the following combinatorial meaning:

$$\begin{aligned}\frac{1}{q-1}\Big(d_\lambda - \frac{t^m - 1}{t-1}\Big) &= \frac{1}{q-1}\sum_{i=1}^{m} t^{m-i}(q^{\lambda_i} - 1)\\ &= \sum_{i=1}^{m}\sum_{j=1}^{\lambda_i} t^{m-i} q^{j-1} = \sum_{s=(i,j)\in D(\lambda)} t^{m-i} q^{j-1}.\end{aligned} \tag{6.74}$$

Similarly, as for the eigenvalue $d^\circ_{\lambda^c}$ in the equation

$$D^\circ_y P^\circ_{\lambda^c}(y) = d^\circ_{\lambda^c} P^\circ_{\lambda^c}(y), \quad d^\circ_{\lambda^c} = \sum_{j=1}^{n} q^{n-j} t^{\lambda^c_j}, \tag{6.75}$$

we have

$$\frac{1}{t-1}\Big(d^\circ_{\lambda^c} - \frac{q^n - 1}{q-1}\Big) = \sum_{j=1}^{n}\sum_{i=1}^{\lambda^c_i} t^{i-1} q^{n-j} = \sum_{s=(i,j)\in D(\lambda)^c} t^{m-i} q^{j-1}, \tag{6.76}$$

where $D(\lambda)^{\mathrm{c}}$ stands for the complement of $D(\lambda)$ in the rectangle $\{1,\ldots,m\}\times\{1,\ldots,n\}$. Since

$$\sum_{s=(i,j)\in(n^m)} t^{m-i}q^{j-1}=\frac{t^m-1}{t-1}\frac{q^n-1}{q-1}, \tag{6.77}$$

the existence of a formula in the form (6.72) is equivalent to

$$\begin{aligned}\Big(\frac{1}{q-1}\Big(D_x-\frac{t^m-1}{t-1}\Big)+\frac{1}{t-1}\Big(D_y^{\circ}-\frac{q^n-1}{q-1}\Big)\Big)\Psi_{m,n}(x;y)\\ =\frac{t^m-1}{t-1}\frac{q^n-1}{q-1}\Psi_{m,n}(x;y),\end{aligned} \tag{6.78}$$

namely,

$$\Big(\frac{1}{q-1}D_x+\frac{1}{t-1}D_y^{\circ}\Big)\Psi_{m,n}(x;y)=\frac{t^mq^n-1}{(t-1)(q-1)}\Psi_{m,n}(x;y). \tag{6.79}$$

**Proposition 6.2** *For two sets of variables $x=(x_1,\ldots,x_m)$ and $y=(y_1,\ldots,y_n)$, the polynomial*

$$\Psi_{m,n}(x;y)=\prod_{i=1}^{m}\prod_{j=1}^{n}(x_i+y_j)\in\mathbb{C}[x,y]^{\mathfrak{S}_m\times\mathfrak{S}_n} \tag{6.80}$$

*satisfies the kernel identity*

$$\Big(\frac{1}{q-1}D_x+\frac{1}{t-1}D_y^{\circ}\Big)\Psi_{m,n}(x;y)=\frac{t^mq^n-1}{(t-1)(q-1)}\Psi_{m,n}(x;y). \tag{6.81}$$

***Proof*** This kernel identity is equivalent to the following identity of rational functions:

$$\begin{aligned}&\frac{1}{q-1}\sum_{i=1}^{m}\prod_{j\neq i}\frac{tx_i-x_j}{x_i-x_j}\prod_{l=1}^{n}\frac{qx_i+y_l}{x_i+y_l}\\ &+\frac{1}{t-1}\sum_{k=1}^{n}\prod_{l\neq k}\frac{qy_k-y_l}{y_k-y_l}\prod_{j=1}^{m}\frac{x_j+ty_k}{x_j+y_k}=\frac{t^mq^n-1}{(t-1)(q-1)},\end{aligned} \tag{6.82}$$

which can be verified directly by the residue calculus combined with induction on the number of variables. In fact, equality (6.82) for $n=0$ is the same as (4.2). When $n>0$, we regard the left-hand side as a rational function of $y_n$. Then, we see that the residues at $y_n=y_k$ $(k=1,\ldots,n-1)$ and at $y_n=-x_i$ $(i=1,\ldots,m)$ are all zero. We can also verify that the limit as $y_n\to 0$ gives the value of the right-hand side, by using the induction hypothesis of the case $(m,n-1)$. □

Finally, we show that $c_\lambda = 1$ for all $\lambda \subseteq (n^m)$. We denote by $\mathcal{A}_{m,n}$ the set of all $m \times n$ integer matrices $A = (a_{ij})_{1 \le i \le m;\ 1 \le j \le n}$ such that $a_{ij} \in \{0, 1\}$ for all $i, j$. Also, for a pair of multi-indices $(\mu, \nu) \in \mathbb{N}^m \times \mathbb{N}^n$, we denote by $\mathcal{A}_{\mu,\nu}$ the set of all $A = (a_{ij}) \in \mathcal{A}_{m,n}$ such that

$$\sum_{j=1}^{n} a_{ij} = \mu_i \quad (i = 1, \dots, m), \quad \sum_{i=1}^{m} a_{ij} = \nu_j \quad (j = 1, \dots, n). \tag{6.83}$$

Then $\Psi_{m,n}(x; y)$ can be expanded as follows:

$$\begin{aligned}
\Psi_{m,n}(x; y) &= \sum_{A=(a_{ij}) \in \mathcal{A}_{m,n}} \prod_{i=1}^{m} \prod_{j=1}^{n} (x_i^{a_{ij}} y_j^{1-a_{ij}}) \\
&= \sum_{\mu \in \mathbb{N}^m, \nu \in \mathbb{N}^n} (\#\mathcal{A}_{\mu,\nu})\, x^\mu y^{(m^n)-\nu} = \sum_{\mu \in \mathbb{N}^m, \nu \in \mathbb{N}^n} (\#\mathcal{A}_{\mu,(m^n)-\nu})\, x^\mu y^\nu \\
&= \sum_{\mu,\nu \subseteq (n^m)} (\#\mathcal{A}_{\mu,(m^n)-\nu^{\mathrm{c}}})\, m_\mu(x)\, m_{\nu^{\mathrm{c}}}(y).
\end{aligned} \tag{6.84}$$

Since $(m^n) - \nu^{\mathrm{c}} = (\nu'_n, \dots, \nu'_1)$ is the reversal of $\nu' = (\nu'_1, \dots, \nu'_n)$, we obtain

$$\Psi_{m,n}(x; y) = \sum_{\mu,\nu \subseteq (n^m)} (\#\mathcal{A}_{\mu,\nu'})\, m_\mu(x)\, m_{\nu^{\mathrm{c}}}(y). \tag{6.85}$$

We now look at the coefficients of $m_\mu(x) m_{\mu^{\mathrm{c}}}(y)$ for partitions $\mu \subseteq (n^m)$.

**Lemma 6.2** *For each partition $\mu \subseteq (n^m)$, $\#\mathcal{A}_{\mu,\mu'} = 1$.*

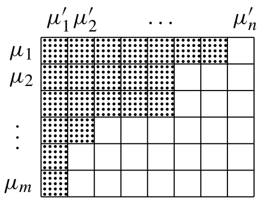
***Proof*** Define $A = (a_{ij})_{1 \le i \le m, 1 \le j \le n}$ by

$$a_{ij} = 1 \quad (1 \le j \le \mu_i), \qquad a_{ij} = 0 \quad (\mu_i < j \le n) \tag{6.86}$$

for all $i \in \{1, \dots, m\}$, so that

$$\left\{ s = (i, j) \in \{1, \dots, m\} \times \{1, \dots, n\} \mid a_{ij} = 1 \right\} = D(\mu). \tag{6.87}$$

Then, one can verify that this matrix $A$ it is the only element of $\mathcal{A}_{\mu,\mu'}$. □




(6.88)

This lemma implies that, for each partition $\mu \subseteq (n^m)$, the coefficient of $x^\mu y^{\mu^c}$ in the expansion of $\Psi_{m,n}(x; y)$ is precisely 1. In the right-hand side of (6.72), the monomial $x^\mu y^{\mu^c}$ arises only if there exists a partition $\lambda \in (n^m)$ such that $\mu \leq \lambda$ and $\mu^c \leq \lambda^c$. One can directly verify that the condition $\mu^c \leq \lambda^c$ implies $\mu' \leq \lambda'$, and hence $\mu \geq \lambda$. Together with $\mu \leq \lambda$, we obtain $\lambda = \mu$. This implies that the monomial $x^\mu y^{\mu^c}$ arises only from the term $P_\mu(x)P^\circ_{\mu^c}(y)$. This also means that the coefficient of $x^\mu y^{\mu^c}$ on the right-hand side is given by $c_\mu$. Hence we have $c_\mu = 1$ for all partitions $\mu \subseteq (n^m)$. This completes the proof of the dual Cauchy formula (6.49) of Theorem 6.5, and also the proof of (6.48).

# Chapter 7
# Littlewood–Richardson Coefficients and Branching Coefficients

**Abstract** The Littlewood–Richardson coefficients are the structure constants for the multiplication of Macdonald polynomials. On the other hand, the branching coefficients describe the expansion of Macdonald polynomials by products of Macdonald polynomials in smaller dimensions. We explain here that these two types of coefficients are intimately related to each other through the Cauchy formula for Macdonald polynomials. We also present a commuting family of $q$-difference operators of row type for which Macdonald polynomials are joint eigenfunctions, and explain how they are related to the Pieri formula of row type.

## 7.1 Littlewood–Richardson Coefficients and Branching Coefficients

For a pair of partitions $\mu, \nu \in \mathcal{P}_n$, one can expand the product $P_\mu(x)P_\nu(x)$ of Macdonald polynomials as a linear combination of Macdonald polynomials $P_\lambda(x)$ ($\lambda \in \mathcal{P}_n$):

$$P_\mu(x)P_\nu(x) = \sum_{\lambda\in\mathcal{P}_n;\ |\lambda|=|\mu|+|\nu|} c^\lambda_{\mu,\nu}\, P_\lambda(x) \tag{7.1}$$

with coefficients $c^\lambda_{\mu,\nu} = c^\lambda_{\mu,\nu}(q,t) \in \mathbb{Q}(q,t)$. These expansion coefficients $c^\lambda_{\mu,\nu}$ are called the *Littlewood–Richardson coefficients* (or *Clebsch–Gordan coefficients*). We remark that the coefficient $c^\lambda_{\mu,\nu} \in \mathbb{Q}(q,t)$ for $\lambda, \mu, \nu \in \mathcal{P}$ does not depend on the choice of $n$ such that $\lambda, \mu, \nu \in \mathcal{P}_n$, thanks to the stability of Macdonald polynomials of Exercise 4.2 with respect to the number of variables.

If $\nu = (1^r)$ is a single column ($r = 0, 1, 2, \ldots$), the coefficients $c^\lambda_{\mu,(1^r)}$ are nothing but the Pieri coefficients $\psi'_{\lambda/\mu} = \psi_{\lambda'/\mu'}(t,q)$ of Theorem 6.3 since $P_{(1^r)}(x) = e_r(x)$:

$$P_\mu(x)e_r(x) = \sum_{\substack{\lambda\supseteq\mu;\ |\lambda/\mu|=r \\ \lambda/\mu:\ \text{v-strip}}} \psi'_{\lambda/\mu}\, P_\lambda(x) \quad (r = 0, 1, \ldots, n), \tag{7.2}$$



M. Noumi, *Macdonald Polynomials*, SpringerBriefs in Mathematical Physics,
https://doi.org/10.1007/978-981-99-4587-0_7

where the sum is over the *vertical strips* $\lambda/\mu$ with $|\lambda/\mu| = r$. Namely, $c^{\lambda}_{\mu,(1^r)} = \psi_{\lambda'/\mu'}(t,q)$ if $\lambda/\mu$ is a vertical strip with $|\lambda/\mu| = r$, and $c^{\lambda}_{\mu,(1^r)} = 0$ otherwise.

These Littlewood–Richardson coefficients $c^{\lambda}_{\mu,\nu}$ are closely related to the *branching coefficients* $b^{\lambda}_{\mu,\nu}$ to be defined below. We expand the Macdonald polynomials $P_\lambda(x, y)$ ($\lambda \in \mathcal{P}_{m+n}$) in $m+n$ variables $(x, y) = (x_1, \ldots, x_m, y_1, \ldots, y_n)$ in terms of the Macdonald polynomials $P_\mu(x)$ of $m$ variables $x$ and $P_\nu(y)$ of $n$ variables y:

$$P_\lambda(x, y) = \sum_{\mu\in\mathcal{P}_m,\, \nu\in\mathcal{P}_n} b^{\lambda}_{\mu,\nu}\, P_\mu(x) P_\nu(y) \tag{7.3}$$

with some coefficients $b^{\lambda}_{\mu,\nu} = b^{\lambda}_{\mu,\nu}(q,t) \in \mathbb{Q}(q,t)$. The expansion coefficients $b^{\lambda}_{\mu,\nu}$ are called the *branching coefficients*. Note that $b^{\lambda}_{\mu,\nu} = 0$ unless $|\lambda| = |\mu| + |\nu|$. We also remark that the branching coefficient $b^{\lambda}_{\mu,\nu} \in \mathbb{Q}(q,t)$ for $\lambda, \mu, \nu \in \mathcal{P}$ does not depend on the choice of $m, n$ as far as $\mu \in \mathcal{P}_m$, $\nu \in \mathcal{P}_n$ and $\lambda \in \mathcal{P}_{m+n}$.

When $n = 1$, the branching coefficients are expressed by the Pieri coefficients $\psi_{\lambda/\mu}(q,t)$ defined in (6.6).

**Theorem 7.1** *For each $\lambda \in \mathcal{P}_n$, the Macdonald polynomial $P_\lambda(x)$ in $n$ variables $x = (x_1, \ldots, x_n)$ is expanded in the form*

$$P_\lambda(x) = \sum_{\substack{\mu\subseteq\lambda \\ |\lambda/\mu|:\,\text{h-strip}}} \psi_{\lambda/\mu}\, P_\mu(x')\, x_n^{|\lambda/\mu|}, \quad x' = (x_1, \ldots, x_{n-1}), \tag{7.4}$$

*in terms of the Macdonald polynomials $P_\mu(x')$ in $n-1$ variables and $x_n$, where the sum is over all horizontal strips $\lambda/\mu$ with $\ell(\mu) \le n-1$ and the coefficients $\psi_{\lambda/\mu}$ are given by $\psi_{\lambda/\mu} = \psi_{\lambda/\mu}(q,t)$.*

In terms of the branching coefficients, this means that $b^{\lambda}_{\mu,(l)} = \psi_{\lambda/\mu}(q,t)$ if $\lambda/\mu$ is a horizontal strip with $|\lambda/\mu| = l$, and $b^{\lambda}_{\mu,(l)} = 0$ otherwise. Theorem 7.1 will be proved in the next section. Note also that Theorem 7.1 is a generalization of Theorem 3.3 of recurrence for Schur functions.

**Remark 7.1** As we explained in Sect. 3.9, for each partition $\lambda \in \mathcal{P}_n$, there exists an irreducible polynomial representation of $\mathrm{GL}_n = \mathrm{GL}_n(\mathbb{C})$ with highest weight $\lambda$, which we denote by $V_n(\lambda)$; the character of $V_n(\lambda)$ is the Schur function $s_\lambda(x)$ in $x = (x_1, \ldots, x_n)$. In this context where $t = q$, the Littlewood–Richard coefficients $c^{\lambda}_{\mu,\nu}$ ($\lambda, \mu, \nu \in \mathcal{P}_n$) are non-negative integers, and they represent the multiplicities of $V_n(\lambda)$ in the irreducible decomposition of the tensor product representation $V_n(\mu) \otimes V_n(\nu)$. In fact, for any pair $\mu, \nu \in \mathcal{P}_n$, we have an isomorphism

$$V_n(\mu) \otimes V_n(\nu) \simeq \bigoplus_{\lambda\in\mathcal{P}_n} V_n(\lambda)^{\oplus c^{\lambda}_{\mu,\nu}} \tag{7.5}$$

of $\mathrm{GL}_n$-modules. On the other hand, for $\lambda \in \mathcal{P}_{m+n}$, $\mu \in \mathcal{P}_m$, $\nu \in \mathcal{P}_n$, the branching coefficients $b^{\lambda}_{\mu,\nu}$ are non-negative integers, and they represent the multiplicity of $V_m(\mu) \otimes V_n(\nu)$ in the restriction of $V_{m+n}(\lambda)$ from $\mathrm{GL}_{m+n}$ to the subgroup $\mathrm{GL}_m \times \mathrm{GL}_n$. In fact, for each $\lambda \in \mathcal{P}_{m+n}$, we have an isomorphism

$$\mathrm{Res}\!\downarrow^{\mathrm{GL}_{m+n}}_{\mathrm{GL}_m \times \mathrm{GL}_n} (V_{m+n}(\lambda)) \simeq \bigoplus_{\mu \in \mathcal{P}_m,\, \nu \in \mathcal{P}_n} \big(V_m(\mu) \otimes V_n(\nu)\big)^{\oplus b^{\lambda}_{\mu,\nu}} \tag{7.6}$$

of $\mathrm{GL}_m \times \mathrm{GL}_n$-modules.

## 7.2 Relation Between $c^{\lambda}_{\mu,\nu}$ and $b^{\lambda}_{\mu,\nu}$

Recall that the Macdonald polynomials have the kernel functions of Cauchy type, and of dual Cauchy type (the Cauchy formula and the dual Cauchy formula): For two sets of variables $z = (z_1, \dots, z_M)$ and $w = (w_1, \dots, w_N)$,

$$\begin{aligned} \Pi_{M,N}(z; w) &= \prod_{k=1}^{M} \prod_{l=1}^{N} \frac{(t z_k w_l; q)_\infty}{(z_k w_l; q)_\infty} = \sum_{\ell(\lambda) \le \min\{M,N\}} b_\lambda P_\lambda(z) P_\lambda(w), \\ \Pi^{\vee}_{M,N}(z; w) &= \prod_{k=1}^{M} \prod_{l=1}^{N} (1 + z_k w_l) = \sum_{\lambda \subseteq (N^M)} P_\lambda(z) P^{\circ}_{\lambda'}(w), \end{aligned} \tag{7.7}$$

where $P^{\circ}_{\lambda'}(w) = P_{\lambda'}(w; t, q)$. In what follows, we denote by $\cdot^{\circ} : \mathbb{Q}(q,t) \to \mathbb{Q}(q,t)$ the involutive automorphism such that $q^{\circ} = t$ and $t^{\circ} = q$.

**Theorem 7.2** *Let $\mu \in \mathcal{P}_m$, $\nu \in \mathcal{P}_n$ and $\lambda \in \mathcal{P}_{m+n}$. Then we have*

$$(1)\ \ b_\lambda b^{\lambda}_{\mu,\nu} = b_\mu b_\nu c^{\lambda}_{\mu,\nu}, \qquad (2)\ \ b^{\lambda}_{\mu,\nu} = (c^{\lambda'}_{\mu',\nu'})^{\circ}. \tag{7.8}$$

***Proof*** (1) Setting $M = m + n$ in (7.7), we suppose that $N \ge M$. Then we have

$$\begin{aligned} \Pi_{m+n,N}(x, y; w) &= \sum_{\lambda \in \mathcal{P}_N} b_\lambda P_\lambda(x, y) P_\lambda(w) \\ &= \sum_{\lambda \in \mathcal{P}_N} \sum_{\mu \in \mathcal{P}_m,\, \nu \in \mathcal{P}_n} b_\lambda b^{\lambda}_{\mu,\nu} P_\mu(x) P_\nu(y) P_\lambda(w). \end{aligned} \tag{7.9}$$

On the other hand,

$$\begin{aligned}\Pi_{m,N}(x;w)\Pi_{n,N}(y;w) &= \sum_{\mu\in\mathcal{P}_m} b_\mu P_\mu(x)P_\mu(w) \sum_{\nu\in\mathcal{P}_n} b_\nu P_\nu(y)P_\nu(w) \\ &= \sum_{\mu\in\mathcal{P}_m,\,\nu\in\mathcal{P}_n} b_\mu b_\nu P_\mu(x)P_\nu(y)P_\mu(w)P_\nu(w) \\ &= \sum_{\mu\in\mathcal{P}_m,\,\nu\in\mathcal{P}_n} b_\mu b_\nu P_\mu(x)P_\nu(y) \sum_{\lambda\in\mathcal{P}_N} c^\lambda_{\mu,\nu} P_\lambda(w) \\ &= \sum_{\mu\in\mathcal{P}_m,\,\nu\in\mathcal{P}_n} \sum_{\lambda\in\mathcal{P}_N} b_\mu b_\nu c^\lambda_{\mu,\nu} P_\mu(x)P_\nu(y)P_\lambda(w).\end{aligned} \tag{7.10}$$

Since $\Pi_{m+n,N}(x,y;w) = \Pi_{m,N}(x;w)\Pi_{n,N}(y;w)$, we obtain $b_\lambda b^\lambda_{\mu,\nu} = b_\mu b_\nu c^\lambda_{\mu,\nu}$.
(2) Setting $M = m+n$ in (7.7), we have

$$\begin{aligned}\Pi^\vee_{m+n,N}(x,y;w) &= \sum_{\lambda\subseteq(N^{m+n})} P_\lambda(x,y)P^\circ_{\lambda'}(w) \\ &= \sum_{\lambda'\in\mathcal{P}_N} \sum_{\mu\in\mathcal{P}_m,\,\nu\in\mathcal{P}_n} b^\lambda_{\mu,\nu} P_\mu(x)P_\nu(y)P^\circ_{\lambda'}(w)\end{aligned} \tag{7.11}$$

where $\circ$ denotes the operation of exchanging the parameters $q$ and $t$. On the other hand,

$$\begin{aligned}\Pi^\vee_{m,N}(x;w)\,\Pi^\vee_{n,N}(y;w) &= \sum_{\mu\subseteq(N^m)} P_\mu(x)P^\circ_{\mu'}(w) \sum_{\nu\subseteq(N^n)} P_\nu(y)P^\circ_{\nu'}(w) \\ &= \sum_{\mu\subseteq(N^m),\,\nu\subseteq(N^n)} P_\mu(x)P_\nu(y)P^\circ_{\mu'}(w)P^\circ_{\nu'}(w) \\ &= \sum_{\mu\subseteq(N^m),\,\nu\subseteq(N^n)} \sum_{\lambda'\subseteq(N^{m+n})} \left(c^{\lambda'}_{\mu',\nu'}\right)^\circ P_\mu(x)P_\nu(y)P^\circ_{\lambda'}(w).\end{aligned} \tag{7.12}$$

Since $\Pi^\vee_{m+n,N}(x,y;w) = \Pi^\vee_{m,N}(x;w)\Pi^\vee_{n,N}(y;w)$, we obtain $b^\lambda_{\mu,\nu} = \left(c^{\lambda'}_{\mu',\nu'}\right)^\circ$. □

***Proof (of Theorem* 7.1*)*** By the Pieri rule (7.2), for $\lambda, \mu \in \mathcal{P}$ and $k \geq 0$ we have

$$c^\lambda_{\mu,(1^k)} = \psi'_{\lambda/\mu} = \psi_{\lambda'/\mu'}(t,q) = \psi^\circ_{\lambda'/\mu'} \tag{7.13}$$

if $\lambda/\mu$ is a vertical strip with $|\lambda/\mu| = k$, and $c^\lambda_{\mu,(1^r)} = 0$ otherwise, Then, by Theorem 7.2, we obtain

$$b^\lambda_{\mu,(k)} = (c^{\lambda'}_{\mu',(1^k)})^\circ = \psi_{\lambda/\mu} = \psi_{\lambda/\mu}(q,t) \tag{7.14}$$

if $\lambda/\mu$ is a horizontal strip with $|\lambda/\mu| = k$, and $b^\lambda_{\mu,(k)} = 0$ otherwise. This implies the branching rule of (7.4). □

## 7.3 Explicit Formula for $b_\lambda$

In this section, we derive the explicit formula (6.46) for the coefficients $b_\lambda$ from the compatibility of the Cauchy formula and the dual Cauchy formula for Macdonald polynomials,

We consider the Cauchy formula for two sets of variables $x = (x_1, \ldots, x_m)$ and $y = (y_1, \ldots, y_n)$, assuming that $n \geq m$:

$$\Pi_{m,n}(x; y) = \prod_{i=1}^{m}\prod_{j=1}^{n} \frac{(tx_i y_j; q)_\infty}{(x_i y_j; q)_\infty} = \sum_{\lambda \in \mathcal{P}_m} b_\lambda P_\lambda(x) P_\lambda(y). \tag{7.15}$$

Setting $y = t^{\delta^{(n)}} u = (t^{n-1}u, t^{n-2}u, \ldots, u)$, $\delta^{(n)} = (n-1, n-2, \ldots, 0)$, we obtain

$$\sum_{\lambda \in \mathcal{P}_m} u^{|\lambda|} b_\lambda P_\lambda(x) P_\lambda(t^{\delta^{(n)}}) = \prod_{i=1}^{m} \frac{(t^n x_i u; q)_\infty}{(x_i u; q)_\infty}, \tag{7.16}$$

or, equivalently

$$\sum_{\lambda \in \mathcal{P}_m} (ut^{-n})^{|\lambda|} b_\lambda P_\lambda(x) P_\lambda(t^{\delta^{(n)}}) = \prod_{i=1}^{m} \frac{(x_i u; q)_\infty}{(t^{-n} x_i u; q)_\infty}. \tag{7.17}$$

On the other hand, specializing the dual Cauchy formula for variables $x = (x_1, \ldots, x_m)$ and $z = (z_1, \ldots, z_N)$,

$$\Pi^\vee_{m,N}(x; z) = \prod_{i=1}^{m}\prod_{j=1}^{N}(1 + x_i z_j) = \sum_{\lambda \subseteq (N^m)} P_\lambda(x) P^\circ_{\lambda'}(z), \tag{7.18}$$

by $z = -q^{\delta^{(N)}} u = (-q^{N-1}u, \ldots, -u)$, we obtain

$$\sum_{\lambda \subseteq (N^m)} (-u)^{|\lambda|} P_\lambda(x) P^\circ_{\lambda'}(q^{\delta^{(N)}}) = \prod_{i=1}^{m} (x_i u; q)_N. \tag{7.19}$$

Comparing (7.17) and (7.19), we set $t = q^{-\frac{N}{n}}$ so that

$$\sum_{\lambda \in \mathcal{P}_m} (ut^{-n})^{|\lambda|} b_\lambda P_\lambda(x) P_\lambda(t^{\delta^{(n)}}) = \sum_{\lambda \subseteq (N^m)} (-u)^{|\lambda|} P_\lambda(x) P^\circ_{\lambda'}(q^{\delta^{(N)}}). \tag{7.20}$$

Note that the genericity condition on $t = q^{-\frac{N}{n}}$ for the existence of relevant Macdonald polynomials is fulfilled by infinitely many $N/n \in \mathbb{Q}$ (taking distinct pairs of primes $n, N > m$ for example). From (7.20), we have

$$t^{-n|\lambda|}b_\lambda P_\lambda(\delta^{(n)}) = (-1)^{|\lambda|}P^\circ_{\lambda'}(q^{\delta^{(N)}}), \quad \text{i.e.}$$
$$b_\lambda = (-1)^{|\lambda|}t^{n|\lambda|}\frac{P^\circ_{\lambda'}(q^{\delta^{(N)}})}{P_\lambda(t^{\delta^{(n)}})} \tag{7.21}$$

for $\lambda \in \mathcal{P}_m$ with $\lambda_1 \leq N$, and $b_\lambda P_\lambda(\delta^{(n)}) = 0$ for $\lambda \in \mathcal{P}_m$ with $\lambda_1 > N$, in this specialization. By Theorem 6.1 we already know the explicit formulas

$$P_\lambda(t^{\delta^{(n)}}) = \prod_{s\in\lambda}\frac{t^{l'_\lambda(s)} - t^n q^{a'_\lambda(s)}}{1 - t^{l_\lambda(s)+1}q^{a_\lambda(s)}},$$
$$P^\circ_{\lambda'}(q^{\delta^{(N)}}) = \prod_{s\in\lambda}\frac{q^{a'_\lambda(s)} - q^N t^{l'_\lambda(s)}}{1 - q^{a_\lambda(s)+1}t^{l_\lambda(s)}}, \tag{7.22}$$

where we used $l_{\lambda'}(s) = a_\lambda(s)$, $a_{\lambda'}(s) = l_\lambda(s)$, $l'_{\lambda'}(s) = a'_\lambda(s)$, $a'_{\lambda'}(s) = l'_\lambda(s)$. Note that, for $\lambda \in \mathcal{P}_m$ with $\lambda_1 > N$, $P_\lambda(t^{\delta^{(n)}}) = 0$ since $t^{l'_\lambda(s)} - t^n q^{a'_\lambda(s)} = 1 - t^n q^N = 0$ at $s = (1, N+1) \in \lambda$. Hence, we obtain

$$b_\lambda = (-1)^{|\lambda|}t^{n|\lambda|}\prod_{s\in\lambda}\frac{q^{a'_\lambda(s)} - q^N t^{l'_\lambda(s)}}{1 - q^{a_\lambda(s)+1}t^{l_\lambda(s)}}\,\frac{1 - t^{l_\lambda(s)+1}q^{a_\lambda(s)}}{t^{l'_\lambda(s)} - t^n q^{a'_\lambda(s)}}$$
$$= \prod_{s\in\lambda}\frac{1 - t^{l_\lambda(s)+1}q^{a_\lambda(s)}}{1 - t^{l_\lambda(s)}q^{a_\lambda(s)+1}} \tag{7.23}$$

under the specialization $t = q^{-\frac{N}{n}}$. Since (7.23) is valid for infinitely many values of $t$, this formula gives the expression of $b_\lambda$ as a rational function of $(q, t)$.

**Remark 7.2** In view of the stability (4.31) of $P_\lambda(x)$ with respect to the number of variables, Macdonald [20] introduces *Macdonald functions* $P_\lambda(x) = P_\lambda(x|q, t)$ in infinite variables $x = (x_i)_{i\geq 1} = (x_1, x_2, \ldots)$. Letting $M \to \infty$ and $N \to \infty$ in (7.7), we have

$$\Pi_\infty(z; w) = \prod_{i\geq 1}\prod_{j\geq 1}\frac{(tx_i y_j; q)_\infty}{(x_i y_j; q)_\infty} = \sum_{\lambda\in\mathcal{P}} b_\lambda P_\lambda(x)P_\lambda(x),$$
$$\Pi^\vee_\infty(x; y) = \prod_{i\geq 1}\prod_{j\geq 1}(1 + x_i y_j) = \sum_{\lambda\in\mathcal{P}} P_\lambda(x)P^\circ_{\lambda'}(y), \tag{7.24}$$

with sums over all partitions $\lambda \in \mathcal{P}$, where $P^\circ_\mu(x) = P_\mu(x|t, q)$. In terms of the power sums $p_k(x) = x_1^k + x_2^k + \cdots$, the kernel functions $\Pi(x; y)$ and $\Pi^\vee(x; y)$ are expressed as

$$\Pi_\infty(x;y) = \exp\Big(\sum_{k=1}^{\infty}\frac{1}{k}\frac{1-t^k}{1-q^k}p_k(x)p_k(y)\Big),$$
$$\Pi_\infty^\vee(x;y) = \exp\Big(\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k}p_k(x)p_k(y)\Big). \tag{7.25}$$

A big advantage of passing to infinite variables is that the power sums $p_k = p_k(x)$ $(k = 1, 2, \ldots)$ become *algebraically independent*. Denoting $\Lambda = \mathbb{C}[p_1, p_2, \ldots]$ the ring of symmetric functions in infinite variables, Macdonald introduces the algebra automorphism $\omega_{q,t} : \ \Lambda \to \Lambda$ by

$$\omega_{q,t}(p_k) = (-1)^{k-1}\frac{1-q^k}{1-t^k}p_k \qquad (k = 1, 2, \ldots) \tag{7.26}$$

in terms of the power sums, so that $\omega^y_{q,t}(\Pi(x;y)) = \Pi^\vee(x;y)$, where $\omega^y_{q,t}$ denotes the automorphism $\omega_{q,t}$ acting on $y$ variables. This implies

$$\sum_{\lambda\in\mathcal{P}} b_\lambda P_\lambda(x)\omega^y_{q,t}(P_\lambda(y)) = \sum_{\lambda\in\mathcal{P}} P_\lambda(x)P^\circ_{\lambda'}(y), \tag{7.27}$$

and hence

$$b_\lambda\, \omega_{q,t}(P_\lambda) = P^\circ_{\lambda'}. \qquad (\lambda \in \mathcal{P}). \tag{7.28}$$

In Macdonald [20], the explicit formula (6.46) for $b_\lambda$ is proved by a somewhat tricky argument based on the compatibility of $b_\lambda\, \omega_{q,t}(P_\lambda) = P^\circ_{\lambda'}$ with the evaluation formula of $P_\lambda(t^{n-1}, t^{n-1}, \ldots, 1)$ in $n$-variables.

## 7.4 Tableau Representation of $P_\lambda(x)$

We already know that the Macdonald polynomials $P_\lambda(x)$ $(\lambda \in \mathcal{P}_n)$ of $n$ variables $x = (x_1, \ldots, x_n)$ satisfy the recurrence formulas

$$P_\lambda(x_1, \ldots, x_{n-1}, x_n) = \sum_{\substack{\mu\in\mathcal{P}_{n-1} \\ \mu\subseteq\lambda,\ \lambda/\mu:\,\text{h-strip}}} \psi_{\lambda/\mu}\ P_\mu(x_1, \ldots, x_{n-1})\, x_n^{|\lambda/\mu|} \tag{7.29}$$

with respect to the number of variables, where the sum is taken over all partitions $\mu \in \mathcal{P}_{n-1}$ such that $\mu \subseteq \lambda$ and $\lambda/\mu$ is a horizontal strip. Note that $\psi_{\lambda/\mu} = 0$ unless $\lambda/\mu$ is a horizontal strip. Repeating this procedure, one can express $P_\lambda(x)$ as a sum

$$P_\lambda(x) = \sum_{\emptyset=\lambda^{(0)}\subseteq\lambda^{(1)}\subseteq\cdots\subseteq\lambda^{(n)}=\lambda}\ \prod_{k=1}^{n}\psi_{\lambda^{(k)}/\lambda^{(k-1)}}\ x_k^{|\lambda^{(k)}/\lambda^{(k-1)}|} \tag{7.30}$$

over all weakly increasing sequences $\lambda^{(k)}$ $(k = 0, 1, \ldots, n)$ of partitions connecting $\emptyset$ and $\lambda$ by $n$ steps such that the skew partitions $\lambda^{(k)}/\lambda^{(k-1)}$ $(k = 1, \ldots, n)$ are horizontal strips. This representation can be interpreted as the sum

$$P_\lambda(x) = \sum_{T \in \mathrm{SSTab}_n(\lambda)} \psi_T \, x^{\mathrm{wt}(T)}, \quad \psi_T = \prod_{k=1}^{n} \psi_{\lambda^{(k)}/\lambda^{(k-1)}}, \tag{7.31}$$

over all semi-standard tableaux of shape $\lambda$. Here the coefficients $\psi_T$ are expressed as

$$\psi_T = \prod_{1 \le i \le j < k \le n} \frac{(t^{j-i+1} q^{\lambda_i^{(k-1)} - \lambda_j^{(k-1)}}; q)_{\lambda_i^{(k)} - \lambda_i^{(k-1)}}}{(t^{j-i} q^{\lambda_i^{(k-1)} - \lambda_j^{(k-1)} + 1}; q)_{\lambda_i^{(k)} - \lambda_i^{(k-1)}}} \cdot \frac{(t^{j-i} q^{\lambda_i^{(k-1)} - \lambda_{j+1}^{(k)} + 1}; q)_{\lambda_i^{(k)} - \lambda_i^{(k-1)}}}{(t^{j-i+1} q^{\lambda_i^{(k-1)} - \lambda_{j+1}^{(k)}}; q)_{\lambda_i^{(k)} - \lambda_i^{(k-1)}}}. \tag{7.32}$$

## 7.5 Macdonald–Ruijsenaars Operators of Row Type (Overview)

In this section, we give an overview of the commuting family of $q$-difference operators of row type for which Macdonald polynomials are joint eigenfunctions (for the details, see Noumi–Sano [26]). We also explain how they are related to the Pieri formula of row type.

### 7.5.1 *q-Difference Operators $H_x^{(l)}$ of Row Type*

Let $\mathcal{R} = \mathbb{C}[D_x^{(1)}, \ldots, D_x^{(n)}]$ be the commutative ring generated by the Macdonald–Ruijsenaars $q$-difference operators. Then, for each symmetric polynomial $f(\xi) \in \mathbb{C}[\xi]^{\mathfrak{S}_n}$, $\xi = (\xi_1, \ldots, \xi_n)$, there exists a unique $q$-difference operator $L_x \in \mathcal{R}$ such that

$$L_x P_\lambda(x) = f(t^\delta q^\lambda) P_\lambda(x) \quad (\lambda \in \mathcal{P}_n). \tag{7.33}$$

(Express $f$ as $f = F(e_1, \ldots, e_n)$ by a polynomial of $e_1, \ldots, e_n$. Then the operator $L_x$ is given by $L_x = F(D_x^{(1)}, \ldots, D_x^{(n)})$.) This correspondence $L_x \to f$ defines an isomorphism $\mathbb{C}[D_x^{(1)}, \ldots, D_x^{(n)}] \xrightarrow{\sim} \mathbb{C}[\xi]^{\mathfrak{S}_n}$ of commutative $\mathbb{C}$-algebras (a variation of the *Harish–Chandra isomorphism*).

For each $l = 0, 1, 2, \ldots$, we define a $q$-difference operator $H_x^{(l)}$ by

$$H_x^{(l)} = \sum_{\mu\in\mathbb{N}^n;\,|\mu|=l} \frac{\Delta(q^\mu x)}{\Delta(x)} \prod_{i,j=1}^{n} \frac{(tx_i/x_j;q)_{\mu_i}}{(qx_i/x_j;q)_{\mu_i}} T_{q,x}^{\mu}, \tag{7.34}$$

where $T_{q,x}^{\mu} = T_{q,x_1}^{\mu_1}\cdots T_{q,x_n}^{\mu_n}$. Then it is known that $H_x^{(l)} \in \mathbb{C}[D_x^{(1)},\ldots,D_x^{(n)}]$, and that

$$H_x^{(l)} P_\lambda(x) = g_l(t^\delta q^\lambda) P_\lambda(x) \quad (l = 0, 1, 2\ldots), \tag{7.35}$$

where $g_l(\xi)$ denotes the Macdonald polynomials attached to $(l)$ of a single row:

$$g_l(\xi) = \sum_{\mu\in\mathbb{N}^n;\,|\mu|=l} \frac{(t;q)_{\mu_1}\cdots(t;q)_{\mu_n}}{(q;q)_{\mu_1}\cdots(q;q)_{\mu_n}} \xi_1^{\mu_1}\cdots\xi_n^{\mu_n} = \frac{(t;q)_l}{(q;q)_l} P_{(l)}(\xi) \tag{7.36}$$

for $l = 0, 1, 2, \ldots$. In view of the generating function

$$G(\xi;u) = \prod_{i=1}^{n} \frac{(t\xi_i u;q)_\infty}{(\xi_i u;q)_\infty} = \sum_{l=0}^{\infty} g_l(\xi) u^l \tag{7.37}$$

of Macdonald polynomials of single rows, we introduce the generating function $H_x(u) = \sum_{l=0}^{\infty} u^l H_x^{(l)}$. Then we have

$$H_x(u) P_\lambda(x) = P_\lambda(x) \prod_{i=1}^{n} \frac{(t^{n-i+1}q^{\lambda}u;q)_\infty}{(t^{n-i}q^{\lambda}u;q)_\infty}. \tag{7.38}$$

Also, it is known that $H_x(u)$ satisfies the kernel identities

$$\begin{aligned} H_x(u)\Pi_{m,n}(x;y) &= \frac{(t^{m-n}u;q)_\infty}{(u;q)_\infty} H_y(t^{m-n}u)\Pi_{m,n}(x;y), \\ (u;q)_\infty H_x(u)\Pi_{m,n}^{\vee}(x;y) &= (t^m q^n u;q)_\infty D_y^{\circ}(u)\Pi_{m,n}^{\vee}(x;y). \end{aligned} \tag{7.39}$$

### *7.5.2 Wronski Relations*

As we proved in Sect. 4.5, the two generating functions $E(x;u)$ and $G(x;u)$ satisfy $E(x;-u)G(x;u) = E(x;-tu)G(x;qu)$. This means that $e_r(x)$ and $g_l(x)$ are related through the recurrence relations

$$\sum_{r+l=k} (-1)^r (1-t^r q^l) e_r(\xi) g_l(\xi) = 0 \quad (k = 1, 2, \ldots) \tag{7.40}$$

of *Wronski type*. One can verify that the operators $H_x^{(l)}$ $(l = 0, 1, 2, \ldots)$ defined above satisfy the Wronski relations

$$\sum_{r+l=k} (-1)^r (1-t^r q^l) D_x^{(r)} H_x^{(l)} = 0 \quad (k=1,2,\ldots). \tag{7.41}$$

From this, it follows that $H_x^{(l)} \in \mathbb{C}[D_x^{(1)}, \ldots, D_x^{(n)}]$ and that $H_x^{(l)}$ are diagonalized by the Macdonald polynomials as in (7.35).

### 7.5.3 Pieri Formula of Row Type

In the same way as we obtained the Pieri formula of column type from $D_x^{(r)}$, we can derive the Pieri formula of row type from

$$H_x^{(l)} = \sum_{|\nu|=l} H_\nu(x) T_{q,x}^\nu, \quad H_\nu(x) = \frac{\Delta(q^\nu x)}{\Delta(x)} \prod_{i,j=1}^n \frac{(tx_i/x_j;q)_{\nu_i}}{(qx_i/x_j;q)_{\nu_i}}. \tag{7.42}$$

In fact we have

$$\sum_{|\nu|=l} H_\nu(t^\delta q^\mu) \widetilde{P}_\lambda(t^\delta q^{\mu+\nu}) = g_l(t^\delta q^\lambda) \widetilde{P}_\lambda(t^\delta q^\mu) \qquad (\lambda, \mu \in \mathcal{P}_n). \tag{7.43}$$

Since $H_\nu(t^\delta q^\mu) = 0$ unless $(\mu+\nu)/\mu$ is a horizontal strip, we obtain

$$\sum_{|\nu|=l} H_\nu(t^\delta q^\mu) \widetilde{P}_{\mu+\nu}(t^\delta q^\lambda) = g_l(t^\delta q^\lambda) \widetilde{P}_\mu(t^\delta q^\lambda) \qquad (\lambda, \mu \in \mathcal{P}_n), \tag{7.44}$$

and hence

$$\sum_{|\nu|=l} H_\nu(t^\delta q^\mu) \widetilde{P}_{\mu+\nu}(x) = g_l(x) \widetilde{P}_\mu(x) \qquad (\mu \in \mathcal{P}_n). \tag{7.45}$$

This means that

$$g_l(x) \widetilde{P}_\mu(x) \sum_{|\lambda/\mu|=l} H_{\lambda/\mu} \widetilde{P}_\lambda(x), \quad H_{\lambda/\mu} = H_{\lambda-\mu}(t^\delta q^\mu) \qquad (\mu \in \mathcal{P}_n), \tag{7.46}$$

where the sum is over all $\lambda \in \mathcal{P}_n$, $\mu \subseteq \lambda$, such that the skew diagram $\lambda/\mu$ is a horizontal strip with $|\lambda/\mu| = l$. Hence we obtain

$$g_l(x) P_\mu(x) = \sum_{|\lambda/\mu|=l} \varphi_{\lambda/\mu} P_\lambda(x), \qquad \varphi_{\lambda/\mu} = \frac{a_\mu}{a_\lambda} H_{\lambda/\mu}. \tag{7.47}$$

Since $g_l(x) = b_{(l)} P_{(l)}(x)$, this means that

$$P_\mu(x)P_{(l)}(x)=\sum_{|\lambda/\mu|=l} c^\lambda_{\mu,(l)}\,P_\lambda(x),\qquad c^\lambda_{\mu,(l)}=\frac{1}{b_{(l)}}\frac{a_\mu}{a_\lambda}H_{\lambda/\mu}. \tag{7.48}$$

The corresponding branching coefficients are also determined as

$$\psi_{\lambda/\mu}=b^\lambda_{\mu,(l)}=\frac{b_\mu b_{(l)}}{b_\lambda}c^\lambda_{\mu,(l)}=\frac{a_\mu b_\mu}{a_\lambda b_\lambda}H_{\lambda/\mu}. \tag{7.49}$$

The coefficients $H_{\lambda/\mu}=H_{\lambda-\mu}(t^\delta q^\mu)$ are explicitly computed as follows:

$$\begin{aligned}H_{\lambda/\mu}&=t^{n(\lambda)-n(\mu)}\prod_{i\ge 1}(t^{n-i+1}q^{\mu_i};q)_{\lambda_i-\mu_i}\\&\quad\cdot\frac{\prod_{i<j}(t^{j-i-1}q^{\mu_i-\lambda_j+1};q)_{\lambda_j-\mu_j}}{\prod_{i\le j}(t^{j-i}q^{\lambda_i-\lambda_j+1};q)_{\lambda_j-\mu_j}}\frac{\prod_{i\le j}(t^{j-i+1}q^{\mu_i-\mu_j};q)_{\mu_j-\lambda_{j+1}}}{\prod_{i\le j}(t^{j-i+1}q^{\lambda_i-\mu_j};q)_{\mu_j-\lambda_{j+1}}}.\end{aligned} \tag{7.50}$$

In combinatorial terms, we have

$$\begin{aligned}H_{\lambda/\mu}&=\frac{t^{n(\lambda)}\prod_{i\ge1}(t^{n-i+1};q)_{\lambda_i}}{t^{n(\mu)}\prod_{i\ge1}(t^{n-i+1};q)_{\mu_i}}\\&\quad\cdot\frac{\prod_{s\in\mu\cap C_{\lambda/\mu}}(1-t^{l_\mu(s)}q^{a_\mu(s)+1})}{\prod_{s\in\lambda\cap C_{\lambda/\mu}}(1-t^{l_\lambda(s)}q^{a_\lambda(s)+1})}\frac{\prod_{s\in\mu\setminus C_{\lambda/\mu}}(1-t^{l_\mu(s)+1}q^{a_\mu(s)})}{\prod_{s\in\lambda\setminus C_{\lambda/\mu}}(1-t^{l_\lambda(s)+1}q^{a_\lambda(s)})},\end{aligned} \tag{7.51}$$

where $C_{\lambda/\mu}$ denotes the union of columns intersecting with the horizontal strip $\lambda/\mu$. Combining this with

$$a_\lambda=\frac{t^{n(\lambda)}(t^{n-i+1};q)_{\lambda_i}}{\prod_{s\in\lambda}(1-t^{l_\lambda(s)+1}q^{a_\lambda(s)})},\quad b_\lambda=\prod_{s\in\lambda}\frac{1-t^{l_\lambda(s)+1}q^{a_\lambda(s)}}{1-t^{l_\lambda(s)}q^{a_\lambda(s)+1}}, \tag{7.52}$$

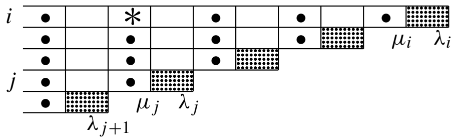
by (7.49) we obtain

$$\psi_{\lambda/\mu}=\prod_{s\in\lambda\setminus C_{\lambda/\mu}}\frac{1-t^{l_\lambda(s)}q^{a_\lambda(s)+1}}{1-t^{l_\lambda(s)+1}q^{a_\lambda(s)}}\prod_{s\in\mu\setminus C_{\lambda/\mu}}\frac{1-t^{l_\mu(s)+1}q^{a_\mu(s)}}{1-t^{l_\mu(s)}q^{a_\mu(s)+1}}. \tag{7.53}$$

In terms of the components of $\lambda$, $\mu$, we recover the formula of (6.6), namely

$$\begin{aligned}\psi_{\lambda/\mu}&=\prod_{1\le i\le j\le\ell(\mu)}\frac{(t^{j-i}q^{\lambda_i-\mu_j+1};q)_{\mu_j-\lambda_{j+1}}}{(t^{j-i+1}q^{\lambda_i-\mu_j};q)_{\mu_j-\lambda_{j+1}}}\frac{(t^{j-i+1}q^{\mu_i-\mu_j};q)_{\mu_j-\lambda_{j+1}}}{(t^{j-i}q^{\mu_i-\mu_j+1};q)_{\mu_j-\lambda_{j+1}}}\\&=\prod_{1\le i\le j\le\ell(\mu)}\frac{(t^{j-i}q^{\lambda_i-\lambda_{j+1}+1};q)_{\lambda_i-\mu_i}}{(t^{j-i+1}q^{\lambda_i-\lambda_{j+1}};q)_{\lambda_i-\mu_i}}\frac{(t^{j-i+1}q^{\mu_i-\mu_j};q)_{\lambda_i-\mu_i}}{(t^{j-i}q^{\mu_i-\mu_j+1};q)_{\lambda_i-\mu_i}}.\end{aligned} \tag{7.54}$$

The relationship between the explicit formulas (7.53) and (7.54) for the coefficient $\psi_{\lambda/\mu}$ can be read off from the picture below:




(7.55)

# Chapter 8
# Affine Hecke Algebra and $q$-Dunkl Operators (Overview)

**Abstract** In this chapter, we give an overview of the *Macdonald–Cherednik theory* of Macdonald polynomials based on the affine and double affine Hecke algebras, taking the example of type $A_{n-1}$. (For a more comprehensive exposition, see Macdonald [22].) We explain how the commuting family of Macdonald–Ruijsenaars operators arise naturally in the framework of affine Hecke algebras. We also show how the self-duality of Macdonald polynomials can be established by means of the Cherednik involution of the double affine Hecke algebra.

## 8.1 Affine Weyl Groups and Affine Hecke Algebras

We denote by $W = \mathfrak{S}_n$ the symmetric group of degree $n$ (*Weyl group* of type $A_{n-1}$), following the convention of Macdonald–Cherednik theory for general root systems.[1] In this chapter, we denote by $\tau_i = T_{q,x_i}$ $(i = 1, \ldots, n)$ the $q$-shift operators in variables $x_i$, in order to avoid the conflict with the generators $T_i$ of Hecke algebras. Setting $\tau = (\tau_1, \ldots, \tau_n)$, we denote by $\mathcal{D}_{q,x} = \mathbb{C}(x)[\tau^{\pm 1}]$ the algebra of $q$-difference operators in $x$ variables with rational coefficients, and by $\mathcal{D}_{q,x}[W]$ the algebra of all operators of the form

$$\begin{aligned} A_x &= \sum_{\mu \in P,\, w \in W} a_{\mu,w}(x)\, \tau^{\mu}\, w \quad \text{(finite sum)}, \\ a_{\mu,w}(x) &\in \mathbb{C}(x) \quad (\mu \in P,\ w \in W), \end{aligned} \tag{8.1}$$

called the *$q$-difference-reflection operators*, where $P = \mathbb{Z}^n$ (*weight lattice* of $\mathrm{GL}_n$), and for $\mu = (\mu_1, \ldots, \mu_n) \in P$, $\tau^{\mu} = \tau_1^{\mu_1} \cdots \tau_n^{\mu_n}$. Through their natural action on rational functions, we regard $\mathcal{D}_{q,x}[W]$ as a subalgebra of $\mathrm{End}_{\mathbb{C}}(\mathbb{C}(x))$.

We denote by $\tau^P = \{\tau^{\mu} \mid \mu \in P\}$ the group of $q$-shift operators (*translations*) with respect to $P$, and define the *extended affine Weyl group* $\widetilde{W}$ by

[1] We define a version of $q$-Dunkl operators from which Macdonald–Ruijsenaars operators directly arise. Notice that our convention of $q$-Dunkl operators and nonsymmetric Macdonald polynomials is different from that of Macdonald's monograph [22].

 
M. Noumi, *Macdonald Polynomials*, SpringerBriefs in Mathematical Physics,
https://doi.org/10.1007/978-981-99-4587-0_8

$$\widetilde{W} = \tau^P \rtimes W = \{\tau^\mu w \mid \mu \in P, w \in W\},$$
$$w\tau^\mu = \tau^{w.\mu} w \quad (\mu \in P,\ w \in W), \tag{8.2}$$

which is an extension of the standard *affine Weyl group* $W^{\mathrm{aff}} = \tau^Q \rtimes W$ with the group of translations by $Q = \{\mu \in P \mid |\mu| = \mu_1 + \cdots + \mu_n = 0\}$ (*root lattice*). Denoting the canonical basis of $P$ by $\varepsilon_i$ $(i = 1, \ldots, n)$, we use the notation $\alpha_1 = \varepsilon_1 - \varepsilon_2,\ \alpha_2 = \varepsilon_2 - \varepsilon_3,\ \ldots,\ \alpha_{n-1} = \varepsilon_{n-1} - \varepsilon_n$ for the *simple roots*, so that $Q = \mathbb{Z}\alpha_1 \oplus \cdots \oplus \mathbb{Z}\alpha_{n-1} \subseteq P$. The idea of (affine) Hecke algebras is to construct *$t$-deformations* of all groups

$$W = \mathfrak{S}_n \subseteq W^{\mathrm{aff}} = \tau^Q \rtimes W \subseteq \widetilde{W} = \tau^P \rtimes W \subset \mathcal{D}_{q,x}[W] \tag{8.3}$$

within the algebra $\mathcal{D}_{q,x}[W]$ of $q$-difference-reflection operators. Note that the group-ring of $\widetilde{W}$

$$\mathbb{C}[\widetilde{W}] = \mathbb{C}[\tau^P \rtimes W] = \mathbb{C}[\tau^{\pm 1}][W] \subseteq \mathcal{D}_{q,x}[W] \tag{8.4}$$

is the ring of $q$-difference-reflection operators with constant coefficients.

We denote by

$$s_1 = (12),\ s_2 = (23),\ \ldots,\ s_{n-1} = (n-1, n) \tag{8.5}$$

the adjacent transpositions in $W = \mathfrak{S}_n$ (*simple reflections*) so that $W = \langle s_1, \ldots, s_{n-1} \rangle$. Note that $s_i$ acts on the $x$ variables by exchanging $x_i$ and $x_{i+1}$. Besides these generators, we use the *affine reflection* $s_0$ and the *diagram automorphism* $\omega$ by setting

$$s_0 = \tau_1^{-1}\tau_n(1, n) \in W^{\mathrm{aff}}, \quad \omega = \tau_n(n, n-1, \ldots, 1) = \tau_n s_{n-1} \cdots s_1 \in \widetilde{W}, \tag{8.6}$$

where $(1, n)$ stands for the transposition of 1 and $n$, and $(n, n-1, \ldots, 1)$ for the cyclic permutation $n \to n-1 \to \cdots \to 1 \to n$. These $s_0$ and $\omega$ are characterized as field automorphisms of $\mathbb{C}(x)$ acting on the $x$-variables by

$$s_0(x_1) = qx_n, \quad s_0(x_i) = x_i \ \ (i = 2, \ldots, n-1), \quad s_0(x_n) = q^{-1}x_1$$
$$\omega(x_1) = qx_n, \quad \omega(x_2) = x_1, \quad \ldots, \quad \omega(x_n) = x_{n-1}. \tag{8.7}$$

If fact, it is known that the three groups in (8.3) are generated by these operators as

$$W = \langle s_1, \ldots, s_{n-1} \rangle \subseteq W^{\mathrm{aff}} = \langle s_0, s_1, \ldots, s_{n-1} \rangle \subseteq \widetilde{W} = \langle s_0, s_1, \ldots, s_{n-1}, \omega \rangle, \tag{8.8}$$

with the fundamental relations:

$$\begin{aligned}
&(0) \quad s_i^2 = 1 \quad (i = 0, 1, \ldots, n-1),\\
&(1) \quad s_i s_j = s_j s_i \quad (j \not\equiv i, i \pm 1 \mod n),\\
&(2) \quad s_i s_j s_i = s_j s_i s_j \quad (j \equiv i \pm 1 \mod n),\\
&(3) \quad \omega s_i = s_{i-1}\omega \quad (i = 1, \ldots, n-1), \qquad \omega s_0 = s_{n-1}\omega.
\end{aligned} \tag{8.9}$$

In terms of these generators, the $q$-shift operators $\tau_i$ $(= T_{q,x_i})$ are expressed as follows:

$$\tau_1 = s_1 \cdots s_{n-1}\omega, \quad \tau_2 = s_2 \cdots s_{n-1}\omega s_1, \quad \cdots, \quad \tau_n = \omega s_1 \cdots s_{n-1}. \tag{8.10}$$

We now define the $q$-difference-reflection operators $T_i$ $(i = 0, 1 \ldots, n-1)$, called the *Demazure–Lusztig operators*, by

$$T_i = t^{-\frac{1}{2}} \frac{1 - t x_i/x_{i+1}}{1 - x_i/x_{i+1}}(s_i - 1) + t^{\frac{1}{2}} = t^{-\frac{1}{2}} \frac{1 - t x_i/x_{i+1}}{1 - x_i/x_{i+1}} s_i + \frac{t^{\frac{1}{2}} - t^{-\frac{1}{2}}}{1 - x_i/x_{i+1}} \tag{8.11}$$

for $i = 1, \ldots, n-1$ and

$$T_0 = t^{-\frac{1}{2}} \frac{1 - t q x_n/x_1}{1 - q x_n/x_1}(s_0 - 1) + t^{\frac{1}{2}} = t^{-\frac{1}{2}} \frac{1 - t q x_n/x_1}{1 - q x_n/x_1} s_0 + \frac{t^{\frac{1}{2}} - t^{-\frac{1}{2}}}{1 - q x_n/x_1}. \tag{8.12}$$

Note that $x_i/x_{i+1} = x^{\alpha_i}$ $(i = 1, \ldots, n-1)$ correspond to the simple roots, and $q x_n/x_1 = x^{\alpha_0}$ to the *simple affine root* $\alpha_0 = \gamma - \varepsilon_1 + \varepsilon_n$ with the convention $x^\gamma = q$, where we denoted the null root by $\gamma$ to avoid the conflict with the notation of staircase partition $\delta$.

**Theorem 8.1** *The operators $T_i$ $(i = 0, 1, \ldots, n-1)$ in $\mathcal{D}_{q,x}[W]$ together with $\omega$ satisfy the following relations*:

$$\begin{aligned}
&(0) \quad (T_i - t^{\frac{1}{2}})(T_i + t^{-\frac{1}{2}}) = 1 \quad (i = 0, 1, \ldots, n-1),\\
&(1) \quad T_i T_j = T_j T_i \quad (j \not\equiv i, i \pm 1 \mod n),\\
&(2) \quad T_i T_j T_i = T_j T_i T_j \quad (j \equiv i \pm 1 \mod n),\\
&(3) \quad \omega T_i = T_{i-1}\omega \quad (i = 1, \ldots, n-1), \quad \omega T_0 = T_{n-1}\omega.
\end{aligned} \tag{8.13}$$

In this way, we obtain $t$-deformations of the group-rings of $W$, $W^{\text{aff}}$, $\widetilde{W}$ in $\mathcal{D}_{q,x}[W]$ as follows:

$$\begin{array}{cc}
\mathbb{C}[W] = \mathbb{C}\langle s_1, \ldots, s_{n-1}\rangle & H[W] = \mathbb{C}\langle T_1, \ldots, T_{n-1}\rangle,\\
\cap\!\mid & \cap\!\mid\\
\mathbb{C}[W^{\text{aff}}] = \mathbb{C}\langle s_0, s_1, \ldots, s_{n-1}\rangle & H[W^{\text{aff}}] = \mathbb{C}\langle T_0, T_1, \ldots, T_{n-1}\rangle,\\
\cap\!\mid & \cap\!\mid\\
\mathbb{C}[\widetilde{W}] = \mathbb{C}\langle s_0, s_1, \ldots, s_{n-1}, \omega^{\pm 1}\rangle & H[\widetilde{W}] = \mathbb{C}\langle T_0, T_1, \ldots, T_{n-1}, \omega^{\pm 1}\rangle.
\end{array} \tag{8.14}$$

Here $H(W)$ denotes the *Hecke algebra* associated with the Weyl group $W$; $H(W^{\text{aff}})$ and $H(\widetilde{W})$ are called (extended) *affine Hecke algebras*. Note that the fundamental relations $s_i^2 = 1$ are replaced by the quadratic relations $(T_i - t^{\frac{1}{2}})(T_i + t^{-\frac{1}{2}}) = 0$, and hence $T_i^{-1} = T_i - (t^{\frac{1}{2}} - t^{-\frac{1}{2}})$.

## 8.2 $q$-Dunkl Operators

In view of the two expressions

$$\mathbb{C}[\widetilde{W}] = \mathbb{C}\langle s_0, s_1, \ldots, s_{n-1}, \omega^{\pm 1}\rangle = \mathbb{C}[\tau^{\pm 1}][W] \tag{8.15}$$

of the group-ring of $\widetilde{W}$, it would be natural to ask what are the "translations" in $H(\widetilde{W})$. Imitating the formulas (8.10) for $\tau_i$ $(i = 1, \ldots, n)$, we define the *$q$-Dunkl operators* (or *Cherednik operators*) $Y_1, \ldots, Y_n \in H(\widetilde{W})$ by

$$Y_1 = T_1 \cdots T_{n-1}\omega, \quad Y_2 = T_2 \cdots T_{n-1}\omega T_1^{-1}, \quad \ldots, \quad Y_n = \omega T_1^{-1} \cdots T_{n-1}^{-1}. \tag{8.16}$$

Notice here that $T_i$ are replaced by their inverses $T_i^{-1}$ when they are located to the right side of $\omega$.

**Theorem 8.2** *The $q$-Dunkl operators $Y_1, \ldots, Y_n \in H(\widetilde{W})$ commute with each other. Furthermore, they generate a commutative subalgebra $\mathbb{C}[Y^{\pm 1}] = \mathbb{C}[Y_1^{\pm 1}, \ldots, Y_n^{\pm 1}] \subseteq H(\widetilde{W})$ isomorphic to the algebra of Laurent polynomials in $n$ variables.*

One can directly verify the commutativity $Y_i Y_j = Y_j Y_i$ $(i, j \in \{1, \ldots, n\})$ by the definition (8.16) and the fundamental relations of $T_i, \omega$ in (8.13). With this "translation subalgebra" of $q$-Dunkl operators, the extended affine Hecke algebra is expressed in the form

$$H(\widetilde{W}) = \mathbb{C}[Y^{\pm 1}] \otimes H(W) = \bigoplus_{w \in W} \mathbb{C}[Y^{\pm 1}]\, T_w, \tag{8.17}$$

where, for each $w \in W$, $T_w$ is the element defined as $T_w = T_{s_{i_1}} \cdots T_{s_{i_l}}$ in terms of a reduced (shortest) expression $w = s_{i_1} \cdots s_{i_l}$ of $w$; this definition does not depend on the choice of the reduced expression (Iwahori–Matsumoto Lemma). From this, we see that one can take $T_1, \ldots, T_{n-1}$ and the $q$-Dunkl operators $Y_1^{\pm 1}, \ldots, Y_n^{\pm 1}$ for the generators of the extended affine Hecke algebra:

$$H(\widetilde{W}) = \mathbb{C}\langle T_0, T_1, \ldots, T_{n-1}; \omega^{\pm 1}\rangle = \mathbb{C}\langle T_1, \ldots, T_n; Y_1^{\pm 1}, \ldots Y_n^{\pm 1}\rangle. \tag{8.18}$$

The following theorem is the key for relating $q$-Dunkl operators with Macdonald–Ruijsenaars operators.

**Theorem 8.3** (Bernstein) *The center $\mathcal{Z}H(\widetilde{W})$ of the extended affine Hecke algebra is precisely the $W$-invariant part of the commutative algebra of $q$-Dunkl operators*:

$$\mathcal{Z}H(\widetilde{W}) = \mathbb{C}[Y^{\pm 1}]^W = \left\{ f(Y) \mid f(\xi) \in \mathbb{C}[\xi^{\pm 1}]^W \right\}, \quad \xi = (\xi_1, \ldots, \xi_n). \quad (8.19)$$

## 8.3 From $q$-Dunkl Operators to Macdonald–Ruijsenaars Operators

Let $A_x \in \mathcal{D}_{q,x}[W]$ be a $q$-difference-reflection operator in the form

$$\begin{aligned} A_x &= \sum_{\mu \in P,\, w \in W} a_{\mu,w}(x)\, \tau^\mu\, w \quad \text{(finite sum)}, \\ a_{\mu,w}(x) &\in \mathbb{C}(x) \quad (\mu \in P,\ w \in W). \end{aligned} \quad (8.20)$$

If $\varphi(x)$ is a symmetric ($W$-invariant) function, $A_x$ acts on $\varphi(x)$ as a $q$-difference operator. Since $w\varphi(x) = \varphi(x)$ $(w \in W)$, we have in fact

$$\begin{aligned} A_x \varphi(x) &= \sum_{\mu \in P,\, w \in W} a_{\mu,w}(x)\, \tau^\mu \varphi(x) = L_x \varphi(x), \\ L_x &= \sum_{\mu \in P,\, w \in W} a_{\mu,w}(x)\, \tau^\mu. \end{aligned} \quad (8.21)$$

In order to describe the action of $q$-Dunkl operators, we set

$$c(z) = t^{-\frac{1}{2}} \frac{1 - tz}{1 - z}, \quad d_\pm(z) = t^{\pm\frac{1}{2}} - c(z) \quad (8.22)$$

so that

$$\begin{aligned} T_i^{\pm 1} &= c(x_i/x_{i+1}) s_i + d_\pm(x_i/x_{i+1}) \quad (i = 1, \ldots, n-1), \\ T_0^{\pm 1} &= c(qx_n/x_1) s_i + d_\pm(qx_n/x_1). \end{aligned} \quad (8.23)$$

Note also that $c(z)$ satisfies $c(z) + c(z^{-1}) = t^{\frac{1}{2}} + t^{-\frac{1}{2}}$.

**Example: Case $n = 2$**

In this case, we have two $q$-Dunkl operators

$$\begin{aligned} Y_1 &= T_1 \omega = c(x_1/x_2) s_1 \omega + d_+(x_1/x_2) \omega = c(x_1/x_2) \tau_1 + d_+(x_1/x_2) \tau_2 s_1, \\ Y_2 &= \omega T_1^{-1} = \omega c(x_1/x_2) s_1 + \omega d_-(x_1/x_2) = c(qx_2/x_1) \tau_2 + d_-(qx_2/x_1) \tau_2 s_1. \end{aligned} \quad (8.24)$$

Then, we compute

$$\begin{aligned} Y_1 + Y_2 &= c(x_1/x_2)\tau_1 + c(qx_2/x_1)\tau_2 + (d_-(qx_2/x_1) + d_+(x_1/x_2))\tau_2 s_1 \\ Y_2 Y_1 &= \omega^2 = \tau_1\tau_2. \end{aligned} \tag{8.25}$$

Since

$$c(qx_2/x_1) + d_-(qx_2/x_2) + d_+(x_1/x_2) = t^{-\frac{1}{2}} + t^{\frac{1}{2}} - c(x_1/x_2) = c(x_2/x_1), \tag{8.26}$$

for any symmetric function $\varphi(x) = \varphi(x_1, x_2)$, we have

$$\begin{aligned} (Y_1 + Y_2)\varphi(x) &= (c(x_1/x_2)\tau_1 + c(x_2/x_1)\tau_2)\varphi(x) \\ &= t^{-\frac{1}{2}}\Big(\frac{1 - tx_1/x_2}{1 - x_1/x_2}\tau_1 + \frac{1 - tx_2/x_1}{1 - x_2/x_1}\tau_2\Big)\varphi(x) \\ &= t^{-\frac{1}{2}} D_x^{(1)}\varphi(x) \\ (Y_2Y_1)\varphi(x) &= \tau_1\tau_2\varphi(x) = t^{-1}D_x^{(2)}\varphi(x), \end{aligned} \tag{8.27}$$

where $D_x^{(1)}$, $D_x^{(2)}$ are the Macdonald–Ruijsenaars operators in two variables.

---

For each $f(\xi) \in \mathbb{C}[\xi^{\pm 1}]$, there exists a unique $q$-difference operator $L_x^f \in \mathcal{D}_{q,x}$ such that

$$f(Y)\varphi(x) = L_x^f\varphi(x) \tag{8.28}$$

for any symmetric function $\varphi(x)$; express $A_x = f(Y)$ in the form (8.20), and take $L_x = L_x^f$ as in (8.21).

**Theorem 8.4** *For any $f \in \mathbb{C}[\xi^{\pm 1}]^W$, $L_x^f \in \mathcal{D}_{q,x}$ is a $W$-invariant $q$-difference operator. Furthermore, $L_x^f$ ($f \in \mathbb{C}[\xi^{\pm 1}]^W$) commute with each other: $L_x^f L_x^g = L_x^g L_x^f$ for any $f, g \in \mathbb{C}[\xi^{\pm 1}]^W$.*

Let's take the elementary symmetric functions $e_r(\xi) \in \mathbb{C}[\xi^{\pm 1}]^W$ ($r = 1, \dots, n$) for $f(\xi)$. Then, one can show that the $q$-Dunkl operators

$$e_r(Y) = \sum_{1 \le i_1 < \cdots < i_r \le n} Y_{i_1} \cdots Y_{i_r} \tag{8.29}$$

induce a commuting family of $W$-invariant $q$-difference operators $L_x^{e_r}$ of the form

$$L_x^{e_r} = \Big(\prod_{\substack{1 \le i \le r \\ r+1 \le j \le n}} t^{-\frac{1}{2}}\frac{1 - tx_i/x_j}{1 - x_i/x_j}\Big)\tau_1 \cdots \tau_r + \cdots . \tag{8.30}$$

This implies that $L_x^{e_r}$ are constant multiples of $D_x^{(r)}$ respectively,

$$L_x^{e_r} = t^{-\frac{1}{2}(n-1)r} D_x^{(r)} \quad (r = 0, 1, \dots, n), \tag{8.31}$$

and also that they are diagonalized by the Macdonald polynomials:

$$L_x^{e_r} P_\lambda(x) = e_r(t^\rho q^\lambda) P_\lambda(x) \quad (\lambda \in \mathcal{P}_n), \tag{8.32}$$

where $\rho = \frac{1}{2} \sum_{i=1}(n - 2i + 1)\varepsilon_i = \delta - \frac{1}{2}(n-1)(1^n)$.
To summarize: There is an isomorphism of commutative algebras

$$\mathcal{Z}H(\widetilde{W}) = \mathbb{C}[Y^{\pm 1}]^W \xrightarrow{\sim} \mathbb{C}[D_1^{(1)}, \dots, D_x^{(n)}, (D_x^{(n)})^{-1}] : \quad f(Y) \to L_x^f \tag{8.33}$$

from the algebra of symmetric $q$-Dunkl operators to the algebra of Macdonald–Ruijsenaars operators. Furthermore, for all $f(\xi) \in \mathbb{C}[\xi^{\pm 1}]^W$, we have

$$f(Y)P_\lambda(x) = L_x^f P_\lambda(x) = f(t^\rho q^\lambda) P_\lambda(x) \quad (\lambda \in \mathcal{P}_n). \tag{8.34}$$

## 8.4 Nonsymmetric Macdonald Polynomials

One can directly check that the operators $T_i$ $(i = 0.1, \dots, n-1)$ stabilize the algebra $\mathbb{C}[x^{\pm 1}]$ of Laurent polynomials in $x = (x_1, \dots, x_n)$. Hence $\mathbb{C}[x^{\pm 1}]$ can be regarded as a left $H(\widetilde{W})$-module. It would be natural to expect that the commutative subalgebra $\mathbb{C}[Y^{\pm 1}]$ of $H(\widetilde{W})$ can be simultaneously diagonalized on $\mathbb{C}[x^{\pm 1}]$. In fact, the $q$-Dunkl operators have common eigenfunctions $E_\mu(x)$ $(\mu \in P)$, called the *nonsymmetric Macdonald polynomials*, parameterized by the weight lattice $P$.

In the following, we denote by

$$\begin{aligned} P_+ &= \{\lambda = (\lambda_1, \dots, \lambda_n) \in P \mid \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n\} \\ &= \mathbb{N}\varpi_1 \oplus \cdots \oplus \mathbb{N}\varpi_{n-1} \oplus \mathbb{Z}\varpi_n \end{aligned} \tag{8.35}$$

the cone of *dominant integral weights*, where, for $r = 1, \dots, n$, $\varpi_r = (1^r) = \varepsilon_1 + \cdots + \varepsilon_r$ (*fundamental weights*). Then, for each $\mu \in P$, there exists a unique $\mu_+ \in P_+$ in the $W$-orbit of $\mu$: $W.\mu \cap P_+ = \{\mu_+\}$. For the diagonalization of the $q$-Dunkl operators, we make use of the partial order

$$\mu \preceq \nu \quad \Longleftrightarrow \quad \mu_+ < \nu_+ \text{ or } (\mu_+ = \nu_+ \text{ and } \mu \leq \nu) \tag{8.36}$$

defined by applying the dominance order in two steps. For each $\mu \in P$, we denote by $w_\mu$ the shortest element among all $w \in W$ such that $w.\mu_+ = \mu$, and set $\rho_\mu = w_\mu.\rho \in P$.

**Theorem 8.5** *Assume that $t \in \mathbb{C}^*$ is generic. Then, for each $\mu \in P$ there exists a unique Laurent polynomial $E_\mu(x) \in \mathbb{C}[x^{\pm 1}]$ such that*

$$(1) \quad f(Y)E_\mu(x) = f(t^{\rho_\mu} q^\mu)E_\mu(x) \ \textit{for all} \ f(\xi) \in \mathbb{C}[\xi^{\pm 1}], \tag{8.37}$$

$$(2) \quad E_\mu(x) = x^\mu + (\textit{lower-order terms with respect to } \preceq). \tag{8.38}$$

Then, regarded as a $H(\widetilde{W})$-module, the algebra of Laurent polynomials $\mathbb{C}[x^{\pm 1}]$ is decomposed into irreducible components as follows:

$$\mathbb{C}[x^{\pm 1}] = \bigoplus_{\lambda \in P_+} V(\lambda), \quad V(\lambda) = \bigoplus_{\mu \in W.\lambda} \mathbb{C}E_\mu(x). \tag{8.39}$$

Furthermore, for each $\lambda \in P_+$, we have

$$V(\lambda)^{H(W)} = \left\{ v \in V(\lambda) \mid T_i v = t^{\frac{1}{2}} v \quad (i = 1, \ldots, n-1) \right\} = \mathbb{C}P_\lambda(x), \tag{8.40}$$

where $P_\lambda(x)$ is the Macdonald (Laurent) polynomial attached to $\lambda \in P_+$; if we take $l \in \mathbb{Z}$ and $\mu \in \mathcal{P}_n$ such that $\lambda = \mu + (l^n)$, then $P_\lambda(x)$ is expressed as $P_\lambda(x) = (x_1 \ldots x_n)^l P_\mu(x)$ in terms of the Macdonald polynomial $P_\mu(x)$ attached to a partition $\mu \in \mathcal{P}_n$.

In this picture, for each $\lambda \in P_+$ the Macdonald polynomial $P_\lambda(x)$ is expressed as a linear combination of nonsymmetric Macdonald polynomials $E_\mu(x)$ $(\mu \in W.\lambda)$ with explicitly determined coefficients. Also, $P_\lambda(x)$ is obtained by applying the *symmetrizer* $\sum_{w \in W} t^{\frac{1}{2}\ell(w)} T_w$ of the Hecke algebra $H(W)$ to $E_\lambda(x)$.

## 8.5 Double Affine Hecke Algebra and Cherednik Involution

The algebra $\mathcal{D}_{q,x}[W]$ contains the subalgebra

$$\mathbb{C}[x^{\pm 1}; \tau^{\pm}][W] = \mathbb{C}\langle x_1^{\pm 1}, \ldots, x_n^{\pm 1}; s_1, \ldots, s_{n-1}; \tau_1^{\pm 1}, \ldots, \tau_n^{\pm 1} \rangle \subseteq \mathcal{D}_{q,x}[W] \tag{8.41}$$

of $q$-difference-reflection operators with Laurent polynomial coefficients. This algebra can be thought of as a $q$-version of $\mathbb{C}[x; \partial_x][W]$ (crossed product of the Heisenberg algebra and the Weyl group). One can consider the $t$-deformation of this algebra

$$\begin{aligned} DH(\widetilde{W}) &= \mathbb{C}[x^{\pm 1}] \otimes H(W) \otimes \mathbb{C}[Y^{\pm 1}] \\ &= \mathbb{C}\langle x_1^{\pm 1}, \ldots, x_n^{\pm 1}; T_1, \ldots, T_{n-1}; Y_1^{\pm 1}, \ldots, Y_n^{n-1} \rangle \subseteq \mathcal{D}_{q,x}[W], \end{aligned} \tag{8.42}$$

called the *double affine Hecke algebra*. This algebra consists of all operators of the form

$$A = \sum_{\mu,\nu\in P;\ w\in W} a_{\mu,w,\nu}\, x^{\mu}\, T_w\, Y^{\nu} \quad \text{(finite sum)} \quad (a_{\mu,w,\nu} \in \mathbb{C}). \tag{8.43}$$

Also, the commutation relations between $T_i$ and $x_j$, $Y_j$ are given by

$$T_i x_i T_i = x_{i+1} \quad (i = 1, \ldots, n-1), \quad T_i x_j = x_j T_i \quad (j \neq i, i+1), \tag{8.44}$$

and

$$T_i Y_{i+1} T_i = Y_i \quad (i = 1, \ldots, n-1), \quad T_i Y_j = Y_j T_i \quad (j \neq i, i+1). \tag{8.45}$$

**Theorem 8.6** (Cherednik) *There exists a unique involutive anti-homomorphism* $\phi : DH(\widetilde{W}) \to DH(\widetilde{W})$ *such that*

$$\begin{aligned} &\phi(x_i) = Y_i^{-1}, \quad \phi(Y_i) = x_i^{-1} \quad (i = 1, \ldots, n), \\ &\phi(T_i) = T_i \quad (i = 1, \ldots, n-1). \end{aligned} \tag{8.46}$$

This anti-involution $\phi$ is called the *Cherednik involution* ($\phi(1) = 1$, $\phi(ab) = \phi(b)\phi(a)$, $\phi^2 = 1$).

We define the *expectation value* $\langle \cdot \rangle : DH(\widetilde{W}) \to \mathbb{C}$ by

$$\langle A \rangle = A(1)\big|_{x=t^{-\rho}} = \sum_{\mu,\nu\in P; w\in W} a_{\mu,w,\nu} t^{-\langle\rho,\mu\rangle} t^{\frac{1}{2}\ell(w)} t^{\langle\rho,\nu\rangle}, \tag{8.47}$$

where $\ell(w)$ the length of $w$ (i.e. the number of inversions). We also define a scalar product (bilinear form)

$$\langle\ ,\ \rangle : \quad DH(\widetilde{W}) \times DH(\widetilde{W}) \to \mathbb{C} \tag{8.48}$$

by

$$\langle A, B \rangle = \langle \phi(A) B \rangle \in \mathbb{C} \qquad (A, B \in DH(\widetilde{W})). \tag{8.49}$$

By the definition of the Cherednik involution, we have

$$\langle \phi(A) \rangle = \langle A \rangle, \quad \langle A, B \rangle = \langle B, A \rangle. \tag{8.50}$$

(This bilinear form is a variation of Fisher's scalar product.)

We apply formula (8.50) to Macdonald polynomials $A = P_\lambda(x)$ and $B = P_\mu(x)$ ($\lambda, \nu \in P_+$).

$$\begin{aligned} \langle P_\lambda(x), P_\mu(x) \rangle &= \langle \phi(P_\lambda(x)) P_\mu(x) \rangle = \langle P_\lambda(Y^{-1}) P_\mu(x) \rangle \\ &= \langle P_\lambda(t^{-\rho} q^{-\mu}) P_\mu(x) \rangle = P_\lambda(t^{-\rho} q^{-\mu}) P_\mu(t^{-\rho}). \end{aligned} \tag{8.51}$$

Since $\langle P_\lambda(x), P_\mu(x)\rangle = \langle P_\mu(x), P_\lambda(x)\rangle$, we have

$$P_\lambda(t^{-\rho}q^{-\mu})P_\mu(t^{-\rho}) = P_\mu(t^{-\rho}q^{-\lambda})P_\lambda(t^{-\rho}) \qquad (\lambda, \mu \in P_+), \tag{8.52}$$

and hence

$$\frac{P_\lambda(t^{-\rho}q^{-\mu})}{P_\lambda(t^{-\rho})} = \frac{P_\mu(t^{-\rho}q^{-\lambda})}{P_\mu(t^{-\rho})}. \tag{8.53}$$

By the property $P_\lambda(x; q, t) = P_\lambda(x; q^{-1}, t^{-1})$ of Macdonald polynomials, from (8.53) we obtain

$$\frac{P_\lambda(t^{\rho}q^{\mu})}{P_\lambda(t^{\rho})} = \frac{P_\mu(t^{\rho}q^{\lambda})}{P_\mu(t^{\rho})} \qquad (\lambda, \mu \in P_+). \tag{8.54}$$

Since $\rho = \delta - \frac{1}{2}(n-1)(1^n)$, this formula is identical to the self-duality we discussed in Chap. 6.

# Notation

| | |
|---|---|
| $\mathbb{N} = \mathbb{Z}_{\geq 0} = \{0, 1, 2, \ldots\}$ | set of natural numbers (nonnegative integers) |
| $\mathbb{Z} = \{0, \pm 1, \pm 2, \ldots\}$ | set of integers |
| $\mathbb{Q}, \mathbb{R}, \mathbb{C}$ | sets of rational, real, complex numbers |
| $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$ | multiplicative group of nonzero complex numbers |
| $R[x] = R[x_1, \ldots, x_n]$ | ring of polynomials in variables $x = (x_1, \ldots, x_n)$ with coefficients in a ring $R$ |
| $R[x^{\pm 1}] = R[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$ | ring of Laurent polynomials in $x = (x_1, \ldots, x_n)$ with coefficients in a ring $R$ |
| $R[\![x]\!] = R[\![x_1, \ldots, x_n]\!]$ | ring of formal power series in $x = (x_1, \ldots, x_n)$ |
| $R^G$ | subring of $G$-invariant elements in a ring $R$ on which a group $G$ acts by ring automorphisms |
| $K(x) = K(x_1, \ldots, x_n)$ | field of rational functions in $x = (x_1, \ldots, x_n)$ with coefficients in a field $K$ |
| $K[G] = \bigoplus_{g \in G} Kg$ | group-ring of a group $G$ with coefficients in a field $K$ |
| $\mathfrak{S}_n$ | symmetric group of degree $n$, group of permutations of $\{1, \ldots, n\}$ [1.1] |
| $\mathrm{sgn}(\sigma)$ | sign of a permutation $\sigma \in \mathfrak{S}_n$ [2.2] |
| $\ell(\sigma)$ | number of inversions of a permutation $\sigma \in \mathfrak{S}_n$ [2.2] |
| $\mathrm{GL}_n = \mathrm{GL}_n(\mathbb{C})$ | general linear group of degree $n$, group of $\mathbb{C}$-automorphisms of $\mathbb{C}^n$ [2.4, 3.9] |
| $P = \bigoplus_{i=1}^n \mathbb{Z}\varepsilon_i$ | weight lattice of $\mathrm{GL}_n$ [2.4, 8.4] |
| $P_+ = \bigoplus_{i=1}^n \mathbb{N}\varpi_i$ | cone of dominant integral weights [4.1, 8.4] |
| $T_{q,x_i}$ $(i = 1, \ldots, n)$ | $q$-shift operator in $x_i$ [1.1, 3.8, 4.1] |
| $T_{q,x}^I = \prod_{i \in I} T_{q,x_i}$ | $q$-shift operator in $x_i$ $(i \in I)$ [5.3] |
| $\mathcal{D}_{q,x} = \mathbb{C}(x)[T_{q,x}^{\pm 1}]$ | ring of $q$-difference operators with coefficients in $\mathbb{C}(x)$ [4.1] |



M. Noumi, *Macdonald Polynomials*, SpringerBriefs in Mathematical Physics,
https://doi.org/10.1007/978-981-99-4587-0

| | |
|---|---|
| $(\alpha)_k = \prod_{i=0}^{k-1}(\alpha + i)$ | shifted factorial ($k \in \mathbb{N}$) [4.4] |
| ${}_{r+1}F_r$ $(r = 0, 1, 2, \ldots)$ | generalized hypergeometric series [4.4] |
| $(a; q)_k = \prod_{i=0}^{k-1}(1 - q^i a)$ | $q$-shifted factorial ($k \in \mathbb{N}$) [4.2] |
| $(a; q)_\infty = \prod_{i=0}^{\infty}(1 - q^i a)$ | $q$-infinite product ($\lvert q\rvert < 1$) [4.4] |
| ${}_{r+1}\phi_r$ $(r = 0, 1, 2, \ldots)$ | $q$-hypergeometric series, $q$-analogue of ${}_{r+1}F_r$ [4.4] |
| $\phi_D$ | $q$-analogue of Lauricella's hypergeometric series $F_D$ in $n$ variables [4.5] |
| $\mathcal{P}$ | set of all partitions $\lambda = (\lambda_1, \lambda_2, \ldots)$ [1.1] |
| $\mathcal{P}_n$ | set of partitions $\lambda = (\lambda_1, \ldots, \lambda_n)$ with $\ell(\lambda) \le n$ [1.1] |
| $\lambda = (\lambda_1, \lambda_2, \ldots)$ | partition of $\lvert\lambda\rvert = \lambda_1 + \lambda_2 + \cdots$ [1.1] |
| $\lambda' = (\lambda'_1, \lambda'_2, \ldots)$ | conjugate (transpose) of a partition $\lambda$ [2.4] |
| $\ell(\lambda)$ | number of nonzero parts of a partition $\lambda$ [1.1] |
| $a_\lambda(s), l_\lambda(s)$ | arm and leg lengths of $s \in \lambda$ [3.2] |
| $c_\lambda(s), h_\lambda(s)$ | content and hook length of $s \in \lambda$ [3.2] |
| $n(\lambda)$ | $\sum_{i\ge1}(i-1)\lambda_i = \sum_{j\ge1}\binom{\lambda'_j}{2}$ [3.2, 6.1] |
| $\delta = (n-1, n-2, \ldots, 0)$ | staircase partion of $n-1$ parts |
| $M = \{\lambda_i + n - i \mid i = 1, \ldots, n\}$ | Maya diagram attached to $\lambda \in \mathcal{P}_n$ [3.1] |
| $\mu \le \nu$ | dominance order of multi-indices $\mu, \nu \in \mathbb{N}^n$ [2.4] |
| $e_k(x), h_k(x), p_k(x)$ | elementary, complete homogenous and power sum symmetric functions [2.1] |
| $m_\lambda = m_\lambda(x)$ $(\lambda \in \mathcal{P}_n)$ | monomial symmetric function of type $\lambda$ [1.1, 2.4] |
| $\Delta(x) = \prod_{1\le i<j\le n}(x_i - x_j)$ | difference product, Vandermonde determinant [2.2] |
| $\Delta_\mu(x) = \det(x_i^{\mu_j})_{i,j=1}^n$ | alternating polynomial of monomial type $\mu$ [3.2] |
| $s_\lambda = s_\lambda(x) = \Delta_{\lambda+\delta}(x)/\Delta(x)$ | Schur function attached to a partition $\lambda$ [1.1, 3.1] |
| $\mathrm{SSTab_n}(\lambda)$ | set of semi-standard tableaux of shape $\lambda$ [3.1] |
| $P_\lambda(x) = P_\lambda(x; q, t)$ | Macdonald polynomial [1.1, 4.1] |
| $\widetilde{P}_\lambda(x) = P_\lambda(x)/P_\lambda(t^\delta)$ | normalized Macdonald polynomial [6.1] |
| $P_\lambda^{(\beta)}(x)$ | Jack polynomial $\lim_{q\to1} P_\lambda(x; q, q^\beta)$ [1.1, 5.6] |
| $D_x$ | Macdonald–Ruijsenaars $q$-difference operator (of first order) [1.1, 4.1] |
| $d_\lambda = \sum_{i=1}^n t^{n-i}q^{\lambda_i}$ | eigenvalues of $D_x$ ($\lambda \in \mathcal{P}_n$) [1.1, 4.1] |
| $w(x) = w(x; q, t)$ | weight function for the orthogonality [5.1] |
| $\langle f, g\rangle$ | scalar product with weight function $w(x)$ [5.1] |
| $D_x^{(r)}$ $(r = 0, 1, \ldots, n)$ | higher-order Macdonald–Ruijsenaars $q$-difference operators [5.3] |
| $D_x(u) = \sum_{r=0}^n(-u)^r D_x^{(r)}$ | generating function for $D_x^{(r)}$ $(r = 0, 1, \ldots, n)$ [5.3] |
| $A_I(x)$ | coefficient $t^{\binom{\lvert I\rvert}{2}}\prod_{i\in I, j\notin I}\frac{tx_i - x_j}{x_i - x_j}$ of $D_x^{(r)}$ [5.3] |
| $\psi_{\lambda/\mu} = \psi_{\lambda/\mu}(q, t)$ | Pieri coefficient for a horizontal strip $\lambda/\mu$ [6.1] |
| $\psi'_{\lambda/\mu} = \psi_{\lambda'/\mu'}(t, q)$ | Pieri coefficient for a vertical strip $\lambda/\mu$ [6.1] |
| $\Pi_{m,n}(x; y)$ | $\prod_{i=1}^m\prod_{j=1}^n\frac{(tx_iy_j; q)_\infty}{(x_iy_j; q)_\infty}$ (Cauchy kernel) [6.5] |
| $\Pi_{m,n}^\vee(x; y)$ | $\prod_{i=1}^m\prod_{j=1}^n(1 + x_iy_j)$ (dual Cauchy kernel) [6.5] |
| $\Psi_{m,n}(x; y)$ | $\prod_{i=1}^m\prod_{j=1}^n(x_i + y_j)$ [6.5] |

| | |
|---|---|
| $b_\lambda = b_\lambda(q,t)$ | coefficients of the Cauchy formula ($\lambda \in \mathcal{P}$) [6.5] |
| $c^\lambda_{\mu,\nu} = c^\lambda_{\mu,\nu}(q,t)$ | Littlewood–Richardson coefficient ($\lambda, \mu, \nu \in \mathcal{P}$) [7.1] |
| $b^\lambda_{\mu,\nu} = b^\lambda_{\mu,\nu}(q,t)$ | branching coefficient ($\lambda, \mu, \nu \in \mathcal{P}$)[7.1] |
| $W, H(W)$ | Weyl group and its Hecke algebra ($W = \mathfrak{S}_n$) [8.1] |
| $W^{\mathrm{aff}}, \widetilde{W}$ | affine and extended affine Weyl groups [8.1] |
| $H(W^{\mathrm{aff}}), H(\widetilde{W})$ | affine and extended affine Heck algebras [8.1] |
| $T_i\ (i = 0, 1, \ldots, n)$ | Demazure–Lustzig operators [8.1] |
| $s_i\ (i = 0, 1, \ldots, n)$ | simple reflections [8.1] |
| $\omega$ | diagram rotation [8.1] |
| $Y_i\ (i = 1, \ldots, n)$ | $q$-Dunkl operators [8.2] |
| $E_\mu(x)\ (\mu \in P)$ | nonsymmetric Macdonald polynomials [8.2]. |

# References

1. G.E. Andrews, R. Askey, R. Roy, *Special Functions, Encyclopedia of Mathematics and its Applications*, vol. 71 (Cambridge University Press, 1999), xvi+664 pp
2. F. Atai, M. Noumi, Eigenfunctions of the van Diejen model generated by gauge and integral transformations. Adv. Math. **412**, Paper No.108816, 60 pp (2023)
3. I. Cherednik, Double affine Hecke algebras, Knizhnik–Zamolodchikov equations, and Macdonald's operators. Internat. Math. Res. Not. 171–180 (1992)
4. I. Cherednik, Double affine Hecke algebras and Macdonald's conjectures. Ann. Math. **141**, 191–216 (1995)
5. I. Cherednik, *Double Affine Hecke Algebras*. London Mathematical Society Lecture Note Series, vol. 319 (Cambridge University Press, 2005), xii+434 pp
6. L. Euler, *Introductio in Analysin Infinitorum*. Tomus Primus, 1748. Leonhardi Euleri Operat Omina I, 8
7. R. Gangolli, V.S. Varadarajan, *Harmonic Analysis of Spherical Functions on Real Reductive Groups*. Ergebnisse der Mathematik und ihrer Grenzgebiete, vol. 101 (Springer, Berlin, 1988), xiv+365 pp
8. G. Gasper, M. Rahman, *Basic Hypergeometric Series*, 2nd ed. (Cambridge University Press, 2004), xxvi+428 pp
9. R. Goodman, N.R. Wallach, *Symmetry, Representations and Invariants*. Graduate Texts in Mathematics, vol. 255 (Springer, 2009), xx+716 pp
10. J. Haglund, *The $q,t$-Catalan Numbers and the Space of Diagonal Harmonics*. With an Appendix on the Combinatorics of Macdonald Polynomials. University Lecture Series, vol. 41 (American Mathematical Society, 2008), viii+167 pp
11. G. Heckman, H. Schlichtkrull, *Harmonic Analysis and Special Functions on Symmetric Spaces*. Perspectives in Mathematics, vol. 16 (Academic Press, Inc., 1994), xii+225 pp
12. R. Koekoek, P.A. Lesky, R.F Swarttouw, *Hypergeometric Orthogonal Polynomials and Their $q$-analogues*. Springer Monographs in Mathematics (Springer, 2010), xx+578 pp
13. Y. Komori, K. Hikami, Quantum integrability of the generalized elliptic Ruijsenaars models. J. Phys. A **30**, 4341–4364 (1997)
14. T. Koornwinder, Self-duality for $q$-ultraspherical polynomials associated with root system $A_n$, handwritten manuscript, 2 Feb. 1988, 17 pp. (Tom Koornwinder's Home Page: https://staff.fnwi.uva.nl/t.h.koornwinder/art/informal/)



M. Noumi, *Macdonald Polynomials*, SpringerBriefs in Mathematical Physics,
https://doi.org/10.1007/978-981-99-4587-0

15. T.H. Koornwinder, J.V. Stokman (eds.), *Multivariable Special Functions*. Encyclopedia of Special Functions: The Askey-Bateman Project, vol. 2 (Cambridge Univ. Press, 2021), xii+427 pp
16. E. Langmann, Explicit solutions of the (quantum) elliptic Calogero-Sutherland model. Ann. Henri Poincaré **15**, 755–791 (2014)
17. E. Langmann, M. Noumi, J. Shiraishi, Basic properties of non-stationary Ruijsenaars functions. SIGMA **16**, Paper No. 105, 26 pp (2020)
18. E. Langmann, M. Noumi, J. Shiraishi, Construction of eigenfunctions for the elliptic Ruijsenaars difference operators. Commun. Math. Phys. **391**, 901–950 (2022)
19. I.G. Macdonald, A new class of symmetric functions. Sém. Lothar. Combin. **20**, Paper B201, 41 pp (1988)
20. I.G. Macdonald, *Symmetric Functions and Hall Polynomials*, 2nd ed. (Oxford University Press, 1995), x+475 pp
21. I.G. Macdonald, *Symmetric Functions and Orthogonal Polynomials*. University Lecture Series, vol. 12 (American Mathematical Society, 1998), xvi+53 pp
22. I.G. Macdonald, *Affine Hecke Algebras and Orthogonal Polynomials*. Cambridge Tracts in Mathematics, vol. 157 (Cambridge University Press, 2003), x+175 pp
23. K. Mimachi, A new derivation of the inner product formula for the Macdonald symmetric polynomials. Compositio Math. **113**, 117–122 (1998)
24. M. Noumi, Macdonald's symmetric polynomials as zonal spherical functions on some quantum homogeneous spaces. Adv. Math. **123**, 16–77 (1996)
25. M. Noumi, *Painlevé Equations Through Symmetry*. Translations of Mathematical Monographs, vol. 223 (American Mathematical Society, 2004), vii+156 pp
26. M. Noumi, A. Sano, An infinite family of higher-order difference operators that commute with Ruijsenaars operators of type $A$. Lett. Math. Phys. **111**, Paper No. 91. 17 pp (2021)
27. M. Noumi, J. Shiraishi, A direct approach to the bispectral problem for the Ruijsenaars–Macdonald $q$-difference operators (arXiv:1206.5364, 44 pages)
28. M. Noumi, T. Sugitani, Quantum symmetric spaces and related $q$-orthogonal polynomials, in *Group Theoretical Methods in Physics* (*Toyonaka, 1994*), pp. 28–40, (World Scientific Publishing Co Pte Ltd., 1995)
29. A. Ram, M. Yip, A combinatorial formula for Macdonald polynomials. Adv. Math. **226**, 309–331 (2011)
30. S.N.M. Ruijsenaars, Complete integrability of relativistic Calogero-Moser system and elliptic function identities. Commun. Math. Phys. **110**, 181–213 (1987)
31. B.E. Sagan, *The Symmetric Group—Representations, Combinatorial Algorithms, and Symmetric Functions*. Graduate Texts in Mathematics, vol. 203, 2nd ed. (Springer, 2001), xvi+238 pp
32. A.N. Sergeev, A.P. Veselov, Deformed Macdonald-Ruijsenaars operators and super Macdonald polynomials. Commun. Math. Phys. **288**, 653–675 (2009)
33. J.F. van Diejen, Integrability of difference Calogero-Moser system. J. Math. Phys. **35**, 2983–3004 (1994)
34. J.F. van Diejen, T. Görbe, Eigenfunctions of a discrete elliptic integrable particle model with hyperoctahedral symmetry. Commun. Math. Phys. **392**, 279–305 (2022)