Chapter 7
Algebraic Geometry

In this chapter, we will study algebraic geometry and its surroundings. First, we will learn about algebraic sets and manifolds. A manifold is like a globe (an open set family) that has many local maps (open sets). Each open set must be a one-to-one correspondence with the open set of the same dimensional Euclidean space, which allows us to define local variables and local coordinates. The resolution of singular points, referred to as blow-ups, denotes the process of updating local coordinates containing singular points to other local coordinates. The Watanabe-Bayes theory aims to obtain a standard form called normal crossing for each local coordinate. In the regular case, the dimension d of the parameter is twice the real logarithmic threshold λ in the general case. This value of λ can be obtained by resolving singular points. In fact, the resolution of singular points is not directly related to the Watanabe-Bayes theory. This point is often misunderstood in Watanabe-Bayes theory. Based on the Hironaka theorem, whether there are singular points or not, in each local coordinate, we transform the average log-likelihood to normal crossing.

Readers who are learning algebraic geometry for the first time may not understand what is written here at all. In such a case, as mentioned in the "Introduction", I recommend slowly reading while writing the formulas in each section. If you still do not understand, I recommend repeating the same thing tomorrow and the day after. Eventually, you should feel more comfortable.

7.1 Algebraic Sets and Analytical Sets

Hereafter, we denote the set of real-number-coefficient polynomials with variables $x = (x_1, \ldots, x_d)$ as $\mathbb{R}[x_1, \ldots, x_d]$ or $\mathbb{R}[x]$. At this time, using the subset J of $\mathbb{R}[x]$ (assuming it is not an empty set), we define the set I that can be written as

J. Suzuki, *WAIC and WBIC with R Stan*,
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$$I = \left\{ \sum_i f_i(x) g_i(x) \mid f_i(x) \in J, g_i(x) \in \mathbb{R}[x] \right\}$$

as the *ideal* of $\mathbb{R}[x]$. We also say that set J generates the ideal I.

Example 58 For $J = \{x + y^2, x - y^2, y^3\} \subseteq \mathbb{R}[x, y]$, $I = \{x f(x, y) + y^2 g(x, y) \mid f, g \in \mathbb{R}[x, y]\}$ is the ideal generated by J. In other words, the same ideal is generated by $J = \{x, y^2\}$. While $I_1 = \{x f(x, y) \mid f \in \mathbb{R}[x, y]\}$ and $I_2 = \{y^2 g(x, y) \mid g \in \mathbb{R}[x, y]\}$ are ideals, we have

$$I \ni x - y^2 \notin I_1 \cup I_2.$$

■

The set of common zeros of the elements of the ideal I is given by

$$V(I) = \left\{ x \in \mathbb{R}^d \mid f(x) = 0, f \in I \right\} \subseteq \mathbb{R}^d$$

and it is called the *algebraic set* determined by the ideal I in $\mathbb{R}[x]$. If the algebraic set V can be expressed as the union of two distinct non-empty algebraic sets V_1 and V_2, it is said to be *reducible*. Otherwise, it's called *irreducible*. Hereafter, we assume that $V(I)$ is irreducible, and for simplicity, we will refer to it simply as V. In contrast with the forthcoming projective space $\mathbb{P}^d$, we sometimes denote the d-dimensional Euclidean space $\mathbb{R}^d$ as the affine space $\mathbb{A}^d$ according to tradition. When given the algebraic set V, note that the subset of $\mathbb{R}[x]$

$$I(V) = \{f \in \mathbb{R}[x] \mid f(x) = 0, x \in V\}$$

forms an ideal in $\mathbb{R}[x]$.

Example 59 The algebraic set of the ideal I generated by $J = \{x^2 + y^2\}$ is

$$V = V(I) = \left\{ (x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 0 \right\} = \{(0, 0)\}$$

Consequently, $I(V)$ is

$$I(V) = \{f \in \mathbb{R}[x, y] \mid f(0, 0) = 0\} = \left\{ x, y, x^2, y^2, xy, \ldots \right\}$$

and it becomes the ideal generated by $J = \{x, y\}$.

Similarly, we can construct an *analytic set*

$$\{x \in U \mid f(x) = 0\}$$

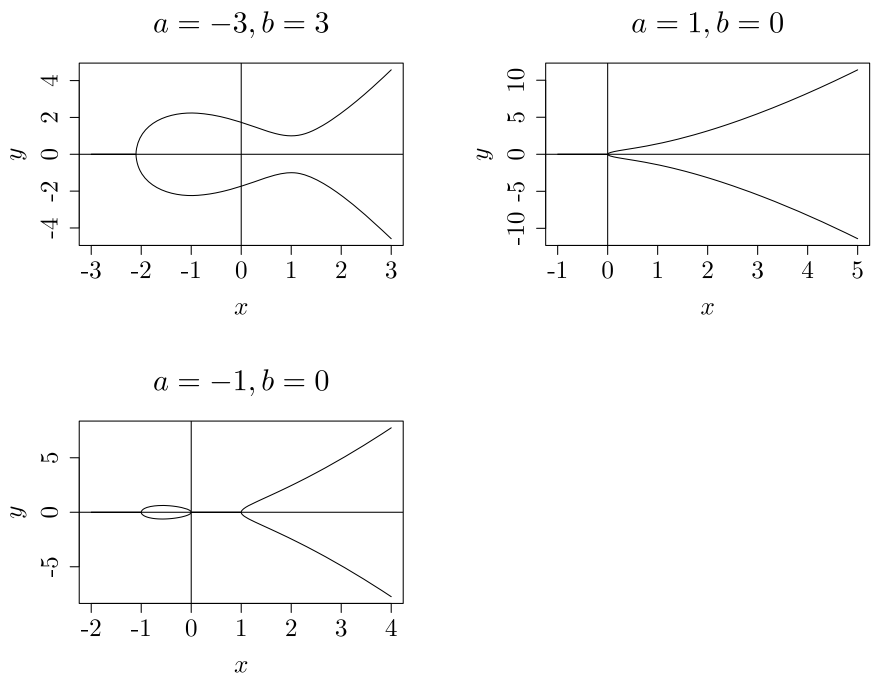

Fig. 7.1 The case where the elliptic curve $y^2 = x^3 + ax + b$ does not have a singularity (refer to Sect. 6.3). The above three types are considered typical

using an analytic function f with an open set $U \subseteq \mathbb{R}^d$ as its domain. Using multiple analytic functions $f_1, \dots, f_d : U \to \mathbb{R}$ that share the domain U, we can also define an analytic set

$$\{x \in U \mid f_1(x) = 0, \dots, f_d(x) = 0\}.$$

Also, if $f : U \to V$ is defined as $f = (f_1, \dots, f_d)$ by analytic functions $f_1, \dots, f_d : U \to \mathbb{R}$ with U, V being open sets in $\mathbb{R}^d$, we call f an *analytic map*.

In this chapter, we mainly focus on algebraic sets defined by a single irreducible (i.e., cannot be factored further) polynomial $0 \not\equiv f \in \mathbb{R}[x]$

$$V(f) = \{x \in \mathbb{R}^d \mid f(x) = 0\} \subseteq \mathbb{R}^d$$

or analytic sets constituted by a single analytic function $f : U \to \mathbb{R}$

$$V(f) = \{x \in U \mid f(x) = 0\} \subseteq U.$$

Example 60 *(Elliptic curve)* For $a, b \in \mathbb{R}$, a curve on a plane determined by the polynomial of 2 variables $f(x, y) = y^2 - x^3 - ax - b = 0$ is called an elliptic curve. Try to draw the outline of the algebraic set

$$V(f) = \{(x, y) \in \mathbb{R}^2 \mid f(x, y) = 0\}$$

for $(a, b) = (-3, 3), (1, 0), (-1, 0)$ using the R language (Fig. 7.1). The following code is used: ■

```
a <- -3
b <- 3
x.min <- -3
x.max <- 3     # Non-singular case (1)
# a <- 1; b <- 0; x.min <- -1; x.max <- 5    # Non-singular case (2)
# a <- -1; b <- 0; x.min <- -2; x.max <- 4   # Non-singular case (3)
# a <- 0; b <- 0; x.min <- -1; x.max <- 5    # Case including singularity
     (1)
# a <- -3; b <- 2; x.min <- -3; x.max <- 3   # Case including singularity
     (2)
f <- function(x) sqrt(max(x^3+a*x+b,0))
x.seq <- seq(x.min,x.max,0.001)
y.seq <- NULL
for(x in x.seq) y.seq <- c(y.seq,f(x))
y.max <- max(y.seq)
plot(0,xlab="x", ylab="y",xlim=c(x.min,x.max), ylim=c(-y.max,y.max),type=
     "n",
     main=paste("a=",a,", b=",b))
lines(x.seq,y.seq)
lines(x.seq,-y.seq)
abline(h=0)
abline(v=0)
```

7.2 Manifold

In this section, we define *topological spaces* and (analytic) *manifolds*. Let M be a set. When a set $\mathcal{U}$ consisting of subsets of M is defined to satisfy the following three conditions, M is called a topological space, $\mathcal{U}$ is called a *family of open sets*, and the elements of $\mathcal{U}$ are called open sets.[1]

1. $\mathcal{U}$ includes the entire set M and the empty set $\{\}$ as elements.
2. The union of any number of elements of $\mathcal{U}$ (open sets of M) is an element of $\mathcal{U}$.
3. The intersection of any finite number of elements of $\mathcal{U}$ (open sets of M) is an element of $\mathcal{U}$.

Furthermore, for any $x \neq y \in M$, when there exist $U, V \in \mathcal{U}$ such that $x \in U, y \in V$, and $U \cap V = \{\}$, that topological space M is called a *Hausdorff space*.

Example 61 *(Distance space)* If a distance $d(x, y)$, $x, y \in M$ is defined for the set M, we can define an open set $B(\epsilon, x) := \{y \in M \mid dist(x, y) < \epsilon\}$ using it, so the family of open sets can be defined as

[1] Including distance metrics such as Euclidean distance to define open sets (metric spaces).

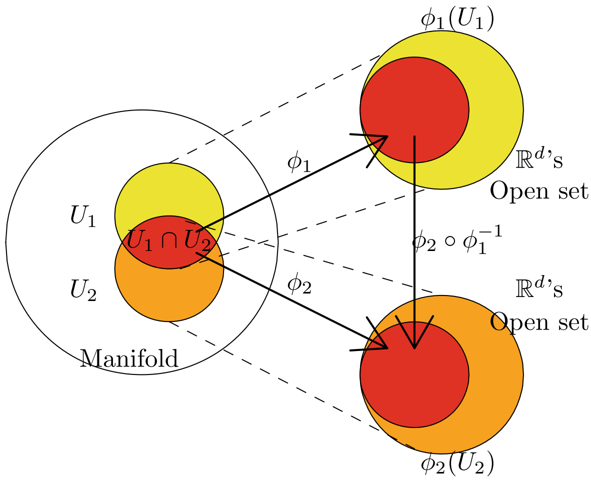

Fig. 7.2 The images of $\phi_1(U_1 \cap U_2)$ and $\phi_2(U_1 \cap U_2)$ by ϕ_1 and ϕ_2 are connected by $\phi_2 \circ \phi_1^{-1}$ and $\phi_1 \circ \phi_2^{-1}$ (coordinate transformation), which are analytic mappings

$$\mathcal{U} = \{B(\epsilon, x) \mid \epsilon > 0, x \in M\}.$$

For $x, y \in M$, $x \neq y$, by taking $\epsilon > 0$ sufficiently small, we can make $B(\epsilon, x) \cap B(\epsilon, y) = \{\}$, so if the topological space M is a distance space, it is Hausdorff. ■

In the following, we define a manifold.[2] Let M be a Hausdorff topological space. When a bijection (mapping one-to-one and onto) from an open set to another open set is continuous in both directions, this mapping is said to be *homeomorphic*.

1. For each open set U of the family of open sets of M, there exists a ϕ such that $U \to \phi(U) \subseteq \mathbb{R}^d$ is homeomorphic.
2. For such pairs (U, ϕ) and $(\tilde{U}, \tilde{\phi})$, when $U \cap \tilde{U}$ is not empty, the *coordinate transformation*

$$\phi \circ \tilde{\phi}^{-1}(U \cap \tilde{U}) : \phi(U \cap \tilde{U}) \to \tilde{\phi}(U \cap \tilde{U})$$

is an analytic mapping (see Fig. 7.2).

In this case, M is said to be a *d-dimensional analytic manifold*. At this time, each element u of U can be treated as if it were an element $\phi(u)$ of the open set $\phi(U)$ of $\mathbb{R}^d$. This $\phi(u) \in \mathbb{R}^d$ is called a *local variable*, and the coordinates constructed by them are called *local coordinates*. By using local coordinates, it is possible to treat a point on U as if it were a point in $\mathbb{R}^d$. In addition, such a set consisting of (U, ϕ) is called a *local coordinate system* of M.

Example 62 $\mathbb{A}^d$ is a (trivial) d-dimensional manifold. With the identity mapping $id : \mathbb{R}^d \to \mathbb{R}^d$, $S = \{(\mathbb{R}^d, id)\}$ forms a coordinate neighborhood system. It may be

[2] A topological space being Hausdorff is a necessary condition for the existence of a partition of unity (Sect. 7.5)

divided into multiple local coordinates. When $d = 1$, define $U_i = (i-1, i+1)$ and ϕ_i for each $i \in \mathbb{Z}$ as

$$\phi_i : U_i \ni x \mapsto x - i \in (-1, 1).$$

Then, ϕ_i becomes a local coordinate system of U_i, and $S = \{(U_i, \phi_i) \mid i \in \mathbb{Z}\}$ gives a coordinate neighborhood system. In this case, for $x \in (0, 1)$,

$$\phi_{i+1} \circ \phi_i^{-1}(x) = x - 1$$

becomes the coordinate transformation. And for each i, $x - i \in (-1, 1)$ can be used as a local variable. ■

As a typical example of a manifold, we consider the projective space. For each $(x_0, x_1, \dots, x_d), (x'_0, x'_1, \dots, x'_d) \in \mathbb{R}^{d+1} \backslash \{(0, \dots, 0)\}$, when there exists a $t \in \mathbb{R}$ such that

$$(x_0, x_1, \dots, x_d) = t(x'_0, x'_1, \dots, x'_d) \ ,$$

an equivalence relationship exists between them. When each class is denoted as $[x_0 : x_1 : \dots : x_d]$, the set of elements is written as $\mathbb{P}^d$, which is called a *d-dimensional projective space*.

Example 63 $\mathbb{P}^d$ is a d-dimensional manifold. When the value of the ith coordinate is not zero, by dividing the values of other coordinates by its value, we get $U_i := \{[x_0 : x_1 : \dots : x_{i-1} : 1 : x_{i+1} : \dots : x_d] \in \mathbb{P}^d\}$. The coordinate transformation

$$\phi_i : U_i \ni [x_0 : x_1 : \dots : x_{i-1} : 1 : x_{i+1} : \dots : x_d] \mapsto (x_0, x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_d) \in \mathbb{A}^d$$

from $\phi_i(U_i \cap U_j)$ to $\phi_j(U_i \cap U_j)$ becomes

$$\begin{aligned} &\phi_j \circ \phi_i^{-1}(x_0, x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_d) \\ &= \begin{cases} \left(\frac{x_0}{x_j}, \frac{x_1}{x_j}, \dots, \frac{x_{i-1}}{x_j}, \frac{1}{x_j}, \frac{x_{i+1}}{x_j}, \dots, \frac{x_{j-1}}{x_j}, \frac{x_{j+1}}{x_j}, \dots, \frac{x_d}{x_j}\right), \ i < j \\ \left(\frac{x_0}{x_j}, \frac{x_1}{x_j}, \dots, \frac{x_{j-1}}{x_j}, \frac{x_{j+1}}{x_j}, \dots, \frac{x_{i-1}}{x_j}, \frac{1}{x_j}, \frac{x_{i+1}}{x_j}, \dots, \frac{x_d}{x_j}\right), \ j < i \end{cases} , \end{aligned} \tag{7.1}$$

(Exercise 70(a)) and $S = \{(U_i, \phi_i) \mid i = 0, 1, \dots, d\}$ forms a coordinate neighborhood system. And for each i,

$$(x_0, x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_d) \in \mathbb{R}^d$$

can be used as local coordinates. Particularly, in the case of $d = 1$, it becomes

$$\phi_x : U_x \ni [1 : u_x] \mapsto u_x \in \mathbb{A}^1$$

and

$$\phi_y : U_y \ni [u_y : 1] \mapsto u_y \in \mathbb{A}^1.$$

And when $u_x, u_y \neq 0$, the coordinate transformation is given by (Exercise 70(b))

$$\phi_{xy} : \phi_x(U_x \cap U_y) \ni u_x \mapsto \frac{1}{u_x} = u_y \in \phi_y(U_x \cap U_y)$$

and

$$\phi_{yx} : \phi_y(U_x \cap U_y) \ni u_y \mapsto \frac{1}{u_y} = u_x \in \phi_x(U_x \cap U_y)$$

from $[1 : u_x] = [u_y : 1]$. ■

When looking at a certain country on a globe, one cannot see the country on the other side of the earth unless the globe is rotated. It may be interpreted that the globe is made by pasting together multiple maps.[3]

7.3 Singular Points and Their Resolution

Next, we define singular points on an algebraic set V. If the ideal $I(V)$ is generated by polynomials $f_1, \ldots, f_m$, and the rank of the matrix

$$\left(\frac{\partial f_i(x_1, \ldots, x_d)}{\partial x_j}\right)_{i=1,\ldots,m,\, j=1,\ldots,d}$$

is constant for every $x = (x_1, \ldots, x_d) \in V$, then V is said to be non-singular. If there exists an $x \in V$ where the rank is smaller, then x is called a singular point. Especially when $m = 1$ and

$$f_1(x_1, \ldots, x_d) = \frac{\partial f_1}{\partial x_1}(x_1, \ldots, x_d) = \cdots = \frac{\partial f_1}{\partial x_d}(x_1, \ldots, x_d) = 0, \tag{7.2}$$

$(x_1, \ldots, x_d) \in V$ is called a singular point of V. If V has no singular points, it's said to be non-singular.

Example 64 From Example 59, $I(V)$ is generated by $J = \{x, y\}$, thus

$$\begin{bmatrix} \frac{\partial f_1(x,y)}{\partial x} & \frac{\partial f_1(x,y)}{\partial y} \\ \frac{\partial f_2(x,y)}{\partial x} & \frac{\partial f_2(x,y)}{\partial y} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

holds for all $(x, y) \in V$. Therefore, V is non-singular.

[3] In this sense, manifolds are sometimes called "atlases", and individual (U_i, ϕ_i) are called "charts".

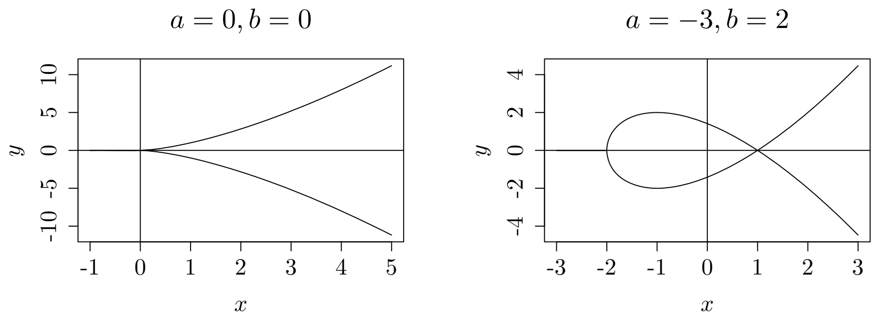

Fig. 7.3 When the elliptic curve $y^2 = x^3 + ax + b$ has a singular point. In the case of $(a, b) = (0, 0)$ (left), $(x, y) = (0, 0)$ becomes a singular point, and in the case of $(a, b) = (-3, 2)$ (right), $(x, y) = (1, 0)$ becomes a singular point. In each case, it can be seen that a tangent line cannot be drawn (as it is sharp), and multiple tangent lines can be drawn. Conversely, a non-singular algebraic curve can draw a single tangent line at any point, and can be said to be smooth

Example 65 *(Singular elliptic curve)* In Example 60, we look for the condition for having a singular point. As (7.3) specifically becomes

$$\begin{aligned} & y^2 = x^3 + ax + b\ ,\ 3x^2 + a = 0\ ,\ 2y = 0 \\ \iff & x(-\frac{a}{3} + a) + b = 0\ ,\ 3x^2 + a = 0\ ,\ y = 0 \\ \iff & \begin{cases} a = b = 0, & (x, y) = (0, 0) \\ a \neq 0, 4a^3 + 27b^2 = 0, & (x, y) = (-\frac{3b}{2a}, 0) \end{cases}\ , \end{aligned}$$

it is found that when

$$4a^3 + 27b^2 = 0 \tag{7.3}$$

$(x, y) = (0, 0)$ or $(-\dfrac{3b}{2a}, 0)$ becomes a singular point (there are no others). As $(a, b) = (0, 0), (-3, 2)$ satisfy (7.3), using the same code as in Example 60, we draw its outline (Fig. 7.3). It can be seen that at the singular point $(x, y) = (0, 0)$ in the former case, no tangent line can be drawn (as it is sharp), and at the singular point $(x, y) = (1, 0)$ in the latter case, multiple tangent lines can be drawn. ■

Next, we define the blow-up of $\mathbb{A}^2$ centered at the origin. First, we introduce a subset

$$U := \{(x, y, [x' : y']) \in \mathbb{A}^2 \times \mathbb{P}^1 \mid xy' = x'y\} \tag{7.4}$$

of $\mathbb{A}^2 \times \mathbb{P}^1$. In other words, the set U consists of two types of elements: $(x, y) \times [x : y]$ for $(x, y) \in \mathbb{A}^2 - \{(0, 0)\}$, and $(0, 0) \times [x' : y']$ for $[x' : y'] \in \mathbb{P}^1$. Furthermore, U can be written as $U_x \cup U_y$ using the set

$$U_x := \{(x, y, [x' : y']) \in U \mid x' \neq 0\}$$

$$U_y := \{(x, y, [x' : y']) \in U \mid y' \neq 0\} ,$$

and it becomes a manifold. In fact,

$$\phi_x : U_x \ni (u_x, u_x v_x, [1 : v_x]) \mapsto (u_x, v_x) \in \mathbb{A}^2$$

and

$$\phi_y : U_y \ni (u_y v_y, v_y, [u_y : 1]) \mapsto (u_y, v_y) \in \mathbb{A}^2$$

become a homeomorphism (1 to 1 and onto mapping that is continuous in both directions). Also, we can confirm that

$$(u_x, u_x v_x, [1 : v_x]) = (u_y v_y, v_y, [u_y : 1]) \Longrightarrow u_x = u_y v_y, v_y = u_x v_x, v_x u_y = 1$$

holds, so the coordinate transformation is given by

$$\phi_x(U_x \cap U_y) \ni (u_x, v_x) \mapsto (\frac{1}{v_x}, u_x v_x) = (u_y, v_y) \in \phi_y(U_x \cap U_y)$$
$$\phi_y(U_x \cap U_y) \ni (u_y, v_y) \mapsto (u_y v_y, \frac{1}{u_y}) = (u_x, v_x) \in \phi_x(U_x \cap U_y).$$

At this time, the projection $\pi : U \to \mathbb{A}^2$, which corresponds only to the $\mathbb{A}^2$ component of the assembled U excluding the $\mathbb{P}^1$ component, gives the isomorphism

$$\pi : U - \{(0, 0)\} \times \mathbb{P}^1 \ni (x, y, [x : y]) \mapsto (x, y) \in \mathbb{A}^2 - \{(0, 0)\} ,$$

when excluding the origin.

Next, we consider the algebraic set $V \subseteq \mathbb{A}^d$ with $d \geq 2$. Consider the subset U' of $U \cap (V \times \mathbb{P}^{d-1})$ restricted to $\mathbb{A}^d \times \mathbb{P}^{d-1}$, where $\pi(U') = V - (0, 0)$. Here, U' does not include elements of $(0, 0) \times \mathbb{P}^{d-1}$. Therefore, it does not generally become an algebraic set. For example,

$$\{(x, y) \in \mathbb{A}^2 \mid y^2 = x^3 + x^2\} - \{(0, 0)\}$$

is not an algebraic set. However, adding $(0, 0)$ to it yields an algebraic set $(x, y) \in \{\mathbb{A}^2 \mid y^2 = x^3 + x^2\}$. In this way, for a subset U' of $\mathbb{A}^d \times \mathbb{P}^{d-1}$, the smallest algebraic set containing it is called its closure, and it is written as $\overline{U'}$. Then, the projection $\pi : U \to \mathbb{A}^d$ is called the *origin-centered blow-up* of the algebraic set V, and the manifold $\overline{U'} = \overline{\pi^{-1}(V - (0, 0))}$ is called the *strict pullback* of V.

Example 66 In Example 65, when $(a, b) = (0, 0)$, it has a singularity only at the origin. If the local coordinates of U_x, U_y are $(u_x, v_x), (u_y, v_y)$, they can be constructed as

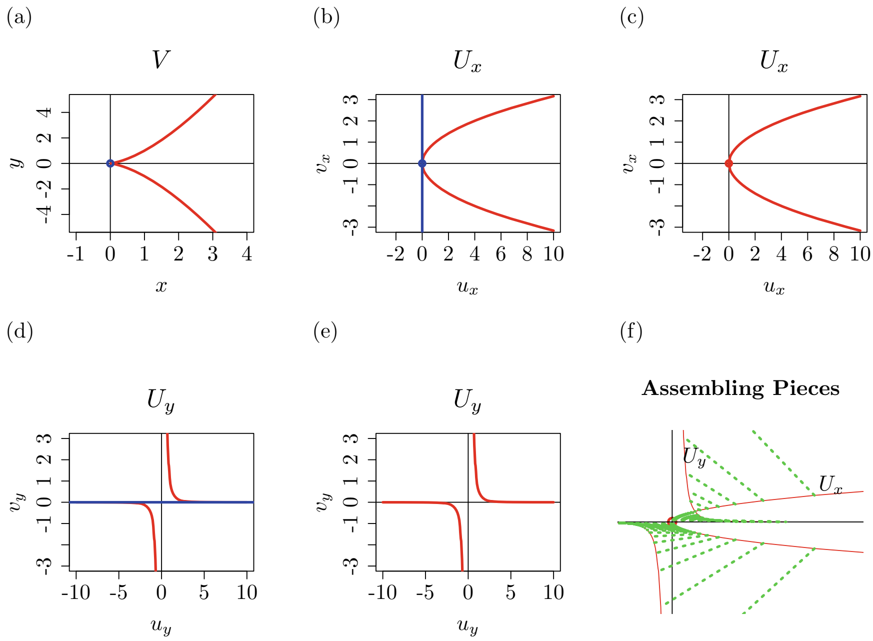

Fig. 7.4 (a) The elliptic curve $y^2 = x^3$ has a singularity at the origin. (b) It corresponds to the open set U_x. (c) Removing the blue part (curve) corresponding to the singularity in V and taking the closure (adding the origin) results in a curve without a singularity. (d) It corresponds to the open set U_y. (e) Removing the blue curve (point) corresponding to the singularity in V results in a curve without a singularity. (f) The open sets U_x, U_y correspond, except at (0, 0)

$$f(x, y) = f(u_x, u_x v_x) = u_x^2(v_x^2 - u_x)$$

$$f(x, y) = f(u_y v_y, v_y) = v_y^2(1 - u_y^3 v_y).$$

In the former case, $(x, y) = (0, 0) \iff u_x = 0$, and in the latter case, $(x, y) = (0, 0) \iff v_y = 0$, so the U' satisfying $\pi(U') = V - \{(0, 0)\}$ is $v_x^2 - u_x = 0$ when $x \neq 0$ ($u_x \neq 0$), and $1 - u_y^3 v_y = 0$ when $y \neq 0$ ($v_y \neq 0$). Taking closures of them, namely, $v_x^2 = u_x$ when $x \neq 0$, and $u_y^3 v_y = 1$ when $y \neq 0$ become the strict pull-backs, both of which are non-singular. Indeed, for $f(u_x, v_x) = v_x^2 - u_x$, $\frac{\partial f}{\partial u_x} = -1$, $\frac{\partial f}{\partial v_x} = 2v_x$, and it is impossible to make these three expressions zero simultaneously. The same is true for $f(u_y, v_y) = 1 - u_y^3 v_y$. Thus, V can be expressed in either of the local coordinates $\{(u_x, v_x) \mid v_x^2 = u_x\}$, $\{(u_y, v_y) \mid 1 = u_y^3 v_y\}$, and when it can be written in both, they correspond by the coordinate transformation $v_x u_y = 1$, $u_x = u_y v_y$, $v_y = u_x v_x$. In Fig. 7.4(c), (e), U_x, U_y are open sets without singularities. Also, the places where either of the local coordinates U_x, U_y can be written (except the origin) are shown in Fig. 7.4(f). ∎

We would like to explain why generating two curves (c) (e) from the elliptic curve in Fig. 7.4(a) can be said to have resolved the singularity. First of all, (c) corresponds to $V - \{(0,0)\}$ represented in the local coordinates (u_x, v_x) $(u_x \neq 0)$. And its pullback (the inverse image of π) is a subset of U_x. Similarly, the pullback of (e) represented in the local coordinates (u_y, v_y) $(v_y \neq 0)$ is a subset of U_y. Therefore, the strict pullback is given as a manifold as a whole. The original elliptic curve included singular points, but after finite blow-ups, when seen as a manifold, it turns out that there are no singular points in any local coordinates.

Note that if the algebraic set is not $(x, y) \in \mathbb{A}^2 \mid y^2 = x^3$, but $(x, y) \in \mathbb{A}^2 \mid y^2 = x^5$, singular points cannot be resolved with a single blow-up. For the obtained local coordinates, another blow-up is performed. In this case, the local coordinates of the manifold are further divided. In the case of $d = 2$, it is known that singular points can be resolved by repeating this process a finite number of times (Hironaka's theorem). For the general $d \geq 3$, it is necessary to apply the general blow-up introduced in the next section.

Example 67 $y^2 = x^5$ has a singular point only at the origin. If the local coordinates of U_x, U_y are $(u_x, v_x), (u_y, v_y)$, they can each be written as

$$f(u_x, u_x v_x) = (u_x v_x)^2 - u_x^5 = u_x^2(v_x^2 - u_x^3)$$

and

$$f(u_y v_y, v_y) = v_y^2 - (u_y v_y)^5 = v_y^2(1 - u_y^5 v_y^3).$$

The term $v_x^2 - u_x^3$ in the former has a singular point at $(u_x, v_x) = (0, 0)$, and the term $1 - u_y^5 v_y^3$ in the latter is non-singular. Indeed, the term $v_x^2 - u_x^3$ can resolve the singular point if another blow-up is performed using the method in Example 66. Also, $1 - u_y^5 v_y^3$ becomes zero only when $u_y = 0$ or $v_y = 0$ when differentiating with respect to u_y, v_y, but in either case $1 - u_y^5 v_y^3$ does not become zero. ∎

Furthermore, in the case where the singular point is not the origin, as in the following example, perform a parallel shift of the coordinates and then blow up.

Example 68 In Example 65, when $(a, b) = (-3, 2)$, it has a singular point only at $(1, 0)$. If $x \mapsto x + 1$, then

$$y^2 - x^3 + 3x - 2 = y^2 - (x-1)^2(x+2) \mapsto y^2 - (x+1-1)^2(x+1+2) = y^2 - x^2(x+3)$$

can be achieved, so consider the parallel-translated origin passing $y^2 = x^3 + 3x^2$. If the local coordinates of U_x, U_y are $(u_x, v_x), (u_y, v_y)$, they can each be written as

$$f(u_x, u_x v_x) = (u_x v_x)^2 - u_x^3 - 3u_x^2 = u_x^2(v_x^2 - u_x - 3)$$

$$f(u_y v_y, v_y) = v_y^2 - (u_y v_y)^3 - 3(u_y v_y)^2 = v_y^2(1 - u_y^3 v_y - 3u_y^2).$$

$v_x^2 - u_x - 3 = 0$, $1 - u_y^3 v_y - 3u_y^2 = 0$ are the strict pullbacks, and both are non-singular. ∎

7.4 Hironaka's Theorem

The theory of resolving singularities in the previous section was constructed by Heisuke Hironaka in 1964.

Proposition 27 (Hironaka [4, 5]) *Let f be an analytic function from a neighborhood of the origin in $\mathbb{R}^d$ to $\mathbb{R}$, with $f(0) = 0$ and not a constant function. Then, there exists a manifold U, an open set V of $\mathbb{R}^d$ containing the origin, and an analytic map $g : U \to V$ that satisfy the following conditions:*

1. *For any compact set K of V, $g^{-1}(K)$ is a compact set of U.*
2. *Let $V_0 := \{x \in V \mid f(x) = 0\}$ and $U_0 := \{u \in U \mid f(g(u)) = 0\}$, then g gives an isomorphism*[4] *of $U\backslash U_0$ and $V\backslash V_0$.*
3. *For each $P \in U_0$, there exists local coordinates $(u_1, \ldots, u_d)$ of U with P as the origin, and using a multi-index $\kappa = (\kappa_1, \ldots, \kappa_d) \in \mathbb{N}^d$ and a sign $S \in \{-1, 1\}$, it can be written as*
$$f(g(u)) = S u_1^{\kappa_1} \ldots u_d^{\kappa_d}. \tag{7.5}$$
4. *The Jacobian of $x = g(u)$ can be written as an analytic function $b(u) \neq 0$, using a multi-index $h = (h_1, \ldots, h_d) \in \mathbb{N}^d$,*
$$g'(u) = b(u) u_1^{h_1} \ldots u_d^{h_d}. \tag{7.6}$$

A representation by local coordinates as in (7.5) is called a *normal crossing*. In this book, we do not prove Hironaka's theorem, but instead perform blow-ups for some specific manifolds, not algebraic sets, to find normal crossings.

Regarding Proposition 27, there are two points to note. First, in this book, we consider $K(\theta) = \mathbb{E}_X[\log \frac{p(X|\theta_*)}{p(X|\theta)}]$, $\theta \in \Theta$ as the function f. That is, it is only applied in the neighborhood of each $\theta_* \in \Theta_*$. Second, the domain of the function g is U, or it is expressed as a function of local variables. The same function symbol g is used even if the local coordinates are different, but the correspondence for each local coordinate is described.

This may be a bit late to mention,[5] in Watanabe's Bayesian theory, (so-called) singularity resolution is not used. Whether a certain $\theta_* \in \Theta_*$ is regular or not, the normal crossing form is sought. Hironaka's theorem guarantees that a normal crossing can be obtained whether it is singular or non-singular. In this sense, it may be said that there is no relationship between whether a point is regular in statistics and whether it is singular in algebraic geometry.

In the following chapters, Hironaka's theorem will be applied in the neighborhood of $\theta_* \in \Theta_*$.

---

[4] Maintains the same structure as the analytic manifold.

[5] In the field of algebraic geometry, it seems that Proposition 27 is called the singularity resolution theorem, and the process of finding the normal crossing form is called singularity resolution. However, in this book, such a description is used for the understanding of beginners.

$$\begin{tikzcd}
y^2 - x^3 \arrow[r, "{(x,y)=(u_1,u_1u_2)}"] \arrow[d, "{(x,y)=(u_1u_2,u_2)}"] & -u_1^2(u_1 - u_2^2) \arrow[r, "{(u_1,u_2)=(v_1,v_1v_2)}"] \arrow[d, "{(u_1,u_2)=(v_1v_2,v_2)}"] & -v_1^3(1 - v_1v_2^2) \\
u_2^2(1 - u_1^3u_2) & -v_1^2v_2^3(v_1 - v_2) \arrow[r, "{(v_1,v_2)=(w_1,w_1w_2)}"] \arrow[d, "{(v_1,v_2)=(w_1w_2,w_2)}"] & -w_1^6w_2^3(1 - w_2) \\
 & w_1^2w_2^6(1 - w_1) &
\end{tikzcd}$$

$$(x, y) = \begin{cases} (u_1u_2, u_2) \\ (u_1, u_1u_2) = \begin{cases} (v_1, v_1^2v_2) \\ (v_1v_2, v_1v_2^2) = \begin{cases} (w_1^2w_2, w_1^3w_2^2) \\ (w_1^2w_2, w_1w_2^3) \end{cases} \end{cases} \end{cases}$$

Fig. 7.5 By variable transformation, (x, y) is expressed in local coordinates (four types in this example), and the normal crossing of $y^2 - x^3$ is found

Example 69 When Hironaka's theorem is applied, Examples 66, 67, and 68 become as follows. For instance, if it is $y^2 - x^3$, the procedure is as shown in Fig. 7.5.

$f(x,y)$	$g(u)$	$g'(u)$	$f(g(u))$	S	κ	h
$y^2 - x^3$ (Example 66)	$(u_1, u_1^2u_2)$	u_1^2	$-u_1^3(1 - u_1u_2^2)$	-1	(3, 0)	(2, 0)
	(u_1u_2, u_2)	u_2	$u_1^2(1 - u_1^3u_2)$	1	(0, 2)	(0, 1)
	$(u_1u_2^2, u_1u_2^3)$	$u_1u_2^4$	$u_1^2u_2^6(1 - u_1)$	1	(2, 6)	(1, 4)
	$(u_1^2u_2, u_1^3u_2^2)$	$u_1^4u_2^2$	$-u_1^6u_2^3(1 - u_2)$	-1	(6, 3)	(4, 2)
$y^2 - x^5$ (Example 67)	(u_1u_2, u_2)	u_2	$u_2^2(1 - u_1^5u_2^3)$	1	(0, 2)	(1, 0)
	$(u_1u_2, u_1u_2^2)$	$u_1u_2^2$	$u_1^2u_2^4(1 - u_1^3u_2)$	1	(2, 4)	(1, 2)
	$(u_1u_2^2, u_1^2u_2^5)$	$u_1^2u_2^6$	$u_1^4u_2^{10}(1 - u_1)$	1	(4, 10)	(2, 6)
	$(u_1^2u_2, u_1^5u_2^3)$	$u_1^6u_2^3$	$-u_1^{10}u_2(1 - u_2)$	-1	(10, 1)	(6, 3)
	$(u_1, u_1^3u_2)$	u_1^3	$-u_1^5(1 - u_1u_2^2)$	-1	(5, 0)	(3, 0)
$y^2 - x^3 - 3x^2$ (Example 68)	$(\frac{u_1}{\sqrt{3}}, u_1u_2)$	$\frac{u_1}{\sqrt{3}}$	$-u_1^2(1 - u_2^2 + \frac{u_1}{3\sqrt{3}})$	-1	(2, 0)	(1, 0)
	(u_1u_2, u_2)	u_2	$u_2^2(1 - u_1^3u_2 - 3u_1^2)$	1	(0, 2)	(0, 1)

■

In the first example of Example 69, $-u_1^3(1 - u_1u_2^2) \approx -u_1^3$, $1 - u_1u_2^2$ becomes 1 near the origin. Even if a polynomial that becomes 1 at the origin is multiplied in this way, a normal crossing can be obtained. As shown in Example 69, a normal crossing cannot be obtained by performing a variable transformation within the range of a rational map. In general, it becomes the product of a normal crossing and an analytic function that does not become 0. If the variable transformation is performed within the range of an analytic map, it becomes a normal crossing. For example, in the case of Example 66, it is sufficient to apply an analytic map that makes $u_1(1-u_1u_2^2)^{1/3}$ a single variable.

In the previous section, we introduced the blow-up centered at the origin for $d=2$, but for $d=3$, it becomes

$$\phi_x : (u_x, u_xu_y, u_xu_z, [1 : u_y : u_z])$$

$$\phi_y : (u_xu_y, u_y, u_yu_z, [u_x : 1 : u_z])$$

$$\phi_z : (u_xu_z, u_yu_z, u_z, [u_x : u_y : 1]),$$

and it is extended to the general $d > 2$. However, there may be cases where the normal crossing claimed in Hironaka's theorem cannot be obtained with the blow-up centered at the origin. From here on, we will introduce the *blow-up centered at the ideal*. The blow-up centered at the origin was

$$U = \{(0,\dots,0)\} \times \mathbb{P}^{d-1} \cup \{(x_1,\dots,x_d,[x_1 : \cdots : x_d]) \mid (x_1,\dots,x_d) \neq (0,\dots,0)\}$$

for the general d, but the blow-up centered at the ideal uses the ideal $I \subseteq \mathbb{R}[x]$ generated by $f_1,\dots,f_m \in \mathbb{R}[x]$, and it is set to be

$$\begin{aligned} U =& V(I) \times \mathbb{P}^{m-1} \cup \{(x_1,\dots,x_d,[f_1(x_1,\dots,x_d) : \cdots : \\ & f_m(x_1,\dots,x_d)]) \mid (x_1,\dots,x_d) \notin V(I)\}. \end{aligned}$$

Note that

$$f_1(x_1,\dots,x_d) = 0,\dots,f_m(x_1,\dots,x_d) = 0 \Leftrightarrow (x_1,\dots,x_n) \in V(I).$$

In the blow-up centered at the ideal, it is not necessary to use all of $x_1,\dots,x_d$. The blow-up centered at the origin is equivalent to the blow-up by the ideal $(f_1(x) = x_1,\dots,f_d(x) = x_d)$. In other words, the blow-up centered at the ideal is a generalization of the blow-up centered at the origin. In Example 70, we perform a blow-up using z, x in (1), $y+\alpha_1, y$ in (2), and β_1, y in (4) as the generators of the ideal.

Example 70 For the function

$$f(x,y,z) = (xy+z)^2 + x^2y^4\ ,$$

we seek a normal crossing representation by local coordinates as shown in Fig. 7.6. Then, we define the mapping from each coordinate $U_i = (u_i, v_i, w_i)$, $i = 1,2,3,4$ to $\mathbb{R}^3$ in

$$
\begin{tikzcd}
(1) \arrow[r, "z=\alpha_1 x"] \arrow[dr, "x=\alpha_2 z"'] & (2) \arrow[r, "y+\alpha_1=\beta_1 y"] \arrow[dr, "y=\beta_2(y+\alpha_1)"'] & (4) \arrow[r, "\beta_1=\gamma_1 y"] \arrow[dr, "y=\gamma_2\beta_1"'] & (6) \\
& (3) & (5) & (7)
\end{tikzcd}
$$

$$
\begin{array}{llll}
(1) & f=(xy+z)^2+x^2y^4 & (5) & f=x^2(y+\alpha_1)^2\{1+{\beta_2}^4(y+\alpha_1)^2\} \\
(2) & f=x^2\{(y+\alpha_1)^2+y^4\} & (6) & f=x^2y^4(1+\gamma_1^2) \\
(3) & f=z^2\{(1+\alpha_2 y)^2+{\alpha_2}^2y^4\} & (7) & f=x^2\gamma_2^2{\beta_1}^4(1+{\gamma_2}^2) \\
(4) & f=x^2y^2({\beta_1}^2+y^2) & &
\end{array}
$$

Fig. 7.6 The normal crossing of the function $f(x, y, z) = (xy + z)^2 + x^2y^4$ is shown in (3) (5) (6) (7), and for this, the variables $\alpha_1, \alpha_2, \beta_1, \beta_2, \gamma_1, \gamma_2$ are introduced

$$
\begin{cases}
(x, y, z) = (u_1w_1, v_1, w_1) \\
(x, y, z) = (u_2, v_2w_2, u_2(1 - v_2)w_2) \\
(x, y, z) = (u_3, v_3, u_3v_3(v_3w_3 - 1)) \\
(x, y, z) = (u_4, v_4w_4, u_4v_4w_4(w_4 - 1)).
\end{cases}
$$

However, let $\alpha_1, \alpha_2, \beta_1, \beta_2, \gamma_1, \gamma_2$ be the values defined in Fig. 7.6, and let

$$
\begin{aligned}
&(u_1, v_1, w_1), (u_2, v_2, w_2), (u_3, v_3, w_3), (u_4, v_4, w_4) \\
=&(\alpha_2, y, z), (x, \beta_2, y + \alpha_1), (x, y, \gamma_1), (x, \gamma_2, \beta_1),
\end{aligned}
$$

respectively. For example, in the local coordinates U_1, we have $f(g(u_1, v_1, w_1)) = w_1^2(u_1v_1 + 1)^2 + u_1^2v_1^4$, and the Jacobian is obtained from

$$
\begin{bmatrix}
\frac{\partial x}{\partial u_1} & \frac{\partial x}{\partial v_1} & \frac{\partial x}{\partial w_1} \\
\frac{\partial y}{\partial u_1} & \frac{\partial y}{\partial v_1} & \frac{\partial y}{\partial w_1} \\
\frac{\partial z}{\partial u_1} & \frac{\partial z}{\partial v_1} & \frac{\partial z}{\partial w_1}
\end{bmatrix}
=
\begin{bmatrix}
w_1 & 0 & u_1 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}
$$

as $g'(u_1, v_1, w_1) = |w_1|$. The same calculations can be done for the others, resulting in the following.

i	U_i	$f(g(u_i, v_i, w_i))$	$g'(u_i, v_i, w_i)$	$(\kappa_1, \kappa_2, \kappa_3)$	(h_1, h_2, h_3)
1	U_1	$w_1^2\{(u_1v_1 + 1)^2 + u_1^2v_1^4\}$	w_1	(0, 0, 2)	(0, 0, 1)
2	U_2	$u_2^2w_2^2(1 + v_2^4w_2^2)$	u_2w_2	(2, 0, 2)	(1, 0, 1)
3	U_3	$u_3^2v_3^4(w_3^2 + 1)$	$u_3v_3^2$	(2, 4, 0)	(1, 2, 0)
4	U_4	$u_4^2v_4^2w_4^4(1 + v_4^2)$	$u_4v_4w_4^2$	(2, 2, 4)	(1, 1, 2)

Also, the Jacobian $g'(u_i, v_i, w_i) \neq 0$ is the necessary and sufficient condition for the local coordinates and x, y, z are isomorphic, so it would be good to perform the pasting according to each condition of

$$g'(u_1, v_1, w_1) \neq 0 \Longleftrightarrow w_1 \neq 0 \Longleftrightarrow z \neq 0$$

$$g'(u_2, v_2, w_2) \neq 0 \Longleftrightarrow u_2 w_2 \neq 0 \Longleftrightarrow xy + z \neq 0$$

$$g'(u_3, v_3, w_3) \neq 0 \Longleftrightarrow u_3 v_3 \neq 0 \Longleftrightarrow xy \neq 0$$

$$g'(u_4, v_4, w_4) \neq 0 \Longleftrightarrow u_4 v_4 w_4 \neq 0 \Longleftrightarrow xy \neq 0.$$

Both of them can be seen to correspond to (7.5) and (7.6). ■

Example 71 The normal crossing representation of the function $f(x, y, z, w) = (xy + zw)^2 + (xy^2 + zw^2)^2$ by local coordinates is shown in Fig. 7.7. We try to perform the blow-up from Eq. (7.3) in two types of procedures. Although the obtained local coordinates are different, both of them are normal crossings obtained in (5) (7) (9) (10) (11). In addition, we introduce the variables $\xi_1, \xi_2, \xi_3, \alpha_1, \alpha_2, \beta_1, \beta_2, \gamma_1, \gamma_2, \delta_1, \delta_2$. Also, the blow-up between (1) and (2) is symmetrical for (x, z), (y, w), so we only performed the former because the same result can be obtained either by $y = \xi_1 w$ or $w = \xi_2 y$. ■

At the end of this section, we omit the proof but present a useful generalization for Bayesian theory by Watanabe. The specific application will be discussed in the next chapter.

Proposition 28 *(Simultaneous normal crossing[4, 5, 13]) Let $f_0, f_1, \ldots, f_m$ be analytic functions from a neighborhood of the origin of $\mathbb{R}^d$ to $\mathbb{R}$, where for each $i = 0, 1, \ldots, m$ we have $f_i(0) = 0$ and they are not constant functions. In this case, there exist a manifold U, an open set V in $\mathbb{R}^d$ containing the origin, and an analytic map $g : U \to V$ that satisfy the following properties:*

1. *For any compact set K in V, $g^{-1}(K)$ is a compact set in U.*
2. *g gives an isomorphism between $U \backslash U_0$ and $V \backslash V_0$, where $V_0 := \cup_{i=1}^m x \in V \mid f_i(x) = 0$ and $U_0 := \cup_{i=1}^m u \in U \mid f_i(g(u)) = 0$.*
3. *For each $P \in U_0$, there exist local coordinates $(u_1, \ldots, u_d)$ of U centered at P, multi-indices $\kappa(i) = (\kappa_1(i), \ldots, \kappa_d(i)) \in \mathbb{N}^d$, $i = 0, 1, \ldots, m$, analytic functions a_i, $1 \leq i \leq m$, and a sign $S \in -1, 1$ such that we can write*

$$f_0(g(u)) = Su^{\kappa(0)}\ ,\ f_1(g(u)) = a_1(u)u^{\kappa(1)}\ ,\ \ldots\ ,\ f_m(g(u)) = a_m(u)u^{\kappa(m)}.$$

   Here, $u^{\kappa(i)} := u_1^{\kappa_1(i)} \ldots u_d^{\kappa_d(i)}$.
4. *The Jacobian (determinant) of $x = g(u)$ can be written as $g'(u) = b(u)u^h$, where $b(u) \neq 0$ is an analytic function that is not zero, and $h = (h_1, \ldots, h_d) \in \mathbb{N}^d$ is a multi-index.*

$$
\begin{tikzcd}
(1) \arrow[r, "y = \xi_1 w"] \arrow[dr, "w = \xi_2 y"'] & (2) \arrow[r, "\xi_3 = x\xi_1 + z"] & (3) \\
 & {} &
\end{tikzcd}
$$

$$
\begin{tikzcd}
(3) \arrow[r, "\xi_3 = \alpha_1 w"] \arrow[dr, "w = \alpha_2 \xi_3"'] & (4) \arrow[r, "\alpha_1 = \beta_1 x"] \arrow[dr, "x = \beta_2 \alpha_1"'] & (6) \arrow[r, "\beta_1 = \gamma_1 \xi_1"] \arrow[dr, "\xi_1 = \gamma_2 \beta_1"'] & (8) \arrow[r, "\xi_1 - 1 = \delta_1 \gamma_1"] \arrow[dr, "\gamma_1 = \delta_2(\xi_1 - 1)"'] & (10) \\
 & (5) & (7) & (9) & (11)
\end{tikzcd}
$$

(1)	$(xy + zw)^2 + (xy^2 + zw^2)^2$	(7)	$\alpha_1^2 w^4\{1 + (\beta_2\xi_1^2 + w - \beta_2\xi_1)^2\}$
(2)	$w^2\{(x\xi_1 + z)^2 + w^2(x\xi_1^2 + z)^2\}$	(8)	$w^4 x^2 \xi_1^2\{\gamma_1^2 + (\xi_1 + \gamma_1 w - 1)^2\}$
(3)	$w^2\{\xi_3^2 + w^2(x\xi_1^2 + \xi_3 - x\xi_1)^2\}$	(9)	$x^2 w^4 \beta_1^2\{1 + (\beta_1\gamma_2^2 + w - \gamma_2)^2\}$
(4)	$w^4\{\alpha_1^2 + (x\xi_1^2 + \alpha_1 w - x\xi_1)^2\}$	(10)	$x^2 w^4\{1 + \delta_1\gamma_1\}^2\gamma_1^2\{1 + (\delta_1 + w)^2\}$
(5)	$\alpha_2^2\xi_3^4\{1 + \alpha_2^2(x\xi_1^2 + \xi_3 - x\xi_1)^2\}$	(11)	$w^4 x^2 \xi_1^2(\xi_1 - 1)^2\{\delta_2^2 + (1 + \delta_2 w)^2\}$
(6)	$x^2 w^4\{\beta_1^2 + (\xi_1^2 + \beta_1 w - \xi_1)^2\}$		

$$
\begin{tikzcd}
(3) \arrow[r, "\xi_3 = \alpha_1 \xi_1"] \arrow[dr, "\xi_1 = \alpha_2 \xi_3"'] & (4) \arrow[r, "\alpha_1 = \beta_1 x"] \arrow[dr, "x = \beta_2 \alpha_1"'] & (6) \arrow[r, "\beta_1 = \gamma_1(\xi_1 - 1)"] \arrow[dr, "\xi_1 - 1 = \gamma_2 \beta_1"'] & (8) \arrow[r, "w = \delta_1 \gamma_1"] \arrow[dr, "\gamma_1 = \delta_2 w"'] & (10) \\
 & (5) & (7) & (9) & (11)
\end{tikzcd}
$$

(4)	$w^2\xi_1^2\{\alpha_1^2 + w^2(x\xi_1 + \alpha_1 - x)^2\}$	(8)	$w^2\xi_1^2 x^2(\xi_1 - 1)^2\{\gamma_1^2 + w^2(\gamma_1 + 1)^2\}$
(5)	$w^2\xi_3^2\{1 + w^2(x\alpha_2^2\xi_3 + 1 - x\alpha_2)^2\}$	(9)	$x^2 w^2(1 + \gamma_2\beta_1)^2\beta_1^2\{1 + (1 + \gamma_2)^2 w^2\}$
(6)	$x^2 w^2 \xi_1^2\{\beta_1^2 + w^2(\xi_1 + \beta_1 - 1)^2\}$	(10)	$x^2\delta_1^2\gamma_1^4\xi_1^2(\xi_1 - 1)^2\{1 + \delta_1^2(\gamma_1 + 1)^2\}$
(7)	$w^2\xi_1^2\alpha_1^2\{1 + w^2(\beta_2\xi_1 + 1 - \beta_2)^2\}$	(11)	$w^4 x^2\xi_1^2(\xi_1 - 1)^2\{\delta_2^2 + (1 + \gamma_1)^2\}$

Fig. 7.7 The normal crossing representation of the function $f(x, y, z, w) = (xy + zw)^2 + (xy^2 + zw^2)^2$ by local coordinates. We tried to perform the blow-up from Eq. (7.3) in two types of procedures. Although the obtained local coordinates are different, both are normal crossings obtained in (5) (7) (9) (10) (11)

7.5 Local Coordinates in Watanabe Bayesian Theory

As mentioned in Chap. 2, when a statistical model $p(\cdot|\theta)\theta \in \Theta$ is given, let Θ denote the set of $\theta \in \Theta$ that minimizes the Kullback-Leibler divergence with respect to the true distribution $q(\cdot)$:

$$\mathbb{E}\left[\log \frac{q(X)}{p(X|\theta)}\right].$$

In the general case without assuming regularity, Θ_* may contain multiple elements. Furthermore, since we assume finite relative variance in this book, according to

Proposition 1.3 (1), the statistical model is homogeneous, and the distribution $p(\cdot|\theta_*)$ does not depend on $\theta_* \in \Theta_*$. Hence, the function

$$K(\theta) = \mathbb{E}[\log \frac{p(X|\theta_*)}{p(X|\theta)}]$$

does not depend on $\theta_* \in \Theta_*$. In Chap. 8, we assume that $K(\cdot)$ is an analytic function. Thus, Θ_* is an analytic set. For each $\theta_* \in \Theta_*$, we shift the coordinates by the amount corresponding to θ_* and apply Proposition 27 with each of them as the origin. Then, using the local coordinates of the corresponding manifold in the neighborhood of θ_*, we express $K(\theta)$ in the form of a normal crossing.

In Chap. 5, we mentioned that we only need to remove regularity constraints on Θ contained in $B(\epsilon_n, \theta_*)$, $\theta_* \in \Theta_*$. Here, it is essential to note that the posterior distribution we want to derive in Watanabe Bayesian theory is with respect to $\Theta_m := \bigcup_{\theta_* \in \Theta_*} B(\epsilon_n, \theta_*)$, not Θ.

For each $\theta_* \in \Theta_*$, when we take θ_* as the origin, the mapping from Proposition 27 which is $g : U \to V$ can be denoted as: $g : U(\theta_*) \to V(\theta_*)$ By patching these together, it is given by: $g : U \to V$, where $U := \bigcup_{\theta_* \in \Theta_*} U(\theta_*)$ and $V := \bigcup_{\theta_* \in \Theta_*} V(\theta_*)$. In this context, rather than setting g for each θ_* with $\theta_* = 0$, a common g adjusted by θ_* is utilized. Therefore, for each $\theta_* \in \Theta_*$, we determine the normal intersection of $f(g(u) - \theta_*)$. If V is compact, from Proposition 27.1, U is also compact. This implies U can be covered by a finite union of open sets. Notably, by merging several open sets, each can include the point u such that $g(u) = 0$.

Without loss of generality, each local coordinate of the open set can be taken as a cube of size 2 centered at some element of $g^{-1}(\Theta_*)$. Furthermore, we can partition each into 2^d pieces, and adjust their signs to set each local coordinate to $[0, 1)^d$. Such variable transformations change the Jacobian. As long as it doesn't become zero within the local coordinate, it doesn't impact discussions in the next chapter. Ultimately, each of the finitely obtained open sets is denoted as U_α.

Additionally, by taking a sufficiently large n, we have $\Theta_n \subseteq V$ And U can be restricted to $g^{-1}(\Theta_n)$.

Finally, each parameter $\theta \in \Theta_m$ can generally be written in multiple local coordinates. And usually, open sets of manifolds overlap, so in the next chapter, without losing generality, we assume the following: That is, construct a C^∞ class function $\rho_\alpha : g^{-1}(\Theta_m) \to [0, 1]$ that satisfies the following three conditions:

1. $0 \le \rho_\alpha(u) \le 1$
2. $\operatorname{supp} \rho_\alpha \subseteq U_\alpha$
3. $\sum_\alpha \rho_\alpha(u) = 1$

This is called a partition of unity[6]. The support $\operatorname{supp} \rho_\alpha$ is defined as the smallest closed set containing points $u \in g^{-1}(\Theta_m)$ where $\rho_\alpha(u) > 0$. For example, for $u \in U_\alpha$ and

[6] As in Murakami [18], the partition of unity is an existing concept in manifold theory.

$$\sigma_\alpha(u) = \begin{cases} \prod_{i=1}^d \exp\left(-\frac{1}{1-u_i}\right), & 0 \le u_i < 1, i = 1, \ldots, d \\ 0, & \text{otherwise} \end{cases}$$

you can set $\rho_\alpha(u) = \frac{\sigma_\alpha(u)}{\sum_{\alpha'} \sigma_{\alpha'}(u)}$.

Exercises 67–74

67. As with the one-variable polynomial $\mathbb{R}[x]$ with real coefficients, ideals can be defined for the set of all integers $\mathbb{Z}$. What kind of a set is the ideal I generated by $J = 2, 3$?
68. For the elliptic curve $y^2 = x^3 + ax + b$,

    (a) Run the program from Example 60 with the settings below, and output the elliptic curve.

```
a <- 0; b <- 0; x.min <- -1; x.max <- 5    # the first one
a <- -3; b <- 2; x.min <- -3; x.max <- 3   # the second one
```

    (b) Demonstrate that the condition for including a singular point is (7.3). Determine whether each of the following is singular or non-singular:

$$(a, b) = (-3, 3), (1, 0), (-1, 0), (0, 0), (-3, 2).$$

    Also, where are the singular points for each singular elliptic curve?

69. With regards to the Hausdorff property of topological spaces, demonstrate the following:

    (a) A metric space M is Hausdorff. [Hint] Use the triangle inequality $dist(x, y) \le dist(x, z) + dist(y, z)$, $x, y, z \in M$.
    (b) Consider the set of all integers $\mathbb{Z}$ as the whole set M. Initially, only include $2n+1$ and $2n-1, 2n, 2n+1$ for each $n \in \mathbb{Z}$ in $\mathcal{U}$, and generate elements of $\mathcal{U}$ to satisfy the second and third properties of a topological space. In this case, M is not Hausdorff.

70. Derive the coordinate transformation for each of the following manifolds:

    (a) For $\mathbb{P}^d = \{(U_i, \phi_i)\}_{i=0,1,\ldots,d}$, the coordinate transformation of

$$U_i := \{[x_0 : x_1 : \cdots : x_{i-1} : 1 : x_{i+1} : \cdots : x_d] \in \mathbb{P}^d\}$$

$$\begin{aligned} \phi_i : U_i \ni [x_0 : x_1 : \ldots : x_{i-1} : 1 : x_{i+1} : \cdots : x_d] \\ \mapsto (x_0, x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_d) \in \mathbb{A}^d \end{aligned}$$

from $\phi_i(U_i \cap U_j)$ to $\phi_j(U_i \cap U_j)$ is given by (7.1).

(b) For the blow-up at the origin of $\mathbb{A}^2$, the coordinate transformation of (U_x, ϕ_x), (U_y, ϕ_y) is given by

$$\phi_x(U_x \cap U_y) \ni (u_x, v_x) \mapsto (\frac{1}{v_x}, u_x v_x) = (u_y, v_y) \in \phi_y(U_x \cap U_y)$$

$$\phi_y(U_x \cap U_y) \ni (u_y, v_y) \mapsto (u_y v_y, \frac{1}{u_y}) = (u_x, v_x) \in \phi_x(U_x \cap U_y).$$

71. Show that the set

$$\{(x, y) \times [x' : y'] \in \mathbb{A}^2 \times \mathbb{P}^1 \mid xy' = x'y\}$$

matches the set below.

$$\{(x, y) \times [x' : y'] \in \mathbb{A}^2 \times \mathbb{P}^1 \mid [x : y] = [x' : y'] \text{ or } (x, y) = (0, 0)\}.$$

72. In Example 69, a normal crossing is obtained using five local coordinates for $y^2 - x^3$. Construct a figure for $y^2 - x^5$ similar to Fig. 7.5.
73. In Example 70, a normal crossing is obtained for four local coordinates. Explain the operations up to obtaining the table below.

i	U_i	$f(g(u_i, v_i, w_i))$	$g'(u_i, v_i, w_i)$	$(\kappa_1, \kappa_2, \kappa_3)$	(h_1, h_2, h_3)
1	U_1	$w_1^2\{(u_1 v_1 + 1)^2 + u_1^2 v_1^4\}$	w_1	(0, 0, 2)	(0, 0, 1)
2	U_2	$u_2^2 w_2^2 (1 + v_2^4 w_2^2)$	$u_2 w_2$	(2, 0, 2)	(1, 0, 1)
3	U_3	$u_3^2 v_3^4 (w_3^2 + 1)$	$u_3 v_3^2$	(2, 4, 0)	(1, 2, 0)
4	U_4	$u_4^2 v_4^2 w_4^4 (1 + v_4^2)$	$u_4 v_4 w_4^2$	(2, 2, 4)	(1, 1, 2)

74. In Example 71, the operations to obtain the local coordinates are performed in two ways. For each of the local coordinates (5) (7) (9) (10) (11) in the first method of Fig. 7.7, calculate the Jacobian $|g'(\cdot, \cdot, \cdot, \cdot)|$.