Surinder Pal Singh Kainth

A Comprehensive Textbook on Metric Spaces

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Dedicated to the authors who don't dedicate their work to themselves

Foreword

It gives me immense pleasure to write the Foreword to this book on metric spaces. I first met the author Surinder Pal Singh Kainth in early 2007 when both of us were graduate students at the Indian Institute of Technology Bombay. We would often discuss several topics over the dinner table at the common hostel there! And among many topics, mathematical discussions were of course part of it! Since both of us were young students, we often talked about books by different authors. It was at that time, he expressed a feeling of writing some good textbooks. After a year, I left IIT Bombay and the conversations were paused. Eventually after a few more years we found ourselves again in the same neighborhood due to the choices of our respective workplaces!

Around five years back when Surinder showed me a preliminary version of this book, I was very much exhilarated that he is exactly on focus! The preliminary version itself looked pretty good for me and I was expecting that the book should be ready in a few months. However, Surinder took immense care and invested a lot of hard work to polish the old version repeatedly. The present form is obtained after many revisions by him. I think this book would serve as an excellent introduction to metric spaces. It contains plenty of materials to master the topic. The author has written this book by keeping undergraduate and postgraduate students of Indian universities in mind. In my opinion, he has been successful to shape it in a way that would be accessible to the target audience. The book will also serve as a reference to the experienced college and university teachers as well. It would also complement any basic course on real analysis or general topology.

The book has ten chapters and four appendices to cover necessary background material. All chapters are full of examples and exercises to inculcate the thought process of the students. Surinder has good expertise on real analysis and he has certainly shown his signature at different places of the book. Even though there are many classic texts on the topic in the literature, this book has the charm in its own right. I really like the idea of devoting a whole chapter on Cantor sets. This is something that I hope to be a useful reference for the students.

Overall, the book would serve both as a text and as a reference book to the mathematical community. It is informative and self-contained. I am extremely happy to see this book published. Finally, I would like to note a quote that is attributed to Brian Herbert: ‘the capacity to learn is a gift; the ability to learn is a skill; the willingness to learn is a choice’. As per as one would like to learn a basic introduction around metric spaces, this book may be offered as a gift to the interested reader. It would develop the mathematical skill to the reader, and would provide future readers a choice to look at the right place to learn!

With warm wishes to the author and the readers of this book.

Krishnendu Gongopadhyay
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Preface

The notion of a metric space was introduced by Frećhet in his Ph.D. dissertation in 1906. It generalizes the concept of distance to more general spaces than the Euclidean ones, leading to the corresponding extensions of convergence and continuity. Therefore, a course on metric spaces is a gateway to mathematical abstraction. Yet, there are only a few books completely devoted to this topic.

The textbooks on real, complex, or functional analysis often pass this subject quite rapidly. It deserves much more attention than that. A thorough course on metric spaces makes a better foundation in abstract analysis and eases other advanced courses such as topology, functional analysis, and measure theory. This book intends to provide such a comprehensive course for upper undergraduates and graduates. I present a smooth take off from the basic real analysis onto general metric spaces.

This book is motivated by my classroom experiences, where I often postpone the notion of countable and uncountable sets until the basic topology is discussed. Therefore, I present convergence, continuity, completeness, compactness, and connectedness before the notions of cardinality and separability. The last two chapters of this textbook are treatises on homeomorphisms and the Cantor set, which makes it different from any other available text on metric spaces. It also addresses set-theoretic matters in fairer details.

The following is a brief description of the contents and style of this textbook, which makes it quite pedagogical and different, as compared to the other literature on the same subject.

Every chapter of this book presents a single concept which is further unfolded and elaborated through related sections and subsections. The particular cases of Euclidean spaces and normed linear spaces are also discussed for every fundamental notion. Their analogies and distinctions with general metric spaces are elaborated by subsuming a variety of examples and exercises.

The prerequisites are very few. It is assumed that the reader is familiar with the basics of sets, relations, and functions. A basic understanding of vector spaces is required for the topology of normed linear spaces. Differentiation of real functions is required for a few results, such as some applications of the Banach contraction principle and Lipschitz continuity.

I have explained the metric space of reals before the topology of metric spaces. Chapter 1 is a self-contained quick review of the basics of real analysis such as the completeness property, continuity, decimal expansions, sequences and series of real numbers; along with the uniform convergence of sequences and series of real functions. However, it is not aimed at providing every little detail of basic real analysis.

Chapter 2 presents a vast collection of metric spaces, including the particular cases of normed spaces and sequence spaces. In Chap. 3, we provide a thorough analysis of the basic notions such as open sets, limit points, closures, subspaces, and continuity. We also discuss strong pointwise convergence which provides a necessary and suffcient condition for the continuity of the pointwise limit of a sequence of continuous functions.

To ensure a deep understanding of the very fundamental ideas of basic topology, I present Chaps. 2 and 3 in greater detail, as compared to most of the standard textbooks. I believe that at the undergraduate level it takes time to develop a taste for abstraction.

The Banach contraction principle, various characterizations of completeness, completion of a metric, and the particular cases of some Banach spaces are presented in Chap. 4. Chapter 5 addresses compactness, its characterizations, and relationship with continuity. It presents uniform continuity in greater detail and a section on Lipschitz continuity, including some necessary and sufficient conditions for the uniform continuity of real functions. In the exercises sections, we outline some recent results regarding Atsuji spaces, strong uniform continuity, and Cauchy continuous maps, which haven't yet made their way into any textbook.

This textbook offers a pedagogical treatment to connectedness. To my experience, the notion of connectedness remains unnatural and unintuitive until path connectedness is discussed on Euclidean spaces. Even in history, path connectedness appears before connectedness. Therefore, I start with path connectedness and its significance. It is followed by connectedness and its characterizations, connected components, and some miscellaneous topics such as locally (path) connected and totally disconnected spaces.

In Chap. 7, we present the notions of countable sets, cardinality, and some applications to topology. It includes a section on the set of discontinuity of a function, which presents the cases of monotone functions and functions between metric spaces. Chapter 8 deals with separability, Polish spaces, perfect sets, Cantor–Bendixon theorem, Baire category, and equicontinuity.

The chapter on homeomorphisms presents equivalence of metrics and several extension theorems, which includes the results by Tietze, Kuratowski, and Lavrentiev. It also presents the cases of normed spaces, particularly the finite-dimensional spaces.

Apart from the basic introduction to the Cantor set, Chap. 10 includes its infinite product representation through a weaker version of Tychonoff's theorem, various embeddings and characterizations of the Cantor set, along with some miscellaneous topics such as the Cantor function and the Cantor's leaky tent. This chapter presents the results by Alexandroff Hausdorff and Brouwer, along with a variety of their applications. That includes a continuous real function which interpolated every bounded sequence.

There are four appendices in this textbook. The first one deals with the axiomatic set theory, which presents Zermelo–Fraenkel axioms along with the axiom of choice. In this appendix, I present a detailed discussion on the choice axiom which includes proofs of various standard results without this axiom.

The second appendix is a further discussion on continuous functions. It includes Weierstrass' approximation theorem, a standard example of a continuous but nowhere differentiable function, and the Banach–Mazurkiewicz theorem, which states that 'most' continuous functions are nowhere differentiable.

The third appendix is special, which offers some tricky proofs in terms of simple two-player games. It establishes the uncountability of reals and perfect sets through such games. It also presents the Banach Mazur game to prove the Baire category theorem. The last appendix gives a glimpse into the general topology.

There are 966 exercises in this textbook, to further explore and elaborate on various concepts. The ones at the beginning of the exercise section motivate the readers. A few exercises address serious alluring questions, arising from theoretic discussions and deliver the aesthetics of reasoning.

Almost every section contains some unconventional exercises never seen before, which also impart the art of questioning. I cherished generating such questions and discussing them with my students. I hope the readers would also enjoy such questions.

I believe that the examples in a textbook should motivate the readers, rather than entangling them in new types of unnecessary technicalities. Therefore, I have refrained from including ideas of marginal value in the examples. I have presented these in the exercises, along with their solutions at the end of the corresponding chapters.

Most of the exercise sections are split into two parts by a light horizontal line. The exercises after that line may be avoided in the first reading, while the ones before that are essential to grasp the gist of the subject. At the end of every chapter, I provide brief hints and solutions to selected exercises. I have provided some of these in greater detail than others.

This book contains various remarks about the history of basic notions and results. I have also presented a few open problems. Several recent contributions have been discussed and cited in the remarks throughout the textbook. A list of 144 references, along with an index, is given at the end. These references include various expository articles, and I hope that it will develop the habit of looking into the journal literature among the students.

Certain sections of this book contain more details than the requirements of standard undergraduate courses on metric spaces. I have dived into those ideas due to their pedagogical value, and have kept them at the end of the corresponding chapters.

Special care has been taken to keep the exposition easy and interesting enough for the beginners in analysis. The presentation of the material is student-friendly, albeit a fairly thorough one sometimes. The notations used are almost standard, with a few exceptions.

I would appreciate receiving any kind of comments and suggestions for further improvements to this book at sps@pu.ac.in or spsingh.math@gmail.com. Many sections of this book contain ideas which were never seen before in any textbook. These consist of fundamentally new results, new proofs of old results, and almost every section contains such exercises. Interested readers can explore the results with following numbers: 2.17, 3.32, 5.38, 6.15, 6.36, 7.16, 7.20, 7.23, 7.41, 9.22, 9.23, 9.28, 9.35, 9.36, 9.37, 10.4, 10.12, 10.14, 10.26, 10.28, 10.33, B.10; Sects. 7.2, 7.3.2, 10.5.2, 10.5.3, A.3, A.4; and Appendix C.

It is pertinent to mention that although this book is intended to be an undergraduate text, a few of its contents are not usually taught at the undergraduate level. Their purpose is to give a glimpse of a wider scope of the subject and to motivate the readers to further explore the subject. Instructors may assign some undergraduate projects on metric spaces, by combining some of such topics and the references cited throughout this book.

Several courses on analysis may be designed from this textbook. I recommend a semester-long course on metric spaces, consisting of at least the following Sects: 1.1, 1.2, 1.4, 2.1, 2.2, 3.1–3.5, 4.1–4.3, 5.1–5.3, 6.1, 6.2, 7.1, 7.2, 7.3.1, 8.1–8.3, 9.1, 9.2, and 10.1. Many sections of this book and the references therein can be helpful in assigning undergraduate projects.

I have benefited a lot from the works cited in the references. I am highly thankful to my alma maters Indian Institution of Technology Bombay and Panjab University Chandigarh where I learnt and have grown in analysis. I would like to thank Prof. Krishnendu Gongopadhyay, Prof. Amin Sofi, and Prof. H. L. Vasudeva, each of whom read through portions of this book and offered important and productive suggestions for revision. It is Prof. Amin Sofi who suggested including Theorems 9.19, 9.25, 10.19, and Example 6.36. I am thankful to Prof. Gerald Beer for his suggestions to include results on strong uniform continuity, Cauchy continuity, and strong pointwise convergence. He sent me an elementary proof of his 2009 result, which is now presented in Theorem 3.21. I would also like to express my sincere thanks to Prof. Krishnendu Gongopadhyay for motivating me throughout this journey, and for writing a foreword to this book.

I would like to take a heartfelt moment to express my deepest gratitude to Dr. Narinder Singh for stepping in at short notice for proofreading. He ensured that the manuscript gleams brighter than ever. Thank you for your invaluable contribution. I am thankful to Springer's editors and anonymous reviewers for their cooperation in the realization of this book. Lastly, I am indebted to my family for their unwavering support and understanding, without which this book would have never been possible.

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September 16, 2023

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Contents

Chapter 1
Real Analysis

This chapter is a review of the basics of real numbers and real functions. It presents a thorough discussion on the completeness property, continuity, decimal expansions, sequences, and series of real numbers, along with the uniform convergence of sequences and series of real functions. However, it is not aimed at providing every little detail of the basic real analysis. We present only those notions which are to be used or extended in the subsequent chapters on general metric space.

1.1 The Real Number System

We assume that the reader is familiar with the basic order and algebraic properties of the set of real numbers, denoted by $\mathbb{R}$. A formal construction of $\mathbb{R}$ is postponed to Sect. 4.4. Let $\mathbb{Z}$, $\mathbb{N}$, and $\mathbb{Q}$ denote the sets of integers, positive integers, and rational numbers, respectively. Positive integers are known as natural numbers. The terms reals and rationals are also common for real and rational numbers, respectively.

The set of rational numbers is inadequate, in the sense that there exists no rational number whose square is 2. The situation is somewhat like the case of natural numbers, as there no natural number n satisfies $n + 2 = 1$. As a remedy, the set $\mathbb{N}$ is enlarged to $\mathbb{Z}$, which comprises all negative integers along with zero, to make such type of equations solvable. Similarly, the set $\mathbb{Q}$ is 'enlarged' as $\mathbb{R}$, in order to have some 'additional properties'.

Therefore we consider $\mathbb{R}$ as a superset of the set of rational numbers, consistent with the order and algebraic properties of $\mathbb{Q}$ and having an additional property, known as the *least upper bound property* of $\mathbb{R}$. This property, also known as the *completeness property* of $\mathbb{R}$, will be presented soon. First we define intervals.

Definition 1.1 A subset I of $\mathbb{R}$ is called an *interval* if it contains all real numbers between any two numbers of I. In other words, $I \subset \mathbb{R}$ is an interval, if $a, b \in I$ and $a < c < b$ imply $c \in I$.

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Above definition includes all open, closed, semi-open, bounded, unbounded intervals, singleton sets, and even the empty set. The reader is presumed to be familiar with the standard notations (a, b), $[a, b]$, and $(a, b]$, etc, to denote different types of intervals.

Definition 1.2 Let S be a nonempty subset of $\mathbb{R}$. Then

(a) S is said to be *bounded above* if there exists some $u \in \mathbb{R}$ such that

$$x \leq u \text{ for all } x \in S.$$

In this case, u is called an *upper bound* of S.

(b) $l \in \mathbb{R}$ is said to be the *least upper bound* of S if l is an upper bound of S and $l \leq u$, for every upper bound u of S. Another term for the least upper bound is *supremum*.

The *least upper bound property of reals* states that every nonempty and bounded above subset of $\mathbb{R}$ has a least upper bound in $\mathbb{R}$. It is immediate that no set can have multiple least upper bounds.

Analogously, we define *bounded below* sets, their *lower bounds* and the *infimum* or the *greatest lower bound.* The supremum (infimum) of a set S, if exists, is denoted by $\sup S$ ($\inf S$). The analogous greatest lower bound property is equivalent to the least upper bound property. We provide such details in the exercises section. A subset S of $\mathbb{R}$ is said to be *bounded* if it is bounded above as well as bounded below.

Examples 1.3 (a) $\{1, 2, 3\}$ is bounded with $\inf\{1, 2, 3\} = 1$ and $\sup\{1, 2, 3\} = 3$.
(b) The supremum of a set may not be its element, as $\sup(0, 1) = 1$.

The following is an important consequence of the completeness property of $\mathbb{R}$.

Theorem 1.4 *(Archimedean property) Let $x, y \in \mathbb{R}$ such that $x > 0$. Then there exists a natural number n such that $nx > y$.*

Proof Assume that $nx \leq y$ for all $n \in \mathbb{N}$. Then $S := \{nx : n \in \mathbb{N}\}$ is a nonempty set bounded above by y. By least upper bound property, S has a supremum, say l.

Since $x > 0$, we have $l - x < l$. Therefore $l - x$ is not an upper bound of S. Hence there exists some $m \in \mathbb{N}$ such that $l - x < mx$. This implies $l < (m + 1)x$ and the latter belongs to S. Therefore $l \neq \sup S$, a contradiction. □

Corollary 1.5 *If $x \in \mathbb{R}$, then there exists a unique integer m such that $m - 1 \leq x < m$.*

Proof Applying Archimedean property on 1 and x, we obtain a natural number n such that $n > x$. Similarly, there exists $k \in \mathbb{N}$ such that $k > -x$. Thus $-k < x < n$. If $I_j := \{r \in \mathbb{R} : j - 1 \leq r < j\}$, we have $x \in \cup_{j=-k+1}^{n} I_j$. Hence there exists a unique integer m such that $x \in I_m$. □

The above corollary allows us to define the greatest integer function, as under.

Notation 1.6 For every real number x, let $[x]$ denote the *greatest integer less than or equal to* x.

Corollary 1.7 *The set of natural numbers is unbounded.*

Proof Assume that $\mathbb{N}$ is bounded and let s denote its supremum. Applying Corollary 1.5, we obtain $s < n$ for some $n \in \mathbb{N}$. Therefore, s is not an upper bound for $\mathbb{N}$, a contradiction. $\square$

Corollary 1.8 *If $x < y$ are reals, then there exists a rational number r such that $x < r < y$.*

Proof The Archimedean property on $y - x (> 0)$ and 1, provides a positive integer n such that $n(y - x) > 1$. Applying Corollary 1.5, there exists some $m \in \mathbb{Z}$ such that $m - 1 \leq nx < m$. Therefore, we conclude that $nx < m \leq nx + 1 < ny$. Hence $x < \frac{m}{n} < y$. $\square$

Remark 1.9 The above corollary is often stated as *rationals are dense in reals.* As a geometric interpretation of the above proof, note that we first pick some $n \in \mathbb{N}$ such that $y - x > 1/n$, that is, $x + (1/n) < y$. Then the function $f : \mathbb{R} \longrightarrow \mathbb{R}$ defined as $f(t) := nt$ maps the interval (x, y) onto (nx, ny), which is at least a unit apart, as $nx < nx + 1 < ny$. Therefore there exists an integer m such that $nx < m < ny$.

We wind up this section with the following property of natural numbers, known as the *well-ordering principle:*

Every nonempty subset of $\mathbb{N}$ contains its infimum.

It is an immediate consequence of the completeness property of $\mathbb{R}$. However, we shall provide an alternative treatment to it in Appendix A.3.

Exercise 1.1 Prove that the supremum of a nonempty set, if it exists, is unique.

Exercise 1.2 If $S \subset \mathbb{R}$ contains one of its upper bounds u, prove that $\sup S = u$.

Exercise 1.3 Let $S \subset \mathbb{R}$ be a bounded set with at least two points. Prove that

(a) both $\inf S$ and $\sup S$ exist in $\mathbb{R}$ with $\inf S < \sup S$.
(b) $\inf S < \inf S_0 < \sup S_0 < \sup S$, for every nonempty subset S_0 of S.

Exercise 1.4 Let $S \subset \mathbb{R}$ be bounded above and $s \in \mathbb{R}$. Prove that $s = \sup S$ if and only if for every $\epsilon > 0$, there exists $x_\epsilon \in S$ such that $s < x_\epsilon + \epsilon$. Does a similar result hold for infimums?

Exercise 1.5 Let S be a nonempty set of real numbers which is bounded above. Prove that the set $-S := \{-x : x \in S\}$ is bounded below and $\inf(-S) = -\sup S$.

Exercise 1.6 If $\emptyset \neq S \subset \mathbb{R}$ and S is bounded below, prove that S has a unique infimum in $\mathbb{R}$.

Exercise 1.7 Let S be nonempty bounded above set of $\mathbb{R}$ and U be the set of upper bounds of S. Prove that U is bounded below and $\inf U = \sup S$. State and prove an analogous result, when S is nonempty and bounded below.

Exercise 1.8 Let S be any nonempty bounded above set of real numbers, U be the set of upper bounds of S and L be the set of lower bounds of U. Prove the following:

(a) $L \cup U = \mathbb{R}$.
(b) $l \leq u$, for every $l \in L$ and $u \in U$.
(c) $L \cap U$ cannot contain two distinct real numbers.
(d) $L \cap U = \{s\}$, for some $s \in \mathbb{R}$. Further, $s = \sup S = \sup L = \inf U$.

Exercise 1.9 Let $x, y \in \mathbb{R}$ be such that $x < y$. Prove that

(a) there exists an irrational number s such that $x < s < y$.
(b) there are infinitely many rational numbers between x and y.
(c) there are infinitely many irrational numbers between x and y.

Exercise 1.10 Let S be a nonempty bounded subset of real numbers, $s := \sup S$, and $t \in \mathbb{R} \setminus S$. Prove that $\sup(S \cup \{t\}) = \max\{\sup S, t\}$.

Exercise 1.11 If A and B are nonempty bounded subsets of $\mathbb{R}$, prove that so is $A \cup B$ and $\sup(A \cup B) = \max\{\sup A, \sup B\}$.

Exercise 1.12 Prove that every finite subset of $\mathbb{R}$ contains its supremum.

Exercise 1.13 Prove that there exists no rational number r with $r^2 = 2$.

Exercise 1.14 Let $E := \{x \in \mathbb{R} : x^2 < 2\}$ and $s := \sup E$. Prove that $s^2 = 2$.

Exercise 1.15 Let $a \in \mathbb{Z}$ and $b \in \mathbb{N}$. Prove that there are integers q and r such that $a = bq + r$ and $0 \leq r < b$.

Exercise 1.16 Prove or disprove: $0 < \inf \left\{ \left| \frac{m}{n} - \sqrt{2} \right| : m \in \mathbb{Z} \right\} < \frac{1}{n}$ for all $n \in \mathbb{N}$.

1.2 Sequences of Real Numbers

Let us start with a question. What is the next term of the sequence $1, 3, 5, \ldots$?
(a) 10 (b) 7 (c) 61 (d) None of these.

The most common answer is 7. I insist that the next term is 61. If you do not agree, verify yourself! The terms of this sequence follow the formula:

$$9n^3 - 54n^2 + 101n - 55.$$

Surprising! In fact for any finitely many reals $a_1, \ldots, a_m$, one can find a polynomial function $f(n)$ such that $f(n) = a_n$ for all $n = 1, \ldots, m$. In this sense, given any finitely many terms of a sequence, any arbitrary real number can be assigned as its next term. Therefore *to define a sequence, all the terms must be well-defined.*

Just a pattern of the first few terms is not sufficient to refer to a sequence. This enforces us the following definition.

Definition 1.10 A *sequence* of reals is defined to be a function from $\mathbb{N}$ into $\mathbb{R}$.

In general, a sequence in any nonempty set X is a function $f : \mathbb{N} \longrightarrow X$. Therefore, *a sequence is an infinite list, precisely defined.* There may not be any recognizable pattern in a sequence.

Instead of $f(n)$, we write f_n or more commonly x_n and say that $\{x_n\}$ is a sequence.

Examples 1.11 Each of $\{n\}$, $\{2n - 1\}$ and $\{(-1)^n\}$ are sequences. If p_n denotes the n^{th} prime, then $\{p_n\}$ is a sequence.

Definition 1.12 A sequence $\{y_k\}_k$ is said to be a *subsequence* of a sequence $\{x_n\}_n$ if there exists a strictly increasing sequence $\{n_k\}_k$ of positive integers such that

$$y_k = x_{n_k} \text{ for all } k \in \mathbb{N}.$$

1.2.1 Convergence of a Sequence

A sequence $\{x_n\}$ of real numbers is said to be *convergent* if there exists some $x_0 \in \mathbb{R}$ such that for every $\epsilon > 0$, there exists some positive integer N such that

$$|x_n - x_0| < \epsilon \text{ for all } n \geq N.$$

In this case, we also say that $\{x_n\}$ converges to x_0. Further, x_0 is called the *limit* of $\{x_n\}$ and we write $x_n \longrightarrow x_0$ as well as $x_0 = \lim_{n\to\infty} x_n$. Another popular notation for this purpose is $x_n \longrightarrow x_0$, as $n \longrightarrow \infty$.

Examples 1.13 (a) If $c \in \mathbb{R}$ and $x_n = c$ for all n, then $x_n \longrightarrow c$.

(b) The sequence $\{1/n\}$ converges to 0. To see this, let $\epsilon > 0$ be given. Applying Theorem 1.4, there exists some $N \in \mathbb{N}$ such that $N\epsilon > 1$. Hence we obtain $1/n < \epsilon$ for all $n \geq N$.

Proposition 1.14 *Every convergent sequence of real numbers has a unique limit.*

Proof Suppose $\{x_n\}$ is a convergent sequence with limits x' and x'' such that $x' \neq x''$. Let $\epsilon := |x' - x''|/2$. Since $x' \neq x''$, we have $\epsilon > 0$. Since $\{x_n\}$ converges to x and x', there are positive integers N' and N'' such that

$$|x_n - x'| < \epsilon \text{ for all } n \geq N'.$$
$$\text{and } |x_n - x''| < \epsilon \text{ for all } n \geq N''.$$

Let $N := \max\{N', N''\}$. Then for all $n \geq N$, we obtain

$$|x' - x''| \leq |x' - x_n| + |x_n - x''| < 2\epsilon = |x' - x''|,$$

a contradiction. Hence the result. □

Proposition 1.15 *Every convergent sequence of reals is bounded.*

Proof Let $\{x_n\}$ be a sequence, convergent to some $x \in \mathbb{R}$. Then there exists $N \in \mathbb{N}$ such that $|x_n - x| < 1$ for all $n \geq N$. The triangle inequality ensures that for each $n \geq N$, we have

$$\big||x_n| - |x|\big| \leq |x_n - x| < 1 \text{ which implies } |x_n| \leq |x| + 1.$$

Therefore, for every $n \in \mathbb{N}$, we have $|x_n| \leq \max\big\{|x_1|, |x_2|, \ldots, |x_N|, |x| + 1\big\}$. Hence the sequence is bounded. □

A *tail of a sequence* $\{x_n\}$ is defined to be the sequence after particular term of $\{x_n\}$, that is $\{x_n\}_{n\geq N}$ for some $N \in \mathbb{N}$. Further, we say that $\{x_n\}$ lies *eventually* in a set E, if E contains a tail of $\{x_n\}$. A *neighborhood of a real* $c \in \mathbb{R}$ is defined as a set containing an open interval centered at c. Hence $x_n \longrightarrow x_0$ if and only if $\{x_n\}$ lies eventually in every neighborhood of x_0.

Non-convergent sequences are called *divergent.* In particular, a sequence $\{x_n\} \subset \mathbb{R}$ is called

(a) *divergent to* $+\infty$ if for every $A \in \mathbb{R}$, there exists some positive integer N such that $x_n > A$ for all $n \geq N$.
(b) *divergent to* $-\infty$ if for every $A \in \mathbb{R}$, there exists some positive integer N such that $x_n < A$ for all $n \geq N$.

In the above two cases, we write $x_n \longrightarrow +\infty$ and $x_n \longrightarrow -\infty$, respectively.

1.2.2 Algebra of Limits

The next two results facilitate in assessing the convergence of some specific types of sequences. These results significantly expand our collection of convergent sequences.

Theorem 1.16 *Let $a, b, c \in \mathbb{R}$ and $\{a_n\}$ and $\{b_n\}$ be sequences of real numbers such that $a_n \longrightarrow a$ and $b_n \longrightarrow b$. Then*

(a) $\{ca_n\} \longrightarrow ca$.
(b) $\{a_n + b_n\} \longrightarrow a + b$.
(c) $a_n b_n \longrightarrow ab$.
(d) If $a \neq 0$, then there exists $N \in \mathbb{N}$ such that $a_n \neq 0$ for all $n \geq N$. Further

$$\left\{\frac{1}{a_n}\right\}_{n\geq N} \longrightarrow \frac{1}{a}.$$

Proof Let $\epsilon > 0$ be given.

(a) Since $a_n \longrightarrow a$, there exists some $N \in \mathbb{N}$ such that $|a_n - a| < \frac{\epsilon}{|c|+1}$ for all $n \geq N$. Then for all $n \geq N$, we obtain

$$|ca_n - ca| = |c||a_n - a| \leq |c|\frac{\epsilon}{|c|+1} < \epsilon.$$

(b) Since $a_n \longrightarrow a$ and $b_n \longrightarrow b$, there exist $N_1, N_2 \in \mathbb{N}$ such that

$$|a_n - a| < \frac{\epsilon}{2} \text{ for all } n \geq N_1 \text{ and } |b_n - b| < \frac{\epsilon}{2} \text{ for all } n \geq N_2.$$

Let $N := \max\{N_1, N_2\}$. Then for all $n \geq N$, we obtain

$$|(a_n + b_n) - (a + b)| \leq |a_n - a| + |b_n - b| \leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

(c) Since $\{a_n\}$ is convergent, by Proposition 1.15, it is bounded. So there exists some $M > 0$ such that $|a_n| < M$, for each n. Further, as $a_n \longrightarrow a$ and $b_n \longrightarrow b$, there are positive integers N_1 and N_2 such that

$$|a_n - a| < \frac{\epsilon}{2(|b|+1)} \text{ for all } n \geq N_1$$
$$\text{and } |b_n - b| < \frac{\epsilon}{2(M+1)} \text{ for all } n \geq N_2.$$

Let $N := \max\{N_1, N_2\}$. Therefore, for all $n \geq N$, we obtain

$$\begin{aligned}|a_n b_n - ab| &\leq |a_n b_n - a_n b| + |a_n b - ab| = |a_n||b_n - b| + |a_n - a||b| \\ &\leq M\frac{\epsilon}{2(M+1)} + \frac{\epsilon}{2(|b|+1)}|b| < \epsilon.\end{aligned}$$

(d) Since $a \neq 0$, there exists some $N' \in \mathbb{N}$ such that for all $n \geq N'$, we have

$$\big||a_n| - |a|\big| \leq |a_n - a| < \frac{|a|}{2} \text{ which implies } |a_n| > \frac{|a|}{2}.$$

Further, as $a_n \longrightarrow a$, there exists $N'' \in \mathbb{N}$ such that $|a_n - a| < a^2\epsilon/2$ for all $n \geq N''$. Let $N := \max\{N', N''\}$. Then for all $n \geq N$, we obtain

$$\left|\frac{1}{a_n} - \frac{1}{a}\right| \leq \frac{|a_n - a|}{|a.a_n|} < \frac{2}{a^2}|a_n - a| < \epsilon.$$

Since $\epsilon > 0$ is arbitrary, the results hold. □

Theorem 1.17 *(Squeeze Rule) Let $\{a_n\}$, $\{b_n\}$ and $\{c_n\}$ be sequences of real numbers such that $\{a_n\}$ and $\{b_n\}$ are convergent to the same limit and*

$$a_n \leq c_n \leq b_n \textit{ for all } n \in \mathbb{N}.$$

Then $\{c_n\}$ *is also convergent and* $\lim_{n\to\infty} c_n = \lim_{n\to\infty} a_n$.

Proof Let $\epsilon > 0$ be given and write $c := \lim_{n\to\infty} a_n (= \lim_{n\to\infty} b_n)$. Then there exist positive integers N_1 and N_2 such that

$$|a_n - c| < \epsilon \text{ for all } n \geq N_1$$
$$\text{and } |b_n - c| < \epsilon \text{ for all } n \geq N_2.$$

Let $N := \max\{N_1, N_2\}$. Then $c - \epsilon < a_n \leq c_n \leq b_n < c + \epsilon$ for all $n \geq N$. Therefore $|c_n - c| < \epsilon$ for all $n \geq N$ and hence $c_n \longrightarrow c$. □

1.2.3 Bounded Monotone Sequences

Definitions 1.18 A sequence $\{x_n\}$ of real numbers is said to be

(a) (i) *monotonically increasing* if $x_n \leq x_{n+1}$ for all $n \in \mathbb{N}$.
 (ii) *monotonically decreasing* if $x_n \geq x_{n+1}$ for all $n \in \mathbb{N}$.
 (iii) *monotone* if it is either a monotonically increasing or a monotonically decreasing sequence.

(b) (i) *bounded below* if there exists some $l \in \mathbb{R}$ such that $x_n > l$ for all $n \in \mathbb{N}$.
 (ii) *bounded above* if there exists some $u \in \mathbb{R}$ such that $x_n < u$ for all $n \in \mathbb{N}$.
 (iii) *bounded* if it is bounded below as well as bounded above.

Theorem 1.19 *(Monotone Subsequence Theorem) Every sequence of real numbers contains a monotone subsequence.*

Proof Let $\{x_n\}$ be a sequence of reals. Consider its 'peaks', that is, the set

$$S := \{n \in \mathbb{N} : x_n > x_m \text{ for all } m > n\}.$$

If S is an infinite subset of $\mathbb{N}$, one can write $S := \{n_k : k \in \mathbb{N}\}$ such that $n_k < n_{k+1}$ for all $k \in \mathbb{N}$. Since $n_k \in S$ and $n_{k+1} > n_k$, we obtain $x_{n_k} > x_{n_{k+1}}$ for all $k \in \mathbb{N}$. Hence we obtain a strictly decreasing subsequence $\{x_{n_k}\}$ of $\{x_n\}$.

Now assume that S is a finite set. If $S = \emptyset$, let $N = 1$. Otherwise, let N denote the largest element of S. Then $n \notin S$ for all $n > N$. Fix any $m_1 > N$. Since $m_1 \notin S$, there exists $m_2 > m_1$ such that $x_{m_2} \geq x_{m_1}$. Since $m_2 > m_1$, as earlier there exists some $m_3 > m_2$ such that $x_{m_3} \geq x_{m_2}$. Inducting like this, we obtain a monotonically increasing subsequence $\{x_{m_k}\}$ of $\{x_n\}$. □

Theorem 1.20 *Every monotonically increasing sequence of real numbers that is bounded above is convergent and convergent to its least upper bound.*

Proof Let $\{x_n\}$ be a monotonically increasing and bounded above sequence of reals. Write $S := \{x_n : n \in \mathbb{N}\}$. Applying least upper bound property, let $s := \sup S$. We claim that $x_n \longrightarrow s$.

Let $\epsilon > 0$ be given. Then $s - \epsilon$ is not an upper bound of S. So there exists $N \in \mathbb{N}$ such that $x_N > s - \epsilon$. Now for all $n \geq N$, using monotonicity, we have $s - \epsilon < x_N \leq x_n$. Since s is an upper bound of S, each $x_n \leq s$. Therefore, $s - \epsilon < x_n \leq s < s + \epsilon$ for all $n \geq N$. Hence the result. □

Corollaries 1.21 *(a) Every monotonically decreasing sequence that is bounded below is convergent and convergent to its greatest lower bound.*
(b) Every bounded monotone sequence is convergent.

Proof (a) Let $\{x_n\}$ be a bounded below monotonically decreasing sequence of reals. Then $\{-x_n\}$ is bounded above and monotonically increasing. Now apply Theorem 1.20.
(b) This part follows by (a) and Theorem 1.20. □

Theorem 1.22 *(Bolzano-Weierstrass) Every bounded sequence of reals has a convergent subsequence.*

Proof Apply Theorem 1.19 and Corollaries 1.21. □

A sequence of sets $\{A_n\}$ will be called a *nested decreasing sequence* if $A_n \supset A_{n+1}$ for all $n \in \mathbb{N}$.

Theorem 1.23 *(Nested Interval Property) Intersection of any nested decreasing sequence of closed and bounded intervals is nonempty.*

Proof Let $\{[a_n, b_n]\}$ be a nested decreasing sequence of closed and bounded intervals. Then $[a_n, b_n] \supset [a_{n+1}, b_{n+1}]$ for all $n \in \mathbb{N}$. Therefore, $S := \{a_n : n \in \mathbb{N}\}$ is a nonempty subset of reals, bounded above by b_k for all $k \in \mathbb{N}$. If $a := \sup S$, then $a \leq b_k$ for all $k \in \mathbb{N}$. Hence we obtain $a_n \leq a_{n+1} \leq a \leq b_{n+1} \leq b_n$ for all $n \in \mathbb{N}$. Consequently, $a \in \bigcap_{n=1}^{\infty} [a_n, b_n]$. □

1.2.4 Cauchy Sequences

The major issue in determining the convergence of a sequence is the requirement of having a limit to apply the definition. A first step in this direction is to verify whether its terms are getting arbitrarily closer to each other eventually or not.

Definition 1.24 A sequence $\{x_n\}$ of reals is said to be *Cauchy* if for every $\epsilon > 0$ there exists some $N \in \mathbb{N}$ such that

$$|x_n - x_m| < \epsilon \text{ for all } n, m \geq N.$$

Proposition 1.25 *Every convergent sequence of reals is a Cauchy sequence.*

Proof Let $\{x_n\}$ be a sequence of real numbers such that $x_n \longrightarrow x$ for some $x \in \mathbb{R}$. Let $\epsilon > 0$ be given. Then there exists some $N \in \mathbb{N}$ such that $|x_n - x| < \epsilon/2$ for all $n \geq N$. Consequently, for every $n, m \geq N$, we obtain

$$|x_n - x_m| \leq |x_n - x| + |x - x_m| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

Hence $\{x_n\}$ is a Cauchy sequence. □

Proposition 1.26 *Every Cauchy sequence of reals is bounded.*

Proof Let $\{x_n\}$ be a Cauchy sequence of reals. Then there exists $N \in \mathbb{N}$ such that

$$|x_n - x_m| < 1 \text{ for all } n, m \geq N.$$

For every $n \geq N$, we have $\big||x_n| - |x_N|\big| \leq |x_n - x_N| < 1$, which implies that $|x_n| \leq |x_N| + 1$. Let

$$M := \max\left\{|x_i| : i = 1, \ldots, N\right\} + 1.$$

Then $|x_n| \leq M$ for all $n \in \mathbb{N}$. Hence $\{x_n\}$ is bounded. □

Proposition 1.27 *A Cauchy sequence of reals is convergent if it has a convergent subsequence.*

Proof Let $\{x_n\}$ be a Cauchy sequence of real numbers, having a subsequence $\{x_{n_k}\}$ which is convergent to some $x \in \mathbb{R}$. We claim that $x_n \longrightarrow x$. Let $\epsilon > 0$ be given. Then there exist positive integers N and K such that

$$|x_n - x_m| < \frac{\epsilon}{2} \text{ for all } n, m \geq N$$
$$\text{and } |x_{n_k} - x| < \frac{\epsilon}{2} \text{ for all } k \geq K.$$

Let $p \in \mathbb{N}$ such that $n_p \geq \max\{N, n_K\}$. Then for all $n \geq n_p$, we obtain

$$|x_n - x| \leq |x_n - x_{n_p}| + |x_{n_p} - x| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

Hence $\{x_n\}$ converges to x. □

Theorem 1.28 *(Cauchy criterion) Every Cauchy sequence of real numbers is convergent to a real number.*

Proof Let $\{x_n\}$ be a Cauchy sequence of reals. Applying Proposition 1.26 and Theorem 1.22, $\{x_n\}$ is bounded and hence has a convergent subsequence, say $\{x_{n_k}\}$. Applying Proposition 1.27, we conclude that $\{x_n\}$ is convergent in $\mathbb{R}$. □

Remark 1.29 The sequence $\{\sin n\}$ is quite alluring. It is a serious exercise to show that for every $x \in [-1, 1]$, a subsequence of $\{\sin n\}$ converges to x (see [1]).

History Notes 1.30 The modern definitions of convergence and continuity are often considered difficult by beginner in analysis. That is also quite expected. After the emergence of Calculus through Newton and Leibniz in the late seventeenth century, it took mathematicians around 150 years to develop these rigorous notions. In 1734, Bishop George Berkeley argued against science by claiming that calculus rests on an inconsistent foundation of infinitesimals (see [2]). No eighteenth century mathematician could completely respond to his criticism.

In the third decade of the nineteenth century, Cauchy presented a purely verbal definition of limits which is close to the modern epsilon-delta rigour (see [3, p. 185, 189]). Abel wrote, "Cauchy is crazy, and there is no way of getting along with him, even though right now he is the only one who knows how mathematics should be done. What he is doing is excellent, but very confusing". For more on Cauchy and the language of $\epsilon - \delta$, we refer [4, p. 144]. For a quick overview of the history of calculus, we refer [5].

Exercise 1.17 Is the converse of Proposition 1.15 true?

Exercise 1.18 Prove that every subsequence of a convergent sequence is convergent and is convergent to the same limit.

Exercise 1.19 Let $\{x_n\}$ be a sequence of reals and α be a fixed positive real. Prove that $x_n \longrightarrow x$ if and only if for every $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that $|x_n - x| < \alpha\epsilon$ for all $n > N$.

Exercise 1.20 Let $\{x_n\}$ be a sequence of reals and $x \in \mathbb{R}$. Prove the following:

(a) If $\lim\limits_{n\to\infty} x_n = x$, then $\lim\limits_{n\to\infty} x_{n+1} = x$.
(b) If $\lim\limits_{n\to\infty} x_{2n} = x = \lim\limits_{n\to\infty} x_{2n+1}$, then $\lim\limits_{n\to\infty} x_n = x$.

Exercise 1.21 Prove that inserting (removing) any finite number of terms

(a) at (from) the beginning of a sequence doesn't affect its convergence/divergence.
(b) anywhere in a sequence doesn't affect its convergence/divergence.

Exercise 1.22 Let $\{x_n\}$ be a monotonically increasing and unbounded sequence of real numbers. Prove that $\{x_n\} \longrightarrow +\infty$.

Exercise 1.23 Let $n_0 \in \mathbb{Z}$ and $\{n_k\}$ be any sequence in the set $\{0, 1, 2, \ldots, 9\}$. Write

$$x_k := n_0 + \frac{n_1}{10} + \cdots + \frac{n_k}{10^k} \text{ for all } k \in \mathbb{N}.$$

Prove that the sequence $\{x_k\}$ is convergent in $\mathbb{R}$.

Exercise 1.24 Let $a_n \longrightarrow a$ in $\mathbb{R}$. Prove the following:

(a) If $a > 0$, then there exists some $N \in \mathbb{N}$ such that $a_n > 0$ for all $n > N$.
(b) If $a > l$, then there exists some $N \in \mathbb{N}$ such that $a_n > l$ for all $n > N$.
(c) If $a \neq 0$, then there exists some $N \in \mathbb{N}$ such that $a_n \neq 0$ for all $n > N$.

Exercise 1.25 Let $\{x_n\}$ be a sequence of reals and $a \in \mathbb{R}$ be such that $\{x_n\}$ is not convergent to a. Prove that for all sufficiently small $\epsilon > 0$, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $|x_{n_k} - a| > \epsilon$ for all $k \in \mathbb{N}$.

Exercise 1.26 Let $a_n \longrightarrow a$ and $b_n \longrightarrow b$ in $\mathbb{R}$. Prove the following:

(a) If $a_n \leq b_n$ for all $n \in \mathbb{N}$, then $a \leq b$.
(b) If $a_n < b_n$ for all $n \in \mathbb{N}$, then $a < b$ may not hold.
(c) If $a < b$, then there exists some $N \in \mathbb{N}$ such that $a_n < b_n$ for all $n > N$.

Exercise 1.27 If $|a_n| \longrightarrow 0$, prove that $a_n \longrightarrow 0$. Is the converse true?

Exercise 1.28 If $a_n \longrightarrow a$, prove that $|a_n| \longrightarrow |a|$. Is the converse true?

Exercise 1.29 If $a_n \longrightarrow 0$ and $\{b_n\}$ is bounded, prove that $a_n b_n \longrightarrow 0$.

Exercise 1.30 Let $\{a_n\}$ be a sequence of positive reals, convergent to some $a \in \mathbb{R}$. Prove that $a_n^{\frac{1}{k}} \longrightarrow a^{\frac{1}{k}}$ for all $k \in \mathbb{N}$.

Exercise 1.31 Prove that $\lim_{n\to\infty} n^{1/n} = 1$.

Exercise 1.32 Prove that every subsequence of a Cauchy sequence is also Cauchy.

Exercise 1.33 Prove that the termwise sum as well as the termwise product of finitely many Cauchy sequences is also a Cauchy sequence.

Exercise 1.34 Show that $\{\sqrt{n}\}$ is not Cauchy, while $\lim_{n\to\infty} |\sqrt{n+m} - \sqrt{n}| = 0$ for all $m \in \mathbb{N}$.

Exercise 1.35 Let $\{a_n\}$ be a sequence of positive reals such that $a_n \longrightarrow 0$, $b_0 \in \mathbb{R}$ and $\{b_n\}$ be any sequence of real numbers. Prove that $b_n \longrightarrow b_0$ if and only if there exists $c > 0$ and $N \in \mathbb{N}$ such that $|b_n - b_0| < ca_n$ for all $n > N$.

Exercise 1.36 Let S be a nonempty bounded subset of reals and $s := \sup S$. Prove that there exists a monotonically increasing sequence $\{s_n\}$ in S such that $s_n \longrightarrow s$.

Exercise 1.37 Let $\{x_n\} \longrightarrow x$ in $\mathbb{R}$. Prove that $\{x_n\}$ is bounded and

$$\inf\{x_n : n \in \mathbb{N}\} \leq \lim_{n\to\infty} x_n \leq \sup\{x_n : n \in \mathbb{N}\}.$$

Exercise 1.38 Let $\{x_n\}$ be a convergent sequence of positive integers. Prove that the sequence $\{x_n\}$ is *eventually constant,* i. e., there exists $m \in \mathbb{N}$ such that $\{x_n\}_{n\geq k}$ is a constant sequence.

Exercise 1.39 Let $\{p_n\}$ be a sequence of reals such that $\lim_{n\to\infty}(p_{n+1} - p_n) = 0$.

(a) If each $p_n \in \mathbb{Z}$, prove that $\{p_n\}$ is eventually constant.

(b) Show that, in general, the sequence $\{p_n\}$ may not even converge.

Exercise 1.40 Let $\{x_n\} \longrightarrow x$, $\sigma : \mathbb{N} \longrightarrow \mathbb{N}$ be a bijection and $y_n := x_{\sigma(n)}$ for all $n \in \mathbb{N}$. Prove that $\{y_n\} \longrightarrow x$. That is, any permutation of a convergent sequence is convergent and is convergent to the same limit.

Exercise 1.41 Consider a collection of sequences $\{\{x_n^a\}_n : a \in \mathbb{R}\}$, where

$$x_n^a := a + \frac{1}{n} \text{ for all } a \in \mathbb{R} \text{ and for all } n \in \mathbb{N}.$$

Prove that the set $\{x_n^a : n \in \mathbb{N}\} \cap \{x_n^b : n \in \mathbb{N}\}$ is finite, for all $a \neq b$.

Exercise 1.42 For every $a \in \mathbb{R}$, let $f_a : \mathbb{N} \longrightarrow \mathbb{R}$ be a sequence, convergent to a. Prove that the set $f_a(\mathbb{N}) \cap f_b(\mathbb{N})$ is finite, for all distinct reals a and b.

Exercise 1.43 If $\alpha \in \mathbb{R}$, prove that $\lim_{n\to\infty} \frac{\alpha^n}{n!} = 0$.

Exercise 1.44 Prove that $\lim_{n\to\infty} \sqrt[n]{n!} = +\infty$.

Exercise 1.45 (Cauchy) If $a_n \longrightarrow l$, prove that $\lim\limits_{n\to\infty} \dfrac{a_1 + \cdots + a_n}{n} = l$. Is the converse true?

Exercise 1.46 Prove that $\lim_{n\to\infty} \frac{1}{n}\left(1 + \cdots + \frac{1}{n}\right) = 0$.

Exercise 1.47 Let S be any infinite subset of $\mathbb{N}$. Show that S can be written as $\{n_k : k \in \mathbb{N}\}$ such that $n_k < n_{k+1}$ for all $k \in \mathbb{N}$.

Exercise 1.48 Does there exist any monotone sequence $\{x_n\}$ such that

$$\{x_n : n \in \mathbb{N}\} = \left\{\frac{1}{n} : n \in \mathbb{N}\right\} \bigcup \left\{1 + \frac{1}{n} : n \in \mathbb{N}\right\}?$$

Exercise 1.49 The *upper and lower limits* (or limit superior and limit inferior) of a sequence $\{a_n\}$ of reals are defined as

$$a^* := \inf_{n\geq 1} \sup_{k\geq n} a_k \text{ and } a_* := \sup_{n\geq 1} \inf_{k\geq n} a_k,$$

respectively. The extended reals a^* and a_* are denoted by $\limsup_{n\longrightarrow\infty} a_n$ and $\liminf_{n\longrightarrow\infty} a_n$, respectively. Prove the following:

(a) Find a_*, a^* if for all $n \in \mathbb{N}$, a_n equals (i) $(-1)^{n+1}$, (ii) $\frac{(-1)^{n+1}}{n}$, and (iii) $(-n)^n$.
(b) If $\{a_n\}$ is bounded, then $\{a_n\}$ has subsequences convergent to a^* and a_*.
(c) If $\{a_n\}$ is bounded, then $\{a_n\}$ is convergent if and only if $a^* = a_*$.
(d) If A is the set of its subsequential limits of $\{a_n\}$, then $a^* = \sup A$ and $a_* = \inf A$.

Exercise 1.50 (Erdős-Szekeres) If $f : \{1, \ldots, k^2 + 1\} \longrightarrow \mathbb{R}$ is an injective map, prove that there exist integers $n_1 < \cdots < n_{k+1}$ such that either $f(n_1) \leq \cdots \leq f(n_{k+1})$ or $f(n_1) \geq \cdots \geq f(n_{k+1})$.

1.3 Series Convergence

The addition of any finite collection of numbers is well known. Can we add up some infinite collection of numbers? A natural idea since antiquity is the following:

Let $\{a_n\}$ be a sequence of real numbers. To sum up $a_1, a_2, a_3, \ldots,$ consider the corresponding *partial sums* defined as

$$S_n := a_1 + a_2 + \cdots + a_n \text{ for all } n \in \mathbb{N}.$$

The series $\sum_{n=1}^{\infty} a_n$ is said to be *convergent* if the sequence $\{S_n\}$ is convergent. In this case, we denote $\lim_{n\to\infty} S_n$ by $\sum_{n=1}^{\infty} a_n$; and say that the series converges to this limit. It is also known as the *sum of the series.*

Examples 1.31 The geometric series $\sum_{n=0}^{\infty} \frac{1}{2^n}$ is convergent to 2.

Proof Let S_n denote the n^{th} partial sum of the given series. That is,

$$S_n = 1 + \frac{1}{2} + \frac{1}{2^2} + \cdots + \frac{1}{2^n} \text{ for all } n \in \mathbb{N}.$$

Then

$$2S_n = 2 + 1 + \frac{1}{2} + \cdots + \frac{1}{2^{n-1}} \text{ for all } n \in \mathbb{N}.$$

Subtracting these equations, we obtain $S_n = 2 - 2^{-n}$ for all $n \in \mathbb{N}$. So, $S_n \longrightarrow 2$, as $n \longrightarrow \infty$. Hence the given series converges to 2. □

First, we provide a necessary condition for the convergence of a series.

Theorem 1.32 *If* $\sum_{n=1}^{\infty} a_n$ *converges, then* $\{a_n\} \longrightarrow 0$.

Proof Let $S := \sum_{n=1}^{\infty} a_n$ and $S_n := \sum_{k=1}^{n} a_k$ for all $n \in \mathbb{N}$. Then we have $a_n = S_n - S_{n-1}$ for all $n > 1$. Since $S_n \longrightarrow S$, we have $\lim_{n\to\infty} S_{n-1} = S$. Hence we obtain $\lim_{n\to\infty} a_n = \lim_{n\to\infty}(S_n - S_{n-1}) = S - S = 0$. □

Note that the necessary condition in the above theorem is not sufficient.

Example 1.33 The series $\sum_{n=1}^{\infty} \frac{1}{n}$ is not convergent.

Proof Assume that the series $\sum_{n=1}^{\infty} \frac{1}{n}$ is convergent in $\mathbb{R}$. Let S denote its sum and $S_n := \sum_{k=1}^{n} \frac{1}{k}$ for all $n \in \mathbb{N}$. Then

$$\begin{aligned} S = \lim_{n\to\infty} S_n = \lim_{n\to\infty} S_{2n} &= \lim_{n\to\infty} \left(1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \cdots + \frac{1}{2n}\right) \\ &= \lim_{n\to\infty} \left(\frac{1}{2} + \left(\frac{1}{2} + \frac{1}{2}\right) + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \frac{1}{6}\right) + \cdots + \left(\frac{1}{2n-1} + \frac{1}{2n}\right)\right) \end{aligned}$$

$$\geq \lim_{n\to\infty}\left(\frac{1}{2}+\left(\frac{1}{2}+\frac{1}{2}\right)+\left(\frac{1}{4}+\frac{1}{4}\right)+\left(\frac{1}{6}+\frac{1}{6}\right)+\cdots+\left(\frac{1}{2n}+\frac{1}{2n}\right)\right)$$
$$=\lim_{n\to\infty}\left(\frac{1}{2}+S_n\right)=\frac{1}{2}+S.$$

Hence $S \geq \frac{1}{2}+S$, a contradiction. □

Theorem 1.34 *(Cauchy criterion) A series $\sum_{n=1}^{\infty} x_n$ of reals is convergent if and only if for every $\epsilon > 0$ there exists some $N \in \mathbb{N}$ such that*

$$\left|\sum_{n=n_1+1}^{n_2} x_n\right| < \epsilon \text{ for all } n_2 > n_1 > N. \tag{1.1}$$

Proof The required result is a direct application of Proposition 1.25 and Theorem 1.28 on the sequence of partial sums of $\sum_{n=1}^{\infty} x_n$. Let $S_n := \sum_{k=1}^{n} x_k$ for all $n \in \mathbb{N}$.

Note that $\sum_{n=1}^{\infty} x_n$ is convergent if and only if $\{S_n\}$ is convergent if and only if $\{S_n\}$ is Cauchy if and only if the result (1.1) holds. □

Definition 1.35 A series $\sum_{n=1}^{\infty} x_n$ of reals is said to be *absolutely convergent* if the series $\sum_{n=1}^{\infty} |x_n|$ is convergent.

Note that the partial sums of the series $\sum_{n=1}^{\infty} |x_n|$ form a monotonically increasing sequence of reals. Hence this series is either convergent or divergent to $+\infty$. Therefore we write $\sum_{n=1}^{\infty} |x_n| < \infty$, if the series $\sum_{n=1}^{\infty} x_n$ is absolutely convergent.

Theorem 1.36 *Absolutely convergent series of real numbers are convergent.*

Proof Let $\sum_{n=1}^{\infty} x_n$ be any absolutely convergent series of real numbers. Then for $\epsilon > 0$, there exists some $N \in \mathbb{N}$ such that

$$\sum_{n=n_1+1}^{n_2} |x_n| < \epsilon \text{ for all } n_2 > n_1 > N.$$

The triangle inequality implies $|\sum_{n=n_1+1}^{n_2} x_k| \leq \sum_{n=n_1+1}^{n_2} |x_k|$. Therefore,

$$\left|\sum_{n=n_1+1}^{n_2} x_n\right| < \epsilon \text{ for all } n_2 > n_1 > N.$$

Applying Theorem 1.34, we conclude that the series $\sum_{n=1}^{\infty} x_n$ is convergent. □

Not every convergent series is absolutely convergent (see Exercise 1.67).

Theorem 1.37 *(Comparison test) Let $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ be series of non-negative terms such that $a_n \leq b_n$ for all $n \in \mathbb{N}$. If $\sum_{n=1}^{\infty} b_n$ is convergent, then so is $\sum_{n=1}^{\infty} a_n$.*

Proof Let $\{A_n\}$ and $\{B_n\}$, respectively, denote the sequences of partial sums of the series $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$. By hypothesis, both are monotonically increasing sequences of reals such that $A_n \leq B_n$ for every $n \in \mathbb{N}$.

Since $\sum_{n=1}^{\infty} b_n$ is convergent, the sequence $\{B_n\}$ is bounded above. So the sequence $\{A_n\}$ is also bounded above and thus convergent. Hence the series $\sum_{n=1}^{\infty} a_n$ is also convergent. □

Several other common tests for series convergence will be presented in the exercises. It is important to note that such tests for series convergence do not provide the limit of the series. Computing the exact limit is a much tedious task, in general, as compared to establishing the convergence of the series.

It is also interesting to note that the prime harmonic series $\sum_p \frac{1}{p}$ is divergent (see [6]). Another interesting series will be presented in Exercise 1.93.

There are various simple looking series and sequences for which the question of convergence is open till date. Here we provide a few such cases.

Open Questions 1.38 (a) Does the series $\sum_{n=1}^{\infty}(n^3 \sin^2 n)^{-1}$ converge?
(b) Does the sequence $\{(n^2 \sin n)^{-1}\}$ converge?
(c) If p_k denote the k^{th} prime, does the series $\sum_{n=1}^{\infty} \frac{(-1)^k k}{p_k}$ converge?

Exercise 1.51 Prove that removing (inserting) finitely many terms from (in) a series does not affect its convergence or non-convergence. However, in case of convergence, it may affect the sum.

Exercise 1.52 Let $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ be convergent series of real numbers and $\alpha, \beta \in \mathbb{R}$. Prove that the series $\sum_{n=1}^{\infty}(\alpha a_n + \beta b_n)$ is also convergent and $\sum_{n=1}^{\infty}(\alpha a_n + \beta b_n) = \alpha \sum_{n=1}^{\infty} a_n + \beta \sum_{n=1}^{\infty} b_n$.

Exercise 1.53 Prove that the series $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ is divergent.

Exercise 1.54 Apply comparison test to conclude that $\sum_{n=1}^{\infty} \frac{1}{n}$ is divergent.

Exercise 1.55 Prove that no hypothesis in Theorem 1.37 is redundant.

Exercise 1.56 If $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are series of non-negative terms such that $\sum_{n=1}^{\infty}(a_n + b_n)$ is convergent, prove that both $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are convergent. Also prove that no hypothesis here is redundant.

Exercise 1.57 (Geometric series) If $a \in \mathbb{R}$ and $r \in (-1, 1)$, prove that the series $\sum_{n=0}^{\infty} ar^n$ is convergent.

Exercise 1.58 Let $0 < r < 1$ and $\{x_n\}$ be a sequence of reals such that $|x_{n+1} - x_n| < r^n$ for all $n \in \mathbb{N}$. Prove that $\{x_n\}$ is a convergent sequence.

Exercise 1.59 (Integral test) Let $\sum_{n=1}^{\infty} a_n$ be a series of non-negative reals and $f : [1, \infty] \longrightarrow \mathbb{R}$ be a non-increasing function such that $f(n) = a_n$, for each $n \in \mathbb{N}$. Prove that the series $\sum_{n=1}^{\infty} a_n$ converges if and only if $\int_1^{\infty} f(x)dx$ converges.

Exercise 1.60 (p-test) Prove that the series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges for $p > 1$ and diverges for $p \leq 1$.

Exercise 1.61 (Limit test) Let $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ be series of non-negative reals such that $\lim_{n\to\infty} \frac{a_n}{b_n}$ exists in $[0, \infty]$. Let l denote this limit. Prove the following assertions:

(a) If $0 < l < \infty$, then $\sum_{n=1}^{\infty} a_n$ converges if and only if $\sum_{n=1}^{\infty} b_n$ converges.
(b) If $l = 0$, then $\sum_{n=1}^{\infty} a_n$ converges, if $\sum_{n=1}^{\infty} b_n$ converges.
(c) If $l = \infty$, then $\sum_{n=1}^{\infty} b_n$ converges, if $\sum_{n=1}^{\infty} a_n$ converges.

Exercise 1.62 Does the converse of the second assertion in Exercise 1.61 hold?

Exercise 1.63 (Root test) Let $\sum_{n=0}^{\infty} a_n$ be a series of non-negative reals such that $\lim_{n\to\infty} a_n^{1/n}$ exists in $[0, \infty]$. Let l denote this limit. Prove the following assertions:

(a) If $l < 1$, the series is convergent,
(b) If $l > 1$, the series is divergent, and
(c) If $l = 1$, the series is may or may not be convergent.

Exercise 1.64 (Ratio test) Let $\sum_{n=0}^{\infty} a_n$ be a series of non-negative reals such that $\lim_{n\to\infty} \frac{a_{n+1}}{a_n}$ exists in $[0, \infty]$. Let l denote this limit. Prove the following assertions:

(a) If $l < 1$, the series is convergent,
(b) If $l > 1$, the series is divergent and
(c) If $l = 1$, the series is may or may not be convergent.

Exercise 1.65 (Leibniz test) Prove that a series $\sum_{n=0}^{\infty} (-1)^{n+1} u_n$ is convergent if each $u_n \geq 0$, $\{u_n\}$ is a monotonically decreasing sequence and $\lim_{n\to\infty} u_n = 0$.

Exercise 1.66 Show that no hypothesis in the Leibniz test is redundant.

Exercise 1.67 Prove that the series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ and $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\sqrt{n}}$ are convergent, but not absolutely. (Such series are known as *conditionally convergent series*.)

Exercise 1.68 Let $\{a_n\}$ and $\{b_n\}$ be sequences of reals such that both $\sum_{n=1}^{\infty} a_n^2$ and $\sum_{n=1}^{\infty} b_n^2$ are convergent. Prove that $\sum_{n=1}^{\infty} a_n b_n$ is also convergent.

Exercise 1.69 Discuss the convergence of the series $\sum_{n=1}^{\infty} \frac{n}{4^n+1}$ and $\sum_{n=1}^{\infty} \frac{1}{n5^n+n^5}$.

Exercise 1.70 Prove that the series $\sum_{n=2}^{\infty} \frac{1}{n(\log n)^2}$ is convergent.

Exercise 1.71 Examine the following sequences for convergence:

$$\frac{1}{n}, (-1)^n, \frac{3n-1}{5n+2}, \left(\frac{2n}{n+1}\right)^3, \log n, 2 - \frac{1}{5^n}.$$

Exercise 1.72 Does any of the following the series converge:

$$\sum_{n=1}^{\infty}(\sqrt[3]{n^3+1}-n) \text{ and } \sum_{n=1}^{\infty}(\sqrt{n^4+1}-\sqrt{n^4-1})?$$

Exercise 1.73 Discuss the convergence of the following series:

$$\sum_{n=1}^{\infty}\frac{n}{4^n}, \sum_{n=1}^{\infty}\frac{\cos n}{n^3+n-1}, \sum_{n=1}^{\infty}\frac{1}{n5^n}, \sum_{n=1}^{\infty}\frac{n^2}{2^n}, \sum_{n=1}^{\infty}\frac{n!}{2^n} \text{ and } \sum_{n=1}^{\infty}\frac{n!}{n^n}.$$

Exercise 1.74 Discuss the convergence of the following series:

$$\sum_{n=1}^{\infty}\frac{2n-1}{n(n+1)(n+2)}, \sum_{n=1}^{\infty}\frac{n^2}{(3n+1)(2n^2+3)} \text{ and } \sum_{n=1}^{\infty}\frac{(n-1)^{n-1}}{n^n}.$$

Exercise 1.75 Discuss the convergence of the following series:

$$\sum_{n=1}^{\infty}\frac{\sin n^2}{3n^5+1}, \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{\sqrt{n}+1} \text{ and } \sum_{n=2}^{\infty}\frac{(-1)^{n+1}}{\log n}.$$

Exercise 1.76 Prove that the following series are convergent

$$\sum_{n=1}^{\infty}\sin\frac{1}{n^2}, \sum_{n=1}^{\infty}\sin\frac{1}{n} \text{ and } \sum_{n=1}^{\infty}\frac{1}{\sqrt{n}}\sin\frac{1}{n}.$$

Exercise 1.77 If $a \in \mathbb{R} \setminus \mathbb{Z}$, examine the convergence of the series

$$\frac{1}{a}+\frac{1}{a+1}-\frac{1}{a+2}+\frac{1}{a+3}+\frac{1}{a+4}-\frac{1}{a+5}+\frac{1}{a+6}-\frac{1}{a+7}+\frac{1}{a+8}-\frac{1}{a+9}+\ldots.$$

Exercise 1.78 Let $\{a_n\}$ be a monotonically decreasing sequence of positive real numbers. Prove that $\sum_{n=1}^{\infty}\frac{a_n}{n^2}$ is convergent.

Exercise 1.79 Let $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ be convergent series of reals. Show that

(a) $\sum_{n=1}^{\infty} a_n b_n$ is convergent, if both $a_n, b_n \geq 0$ for all $n \in \mathbb{N}$.
(b) $\sum_{n=1}^{\infty} a_n b_n$ may not be convergent, in general.

Exercise 1.80 (Dirichlet's test) Let $\{a_n\}$, $\{b_n\}$ be sequences of reals satisfying

(a) $\{a_n\}$ is monotone and convergent to 0.
(b) There exists $M > 0$ such that $|\sum_{n=1}^{m} b_n| \leq M$ for all $m \in \mathbb{N}$.

Prove that the series $\sum_{n=1}^{\infty} a_n b_n$ is convergent. Deduce Leibniz's test from it.

Exercise 1.81 (Abel's test) Let $\{a_n\}, \{b_n\}$ be sequences of real numbers such that $\{a_n\}$ is bounded monotone and $\sum_{n=1}^{\infty} b_n$ is convergent. Prove that the series $\sum_{n=1}^{\infty} a_n b_n$ is convergent.

Exercise 1.82 (Cauchy condensation test) Let $\{a_n\}$ be a monotonically decreasing sequence of non-negative reals. Prove that $\sum_{n=1}^{\infty} a_n$ converges if and only if $\sum_{k=0}^{\infty} 2^k a_{2^k}$ converges.

Exercise 1.83 Does there exist a convergent series of positive reals $\sum_{n=1}^{\infty} a_n$ such that $\sum_{n=1}^{\infty} a_n^2$ does not converge?

Exercise 1.84 Let $\sum_{n=1}^{\infty} a_n$ be a convergent series of non-negative reals.

(a) If the sequence $\{a_n\}$ is monotonically decreasing, prove that $\lim_{n\to\infty} na_n = 0$.
(b) Is $\lim_{n\to\infty} na_n = 0$ always true?

Exercise 1.85 Let $\{a_n\}$ be a sequence of positive reals such that $\sum_{n=1}^{\infty} a_n^2 < \infty$. Which of the following series is/are convergent?

(a) $\sum_{n=1}^{\infty} \frac{a_n}{n}$.
(b) $\sum_{n=1}^{\infty} a_n^p$ for all $p > 2$.
(c) $\sum_{n=1}^{\infty} a_n^p$ for all $1 < p < 2$.

Exercise 1.86 Let $\sum_{n=1}^{\infty} a_n$ be a convergent series of positive real numbers. Is any of the series $\sum_{n=1}^{\infty} \frac{\sqrt{a_n}}{n}$ or $\sum_{n=1}^{\infty} \sqrt{\frac{a_n}{n}}$ convergent?

1.4 Decimal and General Expansions

In high school mathematics, real numbers are often introduced through decimal expansions. We will now provide a rigorous justification for such expansions along with some generalizations.

Theorem 1.39 *If $x \in \mathbb{R}$, then there exist an integer n_0 and a sequence $\{n_k\}$ from the set $\{0, 1, 2, \ldots, 9\}$ such that*

$$\sum_{n=0}^{\infty} \frac{n_k}{10^k} = x.$$

Proof Applying Corollary 1.5, let $n_0 \in \mathbb{Z}$ such that $n_0 \le x < n_0 + 1$. Further, let n_1 be the largest integer such that $n_0 + \frac{n_1}{10} \le x$. Note that $0 \le n_1 \le 9$. Assume that $n_1, \ldots, n_k$ have been chosen as the maximal integers from the set $\{0, 1, \ldots, 9\}$ such that $n_0 + \frac{n_1}{10} + \cdots + \frac{n_k}{10^k} \le x$. Let n_{k+1} be the largest integer such that

$$n_0 + \frac{n_1}{10} + \cdots + \frac{n_k}{10^k} + \frac{n_{k+1}}{10^{k+1}} \le x.$$

Then $n_{k+1} \in \{0, \ldots, 9\}$, otherwise the maximality of n_k is violated. Then for $x_k := n_0 + \frac{n_1}{10} + \cdots + \frac{n_k}{10^k}$; $k \in \mathbb{N}$, $\{x_k\}$ defines a monotonically increasing sequence of reals, bounded above by x. Therefore, it has a limit, say s. Since each $x_k \leq x$, we have $s \leq x$. We claim that $s = x$.

If possible, assume that $s < x$. Since $\frac{1}{10^k} \longrightarrow 0$, there exists some $k \in \mathbb{N}$ such that $s + \frac{1}{10^k} < x$, which implies that at the k^{th} iteration, n_k was not the largest possible integer satisfying our requirements above, a contradiction. Hence the result. □

Definition 1.40 Let x be a real number. If $x \geq 0$, then a series $\sum_{n=0}^{\infty} \frac{n_k}{10^k}$, where n_k's are as in Theorem 1.39, is called a *decimal expansion* of x and is denoted as $n_0.n_1n_2\ldots$.

If $x < 0$, the *decimal expansion* of x is defined as $-n_0.n_1n_2\ldots$, where $n_0.n_1n_2\ldots$ is a decimal expansion of $|x|$. Decimal expansions are also known as *decimal representations*.

If x has an infinite recurring decimal expansion in which a finite block of digits $a_1a_2\ldots a_n$ repeat infinitely often, we write $\overline{a_1a_2\ldots a_n}$ to denote its indefinite recurrence. For example, it can be verified that the decimal representations of $\frac{1}{3}$ and $\frac{17}{99}$ are $0.\overline{3}$ and $0.\overline{17}$, respectively.

Remarks 1.41 Note that every real with a finite decimal expansion has an infinite decimal expansion too. In that case, the above proof of Theorem 1.39 will produce only the finite decimal representation.

Examples 1.42 Note that $0.\overline{9} = 1$, as

$$0.\overline{9} = \frac{9}{10} + \frac{9}{10^2} + \cdots + \frac{9}{10^n} + \cdots = \frac{9/10}{1-(1/10)} = 1.$$

Similarly, one can verify that $2.75\overline{9} = 2.76$. and $18.172\overline{9} = 18.173$.

Theorem 1.43 *If $m \in \mathbb{N} \setminus \{1\}$ and $x \in \mathbb{R}$, then there exist $n_0 \in \mathbb{Z}$ and a sequence $\{n_k\}$ in $\{0, 1, 2, \ldots, m-1\}$ such that*

$$\sum_{n=0}^{\infty} \frac{n_k}{m^k} = x.$$

Proof The proof is analogous to Theorem 1.39. The only difference is that here we split a unit into m equal parts, instead of 10. □

Let $x \in \mathbb{R}$. If $x \geq 0$, then $x = (n_0.n_1n_2\ldots n_k\ldots)_m$ defines the *m-base representation/expansion* of x, here $n_0, n_1, n_2, \ldots$ are given by Theorem 1.43. As in Definition 1.40, we obtain the m-base expansion of a negative real x by prefixing the negative sign before the m-base expansion of $|x|$.

If $m = 2, 3$ then these are known as the *binary, ternary representation* of x, respectively. We shall deal with ternary representations, extensively in Chap. 10.

Theorem 1.44 *A real number is rational if and only if for any $m > 1$, its m-base representation is either finite or repeating.*

Proof The converse is trivial, by summing up the geometric series analogous to Examples 1.42, if required. To prove the necessity part, note that it is enough to establish that only for positive real numbers.

Let p/q be a rational number, where $p \in \mathbb{N}$ and $q \in \mathbb{N}$. Dividing p by q, write $p = a_0 q + r_0$ for some $a_0 \in \mathbb{Z}$ and $r_0 \in \{0, \ldots, q-1\}$. Similarly, there exist integers $a_1 \in \mathbb{Z}$ and $r_1 \in \{0, \ldots, q-1\}$ such that $mr_0 = a_1 q + r_1$. Inductively we obtain sequences $\{a_k\}$ and $\{r_k\}$ of integers such that

$$mr_{k-1} = a_k q + r_k \text{ and } 0 \le r_k < 1 \text{ for all } k \in \mathbb{N}.$$

Therefore we obtain $(a_0.a_1 \ldots a_k \ldots)_m$, as an m-base expansion of p/q. Since the sequence $\{r_k\}$ belongs to a finite set, there exist integers $k_1 < k_2$ such that $r_{k_1} = r_{k_2}$. Hence the finite block of digits $a_{k_1}, a_{k_2}, \ldots, a_{k_2-1}$ repeats in the m-base representation of p/q. □

Exercise 1.87 Obtain the binary, ternary, and decimal expansions of $1/2$, $1/3$, $1/4$, and $1/5$. In case of finite representations, write down the infinite ones too.

Exercise 1.88 Let $a \in \mathbb{R}$. Prove that there exists a sequence of rationals $\{r_n\}$ and a sequence of irrationals $\{i_n\}$ such that $\{r_n\} \longrightarrow a$ and $\{i_n\} \longrightarrow a$.

Exercise 1.89 For $y \in \mathbb{R}$, let $[y]$ denote the *greatest integer less than or equal to* y. If $\{x_n\}$ is a sequence of reals, convergent to some x, prove that the sequence $\{[10(x_n - [x_n])]\}_n$ is eventually constant.

Exercise 1.90 Does there exist a non-zero polynomial P in two variables which satisfies the equation $P([x], [2x]) = 0$ for all $x \in \mathbb{R}$?

Exercise 1.91 For $x \in \mathbb{Q}$, let $l(x)$ be the length of the smallest repeating block in the decimal expansion of x. For example, $l(1/3) = 1$ and $l(13/99) = 2$ etc. If $S := \{x \in [0, 1] \cap \mathbb{Q} : l(x) = 8\}$, evaluate the finite sum $\sum_{x \in S} x$.

Exercise 1.92 Let $a, b \in \mathbb{R}$. Write an algorithm to find the decimal expansion of $a + b$, using the decimal expansions of a and b.

Exercise 1.93 (Kempner 1914) Is the harmonic subseries of $\sum_{n=1}^{\infty} \frac{1}{n}$ convergent, if the integers n containing 9 are excluded from the sum? Is the same true for any other digit, apart from 9?

1.5 Continuity

A function $f : \mathbb{R} \longrightarrow \mathbb{R}$ is said to have a *limit* at a point $c \in \mathbb{R}$ if there exist some $l \in \mathbb{R}$ such that for every $\epsilon > 0$ there exists some $\delta > 0$ such that

$$|f(x) - l| < \epsilon \text{ for all } x \in \mathbb{R}, \text{ satisfying } 0 < |x - c| < \delta.$$

In this case, we say that l is the limit of f at c and write $\lim_{x \to c} f(x) = l$. Further, the function f is said to be

(a) *continuous at a point* $c \in \mathbb{R}$, if $\lim_{x \to c} f(x) = f(c)$.
(b) *a continuous function,* if it is continuous at every point $c \in \mathbb{R}$.

Remark 1.45 Any student of calculus knows that it is not necessary for f to be defined at c in order to discuss its limit at c. But what is the minimum requirement from the domain of f for this purpose? We shall discuss the most general situation in Definition 3.23. Similarly, continuity can be discussed on any subset of the domain. In this chapter, we shall avoid all such technical jargon, which often distracts the first-time readers.

First we present the sequential approach to the limit of a real function.

Theorem 1.46 *Let* $f : \mathbb{R} \longrightarrow \mathbb{R}$ *and* $x_0, l \in \mathbb{R}$. *Then* $\lim_{x \to x_0} f(x) = l$ *if and only if for every sequence* $\{x_n\}$ *in* $\mathbb{R} \setminus \{x_0\}$ *such that* $x_n \longrightarrow x_0$, *we have* $f(x_n) \longrightarrow l$.

Proof First, we assume that $\lim_{x \to x_0} f(x) = l$. Let $\epsilon > 0$ be given. Then there exists some $\delta > 0$ such that

$$|f(x) - l| < \epsilon \text{ whenever } 0 < |x - x_0| < \delta.$$

Let $\{x_n\}$ be a sequence in $\mathbb{R} \setminus \{x_0\}$, convergent to x_0. Then there exists a positive integer N such that $0 < |x_n - x_0| < \delta$ for all $n \geq N$. Due to our choice of δ, we obtain $|f(x_n) - l| < \epsilon$ for all $n \geq N$. Hence $f(x_n) \longrightarrow l$.

Conversely, assume that $\lim_{x \to x_0} f(x) \neq l$. Then there exists some $\epsilon > 0$ such that for every $\delta > 0$ the following assertion is not satisfied:

$$|f(x) - l| < \epsilon \text{ for all } x \text{ satisfying } 0 < |x - x_0| < \delta.$$

In particular, for each $n \in \mathbb{N}$, taking $\delta_n := 1/n$, one can choose some $x_n \in \mathbb{R}$ such that $|x_n - x_0| < 1/n$, while $|f(x_n) - l| \geq \epsilon$. Therefore $\{f(x_n)\}$ does not converge to l, while $x_n \longrightarrow x_0$, a contradiction to the hypothesis. □

Corollary 1.47 *Let* $f : \mathbb{R} \longrightarrow \mathbb{R}$ *and* $x_0 \in \mathbb{R}$. *Then* f *is continuous at* x_0 *if and only if* $f(x_n) \longrightarrow f(x_0)$, *whenever* $x_n \longrightarrow x_0$.

Definition 1.48 Let A be any set. The *characteristic function* of A or the *indicator function* of A, denoted by χ_A, is defined as follows:

$$\chi_A(x) := \begin{cases} 1 & \text{if } x \in A, \\ 0 & \text{if } x \in \mathbb{R} \setminus A. \end{cases}$$

Next we present two important functions, due to Gustav Lejeune Dirichlet (1829) and Carl Johannes Thomae (1875), respectively.

Example 1.49 The *Dirichlet function,* defined as $f_D := \chi_{\mathbb{Q}}$ is discontinuous everywhere.

Proof Let $x \in \mathbb{R}$. Consider a sequence $\{r_n\}$ of rationals and $\{i_n\}$ of irrationals such that $\{r_n\} \longrightarrow x$ and $\{i_n\} \longrightarrow x$. Note that $f_D(r_n) = 1$ and $f_D(i_n) = 0$ for all $n \in \mathbb{N}$. Hence $\{f_D(r_n)\} \longrightarrow 1$, while $\{f_D(i_n)\} \longrightarrow 0$. Applying Corollary 1.47, we conclude that f is discontinuous at x. □

Example 1.50 The *Thomae function* $f : \mathbb{R} \longrightarrow \{0\} \cup \{1/n : n \in \mathbb{N}\}$, defined as follows, is continuous on $\mathbb{R} \setminus \mathbb{Q}$ and discontinuous on $\mathbb{Q}$:

$$f(x) := \begin{cases} \frac{1}{n} & \text{if } x \in \mathbb{Q} \text{ and } |x| = \frac{m}{n} \text{ with } (m, n) = 1, \\ 0 & \text{otherwise.} \end{cases}$$

Proof Let $r \in \mathbb{Q}$. Consider a sequence $\{i_n\}$ of irrationals such that $\{i_n\} \longrightarrow r$. If f is continuous at r, we have $\{f(i_n)\} \longrightarrow f(r)$. That is $\{0\} \longrightarrow f(r)$. Hence $f(r) = 0$, which is impossible, as $r \in \mathbb{Q}$. Hence f is discontinuous at r.

Pick any $x \in \mathbb{R} \setminus \mathbb{Q}$. Then $f(x) = 0$. Let $\epsilon > 0$ be given. Pick any $n \in \mathbb{N}$ such that $1/n < \epsilon$. Note that there are only finitely many rationals in the interval $(x - 1, x + 1)$ with denominator less than n. Therefore there exists some $\delta > 0$ such that in $(x - \delta, x + \delta)$, every rational has denominator greater than or equal to n. Consequently, for every $y \in (x - \delta, x + \delta)$, we have $|f(y) - f(x)| = |f(y)| \le 1/n < \epsilon$. Hence f is continuous at x. □

The Thomae function has several other names such as the *popcorn function*, the *raindrop function*, the *countable cloud function*, the *ruler function,* and the *modified Dirichlet function.*

Remarks 1.51 In Corollary 8.39(b), we shall prove that there exists no $\mathbb{R} \longrightarrow \mathbb{R}$ function, which is continuous at every rational number and discontinuous at every irrational number.

Theorem 1.52 *(Algebra of Limits) Let $f, g : \mathbb{R} \longrightarrow \mathbb{R}$ and $c \in \mathbb{R}$ be such that* $\lim_{x\to c} f(x) = l$ *and* $\lim_{x\to c} g(x) = l'$, *for some $l, l' \in \mathbb{R}$. Then*

(a) $\lim_{x\to c} \alpha f(x) = \alpha l$ *for all* $\alpha \in \mathbb{R}$,
(b) $\lim_{x\to c}(f(x) + g(x)) = l + l'$,
(c) $\lim_{x\to c} f(x)g(x) = ll'$, *and*
(d) $\lim_{x\to c} \frac{1}{f(x)} = \frac{1}{l}$, *provided* $l \neq 0$.

Proof One can use Theorem 1.46 along with Theorem 1.16, to prove all these parts. Let $\{x_n\}$ be a sequence in $\mathbb{R} \setminus \{c\}$ such that $x_n \longrightarrow c$. By Theorem 1.46, we have $f(x_n) \longrightarrow l$ and $g(x_n) \longrightarrow l'$. Applying Theorem 1.16, we have $\alpha f(x_n) \longrightarrow \alpha l$, $(f+g)(x_n) \longrightarrow l + l'$ and $f(x_n)g(x_n) \longrightarrow ll'$.

Further if $l \neq 0$, by Theorem 1.16, there exists some $k \neq 0$ such that $f(x_n) \neq 0$ for all $n > k$ and $\{1/f(x_n)\}_{n>k} \longrightarrow 1/l$. Hence $\lim_{x\to c} \frac{1}{f(x)} = \frac{1}{l}$. □

Corollaries 1.53 *Let $c \in \mathbb{R}$ and $f, g : \mathbb{R} \longrightarrow \mathbb{R}$ be continuous at c. Then*

(a) αf is continuous at c for all $\alpha \in \mathbb{R}$,
(b) $f + g$ is continuous at c,
(c) fg is continuous at c, and
(d) $1/f$ is continuous at c, provided $f(c) \neq 0$.

Proposition 1.54 *If f is continuous at c and g is continuous at $f(c)$, then the composition function $g \circ f$ is continuous at c.*

Proof Let $\epsilon > 0$ be given. Since g is continuous at $f(c)$, there exists some $\rho > 0$ such that $|g(y) - g(f(c))| < \epsilon$ whenever $|y - f(c)| < \rho$. Since f is continuous at c, there exists some $\delta > 0$ such that $|f(x) - f(c)| < \rho$ whenever $|x - c| < \delta$.

Now if x satisfies $|x - c| < \delta$, then $|f(x) - f(c)| < \rho$ and hence $|g(f(x)) - g(f(c))| < \epsilon$. This proves that $g \circ f$ is continuous at c. □

History Notes 1.55 The term 'function' was first used by Leibniz in 1692. In [7], twenty-one historical definitions of the concept of function are presented, in their original languages. This includes the definitions by seventeen authors ranging from Joh. Bernoulli (1718) to N. Bourbaki (1939), without any additional comment. For a radical and historical approach toward real analysis, we recommend [8, 9], respectively.

Exercise 1.94 Prove that the limit of a function at a point, if it exists, is unique.

Exercise 1.95 Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be a function, $c \in \mathbb{R}$ and α be a fixed positive real. Prove that $\lim_{x\to c} f(x) = l$ if and only if for every $\epsilon > 0$, there exists some $\delta > 0$ such that $|f(x) - l| < \alpha\epsilon$ whenever $0 < |x - c| < \delta$.

Exercise 1.96 Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ and $c \in \mathbb{R}$ be such that $\lim_{x\to c} f(x) = l$. (A *deleted neighborhood* of c is defined as $U \setminus \{c\}$, where U is a neighborhood of c.)

(a) If $l > 0$, prove that $f(x) > 0$, in a deleted neighborhood of c,
(b) If $l > l'$, prove that $f(x) > l'$, in a deleted neighborhood of c.

Exercise 1.97 Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ and $c \in \mathbb{R}$. Prove that the following are equivalent:

(a) f is continuous at c.
(b) For every $\epsilon > 0$ there exists some $\delta > 0$ such that $f((c - \delta, c + \delta)) \subset (f(c) - \epsilon, f(c) + \epsilon)$.
(c) For every neighborhood J of $f(c)$, there exists some neighborhood I of c such that $f(I) \subset J$.

Exercise 1.98 Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ and $c \in \mathbb{R}$. Prove that the following are equivalent:

(a) f has a limit l at c.
(b) For every $\epsilon > 0$ there exists some $\delta > 0$ such that $f\big((c-\delta, c+\delta) \setminus \{c\}\big) \subset (l-\epsilon, l+\epsilon)$.
(c) For every neighborhood J of l, there exists some neighborhood I of c such that $f(I \setminus \{c\}) \subset J$.

Exercise 1.99 Prove the following function f is continuous only at 0. Also prove that it has no limits at any other point.

$$f(x) := \begin{cases} x \text{ if } x \in \mathbb{Q}, \\ 0 \text{ if } x \in \mathbb{R} \setminus \mathbb{Q}. \end{cases}$$

Exercise 1.100 Discuss the continuity of the real functions f and g defined as

$$f(x) := \begin{cases} \sin\frac{1}{x} & \text{if } x \neq 0, \\ 0 & \text{if } x = 0 \end{cases} \quad \text{and } g(x) := \begin{cases} x\sin\frac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0. \end{cases}$$

Exercise 1.101 Obtain the set of discontinuities of the function f defined as

$$f(x) := \begin{cases} x^4 & \text{if } x \in \mathbb{Q}, \\ 2x^2 - 1 & \text{if } x \in \mathbb{R} \setminus \mathbb{Q}. \end{cases}$$

Exercise 1.102 Show that the Thomae function is not differentiable anywhere and has a strict local maximum at every rational number.

Exercise 1.103 Prove that every polynomial, in one variable, with real coefficients is a continuous function on $\mathbb{R}$. What can you conclude about the continuity of rational functions?

Exercise 1.104 Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ and $c \in \mathbb{R}$ be such that $\lim_{x \to c} f(x)$ exists. Prove that f is bounded in a neighborhood of c.

Exercise 1.105 Write a proof of Proposition 1.54 using Corollary 1.47.

Exercise 1.106 Without using Theorem 1.46, prove Theorem 1.52.

Exercise 1.107 Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be a continuous function with $f(x+y) = f(x) + f(y)$ for all $x, y \in \mathbb{R}$. Prove that there exists some $r \in \mathbb{R}$ such that $f(x) = rx$, for each $x \in \mathbb{R}$.

Exercise 1.108 Let $f, g : \mathbb{R} \longrightarrow \mathbb{R}$ and D_f, D_g denote the set of discontinuities of f and g, respectively,. Prove that $D_{f+g} \subset D_f \cup D_g$. Can strict inclusion occur here?

Exercise 1.109 Let I be any nonempty interval. Prove that there exists a continuous surjective function $f : \mathbb{R} \longrightarrow I$. (Several other results of this form will be discussed in Exercises 5.35 and 7.14-7.16.)

Exercise 1.110 Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be continuous at 0. Prove that there exists $n \in \mathbb{N}$ such that the map $x \longmapsto [10(f(x) - [f(x)])]$ is constant on $[-n^{-1}, n^{-1}]$.

Exercise 1.111 Let $f(x) := x - \tan^{-1} x$ for all $x \in \mathbb{R}$. Let x_1 be a positive real and $x_{n+1} := f(x_n)$ for all $n \in \mathbb{N}$. Prove that $\lim_{n\to\infty} x_n = 0$.

Exercise 1.112 Let $f : [0, 1] \longrightarrow [0, \infty)$ be a monotonically increasing function such that $f(0) > 0$ and $f(x) \neq x$ for all $x \in [0, 1]$. Prove that $f(1) > 1$.

1.6 Uniform Convergence

Since sequences and series of functions often arise in science and engineering, it becomes important to ask the following question:

If each term in a convergent sequence (series) of functions satisfies a particular property, can we expect the same property from the limiting function?

That property can be continuity, existence of limits, differentiability or integrability, etc. There are two common notions of convergence of such sequences, defined as follows.

Definition 1.56 Let E be any set. A sequence $\{f_n\}$ of $E \longrightarrow \mathbb{R}$ functions is called

(a) *pointwise convergent* to a function f on E, if for every $x \in E$, the sequence of scalars $\{f_n(x)\}$ is convergent. In other words, if for every $x \in E$ and for every $\epsilon > 0$, there exists a positive integer $N(x)$ such that

$$|f_n(x) - f(x)| < \epsilon \text{ for all } n \geq N(x).$$

(b) *uniformly convergent* to a function f on E, if for every $\epsilon > 0$ there exists a positive integer N (independent of $x \in E$) such that

$$|f_n(x) - f(x)| < \epsilon \text{ for all } n \geq N \text{ and for all } x \in E.$$

In these cases, we write $f_n \longrightarrow f$ pointwise/uniformly on E, respectively.

It is immediate that the uniform convergence of a sequence implies its pointwise convergence. However, the converse is not true. Also the pointwise convergence does not preserve continuity (see also Theorems 3.31 and 3.32).

Examples 1.57 Let $\{f_n\}$ be a sequence of $[0, 1] \longrightarrow \mathbb{R}$ functions, defined as

$$f_n(x) := x^n \text{ for all } x \in [0, 1] \text{ and for all } n \in \mathbb{N}.$$

Then $\{f_n\}$ is pointwise convergent to $\chi_{\{1\}}$ on [0, 1], each f_n is continuous on [0, 1], while its pointwise limit $\chi_{\{1\}}$ is discontinuous at 1. Further, $\{f_n\}$ is not uniformly convergent on [0, 1].

Proof Let f denote the pointwise limit $\chi_{\{1\}}$ of $\{f_n\}$ on [0, 1]. We only prove the last assertion and leave the rest to the reader. Assume that the convergence is uniform. Then for $\epsilon = \frac{1}{3}$, there exists a positive integer N such that

$$\left|f_n(x) - f(x)\right| < \frac{1}{3} \text{ for all } n \geq N \text{ and for all } x \in [0, 1].$$

That ensures that $x^n \in (f(x) - \frac{1}{3}, f(x) + \frac{1}{3})$ for all $n \geq N$ and for all $x \in [0, 1]$, In particular, for $n = N$ we obtain

$$x^N \in \left(-\frac{1}{3}, \frac{1}{3}\right) \text{ for all } x \in [0, 1).$$

Since f_N is continuous at 1, there exists some $\delta > 0$ such that $x^N \in \left(\frac{2}{3}, \frac{4}{3}\right)$ for all $x \in (1 - \delta, 1]$. Therefore for all $x \in (1 - \delta, 1)$, we obtain $x^N \in \left(-\frac{1}{3}, \frac{1}{3}\right) \cap \left(\frac{2}{3}, \frac{4}{3}\right) = \emptyset$, a contradiction. □

Definition 1.58 A series $\sum_{n=1}^{\infty} f_n$ of real valued functions on a set E is said to be *pointwise (uniformly) convergent on* E, if the sequence of partial sums $\{\sum_{k=1}^{n} f_k(x)\}_n$ is convergent pointwise (uniformly) on E. In this case, we write $\sum_{n=1}^{\infty} f_n = f$ pointwise (uniformly) on E.

1.6.1 Necessary and Sufficient Conditions

Theorem 1.59 *Let $\{f_n\}$ be a sequence of real valued functions, pointwise convergent to f on a set E. Write*

$$M_n := \sup\left\{|f_n(x) - f(x)| : x \in E\right\} \textit{ for all } n \in \mathbb{N}.$$

Then $f_n \longrightarrow f$ uniformly on E if and only if $\lim_{n\to\infty} M_n = 0$.

Proof Let $\epsilon > 0$ be given. Suppose $f_n \longrightarrow f$ uniformly on E. Then there exists some $N_1 \in \mathbb{N}$ such that $|f_n(x) - f(x)| < \epsilon$ for all $n > N_1$ and for all $x \in E$. Consequently, we obtain $M_n \leq \epsilon$ for all $n > N_1$. Hence $M_n \longrightarrow 0$.

Conversely, if $M_n \longrightarrow 0$, then there exists some $N_2 \in \mathbb{N}$ such that $M_n \leq \epsilon$ for all $n > N_2$. Therefore for every $n > N_2$ and for all break $x \in E$, we have $|f_n(x) - f(x)| \leq M_n < \epsilon$. Consequently, $f_n \longrightarrow f$ uniformly on E. □

Analogous to the case of sequences of real numbers, we also have a Cauchy criterion for the uniform convergence of sequences of real valued functions.

Theorem 1.60 *(Cauchy criterion) A sequence $\{f_n\}$ of real valued functions is uniformly convergent on a set E if and only if for every $\epsilon > 0$ there exists some $N \in \mathbb{N}$ such that*

$$|f_{n_2}(x) - f_{n_1}(x)| < \epsilon \text{ for all } n_2 > n_1 \geq N \text{ and for all } x \in E.$$

Proof Assume that $\{f_n\}$ is uniformly convergent to f on E. Let $\epsilon > 0$ be given. Then there exists a positive integer N (independent of $x \in E$) such that

$$|f_n(x) - f(x)| < \frac{\epsilon}{2} \text{ for all } n \geq N \text{ and for all } x \in E.$$

Then for all $n_2 > n_1 \geq N$ and for all $x \in E$, we have

$$|f_{n_2}(x) - f_{n_1}(x)| \leq |f_{n_2}(x) - f(x)| + |f(x) - f_{n_1}(x)| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

Conversely, let $\epsilon > 0$ be given. Assume that there exists some $N \in \mathbb{N}$ such that

$$|f_{n_2}(x) - f_{n_1}(x)| < \epsilon \text{ for all } n_2 > n_1 \geq N \text{ and for all } x \in E. \tag{1.2}$$

Then for each $x \in E$, the sequence $\{f_n(x)\}$ is a Cauchy sequence of reals and hence convergent. Define $f(x) := \lim_{n\to\infty} f_n(x)$ for all $x \in E$. Passing limit $n_2 \longrightarrow \infty$ in (1.2) and replacing n_1 with n, we obtain

$$|f_n(x) - f(x)| \leq \epsilon \text{ for all } n \geq N \text{ and for all } x \in E.$$

Hence $\{f_n\}$ is uniformly convergent to f on E. □

Corollary 1.61 *A series $\sum_{n=1}^{\infty} f_n$ of real valued functions is uniformly convergent on a set E if and only if for every $\epsilon > 0$ there exists some $N \in \mathbb{N}$ such that*

$$\left| \sum_{n=n_1+1}^{n_2} f_n(x) \right| < \epsilon \text{ for all } n_2 > n_1 \geq N \text{ and for all } x \in E.$$

Proof Apply Theorem 1.60 on the sequence of partial sums of $\sum_{n=1}^{\infty} f_n$. □

We now present a sufficient condition for the uniform convergence of a series of functions, in fact for the uniform convergence of the corresponding series of absolute terms. Two other sufficient conditions for uniform convergence of sequences of functions will be provided in Theorem 5.15 and Exercise 5.19.

Theorem 1.62 *(Weierstrass M-test) Let $\sum_{n=1}^{\infty} f_n$ be a series of real valued functions on a set E. Suppose that there exists a sequence of positive reals $\{M_n\}$ with $\sum_{n=1}^{\infty} M_n < \infty$ such that*

$$|f_n(x)| \leq M_n \text{ for all } x \in E \text{ and for all } n \in \mathbb{N}.$$

Then the series $\sum_{n=1}^{\infty} f_n$ *is uniformly convergent on* E.

Proof Let $\epsilon > 0$ be given and $P_n(x) := \sum_{k=1}^{n} f_k(x)$ for all $x \in E$ and for all $n \in \mathbb{N}$. Since $\sum_{n=1}^{\infty} M_n < \infty$, there exists $N \in \mathbb{N}$ such that $\sum_{k=n_1+1}^{n_2} M_k < \epsilon$ for all $n_2 > n_1 > N$. for all $x \in E$ and for all $n_2 > n_1 > N$, we obtain

$$|P_{n_2}(x) - P_{n_1}(x)| = |\sum_{k=n_1+1}^{n_2} f_k(x)| \leq \sum_{k=n_1+1}^{n_2} M_k < \epsilon.$$

This concludes that the sequence $\{P_n\}$ is uniformly Cauchy on E. Applying Theorem 1.60, $\{P_n\}$ is uniformly convergent on E. Hence the result. □

1.6.2 Notes and Remarks

In Example 1.57, we have observed that the pointwise limit of a sequence of continuous functions may not be continuous. A natural question is that how much discontinuous it can be? In particular, does there exist a sequence $\{f_n\}$ of continuous functions on $[0, 1]$, pointwise convergent to a function f such that f is discontinuous at every point of $[0, 1]$? The answer is in the negative, as we have the following result from [10, Corollary 5.17, p. 78].

Theorem 1.63 *Let f be the pointwise limit of a sequence of real valued continuous functions on an interval* $[a, b]$. *Then the set of points of continuity of f is dense in* $[a, b]$, *that is, every nonempty open interval inside* $[a, b]$ *contains a point of continuity of* f.

A function $f : [a, b] \longrightarrow \mathbb{R}$ such that f is the pointwise limit of a sequence of continuous functions, is known as a *Baire class one function*. Therefore, every continuous function is a Baire class one function.

As a consequence of the above theorem, it follows that the derivative of an everywhere differentiable function is continuous on a dense subset of its domain and-courtesy Darboux-has the intermediate value property.

Further, if f is the pointwise limit of a sequence of Baire class one functions, it is known as a *Baire class two function.* Similarly, there are Baire class three functions and so on. A thorough discussion of Baire class one functions is available in [10, Chap. 5].

Exercise 1.113 Let $\{f_n\} \longrightarrow f$ and $\{g_n\} \longrightarrow g$ uniformly on a set E. Prove that $\{f_n + g_n\} \longrightarrow f + g$ uniformly on E.

Exercise 1.114 For each $n \in \mathbb{N}$, let $f_n : [0, 1] \longrightarrow \mathbb{R}$ be the piecewise linear function, whose graph is given by the segments joining $(0, 0)$, $(\frac{1}{2n}, 2n)$, $(\frac{1}{n}, 0)$ and $(1, 0)$ in the $\mathbb{R}^2$ plane. Prove that $\{f_n\}$ is pointwise convergent on $[0, 1]$, but not uniformly convergent on $[0, 1]$.

Exercise 1.115 Let E be a finite set and $\{f_n\}$ be a sequence of real valued functions on E. Prove that $\{f_n\}$ converges pointwise on E if and only if $\{f_n\}$ converges uniformly on E.

Exercise 1.116 If $\{a_{m,n} : m, n \in \mathbb{N}\}$ is a subset of real numbers, is the following true $\lim_{m\to\infty} \lim_{n\to\infty} a_{m,n} = \lim_{n\to\infty} \lim_{m\to\infty} a_{m,n}$?

Exercise 1.117 Prove that the following sequence of functions $\{f_n\}$ is convergent pointwise, but not uniformly convergent on (0, 1).

$$f_n(x) := \begin{cases} 1 \ ; x \in (0, 1/n), \\ 0 \ ; x \in [1/n, 1). \end{cases}$$

Exercise 1.118 If $f_n(x) := nx^n$ for all $x \in [0, 1)$ and for all $n \in \mathbb{N}$, prove that $\{f_n\}$ is convergent pointwise, but not uniformly on [0, 1).

Exercise 1.119 Prove that the sequence of functions defined by

$$f_n(x) := \frac{nx}{1 + n^2x^2} \text{ for all } x \in \mathbb{R} \text{ and for all } n \in \mathbb{N}$$

is uniformly convergent on $[a, \infty]$ for all $a > 0$, but not on $[0, \infty]$.

Exercise 1.120 Discuss the convergence of the series $\sum_{n=1}^{\infty} \frac{1}{n(1-x)^n}$.

Exercise 1.121 What is the domain of pointwise convergence of $\sum_{n=1}^{\infty} \frac{\cos(3^n x)}{2^n}$?

Exercise 1.122 Obtain the limit $\lim_{x\to 0} \sum_{n=1}^{\infty} \frac{\sin(nx)}{n^2}$. Justify your answer.

Exercise 1.123 Let $\{f_n\}$ and $\{g_n\}$ be sequences of bounded functions, uniformly convergent on E, to functions f and g, respectively. Prove that the sequence $\{f_n g_n\}$ is uniformly convergent to fg on E.

Exercise 1.124 If a sequence of real functions is a contraction of a convergent sequence of constants, prove that it is uniformly convergent. In other words, let $\{f_n\}$ be a sequence of real functions on E and $\{a_n\}$ be a convergent sequence of real numbers such that

$$|f_n(x) - f_m(x)| \leq |a_n - a_m|, \text{ for each } x \in E \text{ and for each } n, m \in \mathbb{N}.$$

Prove that the sequence $\{f_n\}$ is uniformly convergent on E.

Exercise 1.125 Let $\{f_n\}$ be as in Theorem 1.62. Prove that $\sum_{n=1}^{\infty} |f_n|$, is uniformly convergent on E, if there exists a sequence of positive reals $\{M_n\}$ with $\sum_{n=1}^{\infty} M_n < \infty$ such that

$$|f_n(x)| \leq M_n \text{ for all } x \in E \text{ and for all } n \in \mathbb{N}.$$

Exercise 1.126 Show that the Weierstrass M-test generalizes the comparison test.

Exercise 1.127 Generalize Exercises 1.80 and 1.81 for uniform convergence.

Exercise 1.128 Let $\sum_{n=1}^{\infty} a_n$ be an absolutely convergent series and $\{b_n\}$ be any sequence of reals. Prove that the series $\sum_{n=1}^{\infty} a_n(\sin b_n x + \cos b_n x)$ is uniformly convergent on $\mathbb{R}$.

Exercise 1.129 Let $\{f_n\}_{n=0}^{\infty}$ be a sequence of real valued functions on E with $f_0 \equiv 0$. Suppose there exists a sequence of positive reals $\{M_n\}$ with $\sum_{n=1}^{\infty} M_n < \infty$ such that

$$|f_n(x) - f_{n-1}(x)| \le M_n \text{ for all } x \in E \text{ and for all } n \in \mathbb{N}.$$

Prove that $\{f_n\}$ is uniformly convergent on E.

Exercise 1.130 Let $\{a_n\}$ be a sequence of reals and $\sum_{n=0}^{\infty} a_n x^n$ be convergent at some non-zero real $x = x_0$. Prove that $\sum_{n=0}^{\infty} a_n x^n$ converges uniformly on $[-r, r]$, for all $0 < r < |x_0|$.

1.7 Hints and Solutions to Selected Exercises

1.9 Let r be a rational number such that $x - \sqrt{2} < r < y - \sqrt{2}$. Then $s = r + \sqrt{2}$ is the required irrational.

1.12 Apply induction, along with Exercise 1.10.

1.14 Since $1^2 = 1 < 2$, we have $1 \in E$. Note that E is bounded above by 2. Otherwise there exists some $x \in E$ such that $x > 2$, then $x^2 > 4$, a contradiction. Therefore E is a nonempty and bounded above set. Applying least upper bound property, let $s := \sup E$. We claim that $s^2 = 2$. Define

$$r := s - \frac{s^2 - 2}{s + 2} = \frac{2s + 2}{s + 2}.$$

Then

$$r^2 - 2 = 2\frac{s^2 - 2}{(s + 2)^2}.$$

If $s^2 < 2$, then $r^2 < 2$ and $r > s$. Therefore $r \in E$ and $r > s$ which implies that $s \neq \sup E$, a contradiction.

If $s^2 > 2$, then $r^2 > 2$ and $r < s$. If $x \in E$, then $x < r$, because $x \ge r$ would imply $x^2 \ge r^2 > 2$ and that is impossible as $x \in E$. Hence r is an upper bound for E. This implies $r \ge s$, a contradiction. Hence $s^2 = 2$.

1.28 Apply the triangle inequality $\big||a_n| - |a|\big| \le |a_n - a|$. The converse is not true. For example, consider the sequence $a_n := (-1)^n$ for all $n \in \mathbb{N}$.

1.30 Use the inequality $|a_n^{\frac{1}{k}} - a^{\frac{1}{k}}| \le |a_n - a|^{\frac{1}{k}}$.

1.31 Let $a_n := n^{1/n} - 1$. Then $n = (1 + a_n)^n$. Applying binomial theorem, for all $n \geq 2$, we obtain

$$n = (1 + a_n)^n = 1 + na_n + \frac{n(n-1)}{2}a_n^2 + \cdots + a_n^n > \frac{n(n-1)}{2}a_n^2.$$

Hence $0 \leq a_n < \sqrt{\frac{2}{n-1}}$ for all $n \geq 2$. By Squeeze Rule (1.17), $a_n \longrightarrow 0$.

1.39 The first part is immediate from the definition of convergence. For the second part, let p_n denote the n^{th} partial sum of the series $\sum_{n=1}^{\infty} \frac{1}{n}$.

1.41 Use the fact that $\lim_{n\to\infty} x_n^a = a$ for all $a \in \mathbb{R}$.

1.43 Let $\alpha \in \mathbb{R}$. By Archimedean property, there exists some $N \in \mathbb{N}$ such that $|\alpha| < N$. Then for all $n > N$, we have

$$0 < \frac{|\alpha|^n}{n!} = \frac{|\alpha|^N}{N!}.\frac{|\alpha|}{N+1}.\frac{|\alpha|}{N+2}\cdots\frac{|\alpha|}{n} \leq \frac{|\alpha|^{N+1}}{N!}.\frac{1}{n}.$$

Applying Squeeze Rule (1.17), $\lim_{n\to\infty} \frac{|\alpha|^n}{n!} = 0$. Hence $\lim_{n\to\infty} \frac{\alpha^n}{n!} = 0$.

1.44 It is enough to show that the sequence is monotonically increasing and not bounded above. First assume that there exists some $\alpha > 0$ such that $\sqrt[n]{n!} < \alpha$ for all $n \in \mathbb{N}$. Then we have $n! < \alpha^n$ and thus $\alpha^n/n! > 1$ for all $n \in \mathbb{N}$, a contradiction to Exercise 1.43.

To prove that $\{\sqrt[n]{n!}\}$ is monotonically increasing, let $n \in \mathbb{N}$. Note that

$$\sqrt[n]{n!} < \sqrt[n+1]{(n+1)!} \iff (n!)^{n+1} < [(n+1)!]^n \iff n! < (n+1)^n,$$

which is always true, by the virtue of binomial expansions.

1.45 Let $\epsilon > 0$ be given. Pick any $N_1 \in \mathbb{N}$ such that $|a_n - l| < \frac{\epsilon}{2}$ for all $n \geq N_1$. Then there exists a natural number $N > N_1$ such that

$$\frac{1}{n}\sum_{i=1}^{N_1} |a_i - l| < \frac{\epsilon}{2} \text{ for all } n \geq N.$$

Therefore, for all $n > N$, we obtain

$$\begin{aligned} \left|\frac{a_1 + \cdots + a_n}{n} - l\right| &\leq \frac{1}{n}\sum_{i=1}^{N_1} |a_i - l| + \frac{1}{n}\sum_{i=N_1+1}^{n} |a_i - l| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} \times \frac{n - N_1}{n} < \epsilon. \end{aligned}$$

The converse is false. For example, take $a_n := (-1)^n$.

1.46 Apply Exercise 1.45.

1.47 Inductively apply the well-ordering principle of $\mathbb{N}$.

1.50 For each $i = 1, \ldots, k^2 + 1$, let a_i denote the largest number for which there are $n_1 < \cdots < n_{a_i} \le i$ such that $f(n_1) \le \cdots \le f(n_{a_i})$, and b_i denote the largest number for which there are $n_1 < \cdots < n_{b_i} \le i$ such that $f(n_1) \ge \cdots \ge f(n_{b_i})$.

Let $i < j$. If $f(i) \le f(j)$, then $a_i < a_j$; otherwise $b_i < b_j$. So $(a_i, b_i) \ne (a_j, b_j)$ for all $i \ne j$. In other words, the map $i \longmapsto (a_i, b_i)$ is injective. If $1 \le a_i, b_i \le k$ for all i, then the pairs (a_i, b_i) are at most k^2 in number, which is false. Hence either $a_i > k$ or $b_i > k$ for some i.

1.53 Note that for every $n \in \mathbb{N}$, we have

$$1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \cdots + \frac{1}{\sqrt{n}} \ge \frac{1}{\sqrt{n}} + \frac{1}{\sqrt{n}} + \frac{1}{\sqrt{n}} + \cdots + \frac{1}{\sqrt{n}} = \sqrt{n}.$$

1.54 Use the following observation

$$\begin{aligned} & 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \ldots \\ > & 1 + \frac{1}{2} + \left[\frac{1}{4} + \frac{1}{4}\right] + \left[\frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8}\right] + \ldots \\ = & 1 + \frac{1}{2} + \frac{1}{2} + \ldots . \end{aligned}$$

1.57 Write $S_n := \sum_{k=0}^{n} ar^k$ for all $n \in \mathbb{N}$. Then we have $rS_n = \sum_{k=0}^{n} ar^{k+1}$, which implies that $S_n(1-r) = a(1-r^{n+1})$. Therefore,

$$S_n = \frac{a(1-r^{n+1})}{1-r} \text{ for all } n \in \mathbb{N}.$$

Since $|r| < 1$, we obtain $\{r^{n+1}\} \longrightarrow 0$. So $S_n \longrightarrow \frac{a}{1-r}$. Hence $\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r}$.

1.59 By comparing the upper and the lower areas (upper and lower Riemann sums) with the integral of f, in the interval $[1, n]$, we obtain

$$a_2 + a_3 + \cdots + a_{n+1} \le \int_1^{n+1} f(t)dt \le a_1 + a_2 + \cdots + a_n.$$

By hypothesis, the partial sums as well as the sequence $\left\{\int_1^n f(t)dt\right\}$ is a monotonically increasing of real numbers.

1.62 No. Take $a_n := 1/n^2$ and $b_n := 1$.

1.66 Show that both of the following series are divergent:

$$\begin{aligned} & 1 + \frac{1}{2} - \frac{1}{3} + \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} + \frac{1}{8} - \frac{1}{9} + \ldots \\ \text{and } & 1 - \frac{1}{2^2} + \frac{1}{3} - \frac{1}{2^3} + \frac{1}{5} - \frac{1}{2^4} + \frac{1}{7} - \frac{1}{2^5} + \ldots . \end{aligned}$$

1.67 Apply Leibniz test. Also note that $\sum_{n=0}^{\infty}(-1)^{n+1}/n = \log 2$.
1.68 Since the geometric mean of two reals is less than or equal to their arithmetic mean, we conclude that

$$|a_n b_n| = \sqrt{a_n^2 b_n^2} \leq \frac{a_n^2 + b_n^2}{2} \text{ for all } n \in \mathbb{N}.$$

Applying hypothesis and the comparison test, $\sum_{n=1}^{\infty} a_n b_n$ converges absolutely.
1.72 Yes, both. Rationalize suitably.
1.76 Apply limit test, on the corresponding series of absolute terms with $\sum_{n=1}^{\infty} 1/n^2$, $\sum_{n=1}^{\infty} 1/n$ and $\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$, respectively.
1.79 (a) Since the series $\sum_{n=1}^{\infty} b_n$ is convergent, the sequence $\{b_n\}$ is bounded. Therefore, there exists some $M > 0$ such that $b_n < M$ for all n. Hence

$$\sum_{n=1}^{\infty} a_n b_n \leq M \sum a_n < \infty.$$

(b) For a counter example, take $a_n = b_n = (-1)^{n+1}/\sqrt{n}$ for all $n \in \mathbb{N}$.
1.80 Write $B_n := \sum_{k=1}^{n} a_k$ for all $n \in \mathbb{N}$. Then

$$\sum_{k=1}^{n} a_k b_k = a_n B_n + \sum_{k=1}^{n-1} B_k(a_k - a_{k+1}) \text{ for all } n > 1.$$

Since $a_n \longrightarrow 0$ and $\{B_n\}$ is bounded, we obtain $a_n B_n \longrightarrow 0$. The result follows, because $\sum_{k=1}^{n-1} B_k(a_k - a_{k+1})$ converges absolutely, as

$$\sum_{k=1}^{n-1} |B_k(a_k - a_{k+1})| \leq M \Big| \sum_{k=1}^{n-1} (a_k - a_{k+1}) \Big| = M|a_1 - a_n| \longrightarrow M|a_1|, \text{ as } n \longrightarrow \infty.$$

To deduce Leibniz's test, use $b_n := (-1)^n$ for all $n \in \mathbb{N}$.
1.81 Analogous to Exercise 1.80.
1.82 Let $m \in \mathbb{N}$ and observe the following inequalities

$$\sum_{k=1}^{2^m - 1} a_k = (a_1) + (a_2 + a_3) + (a_4 + a_5 + a_6 + a_7) \ldots$$
$$\leq a_1 + 2a_2 + 4a_4 \ldots \leq \sum_{k=0}^{m-1} 2^k a_{2^k}$$

$$\text{and } \sum_{k=0}^{m-1} 2^k a_{2^k} = (a_1 + a_2) + (a_2 + a_4 + a_4 + a_4) + a_4 \ldots$$

$$\leq (a_1 + a_1) + (a_2 + a_2 + a_3 + a_3) + 2a_4 \ldots \leq 2 \sum_{k=1}^{2^{m-1}} a_k.$$

The result follows by the virtue of the comparison test.

1.83 No, as the Cauchy-Schwarz inequality (2.2) ensures that

$$\sum_{n=1}^{N} a_n^2 \leq \Big(\sum_{n=1}^{N} a_n\Big)^2 \leq \Big(\sum_{n=1}^{\infty} a_n\Big)^2 \text{ for all } N \in \mathbb{N}.$$

1.84 (a) By Cauchy condensation test (1.82), $\sum_{k=0}^{\infty} 2^k a_{2^k}$ converges. For $2^k < n < 2^{k+1}$, we obtain $2^k a_{2^{k+1}} \leq n a_n \leq 2^{k+1} a_{2^k}$. Now apply Squeeze Rule (1.17). For an alternative proof, let $T_n := \sum_{m>n} a_n$. Then $T_n \longrightarrow 0$ and for all $n \in \mathbb{N}$, we have

$$0 \leq n a_{2n} \leq a_{n+1} + \cdots + a_{2n} \leq T_n$$
$$\text{and } 0 \leq (2n+1) a_{2n+1} = 2n a_{2n+1} + a_{2n+1} \leq 2n a_{2n} + a_{2n+1}.$$

Applying Squeeze Rule (1.17), these inequalities ensure that $n a_{2n} \longrightarrow 0$ and $(2n+1) a_{2n+1} \longrightarrow 0$, respectively. Hence $n a_n \longrightarrow 0$.

(b) No. For example, let $\sum_{n=1}^{\infty} a_n$ be the rearrangement of $\sum_{n=1}^{\infty} 2^{-n}$, obtained by interchanging the $(2^k)^{th}$ and k^{th} terms, for all positive odd integers k.

1.85 The series (a) and (b) are convergent, while (c) is not, in general.

(a) Applying Cauchy-Schwarz inequality (2.2), for every $N \in \mathbb{N}$, we obtain

$$\sum_{n=1}^{N} \frac{a_n}{n} \leq \left(\sum_{n=1}^{N} a_n^2\right)\left(\sum_{n=1}^{N} \frac{1}{n^2}\right).$$

(b) Since $\sum_{n=1}^{\infty} a_n^2$ is convergent, there exists $m \in \mathbb{N}$ such that $a_n < 1$ for all $n > m$. If $p > 2$ and $n > m$, then $a_n^p = a_n^2 a_n^{p-2} < a_n^2$. Applying Comparison test (1.37), $\sum_{n=1}^{\infty} a_n^p$ is convergent.

(c) If $p_n = \frac{3}{2}$ and $a_n := n^{-\frac{5}{8}}$, then

$$\sum_{n=1}^{\infty} a_n^2 = \sum_{n=1}^{\infty} n^{-\frac{5}{4}} < \infty \text{ while } \sum_{n=1}^{\infty} a_n^p = \sum_{n=1}^{\infty} n^{-\frac{15}{16}} = \infty.$$

1.86 The series $\sum_{n=1}^{\infty} \frac{\sqrt{a_n}}{n}$ converges by Exercise 1.85(a). However, $\sum_{n=1}^{\infty} \sqrt{\frac{a_n}{n}}$ may not converge. For example, take $a_n := \frac{1}{n(\log n)^2}$.

1.90 Yes. Observing the values of $[x]$ and $[2x]$ for intervals of half length, one can show that for every $x \in \mathbb{R}$, the integer $[2x]$ is either $2[x]$ or $2[x] + 1$. Hence $P(x, y) := (y - 2x)(y - 2x - 1)$ satisfies our requirement.

1.91 Let us first find the number of elements of S. Note that there are 10^8 ways of choosing a block of length 8. In this we also count numbers with repeating blocks of length 2 and 4. Since every number x with a repeating block of length 2 also has a repeating block of length 4, we obtain $|S| := 10^8 - 10^4 = 99990000$.

Now for every $0.\overline{abcgefgh} \in S$, there exists a unique $0.\overline{a'b'c'g'e'f'g'h'} \in S$ such that $x' = 9 - x$. Also $0.\overline{abcgefgh} + 0.\overline{a'b'c'g'e'f'g'h'} = .\overline{9} = 1$. Hence $\sum_{x \in S} x = |S|/2 = 49995000$.

1.93 Yes. The proof is by grouping the sum by number of digits in the denominator. If $a = \{1, \ldots, 9\}$, the number of k-digit positive integers that have no digit equal to a is $8 \times 9^{k-1}$. Moreover each of these numbers is strictly more than 10^{k-1}. Hence the series sum in this case would be $\leq 8 \sum_{k=1}^{\infty} (\frac{9}{10})^{k-1} = 80$. If $a = 0$, the same is at most $9 \sum_{k=1}^{\infty} (\frac{9}{10})^{k-1} = 90$. Ditto for any other digit.

1.106 Apply the direct definition and write proofs analogous to Theorem 1.16. For the third part, note that if f has a limit at c, then f is bounded in a deleted neighborhood of c. The following relation will be useful

$$\begin{aligned} |f(x)g(x) - ll'| &\leq |f(x)g(x) - f(x)l'| + |f(x)l' - ll'| \\ &= |f(x)||g(x) - l'| + |f(x) - l||l'|. \end{aligned}$$

For the last part, note that $l \neq 0$, implies $|f(x)| > |l|/2$, in a deleted neighborhood of c. Now in a deleted neighborhood of c, we have

$$\left| \frac{1}{f(x)} - \frac{1}{l} \right| \leq \frac{|f(x) - l|}{|lf(x)|} < \frac{2}{l^2} |f(x) - l|.$$

1.108 A strict inclusion is possible, e.g. take $f := \chi_{\mathbb{Q} \cap [0,1]}$ and $g := -f$.

1.109 We discuss some particular cases. Other intervals of the same form can be handled by taking compositions with suitable linear maps.

If $I = \mathbb{R}$, then one can take f to be identity map. If $I = [0, +\infty)$ and $(0, +\infty)$, the maps $x \longmapsto x^2$ and $x \longmapsto e^x$, respectively, serve our purpose. For $I = (-\infty, 0]$ and $(-\infty, 0)$, the negative of these maps, that is, $x \longmapsto -x^2$ and $x \longmapsto -e^x$ would work.

If $I = [-1, 1]$, then $x \longmapsto \sin x$ works. If $I = (-1, 1)$, we take $f_1(x) := \frac{2}{\pi} \tan^{-1} x$.

Now consider the case when $I = (0, 1]$. Let f_2 be the linear bijection from $(-1, 1)$ onto $(0, 1)$. Consider the continuous surjection $f_3 : (0, 1) \longrightarrow (0, 1]$ given by $f_3(x) := \sin \pi x$ for all $x \in (0, 1)$. Then $f := f_3 \circ f_2 \circ f_1$ is a continuous surjection from $\mathbb{R}$ onto $(0, 1]$. Taking negative of this function, we obtain a continuous surjection from $\mathbb{R}$ onto $[-1, 0)$.

1.111 Note that $f'(x) = \frac{x^2}{1+x^2}$ for all x. Since $f(0) = 0$ and $f'(x) > 0$ for all $x > 0$ we obtain $f(x) > 0$ for all $x > 0$. Therefore, $x_n > 0$ for all n, and thence $x_{n+1} - x_n = -\tan^{-1} x_n < 0$.

Thus $\{x_n\}$ is a monotonically decreasing sequence of positive reals and hence convergent. Let $l := \lim_{n\to\infty} x_n$. Then $x_{n+1} - f(x_n) = 0$ for all $n \in \mathbb{N}$. Since f is continuous,

$$0 = \lim_{n\to\infty}(x_{n+1} - f(x_n)) = l - f(l) = \tan^{-1} l \text{ which implies } l = 0.$$

1.112 Let $E := \{x \in [0, 1] : f(x) > x\}$. Since $0 \in E \subset [0, 1]$, E is a nonempty subset of $\mathbb{R}$ bounded above by 1. Let $s := \sup E$. We claim that $s \in \sup E$ and $s = 1$.

If $s \notin E$, then $f(s) \leq s$ and hence $f(s) < s$, as $f(x) \neq x$ for all x. So $f(s)$ is not an upper bound of E. So there exists some $x \in E$ such that $f(s) < x$. Since $s = \sup E$, we have $x \leq s$ which implies $f(x) \leq f(s) < x$. Hence $x \notin E$, a contradiction. Thus $s \in \sup E$.

Now assume that $s < 1$. Let $s_n := s + \frac{1}{n}$, for each $n \in \mathbb{N}$. Then for sufficiently large n, the sequence $\{s_n\}$ belongs to $(s, 1]$. So for all sufficiently large n, $f(s_n) < s_n$ and thence

$$s < f(s) \leq f(s_n) < s_n.$$

Since $s_n \longrightarrow s$, by Squeeze Rule (1.17), $\{f(s_n)\} \longrightarrow f(s)$ as well as $\{f(s_n)\} \longrightarrow s$. Hence $f(s) = s$, a contradiction.

1.116 No. For example, take $a_{m,n} := \frac{m}{m+n}$ for all $m, n \in \mathbb{N}$.

1.119 Note that $\lim_{n\to\infty} f_n(x) = 0$ for all $x \in \mathbb{R}$. Also for all $n \in \mathbb{N}$, the function f_n is monotonically increasing in $[0, 1/n]$ and monotonically decreasing in $[1/n, +\infty)$ with $f_n(0) = 0$ and $f_n(1/n) = 1/2$ for all $n \in \mathbb{N}$.

1.120 Applying ratio test on the corresponding absolute series, we obtain absolute convergence in $(-\infty, 0) \cup (2, \infty)$, absolute divergence in $[0, 2] \setminus \{1\}$, conditional convergence at 2 and divergent in $(0, 1) \cup (1, 2)$.

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Chapter 2
Metric Spaces

Exploring the properties of real functions or sequences is just the beginning. A few answers lead to several questions. Can we extend our results from $\mathbb{R}$ to more general spaces, such as the plane $\mathbb{R}^2$ or the three-dimensional space $\mathbb{R}^3$ or to $\mathbb{R}^n$? Sometimes the proofs depend only upon a few properties of the underlying space. The ones which depend only upon the distance function can be extended to metric spaces.

A metric space is defined to be a nonempty set along with a distance function having some particular properties. This chapter presents a vast collection of metric spaces, including the particular cases of normed spaces and sequence spaces. To provide a glimpse into generalizations from reals, we have included a section on convergence of sequences in metric spaces which also contains the case of finite-dimensional Euclidean spaces.

2.1 Introduction

To delve into the concept of 'distance' in general spaces, it is necessary to first define the notion of a 'space.'

Definition 2.1 A *space* is defined to be any nonempty set.

The definition of a space is often avoided in most of the textbooks. Some texts define 'space' as a nonempty set with some additional structure on it, such as a metric space, linear space, normed space. The term 'additional structure' is a bit vague, especially for those who do not know any kind of such 'space'. In that sense, the above definition appears more appropriate.

Definition 2.2 Let X be a nonempty set. A function $d : X \times X \longrightarrow \mathbb{R}$ is said to be a *metric* on X if for every $x, y, z \in X$, we have

(a) $d(x, y) \geq 0$, $d(x, y) = 0$ if and only if $x = y$, (positive definiteness)
(b) $d(x, y) = d(y, x)$ and (symmetry)
(c) $d(x, y) \leq d(x, z) + d(z, y)$. (triangle inequality)

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In this case, (X, d) is called a *metric space* or that X is a metric space with metric d. If there is no ambiguity about the metric, we simply say that X is a metric space.

Examples 2.3 (a) Define $d(x, y) := |x - y|$, for all $x, y \in \mathbb{R}$. Then d is a metric on $\mathbb{R}$, known as the *usual metric.*

(b) Let X be any nonempty set (it could even be the set of English Alphabets) and $d_c : X \times X :\longrightarrow \mathbb{R}$ be defined as follows:

$$d_c(x, y) := \begin{cases} 1 \,; x \neq y, \\ 0 \,; x = y. \end{cases}$$

It can be shown that d_c is a metric on X. In this case, d_c is said to be the *discrete metric* on X and (X, d_c) is called the *discrete metric space.*

Example 2.4 Let (X, d) be any metric space and Y be a nonempty subset of X. Then d is also a metric on Y, known as the *induced metric.* In this case, the metric space Y is called a *subspace* of X. For example, $\mathbb{Q}$ is a subspace $\mathbb{R}$.

In the sequel, if X is a nonempty subset of $\mathbb{R}$, the space X will refer to the metric space X equipped with the usual metric.

Example 2.5 Let $r \in (0, 1]$ and X be the collection of sequences with terms 0 or 1. For any sequences $x = \{x_n\}, y = \{y_n\} \in X$ such that $x \neq y$, define $n(x, y) := \min\{k : x_k \neq y_k\}$ and

$$\rho_0(x, y) := \begin{cases} 0 & ; x = y, \\ \frac{1}{n(x,y)} & ; \text{otherwise} \end{cases} \quad \text{and } \rho_r(x, y) := \begin{cases} 0 & ; x = y, \\ r^{n(x,y)} & ; \text{otherwise.} \end{cases}$$

Then for every $r \in [0, 1]$, the function ρ_r is a metric on X with

$$\rho_r(x, y) \leq \max\{\rho_r(x, z), \rho_r(y, z)\} \text{ for all } x, y, z \in X. \tag{2.1}$$

This above inequality is known as the *strong triangle inequality,* and a metric that satisfies it is referred to as an *ultrametric.*

Proof Note that for $r = 1$, ρ_1 is the discrete metric on X, which clearly satisfies the inequality (2.1). The symmetry and positive definiteness of each ρ_r is trivial. The triangle inequality follows from (2.1), which is immediate if any two of x, y, z are equal.

Assume that $x = \{x_n\}$, $y = \{y_n\}$ and $z = \{z_n\}$ are all different. Then $x_i = z_i$ for all $i < n(x, z)$ and $z_i = y_i$ for all $i < n(z, y)$. Therefore $x_i = y_i$ for all $i < \min\{n(x, z), n(z, y)\}$, and consequently, $n(x, y) \geq \min\{n(x, z), n(z, y)\}$. Hence ρ_r satisfies (2.1), for all $r \in [0, 1)$. □

Definition 2.6 Let X be a nonempty set. A function $d : X \times X \longrightarrow \mathbb{R}$ is said to be a *pseudo-metric* on X if for every $x, y, z \in X$ we have

(a) $d(x, y) \geq 0$ and $d(x, x) = 0$, (positive semi-definiteness)
(b) $d(x, y) = d(y, x)$ and (symmetry)
(c) $d(x, y) \leq d(x, z) + d(z, y)$. (triangle inequality)

Clearly, every metric is a pseudo-metric, while the converse is not true.

Example 2.7 Let $d(x, y) := |x^2 - y^2|$ for all $x, y \in \mathbb{R}$. Then d is a pseudo-metric on $\mathbb{R}$, but not a metric on $\mathbb{R}$.

Remarks 2.8 Some of the requirements in Definition 2.2 are redundant (see Exercise 2.7). An important example of a metric, the *Hausdorff metric* will be provided in Exercise 3.67.

2.1.1 The Euclidean Spaces

Note that the standard Euclidean distance in a plane satisfies all the requirements of a metric, making that plane a metric space. The positive definiteness and the symmetry are obvious. We shall provide a proof for the triangle inequality.

Let $n \in \mathbb{N}$. The n-dimensional real *Euclidean space* $\mathbb{R}^n$ is defined as

$$\mathbb{R}^n := \{(x_1, \ldots, x_n) : x_i \in \mathbb{R}, 1 \leq i \leq n\}.$$

For $x = (x_1, \ldots, x_n)$, $y = (y_1, \ldots, y_n) \in \mathbb{R}^n$ and $r \in \mathbb{R}$, the *sum* $x + y$, *scalar multiplication* rx, *modulus* $|x|$ and the *dot product* $x \cdot y$ are defined as follows:

$$\begin{aligned} x + y &:= (x_1 + y_1, \ldots, x_n + y_n), \\ rx &:= (rx_1, \ldots, rx_n), \\ |x| &:= \sqrt{x_1^2 + \cdots + x_n^2} \\ \text{and } x \cdot y &:= x_1 y_1 + \cdots + x_n y_n. \end{aligned}$$

First we present the Cauchy-Schwarz inequality. This is one of the most fundamental inequality in analysis and has several conceptually different proofs. Here we present the popular one, which can be extended to even more general spaces, namely the inner product spaces (see Theorem 2.33). An alternative proof will be provided in Exercise 2.19.

Theorem 2.9 *(Cauchy-Schwarz inequality) For every $x, y \in \mathbb{R}^n$, we have*

$$|x \cdot y| \leq |x||y|. \tag{2.2}$$

In other words, if $x = (x_1, \ldots, x_n)$, $y = (y_1, \ldots, y_n) \in \mathbb{R}^n$, then

$$|x_1 y_1 + \cdots + x_n y_n| \leq \sqrt{x_1^2 + \cdots + x_n^2}\sqrt{y_1^2 + \cdots + y_n^2}.$$

Further, the equality holds if and only if x and y are linearly dependent, that is, there exist real numbers a and b not both zero such that $ax + by = 0$.

Proof Consider $z := |y|^2 x - (x \cdot y)y$ and observe that

$$0 \leq |z|^2 = z \cdot z = \big(|y|^2 x - (x \cdot y)y\big) \cdot \big(|y|^2 x - (x \cdot y)y\big) = |y|^2\big(|x|^2|y|^2 - |x \cdot y|^2\big). \tag{2.3}$$

If $y = 0$, then (2.2) holds trivially. If $y \neq 0$, then $|y|^2 = y \cdot y > 0$ and therefore by cancelling the positive scalar $|y|^2$ from (2.3), we obtain (2.2).

Suppose there exist real numbers a and b not both zero such that $ax + by = 0$. Without loss of generality, we assume that $a \neq 0$. Then with $x = -by/a$, the equality in (2.2) holds true.

Conversely, assume that the equality holds in (2.2). Using that in (2.3), we obtain $z \cdot z = 0$, which implies $z = 0$. Hence $(y \cdot y)x = (x \cdot y)y$. If $y \neq 0$, then $y \cdot y \neq 0$. Otherwise $0.x + 1.y = 0$. Hence x and y are linearly dependent. □

Corollary 2.10 *(Minkowski's inequality) For every $(x_1, \ldots, x_n), (y_1, \ldots, y_n) \in \mathbb{R}^n$, we have*

$$\sqrt{(x_1 + y_1)^2 + \cdots + (x_n + y_n)^2} \leq \sqrt{x_1^2 + \cdots + x_n^2} + \sqrt{y_1^2 + \cdots + y_n^2}. \tag{2.4}$$

Proof By squaring and canceling, we observe that (2.4) holds if and only if (2.2) is satisfied, which is already true. Hence the result. □

Corollary 2.11 *(Euclidean metric) For every $x = (x_1, \ldots, x_n)$, $y = (y_1, \ldots, y_n) \in \mathbb{R}^n$, define*

$$d_2\big(x, y\big) := \sqrt{|x_1 - y_1|^2 + \cdots + |x_n - y_n|^2}.$$

Then $(\mathbb{R}^n, d_2)$ is a metric space.

Proof Applying Corollary 2.10, d_2 satisfies the triangle inequality. The positive definiteness and symmetry of d_2 are obvious from its definition. □

The above d_2 is known as the *usual metric* or the *Euclidean metric* on $\mathbb{R}^n$. For convenience, we write *metric space* $\mathbb{R}^n$ for the *metric space* $(\mathbb{R}^n, d_2)$. We also write $|x - y|$ for $d_2(x, y)$.

We wind up this section with the space of complex numbers. Various other examples of metric spaces will be discussed in the exercises.

Definition 2.12 The set of *complex numbers* $\mathbb{C}$ is defined to be the two-dimensional Euclidean space $\mathbb{R}^2$, along with an additional multiplication operation given by

$$(x_1, x_2) \times (y_1, y_2) := (x_1 y_1 - x_2 y_2, x_1 y_2 + x_2 y_1) \text{ for all } (x_1, x_2), (y_1, y_2) \in \mathbb{C}.$$

It is conventional to denote $(0, 1)$ by i and (a, b) by $a + ib$.

Remark 2.13 Note that the usual metric on $\mathbb{R}^2$ provides a metric on $\mathbb{C}$, also known as the usual metric on $\mathbb{C}$. Therefore, topologically $\mathbb{C}$ and $\mathbb{R}^2$ are same. The product in $\mathbb{C}$ makes functions on $\mathbb{C}$ quite different from those on $\mathbb{R}^2$, which leads to the Cauchy theory of complex analysis. However, that is not the concern of this textbook. We limit our discussion to the basic algebraic and topological properties of $\mathbb{C}$.

History Notes 2.14 The concept of metric spaces was introduced by Frećhet, under the name 'classes (E)', in his 1906 Ph.D. dissertation. Later Hausdorff coined the term metric space in 1914 and laid the foundations of topology (see [1, p. 253]).

2.1.2 *Balls and Bounded Sets*

Definition 2.15 Let (X, d) be a metric space, $x \in X$ and $r > 0$. The *ball* of radius r centered at x is defined as

$$B_d(x; r) := \{y \in X : d(y, x) < r\}.$$

These balls are also called *open balls.* If there is no ambiguity about the metric, we simply write $B(x; r)$, instead of $B_d(x; r)$. Note that we did not allow balls with radius zero.

Examples 2.16 (a) Under the usual metric on reals, the open balls are open intervals. In particular, for all $x \in \mathbb{R}$ and $r > 0$, we have $B(x; r) = (x - r, x + r)$.

(b) Let (X, d) be a discrete metric space, $x \in X$ and $r > 0$. Then

$$B(x; r) := \begin{cases} X & ; r > 1, \\ \{x\} & ; 0 < r \leq 1. \end{cases}$$

Example 2.17 Let (X, d) be a metric space such that d is an *ultrametric* on X, i.e.

$$d(x, y) \leq \max\{d(x, z), d(y, z)\} \text{ for all } x, y, z \in X.$$

If $x, y, z \in X$ and $r, s > 0$ are arbitrary, then X satisfies the following properties:

(a) Every triangle in X is isosceles, i.e. if $d(x, y) \neq d(y, z)$, then $d(z, x)$ is equal to either $d(x, y)$ or $d(y, z)$.
(b) Every point inside a ball is its center, i.e. $B(x; r) = B(y; r)$ for all $y \in B(x; r)$.
(c) If two balls meet, then one is contained in the other; i.e. if $B(x; r) \cap B(y; s) \neq \emptyset$, then either $B(x; r) \subset B(y; s)$ or $B(y; s) \subset B(x; r)$.

Proof (a) Without loss of generality, we assume that $d(x, y) < d(y, z)$. Then $d(y, z) = d(z, x)$, as

$$\begin{aligned} d(z, x) &\leq \max\{d(z, y), d(y, x)\} = d(y, z) \\ \text{and } d(y, z) &\leq \max\{d(y, x), d(x, z)\} = d(z, x). \end{aligned}$$

(b) Suppose $d(y, x) < r$. If $z \in B(x; r)$, then $d(y, z) \leq \max\{d(y, x), d(x, z)\} < r$ and hence $z \in B(y; r)$. So $B(x; r) \subset B(y; r)$. Interchanging y and x, we obtain $B(x; r) = B(y; r)$.
(c) Without loss of generality, suppose $r \leq s$ and let $z \in B(x; r) \cap B(y; s)$. By (b), we conclude that $B(x; r) = B(z; r) \subset B(z; s) = B(y; s)$. □

Definition 2.18 A subset E of a metric space X is called *bounded* if it is contained in some ball. That is, $E \subset B(x; r)$ for some $x \in X$ and $r > 0$.

Therefore, E is bounded if and only if the set of distance between points of E is bounded above. Analogous to open balls, the closed balls are defined as follows:

Definition 2.19 Let (X, d) be a metric space, $x \in X$ and $r > 0$. The *closed ball* of radius r centered at x is defined as $B[x; r] := \{y \in X : d(y, x) \leq r\}$.

Exercise 2.1 For a metric space (X, d), prove that the following are equivalent:

(a) d is a constant,
(b) X is a singleton set and
(c) $d(x, y) \geq d(x, z) + d(z, y)$ for all $x, y, z \in X$.

Exercise 2.2 If d is a metric on a space X, prove that so is $\sqrt{d}$.

Exercise 2.3 If d is a metric on a space X and $x, y, z \in X$. prove the inequality $|d(x, y) - d(y, z)| \leq d(x, z)$.

Exercise 2.4 Does $(x, y) \longmapsto \left|\frac{1}{x} - \frac{1}{y}\right|$ define a metric on $\mathbb{R} \setminus \{0\}$?

Exercise 2.5 Does any of the following expressions define a metric on $\mathbb{R}$:

$$|x^2 - y^2|, |x - y| + 1 \text{ or } \frac{1}{|x - y| + 1}?$$

Exercise 2.6 Prove that $(x, y) \longmapsto |x - y| + |x^2 - y^2|$ defines a metric on $\mathbb{R}$.

Exercise 2.7 If X is nonempty and $d : X \times X \longrightarrow \mathbb{R}$ such that for all $x, y \in X$,

$$d(x, y) \leq d(x, z) + d(y, z)$$
$$\text{and } d(x, y) = 0 \text{ if and only if } x = y.$$

Prove that $d(x, y) \geq 0$ and $d(x, y) = d(y, x)$ for all $x, y \in X$.

Exercise 2.8 Deduce the triangle inequality in $(\mathbb{R}^n, d_2)$ from Corollary 2.10.

Exercise 2.9 For any $(x_1, x_2), (y_1, y_2) \in \mathbb{R}^2$, define

(a) $d_1\big((x_1, x_2), (y_1, y_2)\big) := |x_1 - y_1| + |x_2 - y_2|$ (the *taxi cab metric*).
(b) $d_\infty\big((x_1, x_2), (y_1, y_2)\big) := \max\{|x_1 - y_1|, |x_2 - y_2|\}$. (the *sup metric*).

Prove that d_1 and d_∞ are metrics on $\mathbb{R}^2$.

Exercise 2.10 Generalize the metrics d_1 and d_∞ of Exercise 2.9 to $\mathbb{R}^n$, and characterize the collection of balls in $\mathbb{R}^n$ with respect to these metrics.

Exercise 2.11 Let (X, d) be a metric space and $E \subset X$. Prove that the following are equivalent:

(a) E is bounded,
(b) there exists some $M > 0$ such that $d(x, y) < M$, for every $x, y \in E$,
(c) for any $x \in X$, there exists $M_x > 0$ such that $d(y, x) < M_x$ for all $y \in E$.

Exercise 2.12 Characterize bounded subsets of discrete metric spaces.

Exercise 2.13 If A and B are bounded subsets of a metric space X, prove that so is $A \cup B$.

Exercise 2.14 Let X be a metric space and $A \subset X$. Prove that A is bounded if and only if the diameter of A is finite, i.e. $\sup\{d(x, y) : x, y \in A\} < \infty$.

Exercise 2.15 Let (X, d) be a metric space and ρ be a pseudo-metric on X. Prove that $d + \rho$ is a metric on X.

Exercise 2.16 Let X be a nonempty set and $\rho_1, \ldots, \rho_n$ be (pseudo-)metrics on X. Prove that $\rho_1 + \cdots + \rho_n$ is also a (pseudo-)metric on X.

Exercise 2.17 Let d be a pseudo-metric on a space X. Define a relation $\sim$ on X as

$$x \sim y \text{ if and only if } d(x, y) = 0.$$

Prove that $\sim$ is an equivalence relation on X. For each $x \in X$, let $[x]$ denote the equivalence class of x under this relation and $X^* := \{[x] : x \in X\}$. Prove that d^* is a metric on X^*, where $d^*([x], [y]) := d(x, y)$ for all $[x], [y] \in X^*$.

Exercise 2.18 Let (X, d) be a metric space. For every $x, y \in X$, define

$$\rho_1(x, y) := \min\{1, d(x, y)\} \text{ and } \rho_2(x, y) := \frac{d(x, y)}{1 + d(x, y)}.$$

Prove that both ρ_1 and ρ_2 are metrics on X. Further show that every subset of X is bounded in (X, ρ_1) as well as in (X, ρ_2).

Exercise 2.19 Prove the Cauchy-Schwarz inequality in $\mathbb{R}^2$, as follows:

(a) Let $a \geq 0$ and $p(t) := at^2 + bt + c$. If $p(t) \geq 0$ for all $t \in \mathbb{R}$, prove that $b^2 \leq 4ac$.
(b) Let $(x_1, x_2), (y_1, y_2) \in \mathbb{R}^2$. Applying (a) with $p(t) := (tx_1 + y_1)^2 + (tx_2 + y_2)^2$, prove that

$$|x_1 y_1 + x_2 y_2| \leq \sqrt{x_1^2 + x_2^2}\sqrt{y_1^2 + y_2^2}.$$

Exercise 2.20 Let X denote the family of real valued functions on the interval [0, 1] and $d(f, g) := \sup\{|f(x) - g(x)| : x \in [0, 1]\}$ for all $f, g \in X$. Prove that d is a metric on X.

Exercise 2.21 Let (X, d) be as in Exercise 2.20. If $f \in X$ and $r > 0$, prove that $B(f; r)$ is the family of all those functions in X whose graphs lie in a band of width r about the graph of f.

Exercise 2.22 In $(\mathbb{R}^n, d_\infty)$, prove that the open balls look like hypercubes. In other words, $B(x; r) = (x_1 - r, x_1 + r) \times \cdots \times (x_n - r, x_n + r)$ for all $x := (x_1, \ldots, x_n) \in \mathbb{R}^n$ and $r \geq 0$.

Exercise 2.23 (Post office metric) Let $p \in \mathbb{R}^2$ be a fixed point and d_2 be the Euclidean metric on $\mathbb{R}^2$. Prove that d defines a metric on $\mathbb{R}^2$, where

$$d(a, b) := d_2(a, p) + d_2(p, b) \text{ for all } a, b \in \mathbb{R}^2.$$

Exercise 2.24 Let $(X_1, \rho_1), \ldots, (X_n, \rho_n)$ denote a finite family of metric spaces. For every $x = (x_1, \ldots, x_n), y = (y_1, \ldots, y_n) \in \prod_{i=1}^n X_i$, define

$$\rho(x, y) := \sqrt{\rho_1^2(x_1, y_1) + \cdots + \rho_n^2(x_n, y_n)}.$$

Prove that ρ is a metric on the Cartesian product $\prod_{i=1}^n X_i$.

Exercise 2.25 Let d be a metric on $\mathbb{R}^n$ and $(X_1, \rho_1), \ldots, (X_n, \rho_n)$ be any finitely many metric spaces. For any $x = (x_1, \ldots, x_n), y = (y_1, \ldots, y_n) \in \prod_{i=1}^n X_i$, define

$$\rho_d(x, y) := d\big(\rho_1(x_1, y_1), \ldots, \rho_n(x_n, y_n), (0, \ldots, 0)\big).$$

Prove that ρ_d is a metric on the Cartesian product $\prod_{i=1}^n X_i$.

Exercise 2.26 Let $\{(X_n, d_n) : n \in \mathbb{N}\}$ be a collection of metric spaces such that $d_n \leq 1$ for all $n \in \mathbb{N}$. Let X denote the Cartesian product $\prod_{n=1}^\infty X_n$, that is, the family of sequences $\{x_n\}$ such that $x_n \in X_n$ for all $n \in \mathbb{N}$. For every $x = \{x_n\}, y = \{y_n\} \in X$, define

$$\rho(x, y) := \sup\left\{\frac{d_n(x_n, y_n)}{n} : n \in \mathbb{N}\right\} \text{ and } \eta(x, y) := \sum_{n=1}^\infty \frac{d_n(x_n, y_n)}{2^n}.$$

Prove that both ρ and η are metrics on X.

Exercise 2.27 Let $\{(X_n, d_n) : n \in \mathbb{N}\}$ be a collection of metric spaces and $X := \prod_{n=1}^\infty X_n$. For any $x = \{x_n\}, y = \{y_n\} \in X$, define

$$d(x, y) := \sum_{n=1}^\infty \frac{1}{2^n} \cdot \frac{d_n(x_n, y_n)}{1 + d_n(x_n, y_n)}.$$

Prove that d is a metric on X. Also provide three other metrics on X.

Exercise 2.28 Let $n \in \mathbb{N} \cup \{0\}$, X be the set of polynomials with degree less than or equal to n and $p^{(i)}$ be the i^{th} derivative of p for every $p \in X$. For each $k \in \mathbb{N}$, define

$$d_k(p, q) := \max\{|p^{(i)}(0) - q^{(i)}(0)| : 1 \leq i < k\} \text{ for all } p, q \in X.$$

Obtain a necessary and sufficient condition in terms of k and n such that d_k is a metric on X.

Exercise 2.29 Let (X, d) be a metric space. For every $x \in X$, define a map $\delta_x : X \longrightarrow \mathbb{R}$ as $\delta_x(y) := d(x, y)$ for all $y \in X$. Let $\delta(X) := \{\delta_x : x \in X\}$. Prove that the map $x \longrightarrow \delta_x$ is a bijection between X and $\delta(X)$.

Exercise 2.30 (p-adic metric) Fix a prime number p. Let $x, y \in \mathbb{Q}$ be arbitrary. If $x = y$, define $d(x, y) := 0$. Otherwise, write $x - y = p^k a/b$, where $a, k \in \mathbb{Z}$ and $b \in \mathbb{N}$ such that p does not divide ab; and define $d(x, y) := p^{-k}$. Prove that d is an ultrametric on $\mathbb{Q}$.

Exercise 2.31 Let $\mathcal{I}$ denote the collection of closed bounded intervals. Define

$$d\big([a, b], [c, d]\big) := \max\big\{|a - c|, |b - d|\big\} \text{ for all } [a, b], [c, d] \in \mathcal{I}.$$

Prove that d is a metric on $\mathcal{I}$.

Exercise 2.32 Does there exist a metric on the space of extended reals $\mathbb{R} \cup \{-\infty, +\infty\}$, which extends the usual metric on $\mathbb{R}$?

Exercise 2.33 Let ∞ denote the (unique) infinity for the set of complex numbers and $\mathbb{C}_\infty := \mathbb{C} \cup \{\infty\}$. Is there a metric on $\mathbb{C}_\infty$, that extends the usual metric on $\mathbb{C}$?

Exercise 2.34 Let (X, d) be a metric space and $y \notin X$. Does there always exist a metric on $X \cup \{y\}$, which extends the metric d?

Exercise 2.35 Does there exist a metric space with two closed balls B_1 and B_2 of radii r_1 and r_2, respectively, such that $B_1 \subset B_2$ and $r_1 > r_2$?

2.2 Convergence in Metric Spaces

Analogous to the case of $\mathbb{R}$, the notions of convergent sequences and Cauchy sequences, in general metric spaces, are defined as follows:

Definition 2.20 A sequence $\{x_n\}$ in a metric space (X, d) is said to be *convergent* in X if there exists some $x_0 \in X$ satisfying the following condition:

for every $\epsilon > 0$, *there exists some* $N \in \mathbb{N}$ *such that* $d(x_n, x_0) < \epsilon$ for all $n \geq N$.

In this case, we say that $\{x_n\}$ *converges to* x_0 and write $x_n \longrightarrow x_0$. We also call x_0 as the *limit* of $\{x_n\}$ and write $x_0 = \lim_{n\to\infty} x_n$.

Definition 2.21 If $x \in X$, a subset U of X is said to be a *neighborhood* of x if

$$U \supset B(x; \delta) \text{ for some } \delta > 0.$$

It is immediate that $x_n \longrightarrow x$ if and only if every neighborhood of x contains all but finitely many terms of $\{x_n\}$.

Definition 2.22 A sequence $\{x_n\}$ in a metric space (X, d) is said to be *Cauchy* if for every $\epsilon > 0$, there exists some $N \in \mathbb{N}$ such that

$$d(x_n, x_m) < \epsilon \text{ for all } n, m \geq N.$$

Subsequences of a sequence in any space are defined naturally, as in Definition 1.12. Various results on metric spaces can be proven analogously to the case of $\mathbb{R}$. Here we present some sample cases. Several other analogous results will be provided in Exercise 2.36.

Theorem 2.23 *In metric spaces, convergent sequences have unique limits.*

Proof If possible, let $\{x_n\}$ be a convergent sequence in a metric space (X, d) with limits x' and x'' such that $x' \neq x''$. Let $\epsilon = d(x', x'')/2$. Then $\epsilon > 0$, as $x' \neq x''$. Since $\{x_n\}$ converges to x' and x'', there are positive integers N' and N'' such that

$$d(x_n, x') < \frac{\epsilon}{2} \text{ for all } n \geq N'$$
$$\text{and } d(x_n, x'') < \frac{\epsilon}{2} \text{ for all } n \geq N''.$$

Let $N := \max\{N', N''\}$. Then for all $n \geq N$, we obtain

$$d(x', x'') \leq d(x', x_n) + d(x_n, x'') < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon = d(x', x''),$$

which is absurd. This completes the proof. □

Analogous to the case of reals, in any metric space, a Cauchy sequence is convergent if it has a convergent subsequence.

Theorem 2.24 *Let* $\{x_n\}$ *be a Cauchy sequence in a metric space* (X, d), $x \in X$ *and* $\{x_{n_k}\}$ *be a subsequence of* $\{x_n\}$ *such that* $\lim_{k\to\infty} x_{n_k} = x$. *Then* $x_n \longrightarrow x$.

Proof Imitating the proof of Proposition 1.27, for every $\epsilon > 0$, there exist some $N, K \in \mathbb{N}$ such that

$$d(x_n, x_m) < \frac{\epsilon}{2} \text{ for all } n, m \geq N$$
$$\text{and } d(x_{n_k}, x) < \frac{\epsilon}{2} \text{ for all } k \geq K.$$

Let $p \in \mathbb{N}$ such that $n_p \geq \max\{N, n_K\}$. Then for all $n \geq n_p$, we have

$$d(x_n, x) \leq d(x_n, x_{n_p}) + d(x_{n_p}, x) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

Hence $\{x_n\}$ converges to x. □

Now we discuss convergence in Euclidean spaces. Note that the m-dimensional Euclidean space $\mathbb{R}^m$ has a natural bijection with the collection of functions from $\{1, \dots, m\}$ into $\mathbb{R}$. Motivated by this and for the sake of convenience, we write $x := \big(x(1), \dots, x(m)\big)$ for every $x \in \mathbb{R}^m$.

Theorem 2.25 *Let $\{x_n\}$ be a sequence in $\mathbb{R}^m$ and $x_0 \in \mathbb{R}^m$ such that*

$$x_n := \big(x_n(1), \dots, x_n(m)\big) \textit{ for all } n \in \mathbb{N} \cup \{0\}.$$

Then

(a) $x_n \longrightarrow x_0$ *if and only if* $x_n(j) \longrightarrow x_0(j)$ *for every* $j = 1, \dots, m$.
(b) $\{x_n\}$ *is Cauchy if and only if* $\{x_n(j)\}$ *is Cauchy, for every* $j = 1, \dots, m$.

Proof We shall prove the first part. The second one is similar. Note that for every $(a(1), \dots, a(m)) \in \mathbb{R}^m$ and $j = 1, \dots, m$, we have

$$|a(j)| \leq \sqrt{\sum_{k=1}^{m} |a(k)|^2} \leq \sum_{k=1}^{m} |a(k)|.$$

Hence for every $j = 1, \dots, m$ and for every $n \in \mathbb{N}$, we have

$$\big|x_n(j) - x_0(j)\big| \leq d_2(x_n, x_0) \leq \sum_{k=1}^{m} \big|x_n(k) - x_0(k)\big|.$$

Let $\epsilon > 0$ be given. If $x_n \longrightarrow x_0$, there exists some $N \in \mathbb{N}$ such that $d_2(x_n, x_0) < \epsilon$ for all $n \geq N$. Hence for every $j = 1, \dots, m$ and for every $n \geq N$, we obtain

$$\big|x_n(j) - x_0(j)\big| \leq d_2(x_n, x_0) < \epsilon.$$

This proves that every $x_n(j) \longrightarrow x_0(j)$ for every $j = 1, \dots, m$.

Conversely, if $x_n(j) \longrightarrow x_0(j)$, for all $j = 1, \dots, m$, there exist $N_j \in \mathbb{N}$ such that

$$|x_n(j) - x_0(j)| < \frac{\epsilon}{m} \text{ for all } n \geq N_j.$$

Let $N_0 := \max\{N_1, \ldots, N_m\}$. Then for every $n \geq N_0$, we obtain

$$d_2(x_n, x_0) \leq \sum_{j=1}^{m} \left|x_n(j) - x_0(j)\right| < \sum_{j=1}^{m} \frac{\epsilon}{m} = \epsilon.$$

Hence $\{x_n\}$ is convergent to x_0. □

Theorem 2.26 *Every Cauchy sequence in $\mathbb{R}^m$ is convergent in $\mathbb{R}^m$.*

Proof Let $\{x_n\}$ be a Cauchy sequence in $\mathbb{R}^m$. Applying Theorem 2.25, $\{x_n(j)\}$ is also Cauchy, for every $j = 1, \ldots, m$. Now Theorems 1.28 and 2.25 ensure that $\{x_n\}$ is convergent in $\mathbb{R}^m$. □

Now we generalize the Bolzano-Weierstrass property, already proved for sequences of real numbers in Theorem 1.22.

Theorem 2.27 *(Bolzano-Weierstrass) Every bounded sequence in $\mathbb{R}^m$ contains a subsequence that converges in $\mathbb{R}^m$.*

Proof Let $\{x_n\}$ be a bounded sequence in $\mathbb{R}^m$. Write $x_n := \big(x_n(1), \ldots, x_n(m)\big)$ for all $n \in \mathbb{N}$. As $\{x_n\}$ is bounded and $|x_n(j)| \leq d_2(x_n, 0)$, for all j, the sequence $\{x_n(j)\}$ is bounded.

Since $\{x_n(1)\}$ is a bounded sequence of reals, by Theorem 1.22, it has a convergent subsequence. Let $\{x_{n_{k_1}}(1)\}$ be that subsequence and $x(1)$ be its limit. This gives us a subsequence $\{x_{n_{k_1}}\}$, of the original sequence. As earlier obtain a subsequence $\{x_{n_{k_2}}(2)\}$ of $\{x_{n_{k_1}}(2)\}$, which is convergent to some real $x(2)$. Continuing like this m-times, we obtain a subsequence $\{x_{n_{k_m}}\}$ of $\{x_n\}$ such that $x_{n_{k_m}}(j) \longrightarrow x(j)$ for every $j = 1, \ldots, m$.

Let $x := \big(x(1), \ldots, x(m)\big)$. Then $x \in \mathbb{R}^m$ and by Theorem 2.25, we conclude that $x_{n_{k_m}} \longrightarrow x$. Hence the result. □

Exercise 2.36 In any metric space, prove that the following assertions hold:

(a) Every convergent sequence is bounded.
(b) Every convergent sequence is Cauchy.
(c) Every Cauchy sequence is bounded.
(d) Every subsequence of a Cauchy sequence is also a Cauchy sequence.
(e) All subsequences of a convergent sequence are convergent to the same limit.
(f) Removing (inserting) any finite number of terms anywhere from (in) a sequence does not affect its convergence.

Exercise 2.37 In Exercise 2.36, show that the converse statements of (a), (b), and (c) are not true, in general.

Exercise 2.38 Let $\{x_n\}$ be a sequence in a metric space (X, d) and $x \in X$. Prove that the following are equivalent:

(a) $x_n \longrightarrow x$,

(b) $d(x_n, x) \longrightarrow 0$, and
(c) For every neighborhood U of x, there exists some a positive integer N_U such that $x_n \in U$ for all $n > N_U$.

Exercise 2.39 Characterize convergent sequences in discrete metric spaces.

Exercise 2.40 Let $a_n \longrightarrow a$ and $b_n \longrightarrow b$, in a Euclidean space $\mathbb{R}^m$. Prove that

(a) $\{ka_n\} \longrightarrow a$, for all scalars $k \in \mathbb{R}$.
(b) $\{a_n + b_n\} \longrightarrow a + b$, and
(c) $\{a_n \cdot b_n\} \longrightarrow a \cdot b$, here $x \cdot y$ represents the dot product of $x, y \in \mathbb{R}^m$.

Exercise 2.41 Let $a_n \longrightarrow a$ in $\mathbb{R}^m$ and $b_n \longrightarrow b$ in $\mathbb{R}$. Prove that $a_n b_n \longrightarrow ab$.

Exercise 2.42 Let $a_n \longrightarrow 0$ in $\mathbb{R}^m$ and $\{b_n\}$ be a bounded sequence of real numbers. Prove that $a_n b_n \longrightarrow 0$.

Exercise 2.43 Let $a_n \longrightarrow a$ in $\mathbb{R}^m$. Prove that $|a_n| \longrightarrow |a|$. Is the converse true?

Exercise 2.44 Let $\{a_n\}$ and $\{b_n\}$ be two Cauchy sequences in $\mathbb{R}^m$. Prove that

(a) $\{ka_n\}$ is a Cauchy sequence, for all scalars $k \in \mathbb{R}$.
(b) $\{a_n + b_n\}$ is a Cauchy sequence.

Exercise 2.45 In discrete metric spaces, prove that

(a) convergent sequences are eventually constant,
(b) Cauchy sequences are eventually constant, and
(c) Cauchy sequences are convergent.

Exercise 2.46 Write a proof for the second part of Theorem 2.25.

Exercise 2.47 Write an alternate proof of Theorem 2.26, using Exercise 2.36(c), Theorems 2.27 and 2.24.

Exercise 2.48 Let X be a metric space, $x \in X$ and $\{x_n\}$ be a sequence in X. If every subsequence of $\{x_n\}$ has a subsequence convergent to x, prove that $x_n \longrightarrow x$.

Exercise 2.49 Let X be a metric space containing two points x and y. If $x_n \longrightarrow x$ and $y_n \longrightarrow y$ in X, then prove that the set $\{x_n : n \in \mathbb{N}\} \cap \{y_n : n \in \mathbb{N}\}$ is finite.

2.3 Normed Linear Spaces

The notion of metric spaces generalizes the space of real numbers, by extending the distance function. Now we discuss normed linear spaces, which also extend the addition and scalar multiplication operations from finite-dimensional Euclidean spaces, along with the distance.

We assume that the reader is familiar with the notion of vector spaces. A few subsequent results will require the notions of algebraic basis and subspace of a vector space. All vector spaces in this book will be considered over the scalar fields $\mathbb{R}$ or $\mathbb{C}$.

Let X be a linear (vector) space over a field $\mathbb{R}$ or $\mathbb{C}$. A function $\|.\| : X \longrightarrow [0, \infty)$ is said to be a *norm* on X if for every $x, y \in X$ and for every scalar k, it satisfies the following conditions:

(a) $\|x\| \geq 0$ ($\|.\|$ is positive)
(b) $\|x\| = 0$ if and only if $x = 0$ ($\|.\|$ is definite)
(c) $\|kx\| = |k|\|x\|$ ($\|.\|$ is homogeneous)
(d) $\|x + y\| \leq \|x\| + \|y\|$. ($\|.\|$ satisfies the triangle inequality)

In this case, we say that $(X, \|.\|)$ is a *normed linear space* or simply a *normed space.* If there is no ambiguity on the norm, we simply write X for $(X, \|.\|)$.

Note that every norm $\|.\|$ on a linear space X induces a metric given by

$$d(x, y) := \|x - y\| \text{ for all } x, y \in X.$$

Therefore every normed linear space is a metric space.

Examples 2.28 (a) If $X = \mathbb{R}$, then $x \longmapsto |x|$ defines a norm on X.
(b) Let $n \in \mathbb{N}$ and $X = \mathbb{R}^n$. For each $x = (x_1, \ldots, x_n) \in X$, define

$$\|x\|_2 := \sqrt{x_1^2 + \cdots + x_n^2}.$$

By Corollary 2.10, one can conclude that $(X, \|.\|_2)$ is a normed linear space.
(c) Let $C[a, b]$ denote the space of continuous real valued functions on a closed bounded interval $[a, b]$. Then

$$\|f\| := \sup \left\{|f(x)| : x \in X\right\} \text{ for all } f \in C[a, b].$$

defines a norm on $C[a, b]$, known as the *uniform norm* or the *supremum norm.*
(d) If Y is a linear subspace of a normed linear space $(X, \|.\|)$, then $(Y, \|.\|)$ is also a normed linear space.

Remark 2.29 In general, a subspace of a metric space (X, d) is a nonempty subset Y of X, equipped with the same metric d. However, in case of normed linear spaces X, the term subspace is reserved only for linear subspaces of X.

Proposition 2.30 *Let ℓ^2 denote the collection of sequences $\{x_n\}$ of real numbers such that $\sum_{n=1}^{\infty} |x_n|^2 < \infty$. Define*

$$\|x\|_2 := \sqrt{\sum_{n=1}^{\infty} |x_n|^2} \textit{ for all } x = \{x_n\} \in \ell^2.$$

Then $(\ell^2, \|.\|_2)$ is a normed linear space.

Proof It is easy to see that the function $\|.\|_2$ satisfies the first two requirements of a norm. To prove the triangle inequality, let $x := \{x_n\}$ and $y := \{y_n\}$ be any two elements of ℓ^2. Applying Corollary 2.10, for every $n \in \mathbb{N}$, we obtain

$$\begin{aligned}\Big(\sum_{k=1}^{n} |x_k + y_k|^2\Big)^{\frac{1}{2}} &\leq \Big(\sum_{k=1}^{n} |x_k|^2\Big)^{\frac{1}{2}} + \Big(\sum_{k=1}^{n} |y_k|^2\Big)^{\frac{1}{2}} \\ &\leq \Big(\sum_{k=1}^{\infty} |x_k|^2\Big)^{\frac{1}{2}} + \Big(\sum_{k=1}^{\infty} |y_k|^2\Big)^{\frac{1}{2}} = \|x\|_2 + \|y\|_2\end{aligned}$$

Passing limit $n \longrightarrow \infty$, we obtain $\|x + y\|_2 \leq \|x\|_2 + \|y\|_2$. Hence the result. □

Above we have generalized Minkowski's inequality, given by Corollary 2.10, to the space ℓ^2. Similarly, one can generalize the Cauchy-Schwarz inequality (2.9). Next we discuss a particular class of normed spaces, known as the inner product spaces.

Definition 2.31 Let X be a linear space over a field $\mathbb{K}$ (either $\mathbb{R}$ or $\mathbb{C}$). An *inner product* on X is a mapping $\langle ., . \rangle : X \times X \longrightarrow \mathbb{K}$ such that for all $x, y, z \in X$ and $\alpha, \beta \in \mathbb{K}$ we have

(a) $\langle x, x \rangle \geq 0$, and $\langle x, x \rangle = 0$ if and only if $x = 0$. (positive definiteness)
(b) $\langle \alpha x + \beta y, z \rangle = \alpha \langle x, z \rangle + \beta \langle y, z \rangle$ (linearity in the first variable)
(c) $\langle x, y \rangle = \overline{\langle y, x \rangle}$ (conjugate linearity in the second variable)

In this case, $(X, \langle . \rangle)$ is known as an *inner product space.*

Examples 2.32 (a) The standard dot product on $\mathbb{R}^n$ is an inner product.
(b) If $X := c_{00}$, then $\big\langle \{x_n\}, \{y_n\} \big\rangle := \sum_{n=1}^{\infty} x_n \overline{y_n}$ defines an inner product on X.

Theorem 2.33 *Let $(X, \langle , \rangle)$ be an inner product space over $\mathbb{K}$, and $x, y \in X$. Then the following hold:*

(a) Cauchy-Schwarz inequality: $|\langle x, y \rangle| \leq \langle x, x \rangle \langle y, y \rangle$, and the equality holds here if and only if x and y are linearly dependent.
(b) $\|x\| := \sqrt{\langle x, x \rangle}$ defines a norm on X.
(c) Parallelogram law: $\|x + y\|^2 + \|x - y\|^2 = 2(\|x\|^2 + \|y\|^2)$.

(d) Polarization identity:

$$\langle x, y\rangle = \begin{cases} \frac{1}{4}\left(\|x+y\|^2 - \|x-y\|^2\right) & \text{if } \mathbb{K}=\mathbb{R} \\ \frac{1}{4}\left(\|x+y\|^2 - \|x-y\|^2 + i\|x+iy\|^2 - i\|x-iy\|^2\right) & \text{if } \mathbb{K}=\mathbb{C}. \end{cases} \tag{2.5}$$

Proof With $z := \langle y, y\rangle x - \langle x, y\rangle y$, (a) can be established analogous to Theorem 2.9. Further, (c) and (d) are routine manipulations. Here we we prove (b) only.

The positive definiteness and homogeneity are immediate. For the triangle inequality, note that the inequality in (a) translates to $|\langle x, y\rangle| \leq \|x\|\|y\|$. Hence

$$\begin{aligned}\|x+y\|^2 = \langle x+y, x+y\rangle = \|x\|^2 + 2Re(\langle x, y\rangle) + \|y\|^2 &\leq \|x\|^2 + 2|\langle x, y\rangle| + \|y\|^2 \\ &\leq \|x\|^2 + 2\|x\|\|y\| + \|y\|^2 = (\|x\| + \|y\|)^2.\end{aligned}$$

This proves that $\|x+y\| \leq \|x\| + \|y\|$. □

Remarks 2.34 In 1935, Jordan-Von Neumann established that if a normed space satisfies the parallelogram law, then its norm is induced by an inner product. In that case, the inner product is given by the polarization identity (2.5). There are 350 characterizations of inner product spaces in the book of Dan Amir, see [2]. For more on inner product spaces, the reader is referred to [3, Chap. VI].

Exercise 2.50 Let X be a normed space, $x, y \in X$, and α be a scalar. Prove that

$$\big|\|x\| - \|y\|\big| \leq \|x-y\| \text{ and } \|\alpha x - \alpha y\| = |\alpha|\|x-y\|.$$

Exercise 2.51 Which vector subspaces of a normed space are bounded subsets?

Exercise 2.52 Let c_{00} be the set of sequences of reals which are eventually zero, that is, real sequences $\{x_n\}$ such that $x_n = 0$ for all sufficiently large n. Define

$$\|\{x_n\}\|_2 := \sqrt{\sum_{n=1}^{\infty} |x_n|^2} \text{ for all } \{x_n\} \in c_{00}.$$

Prove that $(c_{00}, \|.\|_2)$ is a normed linear space.

Exercise 2.53 Let $n \in \mathbb{N}$ and $p \in [1, \infty]$. For every $x = (x_1, \dots, x_n) \in \mathbb{R}^n$, define

$$\|x\|_p := \begin{cases} \left(\sum_{i=1}^{n} |x_i|^p\right)^{\frac{1}{p}} & ; 1 \leq p < \infty, \\ \sup\{|x_1|, \dots, |x_n|\} & ; p = \infty. \end{cases}$$

Prove that $\|.\|_p$ defines a norm on the linear space $\mathbb{R}^n$ over $\mathbb{R}$.

Exercise 2.54 If $x = \{x_k\}, y = \{y_k\} \in \ell^2$, prove that $\sum_{k=1}^{\infty} |x_k y_k| \leq \|x\|_2\|y\|_2$.

Exercise 2.55 Write a proof for the parallelogram law and the polarization identity as given in Theorem 2.33.

Exercise 2.56 Let X be a normed space, $y \in Y \subset X, x \in X$ and α be a scalar. If $dist(x; Y) := \inf\{d(x, y) : y \in Y\}$, prove that $\|kx + y\| \geq |\alpha| \times dist(x; Y)$.

Exercise 2.57 Is there any linear space on which the discrete metric can be induced by a norm?

Exercise 2.58 Show that the metric induced by any norm, on a linear space, is translation invariant.

Exercise 2.59 Is it possible to assign a norm to every linear space over $\mathbb{C}$?

Exercise 2.60 Let d be a translation invariant and homogeneous metric on a vector space X, and $\|x\| := d(x, 0)$ for all $x \in X$. Prove that $(X, \|.\|)$ is a normed space and induces metric d.

Exercise 2.61 Let X be a linear space as well as a metric space. Under what conditions it becomes a normed linear space having topology same as the one given by the metric?

2.4 Sequence Spaces

Let $\mathbb{K}$ be any of $\mathbb{R}$ or $\mathbb{C}$. We start with the following vector spaces over $\mathbb{K}$.

$$\begin{aligned}
c_{00} &:= \text{the space of all sequences over } \mathbb{K} \text{ with only finitely many non-zero terms.}\\
c_0 &:= \text{the space of all sequences over } \mathbb{K}, \text{ convergent to } 0.\\
c &:= \text{the space of all convergent sequences over } \mathbb{K}.
\end{aligned}$$

Let $1 \leq p \leq \infty$. For a sequence $x = \{x_j\}$ over $\mathbb{K}$, define extended real numbers $\|x\|_p$ as follows:

$$\|x\|_p := \begin{cases} \left(\sum_{j=1}^{\infty} |x_j|^p\right)^{\frac{1}{p}} & ; 1 \leq p < \infty, \\ \sup\{|x_j| : j \in \mathbb{N}\} & ; p = \infty. \end{cases}$$

For every $1 \leq p \leq \infty$, let ℓ^p denote the collection of all sequences x over $\mathbb{K}$ with $\|x\|_p < \infty$. It is easy to see that c_{00}, c_0 and c are vector spaces over $\mathbb{K}$. The same is true for $\ell^p (1 \leq p \leq \infty)$.

Theorem 2.35 *ℓ^p is a linear space, for all* $1 \leq p \leq \infty$.

Proof It is evident that each ℓ^p is closed under scalar multiplication. Let $p \in [1, +\infty]$ and $x, y \in \ell^p$. We shall now establish that $x + y \in \ell^p$. Write $x = \{x_n\}$ and $y = \{y_n\}$.

First consider the case when $p = \infty$. By triangle inequality $|x_n + y_n| \le |x_n| + |y_n| \le \|x\|_\infty + \|y\|_\infty$ for all $n \in \mathbb{N}$. Therefore, $\|x + y\|_\infty \le \|x\|_\infty + \|y\|_\infty < \infty$ and hence $x + y \in \ell^\infty$.

Now suppose that $1 \le p < \infty$. Let $z_n := \max\{|x_n|, |y_n|\}$ for all $n \in \mathbb{N}$. Note that $|z_n|^p \le |x_n|^p + |y_n|^p$ which implies $\|z\|_p^p \le \|x\|_p^p + \|y\|_p^p$. Hence $z = \{z_n\} \in \ell^p$. Further note that

$$|x_n + y_n|^p \le \big||x_n| + |y_n|\big|^p \le (2|z_n|)^p = 2^p |z_n|^p.$$

Summing $\sum_{n=1}^\infty$, we obtain $\|x + y\|_p^p \le 2^p \|z\|_p^p < \infty$. Thus $x + y \in \ell^p$. □

Now we claim that $\|.\|_p$ is a norm on the linear space ℓ^p for every $1 \le p \le \infty$. It is easy to see that $\|.\|$ is positive definite and homogeneous. If $p = 1$ or ∞, then the triangle inequality follows immediately from the definition of $\|.\|_p$. We shall establish this inequality for the case $1 < p < \infty$ soon, which needs some further results. Before that, let us discuss the inclusion relations among sequence spaces.

Theorem 2.36 *(Jensen's inequality) Let $1 \le a < b \le \infty$. If $x \in \ell^a$, then $\|x\|_b \le \|x\|_a$. Consequently $\ell^a \subset \ell^b$.*

Proof The consequence is immediate from the inequality. Also for $b = \infty$, the result follows from the definition of $\|.\|_\infty$. Suppose $b < \infty$ and write $x := \{x_n\}$.

First assume that $\|x\|_a \le 1$. Then for every $n \in \mathbb{N}$, we have $|x_n| \le 1$, which implies that $|x_n|^b \le |x_n|^a$. Hence $\|x\|_b^b \le \sum_{n=1}^\infty |x_n|^a = \|x\|_a^a$.

Now for any $x \in \ell^a$, applying the above calculations by replacing x with $x/\|x\|_a$, we conclude that

$$\left\| \frac{x}{\|x\|_a} \right\|_b^b \le \left\| \frac{x}{\|x\|_a} \right\|_a^a,$$

and hence $\|x\|_b \le \|x\|_a$. □

We leave it to the reader to prove that the following chain of inclusion relations holds among sequence spaces, which is proper at every stage:

$$c_{00} \subset \ell^a \subset \ell^b \subset c_0 \subset c \subset \ell^\infty \text{ for all } 1 \le a < b < \infty. \tag{2.6}$$

To establish the triangle inequality for sequence spaces, we present a set of inequalities.

If $p, q \in [1, +\infty]$ satisfy $\frac{1}{p} + \frac{1}{q} = 1$, then these are known as *conjugate exponents* of each other.

Theorem 2.37 *(Young's inequality) Let p, q be conjugate exponents such that $p \in (1, \infty)$. Then*

$$ab \le \frac{a^p}{p} + \frac{b^q}{q} \text{ for all } a, b \in [0, \infty).$$

Moreover, the equality occurs if and only if $a^p = b^q$.

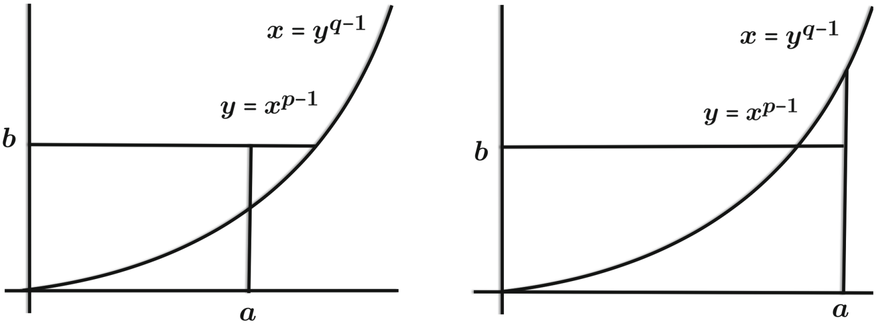
Proof The result is trivial if either $a = 0$ or $b = 0$. Suppose that both a and b are positive real numbers. Also note that

$$p - 1 = p\left(1 - \frac{1}{p}\right) = \frac{p}{q} \text{ and } q - 1 = \frac{q}{p} = \frac{1}{p-1}.$$

Consider the functions f and g on $(0, \infty)$, defined as follows:

$$f(t) := t^{p-1} \text{ and } g(t) := t^{q-1} \text{ for all } t > 0.$$

Since $p - 1$ and $q - 1$ are positive, both f and g are strictly increasing functions from $(0, \infty)$ onto $(0, \infty)$. It can be shown that these are inverses of each other.



Let $a, b \in (0, \infty)$. Then the area of the rectangle $[0, a] \times [0, b]$ is at least the sum of areas of the regions $\{(x, x^{p-1}) : 0 \le x \le a\}$ and $\{(y^{q-1}, y) : 0 \le y \le b\}$. That is

$$ab \le \int_0^a x^{p-1} dx + \int_0^b y^{q-1} dy = \frac{a^p}{p} + \frac{b^q}{q}.$$

Further, the equality occurs here if and only if the area of above rectangle is exactly equal to the sum of areas of those two regions, which is true if and only if $b = a^{p-1}$. Now $b = a^{p-1}$ holds if and only if $b^q = a^{q(p-1)} = a^p$. Hence the result. □

Theorem 2.38 *(Hölder's inequality) Let p, q be conjugate exponents such that $1 \le p \le \infty$, $x = \{x_n\} \in \ell^p$ and $y = \{y_n\} \in \ell^q$. Then $\sum_{n=1}^{\infty} |x_n y_n| \le \|x\|_p \|y\|_q$.*

Proof The result is trivial, if either $p \in \{1, \infty\}$ or either of $\|x\|_p$ or $\|y\|_q$ is zero or infinity. Therefore, without loss of generality, we assume that $1 < p < \infty$, $0 < \|x\|_p < \infty$ and $0 < \|y\|_q < \infty$. Applying Theorem 2.37, for each $n \in \mathbb{N}$, we conclude that

$$\frac{|x_n y_n|}{\|x\|_p \|y\|_q} \le \frac{1}{p}\left(\frac{|x_n|}{\|x\|_p}\right)^p + \frac{1}{q}\left(\frac{|y_n|}{\|y\|_q}\right)^q.$$

Passing summation $\sum_{n=1}^{\infty}$, we obtain

$$\frac{1}{\|x\|_p\|y\|_q}\sum_{n=1}^{\infty}|x_ny_n| \le \frac{1}{p}\sum_{n=1}^{\infty}\left(\frac{|x_n|}{\|x\|_p}\right)^p + \frac{1}{q}\sum_{n=1}^{\infty}\left(\frac{|y_n|}{\|y\|_q}\right)^q = \frac{1}{p}+\frac{1}{q} = 1.$$

Hence we conclude the required inequality. □

For $p = q = 2$, the Hölder's inequality is essentially the Cauchy-Schwarz inequality.

Theorem 2.39 *(Minkowsky's inequality) Let $p \in [1, +\infty]$ and $x, y \in \ell^p$. Then*

$$\|x+y\|_p \le \|x\|_p + \|y\|_p.$$

Proof The result is trivial for the cases $p = 1$ and $p = \infty$. Also incase $\|x + y\|_p = 0$, there is nothing to prove. Suppose $\|x + y\|_p > 0$ and that $1 < p < +\infty$. Applying Theorem 2.35, we obtain $x + y \in \ell^p$.The triangle inequality implies

$$\sum_{n=1}^{\infty}|x_n+y_n|^p \le \sum_{n=1}^{\infty}|x_n+y_n|^{p-1}|x_n| + \sum_{n=1}^{\infty}|x_n+y_n|^{p-1}|y_n|. \tag{2.7}$$

Let q be the conjugate exponent of p. Then $q(p-1) = p$ and consequently

$$\sum_{n=1}^{\infty}\left(|x_n+y_n|^{p-1}\right)^q = \sum_{n=1}^{\infty}|x_n+y_n|^p < \infty.$$

For $r > 0$ and a sequence $a := \{a_n\}$ of complex numbers, we shall denote the sequence $\{|a_n|^r\}$ with simply $|a|^r$. Therefore $|x + y|^{p-1} \in \ell^q$. Also, we have

$$\left\||x+y|^{p-1}\right\|_q = \left(\sum_{n=1}^{\infty}\left(|x_n+y_n|^{p-1}\right)^q\right)^{\frac{1}{q}} = \left(\sum_{n=1}^{\infty}|x_n+y_n|^p\right)^{\frac{1}{q}} = (\|x+y\|_p)^{\frac{p}{q}}.$$

Applying Hölder's inequality, we obtain

$$\sum_{n=1}^{\infty}|x_n+y_n|^{p-1}|x_n| \le \|x\|_p\left\||x+y|^{p-1}\right\|_q = \|x\|_p(\|x+y\|_p)^{\frac{p}{q}}$$
$$\sum_{n=1}^{\infty}|x_n+y_n|^{p-1}|y_n| \le \|y\|_p\left\||x+y|^{p-1}\right\|_q = \|y\|_p(\|x+y\|_p)^{\frac{p}{q}}.$$

Using this in (2.7), we obtain

$$\|x+y\|_p^p \le \left(\|x\|_p + \|y\|_p\right)\left(\|x+y\|_p\right)^{\frac{p}{q}}.$$

Divide it with $\left(\|x+y\|_p\right)^{\frac{p}{q}}$ to conclude the result. □

Remarks 2.40 (a) Let $x_1, \ldots, x_n$ and $y_1, \ldots, y_n$ be non-negative reals. If the ℓ^p-norms of $(x_1, \ldots, x_n)$ and $(y_1, \ldots, y_n)$ coincide for n different positive reals p, then x_i are just a permutation of y_i (see [4]).

(b) The textbook [5] starts with a chapter on basic inequalities. There is also a complete book on inequalities by Hardy, Littlewood, and Polya (see [6]). An essay on a history of inequalities can be found in [7].

(c) We are avoiding an important class of normed spaces called the Lebesgue spaces or the L^p spaces, as these are beyond the scope of this book (see [1, p. 253] or [9, Chaps. 7-8]).

Exercise 2.62 Prove the Hölder's and Minkowsky's inequalities for $p \in \{1, +\infty\}$.

Exercise 2.63 If $1 \leq a < b \leq \infty$ and $x_n \longrightarrow x$ in ℓ^a, prove that $x_n \longrightarrow x$ in ℓ^b.

Exercise 2.64 Suppose $1 \leq a < \infty$ and $x \in \ell^a$. Prove that $\|x\|_\infty \leq \|x\|_a$.

Exercise 2.65 Let X be the space of polynomials over $\mathbb{C}$. Establish a linear bijection between X and c_{00}. Use it to define a norm on X.

Exercise 2.66 Prove the chain of inclusions (2.6) on page 56 and show that all these inclusion are strict.

Exercise 2.67 Let p, q be conjugate exponents such that $p \in (1, \infty)$. Prove that

$$ab \leq \frac{1}{p}.\left(\frac{a}{c}\right)^p + \frac{(bc)^q}{q} \text{ for all } a, b, c \in (0, \infty).$$

Also show that the equality occurs if and only if $a^p = b^q$.

Exercise 2.68 Applying the Jordan-Von Neumann's characterization, as in Remarks 2.34, prove that ℓ^p is an inner product space if and only if $p = 2$.

Exercise 2.69 If $\{a_1, \ldots, a_n\} \subset \mathbb{N}$ satisfy $\sum_{k=1}^n a_k \leq 1$, prove that $\sum_{k=1}^n \frac{1}{a_k} \geq n^2$.

Exercise 2.70 Deduce AM-GM inequality from Young's inequality.

Exercise 2.71 Let $1 \leq p < \infty$ and $x, y \in \ell^p$. Assuming the convexity of the function $t \longmapsto t^p$ on $(0, \infty)$, provide an alternative proof to the inequality $\|x + y\|_p \leq \|x\|_p + \|y\|_p$.

Exercise 2.72 If $x \in \ell^p$ for all $p \in (1, \infty)$, prove that $\|x\|_\infty = \lim_{p\to\infty} \|x\|_p$.

Exercise 2.73 In Exercise 2.72, is the hypothesis that $x \in \ell^p$ for all $p \in (1, \infty)$ redundant?

Exercise 2.74 Prove that c is the linear space spanned by $c_0 \bigcup \{(1, 1, 1, \ldots)\}$.

2.5 Hints and Solutions to Selected Exercises

2.7 For any $x, y \in X$, the hypothesis implies $d(x, y) \le d(x, x) + d(y, x) = d(y, x)$. Similarly, $d(y, x) \le d(x, y)$. Hence $d(y, x) = d(x, y)$. If $d(x, y) < 0$, then

$$0 = d(x, x) \le d(x, y) + d(y, x) = 2d(x, y) < 0,$$

a contradiction. Hence the result.

2.28 Note that $p^{(i)}(0) = 0$ if and only if the $(i + 1)^{th}$ coefficient in p, starting from the constant term, is zero. Therefore d_k is a metric on X if and only if $k \ge n - 1$.

2.35 Yes. For example, in $[-1, 1]$ under usual metric, we have $B[-1; 2] \subset B[0; 1]$.

2.36 All these proofs are analogous to the case of $\mathbb{R}$ (see Theorem 2.23).

2.41 Use the fact that if a sequence converges, then it is bounded. If $|x - y|$ represents the $d_2(x, y)$, apply the following inequality

$$|a_n b_n - ab| \le |a_n b_n - ab_n| + |ab_n - ab| = |a_n - a||b_n| + |a||b_n - b|.$$

2.48 Suppose that $\{x_n\}$ is not convergent to x. Then there exists some $\epsilon > 0$ and a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $d(x_{n_k}, x) \ge \epsilon$ for all $k \in \mathbb{N}$. Therefore, the subsequence $\{x_{n_k}\}$ of $\{x_n\}$ has no subsequence convergent to x, a contradiction.

2.54 Use Theorem 2.9 and imitate the proof of Proposition 2.30.

2.57 Discrete metric on any linear space doesn't satisfy the second assertion of Exercise 2.50.

2.59 Yes. Let X be any linear space and $\mathcal{B}$ be a basis of X. Then every $x \in X$ can be written uniquely as a finite linear combination $x = \sum_i \alpha_i v_i$, where $\alpha_i \in \mathbb{C}$ and $v_i \in \mathcal{B}$. Then $\|x\| := \sum_i |\alpha_i|$ defines a norm on X.

2.61 See Exercise 2.60.

2.63 Apply Theorem 2.36.

2.66 Suppose $1 \le a < b < \infty$ and define $x_n := n^{-\frac{1}{2}(\frac{1}{a}+\frac{1}{b})}$ for all $n \in \mathbb{N}$. Then

$$|x_n|^a = n^{-\frac{1}{2}(1+\frac{a}{b})} > n^{-c}, \text{ where } c \in \left(\frac{1}{2} + \frac{a}{2b}, 1\right).$$
$$\text{and } |x_n|^b = n^{-\frac{1}{2}(1+\frac{b}{a})} < n^{-d}, \text{ where } d \in \left(1, \frac{1}{2} + \frac{b}{2a}\right).$$

Hence $\{x_n\} \in \ell^b \setminus \ell^a$. The strictness of other inclusions is left to the reader.

2.69 Since arithmetic mean is always greatest than the harmonic mean, we obtain

$$\frac{a_1 + \cdots + a_n}{n} \ge \frac{n}{\frac{1}{a_1} + \cdots + \frac{1}{a_n}} \text{ which implies } \sum_{k=1}^{n} \frac{1}{a_k} \ge \frac{n^2}{\sum_{k=1}^{n} a_k} \ge n^2.$$

2.70 Use $p = 2 = q$.

2.71 Write $a := \|x\|_p$ and $b := \|y\|_p$. The result is trivial, if $a = 0$ or $b = 0$. Suppose not. Write $x = \{x_n\|$, $y = \{y_n\}$ and $c := a/(a + b)$. Then for all $n \in \mathbb{N}$, we have

$$|x_n + y_n|^p \le (|x_n| + |y_n|)^p = (a+b)^p \left(\frac{a}{a+b} . \frac{|x_n|}{\|x\|_p} + \frac{b}{a+b} . \frac{|y_n|}{\|y\|_p} \right)^p$$
$$= (a+b)^p \left(c \frac{|x_n|}{\|x\|_p} + (1-c) . \frac{|y_n|}{\|y\|_p} \right)^p$$
$$\le (a+b)^p \left(c \frac{|x_n|^p}{\|x\|_p^p} + (1-c) . \frac{|y_n|^p}{\|y\|_p^p} \right),$$

using the convexity of the map $t \longmapsto t^p$ on $(0, \infty)$. Passing summation $\sum_{n=1}^{\infty}$ above, we conclude that $\|x + y\|_p^p \le (a+b)^p = (\|x\|_p + \|y\|_p)^p$.

2.72 The result is trivial if $x = 0$. Suppose $x \neq 0$. By Theorem 2.36, we already have $\|x\|_\infty \le \|x\|_p$ for all $p > 1$. Therefore, $\|x\|_\infty \le \liminf_{p \longrightarrow \infty} \|x\|_p$. Let p, q be conjugate exponents such that $q < p$. Writing $x := \{x_n\}$, we obtain

$$\|x\|_p = \left(\sum_{n=1}^{\infty} |x_n|^{p-q} |x_n|^q \right)^{\frac{1}{p}} \le \|x\|_\infty^{\frac{p-q}{p}} \left(\sum_{n=1}^{\infty} |x_n|^q \right)^{\frac{1}{p}} = \|x\|_\infty^{1-\frac{q}{p}} \|x\|_q^{\frac{q}{p}}. \quad (2.8)$$

Therefore, we have

$$\limsup_{p \longrightarrow \infty} \|x\|_p \le \limsup_{p \longrightarrow \infty} (\|x\|_\infty^{1-\frac{q}{p}} \|x\|_q^{\frac{q}{p}}) = \|x\|_\infty. \quad (2.9)$$

Finally from (2.8) and (2.9), we conclude that

$$\limsup_{p \longrightarrow \infty} \|x\|_p \le \|x\|_\infty \le \liminf_{p \longrightarrow \infty} \|x\|_p.$$

2.73 No. For example, let $x_n = 1$ for all $n \in \mathbb{N}$. Then $\{x_n\} \in \ell^\infty \setminus \bigcup_{1 \le p < \infty} \ell^p$.

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Chapter 3
Topology

Consider the graphs of the capital letters of the English alphabet in a plane. By stretching and twisting, one can deform U onto S and vice versa. But can you obtain P and S from each other by twisting and stretching only? Topology deals with such questions about continuous deformations. In Sect. 9.2, the notion of homeomorphisms will further reflect upon this idea.

In this chapter, we delve into a thorough discussion on the basic topological notions such as open sets, closed sets, limits points, closures, and boundaries. It contains a section on continuity in metric spaces, which includes the case of Euclidean spaces and the relationship of continuity and uniform convergence. Little sections on subspace topology and topology of normed linear spaces are also included.

3.1 Open Sets and Closed Sets

Let E be a subset of a metric space X and $x \in X$. Then

(a) x is called an *interior point* of E if there exists $\delta > 0$ such that $B(x; \delta) \subset E$.
(b) E is called an *open set* if either $E = \emptyset$ or every $x \in E$ is an interior point of E.
(c) E is said to be a *closed set,* if its complement $E^c (= X \setminus E)$ is open.

The set of interior points of E will be denoted by E^o, known as *the interior of* E and often read as E*-interior.* If E is a subset of a given space X, E^c will denote the set $X \setminus E$. To provide visual clues, we shall usually denote closed sets by F and open sets by O.

Examples 3.1 With respect to the usual metric on $\mathbb{R}$,

(a) 0 is an interior point of $(-1, 1)$, but not of $[0, 1]$.
(b) all open intervals and $(1, 2) \cup (4, 5)$ are open sets.
(c) the singleton sets, $\mathbb{R}$ and all closed intervals are all closed sets.

The family of open subsets of a metric space X is known as the *topology* of X.

S. P. S. Kainth, *A Comprehensive Textbook on Metric Spaces*,
https://doi.org/10.1007/978-981-99-2738-8_3

Theorem 3.2 *Let X be a metric space. Then*

(a) the empty set $\emptyset$ and the space X are open sets,
(b) arbitrary union of open subsets of X is open, and
(c) any finite intersection of open subsets of X is also an open set.

Proof The first two parts follow immediately from the definition of open sets.

For last part, let $O_1, \ldots, O_n$ be any finitely many open sets and $O := \bigcap_{i=1}^{n} O_i$. To prove that O is open, let $x \in O$. Then $x \in O_i$ for all i. Since each O_i is open, there are $\delta_i > 0$ such that $B(x; \delta_i) \subset O_i$ for $i = 1, \ldots, n$. Let $\delta := \min\{\delta_1, \ldots, \delta_n\}$. Then $\delta > 0$ and $B(x; \delta) \subset B(x; \delta_i) \subset O_i$ for all i. Therefore $B(x; \delta) \subset O$. Hence O is an open set. □

Example 3.3 An infinite intersection of open subsets of a metric space may not be open. For example, $\bigcap_{n=1}^{\infty}(-1/n, 1/n) = \{0\}$ is not open in $\mathbb{R}$.

Theorem 3.4 *Open balls in metric spaces are open sets.*

Proof Let X be a metric space, $x \in X$ and $r > 0$. Pick any $y \in B(x; r)$. Let $\delta := r - d(x, y)$. It is enough to prove that $B(y; \delta) \subset B(x; r)$.

Pick any $z \in B(y; \delta)$. Then $d(z, y) < \delta = r - d(x, y)$, which implies that $d(z, x) \leq d(z, y) + d(y, x) < r$. Therefore $z \in B(x; r)$. Hence the result. □

Theorem 3.5 *Let E be a subset of a metric space X. Then E^o is the largest open subset of X, contained in E. In other words,*

(a) $E^o \subset E$,
(b) E^o is open, and
(c) $E^o \supset O$, for all open sets O contained in E.

Proof The first part follows from the definition of E^o. For the second assertion, let x be an interior point of E^o. Then $E^o \supset B(x; r)$ for some $r > 0$. By Theorem 3.4, $B(x; r)$ is an open set. Therefore $B(x; r) = (B(x; r))^o \subset (E^o)^o$. Hence E^o is open.

Finally, let O be an open set such that $O \subset E$. If $x \in O$, then $O \supset B(x; \delta)$, for some $\delta > 0$. Since $O \subset E$, we have $B(x; \delta) \subset O \subset E$. Therefore, $x \in E^o$ and this proves that $O \subset E^o$. □

Note 3.6 Throughout this textbook, unless specified, X will denote an arbitrary metric space, $x \in X$ and all sets will be subsets of X.

Exercise 3.1 Is any of $(0, 1)$, $[0, 1]$, $(0, \infty)$, $\mathbb{N}$, or $\mathbb{Q}$ open or closed in $\mathbb{R}$?

Exercise 3.2 Prove that a set E is open if and only if $E^o = E$.

Exercise 3.3 Prove that a point $x \in E$ is an interior point of E if and only if there exists an open set O such that $x \in O \subset E$.

Exercise 3.4 Prove that open subsets of metric spaces are unions of open balls.

Exercise 3.5 Prove that E^o is the union of all open sets contained in E. Also show that E^o is a union of open balls.

Exercise 3.6 Let $x \in X$ and $N \subset X$. Prove that N is a neighborhood of x if and only if there exists an open set O such that $x \in O \subset N$.

Exercise 3.7 Prove that

(a) every open set containing x is a neighborhood of x and
(b) any set containing an open set O is a neighborhood of every point of O.

Exercise 3.8 In a metric space X, prove that

(a) the empty set $\emptyset$ and the space X are closed,
(b) arbitrary intersection of closed sets is closed, and
(c) any finite union of closed sets is closed.

Exercise 3.9 Is every infinite union of closed sets closed?

Exercise 3.10 Does there exist any non-open subset of a discrete metric space?

Exercise 3.11 In $(\mathbb{R}^2, d_2)$, show that an

(a) infinite intersection of open sets may not be open.
(b) infinite union of closed sets may not be closed.

Exercise 3.12 Let $x = (x_1, x_2) \in \mathbb{R}^2, r > 0$ and d_1, d_2 and d_∞ be the metrics defined as in Corollary 2.11 and Exercise 2.9. Prove that

(a) In the metric space $(\mathbb{R}^2, d_2)$, prove that the squares of the form
    (i) $(x_1 - r, x_1 + r) \times (x_2 - r, x_2 + r) = B_{d_\infty}(x; r)$ are open sets.
    (ii) $[x_1 - r, x_1 + r] \times [x_2 - r, x_2 + r] = B_{d_\infty}[x; r]$ are closed sets.
(b) In the metric space $(\mathbb{R}^2, d_\infty)$, prove that the circular disks of the form
    (i) $B_{d_2}(x; r)$ are open sets.
    (ii) $B_{d_2}[x; r]$ are closed sets.
(c) Prove that O is open in $(\mathbb{R}^2, d_2)$ if and only if O is open in $(\mathbb{R}^2, d_\infty)$.
(d) State and prove similar results for the metric d_1 on $\mathbb{R}^2$.
(e) Generalize the above results to $\mathbb{R}^n$, for any $n \in \mathbb{N}$.

Exercise 3.13 If X is a metric space and $x \in X$, prove that the intersection of all neighborhoods of x is the singleton set $\{x\}$.

Exercise 3.14 In $\mathbb{R}$, prove that closed intervals are closed sets.

Exercise 3.15 In metric spaces, prove that the closed balls are closed sets.

Exercise 3.16 In metric space, prove that all finite sets are closed.

Exercise 3.17 Let X be a metric space. Prove that every subset of X can be written as an intersection of open sets.

Exercise 3.18 Let E and F be subsets of a metric space X. Prove the following

(a) If $E \subseteq F$, then $E^o \subseteq F^o$.
(b) $(E \cap F)^o = E^o \cap F^o$.
(c) $(E \cup F)^o \supseteq E^o \cup F^o$.
(d) In general, $(E \cup F)^o = E^o \cup F^o$ may not hold.

Exercise 3.19 (a) Prove that a sequence $\{x_n\}$ in a metric space X is convergent to some $x \in X$ if and only if for every open set O containing x, contains $\{x_n\}$ eventually.
(b) Prove that a subset O of a metric space X is open if and only if every sequence $\{x_n\} \subset X$ convergent to some $x \in O$, is eventually contained in O.

Exercise 3.20 Does there exist a metric space with exactly three subsets, which are open as well as closed?

3.2 Limit Points and Isolated Points

Definition 3.7 Let E be a subset of a metric space X and $x \in X$. Then x is called

(a) a *limit point* of E, if for every $\epsilon > 0$ the ball $B(x; \epsilon)$ contains a point of $E \setminus \{x\}$,
(b) an *isolated point* of E, if $x \in E$ and x is not a limit point of E.

In other words, x is said to be

(a) a *limit point* of E if $B(x; \epsilon) \cap E \setminus \{x\} \neq \emptyset$ for all $\epsilon > 0$.
(b) an *isolated point* of E if $B(x; \epsilon) \cap E = \{x\}$ for some $\epsilon > 0$.

The set of limit points of a set E is denoted by E', read as E*-prime.*

Example 3.8 Under usual topology on $\mathbb{R}$, consider the subset $E := (0, 1) \cup \{2\}$. Then 2 is an isolated point of E, while the closed interval $[0, 1]$ is the set of limit points of E.

Remarks 3.9 (a) Limit points are also termed as *cluster points* or *accumulation points*.
(b) It is pertinent to note that a limit point of a set may not belong to that set, while an isolated point of a set is always an element of the set.
(c) In Definition 3.23, we shall see that if X and Y are metric spaces, $E \subset X$, $f : E \longrightarrow Y$ and $x \in X$, then $\lim_{y \to x} f(y)$ cannot be considered if x is not a limit point of E.

Theorem 3.10 *Let X be a metric space and $E \subset X$. Then E is closed if and only if $E' \subset E$.*

Proof First assume that E is closed. Then E^c is open. To prove that $E' \subset E$, let $x \in E'$ be arbitrary. If possible, assume that $x \notin E$. Then $x \in E^c$, which is open. Therefore there exists some $\epsilon > 0$ such that $B(x; \epsilon) \subset E^c$ and therefore $B(x; \epsilon) \cap E = \emptyset$. Thus there exist no $y \in B(x; \epsilon) \cap E$ such that $y \neq x$. Hence $x \notin E'$, a contradiction.

Conversely, assume that $E' \subset E$. To prove that E^c is open, pick any $x \in E^c$. Then $x \notin E'$. Consequently, there exists some $\epsilon > 0$ such that $B(x; \epsilon) \cap E = \emptyset$. Thence $B(x; \epsilon) \subset E^c$, proving that x is an interior point of E^c. Hence E^c is open and the result follows. □

Theorem 3.11 *Let (X, d) be a metric space, $E \subset X$ and $x \in X$. Then $x \in E'$ if and only if there exists a sequence of distinct terms in E, convergent to x.*

Proof Assume that $x \in E'$. Let $x_1 \in E \cap B(x; 1) \setminus \{x\}$. Then we choose an element $x_2 \in E \cap B(x; \min\{1/2, d(x, x_1)\}) \setminus \{x\}$. Once $x_1, \ldots, x_n$ are chosen, we choose x_{n+1} such that

$$x_{n+1} \in E \cap B\left(x; \min\left\{\frac{1}{n+1}, d(x, x_n)\right\}\right) \setminus \{x\}.$$

Inducting this way we obtain a sequence $\{x_n\}$ of distinct terms from E such that $d(x, x_n) < 1/n$ for all $n \in \mathbb{N}$. Therefore $x_n \longrightarrow x$.

Conversely, assume that there exists a sequence $\{x_n\}$ of distinct terms from E such that $x_n \longrightarrow x$. Let $\epsilon > 0$ be arbitrary. Therefore there exists an $m \in \mathbb{N}$ such that $x_n \in B(x; \epsilon)$ for all $n > m$. Since the terms of the sequence $\{x_n\}$ are all distinct, $E \cap B(x; \epsilon)$ contains points other than x. Hence $x \in E'$ and the result follows. □

Corollary 3.12 *Let (X, d) be a metric space, $E \subset X$ and $x \in X$. Then $x \in E'$ if and only if every neighborhood of x contains infinitely many points of E.*

Theorem 3.13 (Bolzano-Weierstrass) *Every bounded infinite subset of $\mathbb{R}^m$ has a limit point in $\mathbb{R}^m$.*

Proof Let E be any bounded infinite subset of $\mathbb{R}^m$. Choose a sequence $\{x_n\}$ of distinct terms from E. Applying Theorem 2.27, let $\{x_{n_k}\}$ be a convergent subsequence of $\{x_n\}$. Let $x := \lim_{k\to\infty} x_{n_k}$. Applying Theorem 3.11, we conclude that $x \in E'$. □

In general, a bounded infinite set may not have a limit point.

Examples 3.14 In the following spaces X and $E \subset X$, we have $E' = \emptyset$.

(a) Let X be any infinite discrete metric space and $E = X$.
(b) Let $X := \mathbb{R} \setminus \mathbb{Q}$ and $E := \{\frac{\sqrt{2}}{n} : n \in \mathbb{N}\}$.

Proposition 3.15 *The set of limit points of any set is closed.*

Proof Let E be any subset of a metric space (X, d). By the virtue of Theorem 3.10, it is enough to prove that $(E')' \subset E'$. If $(E')' = \emptyset$, there is nothing to prove. Otherwise, let $x \in (E')'$ and $\epsilon > 0$ be arbitrary.

Since x is a limit point of E', there exists $y \in E'$ such that $y \in B(x; \epsilon) \cap E'$ and $y \neq x$. Let $\rho := \epsilon - d(x, y)$. Then $\rho > 0$. As in Theorem 3.4, we obtain $B(y; \rho) \subset B(x; \epsilon)$.

Since $y \in E'$, the ball $B(y; \rho)$ contains infinitely many points of E. Hence $B(x; \epsilon)$ contains infinitely many points of E. Therefore $x \in E'$. □

Exercise 3.21 In metric spaces, prove that finite sets have no limit points.

Exercise 3.22 Prove that every subset of a finite metric space is open as well as closed.

Exercise 3.23 If $E := \left\{\frac{1}{n} + \frac{1}{m} : m, n \in \mathbb{N}\right\}$, prove that $E' \subset \{0\} \cup \left\{\frac{1}{n} : n \in \mathbb{N}\right\}$.

Exercise 3.24 Obtain the set of limit points of the set $\left\{\frac{1}{n} \sin \frac{1}{n} : n \in \mathbb{N}\right\}$.

Exercise 3.25 Let $\{x_n\}$ be a sequence of $\mathbb{R}$ such that $x_n \longrightarrow \infty$. Prove that the set $\{x_n : n \in \mathbb{N}\}$ has no real limit point.

Exercise 3.26 Let E be a subset of a metric space X. Prove that a point $x \in E$ is an isolated point of E if and only if $O \cap E = \{x\}$ for some open subset O of X.

Exercise 3.27 Prove that $x \in E'$ if and only if there is a sequence in $E \setminus \{x\}$, convergent to x.

Exercise 3.28 Let E and F be subsets of a metric space X such that F is finite. Prove or disprove: $E' = (E \setminus F)'$.

Exercise 3.29 Let (X, d) be a metric space. For every $x \in X$ and $E \subset X$, define $d(x, E) := \inf\{d(x, y) : y \in E\}$. If F is a closed subset of X, prove that $F = \bigcap_{n=1}^{\infty} \left\{x \in X : d(x, F) < \frac{1}{n}\right\}$.

Exercise 3.30 Prove that a subset E of a metric space X is closed if and only if for all $x \in E$, we have $d(x, E) = 0$.

Exercise 3.31 Let E and F be subsets of a metric space X. Prove the following

(a) If $E \subseteq F$, then $E' \subseteq F'$.
(b) $(E \cup F)' = E' \cup F'$.
(c) $(E \cap F)' \subseteq E' \cap F'$.
(d) Also show that, in general, $(E \cap F)' = E' \cap F'$ does not hold.

Exercise 3.32 Let $\{x_n\}$ be a sequence in a metric space X and $E := \{x_n : n \in \mathbb{N}\}$.

(a) If there exists $x \in E$ such that $x_n \longrightarrow x$, prove that E is a closed set.

(b) If $X = \mathbb{R}^n$ under the usual metric, prove that E is not an open set. Is the same true in any metric space?

Exercise 3.33 Let X be a metric space and $\{x_n\}$ be a sequence in X having no convergent subsequence. Prove or disprove:

$$\textit{For each } x \in X, \textit{ there exists } \epsilon_x > 0 \textit{ such that } d(x_n, x) \geq \epsilon_x, \\ \textit{for all but finitely many } n \in \mathbb{N}.$$

Exercise 3.34 Prove that the set of limit points of $\{\sin n : n \in \mathbb{N}\}$ in $\mathbb{R}$ is $[-1, 1]$.

3.3 Closures and Boundaries

Definition 3.16 Let X be a metric space and $E \subset X$. An element $x \in X$ is called

(a) an *adherent point* of E if every ball centered at x contains a point of E
(b) a *boundary point* of E, if x is an adherent point of E as well as of $X \setminus E$.

The set of adherent and boundary points of a set E are denoted by $\overline{E}$ and ∂E, read as *E-closure* and *boundary of* E, respectively. In other words,

(a) $x \in \overline{E}$ if and only if $B(x; \epsilon) \cap E \neq \emptyset$ for all $\epsilon > 0$.
(b) $x \in \partial E$ if and only if $B(x; \epsilon) \cap E \neq \emptyset$ and $B(x; \epsilon) \setminus E \neq \emptyset$ for all $\epsilon > 0$.

That is $\partial E = \overline{E} \cap \overline{X \setminus E}$ and $\overline{E} = E \cup E'$.

Examples 3.17 (a) In $\mathbb{R}$, we have $\overline{[1, 2)} = [1, 2]$ and $\partial[1, 2) = \{1, 2\}$.
(b) Let $X := \mathbb{R}^2$ and $B := B(0; 1)$. Then $\overline{B} = B[0; 1]$ and $\partial B = S(0; 1)$.

Theorem 3.18 *Let (X, d) be a metric space, $E \subset X$ and $x \in X$. Then $x \in \overline{E}$ if and only if there exists a sequence in E, convergent to x.*

Proof Let $x \in \overline{E}$. If $x \in E$, then take $x_n := x$ for all $n \in \mathbb{N}$. Otherwise $x \in E'$. Then apply Theorem 3.11 to choose a sequence in E, convergent to x.

Conversely, let $\{x_n\}$ be a sequence in E, convergent to x. Let $\epsilon > 0$ be given. Then there exists $N \in \mathbb{N}$ such that $d(x_n, x) < \epsilon$ for all $n \geq N$. Thus $B(x, \epsilon) \cap E$ is a nonempty set, as it contains x_N. Consequently, $x \in \overline{E}$. Hence the result. □

Theorem 3.19 *Let X be a metric space and $E \subset X$. Then $\overline{E}$ is the smallest closed subset of X, containing E. That is*

(a) $\overline{E} \supset E$,
(b) $\overline{E}$ *is closed and*
(c) $\overline{E} \subset F$, *whenever F is closed and $E \subset F$.*

Proof The first part follows from the definition of $\overline{E}$. For part (b), let x be a limit point of $\overline{E} = E \cup E'$. Then either x is a limit point of E or x is a limit point of E'. That is $x \in E' \cup (E')' \subset E' \subset \overline{E}$, by Proposition 3.15. Hence $\overline{E}$ is closed, by Theorem 3.10.

Finally, let F be a closed set containing E and $x \in \overline{E} = E \cup E'$. If $x \in E$, then $x \in F$. If $x \in E'$, then $x \in F'$. Since F is closed, we have $F' \subset F$. Hence $x \in F$. □

Proposition 3.20 *Let E be a nonempty and bounded above subset of $\mathbb{R}$ and $s := \sup E$. Then $s \in \overline{E}$. Consequently $s \in E$, provided E is closed.*

Proof Let $\epsilon > 0$ be given. Since $s = \sup E$, the number $s - \epsilon$ is not an upper bound of E. Thus there exists some $x \in E$ such that $s - \epsilon < x$.

Since s is an upper bound of E, we obtain $x \leq s$. Hence $x \in (s - \epsilon, s + \epsilon) \cap E$. Therefore, $(s - \epsilon, s + \epsilon) \cap E \neq \emptyset$ for all $\epsilon > 0$. Consequently, $s \in \overline{E}$. Hence the result. □

Exercise 3.35 With respect to the usual metric on $\mathbb{R}$, find out the interior points, limit points, adherent points, boundary points, interiors, and closures of the following sets. Also state whether these are open, closed, or bounded.

$$(0, 1), [0, 1), (-1, 0) \cup (0, 1), \mathbb{Q}, \mathbb{R} \setminus \mathbb{Q}, [0, \infty), \mathbb{Q} \cap (-\infty, 1)$$
$$\left\{\frac{1}{n} : n \in \mathbb{N}\right\}, \left\{\frac{1}{m} + \frac{1}{n} : m, n \in \mathbb{N}\right\}, \left\{\frac{1}{m} + \frac{1}{n} : m, n \in \mathbb{N}\right\} \cup \{0\}.$$

Exercise 3.36 Obtain the closure of the set $\big\{(x, \sin(1/x)) : x \in (0, 1]\big\} \cup \{(0, 0)\}$.

Exercise 3.37 In $\mathbb{R}$ under usual metric, give an example of

(a) a bounded subset, which is not closed,
(b) a closed subset, which is not bounded,
(c) a bounded sequence, which is not convergent,
(d) an infinite set, with empty interior,
(e) a set, which is neither closed nor open,
(f) a set, which contains no interval but whose closure is an interval,
(g) a set, whose closure is an infinite set containing no interval,
(h) a set, the closure of whose interior is empty,
(i) a set, with exactly two limit points,
(j) a set, with all naturals as limit points.

Exercise 3.38 Let X be a metric space, $x \in X$ and $r > 0$. Prove that $\overline{B(x; r)} \subset B[x; r]$. Also show that a strict inclusion may occur here.

Exercise 3.39 Let A be an arbitrary open subset of a metric space X. Does it imply $(\overline{A})^o = A$?

Exercise 3.40 Let E be a subset of a metric space X. Prove that $x \in X$ is an adherent point of E if and only if $O \cap E \neq \emptyset$, for every open set O containing x.

Exercise 3.41 Prove that $\overline{E}$ is the intersection of all closed sets containing E.

Exercise 3.42 Prove that E is closed if and only if $\overline{E} = E$.

Exercise 3.43 Let X be a metric space, $E \subset X$ and $x \in X$. Prove that $x \in E'$ if and only if $x \in \overline{E \setminus \{x\}}$.

Exercise 3.44 Let $A \subset X$, $x \in X$ and $d(x, A) := \inf\{d(x, y) : y \in A\}$. Prove that $x \in \overline{A}$ if and only if $d(x, A) = 0$.

Exercise 3.45 If E is a subset of a discrete metric space X, obtain E^o, $\overline{E}$ and ∂E.

Exercise 3.46 If $x_n \longrightarrow 0$, what is the closure of $\{kx_n : k, n \in \mathbb{N}\}$ in $\mathbb{R}$?

Exercise 3.47 Prove that the closure of a set never adds up any open set. That is, if A is any subset of a metric space X, then there exist no nonempty open subset O of X, disjoint from A and contained in $\overline{A}$.

Exercise 3.48 If A is any subset of a metric space X, prove that $(\overline{A} \setminus A)^o = \emptyset$.

Exercise 3.49 Let X be any metric space, $I \subset X$ and $J := X \setminus \overline{I}$. Prove that $(\overline{I})^o \cap (\overline{J})^o = \emptyset$.

Exercise 3.50 Let O be an open subset of X and $E \subset X$. Prove that $O \cap \overline{E} \subset \overline{O \cap E}$.

Exercise 3.51 Let E be a subset of a metric space X. Prove that

(a) $\partial E = \overline{E} \cap \overline{X \setminus E}$,
(b) $\partial E \cap E^o = \emptyset$ and
(c) $\overline{E} = \partial E \cup E^o$.

Exercise 3.52 Let E be a subset of a metric space X. Prove that

(a) E is open if and only if $\partial E \cap E = \emptyset$.
(b) E is closed if and only if $\partial E \subset E$.
(c) E is both open as well as closed if and only if $\partial E = \emptyset$.

Exercise 3.53 Let A and B be subsets of a metric space X. Prove that $\overline{A} \setminus \overline{B} \subset \overline{A \setminus B}$. Also show that a strict inclusion may occur.

Exercise 3.54 Discuss the closure and the set of limit points of the following subsets of reals

$$\left\{\frac{1}{n_1} + \cdots + \frac{1}{n_k} : k, n_1, \ldots, n_k \in \mathbb{N}\right\},$$
$$\text{and } \left\{\frac{1}{n_1} + \frac{2}{n_2} + \cdots + \frac{k}{n_k} : k, n_1, \ldots, n_k \in \mathbb{N} \text{ such that } n_1 < n_2 < \cdots < n_k\right\}.$$

Exercise 3.55 Check whether the set $\{\sin n : n \in \mathbb{N}\}$ is open or closed? Find out its interior and closure.

Exercise 3.56 Let F be a closed subset of a metric space X and $E := \overline{F^o}$. Prove that $E \subset F$ and $E^o = F^o$.

Exercise 3.57 If E is an arbitrary subset of a metric space.

(a) Do E and E^o always have same closures?
(b) Do E and $\overline{E}$ always have same interiors?
(c) Does $E^o \subset E'$ always true?

Exercise 3.58 Let E and F be subsets of a metric space. Prove the following:

(a) If $E \subseteq F$, then $\overline{E} \subseteq \overline{F}$.
(b) $\overline{E \cup F} = \overline{E} \cup \overline{F}$.
(c) $\overline{E \cap F} \subseteq \overline{E} \cap \overline{F}$.
(d) Also show that, in general, $\overline{E \cap F} = \overline{E} \cap \overline{F}$ does not hold.

Exercise 3.59 Let X be a metric space, $x \in X$, and $E \subset X$. Prove that the following are equivalent:

(a) $x \in \overline{E} \cap \overline{X \setminus E}$,
(b) x is a boundary point of E and
(c) x is neither an interior point of E nor an interior point of $X \setminus E$.

Exercise 3.60 Let X be a metric space and $A, B \subset X$. Define *exterior* of A as $Ext(A) := X \setminus \overline{A}$. Prove the following:

(a) $Ext(\emptyset) = X$,
(b) $Ext(A) \cap A = \emptyset$,
(c) $Ext(A \cup B) = Ext(A) \cap Ext(B)$ for all $A, B \subset X$,
(d) $Ext(A) \subset Ext(Ext(X \setminus A))$ and a strict inclusion may hold here.

Exercise 3.61 Let A and B be subsets of $\mathbb{R}$. Then $A \times B \subset \mathbb{R}^2$. Which of the following are true? Justify your answer.

(a) $\overline{A \times B} = \overline{A} \times \overline{B}$
(b) $(A \times B)' = A' \times B'$.
(c) $(A \times B)^o = A^o \times B^o$.
(d) $\partial(A \times B) = \partial A \times \partial B$.

Exercise 3.62 For $A, B \subset \mathbb{R}^n$, define $A + B := \{a + b : a \in A, b \in B\}$. Compute $\mathbb{Q} + (0, 1/10)$ and $\mathbb{Q} + A$, if $A (\subset \mathbb{R})$ has a non-empty interior.

Exercise 3.63 Let A and B be subsets of $\mathbb{R}^2$. Prove that $\overline{A} + \overline{B} \subset \overline{A + B}$. Can this inclusion ever be proper?

Exercise 3.64 If $\mathcal{F}$ is a subfield of $\mathbb{C}$ such that $\mathcal{F} \not\subset \mathbb{R}$, obtain the closure of $\mathcal{F}$ in $\mathbb{C}$?

Exercise 3.65 Let d be an ultrametric on X, $x, y \in X$, and $r > 0$. Prove the following:

(a) Open balls in X are closed sets.
(b) The set $\Omega := \{B(x; r) : x \in B[y; r]\}$ is a disjoint collection of open subsets of $B[y; r]$ with $\bigcup_{x \in B[y;r]} B(x; r) = B[y; r]$.
(c) If $B_1, B_2 \in \Omega$ such that $B_1 \neq B_2$, then $dist(B_1, B_2) := \inf\{d(b_1, b_2) : b_1 \in B_1, b_2 \in b_2\} \geq r$.
(d) Closed balls in X are open sets.
(e) $\partial B(x; r) = \emptyset = \partial B[x; r]$.
(f) $\sup\{d(y, z) : y, z \in B(x; r)\} = \sup\{d(x, y) : y \in B(x; r)\} \leq r$.

(Readers interested in more on ultrametric spaces are referred to [1].)

Exercise 3.66 Let $r \in [0, 1)$ and (X, ρ_r) be the ultrametric space of Example 2.5. For any finite tuple $a = (a_1, \ldots, a_n)$; $a_i \in \{0, 1\}$, let $[a]$ denote the collection of sequences with first n terms as $a_1, \ldots, a_n$. Prove the following:

(a) $[a]$ is open. In fact if $r^{n+1} < s \leq r^n$ and x denote a sequence with first n terms as $a_1, \ldots, a_n$, then $[a] = B(x; s)$.
(b) Every open ball in X is some $[a]$. That is, for every $x \in X$ and $s > 0$, there exists some $a = (a_1, \ldots, a_n)$; $a_i \in \{0, 1\}$ such that $B(x; s) = [a]$.

Exercise 3.67 (Hausdorff metric) Let (X, d) be a metric space and $\mathcal{B}(X)$ denote the family of nonempty closed bounded subsets of X. For $A, B \in \mathcal{B}(X)$, define

$$h(A, B) := \max\Big\{\sup\{dist(b, A) : b \in B\},\ \ \sup\{dist(a, B) : a \in A\}\Big\}.$$

Prove that h is a metric on $\mathcal{B}(X)$ such that $h(\{x\}, \{y\}) = d(x, y)$ for all $x, y \in X$.

3.4 Subspace Topology

Let S be the unit open ball in $\mathbb{R}^3$ and P be a plane intersecting it. Then the cross section will be a circular disk. Can one define a metric on the plane P so that the cross section becomes the corresponding open ball?

If (X, d) is a metric space and $\emptyset \neq Y \subset X$, it is easy to see that (Y, d) is also a metric space. In this case, we say that Y is a subspace of X. We now discuss the relationship between open subsets of X and Y. First consider the case of open balls.

Since the same metric d is defined on X as well as Y, in order to avoid confusion, for $x \in Y$ and $r > 0$, we write

$$B_Y(x; r) := \{y \in Y : d(y, x) < r\} \text{ and } B_X(x; r) := \{y \in X : d(y, x) < r\}.$$

Definition 3.21 Let $\emptyset \neq Y \subset X$ and $E \subset Y$. Then E is said to be *open relative to* Y if it is open in the subspace (Y, d). That is, either $E = \emptyset$ or for every $x \in E$, there exists some $r_x > 0$ such that $B_Y(x; r_x) \subset E$.

Similarly, we say that E is *closed relative to* Y if $Y \setminus E$ is open in the subspace (Y, d). The notions like limit points, interior points, adherent points, and boundary points relative to a space are defined on the similar lines.

Clearly, $B_Y(x; r) = B_X(x; r) \cap Y$. Interestingly, a similar result holds for open sets.

Theorem 3.22 *Let (X, d) be a metric space and $G \subset Y \subset X$. Then G is open in (Y, d) if and only if there exists an open subset O of (X, d) such that $O \cap Y = G$.*

Proof Assume that G is open in (Y, d). Then for each $x \in G$, there exists $\delta_x > 0$ such that $B_Y(x; \delta_x) \subset G$. Then $\bigcup_{x \in G} B_Y(x; \delta_x) = G$. Let $O := \bigcup_{x \in G} B_X(x; \delta_x)$. Then O is open in X and

$$O \cap Y = \bigcup_{x \in G} \big(B_X(x; \delta_x) \cap Y\big) = \bigcup_{x \in G} B_Y(x; \delta_x) = G.$$

Conversely, suppose that $G = O \cap Y$ for some open subset O of X. Let $x \in G$. Since $x \in G \subset O$, there exists some $\delta_x > 0$ such that $B_X(x; \delta_x) \subset O$. Therefore,

$$B_Y(x; \delta_x) = B_X(x; \delta_x) \cap Y \subset O \cap Y = G.$$

Hence G is open in Y. □

Exercise 3.68 Prove that the set $\mathbb{Q} \cap (0, 1)$ is open relative to $\mathbb{Q}$ but not relative to $\mathbb{R}$, under the usual metric.

Exercise 3.69 Let $X := \mathbb{R}^3$ be a metric space under the Euclidean metric d_2. Let

$$Y := \{(x, y, z) \in \mathbb{R}^3 : x + y + z = 1\} \text{ and } Z := \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\}.$$

Characterize open subsets and open balls of (Y, d_2) and (Z, d_2).

In the following exercises, let (X, d) be a metric space and $A \subset Y \subset X$ such that $Y \neq \emptyset$.

Exercise 3.70 Prove that A is closed in (Y, d) if and only if there exists a closed subset F of (X, d) such that $A = F \cap Y$.

Exercise 3.71 If A is closed in Y and $\overline{A}$ denotes the closure A in X, prove that $\overline{A} \cap Y = A$.

Exercise 3.72 Prove that a point $x \in Y$ is a limit point of A in (Y, d) if and only if x is a limit point of A in (X, d).

Exercise 3.73 If $\overline{A}$ is the closure of A in X, prove that the closure of A in Y is $Y \cap \overline{A}$.

Exercise 3.74 Prove that the interior of A with respect to X is contained in the interior of A with respect to Y. Show that this inclusion is proper.

Exercise 3.75 Prove that the boundary of A with respect to Y is contained in the intersection of Y with the boundary of A with respect to X. Further show that this inclusion is proper.

3.5 Limits and Continuity

Unless specified, let (X, d_X) and (Y, d_Y) be metric spaces, $E \subset X$ and $f : E \longrightarrow Y$.

Definition 3.23 Let c be a limit point of E in X. Then f is said to have a *limit* at c if there exists an $l \in Y$ such that for every $\epsilon > 0$ there exists a $\delta > 0$ such that

$$d_Y(f(x), l) < \epsilon \text{ for all } x \in E \text{ satisfying } 0 < d_X(x, c) < \delta.$$

In this case, we call l to be the limit of f at c and write $\lim_{x\to c} f(x) = l$.

Definition 3.24 The function f, as above, is said to be *continuous at* $c \in E$ if for every $\epsilon > 0$ there exists some $\delta > 0$ such that

$$d_Y(f(x), f(c)) < \epsilon \text{ for all } x \in E \text{ satisfying } d_X(x, c) < \delta.$$

That is, if for every $\epsilon > 0$ there exists some $\delta > 0$ such that $f(E \cap B(c; \delta)) \subset B(f(c); \epsilon)$.

In addition, if c is a limit point of X, then f is continuous at c if and only if $\lim_{x\to c} f(x) = f(c)$.

Further, f is said to be *continuous,* if it is continuous at every point of its domain.

Remark 3.25 It is emphasized that $\lim_{x\to c} f(x)$ is not defined, if c is not a limit point of E. Further to define continuity at c, the point c must belong to the domain E of f, while the same is not required for limits.

Several results about limits and continuity for real functions can be extended to general metric spaces. First we present an analogue of Theorem 1.46.

Theorem 3.26 *Let* $f : X \longrightarrow Y, c \in X'$ *and* $l \in Y$. *Then* $\lim_{x\to c} f(x) = l$ *if and only if* $f(x_n) \longrightarrow l$, *for every sequence* $\{x_n\}$ *in* $X \setminus \{c\}$ *such that* $x_n \longrightarrow c$.

Proof Let d_X and d_Y, respectively, denote the metrics on X and Y. First assume that $\lim_{x\to c} f(x) = l$. Let $\epsilon > 0$ be given. Then there exists some $\delta > 0$ such that

$$d_Y(f(x), l) < \epsilon \text{ whenever } 0 < d_X(x, c) < \delta.$$

Let $\{x_n\}$ be a sequence in $X \setminus \{c\}$, convergent to c. Then there exists some $m \in \mathbb{N}$ such that $0 < d_X(x_n, c) < \delta$ for all $n > m$. Because of the choice of δ, we obtain $d_Y(f(x_n), l) < \epsilon$ for all $n > m$. Hence $f(x_n) \longrightarrow l$.

Conversely, assume that $\lim_{x\to c} f(x) \neq l$. Then there exists some $\epsilon > 0$ such that for every $\delta > 0$ the following assertion is not satisfied:

$$d_Y(f(x), l) < \epsilon \text{ for all } x \text{ satisfying } 0 < d_X(x, c) < \delta.$$

In particular, for each $n \in \mathbb{N}$, taking $\delta_n := 1/n$, one can choose some $x_n \in X$ such that $d_X(x_n, c) < 1/n$, while $d_Y(f(x_n), l) \geq \epsilon$. Therefore $\{f(x_n)\}$ does not converge to l, while $x_n \longrightarrow c$, a contradiction to the hypothesis. Hence the result. □

Corollary 3.27 *A function $f : X \longrightarrow Y$ is continuous at c if and only if $f(x_n) \longrightarrow f(c)$ in Y, whenever $x_n \longrightarrow c$ in X.*

Proof If c is a limit point of X, then the result follows by Theorem 3.26. Suppose otherwise. Then c is an isolated point of X and thus there exists $\eta > 0$ such that $B(c; \eta) = \{c\}$. In this case, any function f on X satisfies both of the above conditions.

Note that for every $\epsilon > 0$ the constant $\delta := \eta$ satisfies the requirement of continuity of f at c. Also if $x_n \longrightarrow c$, then there exists $m \in \mathbb{N}$ such that $x_n \in B(c; \eta) = \{c\}$ for all $n \geq m$. Therefore, $f(x_n) = f(c)$ for all $n \geq m$ which implies that $f(x_n) \longrightarrow f(c)$. Hence the result. □

Theorem 3.28 *Let $f : X \longrightarrow Y$. Then f is continuous if and only if $f^{-1}(O)$ is open, for every open set $O \subset Y$.*

Proof Let f be continuous and O be an open subset of Y. If $f^{-1}(O) = \emptyset$, the set is trivially open. Otherwise pick any $x \in f^{-1}(O)$. Then $f(x) \in O$. Since O is open there exists some $r > 0$ such that $B(f(x); r) \subset O$. Since f is continuous at x, there exists some $\delta > 0$ such that $f(B(x; \delta)) \subset B(f(x); r)$. Hence $f(B(x; \delta)) \subset B(f(x); r) \subset O$. Therefore $B(x; \delta) \subset f^{-1}(O)$ and hence $f^{-1}(O)$ is open.

Conversely, suppose that $f^{-1}(O)$ is open, for every open set $O \subset Y$. Pick any $x \in X$. Let $\epsilon > 0$ be given. Then for $O := B(f(x); \epsilon)$, the set $f^{-1}(O)$ is open. Since $x \in f^{-1}(O)$, there exists a $\delta > 0$ such that $B(x; \delta) \subset f^{-1}(O)$. Therefore $f(B(x; \delta)) \subset O = B(f(x); \epsilon)$. So f is continuous at x and the result follows. □

3.5.1 The Case of Euclidean Spaces

Now we present the case of limit of functions between finite dimensional spaces.

Theorem 3.29 *Let X be any metric space, $c \in X'$, $n \in \mathbb{N}$ and $f : X \longrightarrow \mathbb{R}^n$ be given by $f = (f_1, \ldots, f_n)$, where each f_j is a real valued function on X. Then $\lim_{x\to c} f(x)$ exists if and only if $\lim_{x\to c} f_j(x)$ exists for every $1 \leq j \leq n$. Further, in this case*

$$\lim_{x\to c} f(x) = \big(\lim_{x\to c} f_1(x), \ldots, \lim_{x\to c} f_n(x)\big).$$

Proof The result is immediate from Theorems 3.26 and 2.25.

First assume that $\lim_{x\to c} f(x) = l$ for some $l = (l_1, \ldots, l_n) \in \mathbb{R}^n$. Let $\{x_n\}$ be any sequence in $X \setminus \{c\}$ convergent to c. Then $f(x_n) \longrightarrow l$. Applying Theorem 2.25, we conclude that $f_j(x_n) \longrightarrow l_j$ for all $j = 1, \ldots, n$. Applying Theorem 3.26, we obtain $\lim_{x\to c} f_j(x) = l_j$ for every $j = 1, \ldots, n$. The converse is similar. □

Corollary 3.30 (Algebra of Limits) *Let X be any metric space, $c \in X'$, $n \in \mathbb{N}$ and $f, g : X \longrightarrow \mathbb{R}^n$. Assume that $\lim_{x\to c} f(x) = l$ and $\lim_{x\to c} g(x) = l'$, for some $l, l' \in \mathbb{R}^n$. Then*

(a) $\lim_{x\to c} \alpha f(x) = \alpha l$ *for all* $\alpha \in \mathbb{R}$.
(b) $\lim_{x\to c} (f(x) + g(x)) = l + l'$.
(c) $\lim_{x\to c} f(x)g(x) = ll'$, *provided* $n = 1$.
(d) *If* $n = 1$ *and* $l \neq 0$, *then* $\lim_{x\to c} \frac{1}{f(x)} = \frac{1}{l}$.

Proof Apply Theorem 3.29 along with Theorem 1.52. □

3.5.2 Continuity and Uniform Convergence

Now we discuss the relationship of uniform convergence with continuity and limits.

Theorem 3.31 *Let $\{f_n\}$ be a sequence of real valued continuous functions on a metric space (X, d), uniformly convergent to a function f on X. Then f is continuous on X.*

Proof Let $c \in X$ and $\epsilon > 0$ be given. Then there exists some $m \in \mathbb{N}$ such that

$$|f_n(x) - f(x)| < \frac{\epsilon}{3} \text{ for all } n \geq m \text{ and for all } x \in X.$$

Since the function f_m is continuous at c, there exists some $\delta > 0$ such that $|f_m(x) - f_m(c)| < \epsilon/3$ for all $x \in B(c; \delta)$. Then for every $x \in B(c; \delta)$, we obtain

$$\begin{aligned} |f(x) - f(c)| &\leq |f(x) - f_m(x)| + |f_m(x) - f_m(c)| + |f_m(c) - f(c)| \\ &< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon. \end{aligned}$$

Hence f is continuous at c. Since $c \in X$ was arbitrary, f is continuous on X. □

Next, we answer a very natural question: What precisely needs to be added to pointwise convergence to preserve continuity?

Theorem 3.32 (Beer, 2009 *[2]*) *Let (X, d_X), (Y, d_Y) be metric spaces, $x_0 \in X$, and $f, f_n : X \longrightarrow Y$; $n \in \mathbb{N}$. Suppose that each f_n is continuous at x_0, and $\{f_n\} \longrightarrow f$ pointwise on X. Then the following are equivalent:*

(a) *f is continuous at x_0.*
(b) *For every $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that for each $n \geq N$, there exists some $\delta_n > 0$ such that $d_X(x, x_0) < \delta_n$ implies $d_Y(f_n(x), f(x)) < \epsilon$.*

(In this case, we say that $\{f_n\}$ is strongly pointwise convergent to f at x_0.)

Proof Assume f is continuous at x_0. Let $\epsilon > 0$ be arbitrary. Then there exists some $N \in \mathbb{N}$ and $\delta > 0$ such that

$$d_Y(f_n(x_0), f(x_0)) < \frac{\epsilon}{3} \text{ for all } n \geq N$$
$$\text{and } d_Y(f(x), f(x_0)) < \frac{\epsilon}{3} \text{ for all } x \in X \text{ such that } d_X(x, x_0) < \delta.$$

Let $n \geq N$ be arbitrary. The continuity of f_n at x_0 ensures a $\delta_n \in (0, \delta)$ such that if $d_X(x, x_0) < \delta_n$, then $d_Y(f_n(x), f_n(x_0)) < \frac{\epsilon}{3}$. Consequently, for $d_X(x, x_0) < \delta_n$,

$$d_Y(f_n(x), f(x)) \leq d_Y(f_n(x), f_n(x_0)) + d_Y(f_n(x_0), f(x_0)) + d_Y(f(x_0), f(x)) < \epsilon.$$

Conversely, assume that $\{f_n\}$ is strongly pointwise convergent to f at x_0. Let $\epsilon > 0$ be arbitrary. Choose $N \in \mathbb{N}$ and a sequence $\{\delta_n\}_{n \geq N}$ of positive reals as in (b). By continuity of f_N at x_0, there exists a positive scalar $\delta \in (0, \delta_N)$ such that $d_X(x, x_0) < \delta$ implies $d_Y(f_N(x), f_N(x_0)) < \frac{\epsilon}{3}$. Then for $d_X(x, x_0) < \delta$,

$$d_Y(f(x_0), f(x)) \leq d_Y(f(x_0), f_N(x_0)) + d_Y(f_N(x_0), f_N(x)) + d_Y(f_N(x), f(x)) < \epsilon.$$

This ensures that f is continuous at x_0. □

Theorem 3.33 *Let (X, d) be a metric space and c be a limit point of X. Let $\{f_n\}$ be a sequence of real valued functions, uniformly convergent to a function f on X. If each f_n has a limit at c, then so does f. Moreover* $\lim_{x \to c} f(x) = \lim_{n \to \infty} \lim_{x \to c} f_n(x)$.

Proof Write $l_n := \lim_{x \to c} f_n(x)$, for each $n \in \mathbb{N}$. Applying the Cauchy criterion for uniform convergence, there exists some $m_0 \in \mathbb{N}$ such that

$$|f_{n_2}(x) - f_{n_1}(x)| < \frac{\epsilon}{3} \text{ for all } x \in E \text{ and for all } n_1, n_2 \geq m_0.$$

Passing limit $x \longrightarrow c$, we obtain $|l_{n_2} - l_{n_1}| \leq \frac{\epsilon}{3}$ for all $n_1, n_2 \geq m_0$. Therefore, $\{l_n\}$ is a Cauchy sequence of real numbers and hence convergent. Let $l := \lim_{n \to \infty} l_n$. Then there exists some $m = m(\epsilon) \in \mathbb{N}$ such that

$$|l_n - l| < \frac{\epsilon}{3} \text{ and } |f_n(x) - f(x)| < \frac{\epsilon}{3} \text{ for all } x \in E \text{ and for all } n \geq m.$$

Since f_m has limit l_m at c, there exists some $\delta > 0$ such that $|f_m(x) - l_m| < \frac{\epsilon}{3}$ for some $x \in X$ such that $0 < d(x, c) < \delta$. Therefore, if $0 < d(x, c) < \delta$, then

$$|f(x) - l| \leq |f(x) - f_m(x)| + |f_m(x) - l_m| + |l_m - l| < \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon.$$

Since $\epsilon > 0$ is arbitrary, the result follows. □

Remarks 3.34 (a) We have the following analogous relation between uniform convergence and Riemann integration (see [3, p. 151, Theorem 7.16]).
If $\{f_n\}$ *is a sequence of Riemann integrable functions, uniformly convergent to* f *on a closed bounded interval* $[a, b]$, *then* f *is also Riemann integrable on* $[a, b]$ *and*

$$\int_a^b f = \lim_{n\to\infty} \int_a^b f_n.$$

(b) In Exercises 3.101-3.102, we discuss upper and lower semi-continuous functions. If f is upper semi-continuous and g is lower semi-continuous such that $f \leq g$, then there exists a continuous h such that $f \leq h \leq g$. Further every upper semi-continuous function is the pointwise limit of a decreasing sequence of continuous functions (see [4, p. 61-63]).

Exercise 3.76 Prove that the constant map between any two metric spaces is continuous.

Exercise 3.77 Suppose (X, d) is a metric space such that $d(x, y) \geq 1$ for all $x, y \in X$. Does there exist any discontinuous function X?

Exercise 3.78 If $f : X \longrightarrow \mathbb{R}$ is continuous at some $c \in X$, prove that f is bounded in a neighborhood of c.

Exercise 3.79 Let X, Y be metric spaces, $c \in X'$ and $f : X \longrightarrow Y$. Suppose that $\lim_{x\to c} f(x) = l$ for some $l \in Y$. Prove the following:

(a) If $Y = \mathbb{R}$ and $l \neq 0$, then $f(x) \neq 0$, in a deleted neighborhood of c.
(b) If $l \neq l_0$, for some $l_0 \in Y$, then $f(x) \neq l_0$, in a deleted neighborhood of c.

Exercise 3.80 Let X, Y and Z be metric spaces, $f : X \longrightarrow Y$, $g : Y \longrightarrow Z$ be functions and $c \in X$ be such that f is continuous at c and g is continuous at $f(c)$. Prove that the composition function $g \circ f$ is continuous at c.

Exercise 3.81 Does there exist a metric d on $\mathbb{R}^n$, such that the identity map $f : (\mathbb{R}^n, d_2) \longrightarrow (\mathbb{R}^n, d)$ is discontinuous?

Exercise 3.82 Let X be any nonempty set and (Y, d) be any metric space. Does there exist a metric ρ on X such that every $f : (X, \rho) \longrightarrow (Y, d)$ becomes a continuous function?

Exercise 3.83 If $p : \mathbb{R}^m \longrightarrow \mathbb{R}$ is a polynomial, prove that p is continuous on $\mathbb{R}$.

Exercise 3.84 Let f and g be $\mathbb{R}^m \longrightarrow \mathbb{R}^n$ functions, continuous at some $c \in \mathbb{R}^m$. Prove that $\alpha f + \beta g$ is continuous at c for all $\alpha, \beta \in \mathbb{R}$,

Exercise 3.85 Let f and g be $\mathbb{R}^m \longrightarrow \mathbb{R}$ functions, continuous at $c \in \mathbb{R}^m$. Prove the following assertions:

(a) fg is continuous at c.
(b) If $f(c) \neq 0$, then $1/f$ is defined in a neighborhood of c.
(c) If $f(c) \neq 0$, then $1/f$ is continuous at c.

Exercise 3.86 Let X be a metric space, $c \in X$ and $f, g : X \longrightarrow \mathbb{R}^n$ be continuous at c. Let $x \cdot y$ denote the usual dot product of the vectors $x, y \in \mathbb{R}^n$. Prove that the function $f \cdot g : X \longrightarrow \mathbb{R}$ defined as $(f \cdot g)(x) := f(x) \cdot g(x)$ for all $x \in X$, is continuous at c.

Exercise 3.87 Define a sequence of $\mathbb{R} \longrightarrow \mathbb{R}$ functions $\{f_n\}$ as

$$f_n(x) := \frac{x^2}{(1+x^2)^n} \text{ for all } x \in \mathbb{R}, n \in \mathbb{N}.$$

Show that $\sum_{n=1}^{\infty} f_n$ is a series of continuous functions and converges pointwise to the discontinuous function f, where $f(0) = 0$ and $f(x) = 1$ for all $x \in \mathbb{R} \setminus \{0\}$.

Exercise 3.88 What is the relationship between strong pointwise convergence and pointwise convergence? Are these two equivalent? Justify your answer.

Exercise 3.89 Let $f : X \longrightarrow Y$ be any function, $A, B \subset X$ and Ω be a collection of subsets of X. Prove the following assertions:

(a) $f(\bigcup_{A\in\Omega} A) = \bigcup_{A\in\Omega} f(A)$.
(b) $f(A) \subset f(B)$, provided $A \subset B$.
(c) $f(A \cap B) \subset f(A) \cap f(B)$.
(d) Show that $f(A \cap B) = f(A) \cap f(B)$ is not true, in general.
(e) Show that $f(A^c) = (f(A))^c$ is not true, in general.

Exercise 3.90 Let $f : X \longrightarrow Y$ be any function, $A \subset Y$ and Ω be a collection of subsets of Y. Prove the following

(a) $f^{-1}(A^c) = (f^{-1}(A))^c$,
(b) $f^{-1}(\bigcup_{A\in\Omega} A) = \bigcup_{A\in\Omega} f^{-1}(A)$,
(c) $f^{-1}(\bigcap_{A\in\Omega} A) = \bigcap_{A\in\Omega} f^{-1}(A)$,

Exercise 3.91 Prove that the following are equivalent:

(a) $f : X \longrightarrow Y$ is continuous.
(b) $f^{-1}(F)$ is closed, for every closed subset F of Y.
(c) $f(\overline{E}) \subset \overline{f(E)}$ for all $E \subset X$.
(d) $\overline{f^{-1}(E)} \subset f^{-1}(\overline{E})$ for all $E \subset Y$.

Exercise 3.92 Let $A \subset X$ and $f : A \longrightarrow Y$. Prove that the following are equivalent:

(a) f is continuous on A.
(b) $f^{-1}(O)$ is open relative to A, for every open set O in Y.

(c) $f^{-1}(F)$ is closed relative to A, for every closed set F in Y.

Exercise 3.93 Let $f : X \longrightarrow Y$ and $c \in X$. Prove that the following are equivalent:

(a) f is continuous at c,
(b) for every $\epsilon > 0$, there exists a $\delta > 0$ such that $f(B(c; \delta)) \subset B(f(c); \epsilon)$,
(c) for every open set O_Y containing $f(c)$, there exists an open set O_X containing c such that $f(O_X) \subset O_Y$,
(d) for every open set O_Y containing $f(c)$, c is an interior point of the set $f^{-1}(O_Y)$,
(e) for every neighborhood N_Y of $f(c)$, $f^{-1}(N_Y)$ is a neighborhood of c,
(f) for every neighborhood N_Y of $f(c)$, there exists a neighborhood N_X of c such that $f(N_X) \subset N_Y$.

Exercise 3.94 Let $f : X \longrightarrow Y$ and $c \in X$. Prove that the following are equivalent:

(a) f has limit l at c,
(b) for every $\epsilon > 0$, there exists a $\delta > 0$ such that $f(B(c; \delta) \setminus \{c\}) \subset B(l; \epsilon)$,
(c) for every open set O_Y containing l, there exists some open set O_X containing c such that $f(O_X \setminus \{c\}) \subset O_Y$,
(d) for every neighborhood N_Y of l, there exists some neighborhood N_X of c with $f(N_X \setminus \{c\}) \subset N_Y$.

Exercise 3.95 Does there exist a sequence $\{f_n\}$ of nowhere continuous functions on $[0, 1]$, pointwise convergent to a function f such that f is continuous on $[0, 1]$?

Exercise 3.96 Does there exist a sequence of continuous functions on $[0, 1]$, pointwise convergent to a function discontinuous at infinitely many points?

Exercise 3.97 Let X, Y be metric spaces, $\{f_n\}$ be a sequence of $X \longrightarrow Y$ continuous functions, uniformly convergent on X to a function $f : X \longrightarrow Y$. Does this imply that f is continuous on X?

Exercise 3.98 Conclude Theorem 3.31 from Theorem 3.33.

Exercise 3.99 State and prove series analogues of Theorems 3.31 and 3.33.

Exercise 3.100 With $\{f_n\}$ of Exercise 1.117, show that $\lim_{x\to 0} \lim_{n\to\infty} f_n(x) \neq \lim_{n\to\infty} \lim_{x\to 0} f_n(x)$, although both of the above double limits are convergent.

Exercise 3.101 Let X be a metric space and $f : X \longrightarrow \mathbb{R}$. Then f is called *upper (lower) semi-continuous* if $\{x : f(x) < \alpha\}$ ($\{x : f(x) > \alpha\}$) is open, for all $\alpha \in \mathbb{R}$. Prove the following:

(a) f is continuous if and only if it is both upper as well as lower semi-continuous.
(b) If $S \subset X$, then the characteristic function χ_S upper (lower) semi-continuous if and only if S is closed (open).
(c) Let Ω be a collection of $X \longrightarrow \mathbb{R}$ functions.

  (i) If each $f \in \Omega$ is a lower semi-continuous function, then so is the mapping $x \longmapsto \sup\{f(x) : f \in \Omega\}$.

(ii) If each $f \in \Omega$ is an upper semi-continuous function, then so is the mapping $x \longmapsto \inf\{f(x) : f \in \Omega\}$.

Exercise 3.102 Let X be a metric space and $f : X \longrightarrow \mathbb{R}$. Prove that the following are equivalent:

(a) f is upper semi-continuous.
(b) If $x_n \longrightarrow x$, then $\limsup_{n\longrightarrow\infty} f(x_n) \leq f(x)$.
(c) The set $\{(x, r) \in X \times \mathbb{R} : r \leq f(x)\}$ is closed in $X \times \mathbb{R}$.

Exercise 3.103 Let $\{a_n\}$ be a sequence of complex numbers and $z_0 \in \mathbb{C} \setminus \{0\}$ such that the series $\sum_{n=0}^{\infty} a_n z_0^n$ is convergent. Prove the following:

(a) $\sum_{n=0}^{\infty} a_n z^n$ converges absolutely on $\{z : |z| < |z_0|\}$.
(b) There exists some $R \in (0, +\infty]$ such that $\sum_{n=0}^{\infty} a_n z^n$ is convergent on $\{z : |z| < R\}$ and non-convergent on $\{z : |z| > R\}$. (This R is called the *radius of convergence* of $\sum_{n=0}^{\infty} a_n z^n$.)
(c) $\sum_{n=0}^{\infty} |a_n z^n|$ converges uniformly on $\{z : |z| \leq r\}$, for every $r < |z_0|$.

Exercise 3.104 **(Sophomore's dream)** Prove the following identities:

$$\int_0^1 \frac{1}{x^x} dx = \sum_{n=1}^{\infty} \frac{1}{n^n} \text{ and } \int_0^1 x^x dx = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^n}.$$

3.6 Topology of Normed Linear Spaces

For this section, let X denote a normed linear space over the scalar field $\mathbb{R}$ or $\mathbb{C}$.

Theorem 3.35 *Let Y be a linear subspace of a normed space X. Then $Y = X$ if and only if $Y^o \neq \emptyset$.*

Proof The necessity is trivial. For the converse, suppose $Y^o \neq \emptyset$. Then Y contains a ball $B(y; r)$ for some $y \in Y$ and $r > 0$. Let $x \in X$ and write

$$z := y + \frac{r}{1 + \|x\|} x.$$

Then $z \in B(y; r) \subset Y$. Since Y is a linear subspace of X, we conclude that $x = \frac{(1+\|x\|)}{r}(z - y) \in Y$. Consequently, $X \subset Y$ and hence $Y = X$. □

In general metric spaces, the open and closed balls are related by the inclusion $\overline{B(x; r)} \subset B[x; r]$ and a strict inclusion may occur. Consider the following example for this purpose.

Example 3.36 If X is a discrete metric space having at least two points x and y, then $\overline{B(x; 1)} = \{x\} \neq X = B[x; 1]$.

However the equality holds here, in case of balls in normed spaces. To establish that we need a few notions, as follows: If X is a normed space, $x \in X$, $E \subset X$ and α be a scalar, we define

$$E + x := \{t + x : t \in E\} \text{ and } \alpha E := \{\alpha t : t \in E\}.$$

Lemma 3.37 *Let X be a normed space, $x \in X$ and $E \subset X$. Then $\overline{E + x} = \overline{E} + x$.*

Proof Let $y \in \overline{E + x}$. Choose a sequence $\{z_n\}$ from $E + x$ such that $z_n \longrightarrow y$. Then $\{z_n - x\}$ is a sequence from E, convergent to $y - x$. Therefore, $y - x \in \overline{E}$ and hence $y \in \overline{E} + x$. Therefore, $\overline{E + x} \subset \overline{E} + x$. The opposite inclusion is analogous. □

Theorem 3.38 *Let X be a normed space, $x \in X$ and $r > 0$. Then $\overline{B(x; r)} = B[x; r]$.*

Proof Note that $B[x; r]$ is a closed set, as $y \longmapsto \|y - x\|$ defines a continuous map on X. Since $B(x; r) \subset B[x; r]$, we have $\overline{B(x; r)} \subset B[x; r]$.

To prove the opposite inclusion, let $y \in B[0; r]$. Let $y_n := y(1 - \frac{1}{n})$ for all $n \in \mathbb{N}$. Then $\|y_n\| = (1 - \frac{1}{n})\|y\| < \|y\| \leq r$ which implies that $y_n \in B(0; r)$ for all $n \in \mathbb{N}$. Since $y_n \longrightarrow y$, we obtain $y \in \overline{B(0; r)}$. Hence $B[0; r] \subset \overline{B(0; r)}$.

Clearly, $B(x; r) = x + B(0; r)$ and $B[x; r] = x + B[0; r]$. Applying Lemma 3.37, we obtain $B[x; r] = B[0; r] + x \subset \overline{B(0; r)} + x = \overline{B(0; r) + x} = \overline{B(x; r)}$. □

Exercise 3.105 If $x_n \longrightarrow x$ in a normed space $(X, \|.\|)$, prove that $\|x_n\| \longrightarrow \|x\|$.

Exercise 3.106 Let $(X, \|.\|)$ be a normed space over $\mathbb{R}$. Prove that addition is a continuous function of $X \times X$ into X and scalar multiplication is continuous from $\mathbb{R} \times X$ into X.

Exercise 3.107 Let X be a normed linear space $x \in X$ and $r > 0$. Prove that $B(x; r) = \big(B(0; 1) + \frac{x}{r}\big)r = x + rB(0; 1)$.

Exercise 3.108 Prove that $(x, y) \longmapsto \min\{1, |x - y|\}$ defines a metric on $\mathbb{R}$. In this space, show that $\overline{B(0; 1)} \neq B[0; 1]$ by establishing $B(0; 1) = (-1, 1)$, $\overline{B(0; 1)} = [-1, 1]$, and $B[0; 1] = \mathbb{R}$.

Exercise 3.109 Let X be a normed space, $x \in X$ and $r > 0$. Prove that $(B[x; r])^o = B(x; r)$.

Exercise 3.110 Let Y be a linear subspace of a normed space X. Prove that $\overline{Y}$ is a normed space under the induced norm.

Exercise 3.111 Let X be a normed space over $\mathbb{C}$, $x \in X$, $E \subset X$ and $\alpha \in \mathbb{C} \setminus \{0\}$.

(a) If E is closed, then prove that so are $E + x$ and αE.
(b) If E is open, then prove that $E + x$ and αE are also open.

Exercise 3.112 Let X be an inner product space, $x_n \longrightarrow x$ and $y_n \longrightarrow y$ in X. Prove that $\langle x_n, y_n\rangle \longrightarrow \langle x, y\rangle$.

Exercise 3.113 Let X be a normed space, $x, y \in X$ and $r, s \in (0, \infty)$. Do we have

$$B(y; s) \cap \overline{B(x; r)} = \emptyset \text{ if and only if } \|x - y\| \geq r + s?$$

For the next two exercises, let Y be a linear subspace of a normed space X. For $a, b \in X$, set $a \sim b$ if and only if $a - b \in Y$. It is easy to see that $\sim$ defines an equivalence relation on X. Let X/Y be the collection of equivalence classes of X, under this relation.

Exercise 3.114 Let Y be closed in X. For every $[x] := x + Y \in X/Y$, define

$$\|[x]\| = |||x + Y||| := \inf\{x + y : y \in Y\}. \tag{3.1}$$

Prove that X/Y is a linear space and $|||.|||$ defines a norm on X/Y. (The norm on X/Y as in (3.1) is known as the *quotient norm* and X/Y is known as the *quotient space* of X with respect to the subspace Y.)

Exercise 3.115 Let Y be a closed subspace of X. Prove that $\{x_n + Y\} \longrightarrow x + Y$ in X/Y if and only if $\{x_n + y_n\} \longrightarrow x$ in X, for some sequence $\{y_n\}$ in Y.

3.7 Hints and Solutions to Selected Exercises

3.8 Apply Theorem 3.2 along with DeMorgan's law.

3.9 No. For example $\bigcup_{n=1}^{\infty}[0, 1 - 1/n] = [0, 1)$ is not a closed subset of reals.

3.15 Let $x \in X$ and $r > 0$. Pick any $y \in X \setminus B[x; r]$. Then $d(y, x) > r$. Let $\delta := d(y, x) - r > 0$. It is enough to show that $B(y; \delta) \subset X \setminus B[x; r]$.

To show this, let $z \in B(y; \delta)$. Then $d(z, y) < \delta = d(y, x) - r$. The triangle inequality implies $d(x, y) \leq d(x, z) + d(z, y)$. Therefore $d(x, z)) \geq d(x, y) - d(z, y) > d(x, y) - \delta = r$. Hence $z \in X \setminus B[x; r]$.

3.17 Write $E := \bigcap_{x\in E^c}\{x\}^c$.

3.27 Apply Theorem 3.11.

3.33 The statement is true. Assume its negation, that is, *there exists* $x \in X$ *such that for all* $\epsilon > 0$ *infinitely many* $n \in \mathbb{N}$ *satisfy* $d(x_n, x) < \epsilon$.

Let $E := \{x_n : n \in \mathbb{N}\}$. Then the above statement implies that $x \in \overline{E}$. Hence there exists a sequence in E, convergent to x.

By Exercise 1.40, rearrangements of convergent sequences are convergent to the same limit. Hence we obtain a subsequence of $\{x_n\}$ convergent to x, a contradiction.

3.38 Consider a ball in the discrete space with radius 1.

3.39 No. For example, take $X := [0, 1]$ and $A := (0, 1)$. Then $(\overline{A})^o = [0, 1] \neq A$.

3.47 If O is such an open set, let $x \in O \subset \overline{A}$. Then $O \cap A \neq \emptyset$, a contradiction.

3.48 Apply Exercise 3.47.

3.49 If possible, let $x \in (\overline{I})^o \cap (\overline{J})^o$. Then there exists an open set O such that $x \in O \subset \overline{I} \cap \overline{J}$. Since $O \subset \overline{I} = X \setminus J$, we obtain $O \cap J = \emptyset$. This is a contradiction as O is an open neighborhood of an adherent point of J.

3.50 Let $x \in O \cap \overline{E}$. We need to show that $x \in \overline{O \cap E}$. This is trivial if $x \in E$. Let $x \in \overline{E} \setminus E = E'$ and U be an open set containing x. Then $U \cap O$ is also an open set containing x. Since $x \in E'$, there exists $y \in (U \cap O) \cap E = U \cap (O \cap E)$ such that $y \neq x$. Hence $x \in (O \cap E)' \subset \overline{O \cap E}$.

3.52 Use the fact that $\partial E = \overline{E} \cap \overline{X \setminus E}$.

3.53 The strict inclusion occurs for $X := \mathbb{R}$, $A := [0, 2]$ and $B := \{1\}$. Let $x \in \overline{A} \setminus \overline{B}$ and U be an open set containing x. Then $U \setminus \overline{B}$ is an open set containing x. Therefore, there exists some $y \in A \cap (U \setminus \overline{B}) \subset (A \setminus B) \cap U$. Consequently, $x \in \overline{A \setminus B}$.

3.54 Let E_n denote the n^{th} set of this exercise. Since $p/q = 1/q + \cdots + 1/q$, so clearly $E_1 = \mathbb{Q} \cap (0, +\infty)$. Therefore, $\overline{E_1} = [0, +\infty) = E_1'$. Similarly, $\overline{E_2} = [0, +\infty) = E_2'$.

3.56 Since $F^o \subset F$, we obtain $E \subset \overline{F} = F$. Hence $E^o \subset F^o$. Let $x \in F^o$. Then there exists an open set O containing x such that $O \subset F$. Note that $O \subset F^o \subset E$ which implies $x \in E^o$. Thus $F^o \subset E^o$ and hence $F^o = E^o$.

3.57 Consider $E := \mathbb{Q}$ under usual metric on $\mathbb{R}$.

3.60 The first two assertions are evident from the definitions. The third one follows from $\overline{A \cup B} = \overline{A} \cup \overline{B}$. The last one holds if and only if $X \setminus \overline{A} \subset X \setminus \overline{(X \setminus \overline{(X \setminus A)})}$. That is $\overline{A} \supset \overline{(X \setminus \overline{(X \setminus A)})}$. With $A_1 := X \setminus A$, we need to show that $\overline{(X \setminus \overline{A_1})} \subset \overline{X \setminus A_1}$. This holds, as $\overline{X \setminus A_1}$ is a closed set and by Exercise 3.53, we have $\overline{X} \setminus \overline{A_1} \subset \overline{X \setminus A_1}$. The strict inclusion holds for $X := \mathbb{R}$ and $A := \mathbb{Q}$.

3.65 (a) Let $y \in \overline{B(x; r)}$ and $s \in (0, r)$. Then there exists $z \in B(y; s) \cap B(x; r)$. By 2.17(c), we obtain $B(y; s) \subset B(x; r)$. Thus $y \in B(x; r)$ and hence $\overline{B(x; r)} = B(x; r)$.

(b) Let $x, x' \in B[y; r]$. If there exists some $z \in B(x; r) \cap B(x'; r)$, applying 2.17(b) we obtain $B(x; r) = B(z; r) = B(x'; r)$. Therefore, Ω is a disjoint collection. Note that the inclusion $B[y; r] \subset \bigcup_{x \in B[y;r]} B(x; r)$ is obvious. Let $x \in B[y; r]$ and $z \in B(x; r)$. Then $B(z; r) = B(x; r) \subset B[y; r]$. The proves the opposite inclusion.

(c) Let x_1, x_2 denote the centers of balls B_1, B_2, respectively. Suppose that $dist(B_1, B_2) < r$. Then there are some $b_1 \in B_1$ and $b_2 \in B_2$ such that $d(b_1, b_2) < r$. Therefore, $d(b_1, x_2) \leq \max\{d(b_1, b_2), d(b_2, x_2)\} < r$. Hence $B_1 = B(x_1; r) = B(b_1; r) = B(x_2; r) = B_2$.

(d) By (b), $B[y; r]$ is a union of open sets and hence open.

(e) Follows from parts (a) and (d).

(f) Write $B := B(x; r)$ and $D := \sup\{d(y, z) : y, z \in B\}$. If $y, z \in B$, then $d(y, z) \leq \max\{d(y, x), d(x, z)\} \leq d(y, x)$. So $D \leq \sup\{d(x, y) : y \in B\}$.

The opposite inequality follows from the definition of D. Hence $D = \sup\{d(x, y) : y \in B\}$.

3.66 We prove the results for $r > 0$. The case of $r = 0$ is similar.

(a) Let $s \in (r^{n+1}, r^n]$ and x be as given. Then $y \in [a]$ if and only if y and x have first n terms common if and only if $\rho_r(y, x) \le r^{n+1} < s$. i.e. $y \in B(x; s)$.

(b) Let $n \in \mathbb{N}$ such that $r^{n+1} < s \le r^n$, and $a = (x_1, \dots, x_n)$. As above $[a] = B(x; s)$.

3.67 Clearly h is positive and symmetric. Let $A, B \in \mathcal{B}(X)$ such that $h(A, B) = 0$. Then

$$\sup\{dist(b, A) : b \in B\} = 0 = \sup\{dist(a, B) : a \in A\}.$$

This implies that $dist(b, A) = 0 = dist(a, B)$ for all $a \in A$ and $b \in B$. Hence $B \subset \overline{A} = A$ and $A \subset \overline{B} = B$. Consequently, $A = B$.

To prove the triangle inequality, let $A, B, C \in \mathcal{B}(X)$, $x \in A$ and $\epsilon > 0$ be arbitrary. Then there exists some $y \in B$ such that $d(x, y) < dist(x, B) + \frac{\epsilon}{2}$. Similarly, there exists some $z \in C$ such that $d(y, z) < dist(y, C) + \frac{\epsilon}{2}$. Therefore

$$\begin{aligned} dist(x, C) &\le d(x, z) \le d(x, y) + d(y, z) \\ &\le dist(x, B) + dist(y, C) + \epsilon \le h(A, B) + h(B, C) + \epsilon. \end{aligned}$$

Hence $\sup\{dist(a, C) : a \in A\} \le h(A, B) + h(B, C) + \epsilon$ for all $\epsilon > 0$, which implies $\sup\{dist(a, C) : a \in A\} \le h(A, B) + h(B, C)$. Similarly, $\sup\{dist(c, A) : c \in C\} \le h(A, B) + h(B, C)$. Hence $h(A, C) \le h(A, B) + h(B, C)$, as required.

3.71 Since A is closed in Y, there exists an open set $F \subset X$ such that $A = F \cap Y$. Pick any $x \in \overline{A} \cap Y$. Then there exists a sequence $\{x_n\}$ in A such that $x_n \longrightarrow x$. Therefore $x \in F$ and thus $x \in Y \cap F = A$.

3.74 Take $X := \mathbb{R}^2$, $Y := \mathbb{R} \times \{0\}$ and $A := [0, 1] \times \{0\}$. The corresponding inclusion $\emptyset \subset (0, 1) \times \{0\}$ is proper.

3.75 Take $X := \mathbb{R}^2$, $Y := \mathbb{R} \times \{0\}$ and $A := [0, 1] \times \{0\}$. The corresponding inclusion $\{0, 1\} \times \{0\} \subset (0, 1) \times \{0\}$ is proper.

3.95 Yes. Consider $f_n : [0, 1] \longrightarrow \mathbb{R}$ defined as follows:

$$f_n(x) := \begin{cases} \frac{1}{n} & ; x \in [0, 1] \cap \mathbb{Q}, \\ 0 & ; x \in [0, 1] \setminus \mathbb{Q}. \end{cases}$$

3.96 Shift and scale copies of $x \longrightarrow x^n$ in $[0, 1]$.

3.103 (a) By hypothesis, $\lim_{n\longrightarrow\infty} a_n z_0^n = 0$. Choose $m \in \mathbb{N}$ such that $|a_n z_0^n| < 1$ for all $n > m$. Then for any $z \in \mathbb{C}$ such that $|z| < |z_0|$, we obtain

$$|a_n z^n| = |a_n z_0^n| \left|\frac{z}{z_0}\right|^n < \left|\frac{z}{z_0}\right|^n \text{ for all } n > m.$$

Applying comparison test, with the geometric series $\sum |z/z_0|^n$, we obtain the required result.

(b) Take $R := \sup\{|z| : \sum_{n=1}^{\infty} a_n z^n \text{ is convergent}\}$.
(c) Fix any $r \in (0, |z_0|)$. As in (a), for sufficiently large n and $|z| \le r$, we obtain

$$|a_n z^n| = |a_n z_0^n| \left|\frac{z}{z_0}\right|^n < \left|\frac{z}{z_0}\right|^n \le \left|\frac{r}{z_0}\right|^n.$$

Apply M-test, along with the convergence of the geometric series $\sum |r/z_0|^n$.

3.104 Note that $x^x = e^{x \log x} = \sum_{n=0}^{\infty} \frac{(x \log x)^n}{n!}$ and that the series on the right converges uniformly on closed bounded intervals, which allows termwise integration. Therefore, with $x = e^{-u/(n+1)}$, we obtain

$$\int_0^1 (x \log x)^n dx = (-1)^n (n+1)^{-(n+1)} \int_0^{+\infty} u^n e^{-u} du = (-1)^n (n+1)^{-(n+1)} n!.$$

Hence we obtain $\int_0^1 x^x dx = \sum_{n=0}^{\infty} \frac{1}{n!} \int_0^1 (x \log x)^n dx = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^n}$. The first integral is analogous.

3.107 Let $y \in x + rB(0; 1)$. Then $y = x + tr$ for some $t \in B(0; 1)$. Thus $\|y - x\| = r\|t\| < r$ which implies $y \in B(x; r)$. All other inclusions are similar.

3.110 It is enough to show that $\overline{Y}$ is a linear subspace of X. Let $x, y \in \overline{Y}$, and a, b be scalars. Then there exist sequences $\{x_n\}$ and $\{y_n\}$ in Y, convergent to x and y, respectively. Therefore, $\{ax_n + by_n\}$ is a sequence in Y, convergent to $ax + by$. Hence $ax + by \in \overline{Y}$.

3.111 Let $y \in (E + x)'$. Choose a sequence $\{z_n\}$ of distinct points from $E + x$ such that $z_n \longrightarrow y$. Then $\{z_n - x\}$ is a sequence of distinct points from E, convergent to $y - x$. If E is closed, we have $y - x \in E$. Hence $y \in E + x$. This proves that $E + x$ is closed.

Let E be open and $y \in E + x$. Then $y - x \in E$. Since E is open there exists some $r > 0$ such that $B(y - x; r) \subset E$. Hence $B(y; r) \subset E + x$. This proves that $E + x$ is an open set. Similarly, one can prove the remaining parts.

3.113 True. By Theorem 3.38, we have $B(y; s) \cap \overline{B(x; r)} = B(y; s) \cap B[x; r]$. Suppose that $\|x - y\| < r + s$. Let $t := 1 - \frac{r}{\|x-y\|}$ and $z := tx + (1-t)y$. Then $\|z - x\| = |1 - t|\|x - y\| = r$ and $\|z - t\| = |t|\|x - y\| = \|x - y\| - r < r + s - r = s$. Hence $z \in B(y; s) \cap B[x; r]$.

Conversely, suppose $B(y; s) \cap B[x; r] \neq \emptyset$. Let $z \in B(y; s) \cap B[x; r]$. Thus $\|z - x\| \le r$ and $\|z - y\| < s$. By triangle inequality, $\|x - y\| < r + s$.

3.115 Suppose $\{x_n + Y\} \longrightarrow x + Y$ in X/Y. Then $|||(x_n - x) + Y||| \longrightarrow 0$, that is, $\|(x_n - x) + y_n\| \longrightarrow 0$, for a sequence $\{y_n\} \subset Y$. Hence $x_n + y_n \longrightarrow x$, in X, for a sequence $\{y_n\} \subset Y$.

Conversely, if $\{x_n + y_n\} \longrightarrow x$ in X, for some sequence $\{y_n\}$ in Y, then

$$|||(x_n + Y) - (x + Y)||| = |||(x_n - x) + Y||| \le \|x_n - x + y_n\| \longrightarrow 0.$$

Hence $\{x_n + Y\} \longrightarrow x + Y$ in X/Y.

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Chapter 4
Completeness

Roughly speaking, a metric space X is complete if every sequence in X, that attempts to converge, finds a buddy in X for that purpose. In other words, X is incomplete if it lacks some 'good' points. However, it is always possible to extend an incomplete space to a complete one, by appending all such missing 'good' points.

This chapter starts with a brief introduction to complete metric spaces, followed by its most important application; the Banach Contraction Principle. Then we provide various characterizations of completeness, in terms of Cantor intersection property and totally bounded sets. The completion of a metric space is discussed in a separate section where we establish that the Cauchy completion of $\mathbb{Q}$ is isometric to its Dedekind completion. Finally, we present various Banach spaces, including the space of continuous functions, and some results regarding absolute and unconditional convergence.

4.1 Introduction

A metric space X is said to be *complete* if every Cauchy sequence in X is convergent in X.

As a convention, we shall call the empty set to be complete. Therefore, a subset S of a metric space (X, d) will be called *complete* if either $S = \emptyset$ or (S, d) forms a complete subspace.

Examples 4.1 (a) Every finite-dimensional Euclidean space is complete.
(b) $(0, 1)$ is not a complete subspace of $\mathbb{R}$.
(c) Every discrete metric space is complete.

Proof (a) By Theorem 2.26, every Cauchy sequence in $\mathbb{R}^m$ is convergent in $\mathbb{R}^m$.
(b) Note that $\{1/n\}$ is a Cauchy sequence, but not convergent in $(0, 1)$.
(c) If $\{x_n\}$ is a Cauchy sequence in a discrete metric space (X, d), then there exists some $N \in \mathbb{N}$ such that for all $n, m \geq N$, we have $d(x_n, x_m) < 1$ which implies $x_n = x_m$. Therefore, $\{x_n\}$ is eventually constant and hence convergent in X. □

S. P. S. Kainth, *A Comprehensive Textbook on Metric Spaces*,
https://doi.org/10.1007/978-981-99-2738-8_4

Theorem 4.2 *If Y is a complete subspace of a metric space X, then Y is a closed subset of X.*

Proof Pick any $x \in \overline{Y}$. Then there exists a sequence $\{y_n\}$ from Y such that $y_n \longrightarrow x$. Since $\{y_n\}$ is convergent in X, it must be a Cauchy sequence. Since Y is complete, there exists some $y \in Y$ such that $y_n \longrightarrow y$. Since $y_n \longrightarrow x$, by Theorem 2.23, $x = y \in Y$. Hence, $\overline{Y} \subset Y$, that is, Y is a closed subset of X. □

Therefore, complete subspaces are closed subsets. The converse is false, as (0, 1) is a closed subset of itself, but not complete. However, closed subspaces of complete spaces are complete.

Theorem 4.3 *If Y is a closed subset of a complete metric space X, then Y is complete.*

Proof If $Y = \emptyset$, then it is complete by definition. Suppose $Y \neq \emptyset$. Let $\{y_n\}$ be a Cauchy sequence in Y. Then $\{y_n\}$ is also a Cauchy sequence in X. Since X is complete, there exists $x \in X$ such that $y_n \longrightarrow x$. Since a sequence from Y converges to x, we have $x \in \overline{Y}$. Since Y is closed, $\overline{Y} = Y$ and thence $x \in Y$. So $\{y_n\}$ converges to a point of Y. Hence, Y is complete. □

We will provide various characterizations of completeness in exercises and in Sect. 4.3.

Remark 4.4 The fundamental fact that any two non-parallel planar lines intersect and the Euclidean construction of an equilateral triangle with a given line segment as its base depends on the completeness property of the Euclidean plane.

Exercise 4.1 Prove that $\mathbb{N}$ is a complete subspace of $\mathbb{R}$, while $\mathbb{Q}$ is not complete.

Exercise 4.2 Prove that completeness is independent of the embedded space, that is, if $E \subset Y \subset X$, then E is complete in X if and only if E is complete in Y.

Exercise 4.3 Let $d(x, y) := |\tan^{-1} x - \tan^{-1} y|$ for all $x, y \in \mathbb{R}$. Prove that d is a metric on $\mathbb{R}$. Is $(\mathbb{R}, d)$ a complete metric space?

Exercise 4.4 Let $X := [1, +\infty)$. Prove that $d(x, y) := \left|\frac{1}{x} - \frac{1}{y}\right|$ for all $x, y \geq 1$ defines a metric on X. Is (X, d) a complete metric space?

Exercise 4.5 Let $\{x_n\}$ be a sequence from a metric space (X, d). Assume that there exists some $x_0 \in X$ such that $x_n \longrightarrow x_0$. Prove that $\{x_n : n \geq 0\}$ is a complete subspace of (X, d).

Exercise 4.6 Prove that a metric space X is complete if and only if every countable closed subset of X is complete.

Exercise 4.7 Prove that a metric space X is complete if and only if every closed ball in X is a complete subspace of X.

Exercise 4.8 Let X be a metric space and $x \in X$. Prove that X is complete if and only if for every closed ball of X, centered at x is a complete subspace.

Exercise 4.9 Let (X, d) be a metric space. Prove that $\sqrt{d}$ is also a metric on X. Is it true that (X, d) is complete if and only if $(X, \sqrt{d})$ is complete?

Exercise 4.10 Let $\{x_n\}$ and $\{y_n\}$ be Cauchy sequences in a complete metric space. Prove that these have same limit if and only if $d(x_n, y_n) \longrightarrow 0$.

Exercise 4.11 Let X be a metric space. Prove that

(a) finite union of complete subspaces of X is a complete subspace.
(b) arbitrary intersection of complete subspaces of X is a complete subspace.

Exercise 4.12 Let (X, d) be a metric space and

$$\rho(x, y) := \frac{d(x, y)}{1 + d(x, y)} \text{ for all } x, y \in X.$$

Let $\{x_n\}$ be a sequence in X and $x \in X$. Prove the following:

(a) ρ is also a metric on X.
(b) $\{x_n\}$ is Cauchy in (X, d) if and only if $\{x_n\}$ is Cauchy in (X, ρ).
(c) $x_n \longrightarrow x$ in (X, d) if and only if $x_n \longrightarrow x$ in (X, ρ).
(d) (X, d) is complete if and only if (X, ρ) is complete.

Exercise 4.13 Let (X, d) be a metric space and

$$\eta(x, y) := \min\{1, d(x, y)\} \text{ for all } x, y \in X.$$

Let $\{x_n\}$ be a sequence in X and $x \in X$. Prove the following:

(a) η is also a metric on X.
(b) $\{x_n\}$ is Cauchy in (X, d) if and only if $\{x_n\}$ is Cauchy in (X, η).
(c) $x_n \longrightarrow x$ in (X, d) if and only if $x_n \longrightarrow x$ in (X, η).
(d) (X, d) is complete if and only if (X, η) is complete.

Exercise 4.14 Prove that a metric space X is complete if and only if every sequence $\{x_n\}$ in X satisfying $\sum_{n=1}^{\infty} d(x_{n+1}, x_n) < \infty$, is convergent in X.

Exercise 4.15 Let X, Y be metric spaces such that Y is complete. Let c be a limit point of X and $\{f_n\}$ be a sequence of $X \longrightarrow Y$ functions, uniformly convergent to a function f on X. If each f_n has a limit at c, then prove that so does f. Further show that

$$\lim_{x \to c} f(x) = \lim_{n \to \infty} \lim_{x \to c} f_n(x).$$

Exercise 4.16 For $x, y \in \mathbb{R} \setminus \mathbb{Q}$, let $d(x, y) = 0$ if $x = y$. Otherwise, define $d(x, y) := \frac{1}{n+1}$, where n is the first index where the continued fraction expansions of x and y differ. Prove that d is a complete metric on $\mathbb{R} \setminus \mathbb{Q}$.

Exercise 4.17 Prove that the ultrametric space (X, ρ_r) of Example 2.5 is complete.

Exercise 4.18 Let X, Y be metric spaces and $f : X \longrightarrow Y$. If f maps Cauchy sequences onto Cauchy sequences, prove that f is a continuous function on X. When does the converse hold?

Exercise 4.19 **(Snipes, 1977** [1]**)** Let (X, d), (Y, ρ) be metric spaces such that Y is complete. Let $A \subset X$ such that $\overline{A} = X$, and let $f : A \longrightarrow Y$ maps Cauchy sequences onto Cauchy sequences. Prove that there exists a continuous function $F : X \longrightarrow Y$ such that $F|_A = f$.

4.2 Banach Contraction Principle

Solving an equation of the form $f(x) = 0$, is an important task in mathematics. It is equivalent to solving an equation of the form $g(x) = x$. If f is a 'nice' function on a complete metric space, by Banach contraction principle, such solutions exist.

Definition 4.5 Let A be any set and $f : A \longrightarrow A$ be a function. A point $a \in A$ is said to be a *fixed point* of f, if $f(a) = a$.

Definition 4.6 Let (X, d) be a metric space. A function $f : (X, d) \longrightarrow (X, d)$ is said to be a *strict contraction,* if it satisfies the Lipschitz condition with some *Lipschitz constant* less than 1, that is, if there exists some $\alpha \in [0, 1)$ such that

$$d\big(f(x), f(y)\big) \leq \alpha d(x, y) \text{ for all } x, y \in X.$$

We now present the *Banach contraction principle,* also known as the *Banach fixed point theorem, the contraction mapping theorem, shrinking lemma, Banach-Cacciopoli principle, and the contraction principle.*

Theorem 4.7 **(Banach, 1922)** *Every strict contraction on a complete metric space has a unique fixed point.*

Proof Let (X, d) be a complete metric space and $f : (X, d) \longrightarrow (X, d)$ be a strict contraction. Then there exists $\alpha \in [0, 1)$ such that

$$d(f(x), f(y)) \leq \alpha d(x, y) \text{ for all } x, y \in X. \tag{4.1}$$

The *uniqueness part* is clear, as if x and y are two distinct fixed points, then

$$d(x, y) = d(f(x), f(y)) \leq \alpha d(x, y) < d(x, y),$$

a contradiction. For the *existence part,* let $x_0 \in X$. We claim that the sequence $\{f^n(x_0)\}$ is Cauchy. Applying induction on (4.1), for any $n \in \mathbb{N}$, we obtain

$$d(f^{n+1}(x_0), f^n(x_0)) \le \alpha d(f^n(x_0), f^{n-1}(x_0)) \le \cdots \le \alpha^n d(f(x_0), x_0) = \alpha^n D_0,$$

where $D_0 := d(f(x_0), x_0)$. Let $\epsilon > 0$ be given. Since $\alpha \in [0, 1)$, the series $\sum_{n=1}^{\infty} \alpha^n$ converges. By the Cauchy criterion, there exists some $m \in \mathbb{N}$ such that

$$\sum_{n=n_1}^{n_2-1} \alpha^n < \frac{\epsilon}{1 + D_0} \text{ for all } n_2 > n_1 > m.$$

Hence, for all $n_2 > n_1 > m$, the triangle inequality ensures that

$$d\big(f^{n_2}(x_0), f^{n_1}(x_0)\big) \le \sum_{n=n_1}^{n_2-1} d\big(f^{n+1}(x_0), f^n(x_0)\big) \le D_0 \sum_{n=n_1}^{n_2-1} \alpha^n < \epsilon. \tag{4.2}$$

Therefore, the sequence $\{f^n(x_0)\}$ is Cauchy and hence convergent. Let $x \in X$ be such that $\{f^n(x_0)\} \longrightarrow x$. By (4.1), the function f is continuous at x. Since $\{f^n(x_0)\} \longrightarrow x$, we have $\{f(f^n(x_0))\} \longrightarrow f(x)$. By Theorem 2.23, we have

$$f(x) = \lim_{n\to\infty} f^{n+1}(x_0) = \lim_{n\to\infty} f^n(x_0) = x.$$

Hence, x is a fixed point of f. □

Corollary 4.8 *Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be a differentiable function such that there exists some $\alpha \in [0, 1)$ such that $|f'| < \alpha$, on $\mathbb{R}$. Then f has a unique fixed point.*

Proof Applying Mean Value Theorem, we obtain

$$|f(y) - f(x)| < \alpha |y - x| \text{ for all } x, y \in \mathbb{R}.$$

Hence, f is a strict contraction and the result follows by Theorem 4.7. □

Example 4.9 The polynomial $x^6 + 7x - 1$ has a unique solution in $[0, 1]$.

Proof Let $f(x) := \frac{1-x^6}{7}$ for all $x \in [0, 1]$. Note that the fixed points of f are essentially roots of the given polynomial. Further, we have

$$|f'(x)| \le \frac{6}{7} < 1 \text{ for all } x \in [0, 1].$$

By Corollary 4.8, f has a unique fixed point in $[0, 1]$. Hence the result. □

Remark 4.10 The proof of Theorem 4.7 provides an iterative method to approximate fixed points. The fixed point of f is essentially the limit of the sequence $\{f^n(a)\}$

for every $a \in X$. Also, from (4.2), we obtain a bound on the error at the n^{th}-iteration, given by

$$d(f^n(x_0), x) \leq D_0 \sum_{k=n}^{\infty} \alpha^k = D_0 \frac{\alpha^n}{1-\alpha} \text{ for all } n \in \mathbb{N}.$$

Remarks 4.11 The Banach Contraction Principle (4.7) has various applications. It is used in the proofs of the inverse function theorem for multivariate functions; and the Picard-Lindelöf theorem about the existence and uniqueness of solutions to certain ordinary differential equations (see [2, p. 221, Theorem 9.24] and [3, p. 12, Theorem 3.1]).

There is a major branch of analysis focused on fixed point theorems. In [4], chapter 15 is devoted to several types of such theorems.

Exercise 4.20 Is $f : (X, d) \longrightarrow (X, d)$ a strict contraction, if

$$d(f(x), f(y)) < d(x, y) \text{ for all } x, y \in X?$$

Exercise 4.21 A function $f : X \longrightarrow X$ is said to be a *shrinking map,* if

$$d(f(x), f(y)) \leq d(x, y) \text{ for all } x, y \in X.$$

Prove that $x \longmapsto x - x^2/2$ is a shrinking map, but not a contraction on $[0, 1]$.

Exercise 4.22 Show that the function $f : [1, +\infty) \longrightarrow [1, +\infty)$ defined as $f(x) := x + \frac{1}{x}$ is not a strict contraction.

Exercise 4.23 Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be defined as $f(x) := x - \log(1 + e^x)$ for all $x \in \mathbb{R}$. Prove that $|f(x) - f(y)| < |x - y|$ for all $x, y \in \mathbb{R}$ and that f has no fixed point.

Exercise 4.24 Check whether the following functions are strict contractions on $\mathbb{R}$ or not? If yes, also find the corresponding fixed points

$$f_1(x) := \sqrt{x^2 + 1},\ f_2(x) := \sqrt{|x| + 1} \text{ and } f_3(x) := \frac{\pi}{2} + x - \tan^{-1} x.$$

Exercise 4.25 In Theorem 4.7, show that the conclusion won't hold if instead of a strict contraction, we assume $d(f(x), f(y)) < d(x, y)$ for all $x, y \in X$.

Exercise 4.26 In Theorem 4.7, show that the completeness of X is not redundant.

Exercise 4.27 Does there exist any incomplete metric space X with a strict contraction $f : X \longrightarrow X$ such that f has two different fixed points?

Exercise 4.28 Let X be a complete metric space and $f : X \longrightarrow X$ be a continuous function such that f^k is a strict contraction, for some $k \in \mathbb{N}$. Prove that f has a unique fixed point.

Exercise 4.29 Let X be a complete metric space, $\sum_{n=1}^{\infty} c_n < \infty$ and $f : X \longrightarrow X$ be a continuous function such that

$$d(f^n(x), f^n(y)) \leq c_n d(x, y) \text{ for all } x, y \in X \text{ and for all } n \in \mathbb{N}.$$

Prove that f has a unique fixed point.

4.3 Characterizations of Completeness

In this section, we provide several characterizations of completeness. These arise from two fundamental concepts: the Cantor Intersection Property and the notion of total boundedness.

4.3.1 Cantor Intersection Property

Motivated by the Nested Interval Property (1.23), we define the Cantor Intersection Property and will prove it to be equivalent to the notion of completeness.

A sequence of sets $\{E_n\}$ is called a *nested decreasing sequence,* if

$$E_n \supset E_{n+1} \text{ for all } n \in \mathbb{N}.$$

Let (X, d) be a metric space and $E \subset X$. The *diameter* of E is defined as

$$diam(E) := \sup\{d(x, y) : x, y \in E\}.$$

Definition 4.12 A metric space X is said to satisfy the *Cantor Intersection Property* if for every nested decreasing sequence $\{F_n\}$ of nonempty closed subsets of X such that $diam(F_n) \longrightarrow 0$, the intersection $\bigcap_{n=1}^{\infty} F_n$ is a singleton set.

We shall prove that the completeness of a metric space is equivalent to the Cantor Intersection Property. For that purpose, the following simple lemma is required.

Lemma 4.13 *Let (X, d) be a metric space and A, B be subsets of X. Then*

(a) $A \subset B$ implies $diam(A) \leq diam(B)$ and
(b) $diam(\overline{A}) = diam(A)$.

Proof Part (a) is immediate from the definition of the diameter of a set. Hence, for part (b), it is enough to prove that $diam(\overline{A}) \leq diam(A)$. Let $\epsilon > 0$ be given.

Pick any $x, y \in \overline{A}$. Then there are some $x', y' \in A$ such that $d(x, x') < \epsilon$ and $d(y, y') < \epsilon$. Hence,

$$d(x, y) \leq d(x, x') + d(x', y') + d(y', y) < 2\epsilon + diam(A).$$

Since $\epsilon > 0$ was arbitrary, we obtain $d(x, y) \leq diam(A)$ for all $x, y \in \overline{A}$. Hence, $diam(\overline{A}) \leq diam(A)$ and the result follows. □

Theorem 4.14 *A metric space X is complete if and only if for every nested decreasing sequence $\{F_n\}$ of nonempty closed subsets of X with $diam(F_n) \longrightarrow 0$, we have $\bigcap_{n=1}^{\infty} F_n \neq \emptyset$.*

Proof Assume that X is a complete metric space. Let $\{F_n\}$ be a nested decreasing sequence of nonempty closed subsets of X such that $diam(F_n) \longrightarrow 0$. For each $n \in \mathbb{N}$, choose $x_n \in F_n$.

We claim that the sequence $\{x_n\}$ is Cauchy. Let $\epsilon > 0$ be given. Since $diam(F_n) \longrightarrow 0$, there exists some $m \in \mathbb{N}$ such that $diam(F_n) < \epsilon$ for all $n \geq m$. For any $n_2 > n_1 \geq m$, we have $F_n \supset F_{n+1}$ which implies $x_{n_1}, x_{n_2} \in F_m$ and therefore, $d(x_{n_2}, x_{n_1}) \leq diam(F_m) < \epsilon$. Hence, $\{x_n\}$ is a Cauchy sequence in X.

Since X is complete, there exists some $x \in X$ such that $x_n \longrightarrow x$. We claim that $x \in \bigcap_{n=1}^{\infty} F_n$. Pick any $n \in \mathbb{N}$. Since F_n is closed, $\overline{F_n} = F_n$. Then, as above, $x_k \in F_n$ for all $k \geq n$. Thus, F_n contains the sequence $\{x_k : k \geq n\}$, convergent to x. Therefore, $x \in \overline{F_n} = F_n$. Hence, $x \in \bigcap_{n=1}^{\infty} F_n$.

Conversely, let $\{x_n\}$ be a Cauchy sequence in X and $\epsilon > 0$ be given. Then there exists some $m \in \mathbb{N}$ such that $d(x_{n_2}, x_{n_1}) < \epsilon$ for all $n_2 > n_1 \geq m$. For each $n \in \mathbb{N}$, consider the set $E_n := \{x_k : k \geq n\}$. Applying Lemma 4.13, we obtain $diam(\overline{E_n}) = diam(E_n) \leq \epsilon$ for all $n \geq m$.

Hence, $\{\overline{E_n}\}$ is a nested decreasing sequence of nonempty closed subsets of X such that $diam(\overline{E_n}) \longrightarrow 0$. By hypothesis $\bigcap_{n=1}^{\infty} \overline{E_n} \neq \emptyset$. Let $x \in \bigcap_{n=1}^{\infty} \overline{E_n}$.

Also, for any $n \geq m$, we have $x, x_n \in \overline{E_m}$ which implies $d(x, x_n) \leq diam(\overline{E_m}) = diam(E_m) \leq \epsilon$. Hence, $x_n \longrightarrow x$ and the result follows. □

Corollary 4.15 *A metric space X is complete if and only if X satisfies the Cantor Intersection Property.*

Proof The converse is immediate by Theorem 4.14. For the only if part, let $\{F_n\}$ be a nested decreasing sequence of nonempty closed subsets of X such that $diam(F_n) \longrightarrow 0$. Again by Theorem 4.14, we have $\bigcap_{n=1}^{\infty} F_n \neq \emptyset$.

Let $x, y \in \bigcap_{n=1}^{\infty} F_n$. Then for all $n \in \mathbb{N}$, we have $x, y \in F_n$ which implies that $0 < d(x, y) \leq diam(F_n)$. Since $diam(F_n) \longrightarrow 0$, we obtain $d(x, y) = 0$. Hence, $x = y$ and the result follows. □

The Cantor Intersection Property is also known as the *nested set property*. Another related characterization of completeness will be provided in Theorem 5.20. Also, see Exercise 7.31.

4.3.2 *Totally Bounded Sets*

Definition 4.16 A subset E of a metric space X is said to be *totally bounded* if for every $\epsilon > 0$, there exist finitely many $x_1, \ldots, x_n \in X$ such that $E \subset \bigcup_{i=1}^{n} B(x_i; \epsilon)$.

Examples 4.17 (a) Any finite space is totally bounded.
(b) The set of natural numbers $\mathbb{N}$ is not totally bounded.
(c) Every subset of a totally bounded set is totally bounded.
(d) The set of real numbers $\mathbb{R}$ is not totally bounded.

Proof (a) If X is a finite metric space, then for every $\epsilon > 0$, we have $X = \bigcup_{x \in X} B(x; \epsilon)$ and hence X is totally bounded.
(b) If $\mathbb{N}$ is totally bounded, there exist finitely natural numbers $n_1 < \cdots < n_k$ such that

$$\mathbb{N} \subset \bigcup_{i=1}^{k} \{x \in \mathbb{N} : |x - n_j| < 1\} = \bigcup_{i=1}^{k} \{n_j\} = \{n_1, \ldots, n_k\}.$$

This implies that $\mathbb{N}$ is bounded above by n_k, a contradiction.
(c) Immediate from Definition 4.16.
(d) If $\mathbb{R}$ is totally bounded, by (c), then so is its subspace $\mathbb{N}$. This contradicts (b). □

Theorem 4.18 *Totally bounded subsets of metric spaces are bounded.*

Proof Let E be a totally bounded subset of a metric space (X, d). Then there are finitely many $x_1, \ldots, x_n \in X$ such that $E \subset \bigcup_{i=1}^{n} B(x_i; 1)$. Let

$$M := \max\{d(x_1, x_i) : i = 1, \ldots, n\} + 1.$$

If $x \in E$, then there exists some i such that $x \in B(x_i; 1)$ and therefore, we have $d(x, x_1) \leq d(x, x_i) + d(x_i, x_1) < M$. That is $x \in B(x_1; M)$. Consequently, $E \subset B(x_1; M)$ and hence E is a bounded subset of X. □

The converse of Theorem 4.18 is not true.

Example 4.19 Let $X = \mathbb{N}$, equipped with the discrete metric. Then X is bounded, but not totally bounded.

However, the converse holds for subspaces of finite-dimensional Euclidean spaces.

Theorem 4.20 *Bounded subsets of* $\mathbb{R}^n$ *are totally bounded.*

Proof Let E be a bounded subset of $\mathbb{R}^n$ and let $\epsilon > 0$ be given. Since E is bounded, there exists some $R > 0$ such that $E \subset B(0; R)$. Then $E \subset B(0; R) \subset \prod_{i=1}^{n} [-R, R]$.

Let $m \in \mathbb{N}$ be such that $\sqrt{n} R/m < \epsilon$. Partition the interval $[-R, R]$ into $2m$ intervals of equal length R/m. Therefore, our hypercube is divided into $(2m)^n$ smaller hypercubes, each having diameter $\sqrt{n} R/m < \epsilon$. Let $x_1, x_2, \ldots, x_{(2m)^n}$ be any arbitrary points, chosen one from each of these hypercubes. Then we have $E \subset \bigcup_{k=1}^{(2m)^n} B(x_i; \epsilon)$. Hence the result. □

Lemma 4.21 *Let* $\{x_n\}$ *be any sequence in a metric space* (X, d), $\{y_n\}$ *be any permutation of* $\{x_n\}$, *and* $x \in X$.

(a) If $\{x_n\}$ *is Cauchy, then so is* $\{y_n\}$.
(b) If $x_x \longrightarrow x$, *then* $y_n \longrightarrow x$.

Proof We prove part (a) only, as (b) is analogous. Let $\epsilon > 0$ be given, and assume that $\{x_n\}$ is Cauchy. Then there exists some $m \in \mathbb{N}$ such that $d(x_{n_2}, x_{n_1}) < \epsilon$ for all $n_2 > n_1 > m$. Since $\{y_n\}$ is a permutation of $\{x_n\}$, there exists a bijection $\tau : \mathbb{N} \longrightarrow \mathbb{N}$ such that

$$y_n = x_{\tau(n)} \text{ for all } n \in \mathbb{N}.$$

Let $M := \max\{\tau^{-1}(1), \ldots, \tau^{-1}(m)\}$. Then for $n_2 > n_1 > M$, both $\tau(n_2)$ and $\tau(n_1)$ are greater than m. Consequently,

$$d(y_{n_2}, y_{n_1}) = d(x_{\tau(n_2)}, x_{\tau(n_1)}) < \epsilon.$$

Hence, $\{y_n\}$ is a Cauchy sequence. □

Theorem 4.22 *A metric space X is totally bounded if and only if every sequence in X has a Cauchy subsequence.*

Proof The converse is easy. Assume that X is not totally bounded. Then there exists some $\epsilon > 0$ such that X cannot be contained in any finite union of balls with radius ϵ. Let $x_1 \in X$. If $x_1, x_2 \ldots, x_n$ are chosen from X, then $X \not\subset \bigcup_{j=1}^{n} B(x_j; \epsilon)$. Choose $x_{n+1} \in X \setminus \bigcup_{j=1}^{n} B(x_j; \epsilon)$. Therefore, for all $n_2 > n_1$, we have $x_{n_2} \notin B(x_{n_1}; \epsilon)$, that is, $d(x_{n_2}, x_{n_1}) \geq \epsilon$. Hence, $\{x_n\}$ has no Cauchy subsequence.

Toward the necessity part, suppose that (X, d) is totally bounded. Let $\{a_n\}$ be a sequence in X and $E := \{a_n : n \in \mathbb{N}\}$. If E is a finite set, at least one term of $\{a_n\}$ repeats infinitely often. Consequently, it has a convergent subsequence, essentially a corresponding constant subsequence.

Now consider the case when E is an infinite set. Being a subset of a totally bounded space, E is totally bounded. So there exists a finite set $F_1 \subset E$ such that $E \subset \bigcup_{x \in F_1} B(x; 1)$. Since E is an infinite set, there exists some $x_1 \in F_1$ such that $E \cap B(x_1; 1)$ is infinite.

Note that the subset $E \cap B(x_1; 1) \setminus \{x_1\}$ of E is infinite and totally bounded. Hence, it contains a finite subset F_2 and is contained in the union $\bigcup_{x \in F_2} B(x; 1/2)$. Thus, there exists $x_2 \in F_2$ such that $E \cap B(x_1; 1) \cap B(x_2; 1/2)$ is an infinite set.

Continuing like this, we obtain a sequence $\{x_n\}$ of distinct terms from E such that

$$x_n \in E \cap \Big(\bigcap_{i<n} B\big(x_i; 1/i\big)\Big) \text{ for all } n \in \mathbb{N}.$$

Therefore, for every $i \geq n \geq 1$, we have $x_i \in B\big(x_n; 1/n\big)$. Hence, if $i > j \geq n \geq 1$, we obtain

$$d(x_i, x_j) \leq d(x_i, x_n) + d(x_n, x_j) \leq \frac{2}{n}.$$

Hence, $\{x_n\}$ is Cauchy in X. Since $\{x_n\}$ is a sequence of distinct terms from E, it is a permutation of some subsequence of $\{a_n\}$, say $\{a_{n_k}\}$. Since $\{x_n\}$ is Cauchy, by Lemma 4.21, $\{a_{n_k}\}$ is also a Cauchy subsequence of $\{a_n\}$. Hence the result. □

Theorem 4.23 *A metric space X is complete if and only if every infinite totally bounded subset of X has a limit point in X.*

Proof Let X be complete and E be an infinite totally bounded subset of X. Choose a sequence $\{x_n\}$ of distinct terms from E. By Theorem 4.22, $\{x_n\}$ has a Cauchy subsequence, say $\{x_{n_k}\}$. Since X is complete, there exists some $x \in X$ such that $x_{n_k} \longrightarrow x$. Hence, x is a limit point of E, in X.

Conversely, let $\{x_n\}$ be a Cauchy sequence in X and $E := \{x_n : n \in \mathbb{N}\}$. If E is a finite set, as in Theorem 4.22, $\{x_n\}$ has a convergence subsequence.

Consider the case when E is an infinite set. By Lemma 4.21 and Theorem 4.22, the set E is totally bounded. By hypothesis, E has a limit point in X, say x. Therefore, there exists a sequence in E, say $\{y_n\}$, convergent to x. Note that this sequence is a permutation of some subsequence of $\{x_n\}$, say $\{x_{n_k}\}$. Applying Lemma 4.21 again, we have $\{x_{n_k}\} \longrightarrow x$.

Hence, in both cases, the Cauchy sequence $\{x_n\}$ has a subsequence, convergent in X. Applying Theorem 2.24, $\{x_n\}$ is convergent in X. Hence the result. □

Note that the Bolzano-Weierstrass Property (2.27) becomes an immediate consequence of Theorems 4.20 and 4.23.

Corollary 4.24 *Every infinite bounded subset of $\mathbb{R}^n$ has a limit point in $\mathbb{R}^n$.*

Note that the direct analogue of Theorem 1.23 is not true in general metric space.

Example 4.25 Consider $X = \mathbb{N}$, equipped with the discrete metric and let $F_n := \{m \in X : m \geq n\}$ for all $n \in \mathbb{N}$. Then $\{F_n\}$ is a nested decreasing sequence of nonempty closed bounded subsets of X while $\bigcap_{n=1}^{\infty} F_n = \emptyset$. (Note that here X is even complete).

However, we have the following generalization of Theorem 1.23.

Theorem 4.26 *A metric space X is complete if and only if the intersection of every nested decreasing sequence of nonempty closed totally bounded subsets of X is nonempty.*

Proof Assume that X is complete and $\{F_n\}$ be a nested decreasing sequence of nonempty closed totally bounded subsets of X. For each $n \in \mathbb{N}$, choose $x_n \in F_n$. Write $E := \{x_n : n \in \mathbb{N}\}$. If E is finite, as in Theorem 4.22, $\{x_n\}$ has a convergent subsequence. Otherwise, by Theorem 4.23, E has a limit point, say x. Hence, $\{x_n\}$ has a some subsequence $\{x_{n_k}\}$, convergent to x. Note that $\{x_{n_k}\}$ is eventually in each F_n. Since each F_n is closed, we obtain $x \in \bigcap_{n=1}^{\infty} F_n$.

Conversely, let $\{x_n\}$ be a Cauchy sequence in X. Define $F_n := \overline{\{x_k : k \geq n\}}$ for all $n \in \mathbb{N}$. Then each F_n is totally bounded. By hypothesis, let $x \in \bigcap_{n=1}^{\infty} F_n$. Since $\{x_n\}$

is Cauchy, $diam(F_n) \longrightarrow 0$. Therefore, for each $k \in \mathbb{N}$, one can choose $n_k \in \mathbb{N}$ such that $d(x_{n_k}, x) < 1/k$. Without loss of generality, suppose $n_k < n_{k+1}$ for all $k \in \mathbb{N}$. Therefore, $\{x_n\}$ has a subsequence $\{x_{n_k}\}$, convergent to x. Consequently, $\{x_n\} \longrightarrow x$. Hence the result. □

Remark 4.27 Totally bounded sets are also known as *pre-compact sets,* as in Theorem 5.26, we shall show that every compact set is totally bounded.

Exercise 4.30 Let E be a nonempty subset of $\mathbb{R}$. Prove that $diam(E) = \sup E - \inf E$.

Exercise 4.31 In Theorem 4.14, is the hypothesis '$diam(F_n) \longrightarrow 0$' redundant?

Exercise 4.32 In Theorem 4.14, prove that the hypothesis, that 'every F_n is closed' can be replaced with 'there exists some $m \in \mathbb{N}$ such that F_n is closed, for every $n > m$.' Also, show that no other hypothesis in Theorem 4.14 is redundant.

Exercise 4.33 Is any hypothesis in Theorem 4.26 redundant?

Exercise 4.34 Use Cantor Intersection Property to conclude that $\mathbb{Q}$ and $(0, 1)$ are not complete subspaces of $\mathbb{R}$.

Exercise 4.35 Let d be a metric on $\mathbb{R}$ such that $(\mathbb{R}, d)$ is a complete metric space. Let $\{F_n\}$ be a nested decreasing sequence of nonempty closed subsets of $(\mathbb{R}, d)$ such that $\{diam(F_n)\}$ is a bounded sequence. Prove or disprove: $\bigcap_{n=1}^{\infty} F_n \neq \emptyset$.

Exercise 4.36 Does there exist any generalization of Exercise 4.56 to $\mathbb{R}^n$?

Exercise 4.37 Let $d(x, y) := \min\{1, |x - y|\}$ for all $x, y \in \mathbb{R}$. Prove that $(\mathbb{R}, d)$ is a bounded metric space, which is not totally bounded.

Exercise 4.38 Prove that the subsets of totally bounded sets are also totally bounded.

Exercise 4.39 If $E \neq \emptyset$, prove that $x_1, \ldots, x_n$ in Definition 4.16 can be taken in E.

Exercise 4.40 Let $\{x_n\}$ be a Cauchy sequence and $E := \{x_n : n \in \mathbb{N}\}$. Prove that every sequence with terms from E is Cauchy.

Exercise 4.41 Write a proof for the second part of Lemma 4.21.

Exercise 4.42 Show that Theorem 4.20, is not true for general metric spaces.

Exercise 4.43 Does there exist a metric space, other than a subspace of some $\mathbb{R}^n$, in which every bounded set is totally bounded?

Exercise 4.44 Does there exist a bounded non-discrete space which is not totally bounded?

Exercise 4.45 Use Theorem 4.23 to show that $\mathbb{Q}$ is not a complete subspace of $\mathbb{R}$.

Exercise 4.46 Prove that a subset E of a metric space is totally bounded if and only if for every $\epsilon > 0$, there exist finitely many sets $E_1, \ldots, E_n$ such that $E = \bigcup_{i=1}^{n} E_i$ and $diam(E_i) < \epsilon$ for all i.

Exercise 4.47 Prove that a subset E of a metric space is totally bounded if and only if for every $\epsilon > 0$, there exist finitely many closed sets $E_1, \ldots, E_n$ such that $E \subset \bigcup_{i=1}^{n} E_i$ and $diam(E_i) < \epsilon$ for all i.

Exercise 4.48 Prove that a subset E of a metric space is totally bounded if and only if $\overline{E}$ is totally bounded.

Exercise 4.49 If E is totally bounded, prove that so is E', that is, the set of limit points of E. Is the converse true?

Exercise 4.50 Prove that the notion of totally bounded sets is independent of the embedded space. That is, if X is a metric space and $E \subset Y \subset X$, then E is totally bounded in X if and only if E is totally bounded in Y.

Exercise 4.51 Prove that a set E is not totally bounded if and only if E has an infinite subspace, in which every set is open as well as closed.

Exercise 4.52 Prove that a subset E of a metric space (X, d) is totally bounded if and only if every infinite subset of E contains a sequence $\{x_n\}$ such that $d(x_{n+1}, x_n) < \frac{1}{2^n}$ for all $n \in \mathbb{N}$.

Exercise 4.53 Prove that a metric space (X, d) is totally bounded if and only if every sequence $\{x_n\}$ in X contains a subsequence $\{x_{n_k}\}$ satisfying $d(x_{n_k}, x_{n_{k+1}}) < \frac{1}{2^k}$ for all $k \in \mathbb{N}$.

Exercise 4.54 Let $X := (0, +\infty)$ and $d(x, y) := \left|\frac{1}{x} - \frac{1}{y}\right|$ for all $x, y \in X$. Prove that every bounded subset of (X, d) is totally bounded.

Exercise 4.55 Does there exist a closed bounded subset of ℓ^∞, which is not totally bounded?

Exercise 4.56 Let $\{I_n\}$ be a sequence of closed intervals such that $I_n \cap I_m \neq \emptyset$ for all $m \neq n$. Prove that $\bigcap_{n=1}^{\infty} I_n \neq \emptyset$.

Exercise 4.57 (Kuratowski, 1930 [5]) Let (X, d) be a metric space. For every subset A of X, define

$$\alpha(A) := \inf\{r \geq 0 : \ A \text{ can be covered by finitely many subsets of } X, \text{ each with diameter } < r\}.$$

(a) Prove that A is totally bounded if and only if $\alpha(A) = 0$. (So $\alpha(A)$ may be termed as the *measure of non-total boundedness* of A.)
(b) Prove that X is complete if and only if for every nested decreasing sequence $\{A_n\}$ of nonempty closed subsets of X such that $\alpha(A_n) \longrightarrow 0$, we have $\bigcap_{n=1}^{\infty} A_n \neq \emptyset$.

4.4 Completion of a Metric Space

If a metric space X is not complete, then it lacks limits of some of its Cauchy sequences. Can we somehow get those limits and find a super-space of X, which is complete?

In this section, we show that such 'complete super-spaces' always exist. We can embed every metric space X as a dense subspace of a complete metric space. For this purpose, we first present the notions of isometry and denseness.

Definition 4.28 A subset E of a metric space X is said to be *dense* in X if $\overline{E} = X$.

Therefore, E is dense in X if and only if $X \subset \overline{E}$. If $A \subset B \subset X$, we shall say that A is *dense in* B if $B \subset \overline{A}$ which holds if and only if $\overline{A} \cap B = B$.

Examples 4.29 (a) The sets $\mathbb{Q}$, $\mathbb{R} \setminus \mathbb{Q}$ and $\mathbb{R} \setminus \mathbb{Z}$ are all dense subsets of $\mathbb{R}$.
(b) $[0, 1]$ is not dense in $\mathbb{R}$.
(c) No proper subspace of a discrete metric space X is dense in X.

Definition 4.30 Let (X, d) and (Y, ρ) be metric spaces. A function $f : X \longrightarrow Y$ is said to be an *isometry* if

$$\rho(f(x), f(y)) = d(x, y) \text{ for all } x, y \in X.$$

Clearly, every isometry is an injective continuous function. In fact, if f is an isometry of X into Y, then Y contains a copy of X. That is, X is embedded into Y.

Definition 4.31 Suppose X is a metric space. A metric space Y is said to be a *completion* of X if Y is complete and X is isometric to a dense subset Y, that is, there exists an isometry $f : X \longrightarrow Y$ such that $\overline{f(X)} = Y$.

Examples 4.32 (a) Every complete metric space is the completion of itself.
(b) $\mathbb{R}$ and $[0, 1]$ are completions of $\mathbb{Q}$ and $(0, 1)$, respectively.

We will show that every metric space has a completion, which is unique upto isometry. The next lemma is required in that direction.

Lemma 4.33 *Let X be a metric space having a dense subset A such that every Cauchy sequence in A is convergent in X. Then X is complete.*

Proof Let $\{x_n\}$ be any Cauchy sequence in X and $\epsilon > 0$ be given. Then there exists some $N_1 \in \mathbb{N}$ such that

$$d(x_n, x_m) < \frac{\epsilon}{2} \text{ for all } n > m \geq N_1.$$

Since $\overline{A} = X$, for every $n \in \mathbb{N}$, one can choose $a_n \in A$ such that $d(x_n, a_n) < \frac{\epsilon}{2^n}$. Then $\{a_n\}$ is a Cauchy sequence in A, as for every $n > m > N_1$, we have

$$d(a_n, a_m) \leq d(a_n, x_n) + d(x_n, x_m) + d(x_m, a_m) < \frac{\epsilon}{2^n} + \frac{\epsilon}{2} + \frac{\epsilon}{2^m} < \epsilon.$$

By hypothesis, $a_n \longrightarrow x$ for some $x \in X$. Thus, there exists $N_2 \in \mathbb{N}$ such that $d(a_n, x) < \frac{\epsilon}{2}$ for all $n > N_2$. Therefore, for all $n > N_2$, we obtain

$$d(x_n, x) \leq d(x_n, a_n) + d(a_n, x) < \frac{\epsilon}{2^n} + \frac{\epsilon}{2} < \epsilon.$$

This proves that $\{x_n\} \longrightarrow x$, in X. Hence the result. □

Theorem 4.34 (Cantor) *Every metric space has a completion.*

Proof Let (X, d) be an arbitrary metric space. The main idea of this proof is that two convergent sequences $\{x_n\}$ and $\{y_n\}$ have the same limit if and only if $d(x_n, y_n) \longrightarrow 0$. Therefore, we first classify Cauchy sequences in X which could converge to the same limit, even in the desirable super-space.

We prove the result in six steps. In the first step, we construct (X^*, d^*), which will be established as the completion of (X, d), in the latter steps.

Step I. *Defining* (X^*, d^*). Let Ω denote the family of Cauchy sequences in X. For sequences $\{x_n\}, \{y_n\} \in X^*$, define

$$\{x_n\} \sim \{y_n\} \text{ if and only if } \lim_{n\to\infty} d(x_n, y_n) = 0.$$

Then $\sim$ is an equivalence relation on Ω. Let X^* denote the set of equivalence classes of X with respect to $\sim$. If $[\{x_n\}]$ denote the equivalence class of the sequence $\{x_n\} \in \Omega$, then $X^* = X/\!\sim\, = \{[\{x_n\}] : \{x_n\} \in \Omega\}$. Define $d^* : X^* \times X^* \longrightarrow [0, \infty)$ as follows:

$$d^*\big([\{x_n\}], [\{y_n\}]\big) := \lim_{n\to\infty} d(x_n, y_n) \text{ for all } [\{x_n\}], [\{y_n\}] \in X^*.$$

Step II. d^* *is well-defined on* X^*. Note that the above definition of d^* uses specific Cauchy sequences $\{x_n\}$ and $\{y_n\}$ representing the classes, on which d^* is defined. Therefore, it is important to ensure that this definition is independent of the representatives sequences from these classes.
Let $\{x'_n\}, \{y'_n\}$ be Cauchy sequences in X such that $\{x'_n\} \sim \{x_n\}$ and $\{y'_n\} \sim \{y_n\}$. Then

$$\lim_{n\to\infty} d(x_n, x'_n) = 0 = \lim_{n\to\infty} d(y_n, y'_n).$$

Applying triangle inequality, we obtain

$$d(x_n, y_n) \leq d(x_n, x'_n) + d(x'_n, y'_n) + d(y'_n, y_n)$$
$$\text{and } d(x'_n, y'_n) \leq d(x'_n, x_n) + d(x_n, y_n) + d(y_n, y'_n).$$

Therefore, $|d(x_n, y_n) - d(x'_n, y'_n)| \leq d(x'_n, x_n) + d(y_n, y'_n)$. Passing limit $n \longrightarrow \infty$, we have

$$\lim_{n\to\infty} d(x_n, y_n) = \lim_{n\to\infty} d(x'_n, y'_n).$$

Step III. *d^* is a metric on X^*.* We leave it for the readers to show that each of the three properties of the metric d^* follows from the corresponding property of d on X.

Step IV. *Embedding X into X^*.* Define $f : X \longrightarrow X^*$ as $f(x) := [\{x_n\}]$, where $x_n = x$ for all $n \in \mathbb{N}$. Then for any $x, y \in X$, we have

$$d^*(f(x), f(y)) = \lim_{n\to\infty} d(x, y) = d(x, y).$$

Therefore, f is an isometry from X into X^*.

Step V. *$f(X)$ is a dense subset of X^** : Let $x^* = [\{x_n\}] \in X^*$, and let $\epsilon > 0$ be given. Since $\{x_n\}$ is Cauchy, there exists $N \in \mathbb{N}$ such that $d(x_n, x_m) < \frac{\epsilon}{2}$ for all $n > m \geq N$. For $y^* := f(x_N)$,

$$d^*(x^*, y^*) = \lim_{n\to\infty} d(x_n, x_N) \leq \frac{\epsilon}{2} < \epsilon.$$

Therefore, we obtain $y^* \in B_{d^*}(x^*; \epsilon) \cap f(X)$ which implies that $x^* \in \overline{f(X)}$. Hence, $f(X)$ is a dense subset of (X^*, d^*).

Step VI. *(X^*, d^*) is a complete metric space.* By Lemma 4.33, it is enough to show that every Cauchy sequence in $f(X)$ is convergent.
Let $\{y_n\}$ be any Cauchy sequence in $f(X)$. Then for each $n \in \mathbb{N}$ there exists some $x_n \in X$ such that $f(x_n) = y_n$. Since f is an isometry, we have

$$d(x_n, x_m) = d^*(f(x_n), f(x_m)) \text{ for all } m, n \in \mathbb{N}.$$

Hence, $\{x_n\}$ is a Cauchy sequence in X. Write $x^* := [\{x_n\}] \in X^*$. We claim that $\lim_{n\to\infty} f(x_n) = x^*$. Let $\epsilon > 0$ be given. Pick any $N \in \mathbb{N}$ such that $d(x_n, x_m) < \epsilon/2$ for all $n > m \geq N$. Then for each $m \geq N$, we obtain

$$d^*(f(x_m), x^*) = \lim_{n\to\infty} d(x_m, x_n) \leq \frac{\epsilon}{2} < \epsilon.$$

This proves that $\lim_{n\to\infty} f(x_n) = x^*$. By Lemma 4.33, X^* is complete.

Therefore, (X^*, d^*) is a completion of (X, d). Hence the result. □

Now we establish that the completion of a metric space is unique up to isometry.

Theorem 4.35 *Let X_1 and X_2 be completions of a metric space X, with isometries $f_1 : X \longrightarrow X_1$ and $f_2 : X \longrightarrow X_2$. Then there exists a unique isometry $\phi : X_1 \longrightarrow X_2$ such that $\phi \circ f_1 \equiv f_2$.*

Proof Since f_1 is an isometry, $f_1^{-1} : f_1(X) \longrightarrow X$ is a surjective isometry. Also, f_2 is an isometry from X onto $f_2(X)$. Therefore, $f_2 \circ f_1^{-1} : f_1(X) \longrightarrow f_2(X)$ is a surjective isometry. Write $g := f_2 \circ f_1^{-1}$. Then

$$g \circ f_1 = (f_2 \circ f_1^{-1}) \circ f_1 = f_2 \circ (f_1^{-1} \circ f_1) = f_2 \circ I_X = f_2,$$

where I_X represents the identity map from X onto itself. Similarly, there exists a surjective isometry $h : f_2(X) \longrightarrow f_1(X)$ such that $h \circ f_2 = f_1$. Hence

$$h \circ (g \circ f_1) = h \circ f_2 = f_1 \text{ and } g \circ (h \circ f_2) = g \circ f_1 = f_2.$$

Consequently, $h \circ g = I_{f_1(X)}$ and $g \circ h = I_{f_2(X)}$. Since $f_1(X)$ is dense in X_1, one can extend the identity map $h \circ g$ to X_1 as $h \circ g = I_{X_1}$. Similarly, $g \circ h$ can be extended to X_2 such that $g \circ h = I_{X_2}$. Therefore, $h = g^{-1}$. With $\phi := h \circ g = I_{X_1}$, the result holds. Hence the result. □

In the exercises section, we shall outline some other procedures to obtain the completion of a metric space. However, due to pedagogical reasons, we advocate the above method.

The Set of Real Numbers

The natural numbers or positive integers are often perceived as 'God' given numbers. Peano, along with a few other mathematicians, proposed an axiomatic approach to define these. We will also obtain the set of natural numbers from a standard set of axioms in Appendix A.3.

The natural binary operations of addition and multiplication on the set of natural numbers $\mathbb{N}$, lead to negative numbers and fractions. One can show that the set $\mathbb{Q}$ of rational numbers (fractions) is a field under these binary operations.

Let $a_n := \sum_{k=0}^{n} \frac{1}{k!}$ for all $n \in \mathbb{N}$. It is well known that $\{a_n\}$ is a Cauchy sequence of rational numbers, which is not convergent to any rational number. Therefore, the metric space $\mathbb{Q}$, under the usual metric, is not complete. Applying Theorems 4.34 and 4.35, $\mathbb{Q}$ has a unique completeness, say $\mathbb{Q}^*$. This defines the set of real numbers.

Dedekind proposed another axiomatic construction of the real number system $\mathbb{R}$, as a field extension of $\mathbb{Q}$ satisfying the least upper bound property (see [2, p. 17, Appendix]).

In Theorem 1.28, starting with the least upper bound property, we established that every Cauchy sequence of reals is convergent in $\mathbb{R}$. Now we establish that the two extensions are isometrically the same. Define $f : \mathbb{Q}^* \longrightarrow \mathbb{R}$ as

$$f(\{r_n\}) := \lim_{n\to\infty} r_n \text{ for all } [\{r_n\}] \in \mathbb{Q}^*.$$

By Theorem 1.28 and the definition of $\mathbb{Q}^*$, it can be shown that f is a well-defined isometry. Hence, $\mathbb{R}$ and $\mathbb{Q}^*$ are isometric.

Remarkably, Dedekind's completion makes an extensive use of the order structure of $\mathbb{Q}$, while Cauchy completion depends only upon the usual metric on $\mathbb{Q}$.

Finally, we invite the attention on the terminology that we inherited for the different number systems so far. Let us ponder upon the dictionary meanings of the following two categories of words: (i) Natural, rational, real; and (ii) Negative, irrational, imaginary, complex.

These words reflect exciting stories. Certain numbers were labeled as positive, but only when people started accepting the negative numbers. When Pythagoreans realized that $\sqrt{2}$ is not a fraction, it was termed as an irrational. Since then all fractional numbers are called rational. Similarly, a class of numbers was labeled as real numbers, when the imaginary number $\sqrt{-1}$ established its existence. Now we deal with even complex numbers very naturally, without finding any 'complexity' about them!

Remarks 4.36 Several proofs of elementary real analysis rely on the completeness property of real numbers. In [6], alternative proofs to some of such results are provided using tagged partitions of elementary real analysis. This inspired the authors of [7] to use dyadic partitions for constructive proofs of the same results. Several results of analysis are equivalent to the completeness property of reals. Such results along with various other characterizations of the completeness property can be found in [8–10]. A thorough discussion on the real number system is given in [11]. Advanced math students, who want to go beyond the standard textbook results about real numbers, are referred to [12]. For the classification of complete metric spaces up to isometry, see [13].

Exercise 4.58 Prove that d^*, as in the proof of Theorem 4.34, is a metric on X^*.

Exercise 4.59 Let X, Y be metric spaces, E be a dense subset of X and f, g be continuous functions from X into Y. Prove that

(a) $f(E)$ is dense in $f(X)$.
(b) If $f(x) = g(x)$ for all $x \in E$, then $f(x) = g(x)$ for all $x \in X$.

Exercise 4.60 Let X, Y be metric spaces such that X is complete. If $f : X \longrightarrow Y$ is an isometry, prove that $f(X)$ is a complete subspace of Y.

Exercise 4.61 Prove that f, defined as on page 105, is a well-defined isometry.

Exercise 4.62 If $\{x_n\}$ and $\{y_n\}$ are Cauchy sequences in a metric space (X, d), does the sequence $\{d(x_n, y_n)\}$ always converge?

Exercise 4.63 Deduce the Least Upper Bound Property from the Nested Interval Property (1.23).

Exercise 4.64 Let (X, d) be any metric space and $a \in X$. For every $x \in X$, define $f_x : X \longrightarrow \mathbb{R}$ such that $f_x(t) := d(x, t) - d(a, t)$ for all $t \in X$. Prove that f_x is a bounded real valued function on X for all $x \in X$. Further show that $\sup\{|f_x(t) - f_y(t)| : t \in X\} = d(x, y)$ for all $x, y \in X$.

Exercise 4.65 Let (X, d) be a metric space and $L^\infty(X)$ denote the collection of all bounded real valued functions on X. Define

$$d^*(f, g) := \sup_{x \in X} |f(x) - g(x)| \text{ for all } f, g \in L^\infty(X).$$

Prove that $(L^\infty(X), d^*)$ is a complete metric space and there exists an isometry $T : X \longrightarrow L^\infty(X)$. Further show that the closure of $T(X)$ inside $L^\infty(X)$ is a completion of X.

Exercise 4.66 Let X, Y be complete metric spaces, X_0 is dense in X, and $f : X_0 \longrightarrow Y$ is an isometry such that $f(X_0)$ is dense in Y. Prove that f can be extended to X such that $f : X \longrightarrow Y$ is a surjective isometry.

Exercise 4.67 Suppose $1 \leq p < \infty$. Prove that the completion of $(c_{00}, \|.\|_p)$ is given by the sequence space ℓ^p.

Exercise 4.68 Prove that the completion of c_{00}, under $\|.\|_\infty$ norm, is c_0. (Recall that c_0 is the space of sequences which converge to 0).

Exercise 4.69 Let (X, d) be a metric space, $\{x_n\}$ be a Cauchy sequence in X, which is not convergent in X, x^* be any point outside X and $X^* := X \cup \{x^*\}$. Let $\mathcal{F}$ denote the collection of closures of tails of $\{x_n\}$. Define

$$d^*(x, y) := \begin{cases} 0 & ; \text{ if } x = y \in X^*, \\ d(x, y) & ; \text{ if } x, y \in X, \\ \sup\{d(z, F) : F \in \mathcal{F}\} & ; \text{ if } x = z \in X \text{ and } y = x^*. \\ \sup\{d(z, F) : F \in \mathcal{F}\} & ; \text{ if } y = z \in X \text{ and } x = x^*. \end{cases}$$

Prove that d^* is a metric on X^* and $x_n \longrightarrow x^*$ in (X^*, d^*).

Exercise 4.70 Let (X, d), x^* and X^* be as in Exercise 4.69. Let $\mathcal{F}$ denote a collection of nonempty closed subsets of X such that

(a) for every $A, B \in \mathcal{F}$, either $A \subset B$ or $B \subset A$,
(b) $\inf\{diam(F) : F \in \mathcal{F}\} = 0$ and
(c) $\bigcap_{F \in \mathcal{F}} F = \emptyset$.

Let d^* be defined on $X^* \times X^*$, as in Exercise 4.69. Prove that d^* is a metric on X^* and the closure of F in (X^*, d^*) is $F \cup \{x^*\}$ for all $F \in \mathcal{F}$.

4.5 Banach Spaces

Recall that every normed linear space $(X, \|.\|)$ is a metric space with respect to the induced metric defined as $d(x, y) := \|x - y\|$ for all $x, y \in X$.

Definition 4.37 A normed space is called *Banach,* if it is complete under the metric induced by its norm. Complete inner product spaces are called *Hilbert spaces.*

Examples 4.38 (a) $\mathbb{R}^n$ and $\mathbb{C}$ are Hilbert spaces, and hence Banach spaces.
(b) The space $(c_{00}, \|.\|_1)$ as defined in Sect. 2.4, is not a Banach space. To see this, let $x_n := (1, 2^{-1}, \dots, 2^{-n}, 0, 0, \dots)$ for all $n \in \mathbb{N}$. We leave it for the reader to show that the sequence $\{x_n\}$ is Cauchy, but not convergent in $(c_{00}, \|.\|_1)$.

Remark 4.39 Every normed space is a metric space and hence has a completion. In fact, there also exists a compatible norm on its completion (see Exercise 4.89).

Now we provide some important classes of Banach spaces. The first one is given by some collections of continuous functions. Let X be a metric space and

$$C(X) := \left\{f : f \text{ is a continuous, complex valued, bounded function on } X\right\}.$$

For a function $f \in C(X)$, the *uniform (supremum) norm* of f is defined as

$$\|f\|_\infty := \sup\left\{|f(x)| : x \in X\right\}.$$

The terminology ‘uniform norm’ is consistent with ‘uniform convergence’ (see Exercise 4.74). It can be shown that $\|.\|_\infty$ is a norm on $C(X)$. Moreover, $(C(X), \|.\|_\infty)$ is a Banach space. In case X is a compact interval $[a, b]$, we shall simply write $C[a, b]$ instead of $C([a, b])$. Unless specified, the default norm on $C(X)$ or $C[a, b]$ will always be the uniform norm.

Theorem 4.40 $(C(X), \|.\|_\infty)$ *is a Banach space.*

Proof Let $\{f_n\}$ be a Cauchy sequence in $C(X)$ and $\epsilon > 0$ be given. Then there exists some $m \in \mathbb{N}$ such that $\|f_{n_2} - f_{n_1}\|_\infty < \epsilon$ for all $n_2 > n_1 \geq m$. Therefore, for all $x \in X$ we have

$$|f_{n_2}(x) - f_{n_1}(x)| \leq \|f_{n_2} - f_{n_1}\|_\infty < \epsilon \text{ for all } n_2 > n_1 \geq m. \tag{4.3}$$

Then for each $x \in X$, $\{f_n(x)\}$ is a Cauchy sequence of complex numbers and hence convergent. Define $f(x) := \lim_{n\to\infty} f_n(x)$ for all $x \in X$. In (4.3), passing limit $n_1 \longrightarrow \infty$ and replacing n_2 with n, we obtain

$$|f_n(x) - f(x)| \leq \epsilon \text{ for all } n \geq m \text{ and for all } x \in X.$$

Hence, $\{f_n\}$ is uniformly convergent to f. By Theorem 3.31, f is continuous on X. Since f_m is bounded, $|f_m| < M$ on X for some $M > 0$. Therefore, $|f(x)| \leq |f(x) - f_m(x)| + |f_m(x)| < \epsilon + M$ for all $x \in X$. So, $f \in C(X)$ and hence the result. □

Theorem 4.41 *For every* $1 \leq p \leq \infty$, *the sequence space* ℓ^p *is a Banach space.*

Proof Let $\mathbb{K}$ be the scalar field of ℓ^p and $\epsilon > 0$ be given. Let $\{x_k\}$ be a Cauchy sequence in ℓ^p. Then there exists some $N \in \mathbb{N}$ such that $\|x_k - x_m\|_p < \epsilon$ for all $k > m \geq N$. Let $x_k(n)$ denote the n^{th} term of the sequence x_k.

First suppose $1 \leq p < \infty$. Then for every $k > m \geq N$ and for all $n \in \mathbb{N}$, we have

$$|x_k(n) - x_m(n)| \leq \left(\sum_{j=1}^{n} |x_k(j) - x_m(j)|^p \right)^{\frac{1}{p}} \leq \|x_k - x_m\|_p < \epsilon. \tag{4.4}$$

Therefore, for all $n \in \mathbb{N}$, $\{x_k(n)\}_k$ is Cauchy in $\mathbb{K}$ and hence there exists $x(n) \in \mathbb{K}$ such that $x_k(n) \longrightarrow x(n)$, as $k \longrightarrow \infty$. Passing limit $m \longrightarrow \infty$ in (4.4),

$$\left(\sum_{j=1}^{n} |x_k(j) - x(j)|^p \right)^{\frac{1}{p}} \leq \epsilon \text{ for all } k \geq N \text{ and for all } n \in \mathbb{N}. \tag{4.5}$$

Write $x := (x(1), \ldots, x(n), \ldots)$. Applying Minkowski's inequality in $\mathbb{R}^n$ and (4.5), for every $n \in \mathbb{N}$, we conclude that

$$\begin{aligned} \left(\sum_{j=1}^{n} |x(j)|^p \right)^{\frac{1}{p}} &\leq \left(\sum_{j=1}^{n} |x(j) - x_N(j)|^p \right)^{\frac{1}{p}} + \left(\sum_{j=1}^{n} |x_N(j)|^p \right)^{\frac{1}{p}} \\ &\leq \epsilon + \left(\sum_{j=1}^{\infty} |x_N(j)|^p \right)^{\frac{1}{p}}. \end{aligned}$$

Passing limit $n \longrightarrow \infty$, we have $\|x\|_p \leq \epsilon + \|x_N\|_p < \infty$. Hence, $x \in \ell^p$. Finally passing limit $n \longrightarrow \infty$ in (4.5), we obtain $\|x_k - x\|_p \leq \epsilon$ for all $k \geq N$. Therefore, $x_k \longrightarrow x$ in ℓ^p.

Now suppose $p = \infty$. As earlier, for every $m > k \geq N$, we have

$$|x_k(n) - x_m(n)| \leq \sup_{j \in \mathbb{N}} |x_k(j) - x_m(j)| = \|x_k - x_m\|_\infty < \epsilon. \tag{4.6}$$

Therefore, for every $n \in \mathbb{N}$, the sequence $\{x_k(n)\}_k$ is Cauchy in $\mathbb{K}$ and hence there exists some $x(n) \in \mathbb{K}$ such that $x_k(n) \longrightarrow x(n)$, as $k \longrightarrow \infty$. Write $x := (x(1), x(2), \ldots, x(n), \ldots)$. Passing limit $m \longrightarrow \infty$ in (4.6), we obtain $\|x_k - x\|_\infty \leq \epsilon$ for all $k \geq N$. As earlier, one can conclude that $x \in \ell^\infty$. Hence, $x_k \longrightarrow x$ in ℓ^∞. □

Next, we will provide a necessary and sufficient condition for a normed space to be complete. Motivated by Theorem 1.36, we define the absolute convergence of a series in normed spaces.

Definition 4.42 A series $\sum_{n=1}^{\infty} x_n$ in a normed linear space $(X, \|.\|)$ is said to be

(a) *convergent* if there exists some $x \in X$ such that the sequence of partial sums $\{\sum_{k=1}^{n} x_k\}_n$ converges to x, in $(X, \|.\|)$. In this case, we write $\sum_{n=1}^{\infty} x_n = x$.

(b) *absolutely convergent* if $\sum_{n=1}^{\infty} \|x_n\|$ is convergent in $\mathbb{R}$.

In general, an absolutely convergent series in a normed linear space may not be convergent.

Example 4.43 Let $X := c_{00}$, denote the collection of sequences of real numbers which are eventually zero. For $x = \{x_n\} \in c_{00}$, define $\|x\| := \sum_{n=1}^{\infty} \frac{|x_n|}{2^n}$. Let $e_n \in c_{00}$ be a sequence with all zeros, but having 1 at the n^{th} place. It can be shown that $(X, \|.\|)$ is a normed space over $\mathbb{R}$, $\sum_{n=1}^{\infty} \|e_n\| = 2 < \infty$, and the series $\sum_{n=1}^{\infty} e_n$ does not converge in $(X, \|.\|)$.

Theorem 4.44 *A normed linear space X is complete if and only if every absolutely convergent series in X is convergent in X.*

Proof The direct implication is analogous to Theorem 1.36. Assume that X is complete. Let $\sum_{n=1}^{\infty} x_n$ be an absolutely convergent series in X and $P_n := \sum_{k=1}^{n} x_k$ for all $n \in \mathbb{N}$. For $\epsilon > 0$, let $N \in \mathbb{N}$ such that $\sum_{n=n_1+1}^{n_2} \|x_n\| < \epsilon$ for all $n_2 > n_1 > N$. Then for $n_2 > n_1 > N$,

$$\|P_{n_2} - P_{n_1}\| = \left\| \sum_{n=n_1+1}^{n_2} x_n \right\| \leq \sum_{n=n_1+1}^{n_2} \|x_n\| < \epsilon.$$

Therefore, $\{P_n\}$ is a Cauchy sequence in X. Since X is complete, $\{P_n\}$ is convergent in X. Hence, $\sum_{n=1}^{\infty} x_n$ is convergent in X.

Conversely, assume that every absolutely convergent series in X is convergent. Let $\{x_n\}$ be a Cauchy sequence in X. By Theorem 2.24, it is enough to prove that it has a convergent subsequence. For each $k \in \mathbb{N}$, choose $n_k \in \mathbb{N}$ such that $\|x_n - x_m\| < \frac{1}{2^k}$ for all $n, m \geq n_k$. Without loss of generality, assume that $n_k < n_{k+1}$ for all $k \in \mathbb{N}$.

Let $y_1 := x_{n_1}$ and $y_k := x_{n_k} - x_{n_{k-1}}$ for all $k > 1$. Then $\sum_{k=1}^{\infty} y_k$ is a series in X with k^{th} partial sum x_{n_k}. Also, $\|y_k\| < \frac{1}{2^{k-1}}$ for all $k > 1$. Consequently

$$\sum_{k=1}^{\infty} \|y_k\| < \|y_1\| + \sum_{k=2}^{\infty} \frac{1}{2^{k-1}} = \|y_1\| + 1.$$

Since $\sum_{k=1}^{\infty} y_k$ is absolutely convergent, by hypothesis, $\sum_{k=1}^{\infty} y_k = y$ for some $y \in X$. Thus, the sequence of partial sums of $\sum_{k=1}^{\infty} y_k$, that is $\{x_{n_k}\}$ is convergent in X. Hence the result. □

Remark 4.45 By the virtue of Theorem 4.44, all the convergence tests for a series of positive terms can be applied to Banach spaces, over the corresponding series of absolute terms.

Let us now consider the following 'proof' for the divergence of the harmonic series.

Suppose that $\sum_{n=1}^{\infty} \frac{1}{n}$ converges and let S denote this sum. Then

$$S = 1 + \frac{1}{2} + \frac{1}{3} + \dots \text{ which implies } \frac{S}{2} = \frac{1}{2} + \frac{1}{4} + \frac{1}{6} + \dots.$$

Also, note that

$$1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \cdots > \frac{1}{2} + \frac{1}{4} + \frac{1}{6} + \frac{1}{8} + \cdots = \frac{S}{2}.$$

Thus, $S > \frac{S}{2} + \frac{S}{2} = S$, a contradiction. Hence, $\sum_{n=1}^{\infty} \frac{1}{n}$ doesn't converge. □

This 'proof' intrinsically assumes that all rearrangements of a convergent series are convergent to a unique limit. This is true for absolutely convergent series of reals, but not in general.

Definition 4.46 A series $\sum_{n=1}^{\infty} b_n$ is called a *rearrangement* of another series $\sum_{n=1}^{\infty} a_n$ if there exists a bijection (also called a permutation) $\sigma : \mathbb{N} \longrightarrow \mathbb{N}$, such that $a_{\sigma(n)} = b_n$ for all $n \in \mathbb{N}$.

Definition 4.47 A series $\sum_{n=1}^{\infty} x_n$ in a normed linear space X is said to be

(a) *unconditionally convergent* if its every rearrangement is convergent in X.
(b) *conditionally convergent* if it is not unconditionally convergent.

Through our next few results, it will be established that the above notion of conditional convergence is consistent with the same in standard calculus courses, where a series of real numbers is called *conditionally convergent* if it is convergent in $\mathbb{R}$, but not absolutely.

Let us recall the *Riemann Rearrangement Theorem.* We omit its proof which can be found in various standard textbooks on real analysis (e.g. see [2, Theorem 3.54, p. 76]).

Theorem 4.48 *Let $\sum_{n=1}^{\infty} a_n$ be a convergent series of reals, which is not absolutely convergent and $\alpha \in [-\infty, +\infty]$. Then there exists a bijection $\sigma : \mathbb{N} \longrightarrow \mathbb{N}$ such that $\sum_{n=1}^{\infty} a_{\sigma(n)} = \alpha$.*

Hence, any convergent series of reals, which is not absolutely convergent is conditionally convergent. For example, the alternating series $\sum_{n=1}^{\infty} (-1)^{n+1}/n$ is one such series.

By Riemann Rearrangement Theorem, every unconditionally convergent series of real numbers is absolutely convergent. The converse is true in all Banach spaces.

Theorem 4.49 *If $\sum_{n=1}^{\infty} x_n$ is an absolutely convergent series in a Banach space X, then $\sum_{n=1}^{\infty} x_n$ is unconditionally convergent in X.*

Proof Let $\sum_{n=1}^{\infty} x_n$ be an absolutely convergent series in a Banach space X and $\sigma : \mathbb{N} \longrightarrow \mathbb{N}$ be any bijection. Let S_n and T_n denote the n^{th} partial sums of $\sum_{n=1}^{\infty} x_n$ and $\sum_{n=1}^{\infty} x_{\sigma(n)}$, respectively. We shall prove that $\lim_{n\to\infty} T_n = \lim_{n\to\infty} S_n$.

By Theorem 4.44, $\lim_{n\to\infty} S_n = \sum_{n=1}^{\infty} x_n$ exists in X. Let $\epsilon > 0$ be given. By hypothesis, there exists $N_1 \in \mathbb{N}$ such that $\sum_{n=N_1}^{\infty} \|x_n\| < \epsilon$. Let

$$N_2 := \max\{\sigma^{-1}(1), \ldots, \sigma^{-1}(N_1)\}.$$

Then for all $n \geq N_2$, both sums T_n and S_n will contain the terms $x_1, \ldots, x_{N_1}$ and hence $\|S_n - T_n\| \leq \sum_{i=N_1+1}^{n} \|x_n\| < \epsilon$. Since $\epsilon > 0$ was arbitrary, we obtain $\lim_{n\to\infty} T_n = \lim_{n\to\infty} S_n$. □

The converse of Theorem 4.49 is not true.

Example 4.50 Consider the sequence space ℓ^2 over $\mathbb{C}$ and $\{e_n : n \in \mathbb{N}\}$ be as usual. Then the series $\sum_{n=1}^{\infty} \frac{1}{n} e_n$ converges unconditionally in ℓ^2, but not absolutely convergent.

Proof The series is not absolutely convergent, as

$$\sum_{n=1}^{\infty} \left\| \frac{1}{n} e_n \right\| = \sum_{n=1}^{\infty} \frac{1}{n} \|e_n\| = \sum_{n=1}^{\infty} \frac{1}{n} = \infty.$$

To see that it is unconditionally convergent, let $\sigma : \mathbb{N} \longrightarrow \mathbb{N}$ be any bijection and $\epsilon > 0$ be given. Let x be the sequence with $1/n$ as its n^{th} term. We claim that the series $\sum_{n=1}^{\infty} \frac{e_{\sigma(n)}}{\sigma(n)}$ converges to x in ℓ^2.

First note that $\|x\|_2^2 \leq \sum_{n=1}^{\infty} \frac{1}{n^2} < \infty$ which implies that $x \in \ell^2$. Since $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is convergent in $\mathbb{R}$, there exists some $N \in \mathbb{N}$ such that $\sum_{n>N} \frac{1}{n^2} < \epsilon$. Let $m := \max\{\sigma^{-1}(1), \ldots, \sigma^{-1}(N)\}$. Then $\sigma(n) \geq N$ for all $n \geq m$. Therefore

$$\left\| \sum_{n=1}^{k} \frac{e_{\sigma(n)}}{\sigma(n)} - x \right\|_2^2 \leq \sum_{n>N} \frac{1}{n^2} < \epsilon \text{ for all } k \geq m.$$

This establishes our claim. □

The converse of Theorem 4.49 is valid on finite-dimensional spaces. It can be established by considering the series of scalars represented by the components of terms of a given series.

A natural question is whether there exist any infinite-dimensional normed space, in which unconditional convergence implies absolute convergence? The answer is in the negative and was given by Dvoretzky and Rogers in [14].

Theorem 4.51 (Dvoretzky and Rogers, 1950) *In every infinite-dimensional Banach space, there exists an unconditionally convergent series which is not absolutely convergent.*

Hence, we obtain the following characterization of finite-dimensional Banach spaces.

Theorem 4.52 *Let X be a Banach space. Then X is finite-dimensional if and only if every unconditionally convergent series in X is absolutely convergent.*

Example 4.53 The Riemann rearrangement theorem (4.48) is not true for series of complex numbers. For example, no rearrangement of $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ converges to i.

Notes and Remarks 4.54 Note that in Definition 4.47(a), we did not assume that all rearrangements of $\sum_{n=1}^{\infty} x_n$ converge to the same limit. It is, in fact, true that if all rearrangements of a series converge, then these converge to the same limit (see [15, p. 99, Corollary 3.11]).

Various characterizations of unconditional convergence can be found in [15, p. 94, Theorem 3.10]. For a treatise on Rearrangements of series in Banach spaces, see [16]. To explore the relationship between convergence and absolute convergence of series in ordered fields, we refer [17]. In [18], there is an individual chapters devoted to $C(X)$. For a treatise devoted to algebraic properties of $C[0, 1]$, see [19].

Exercise 4.71 Does there exist $p \in [1, \infty]$ such that $\{x \in \ell^p : \|x\|_p \leq 1\}$ is not complete?

Exercise 4.72 If $\sum_{n=1}^{\infty} x_n$ is convergent in a Banach space X, prove that $\left\| \sum_{n=1}^{\infty} x_n \right\| \leq \sum_{n=1}^{\infty} \|x_n\|$.

Exercise 4.73 If Y is a complete subspace of a normed space X, prove that Y is a closed subset of X. Is the converse true? Prove that the converse holds if X is a Banach space.

Exercise 4.74 Prove that a sequence of functions $\{f_n\}$ in $C(X)$ is

(a) uniformly convergent to a function f on X if and only if $f_n \longrightarrow f$ in $(C(X), \|.\|_\infty)$.
(b) uniformly Cauchy if and only if $\{f_n\}$ is Cauchy in $(C(X), \|.\|_\infty)$.

Exercise 4.75 **(Cauchy criterion)** Prove that a series $\sum_{n=1}^{\infty} x_n$ in a normed space is convergent if and only if for every $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that $\left\| \sum_{n=n_1+1}^{n_2} x_n \right\| < \epsilon$ for all $n_2 > n_1 > N$.

Exercise 4.76 Let $(X, \|.\|)$ and $\{e_n : n \in \mathbb{N}\}$ be as in Example 4.43. Show that $(X, \|.\|)$ is a normed linear space over $\mathbb{R}$, $\sum_{n=1}^{\infty} \|e_n\| = 2 < \infty$, and $\sum_{n=1}^{\infty} e_n$ does not converge in c_{00}.

Exercise 4.77 In $(\mathbb{R}^n, \|.\|_1)$, prove that unconditional convergence implies absolute convergence.

Exercise 4.78 Does there exist a series of reals, whose all rearrangements are convergent, but not to the same sum?

Exercise 4.79 Let $\{x_n\}$ be a sequence of reals, convergent to 0. Prove that there exists a sequence $\{a_n\}$ from $\{-1, 1\}$ such that $\sum_{n=1}^{\infty} a_n x_n$ is convergent.

Exercise 4.80 Let $\sum_{n=1}^{\infty} x_n$ be a series of reals and $x \in \mathbb{R}$. Assume that there exists a nonempty collection of permutations $\mathcal{P}$ of $\mathbb{N}$ such that $\sum_{n=1}^{\infty} x_{\pi(n)} = x$ for all $\pi \in \mathcal{P}$ and $\sum_{n=1}^{\infty} x_{\pi(n)}$ diverges, for all permutations π of $\mathbb{N}$, outside $\mathcal{P}$. Prove that $\sum_{n=1}^{\infty} x_n$ converges absolutely.

Exercise 4.81 Let $\sum_{n=1}^{\infty} x_n$ be a convergent series in a normed linear space X and $m \in \mathbb{N}$. If $\sum_{n=1}^{\infty} x_{\pi(n)}$ be a rearrangement of this series by shifting every term at most m places forward, prove that $\sum_{n=1}^{\infty} x_{\pi(n)} = \sum_{n=1}^{\infty} x_n$.

Exercise 4.82 Define $\phi(f) := \int_0^1 f$ for all $f \in C[0, 1]$. Prove that ϕ is a continuous map on $(C[0, 1], \|.\|_\infty)$.

Exercise 4.83 For every $f \in C[a, b]$, define $\|f\|_1 := \int_0^1 |f(x)|dx$. Prove that $\|.\|_1$ is a norm on $C[a, b]$. Is it complete?

Exercise 4.84 Prove that all finite-dimensional normed spaces are Banach.

Exercise 4.85 Is c_{00} complete under $\|.\|_p$ norm, for any $p \in [1, \infty]$?

Exercise 4.86 Prove that the linear space of real polynomials on $[a, b]$ is not a Banach space, under the supremum norm.

Exercise 4.87 Prove that both c_0 and c are Banach spaces, under $\|.\|_\infty$ norm.

Exercise 4.88 Let $\sum_{n=1}^{\infty} x_n$ be a series in a normed linear space X. Prove that $\sum_{n=1}^{\infty} x_n$ converges unconditionally to some $x \in X$ if and only if for every $\epsilon > 0$ there exists a finite set $F_0 \subset \mathbb{N}$ such that

$$\Big\| x - \sum_{n \in F} x_n \Big\| < \epsilon \text{ for every finite set } F \text{ satisfying } F_0 \subset F \subset \mathbb{N}.$$

Exercise 4.89 Let $(X, \|.\|)$ be a normed linear space over a field $\mathbb{K}$ ($\mathbb{R}$ or $\mathbb{C}$) and Ω be collection of all Cauchy sequences in X. For every $\{x_n\}, \{y_n\} \in \Omega$, define

$$\{x_n\} \sim \{y_n\} \text{ if and only if } \lim_{n\to\infty} \|x_n - y_n\| = 0.$$

Show that $\sim$ is an equivalence relation on Ω. Let $B = \Omega/\sim$ denote the set of equivalence classes of Ω with respect to the relation $\sim$. For $[\{x_n\}], [\{y_n\}] \in B$ and $k \in \mathbb{K}$, define

$$k * [\{x_n\}] = [\{kx_n\}], [\{x_n\}] \oplus [\{y_n\}] = [\{x_n + y_n\}] \text{ and } \|[\{x_k\}]\|^* = \lim_{n\to\infty} \|x_n\|.$$

Prove the following:

(a) $\oplus$ and $*$ are well-defined on B.
(b) B is a linear space, under the addition $\oplus$ and scalar multiplication $*$.
(c) $\|.\|^*$ is a norm on B.
(d) X is isometric to a dense linear subspace of B.
(e) $(B, \|.\|^*)$ is a Banach space.

4.6 Hints and Solutions to Selected Exercises

4.3 It is trivial that d is a metric on $\mathbb{R}$. But $(\mathbb{R}, d)$ is not complete. For each $n \in \mathbb{N}$, let $x_n := \tan\left(\frac{\pi}{2} - \frac{1}{n}\right)$. Then the sequence $\{x_n\}$ is Cauchy in $(\mathbb{R}, d)$. If there exists some $a \in \mathbb{R}$ such that $\{x_n\} \longrightarrow a$ in $(\mathbb{R}, d)$, then $\tan^{-1} x_n \longrightarrow \tan^{-1} a$, under usual metric on $\mathbb{R}$. The uniqueness of limits implies that $\tan^{-1} a = \frac{\pi}{2}$ which implies $a \notin \mathbb{R}$, a contradiction.

4.4 Under this map, the sequence $\{n\}$ is Cauchy but not convergent.

4.7 Note that $B[x; \epsilon]$ is the inverse image of the closed set $[0, \epsilon]$ under the continuous map $y \longmapsto d(y, x)$. For the converse, use the fact that every Cauchy sequence is bounded and hence belongs to a closed ball.

4.12 (a) We leave it for the readers to show that ρ is a metric on X.

(b) Since $\rho \leq d$, every Cauchy sequence in (X, d) is Cauchy in (X, ρ). To prove the converse, let $\{x_n\}$ be a Cauchy sequence in (X, ρ) and $\epsilon > 0$ be given. Consider $\delta := \min\left\{\frac{1}{2}, \frac{\epsilon}{2}\right\}$. Let $N \in \mathbb{N}$ such that $\rho(x_n, x_m) < \delta$ for all $m, n \geq N$. Then

$$d(x_n, x_m) < \frac{\delta}{1-\delta} \leq \epsilon \text{ for all } m, n \geq N.$$

Hence, $\{x_n\}$ be a Cauchy sequence in (X, d).

(c) As above, the direct implication follows from the fact that $\rho \leq d$. Conversely, let $x_n \longrightarrow x$ in (X, ρ). Then

$$\rho(x_n, x) = \frac{d(x_n, x)}{1 + d(x_n, x)} \longrightarrow 0 \text{ which implies } d(x_n, x) \longrightarrow 0,$$

by Theorem 1.16. Hence, $x_n \longrightarrow x$ in (X, ρ).

(d) Follows from the above two parts.

4.13 Analogous to Exercise 4.12.

4.14 Let X be complete and $\{x_n\}$ be a sequence in X such that $\sum_{n=1}^{\infty} d(x_{n+1}, x_n) < \infty$. Let $\epsilon > 0$ be given. By Cauchy criterion for series convergence, there exists $m \in \mathbb{N}$ such that

$$\sum_{n=n_1+1}^{n_2} d(x_{n+1}, x_n) < \epsilon \text{ for all } n_2 > n_1 > m.$$

Therefore, by triangle inequality, $d(x_{n_2}, x_{n_1}) < \epsilon$ for all $n_2 > n_1 > m$. Hence, the sequence $\{x_n\}$ is a Cauchy sequence in X and thence convergent.

Conversely, let $\{x_n\}$ be a Cauchy sequence in X. For each $k \in \mathbb{N}$, choose $n_k \in \mathbb{N}$ such that $d(x_n, x_m) < \frac{1}{2^k}$ for all $n, m \geq n_k$. Without loss of generality, we assume that $n_k < n_{k+1}$ for all $k \in \mathbb{N}$. Hence, we obtain a subsequence $\{x_{n_k}\}$ of the sequence $\{x_n\}$ such that $d(x_{n_k}, x_{n_{k+1}}) < \frac{1}{2^k}$ for all $k \in \mathbb{N}$.

The triangle inequality implies that $d(x_{n_k}, x_{n_l}) < \frac{1}{2^{k-1}}$, for any $l \geq k$. Therefore, we have $\sum_{k=1}^{\infty} d(x_{n_{k+1}}, x_{n_k}) < \sum_{k=1}^{\infty} \frac{1}{2^{k-1}} = 2 < \infty$, By hypothesis, there exists $x \in X$ be such that $x_{n_k} \longrightarrow x$. Finally, by Theorem 2.24, we obtain $x_n \longrightarrow x$.

4.15 Try arguments analogous to Theorem 3.33.

4.17 We prove the result for $r > 0$. The case of $r = 0$ is similar. Let $\{x_n\}$ be a Cauchy sequence in (X, ρ_r). Then there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that

$$\rho_r(x_{n_k}, x_{n_{k+1}}) < r^k \text{ for all } k \in \mathbb{N}.$$

Hence, first k terms of x_{n_k} and $x_{n_{k+1}}$ are same. Let a_k denote the k^{th} term of the sequence x_{n_k} and $a = \{a_k\}_k$. Then $\{x_{n_k}\} \longrightarrow a$ in X. Applying Theorem 2.24, $x_n \longrightarrow a$.

4.18 The converse holds if X is complete. For necessity, let $x \in X$ be arbitrary and $x_n \longrightarrow x$ in X. Then the sequence $x_1, x, x_2, x, \ldots, x_n, x, \ldots\}$ is Cauchy, and hence so is image sequence $f(x_1), f(x), f(x_2), f(x), \ldots, f(x_n), f(x), \ldots\}$, which has a constant subsequence. Applying Theorem 2.24, the later sequence converges to $f(x)$, and hence so is its subsequence $\{f(x_n)\}$. Thus, f is continuous at x.

4.19 For any $x \in X = \overline{A}$, let $\{x_n\}$ be a Cauchy sequence in A, convergent to x. By hypothesis, $\{f(x_n)\}$ is Cauchy in Y, and hence convergent. Write $F(x) := \lim_{n\to\infty} f(x_n)$. This defines a map $F : X \longrightarrow Y$, which extends f. Now show that F is the required extension.

To see that F is well-defined, let $x \in X$ and $\{a_n\}$ and $\{b_n\}$ be sequences from A, convergent to x. Since $d(a_n, b_m) \leq d(a_n, x) + d(x, b_m)$, we obtain the sequence $a_1, b_1, \ldots, a_n, b_n, \ldots$ is Cauchy. By hypothesis, its image under f is also a Cauchy sequence. So $\{\rho(f(a_n), f(b_n))\}_n$ converges to 0 and hence $\lim_{n\to\infty} f(a_n) = \lim_{n\to\infty} f(b_n)$.

By Exercise 4.18, it is enough to prove that F maps Cauchy sequences onto Cauchy sequences. Let $\{x_n\} \subset X$ be an arbitrary Cauchy sequence.

Then for each $k \in \mathbb{N}$, there exists a sequence $\{a_{k,n}\} \subset A$ such that $\lim_{n\to\infty} a_{k,n} = x_k$, and hence $\lim_{n\to\infty} f(a_{k,n}) = F(x_k)$. Choose a strictly increasing sequence $\{n_k\} \subset \mathbb{N}$ such that for all $n \geq n_k$ and for all $k \in \mathbb{N}$, we have

$$d(a_{k,n_k}, x_k) < \frac{1}{k} \text{ and } \rho(f(a_{k,n_k}), F(x_k)) < \frac{1}{k}. \tag{4.7}$$

Note that for all $k, l \in \mathbb{N}$, we have the inequalities

$$d(a_{k,n_k}, a_{l,n_l}) \leq d(a_{k,n_k}, x_k) + d(x_k, x_l) + d(x_l, a_{l,n_l}) \tag{4.8}$$

$$\rho(F(x_k), F(x_l)) \leq \rho(F(x_k), f(a_{k,n_k})) + \rho(f(a_{k,n_k}), f(a_{l,n_l})) \tag{4.9}$$

$$+ \rho(f(a_{l,n_l}), F(x_l)). \tag{4.10}$$

Applying (4.7) and (4.8), we conclude that $\{a_{k,n_k}\}_k$ is a Cauchy sequence, and hence so is the sequence $\{f(a_{k,n_k})\}_k$. By (4.9)-(4.10), $\{F(x_k)\}_k$ is a Cauchy sequence in Y. Hence the result.

4.20 No. For example, $f:(0,1)\longrightarrow(0,1)$ defined as $f(x):=x^2/2$ is not a strict contraction, although it satisfies the given condition.

4.22 f has no fixed point.

4.25 Take $f:[1,+\infty)\longrightarrow[1,+\infty)$ defined as $f(x):=x+\frac{1}{x}$. Use Mean Value Theorem, to conclude that f satisfies the given condition.

4.26 Take $X:=(0,1]$ and $f(x):=x/2$ for all $x\in X$.

4.35 False. Let $X:=\mathbb{R}$ with the metric defined by

$$d(x,y):=\frac{|x-y|}{1+|x-y|}\text{ for all }x,y\in X.$$

It can be shown that (X,d) is a complete metric space and has the same topology as the usual topology on $\mathbb{R}$. Let $F_n:=[n,+\infty)$ for all $n\in\mathbb{N}$. Note that $diam(F_n)\le 1$ for all $n\in\mathbb{N}$. Hence, $\{F_n\}$ is a nested decreasing sequence of nonempty closed subsets of X such that $\{diam(F_n)\}$ is a bounded sequence, while $\bigcap_{n=1}^{\infty}F_n=\emptyset$.

4.39 Let E be any nonempty totally bounded subset of a metric space X and $\epsilon>0$ be given. Choose $x_1,\ldots,x_n\in X$ such that $E\subset\bigcup_{i=1}^{n}B(x_i;\epsilon/2)$. If $E\cap B(x_i;\epsilon/2)\neq\emptyset$, pick any $y_i\in E\cap B(x_i;\epsilon/2)$. Then it can be shown $E\subset\bigcup_i B(y_i;\epsilon)$.

4.40 Use Lemma 4.21.

4.42 For example, consider any infinite discrete space X. Then X is bounded, but not totally bounded. As for $\epsilon=1$, the ϵ balls will be singletons.

4.43 Take any finite metric space or consider the example in Exercise 4.54.

4.44 Yes. Let $X:=[0,1]\cup\mathbb{N}$ and $d(x,y):=\frac{|x-y|}{1+|x-y|}$ for all $x,y\in X$. Then X is bounded, as $X\subset B(0;2)$. Also, X is not totally bounded, as $d(n,n+1)=1/2$ for all $n\in\mathbb{N}$.

4.46 Take $E_i:=B\big(x_i;\frac{\epsilon}{2}\big)\cap E$.

4.49 Since E is totally bounded, so is $\overline{E}$. Hence, every subset of $\overline{E}$ is totally bounded. In particular, $E'\subset\overline{E}$. Therefore, E' is totally bounded. The converse is false. For example, take $E:=\mathbb{N}$ in the usual space $\mathbb{R}$.

4.50 Apply Exercise 4.46.

4.52 Apply Theorem 4.22.

4.54 Let E be a nonempty bounded subset of X. Then $E\subset B(x;r)$ for some $x,r>0$. Then

$$B(x;r)=\left(\frac{1}{\frac{1}{x}+r},\frac{1}{\frac{1}{x}-r}\right)=\left(x-\frac{x^2r}{1+xr},x+\frac{x^2r}{1-xr}\right)=(a,b),\quad(say).$$

Note that $a>0$ and $\min\left\{\frac{x^2r}{1+xr},\frac{x^2r}{1-xr}\right\}=\frac{x^2r}{1+xr}$. Also, observe that

$$\frac{t^2\epsilon}{1+t\epsilon} > \frac{a^2\epsilon}{1+b\epsilon} \text{ for all } t \in (a,b).$$

Let $\epsilon > 0$ be given. Pick any $n \in \mathbb{N}$ such that $\frac{1}{n} < \frac{a^2\epsilon}{1+b\epsilon}$. Let N be the next integer to $\frac{b-a}{n}$. Partition $[a,b]$ into N equal parts each having length at most $\frac{1}{n}$, say $a = x_0 < x_1 < \cdots < x_n = b$. Consequently, we obtain

$$E \subset (a,b) \subset \bigcup_{i=1}^{N}[x_{i-1}, x_i] \subset \bigcup_{i=1}^{N} B(x_i; \epsilon).$$

4.56 If $\bigcap_{n=1}^{\infty} I_n = \emptyset$, then $\bigcap_{n=1}^{m} I_n = \emptyset$, for some $m \in \mathbb{N}$. Without loss of generality, suppose that $\bigcap_{n=1}^{m-1} I_n \neq \emptyset$. Then $\bigcap_{n=1}^{m-1} I_n$ is a closed interval, say $[a,b]$. Let $I_n = [c,d]$. If $b < c$. Note that there exists some $k < m$ such that b is the right end point of I_k. Then $I_k \cap I_n = \emptyset$, a contradiction. Similarly, we have a contradiction for the case when $d < a$.

4.57 Note that (a) follows from the definition of $\alpha(A)$. In (b), the sufficiency holds by the Cantor Intersection Property. Below we prove the necessity part of (b).

By induction, for every integer $n > 1$, choose finitely many nonempty closed subsets $A_{n,1}, \ldots, A_{n,m_n}$ of A_n with diameters $< \alpha(A_n)$ such that $\bigcup_{i=1}^{m_n} A_{n,i} = A_n$ and $A_{n+1,i}$ is contained in some $A_{n,j}$. Write $A_{1,1} := A_1$. If $A_{n+1,i}$ is contained in $A_{n,j}$, let us call it as a descendant of $A_{n,j}$. Descendant of a descendant will be called a 2-descendant, and so on.

We claim that for every $n \in \mathbb{N}$, the set $A_{1,1}$ has an n-descendant. Otherwise, there exists $N \in \mathbb{N}$ such that $A_{1,1}$ does not have any N-descendant and thus $A_{N+1} = \emptyset$, a contradiction. Hence, there exists a (infinite) nested decreasing sequence of descendants in our construction, say $\{A_{n,i_n}\}$. By Cantor intersection property,

$$\bigcap_{n=1}^{\infty} A_n \supset \bigcap_{n=1}^{\infty} A_{n,i_n} \neq \emptyset.$$

4.62 Yes, because there are always points x and y in the completion of (X^*, d^*) such that $x_n \longrightarrow x$ and $y_n \longrightarrow y$. The continuity of the metric d^* implies that $d(x_n, y_n) = d^*(x_n, y_n) \longrightarrow d^*(x,y)$. Therefore, $\{d(x_n, y_n)\}$ is Cauchy in $\mathbb{R}$ and hence convergent.

4.63 Let E be any nonempty subset of reals which is bounded above. Let $a \in E$ and b be any upper bound of E. Let U denote the set of upper bounds of E and L denote the set of lower bounds of U. Then $a \in L$ and $b \in U$. Let

$$[a_1, b_1] := \begin{cases} [a, \frac{a+b}{2}] \ ; \ \frac{a+b}{2} \in U, \\ [\frac{a+b}{2}, b] \ ; \ \frac{a+b}{2} \in L. \end{cases}$$

Imitating above, we obtain a nested decreasing sequence of closed intervals $\{[a_n, b_n]\}$ such that $a_n \in L$, $b_n \in U$ and $b_n - a_n = \frac{b-a}{2^n}$ for all $n \in \mathbb{N}$. Since

$\mathbb{R}$ is complete, it satisfies the Cantor Intersection Property (4.15). Hence, $\bigcap_{n=1}^{\infty}[a_n, b_n] = \{x\}$ for some $x \in \mathbb{R}$.

Therefore, $\lim_{n\to\infty} a_n = x = \lim_{n\to\infty} b_n$. Since both U and L are closed sets, we obtain $x \in L \cap U$. We claim that x is the least upper bound of E. Since $x \in U$, it is an upper bound of E. If some $u < x$ is an upper bound of E, then x is not a lower bound of U. Consequently, $x \notin L$, a contradiction.

4.64 Note that for with $|f_x(y)| \leq d(x, a)$ for all $y \in X$. Pick $x, y \in X$. If we write $d^*(f_x, f_y) = \sup_{z\in X} |f_x(z) - f_y(z)|$, then

$$d^*(f_x, f_y) = \sup_{z\in X} |d(x, z) - d(y, z)| \leq d(x, y).$$

Also, for $z = x$ or y, we note that $|f_x(z) - f_y(z)| = |d(x, z) - d(y, z)| = d(x, y)$. Hence, $d^*(f_x, f_y) = d(x, y)$.

4.65 Apply the procedure same as l_∞ to conclude that $L^\infty(X)$ is a complete metric space. As in Exercise 4.64, the map $x \longmapsto f_x$ is an isometry from X into $L^\infty(X)$.

4.66 Let d_X and d_Y, respectively, be the metrics on X and Y. For $x \in X = \overline{X_0}$, choose a sequence $\{x_n\} \subset X_0$ such that $x_n \longrightarrow x$. Since f is an isometry on X_0, the sequence $\{f(x_n)\}$ is Cauchy in the complete metric space Y. Define $f(x) := \lim_{n\to\infty} f(x_n)$. It can be shown that f is well-defined and continuous on X.

Also, for any $y \in Y = \overline{f(X_0)}$, one can choose a sequence $\{x_n\} \subset X_0$ such that $f(x_n) \longrightarrow y$. Since f is an isometry on X_0, $\{x_n\}$ is Cauchy in X_0. Since X is complete, there exists some $x \in X$ such that $x_n \longrightarrow x$. Hence, $f(x_n) \longrightarrow f(x)$. Consequently, $y = f(x)$ and hence f is surjective.

Now pick any $y_1, y_2 \in Y$. Then $y_1 = f(x_1)$ and $y_2 = f(x_2)$, for some $x_1, x_2 \in X$. Further, there are sequences $\{x_{1,n}\}, \{x_{2,n}\} \subset X_0$ such that $x_{1,n} \longrightarrow x_1$ and $x_{2,n} \longrightarrow x_2$. Then $f(x_{1,n}) \longrightarrow y_1$ and $f(x_{2,n}) \longrightarrow y_2$. The continuity of d_Y and d_X implies that

$$d_Y(y_1, y_2) = \lim_{n\to\infty} d_Y(f(x_{1,n}), f(x_{2,n})) = \lim_{n\to\infty} d_X(x_{1,n}, x_{2,n}) = d_X(x_1, x_2).$$

Hence, $f : X \longrightarrow Y$ is a surjective isometry.

4.68 It is immediate that $c_{00} \subset c_0$. We need to prove that $(c_0, \|.\|)$ is a complete normed space and that c_{00} is dense in it. Let $\epsilon > 0$ be given.

Let $\{x^{(k)}\}$ be a Cauchy sequence in c_0. Write $x^{(k)} = \{x_n^{(k)}\}_n$ for all $k \in \mathbb{N}$. Since $\{x^{(k)}\}$ is Cauchy in c_0, for every $n \in \mathbb{N}$, $\{x_n^{(k)}\}_k$ is a Cauchy sequence in the scalar field. Therefore, for each n, there exists some scalar x_n such that $x_n^{(k)} \longrightarrow x_n$, as $k \longrightarrow \infty$. Hence, $x^{(k)} \longrightarrow \{x_n\}$ in c_0.

If $x = \{x_n\}_n \in c_0$, there exists $m \in \mathbb{N}$ such that $|x_n| < \epsilon$ for all $n > m$. Let $y_n = x_n$ for all $n \leq m$ and 0 otherwise. Then $y = \{y_n\} \in c_{00}$ and $\|x - y\| < \epsilon$. Hence, c_{00} is dense in $(c_0, \|.\|)$.

4.73 Apply Theorems 4.2 and 4.3.

4.75 Let $S_n := \sum_{k=1}^{n} x_k$ for all $n \in \mathbb{N}$. Note that $\sum_{n=1}^{\infty} x_n$ is convergent if and only if $\{S_n\}$ is convergent if and only if $\{S_n\}$ is Cauchy if and only if the given condition is satisfied.

4.79 Choose a strictly increasing sequence $\{m_n\} \subset \mathbb{N}$ such that $|x_k| < 2^{-n}$ for all $k \geq m_n$. Without loss of generality, suppose that $m_1 = 1$. Since all the terms $x_1, \ldots, x_{m_2-1}$ have magnitude less than $1/2$, one can choose $\{a_1, \ldots, a_{m_2-1}\} \subset \{-1, 1\}$ such that $0 \leq \sum_{k=1}^{p} a_k x_k \leq 1$ for all $p < m_2$. Then we choose $\{a_{m_1}, \ldots, a_{m_2-1}\} \subset \{-1, 1\}$ such that

$$0 \leq \sum_{k=m_2}^{p} a_k x_k \leq \frac{1}{2} \text{ for all } m_2 \leq p < m_3.$$

Continuing like this, and comparing with $\sum_{n=1}^{\infty} 2^{-n}$, we conclude the result.

4.80 Suppose $\sum_{n=1}^{\infty} x_n$ doesn't converge absolutely. Let $\pi \in \mathcal{P}$ and $y \in \mathbb{R} \setminus \{x\}$. Then $\sum_{n=1}^{\infty} x_{\pi(n)}$ is convergent, but not absolutely. Hence, there exists a rearrangement of $\sum_{n=1}^{\infty} x_{\pi(n)}$, which is also a rearrangement of $\sum_{n=1}^{\infty} x_{\pi(n)}$, convergent to y. This contradicts our hypothesis.

4.81 For $n \in \mathbb{N}$, let S_n and P_n denote the n^{th} partial sum of given series and its rearrangement, respectively. Let $\epsilon > 0$ be given and choose $N \in \mathbb{N}$ such that $\|x_k\| < \frac{\epsilon}{2m}$ for all $k > N$. The result follows, as for every $n > N + m$, we have

$$\|S_n - P_n\| = \left\| \sum_{k=n-m+1}^{n} \left(x_k - x_{\pi(k)}\right) \right\| \leq \sum_{k=n-m+1}^{n} \left(\|x_k\| + \|x_{\pi(k)}\|\right) < \epsilon.$$

4.82 Follows from the inequality $|\phi(f) - \phi(g)| \leq \int_0^1 \left|f - g\right| \leq \|f - g\|_\infty$.

4.86 If $f_n(x) := x^n \chi_{[0,1]}$, then $\{f_n\}$ is Cauchy, but not convergent in the given space.

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Chapter 5
Compactness

In this chapter, we discuss the notion of compactness, its characterizations, and its relationship with continuity. We start with a basic introduction to compact sets, their relationship with closed subsets, and the case of Euclidean spaces. Then we present various characterizations of compactness; in terms of finite intersection property, sequential compactness, and totally bounded sets. We provide a thorough discussion on continuity and compactness; particularly on the uniform and Lipschitz continuities. We also present some necessary and sufficient conditions for the uniform continuity of real functions. Some recent results regarding UC spaces, strong uniform continuity and Cauchy continuous maps, which haven't yet made their way into any textbook, are also outlined in the exercises.

5.1 Introduction

In elementary calculus courses, the Extreme Value Theorem for continuous real functions f on a closed and bounded interval $[a, b]$ is often stated without proof. To prove it, note that every $x \in [a, b]$ has an open neighborhood O_x on which f is bounded. In this chapter, we shall show that $[a, b]$ is contained in finitely many such neighborhoods $O_{x_1}, \dots, O_{x_n}$; and hence f is bounded on $[a, b]$. Motivated by this fact, we have the following definitions.

Definitions 5.1 Let X be a metric space, $K \subset X$ and Ω be any collection of subsets of X. We say that

(a) Ω is an *open cover* of K, if each member of Ω is an open set and $K \subset \bigcup_{O \in \Omega} O$.
(b) Ω_0 is a *subcover* of K, with respect to an open cover Ω of K, if $K \subset \bigcup_{O \in \Omega_0} O$ and $\Omega_0 \subset \Omega$.
(c) K is *compact,* if every open cover of K has a finite subcover.

To provide visual clues, compact sets are often denoted by K, the first letter of the German word *'Kompakt'*.

S. P. S. Kainth, *A Comprehensive Textbook on Metric Spaces*,
https://doi.org/10.1007/978-981-99-2738-8_5

Examples 5.2 (a) In any metric space, finite sets are compact.

(b) $\mathbb{R}$ is not compact, as $\{(-n, n) : n \in \mathbb{N}\}$ is an open cover of $\mathbb{R}$, having no finite subcover.

(c) Let $X := \mathbb{N}$, under discrete metric. Then X is not compact. as $\{\{n\} : n \in \mathbb{N}\}$ is an open cover of X, having no finite subcover.

Therefore, the notion of compactness is a generalization of finite sets. In fact, it is a much stronger notion, than boundedness and even total boundedness.

Proposition 5.3 *Compact metric spaces are totally bounded.*

Proof Let X be a compact metric space and $\epsilon > 0$ be given. Since $\{B(x; \epsilon) : x \in X\}$ is an open cover of X, there are finitely many $x_1, \ldots, x_n \in X$ such that $X \subset \bigcup_{i=1}^{n} B(x_i; \epsilon)$. Hence, X is totally bounded. □

Corollary 5.4 *In metric spaces, compact sets are bounded.*

Proof Apply Proposition 5.3 along with Theorem 4.18. □

The converse of Proposition 5.3 is not true.

Example 5.5 Let $X := (0, 1)$, under the usual metric. Then X is bounded as well as totally bounded. But X is not compact, as $\{(1/n, 1) : n \in \mathbb{N}\}$ is an open cover of $(0, 1)$, having no finite subcover.

Soon in Theorem 5.26, we shall establish that a metric space X is compact if and only if X is complete and totally bounded.

The notion of compact sets is independent of the embedded space.

Proposition 5.6 *Let X be a metric space and $K \subset Y \subset X$. Then K is compact relative to X if and only if K is compact relative to Y.*

Proof ($\Rightarrow$) Let Ω_Y be an open cover of K, in Y. Consider the collection

$$\Omega_X := \{G : G \text{ open in X such that } G \cap Y \in \Omega_Y\}.$$

Pick any $U \in \Omega_Y$. Since U is open in Y, by Theorem 3.22, there exists a set $G \in \Omega_X$ such that $U = G \cap Y$. Therefore, $K \subset \bigcup_{U \in \Omega_Y} U \subset \bigcup_{G \in \Omega_X} G$. Hence, Ω_X is an open cover of K, in X.

Since K is compact in X, we have $K \subset \bigcup_{i=1}^{n} G_i$, for finitely many $G_1, \ldots, G_n \in \Omega_X$. For each $i = 1, \ldots, m$, if $U_i := G_i \cap Y$, then $U_i \in \Omega_Y$ and we have

$$K = K \cap Y \subset \bigcup_{i=1}^{n} G_i \cap Y = \bigcup_{i=1}^{n} (G_i \cap Y) = \bigcup_{i=1}^{n} U_i.$$

($\Leftarrow$) Let Ω_X be an open cover of K, in X. Write $\Omega_Y := \{G \cap Y : G \in \Omega_X\}$. By Theorem 3.22, each $U \in \Omega_Y$ is open in Y. Also, since $K \subset Y$ and $K \subset \bigcup_{G \in \Omega_X} G$ we obtain $K \subset (\bigcup_{G \in \Omega_X} G) \cap Y = \bigcup_{G \in \Omega_X} (G \cap Y)$.

Therefore, Ω_Y is an open cover of K, in Y. By hypothesis, $K \subset \bigcup_{i=1}^{n} (G_i \cap Y)$, for finitely many $G_1, \ldots, G_n \in \Omega_X$. Hence, $K \subset \bigcup_{i=1}^{n} G_i$ and the result follows. □

Remarks 5.7 (a) By the virtue of Proposition 5.6, without loss of generality, we may restrict our discussion to compact metric spaces, rather than compact subsets of a metric space.

(b) It is pertinent to mention that most of the authors explicitly write open covers in terms of an indexing set, to denote its arbitrary elements. For example, if Ω is an open cover of some set, then one can write $\Omega = \{O_\alpha : \alpha \in \wedge\}$ for some arbitrary indexing set $\wedge$. We shall be intentionally avoiding this terminology, which often confuses the first-time readers.

(c) Further, it must be noted that not every open cover can be written as a sequence of sets. In Chap. 7, we shall discuss a variety of sets that can be written as a sequence.

5.1.1 Compact Sets and Closed Sets

Theorem 5.8 *Compact subsets of metric spaces are closed.*

Proof Let K be a compact subset of a metric space X. We claim that $K^c (= X \setminus K)$ is an open set. If $K = X$, this is trivial. Otherwise, let $a \in K^c$. For each $b \in K$, let $r_b := d(a, b)/2$. Since each $b \neq a$, we have $r_b > 0$.

Since K is compact and $\{B(b; r_b) : b \in K\}$ is an open cover of K, there are finitely many $b_1, \ldots, b_n \in K$ such that $K \subset \bigcup_{i=1}^n B(b_i; r_{b_i}) = U$ (say).

Then $V := \bigcap_{i=1}^n B(a; r_{b_i})$ is an open set containing a. Since $B(a; r_b) \cap B(b; r_b) = \emptyset$, for each $b \in K$, we have $V \cap U = \emptyset$. Therefore, $V \subset U^c \subset K^c$. Hence, K^c contains a neighborhood V of a. This implies that K^c is open. Hence, K is closed. □

The converse of Theorem 5.8 is not true.

Examples 5.9 The following metric spaces X are closed, but not compact.

(a) $\mathbb{R}$ is a closed subset of itself, but not bounded. Hence, $\mathbb{R}$ is not compact.

(b) As in Example 5.2(c), one can conclude that every infinite discrete metric space is closed and bounded, but not compact.

Theorem 5.10 *Closed subsets of compact sets are compact.*

Proof Let K be a compact and F be a closed subset of a metric space X such that $F \subset K \subset X$. Let Ω be an open cover of F. Then $\{F^c\} \cup \Omega$ is an open cover of K.

Since K is compact, there are finitely many $O_1, \ldots, O_n \in \Omega$ such that $K \subset (\bigcup_{i=1}^n O_i) \cup F^c$. Since $F \subset K$, we have $F \subset K \subset (\bigcup_{i=1}^n O_i) \cup F^c$. Therefore, $F \subset \bigcup_{i=1}^n O_i$. Hence the result. □

Remark 5.11 Alternate proofs of Theorems 5.8 and 5.10 will be suggested in Exercises 5.13 and 5.14, respectively. However, the previous proofs are important for theoretical reasons, as general topological spaces may not be metrizable (see Definition D.4). So the notions of completeness or Cauchy sequences do not exist over there. The above proofs of Theorems 5.8 and 5.10 are open to generalizations, even in ‘suitable’ topological spaces, by replacing balls with suitable open sets.

5.1.2 Compact Subsets of Euclidean Spaces

Definition 5.12 A subset E of $\mathbb{R}^k$ is said to be a *k-cell* if E is a Cartesian product of k closed and bounded intervals.

Note that 1-cells are closed bounded intervals, 2-cells are closed rectangles in $\mathbb{R}^2$ and 3-cells are cuboids in $\mathbb{R}^3$.

Theorem 5.13 *Every k-cell is compact.*

Proof Assume that the result is not true. Then there exists a non-compact k-cell, say I. Let Ω be an open cover of I that has no finite subcover.

Write $I := \prod_{i=1}^{k}[a_i, b_i]$. By bisecting each interval $[a_i, b_i]$, we obtain a partition of I into 2^k k-cells, say $J_1, \ldots, J_{2^k}$. If for each j, the k-cell J_j has a finite subcover $\mathcal{F}_j$, then $\bigcup_{j=1}^{2^k} \mathcal{F}_j$ will form a finite subcover of I, which is not possible. Hence, there exists j_0 such that J_{j_0} has no finite subcover from Ω. Let $I_1 := J_{j_0}$. Note that $diam(I_1) = diam(I)/2$.

Inducting like this, we obtain a nested decreasing sequence of closed bounded k-cells $\{I_n\}$ such that no I_n has a finite subcover from Ω and $diam(I_n) = diam(I)/2^n \longrightarrow 0$. Applying Cantor Intersection Property (4.15), we obtain $\bigcap_{n=1}^{\infty} I_n = \{x\}$ for some $x \in \mathbb{R}^k$.

Since $x \in I \subset \bigcup_{O \in \Omega} O$, there exists some $O_1 \in \Omega$ such that $x \in O_1$. Since O_1 is an open set, there exists some $\delta > 0$ such that $B(x; \delta) \subset O_1$.

Also, as $\lim_{n\to\infty} diam(I_n) = 0$, there exists $N \in \mathbb{N}$ such that $diam(I_N) < \delta$. Hence, we obtain $I_N \subset B(x; \delta) \subset O_1$, a contradiction to the choice of I_N. □

Theorem 5.14 (Heine-Borel) *Let $k \in \mathbb{N}$ and E be a subset of $\mathbb{R}^k$. Then E is compact if and only if E is closed and bounded.*

Proof The direct implication is immediate by Theorem 5.8 and Corollary 5.4. For the converse, suppose that E is closed and bounded. Since E is bounded, $E \subset I$ for some k-cell I. By Theorem 5.13, I is compact. Since E is closed, applying Theorem 5.10, we conclude that E is compact. □

Note that Theorem 5.14 is not true for general metric spaces.

Examples 5.15 (a) $(\sqrt{2}, \sqrt{3}) \cap \mathbb{Q}$ is closed and bounded in $\mathbb{Q}$, but not compact.
(b) Every infinite discrete metric space is closed and bounded, but not compact.

In Theorems 5.26 and 9.52, we shall prove generalizations of the Heine-Borel Theorem (5.14) for arbitrary metric spaces and finite-dimensional normed linear spaces, respectively. Also, see Theorem 8.52.

Exercise 5.1 Prove that the set $\{1/n : n \in \mathbb{N}\}$ is not a compact subset of reals, by obtaining an open cover of this set, having no finite subcover.

Exercise 5.2 Find an open cover of $(-1, 1)$ in $\mathbb{R}$ having no finite subcover. Also, find open covers of $B(0; 1)$ in $(\mathbb{R}^2, d_2)$ and in $(\mathbb{R}^2, d_\infty)$ having no finite subcovers.

Exercise 5.3 If $a \in A \subset \mathbb{R}$ such that $A \setminus \{a\}$ is compact. Prove that A is compact.

Exercise 5.4 Prove that finite intersection of compact sets is compact. Can we replace 'finite' with 'infinite' here?

Exercise 5.5 Prove that finite union of compact sets is compact. Can you replace 'finite' with 'infinite' here?

Exercise 5.6 Let $K := \{1/n : n \in \mathbb{N}\} \cup \{0\} \subset \mathbb{R}$ and Ω be any collection of open intervals covering K. Obtain a finite subcover of K from Ω to conclude that K is compact.

Exercise 5.7 If F is a closed and K is a compact subset of a metric space X, then prove that $F \cap K$ is compact.

Exercise 5.8 Is there a non-discrete space that is bounded but not compact?

Exercise 5.9 Using least upper bound property, prove that every closed and bounded real interval is compact.

Exercise 5.10 Let X and Y be metric spaces, $x \in X$, A and B be compact subsets of X and Y, respectively. Prove that

(a) $\{x\} \times Y$ is a compact subset of $X \times Y$.
(b) If Ω is an open cover of $\{x\} \times B$ in $X \times Y$, then there exists a neighborhood U of X such that $U \times B$ can be covered by only finitely many elements from Ω.
(c) $A \times B$ is compact in $X \times Y$.

5.2 Characterizations of Compact Sets

In this section, we present some equivalent approaches to compactness. These are given by the notions of completeness, total boundedness, finite intersection property, and sequential compactness. The former two have already been discussed in detail, in the previous chapter. Now we discuss the latter two, which will be used to establish several characterizations of compactness at the end of this section.

5.2.1 Finite Intersection Property

Definition 5.16 A collection of closed sets $\mathcal{F}$ is said to have the *finite intersection property* if the intersection of every finite subcollection of $\mathcal{F}$ is nonempty.

Examples 5.17 (a) If $\{F_n\}$ is a nested decreasing sequence of nonempty closed subsets of a metric space, then $\{F_n : n \in \mathbb{N}\}$ has finite intersection property.
(b) Each of the following families of subsets of $\mathbb{R}$ has the finite intersection property: $\{[0, x] : x \in \mathbb{R}\}$, $\{[-r, r] : r \in \mathbb{Q}\}$ and $\{[1/n, n] : n \in \mathbb{N}\}$.

The next result establishes that compactness is equivalent to a generalization of the Cantor Intersection Property.

Theorem 5.18 *A metric space X is compact if and only if every collection of closed subsets of X with the finite intersection property has a nonempty intersection.*

Proof Let X be compact and $\mathcal{F}$ be a collection of closed sets with finite intersection property. Assume that $\bigcap_{F\in\mathcal{F}} F = \emptyset$. Then $X = \bigcup_{F\in\mathcal{F}} F^c$. Since each $F \in \mathcal{F}$ is closed, its complement F^c is open. Since X is compact, there exist finitely many $F_1, \ldots, F_n \in \mathcal{F}$, such that $X \subset \bigcup_{i=1}^{n} F_i^c$. Therefore, $\bigcap_{i=1}^{n} F_i = X^c = \emptyset$, a contradiction.

Conversely, assume that every collection of closed subsets of X with finite intersection property has a nonempty intersection. To prove that X is compact, let Ω be an open cover of X. That is $X = \bigcup_{O\in\Omega} O$. Therefore, $\bigcap_{O\in\Omega} O^c = X^c = \emptyset$. By hypothesis, there exists a finite collection of sets $O_1, \ldots, O_m$ from Ω such that $\bigcap_{i=1}^{m} O_i^c = X^c = \emptyset$. Hence, $X = \bigcup_{i=1}^{m} O_i$. This proves that X is compact. □

Corollary 5.19 *If X is a compact metric space, then X is complete.*

Proof Let $\{F_n\}$ be a nested decreasing sequence of nonempty closed subsets of X such that $diam(F_n) \longrightarrow 0$. If $n_1, \ldots, n_k \in \mathbb{N}$ and $N := \max\{n_1, \ldots, n_k\}$, then $\bigcap_{i=1}^{k} F_{n_i} = F_N \neq \emptyset$. Hence, the collection of closed sets $\{F_n : n \in \mathbb{N}\}$ has finite intersection property. Since X is compact, by Theorem 5.18, $\bigcap_{n=1}^{\infty} F_n \neq \emptyset$. Applying Theorem 4.14, X is complete. □

It is interesting to note that the notion of completeness can also be completely characterized in terms of the finite intersection property.

Theorem 5.20 *A metric space (X, d) is complete if and only if every family of closed subsets of X having finite intersection property that contains sets with arbitrary small diameters, has nonempty intersection.*

Proof The converse follows from Corollary 4.15. Assume that (X, d) is a complete metric space. Let $\mathcal{F}$ be a family of closed subsets of X having finite intersection property such that $\mathcal{F}$ contains sets with arbitrary small diameter.

For every $n \in \mathbb{N}$, choose $A_n \in \mathcal{F}$ such that $diam(A_n) < 1/n$ and define $F_n := \bigcap_{m\leq n} A_m$. By hypothesis, $\{F_n\}$ is a sequence of nonempty closed subsets of X with $diam(F_n) \leq diam(A_n) \longrightarrow 0$. By Corollary 4.15, $\bigcap_{n=1}^{\infty} F_n$ is a singleton, say $\{x\}$.

Pick any $A \in \mathcal{F}$. Then repeating the above procedure with $A \cap A_n$ instead of A_n, we conclude that

$$\emptyset \neq \bigcap_{n=1}^{\infty}(A \cap A_n) = A \cap \Big(\bigcap_{n=1}^{\infty} A_n\Big) = A \cap \{x\}.$$

Hence, $x \in A$. Since $A \in \mathcal{F}$ was arbitrary, we conclude that $\bigcap_{A \in \mathcal{F}} A \neq \emptyset$. □

As an application of Theorem 5.18, we now present a sufficient condition for the uniform convergence of some particular types of monotone sequences of continuous functions (see also Exercise 5.59).

Theorem 5.21 (Ulisse Dini) *Let X be a compact metric space and $\{f_n\}$ be a sequence of $X \longrightarrow \mathbb{R}$ functions, pointwise convergent to f on X such that*

(a) each f_n is a continuous function,
(b) $\{f_n\}$ is a monotone sequence of functions on X, and
(c) f is a continuous function on X.

Then $f_n \longrightarrow f$ uniformly on X.

Proof If $\{f_n\}$ is a monotonically decreasing sequence, let $g_n := f_n - f$, otherwise let $g_n := f - f_n$. Then each g_n is continuous, $g_n \geq g_{n+1}$ on X and $g_n \longrightarrow 0$ pointwise on X. It is enough to prove that $g_n \longrightarrow 0$ uniformly. Let $\epsilon > 0$ be given. Define

$$K_n := \{x \in X : g_n(x) \geq \epsilon\}.$$

Let $x \in X$ be arbitrary. Since $g_n(x) \longrightarrow 0$, there exists some $n_x \in \mathbb{N}$ such that $g_{n_x}(x) < \epsilon$. Therefore, $x \notin K_{n_x}$ and thus $x \notin \bigcap_{n=1}^{\infty} K_n$. Hence, $\bigcap_{n=1}^{\infty} K_n = \emptyset$.

Since each g_n is continuous, each K_n is closed. Since X is compact, by Theorem 5.18, $\bigcap_{n=1}^{N} K_n = \emptyset$ for some $N \in \mathbb{N}$. Further, $g_n \geq g_{n+1}$ implies $K_n \supset K_{n+1}$. Hence, we conclude that $K_N = \bigcap_{n=1}^{N} K_n = \emptyset$.

Therefore, for all $n \geq N$, $K_n = \emptyset$ which implies that $g_n(x) < \epsilon$ for all $x \in X$. Hence, $f_n \longrightarrow f$ uniformly on X. □

It must be noted that no hypothesis of the above theorem is redundant.

Examples 5.22 We here provide a few counter examples. It can be shown that these sequences of functions $\{f_n\}$ are monotone and pointwise convergent on X, but are not uniformly convergent on X. Also, recall that completeness and total boundedness both are weaker than the notion of compactness. In Theorem 5.21,

(a) completeness of X is not redundant. For example, let $X := (0, 1)$ and

$$f_n(x) := \frac{1}{1 + nx} \text{ for all } x \in X \text{ and for all } n \in \mathbb{N}.$$

(b) total boundedness of X is not redundant. For example, let $X := [0, +\infty)$ and

$$f_n(x) := \frac{x}{n} \text{ for all } x \in X \text{ and for all } n \in \mathbb{N}.$$

(c) the continuity of the functions f_n, in a tail of the sequence $\{f_n\}$ is not redundant. For example, let $X := [0, +\infty)$ and for every $n \in \mathbb{N}$, define

$$f_n(x) := \begin{cases} 1 \,; x \in (0, 1/n), \\ 0 \,; x \in \{0\} \cup [1/n, 1]. \end{cases}$$

(d) the continuity of f is not redundant. For example, let X be the usual space $[0, 1]$ and $f_n(x) := x^n$ for all $x \in X$ and for all $n \in \mathbb{N}$.

5.2.2 Sequentially Compact Sets

Definition 5.23 A subset K of a metric space is said to be *sequentially compact* if every sequence in K has a subsequence, convergent in K.

Due to our next result, sequential compactness is also known as *Bolzano-Weierstrass Property*. We shall also prove that this is equivalent to the notion of compactness, in metric spaces.

Theorem 5.24 *Let K be a subset of a metric space X. Then K is sequential compact if and only if every infinite subset of K has a limit point in K.*

Proof Let E be an infinite subset of K. Choose a sequence $\{x_n\}$ of distinct terms from E. If K is sequential compact, then $\{x_n\}$ has a subsequence, convergent to some $x \in K$. Hence, x is a limit point of E in K, as required.

Conversely, let $\{x_n\}$ be a sequence in K and $E := \{x_n : n \in \mathbb{N}\}$. If E is finite, at least one term of $\{x_n\}$ will repeat infinitely often. Consequently, $\{x_n\}$ will have a constant subsequence. In case E is an infinite subset of K, by hypothesis, it has a limit point in K, say x. Thus, there exists a sequence $\{y_n\}$, of distinct elements from E, convergent to x.

Note that $\{y_n\}$ may not be a subsequence of $\{x_n\}$. In that case, let $n_1 \in \mathbb{N}$ such that $x_{n_1} = y_1$. Suppose $n_1, \ldots, n_k$ have been selected. Choose n_{k+1} to be the least integer larger than n_k such that

$$x_{n_{k+1}} \in \{y_n : n \in \mathbb{N}\} \setminus \{x_{n_1}, \ldots, x_{n_k}\}.$$

Note that this is possible since $\{y_n\}$ is a sequence of distinct terms from E. Thus, by induction, we obtain a subsequence $\{x_{n_k}\}$ of $\{x_n\}$, which is also a subsequence of $\{y_n\}$. Since $y_n \longrightarrow x$, we obtain $x_{n_k} \longrightarrow x$. Hence the result. $\square$

As earlier, one can show that the notion of sequential compactness is independent of the embedded space. Therefore, we may restrict to only sequentially compact metric spaces, instead of sequentially compact subsets of metric spaces.

Theorem 5.25 *A metric space X is sequentially compact if and only if X is complete and totally bounded.*

Proof Let X be a sequentially compact metric space. To prove that it is complete, let $\{x_n\}$ be a Cauchy sequence in X. By hypothesis, there exists a subsequence of $\{x_n\}$ convergent to some $x \in X$. Since $\{x_n\}$ is Cauchy, by Theorem 2.24, $\{x_n\} \longrightarrow x$. Therefore, X is complete.

Assume that X is not totally bounded. By Theorem 4.22, there exists a sequence $\{x_n\}$ in X which has no Cauchy subsequence. Therefore, no subsequence of $\{x_n\}$ is convergent in X. Hence, X is not sequentially compact, a contradiction. Hence, X is totally bounded.

Conversely, let X be a complete and totally bounded metric space and $\{x_n\}$ be a sequence in X. By Theorem 4.22, it has a Cauchy subsequence, say $\{x_{n_k}\}$. Since X is complete, this subsequence converges to some point of X. Hence the result. □

Now we present a generalization of Theorem 5.14.

Theorem 5.26 (Heine-Borel) *A metric space is compact if and only if it is complete and totally bounded.*

Proof The direct implication holds by Corollary 5.19 and Proposition 5.3. We prove the converse, which is analogous to Theorem 5.13.

Assume that there exists a complete and totally bounded metric space X, which is not compact. Then there exists an open cover, say Ω, of X which has no finite subcover.

Since X is totally bounded, there are finitely many $x_1, \dots, x_n \in X$ such that $X \subset \bigcup_{i=1}^{n} \overline{B(x_i; 1/2)}$. If for each i, the set $\overline{B(x_i; 1/2)}$ has a finite subcover $\mathcal{F}_i$, then $\bigcup_{i=1}^{n} \mathcal{F}_i$ will form a finite subcover of X, which is not possible. Hence, one can choose i_0 such that $\overline{B(x_{i_0}; 1/2)}$ has no finite subcover from Ω. Let $B_1 := \overline{B(x_{i_0}; 1/2)}$. Note that $diam(B_1) \leq 1$.

Since X is totally bounded, so is its subset B_1. As earlier, choose a closed set $B_2 \subset B_1$ such that $diam(B_2) \leq 1/2$ and B_2 has no finite subcover from Ω.

Inducting like this, we obtain a nested decreasing sequence of closed sets $\{B_n\}$ such that no B_n has a finite subcover from Ω and $diam(B_n) \leq 1/2^{n-1}$ for all $n \in \mathbb{N}$. By Cantor Intersection Property (4.15), we obtain $\bigcap_{n=1}^{\infty} B_n = \{x\}$, for some $x \in X$.

Since $x \in X \subset \bigcup_{O \in \Omega} O$, there exists some $O_x \in \Omega$ such that $x \in O_x$. Since O_x is an open set, there exists some $\delta > 0$ such that $B(x; \delta) \subset O_x$.

Also, as $\lim_{n\to\infty} diam(B_n) = 0$, there exists $N \in \mathbb{N}$ such that $diam(B_N) < \delta$. Therefore, we obtain $B_N \subset B(x; \delta) \subset O_x$, a contradiction to the choice of B_N. □

Winding up some previous results, we present a few characterizations of compactness.

Theorem 5.27 (Borel-Lebesgue) *Let X be any metric space. The following are equivalent:*

(a) X is compact.
(b) X is complete and totally bounded.
(c) Every infinite subset of X has a limit point in X.
(d) Every sequence in X has a subsequence convergent in X.
(e) Every family of closed subsets of X with finite intersection property has a nonempty intersection.

Proof Apply Theorems 5.26, 5.25, 5.24, and 5.18. □

An alternate proof for compactness implies sequential compactness is provided below. It is important for theoretical reasons, as it can be extended to even general topological spaces.

Theorem 5.28 *Let X be a compact metric space. Then every infinite subset of X has a limit point in X.*

Proof Suppose not. Let E be an infinite subset of X such that $E' = \emptyset$. Then for every $x \in X$, one can choose an open set O_x containing x such that O_x contains at most one element of E, namely x. That is $O_x \cap E \subset \{x\}$ for every $x \in X$.

Note that $\{O_x : x \in X\}$ is an open cover of X. Since X is compact, there are $x_1, \ldots, x_n \in X$ such that $X \subset \bigcup_{i=1}^n O_{x_i}$. Therefore, $E = E \cap X \subset \bigcup_{i=1}^n (E \cap O_{x_i}) \subset \{x_1, \ldots, x_n\}$. This is a contradiction, as E is an infinite set. □

Another characterization of compactness will be provided in Theorem 7.27.

Remarks 5.29 There is a notion of Henstock-Kurzweil integration, more general than Riemann, Lebesgue, and improper integrals (see [1]). The well-definedness of this integral is ensured by Cousin's property, which completely characterizes compactness on metric spaces (see [2, Theorems 3-4]). A related characterization of completeness is also given in [2, Theorem 5].

History Notes 5.30 The notion of sequential compactness appeared much earlier than open covers. The sequential compactness of $\mathbb{R}$ was first proved by Bolzano (1817) as a lemma to his proof of Intermediate Value Theorem. Open covers were introduced by Dirichlet in his 1862 lectures, which were published in 1904. Various other mathematicians contributed to this idea, e.g. Heine (1872), Borel (1895), Cousin (1895), Lebesgue (1898), Alexandroff, and Urysohn (1929). For a pedagogical history of compactness, we refer [3].

Exercise 5.11 Let X be a compact metric space, O be an open subset of X and $\mathcal{F}$ be a collection of closed subsets of X such that $\bigcap_{F \in \mathcal{F}} F \subset O$. Prove that there exist finitely many $F_1, \ldots, F_n \in \mathcal{F}$ such that $\bigcap_{i=1}^n F_i \subset O$.

Exercise 5.12 Let X be an infinite discrete space. Prove that X is not compact, by constructing a collection of closed sets with empty intersection and finite intersection property.

Exercise 5.13 Prove Theorem 5.8, using Corollary 5.19 and Theorem 4.2.

Exercise 5.14 Write an alternative proof of Theorem 5.10, using Corollary 5.19, Theorem 4.3, Proposition 5.3, Exercise 4.38 and Theorem 5.26.

Exercise 5.15 Prove that each of the four sequences of functions $\{f_n\}$, given by Examples 5.22 are monotone and pointwise convergent on X, but are not uniformly convergent on X.

Exercise 5.16 Prove that the notion of sequential compactness is independent of the embedded space.

Exercise 5.17 Using sequential compactness, prove that every real valued continuous function on a compact set is bounded.

Exercise 5.18 Let X be a metric space and $E \subset X$. Prove the following:

(a) If X is complete and E is totally bounded, then $\overline{E}$ is compact.
(b) X is totally bounded if and only if its completion is compact.

Exercise 5.19 Let X be a compact metric space. Prove that for every nested decreasing sequence $\{F_n\}$ of nonempty closed subsets of X, we have $\bigcap_{n=1}^{\infty} F_n \neq \emptyset$. Is the converse true?

Exercise 5.20 Let $A, B \subset \mathbb{R}^2$ and $A + B := \{a + b : a \in A, b \in B\}$. Prove or disprove:

(a) If A is an open subset of $\mathbb{R}^2$, then so is $A + B$.
(b) If A and B are closed subsets of $\mathbb{R}^2$, then so is $A + B$.
(c) If A is compact and B is closed in $\mathbb{R}^2$, then $A + B$ is closed in $\mathbb{R}^2$.
(d) If A and B are compact subsets of $\mathbb{R}^2$, then so is $A + B$.

What happens when A and B are subsets of a normed linear space?

Exercise 5.21 If $p : \mathbb{C} \longrightarrow \mathbb{C}$ is a polynomial, prove that $p^{-1}(K)$ is compact for every compact set $K \subset \mathbb{C}$.

Exercise 5.22 An infinite subset S of a metric space X said to be *convergent* to $x \in X$ if every neighborhood of x contains all but finitely many elements of S.

(a) If S converges to x, prove that $x \in S'$. Is the converse true?
(b) Prove that X is compact if and only if every infinite subset of X contains a subset which converges to an element of X.

Exercise 5.23 Without using Theorem 4.22, prove directly from the definitions that sequentially compact sets are totally bounded.

Exercise 5.24 Is $d(x, y) := |\tan^{-1} x - \tan^{-1} y|$ a totally bounded metric on $\mathbb{R}$?

Exercise 5.25 Is the metric space $(\mathbb{R}, d)$ of Exercise 5.24 compact?

Exercise 5.26 Establish the equivalences of Theorem 5.27 for pseudo-metric spaces.

Exercise 5.27 Let X be a metric space. Prove that X is totally bounded if and only if the completion of X is compact.

Exercise 5.28 Let (X, d) and $\alpha(A)$ be as in Exercise 4.57. Prove that X is complete if and only if for every nested decreasing sequence $\{A_n\}$ of nonempty closed subsets of X such that $\alpha(A_n) \longrightarrow 0$, the intersection $\bigcap_{n=1}^{\infty} A_n$ is nonempty and compact.

Exercise 5.29 Does there exist a metric d on $\mathbb{R}^* := [-\infty, +\infty]$ such that $(\mathbb{R}^*, d)$ is a compact metric space and $\mathbb{R}$ under usual metric is a subspace of $(\mathbb{R}^*, d)$?

Exercise 5.30 Is there any metric d on $\mathbb{R}$ making $(\mathbb{R}, d)$ a compact metric space?

5.3 Continuity and Compactness

Theorem 5.31 *Continuous images of compact sets are compact.*

Proof Let X and Y be metric spaces, K be a compact subset of X and $f : K \longrightarrow Y$ be a continuous function. Let Ω be any open cover of $f(K)$. Since $f(K) \subset \bigcup_{O\in\Omega} O$,

$$K \subset f^{-1}\big(\bigcup_{O\in\Omega} O\big) = \bigcup_{O\in\Omega} f^{-1}(O).$$

Since f is continuous and each $O \in \Omega$ is open, $\{f^{-1}(O) : O \in \Omega\}$ is an open cover of K. The compactness of K implies that there are finitely many $O_1, \ldots, O_n \in \Omega$ such that $K \subset \bigcup_{i=1}^{n} f^{-1}(O_i) = f^{-1}\big(\bigcup_{i=1}^{n} O_i\big)$. Therefore, $f(K) \subset \bigcup_{i=1}^{n} O_i$. Hence, $f(K)$ is compact. □

Corollaries 5.32 *(a) Continuous functions on compact sets are bounded.*
(b) Let $f : [a, b] \longrightarrow \mathbb{R}$ be a continuous function. Then f is bounded.

Theorem 5.33 (Extreme Value Theorem) *Let X be a compact metric space and $f : X \longrightarrow \mathbb{R}$ be a continuous function. Then f is a bounded function and it attains its bounds on X.*

Proof Applying Theorem 5.31, Theorem 5.8 and Corollary 5.4, we conclude that $f(X)$ is a compact subset of reals and hence it is closed and bounded. Therefore

$$M := \sup f(X) \text{ and } m := \inf f(X).$$

are real numbers. By Proposition 3.20, we have $M \in f(X)$. Write $g := -f$. Since f is continuous on X, so is g. As above, we obtain $\sup g(X) \in g(X)$. Therefore,

$$-m = -\inf f(X) = \sup g(X) \in g(X) = -f(X).$$

Hence, $m \in f(X)$ and the result follows. □

Theorem 5.34 *Let f be a continuous bijection from a compact metric space X onto another metric space Y. Then f^{-1} is also continuous.*

Proof Since f is a bijection, its inverse f^{-1} is well-defined. Since $(f^{-1})^{-1}(E) = f(E)$, it is enough to show that f maps closed sets onto closed sets.

Let F be a closed subset of X. By Theorem 5.10, F is compact and by Theorem 5.31, $f(F)$ is a compact set. Finally, applying Theorem 5.8, we conclude that $f(F)$ is a closed subset of Y. □

5.3.1 Uniform Continuity

Definition 5.35 A function $f : (X, d_X) \longrightarrow (Y, d_Y)$ is said to be *uniformly continuous,* if for every $\epsilon > 0$ there exists some $\delta > 0$ such that

$$d_Y(f(x), f(y)) < \epsilon \text{ for all } x, y \in X \text{ such that } d_X(x, y) < \delta.$$

Every uniformly continuous function is continuous, but the converse is false.

Example 5.36 Let $f(x) := \frac{1}{x}$ for all $x \in (0, 1)$. Then f is continuous on $(0, 1)$, but not uniformly continuous on $(0, 1)$.

Proof The continuity of f on $(0, 1)$ follows from Corollaries 1.53. Assume that f is uniformly continuous on $(0, 1)$. Then for $\epsilon = 1$, there exists some $\delta > 0$ such that

$$|f(x) - f(y)| < 1 \text{ for all } x, y \in (0, 1) \text{ satisfy } |x - y| < \delta.$$

Let N be a positive integer such that $\frac{1}{N} < \delta$. Note that $\frac{1}{N+2}, \frac{1}{N} \in (0, 1)$ and $\frac{1}{N} - \frac{1}{N+2} < \frac{1}{N} < \delta$, while $|f(\frac{1}{N}) - f(\frac{1}{N+2})| = 2$, a contradiction. □

Next, we will provide some sufficient conditions for the uniform continuity of a function. First, we show that continuous functions on compact spaces are uniformly continuous.

Theorem 5.37 *Let (X, d_X) and (Y, d_Y) be metric spaces such that X is compact, and let $f : X \longrightarrow Y$ be a continuous function. Then f is uniformly continuous.*

Proof Assume that f is not uniformly continuous. Then there exists some $\epsilon > 0$ such that for every $n \in \mathbb{N}$ one can choose $x_n, y_n \in X$ satisfying

$$d_X(x_n, y_n) < \frac{1}{n} \text{ and } d_Y(f(x_n), f(y_n)) \geq \epsilon.$$

Since X is compact, by Theorem 5.26, $\{x_n\}$ has a subsequence $\{x_{n_k}\}$, convergent to some $c \in X$. Note that

$$d_X(y_{n_k}, c) \leq d_X(y_{n_k}, x_{n_k}) + d_X(x_{n_k}, c) \leq \frac{1}{n_k} + d_X(x_{n_k}, c) \longrightarrow 0, \text{ as } k \longrightarrow \infty.$$

Therefore, $\{y_{n_k}\}$ also converges to c. Since f is continuous at c, there exists some $\eta > 0$ such that for all $x \in X$ satisfying $d_X(x, c) < \eta$, we have $d_Y(f(x), f(c)) < \epsilon/2$. Then for sufficiently large k, we obtain $d_X(x_{n_k}, c) < \eta$ and $d_X(y_{n_k}, c) < \eta$. Therefore,

$$d_Y(f(x_{n_k}), f(y_{n_k})) \leq d_Y(f(x_{n_k}), f(c)) + d_Y(f(c), f(y_{n_k})) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon,$$

a contradiction to the choice of x_{n_k}, y_{n_k}. Hence the result. □

An Alternate Proof of Theorem 5.37 Let $\epsilon > 0$ be given. Since f is continuous on X, for every $x \in X$, one can choose some $\delta_x > 0$ such that

$$d_Y(f(y), f(x)) < \frac{\epsilon}{2} \text{ for all } x, y \in X \text{ satisfying } d_X(y, x) < \delta_x.$$

Write $B_x := B(x; \delta_x/2)$ for each $x \in X$. Then $\{B_x : x \in X\}$ is an open cover of X. Since X is compact, there are finitely many $x_1, \ldots, x_n \in X$ such that $X \subset \bigcup_{i=1}^n B_{x_i}$. Let

$$\delta := \frac{1}{2} \min\{\delta_{x_1}, \ldots, \delta_{x_n}\}.$$

Then $\delta > 0$. Now pick any $y, z \in X$ such that $d_X(y, z) < \delta$. Since $X \subset \bigcup_{i=1}^n B_{x_i}$, there exists $m \in \{1, \ldots, n\}$ such that $y \in B_{x_m}$. Therefore, $d_X(y, x_m) < \delta_{x_m}/2$. Also

$$d_X(z, x_m) \leq d_X(z, y) + d_X(y, x_m) < \delta + \frac{\delta_{x_m}}{2} \leq \delta_{x_m}.$$

Hence, we conclude that

$$d_Y\big(f(y), f(z)\big) \leq d_Y\big(f(y), f(x_m)\big) + d_Y\big(f(x_m), f(z)\big) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

Hence, f is uniformly continuous on X. □

Another proof of Theorem 5.37 will be outlined in Exercise 5.57. Next, we provide some necessary and sufficient conditions for uniform continuity.

Theorem 5.38 *Let (a, b) be an interval and $f : (a, b) \longrightarrow \mathbb{R}$ be continuous. Then f is uniformly continuous on (a, b) if and only if the limits $\lim_{x\to a+} f(x)$ and $\lim_{x\to b-} f(x)$ both exist in $\mathbb{R}$. (If $a = -\infty$ or $b = +\infty$, then take the limits $x \longrightarrow -\infty$ and $x \longrightarrow +\infty$, respectively.)*

Proof Assume that f is uniformly continuous on (a, b). For each $m \in \mathbb{N}$, one can choose some $\delta_m > 0$ such that

$$|f(x) - f(y)| < \frac{1}{m} \text{ for all } x, y \in (a, b) \text{ satisfying } |x - y| < \delta_m. \tag{5.1}$$

Suppose that $\lim_{x \to b-} f(x)$ does not exist in $\mathbb{R}$. If $\lim_{x \to b-} f(x) = +\infty$, we can choose a sequence $\{x_n\} \subset (a, b)$ such that $x_n \longrightarrow b$ and $\lim_{n \to \infty} f(x_n) = +\infty$. Pick any $N \in \mathbb{N}$ such that $x_n \in (b - \delta_1, b)$ for all $n > N$. Then by (5.1), we have $diam(\{f(x_n) : n > N\}) \leq 1$. This contradicts $\lim_{n \to \infty} f(x_n) = +\infty$. Hence $\lim_{x \to b-} f(x) \neq +\infty$. Similarly, $\lim_{x \to b-} f(x) \neq -\infty$.

If $\lim_{x \to b-} f(x)$ does not exist even in extended reals, there are sequences $\{x_n\}$ and $\{y_n\}$ inside (a, b) with limit b such that $l_1 := \lim_{n \to \infty} f(x_n)$ and $l_2 := \lim_{n \to \infty} f(y_n)$ both exist in $\mathbb{R}$, but are different. Let $m \in \mathbb{N}$ be such that $\frac{1}{m} < |l_1 - l_2|$.

Pick any $N \in \mathbb{N}$ such that $x_n, y_n \in (b - \delta_m, b)$ for all $n \geq N$. Applying (5.1), we obtain $|f(x_n) - f(y_n)| < \frac{1}{m}$ for all $n \geq N$. Passing limit $n \longrightarrow \infty$, we have $|l_1 - l_2| \leq \frac{1}{m}$, a contradiction. Hence, $\lim_{x \to b-} f(x)$ exists. Similarly, $\lim_{x \to a+} f(x)$ also exists in $\mathbb{R}$.

Conversely, if (a, b) is a bounded open interval, by defining $f(a) := \lim_{x \to a+} f(x)$ and $f(b) := \lim_{x \to b-} f(x)$, we extend f as a continuous function to $[a, b]$. Since $[a, b]$ is a compact set, f is uniformly continuous on $[a, b]$ and hence on (a, b).

Now consider the case when $a \in \mathbb{R}$ and $b = +\infty$. We define $f(a) := \lim_{x \to a+} f(x)$. This extends f as a continuous function to $[a, +\infty)$. Let $l := \lim_{x \to +\infty} f(x)$ and $\epsilon > 0$ be given. Pick any $R \in \mathbb{R}$ such that

$$|f(x) - l| < \frac{\epsilon}{2} \text{ for all } x \geq R.$$

Being continuous, f is uniformly continuous on $[a, R]$. Let $\delta \in (0, 1)$ be such that

$$|f(x) - f(y)| < \frac{\epsilon}{2} \text{ for all } x, y \in [a, R] \text{ such that } |x - y| < \delta.$$

Let $x, y \in (a, +\infty)$ such that $|x - y| < \delta$. If $x, y \in [a, R]$, we have $|f(x) - f(y)| < \frac{\epsilon}{2} < \epsilon$. Otherwise $|f(x) - f(y)| \leq |f(x) - f(R)| + |f(R) - f(y)| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$. The case $a = -\infty$ and $b \in \mathbb{R}$ is similar.

Finally, let $a = -\infty$ and $b = +\infty$. By above cases, f is uniformly continuous on $[0, +\infty)$ and $(-\infty, 0]$. Hence, f is uniformly continuous on $(-\infty, +\infty)$. □

Corollary 5.39 *Let f be a real valued uniformly continuous function on a bounded open interval (a, b). Then f has a continuous extension to $[a, b]$.*

In Example 5.36, we have seen that there are continuous functions, which are not uniformly continuous. Moreover, not every bounded continuous real function on a bounded interval is uniformly continuous.

Example 5.40 The mapping $x \longmapsto \sin\left(\frac{1}{x}\right)$ is bounded and continuous on $(0, 1)$, but not uniformly continuous on $(0, 1)$.

There also exist bounded continuous $\mathbb{R} \longrightarrow \mathbb{R}$ functions, which are not uniformly continuous.

Example 5.41 For each $n \in \mathbb{N}$, define $f_n(x) := n|x|$ for all $x \in [-1/n, +1/n]$. Extend f_n as a periodic function to $\mathbb{R}$, with period $2/n$. Define a function $f : \mathbb{R} \longrightarrow \mathbb{R}$ as follows:

$$f(x) := \begin{cases} f_1(x) \ ; x \in [-1, 1], \\ f_n(x) \ ; x \in [-n, n] \setminus (-n+1, n-1) \text{ for some } n \in \mathbb{N} \setminus \{1\}. \end{cases}$$

It can be shown that f is bounded and continuous, but not uniformly continuous.

Theorem 5.42 *If $f : (X, d_X) \longrightarrow (Y, d_Y)$ is uniformly continuous, then*

(a) f maps Cauchy sequences in X onto Cauchy sequences in Y and
(b) f maps totally bounded subsets of X onto totally bounded subsets of Y.

Proof (a) Let $\{x_n\}$ be a Cauchy sequence in X and $\epsilon > 0$ be given. Then there exists some $\delta > 0$ such that

$$d_Y(f(x), f(y)) < \epsilon \text{ for all } x, y \in X \text{ such that } d_X(x, y) < \delta. \tag{5.2}$$

Let $N \in \mathbb{N}$ be such that $d_X(x_n, x_m) < \delta$ for all $n > m \geq N$. Using (5.2), we conclude that $d_Y(f(x_n), f(x_m)) < \epsilon$ for all $n > m \geq N$. Hence, $\{f(x_n)\}$ is a Cauchy sequence.

(b) Let E be a totally bounded subset of X and $\{y_n\}$ be a sequence from $f(E)$. Then there exists a sequence $\{x_n\}$ from E such that $f(x_n) = y_n$ for all $n \in \mathbb{N}$. Since E is totally bounded in X, by Theorem 4.22, $\{x_n\}$ has a Cauchy subsequence, say $\{x_{n_k}\}$. Applying (a), $\{f(x_{n_k})\}$ is Cauchy. Again by Theorem 4.22, $f(E)$ is totally bounded. □

Both (a) and (b) of Theorem 5.42 are invalid for all continuous maps. See also Corollary 9.35.

Examples 5.43 Let $X := (0, 1)$, $Y := \mathbb{R}$ and $f(x) := 1/x$ for all $x \in (0, 1)$. Then

(a) $\{1/n\}$ is Cauchy in X, while $\{f(1/n)\}$ is not Cauchy in Y.
(b) X is totally bounded, while $f(Y) = (1, \infty)$ is not.

5.3.2 Notes and Remarks

If (X, d) is a metric space, then the following are equivalent: (i) all continuity is uniform, (ii) each open cover has a Lebesgue number (see Exercise 5.61), (iii) disjoint nonempty closed sets have a positive distance between them, (iv) the nonvanishing real valued uniformly continuous functions on them have uniformly continuous reciprocals, and (v) whenever $\{x_n\}$ is a sequence in X with $\{dist(x_n, X \setminus \{x_n\})\} \longrightarrow 0$, then the sequence has a cluster point in X.

Metric spaces satisfying these equivalent properties are called *UC spaces or Atsuji spaces.* These lie between compact and complete spaces. A metrizable space X has a compatible UC metric if and only if X' is compact - see [4] for a simple construction of a good metric. Several characterizations of UC metric spaces can be found in [4–8].

The compactness of the domain is not a redundant hypothesis in Theorems 5.31, 5.33, and 5.37 (see Exercise 5.52). In 1948, Hewitt established that a metric space X is compact if and only if every continuous function from X to $\mathbb{R}$ is bounded (see [9, p. 69]). A stronger notion of uniform continuity will be discussed in Exercises 5.63-5.64. Also, see [10–12].

Exercise 5.31 Let (X, d_X) and $Y, d_Y)$ be metric space, $A \subset X$, $B \subset Y$, and

$$d\big((x, y), (x', y')\big) := d_X(x, x') + d_Y(y, y') \text{ for all } (x, y), (x', y') \in X \times Y.$$

Prove that d is a metric on $X \times Y$. Further, show that $A \times B$ is compact in $X \times Y$ if and only if A and B are compact in X and Y, respectively.

Exercise 5.32 Generalize Exercise 5.31 for any finite Cartesian product of metric spaces.

Exercise 5.33 If $f : \mathbb{R} \longrightarrow \mathbb{R}$ maps compact sets onto compact sets, is it continuous?

Exercise 5.34 Is the continuity hypothesis redundant in Theorem 5.31?

Exercise 5.35 Does there exist any of the following:

(a) A continuous surjective function $f : [0, 1] \longrightarrow (0, 1)$?
(b) A continuous surjective function $f : (0, 1) \longrightarrow (0, 1]$?
(c) A continuous surjective function $f : (0, 1) \longrightarrow [0, 1]$?
(d) A continuous bijection $f : (0, 1) \longrightarrow [0, 1]$?
(e) A continuous function $f : [0, 1] \longrightarrow (0, 1)$ such that $f\big((0, 1]\big) = (0, 1)$?

Exercise 5.36 Let E be a nonempty subset of a metric space (X, d). Prove that the function $x \longmapsto dist(x; E)$ is uniformly continuous on X.

Exercise 5.37 Prove that uniformly continuous functions map totally bounded sets onto totally bounded sets, by directly using Definition 4.16.

Exercise 5.38 Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be a continuous function such that $f(x + 1) = f(x)$ for all $x \in \mathbb{R}$. Show that f is uniformly continuous.

Exercise 5.39 Which of the following functions are uniformly continuous on $(0, 1)$:

(a) $f(x) := \cos x \cos \frac{\pi}{x}$ for all $x \in (0, 1)$,
(b) $g(x) := \sin x \sin \frac{\pi}{x}$ for all $x \in (0, 1)$?

Exercise 5.40 For $x > 0$, define $f(x) := x^2$ and $g(x) := \sqrt{x}$ for all $x > 0$. Without using Theorem 5.38, prove the following:

(a) On $[1, +\infty)$, f is not uniformly continuous, while g is uniformly continuous.
(b) On $(0, 1]$, f is uniformly continuous, while g is not uniformly continuous.

Exercise 5.41 Is the mapping $x \longmapsto 1/x^2$ uniformly continuous on $(0, 1)$?

Exercise 5.42 Is the mapping $x \longmapsto e^x$ uniformly continuous on $\mathbb{R}$?

Exercise 5.43 Consider the functions $f, g : [1, +\infty) \longrightarrow \mathbb{R}$ defined as $f(x) := 1/x$ and $g(x) := 1/x^2$. Is any of these uniformly continuous on $[1, +\infty)$?

Exercise 5.44 Write a proof for Corollary 5.39.

Exercise 5.45 Prove that the composition of two uniformly continuous functions, whenever possible, is also a uniformly continuous function.

Exercise 5.46 Is the product of two uniformly continuous functions also uniformly continuous? Is it true in case of functions on closed bounded intervals?

Exercise 5.47 Let f and g be uniformly continuous real valued functions on a set $E \subset \mathbb{R}$ such that $g(x) \neq 0$ for all $x \in E$. Does it imply that f/g is uniformly continuous on E?

Exercise 5.48 Prove that the functions given by Examples 5.40 and 5.41 are not uniformly continuous.

Exercise 5.49 Prove that a function $f : \mathbb{R} \longrightarrow \mathbb{R}$ is uniformly continuous if and only if for any sequences $\{x_n\}$ and $\{y_n\}$ of reals such that $|x_n - y_n| \longrightarrow 0$, we have $|f(x_n) - f(y_n)| \longrightarrow 0$.

Exercise 5.50 Let $\{f_n\}$ be a sequence of uniformly continuous functions on a metric space X, uniformly convergent to a function f on X. Prove that f is uniformly continuous on X.

Exercise 5.51 Prove or disprove:

(a) Every uniformly continuous function on a totally bounded set is bounded.
(b) Every uniformly continuous function on a bounded set is a bounded function.

Exercise 5.52 Let E be a non-compact subset of $\mathbb{R}$. Prove the following:

(a) There exists a continuous unbounded function on E.
(b) There is a continuous bounded function on E, which has no maximum on E.
(c) If E is bounded, then there exists a continuous function $f : E \longrightarrow \mathbb{R}$, which is not uniformly continuous. Is the boundedness of E redundant?

Exercise 5.53 Let f and g be uniformly continuous $E(\subset \mathbb{R}) \longrightarrow \mathbb{R}$ functions.

(a) If f is bounded on E, prove that f^2 is uniformly continuous on E.
(b) If f and g are bounded on E, prove that fg is uniformly continuous on E.
(c) If there exists some $M > 0$ such that $|g(x)| > M$ for all $x \in E$, prove that $1/g$ is uniformly continuous on E.

Exercise 5.54 If f maps Cauchy sequences onto Cauchy sequences, does it imply that f is uniformly continuous?

Exercise 5.55 Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be continuous. Is it necessary that f maps Cauchy sequences onto Cauchy sequences?

Exercise 5.56 Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be strictly monotone. Is it necessary that f maps Cauchy sequences onto Cauchy sequences?

Exercise 5.57 (Daners, 2015 [13]) Write a proof of Theorem 5.37 by establishing the following assertions:

(a) $F : X \times X \longrightarrow \mathbb{R}$ defined as $F(x, y) := d_Y(f(x), f(y))$ is continuous.
(b) For a fixed $\epsilon > 0$, the set $A_\epsilon := F^{-1}\big([\epsilon, \infty)\big)$ is compact.
(c) If $\delta := \inf\{d_X(x, y) : (x, y) \in A_\epsilon\}$, then $\delta > 0$. Further, there exists some $(a, b) \in A_\epsilon$ such that $F(a, b) = \delta$.
(d) If $d_x(x, y) < \delta$, then $d_Y(f(x), f(y) < \epsilon$.

Exercise 5.58 Prove that for every $f \in C[0, 1]$ and $\epsilon > 0$, there exists a piecewise linear function p on $[0, 1]$ such that $\|p - f\|_\infty < \epsilon$.

Exercise 5.59 Let $f, f_n : [a, b] \longrightarrow \mathbb{R}$ be such that

(a) $f_n \longrightarrow f$ pointwise on $[a, b]$,
(b) f is continuous on $[a, b]$ and
(c) each f_n is monotonically increasing on $[a, b]$.

Prove that $f_n \longrightarrow f$ uniformly on $[a, b]$. Conclusion that f is a monotonically increasing on $[a, b]$. Establish similar results for monotonically decreasing functions.

Exercise 5.60 If f is an upper semi-continuous function on a compact metric space X, prove that f has absolute maximum on X.

Exercise 5.61 Let Ω be an open covering of a compact metric space X. Then there exists a $\delta > 0$ such that for every subset of X having diameter less than δ, there exists an element of Ω containing it. (The number δ is called a *Lebesgue number* for the covering Ω.)

Exercise 5.62 Let X be a metric space in which every open cover of X has a Lebesgue number. Prove that every continuous function on X is uniformly continuous.

Exercise 5.63 (Beer, 2009 [11]**)** Let (X, d_X), (Y, d_Y) be metric spaces, $A \subset X$, and $f : X \longrightarrow Y$. Then f is called *strongly uniformly continuous* on $A \subset X$ if for all $\epsilon > 0$ there exists some $\delta > 0$ such that

$$d_Y(f(x), f(y)) < \epsilon \text{ if } d_X(x, y) < \delta \text{ and } \{x, y\} \cap A \neq \emptyset.$$

(a) Prove that f is continuous at some $x \in X$ if and only if f is strongly uniformly continuous on $\{x\}$.
(b) If f is strongly uniformly continuous on A, then prove that f is uniformly continuous on A.
(c) If f is continuous on X and A is compact, then prove that f is strongly uniformly continuous on A.
(d) If $X := \mathbb{R}^2$, $Y := \mathbb{R}$, $A := \{(1, y) : y \in \mathbb{R}\}$ and $f(x, y) := x(x + y)$ for all $x, y \in \mathbb{R}$, then prove that f is uniformly continuous on A, but not strongly uniformly continuous on A.

Exercise 5.64 (Beer, 2009 [11]**)** Let (X, d_X) and (Y, d_Y) be metric spaces, and let $f : X \longrightarrow Y$. Prove that the following are equivalent:

(a) f maps Cauchy sequences onto Cauchy sequences.
(b) f is strongly uniformly continuous on all totally bounded subsets of X.
(c) f is uniformly continuous on all totally bounded subsets of X.

Exercise 5.65 (Beer, 1986 [10]**)** Let X be a metric space with completion X^*. Applying Exercise 4.19, prove that the following are equivalent:

(a) If Y is a metric space and $f : X \longrightarrow Y$ maps Cauchy sequences onto Cauchy sequences, then f is uniformly continuous on X.
(b) If $f : X \longrightarrow \mathbb{R}$ is maps Cauchy sequences onto Cauchy sequences, then f is uniformly continuous on X.
(c) If Y is a metric space and $f : X^* \longrightarrow Y$ is continuous function, then f is uniformly continuous on X.

Exercise 5.66 (Beer, 2009 [11]) Let (X, d_X) and (Y, d_Y) be metric spaces, $\{x_n\}$ be a Cauchy sequence in X, and let $f : X \longrightarrow Y$. Prove that f is uniformly continuous on $\{x_n : n \in \mathbb{N}\}$ if and only if $\{f(x_n)\}$ is a Cauchy sequence in Y.

5.4 Lipschitz Continuity

Definition 5.44 Let (X, d_X) and (Y, d_Y) be metric spaces. A function $f : X \longrightarrow Y$ is said to be *Lipschitz continuous* on set $E \subset X$, if there exists some $M > 0$ such that

$$d_Y(f(x), f(y)) \leq M d_X(x, y) \text{ for all } x, y \in E.$$

In this case, M is known as a *Lipschitz constant* for f.

Examples 5.45 (a) All linear $\mathbb{R} \longrightarrow \mathbb{R}$ functions are Lipschitz continuous.
(b) Every $\mathbb{R} \longrightarrow \mathbb{R}$ contraction mapping is Lipschitz continuous.
(c) Every isometry is Lipschitz continuous.
(d) Every contraction mapping is Lipschitz continuous.

Example 5.46 The function $f(x) := \sqrt{x}$ is not Lipschitz continuous on $[0, 1]$.

Proof Suppose that there exists some $M > 0$ such that

$$|f(x) - f(y)| \leq M|x - y| \text{ for all } x, y \in [0, 1].$$

In particular, with $y = 0$, we obtain $|\sqrt{x}| \leq M|x|$ for all $x \in [0, 1]$, that is, $x > 1/M^2$ for all $x \in [0, 1]$, a contradiction. □

Theorem 5.47 *Every Lipschitz continuous function is uniformly continuous.*

Proof Let (X, d_X) and (Y, d_Y) be metric spaces, $E \subset X$ and $f : X \longrightarrow Y$ be a Lipschitz continuous function on E. Then there exists some $M > 0$ such that for all $x, y \in E$, the inequality $d_Y(f(x), f(y)) \leq M d_X(x, y)$ is satisfied.

Let $\epsilon > 0$ be given. Fix $\delta := \frac{\epsilon}{M}$. Then for any $x, y \in E$ such that $d_X(x, y) < \delta$, we obtain $d_Y(f(x), f(y)) \leq M\frac{\epsilon}{M} = \epsilon$. Hence the result. □

Therefore, to examine the uniform continuity of a function f, the first step is to examine its Lipschitz continuity. The next two theorems facilitate this for some particular types of differentiable functions.

Theorem 5.48 *Let I be any interval and $f : I \longrightarrow \mathbb{R}$ be a differentiable function. Then f is Lipschitz continuous on I if and only if f' is bounded on I.*

Proof First assume that f is Lipschitz continuous on I. Let $M > 0$ be such that

$$|f(x) - f(y)| \leq M|x - y| \text{ for all } x, y \in I.$$

Then for all $x \in I$, using the continuity of the mapping $t \longmapsto |t|$, we obtain

$$|f'(x)| = \left| \lim_{y \to x} \frac{f(y) - f(x)}{y - x} \right| = \lim_{y \to x} \left| \frac{f(y) - f(x)}{y - x} \right| \leq M.$$

Conversely, assume that there exists some $M > 0$ be such that $|f'| \leq M$, on I. Pick any $x, y \in I$. By Mean Value Theorem on $[x, y]$, we have $f(x) - f(y) = f'(c)(x - y)$ for some $c \in (x, y)$. Hence, $|f(x) - f(y)| \leq M|x - y|$. Since x, y were arbitrary, the result follows. □

Theorem 5.49 *Let I be an open interval and $f : \overline{I} \longrightarrow \mathbb{R}$ be continuous on $\overline{I}$ and differentiable on I. Then f is Lipschitz continuous on $\overline{I}$ if and only if f' is bounded on I.*

Proof If f is Lipschitz continuous on $\overline{I}$, it must be Lipschitz continuous on I. Applying Theorem 5.48, f' is bounded on I.

Conversely, if f' is bounded on I, by Theorem 5.48, f is Lipschitz continuous on I. Therefore, there exists some $M > 0$ such that for all $x, y \in I$, we have

$$|f(x) - f(y)| \leq M|x - y|. \tag{5.3}$$

Since I is an open interval, I can be any of the intervals $(-\infty, +\infty)$, $(a, +\infty)$, $(-\infty, a)$ or (a, b) for some $a, b \in \mathbb{R}$. We prove the result for $I = (a, +\infty)$.The other cases are similar.

If $x = y = a$, (5.3) is trivially satisfied. Therefore, it is enough to prove (5.3) for $x > a$ and $y = a$. Let $y = a$ and choose a sequence $\{y_n\} \subset (a, +\infty)$ such that $y_n \longrightarrow a$. Since f is continuous at a, $f(y_n) \longrightarrow f(a)$. Also, we have

$$|f(x) - f(y_n)| \leq M|x - y_n|, \text{ for all } n \in \mathbb{N}.$$

Passing limit $n \longrightarrow +\infty$ we obtain (5.3) for $x > a$ and $y = a$, as required. □

We wind up this section with a characterization of continuous linear maps on normed spaces.

Theorem 5.50 *Let X and Y be normed spaces and $T : X \longrightarrow Y$ be a linear map. That is,*

$$T(\alpha x + \beta y) = \alpha T(x) + \beta T(y) \textit{ for all } \alpha, \beta \in \mathbb{K} \textit{ and for all } x, y \in X.$$

Then the following are equivalent:

(a) T is continuous at 0.
(b) T is continuous on X.
(c) T is uniformly continuous.
(d) T is Lipschitz continuous on X.
(e) There exists $M > 0$ such that $\|T(x)\| \leq M\|x\|$ for all $x \in X$.

Proof The implications $(d) \Rightarrow (c) \Rightarrow (b) \Rightarrow (a)$ are trivial. Due to linearity of T, it is immediate that $(d) \iff (e)$. We now prove that $(a) \Rightarrow (e)$.

Suppose that T is continuous at 0. Being a linear map, we have $T(0) = T(0 \times 0) = 0 \times T(0) = 0$. Then for every $\epsilon > 0$, there exists some $\delta > 0$ such that $\|T(x) - T(0)\| \leq \epsilon$ whenever $\|x - 0\| \leq \delta$. That is $\|T(x)\| \leq \epsilon$ whenever $\|x\| \leq \delta$.

Let $x \in X \setminus \{0\}$. Then for $y := \delta \frac{x}{\|x\|}$, we have $\|y\| = \delta$ and thus $\|T(y)\| \leq \epsilon$. Consequently, $\|T(x)\| \leq \frac{\epsilon}{\delta}\|x\|$ for all $x \in X$. Hence the result. □

Notes and Remarks 5.51 Let $[a, b]$ be a compact interval. It can be shown that every Lipschitz continuous function $f : [a, b] \longrightarrow \mathbb{R}$ is of *bounded variation*. In other words,

$$\sup\left\{\sum_{i=1}^{n}|f(x_i)-f(x_{i-1})| : a=x_0<x_1<\cdots<x_n=b, n\in\mathbb{N}\right\}<\infty.$$

It is well known that every continuous function of bounded variation on $[a, b]$ is a difference of two monotonically increasing continuous functions (see [14, p. 113, Theorem 6.15]). Hence, every Lipschitz continuous function is a difference of two monotonically increasing continuous functions. For a thorough treatise on Lipschitz functions, the readers are referred to [15].

There is another notion of continuity for real functions, known as absolute continuity (see [16, p. 119]). It lies strictly between the Lipschitz and uniform continuities and has an important role in the theory of Lebesgue integration. The Fundamental Theorem of Calculus for the Lebesgue integral ensures that absolutely continuous functions completely characterize the primitives of Lebesgue integrable functions (see [16, p. 125, Theorem 11]). Along with Theorem 5.49, we conclude the following result:

If $f : [a, b] \longrightarrow \mathbb{R}$ has a bounded derivative on $[a, b]$, then f' is Lebesgue integrable and $\int_a^b f'(t)dt = f(b) - f(a)$.

A better result is available in terms of restricted generalized absolute continuity and the integrals of Arnaud Denjoy, Jaroslav Kurzweil and Ralph Henstock. For details, we refer [17, Theorems 11.3-11.4]. Also, see [17, Theorem 9.17] and [18, p. 209, Theorem 7.3.10].

Exercise 5.67 Let $(X, \|.\|)$ be a normed linear space. Prove that $x \longmapsto \|x\|$ defines a Lipschitz continuous function on X.

Exercise 5.68 If f is continuously differentiable on a closed bounded interval I, prove that f is Lipschitz continuous on I.

Exercise 5.69 Prove or disprove: Every Lipschitz continuous function on a bounded set is a bounded function.

Exercise 5.70 Does $x \longmapsto x^2 \sin\frac{1}{x}$ define a Lipschitz continuous map on $(0, 1)$?

Exercise 5.71 Define $f : [-1, 1] \longrightarrow \mathbb{R}$ as

$$f(x) := \begin{cases} x^2 \sin\frac{1}{x^2} & ; x \neq 0, \\ 0 & ; x = 0. \end{cases}$$

Prove that on $[-1, 1]$, the function f is differentiable and uniformly continuous but f' is not bounded. Conclude that f is not Lipschitz continuous on $[-1, 1]$.

Exercise 5.72 Prove that $g : \mathbb{R} \longrightarrow \mathbb{R}$ is Lipschitz continuous if and only if

$$\sup\left\{\left|\frac{g(x)-g(y)}{x-y}\right| : x, y \in \mathbb{R} \text{ and } x \neq y\right\} < \infty.$$

Exercise 5.73 Let $\{f_n\}$ be a sequence of $\mathbb{R} \longrightarrow \mathbb{R}$ functions, pointwise convergent to f. Assume that for every $n \in \mathbb{N}$, there exists $M_n > 0$ such that M_n is a Lipschitz constant for f_n.

(a) If $M_n \longrightarrow M_0$, for some $M_0 \in \mathbb{R}$, prove that f is Lipschitz continuous with constant M_0.
(b) If $M := \inf_{k \geq 1} \sup_{n \geq k} M_n < +\infty$, prove that f is Lipschitz continuous with constant M.
(c) Can these results be extended to functions between metric spaces?

Exercise 5.74 Is the uniform limit of a sequence of Lipschitz continuous functions always Lipschitz continuous?

Exercise 5.75 A function f is called *locally Lipschitz* if f is Lipschitz continuous in some ball about each point of its domain. Prove the following:

(a) $x \mapsto x^2$ is locally Lipschitz on $\mathbb{R}$ but is not Lipschitz continuous on $\mathbb{R}$.
(b) The restriction of a locally Lipschitz function to each compact subset is Lipschitz.

Exercise 5.76 Let $F : X \longrightarrow Y$ be a linear map between normed linear spaces X and Y. Prove that F is continuous if and only if F maps Cauchy sequences onto Cauchy sequences.

5.5 Hints and Solutions to Selected Exercises

5.7 By Theorem 5.8, K is closed. Therefore, $F \cap K$ is a closed subset of K. By Theorem 5.10, $F \cap K$ is compact.

5.8 Yes. Let $X := [0, 1] \cup \mathbb{N}$ and define

$$d(x, y) := \frac{|x - y|}{1 + |x - y|} \text{ for all } x, y \in X.$$

Then X is bounded, as $X \subset B(0; 2)$. Also note that $d(n, n + 1) = 1/2$ for all $n \in \mathbb{N}$. Hence $\{B(x; 1/2) : x \in X\}$ is an open cover of X with no finite subcover.

5.9 Let $[a, b]$ be a compact interval and Ω be an open cover of $[a, b]$. Applying the least upper bound property on the set $E := \{x \in [a, b] : [a, x]$ has a finite subcover from $\Omega\}$ there exists some $s \in \mathbb{R}$ such that $s = \sup E$. Then show that $s = b$.

5.10 Part (a) is trivial. For (b), apply (a) to obtain a finite subcover from Ω, say Ω_0. For every $b \in B$, we have some $O_b \in \Omega_0$ containing (x, b). Then there exists some neighborhoods U_b and V_b, of x and b, respectively, such that $U_b \times V_b \subset O_b$.

Using compactness of B, there are finitely many $b_1, \ldots, b_n \in B$ such that $B \subset \bigcup_{i=1}^n V_{b_i}$. Then $U := \bigcap_{i=1}^n U_{b_i}$ satisfies our requirements.

To prove (c), let Ω be an open cover of $A \times B$ in $X \times Y$. For every $x \in A$, by (b), choose a neighborhood U_x of x such that $U_x \times B$ has a finite subcover Ω_x from Ω.

Since $\{U_x : x \in A\}$ is an open cover of A, there are finitely many $x_1, \ldots, x_n \in A$ such that $A \subset \bigcup_{i=1}^n U_{x_i}$. Then $\bigcup_{i=1}^n \Omega_{x_i}$ is a finite subcover of $A \times B$.

5.12 Let $F_x := X \setminus \{x\}$ for all $x \in X$. Then $\mathcal{F} = \{F_x : x \in X\}$ a collection of closed sets with empty intersection and finite intersection property. Hence X is not compact.

5.13 Let K be a compact subset of a metric space X. Then K is closed in X. By Corollary 5.19, K is complete. By Theorem 4.2, K is a closed subset of X.

5.14 Let X be a compact metric space and K be a closed subset of X. Since X is compact, by Corollary 5.19, X is complete. By Theorem 4.3, K is complete.

Since X is compact, by Proposition 5.3, it is totally bounded. Since subsets of totally bounded sets are totally bounded, see Exercise 4.38, K is totally bounded. Applying Theorem 5.26, we conclude that K is compact.

5.17 Let X be a compact metric space and $f : X \longrightarrow \mathbb{R}$ be a continuous function. Assume that f is not bounded. Then for each $n \in \mathbb{N}$ there exists some $x_n \in X$ such that each $|f(x_n)| > n$ for all $n \in \mathbb{N}$.

Since X is compact, it is sequentially compact. Therefore, $\{x_n\}$ must have a convergent subsequence, say $\{x_{n_k}\} \longrightarrow x$. Since f is continuous at x, we have $\{f(x_{n_k})\} \longrightarrow f(x)$. So, the sequence $\{f(x_{n_k})\}$ is bounded, a contradiction.

5.22 The converse of (a) is not true, as 0 is a limit point of $S := \{n, 1/n : n \in \mathbb{N}\}$, but S does not converge to 0. Part (b) follows by Theorem 5.27.

5.23 To prove the contrapositive, assume that X is a metric space, which is sequentially complete but not totally bounded. Then there exists some $\epsilon > 0$ such that X is not covered by finitely many sets from $\{B(x; \epsilon) : x \in X\}$.

Pick any $x_1 \in X$. Since $X \not\subset B(x_1; \epsilon)$, we can choose $x_2 \in X$ such that $d(x_2, x_1) \geq \epsilon$. Since $X \not\subset B(x_1; \epsilon) \cup B(x_2; \epsilon)$, choose $x_3 \in X$ such that $d(x_3, x_1) \geq \epsilon$ and $d(x_3, x_2) \geq \epsilon$. Continuing like this, one can choose $x_n \in X$ such that $d(x_n, x_1) \geq \epsilon, \ldots, d(x_n, x_{n-1}) \geq \epsilon$. Then $d(x_n, x_m) \geq \epsilon$, for each $n \neq m$.

Since $\{x_n\}$ is a sequence in X, which is sequentially compact, it must have a convergent subsequence, say $\{x_{n_k}\}$. Being convergent $\{x_{n_k}\}$ must be Cauchy, which is impossible as by our construction $d(x_{n_k}, x_{n_l}) \geq \epsilon$, for all $k \neq l$.

5.24 Yes. Clearly, d is a metric on $\mathbb{R}$. For its total boundedness, let $x \in \mathbb{R}$ and $r > 0$. Then

$$B(x; r) := \{y \in \mathbb{R} : |\tan^{-1} y - \tan^{-1} x| < r\} = \tan(\tan^{-1} x - r, \tan^{-1} x + r).$$

Let $r \in (0, \pi)$. Note that the function $f : \mathbb{R} \longrightarrow (-\pi/2, \pi/2)$ given by $f(x) := \tan^{-1} x$ is a strictly increasing bijection. Therefore, there are $x_1, x_2 \in \mathbb{R}$ be such that $\tan^{-1} x_1 = r - \frac{\pi}{2}$ and $\tan^{-1} x_2 = -r + \frac{\pi}{2}$. Then

$$B(x_1; r) = (-\infty, \tan(\tan^{-1} x_1 + r)) \text{ and } B(x_2; r) = (\tan(\tan^{-1} x_2 - r), +\infty).$$

Set $a := \tan(\tan^{-1} x_2 - r)$ and $b := \tan(\tan^{-1} x_1 + r)$. Since $[a, b] \subset \bigcup_{x\in[a,b]} B(x; r)$, using the compactness of $[a, b]$, there are some finitely $t_1, \ldots, t_n \in [a, b]$ such that $[a, b] \subset \bigcup_{i=1}^{n} B(t_i; r)$. Therefore we obtain $\mathbb{R} = B(x_1; r) \cup \left(\bigcup_{i=1}^{n} B(t_i; r)\right) \cup B(x_2; r)$.

5.25 No. This space is not even complete. For each $n \in \mathbb{N}$, let $x_n := \tan\left(\frac{\pi}{2} - \frac{1}{n}\right)$. Then the sequence $\{x_n\}$ is Cauchy in $(\mathbb{R}, d)$. Suppose there exists some $a \in \mathbb{R}$ such that $\{x_n\} \longrightarrow a$ in $(\mathbb{R}, d)$, then $\tan^{-1} x_n \longrightarrow \tan^{-1} a$, under usual metric on $\mathbb{R}$. The uniqueness of limits, implies $\tan^{-1} a = \frac{\pi}{2}$ which implies $a \notin \mathbb{R}$, a contradiction.

5.28 Applying Exercise 4.57, it is enough to show that if $\{A_n\}$ is as in the statement then $A := \bigcap_{n=1}^{\infty} A_n$ is compact. Let $\epsilon > 0$ be given. Then there exists some N such that $\alpha(A_N) < \epsilon$. Since $A \subset A_N$, we conclude that A can be covered by finitely many subsets of X with diameter $< \epsilon$. Therefore, A is totally bounded. Being a closed subset of the complete space X, A is complete and hence compact.

5.29 No. If yes, then $(\mathbb{R}^*, d)$ must be totally bounded. Hence every subset of it, in particular $\mathbb{R}$, must be totally bounded, a contradiction.

5.31 First part is easy. The converse of the second part holds by Exercise 5.10. For the necessity part, let $f : X \times Y \longrightarrow X$ be the projection map $f(x, y) := x$. It can be verified that f is continuous. By Theorem 5.31, $A = f(A \times B)$ is compact in X. Similarly B is compact.

5.33 No. For example, consider the Dirichlet function.

5.34 No. For example, consider $f(x) := x\chi_{\mathbb{Q}}$ on $\mathbb{R}$.

5.35 (a) No. See Theorem 5.31.

(b) Yes. For example, $f(x) := \sin \pi x$.

(c) Yes. For example, $f(x) := \frac{1}{2}[1 + |\sin 2\pi x|]$.

(d) No. If there exists such a bijection f, then there exists some $c \in (0, 1)$ such that $f(c) = 0$. Let $\eta := \min\{c, 1 - c\}$. Pick any $c_1 \in (c - \delta, c)$ and $c_2 \in (c, c + \delta)$. Since f is not injective, $f(c_1) > 0$ and $f(c_2) > 0$. Let $m := \min\{f(c_1), f(c_2)\}$. Then $m > 0$.

By Intermediate Value Theorem, f assumes every value between $[0, m]$ in both of the domains $[c - \delta, c]$ and $[c, c + \delta]$. Consequently, f is not injective, a contradiction.

(e) No. To prove this, assume that there exists such a function. Let f denote one such function and $m := \inf\{f(x) : x \in (0, 1]\}$ and $M := \sup\{f(x) : x \in (0, 1]\}$.

Since $f\big((0, 1]\big) = (0, 1)$, we obtain $m = 0$ and $M = 1$. Let $\{x_n\}$ and $\{y_n\}$ be sequences in $(0, 1]$ such that $f(x_n) \longrightarrow 0$ and $f(y_n) \longrightarrow 1$.

By Bolzano–Weierstrass Property of reals, both $\{x_n\}$ and $\{y_n\}$ will have subsequences convergent in $[0, 1]$. Without loss of generality, we can assume that both $\{x_n\}$ and $\{y_n\}$ are convergent and let $x_0 := \lim_{n\to\infty} x_n$ and $y_0 := \lim_{n\to\infty} y_n$. Then $x_0, y_0 \in [0, 1]$.

If $x_0 \neq 0$, then as f is continuous at x_0, we have $f(x_0) = \lim_{n\to\infty} f(x_n) = 0$, a contradiction. Therefore, $x_0 = 0$. Similarly, $y_0 = 0$. This leads to $0 = \lim_{n\to\infty} f(x_n) = f(x_0) = f(y_0) = \lim_{n\to\infty} f(y_n) = 1$, a contradiction.

Hence, there exists no continuous map $f : [0, 1] \longrightarrow (0, 1)$ such that $f\big((0, 1]\big) = (0, 1)$.

5.37 Let f be uniformly continuous on a totally bounded subset E of a metric space X and $\epsilon > 0$ be given. Choose $\delta > 0$ as per the uniform continuity of f on E. Since E is totally bounded, there are finitely many $x_1, \ldots, x_n \in E$ such that $E \subset \bigcup_{i=1}^{n} B(x_i; \delta)$. Hence

$$f(E) \subset f\big(\bigcup_{i=1}^{n} B(x_i; \delta)\big) \subset \bigcup_{i=1}^{n} f\big(B(x_i; \delta)\big) \subset \bigcup_{i=1}^{n} B(f(x_i); \epsilon).$$

5.39 Answer: g is uniformly continuous on $(0, 1)$, but not f. Apply Theorem 5.38.

5.40 We prove the first part. The second one is similar. Note that $g'(x)$ is bounded by $1/2$, therefore, g is Lipschitz continuous and hence uniformly continuous.

Suppose that f is uniformly continuous. Then there exists some $\delta > 0$ such that $|x^2 - y^2| < 1/2$, whenever $|x - y| < \delta$. Pick any $n \in \mathbb{N}$ such that $1/n < \delta$. Then $|x^2 - y^2| < 1/2$, whenever $|x - y| \leq 1/n$.
Let $x = n + 1/n$ and $y = n$. Since $|x - y| = 1/n$, we have $|(n + 1/n)^2 - n^2| < 1/2$. This is a contradiction, as

$$\left|\left(n + \frac{1}{n}\right)^2 - n^2\right| = \left(n + \frac{1}{n}\right)^2 - n^2 = 2 + \frac{1}{n^2} > 2.$$

5.43 Yes, both f and g are uniformly continuous on $[1, +\infty)$. To see this, let $\epsilon > 0$ be given. Note that if $x, y \in [1, +\infty)$ satisfy $|x - y| < \epsilon$, we have

$$\left|\frac{1}{x} - \frac{1}{y}\right| = \left|\frac{x - y}{xy}\right| \leq |x - y| < \epsilon.$$

Similarly, if $x, y \in [1, +\infty)$ satisfy $|x - y| < \epsilon/2$, we have

$$\left|\frac{1}{x^2} - \frac{1}{y^2}\right| = \left|\frac{x^2 - y^2}{x^2 y^2}\right| = |x - y|\left|\frac{1}{xy^2} + \frac{1}{x^2 y}\right| \leq 2|x - y| < \epsilon.$$

5.45 Let $f : (X, d_X) \longrightarrow (Y, d_Y)$ and $g : (Y, d_Y) \longrightarrow (Z, d_Z)$ be uniformly continuous functions. Let $\epsilon > 0$. Then there exists $\eta > 0$ such that $d_Z(g(s), g(t)) < \epsilon$ whenever $d_Y(s, t) < \eta$.

Similarly, we find $\delta > 0$ such that $d_Y(f(x), f(y)) < \eta$ whenever $d_X(x, y) < \delta$. If $d_X(x, y) < \delta$, we have $d_Y(f(x), f(y)) < \eta$ and hence $d_Z(g \circ f(x), g \circ f(y)) < \epsilon$.

5.46 No. For example, consider $f(x) := x$ on $[1, +\infty)$. Then f is uniformly continuous, while f^2 is not.

In case of closed bounded intervals, it is true as uniformly continuous functions are continuous and hence so is their product. Being continuous function closed bounded intervals, this product will be uniformly continuous over there.

5.47 No. For example, take $E := \mathbb{R}$, $f \equiv 1$, $g(x) := x$.

5.50 Argue as in Theorem 3.31.

5.51 (a) True. Apply Theorem 5.42 and the fact that every totally bounded set is bounded.

(b) False. For example, take $X = \mathbb{R}$ with discrete metric and $Y = \mathbb{R}$ with usual metric. Define $f(x) := x$ for all $x \in X$. Then $f : X \longrightarrow Y$ is uniformly continuous. For any $\epsilon > 0$ take $\delta = 1$.

5.52 *Case I:* E is bounded. Since E is non-compact, it is not closed. If $a \in E' \setminus E$, then

$$f(x) := \frac{1}{x - a} \text{ for all } x \in E$$

serves as an example for (a) and (c) above. For (b), consider the function

$$g(x) := \frac{1}{1 + (x - a)^2} \text{ for all } x \in E.$$

Case II: E is unbounded. Then $h(x) := x$ for all $x \in E$ serves as an example for (a) and

$$u(x) := \frac{x^2}{1 + x^2} \text{ for all } x \in E$$

serves as an example for (b). The boundedness of E is not redundant in (c), e.g. if $E := \mathbb{Z}$ then any function on E is uniformly continuous (with $\delta < 1$.)

5.53 (a) Pick any $M > 0$ such that $|f| < M$ on E. For $\epsilon > 0$, pick any $\delta > 0$ as per the definition of uniform continuity. Then for $x, y \in E$ such that $|x - y| < \delta$, we have

$$|f^2(x) - f^2(y)| \le \big||f(x)| + |f(y)|\big| \times |f(x) - f(y)| \le 2M\epsilon.$$

Since M is fixed and $\epsilon > 0$ is arbitrary, the result follows.

(b) By (a), $fg = \frac{1}{4}((f + g)^2 - (f - g)^2)$ is uniformly continuous on E.

(c) The result follows from the inequality

$$\left|\frac{1}{g(x)} - \frac{1}{g(y)}\right| \leq \frac{1}{M^2}|g(x) - g(y)|.$$

for all $x, y \in E$

5.54 No. For example, let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be defined as $f(x) := x^2$. Then f maps Cauchy sequences onto Cauchy sequences, as Cauchy sequences of reals are precisely convergent sequences. However, f is not uniformly continuous.

5.55 Yes, as Cauchy sequences in $\mathbb{R}$ are convergent.

5.56 No, e.g. consider the function $f : \mathbb{R} \longrightarrow \mathbb{R}$ and a sequence $\{x_n\}$ defined as

$$f(x) := \begin{cases} x & ; x < 0, \\ x + 1 & ; x \geq 0 \end{cases} \text{ and } x_n := \begin{cases} \frac{1}{m} & ; n = 2m, \\ -\frac{1}{m} & ; n = 2m - 1. \end{cases}$$

Then f is strictly increasing and $\{x_n\}$ is Cauchy, while $\{f(x_n)\}$ is not Cauchy.

5.58 Since $[0, 1]$ is compact, f is uniformly continuous on $[0, 1]$. So there exists $\delta > 0$ such that $|f(x) - f(y)| < \frac{\epsilon}{3}$ whenever $x, y \in [0, 1]$ such that $|x - y| < \delta$.

Consider a partition $0 = x_0 < x_1 < \cdots < x_n = 1$ of $[0, 1]$ such that $x_i - x_{i-1} < \delta$ for every i. Let p be a piecewise linear function on $[0, 1]$ such that $p(x_i) = f(x_i)$ for every i and p is linear on each subinterval $[x_{i-1}, x_i]$.

Let $x \in [0, 1]$ be arbitrary. Then there exists some i such that $x \in [x_{i-1}, x_i]$. Therefore, $|p(x) - f(x_i)| = |p(x) - p(x_i)| \leq |p(x_{i-1}) - p(x_i)| < \epsilon/3$. Hence

$$|p(x) - f(x)| \leq |p(x) - f(x_i)| + |f(x_i) - f(x)| < \frac{\epsilon}{3} + \frac{\epsilon}{3} = \frac{2\epsilon}{3}.$$

Therefore, $\|p - f\|_\infty \leq \frac{2\epsilon}{3} < \epsilon$. Hence $p \in B(f; \epsilon)$ and the result follows.

5.59 We prove the result for monotonically increasing functions. The case of monotonically decreasing functions is similar. Also, note that it is enough to prove that $f_n \longrightarrow f$ uniformly on $[a, b]$. Let $\epsilon > 0$ be given.

Since f is continuous on the compact interval $[a, b]$, it is uniformly continuous on $[a, b]$. Therefore, there exists some $\delta > 0$ such that $|f(x) - f(y)| < \frac{\epsilon}{2}$ for all $|x - y| < \delta$.

Let $a = t_0 < t_1 \cdots < t_k = b$ be a partition of $[a, b]$ such that $t_i - t_{i-1} < \delta$ for all $1 \leq i \leq k$. Since $f_n(t_i) \longrightarrow f(t_i)$, for all $i = 0, \ldots, k$, there exists some $N \in \mathbb{N}$ such that

$$|f_n(t_i) - f(t_i)| < \frac{\epsilon}{2} \text{ for all } 0 \leq i \leq k \text{ and for all } n \geq N. \tag{5.4}$$

Let $n \geq N$ and $t \in [a, b]$. Then such that $t \in [t_{i-1}, t_i]$ for some i. Since f_n is monotone

$$f_n(t_{i-1}) \leq f_n(t) \leq f_n(t_i).$$

By (5.4), we have $f(t_{i-1}) - \frac{\epsilon}{2} \le f_n(t) \le f(t_i) + \frac{\epsilon}{2}$. Further our choice of δ ensures that

$$f(t) - \epsilon < f(t_{i-1}) - \frac{\epsilon}{2} < f_n(t) < f(t_i) + \frac{\epsilon}{2} < f(t) + \epsilon.$$

Consequently, $|f_n(t) - f(t)| < \epsilon$ for all $n \ge N$ and for all $t \in [a, b]$.

5.60 Suppose not. Then there exists a sequence $\{x_n\} \subset X$ such that $f(x_n) \longrightarrow \infty$. Since X is a compact metric space, $\{x_n\}$ has a convergent subsequence. Without loss of generality, suppose that $x_n \longrightarrow x$ for some $x \in X$. By Exercises 3.101 and 3.102, we obtain $\lim_{n\to\infty} f(x_n) = \limsup_{n\longrightarrow\infty} f(x_n) \le f(x) < \infty$, a contradiction.

5.61 If $X \in \Omega$, then any positive number is a Lebesgue number for Ω. So we assume that X is not an element of Ω. The compactness of X ensures the existence of a finite subcollection $\{A_1, \ldots, A_n\} \subset \Omega$ that covers X. For each i, set $C_i := X \setminus A_i$, and define $f : X \longrightarrow \mathbb{R}$ as $f(x) := \frac{1}{n}\sum_{i=1}^{n} dist(x, C_i)$. Then f is continuous on the compact space X and hence attains its minimum, say δ. Since every x is contained in some A_i, we note that $f(x) > 0$ and hence $\delta > 0$. We claim that δ is the desired number.

If Y is a subset of X with $diam(Y) < \delta$, then there exists some $x_0 \in X$ such that $Y \subset B(x_0, \delta)$. Since $f(x_0) \ge \delta$, there exists i such that $d(x_0, C_i) \ge \delta$. But this means that $B(x_0, \delta) \subset A_i$, and therefore, in particular, $Y \subset A_i$.

5.62 Let Y be a metric space and $f : (X, d_X) \longrightarrow (Y, d_Y)$ be continuous. Let $\epsilon > 0$ be given. For each $x \in X$, choose $\delta_x > 0$ such that $f(B_X(x; \delta_x)) \subset B_Y(f(x); \epsilon/2)$. Applying hypothesis, let δ denote the Lebesgue number of the open cover $\{B(x; \delta_x/2) : x \in X\}$ of X. Let $a, b \in X$ such that $d_X(a, b) < \delta$. Then there exists $c \in X$ such that $a, b \in B_X(c; \delta_c)$. Hence

$$d_Y(f(a), f(b)) \le d_Y(f(a), f(c)) + d_Y(f(c), f(b)) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

5.63 Parts (a) and (b) are immediate from the definition. For (c), proceed by contradiction, as in the proof of Theorem 5.37. For (d), note that f is Lipschitz continuous on A. Let $\delta > 0$ be arbitrary. If $y = \frac{1}{\delta}$, then $|f(1, y) - f(1 + \delta, y)| = |1 + y - (1 + \delta)(1 + \delta + y)| = (1 + \delta)^2 - 1 + y\delta > 1$.

5.64 The assertion (b) $\Rightarrow$ (c) is obvious. Below we prove the other two.

$(a) \Rightarrow (b)$: Assume that f is not strongly uniformly continuous on a totally bounded set $A \subset X$. Then there exists some $\epsilon > 0$ and sequences $\{x_n\}$, $\{y_n\}$ in X such that

$$x_n \in A, d_X(x_n, y_n) < \frac{1}{n} \text{ and } d_Y(f(x_n), f(y_n)) \ge \epsilon.$$

Since A is totally bounded, $\{x_n\}$ has a Cauchy subsequence, say $\{x_{n_k}\}$. Then the sequence $x_{k_1}, y_{k_1}, \ldots, x_{k_n}, y_{k_n}, \ldots$ is Cauchy, but not its image.

$(c) \Rightarrow (a)$: Let $\{x_n\}$ be a Cauchy sequence in X. Then $\{x_n : n \in \mathbb{N}\}$ is totally bounded. By (c), f is uniformly continuous on $\{x_n : n \in \mathbb{N}\}$. Hence, $\{f(x_n)\}$ is a Cauchy.

5.65 Note that $(a) \Rightarrow (b)$ is obvious. To prove $(b) \Rightarrow (c)$, let Y be a metric space and $f : X^* \longrightarrow Y$ be a continuous function. Since X^* is complete, f maps Cauchy sequences onto Cauchy sequences. By (b), f is uniformly continuous.

To prove $(c) \Rightarrow (a)$, let Y be a metric space and suppose $f : X \longrightarrow Y$ maps Cauchy sequences onto Cauchy sequences. Let Y^* denote the completion of Y. By Exercise 4.19, let g be a continuous extension of f to X^*. Applying (c), g is uniformly continuous, and hence so is its restriction $f = g|_X$.

5.66 Necessity follows from Theorem 5.42. For converse, write $X_0 := \{x_n : n \in \mathbb{N}\}$. Then its completion X_0^* is a compact set and hence every continuous map from X_0^* into another metric space Y is uniformly continuous. By (c) ⇒ (a) part of Exercise 5.65, f is uniformly continuous on X_0.

5.73 Fix any $x, y \in \mathbb{R}$. Then we have $|f_n(x) - f_n(y)| \leq M_n|x - y|$ for all $n \in \mathbb{N}$.

(a) The result follows by passing limit $n \longrightarrow \infty$ in (5.5).

(b) Since $M < \infty$, for all $\epsilon > 0$ there exists some $N_\epsilon \in \mathbb{N}$ such that $M_n < M + \epsilon$ for all $n \geq N_\epsilon$. Hence, $|f_n(x) - f_n(y)| < (M + \epsilon)|x - y|$ for all $n \geq N_\epsilon$ and for all $x, y \in \mathbb{R}$. Passing limit $n \longrightarrow \infty$, we obtain

$$|f(x) - f(y)| \leq (M + \epsilon)|x - y| \text{ for all } x, y \in \mathbb{R}. \tag{5.5}$$

Since $\epsilon > 0$ is arbitrary, we obtain the result.

(c) This part is analogous to the above ones.

5.74 No. Make the sequence converge uniformly to $x \mapsto \sqrt{x}$.

5.76 If F is continuous, then it is Lipschitz continuous and hence maps Cauchy sequences onto Cauchy sequences. Conversely, assume that F is not continuous. Then we have $\sup\{\|F(x)\| : x \in X\} = \infty$. Thus, there exists a sequence $\{x_n\} \subset X$ such that $\|x_n\| = 1$ and $\|F(x_n)\| > n^2$ for all $n \in \mathbb{N}$. Hence, for $y_n := \frac{x_n}{n}$, we have $y_n \longrightarrow 0$, while $\|F(y_n)\| \longrightarrow \infty$. Therefore, $\{F(y_n)\}$ is not Cauchy.

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Chapter 6
Connectedness

The notion of path connectedness is more intuitive than connectedness. It appears before connectedness, even in history. Motivated by that, we start this chapter with path connectedness and its relationship with continuity; through which we deduce the Intermediate Value Theorem. Then we discuss connectedness and its characterizations, followed by a section on components; which provides insights into connected components and path components. Finally, we present some miscellaneous topics such as local connectedness, quasi-components, and totally disconnected spaces.

6.1 Path Connectedness

Path connectedness is natural for subsets of $\mathbb{R}^m$. Analogously, it can be extended to arbitrary metric spaces. First, we introduce the notion of a path.

Definition 6.1 Let X be a metric space and $x, y \in X$. A *path* from x to y, in X, is defined to be the range of a continuous function $f : [0, 1] \longrightarrow X$ such that $f(0) = x$ and $f(1) = y$.

Therefore, a subset γ of X is said to be a path in X if there exists a continuous function $f : [0, 1] \longrightarrow X$ with range γ. Alternate terms for path are *curve* and *arc*.

Definition 6.2 A metric space X is said to be *path connected* if any two points inside X can be joined by a curve, inside X.

Further a nonempty subset S of a metric space X is said to be path connected if it is a path connected subspace of X.

Examples 6.3 Let X be a metric space. The following are immediate:

(a) Every singleton subset of X is path connected.
(b) Every curve in X is path connected.
(c) Every open ball in $\mathbb{R}^2$ is path connected.
(d) If A and B are non-disjoint path connected subsets of X, then so is $A \cup B$.

Recall that a subset I of $\mathbb{R}$ is defined to be an *interval* if I contains all reals between any two points of I (see Definition 1.1).

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Proposition 6.4 *Every nonempty interval is path connected.*

Proof Let I be a nonempty interval and $a, b \in I$. Define $g : [0, 1] \longrightarrow I$ as

$$g(t) := a + t(b - a) \text{ for all } t \in [0, 1].$$

Then g is path in I from a to b. Hence, I is path connected. □

Remarks 6.5 (a) Unless specified, all the sets of this chapter will be taken as nonempty subsets of an arbitrary metric space.

(b) If A and B are disjoint sets, we shall write $A \dot{\bigcup} B$, instead of $A \bigcup B$, which would intrinsically convey that the sets A and B are disjoint.

(c) It is a standard practice to take arbitrary compact intervals in Definition 6.1, instead of [0, 1]. Due to the natural bijection between any two non-degenerate compact intervals, our choice of the unit interval has no loss of generality.

The following lemma opens up a world of abstraction.

Lemma 6.6 *Let I be an interval. Then I is not a disjoint union of two nonempty sets, closed in I.*

Proof Suppose $I = A \dot{\cup} B$, for some nonempty disjoint sets A and B, closed in I. Pick any $a \in A$ and $b \in B$. Without loss of generality, suppose that $a < b$.

Then $A_1 := A \cap [a, b]$ and $B_1 := B \cap [a, b]$ are nonempty disjoint sets, closed in $[a, b]$ such that $A_1 \dot{\cup} B_1 = [a, b]$. Let $c := \sup A_1$. The definition of c and the fact that $A_1 \dot{\cup} B_1$ is an interval implies that

$$(c - \epsilon, c] \cap A_1 \neq \emptyset \text{ and } (c, c + \epsilon) \cap B_1 \neq \emptyset \text{ for all } \epsilon > 0.$$

Therefore, c is an adherent point of both A_1 and B_1, which are closed in $[a, b]$. Also, $c \in [a, b]$. Hence, $c \in A_1 \cap B_1 = \emptyset$, a contradiction. □

Theorem 6.7 *Let X be a path connected metric space. Then X is not a disjoint union of two nonempty closed subsets of X.*

Proof Suppose there are disjoint nonempty closed sets A and B such that $A \dot{\bigcup} B = X$. Pick any $a \in A$ and $b \in B$. Let $f : [0, 1] \longrightarrow X$ be a continuous map with $f(0) = a$ and $f(1) = b$.

Then $f^{-1}(A)$ and $f^{-1}(B)$ are closed subsets of [0, 1], containing 0 and 1, respectively. Since A and B are disjoint, so are $f^{-1}(A)$ and $f^{-1}(B)$. Moreover, $X = A \dot{\bigcup} B$ implies that $[0, 1] = f^{-1}(X) = f^{-1}(A) \dot{\bigcup} f^{-1}(B)$, a contradiction to Lemma 6.6. □

Now we prove that the only path-connected subsets of $\mathbb{R}$ are nonempty intervals.

Theorem 6.8 *Let $\emptyset \neq I \subset \mathbb{R}$. Then I is path connected if and only if I is an interval.*

Proof The converse holds by Proposition 6.4. Assume that there exists a path connected subset I of $\mathbb{R}$ which is not an interval. Then there are $a, b \in I$ and $c \in \mathbb{R} \setminus I$ such that $a < c < b$.

Since I is path connected, there exists a continuous function $f : [0, 1] \longrightarrow I$ such that $f(0) = a$ and $f(1) = b$. Let

$$A := f^{-1}((-\infty, c] \cap I) \text{ and } B := f^{-1}([c, +\infty) \cap I).$$

Note that $a \in A, b \in B$ and $A \cup B = f^{-1}(I) = [0, 1]$. Since f is continuous, A and B are closed in $[0, 1]$. Since $c \notin I$, we have $c \notin f([0, 1])$ and therefore, $A \cap B = \emptyset$. Hence, A and B are nonempty disjoint sets, closed in $[0, 1]$ with union $[0, 1]$, a contradiction to Lemma 6.6. □

Next, we shall discuss the relationship between continuity and path connectedness. We shall present various consequences and generalizations of the Intermediate Value Theorem, which is also known as the Intermediate Value Property of continuous real functions. A few other generalizations will follow in the next section of this chapter.

First, we establish that continuous image of a path connected space is path connected.

Theorem 6.9 *Let X, Y be metric spaces such that X is path connected and $f : X \longrightarrow Y$ be a continuous function. Then $f(X)$ is a path connected subspace of Y.*

Proof Let $y_1, y_2 \in f(X)$. Then $y_1 = f(x_1)$ and $y_2 = f(x_2)$ for some $x_1, x_2 \in X$. Since X is path connected, there exists a continuous function $\phi : [0, 1] \longrightarrow X$ such that $\phi(0) = x_1$ and $\phi(1) = x_2$. Then $f \circ \phi : [0, 1] \longrightarrow f(X)$ is a continuous function with $(f \circ \phi)(0) = y_1$ and $(f \circ \phi)(1) = y_2$. Hence, $f(X)$ is a path connected subset of Y. □

Corollary 6.10 *Let X be a path connected metric space, $f : X \longrightarrow \mathbb{R}$ be a continuous function and $a, b \in X$ be such that $f(a) < f(b)$. Then for every $l \in (f(a), f(b))$, there exists some $c \in X$ such that $f(c) = l$.*

Proof By Theorems 6.8 and 6.9, $f(X)$ is a path connected subset of $\mathbb{R}$ and hence an interval. Therefore, $f(X) \supset [f(a), f(b)]$, which contains l. Hence, $f(c) = l$, for some $c \in X$. □

As an application of the above corollary, we now show that the continuous injective real valued maps on intervals are strictly monotone with strictly monotone inverses on their range.

Example 6.11 Let I be an interval and $f : I \longrightarrow \mathbb{R}$ be a continuous injective map. Then f is strictly monotone with strictly monotone inverse on $f(I)$.

Proof Let $E := \{(x, y) \in I^2 : x < y\}$. Define $g : E \longrightarrow \mathbb{R}$ as $g(x, y) = f(x) - f(y)$. Note that E is path connected. Since g is continuous, $g(E)$ is also path connected.

If f is not strictly monotone, then g takes positive and negative values on E. Therefore, there exist $(x, y) \in E$ such that $g(x, y) = 0$. This contradicts the fact that f is injective.

Hence, f is strictly monotone on I. It can be shown that $f^{-1} : f(I) \longrightarrow I$ is also strictly monotone, which we leave to the readers. □

Theorem 6.12 (Intermediate Value Theorem) *Let $a < b$ be reals and $f : [a, b] \longrightarrow \mathbb{R}$ be a continuous function. Then f maps intervals onto intervals. In other words, if $f(a) < l < f(b)$, then $f(c) = l$ for some $c \in (a, b)$.*

Proof By Theorem 6.8, $[a, b]$ is path connected. Now apply Corollary 6.10. □

The converse is not true. However, it holds under some additional hypotheses (see Exercise 6.13) A few immediate consequences of the Intermediate Value Theorem are presented below. The first one is a fixed point theorem.

Example 6.13 Let $f : [0, 1] \longrightarrow [0, 1]$ be a continuous function. Then there exists some $x \in [0, 1]$ such that $f(x) = x$.

Proof Let $g(x) := f(x) - x$. If $g(0) = 0$ or $g(1) = 1$, we are done. Otherwise $g(0) > 0$ and $g(1) < 0$. Since g is continuous on $[0, 1]$, the result follows by Intermediate Value Theorem. □

Example 6.14 Let S^1 denote the unit circle in $\mathbb{R}^2$ and $f : S^1 \longrightarrow \mathbb{R}$ be continuous. Then there exists some $z \in S^1$ such that $f(z) = f(-z)$. (There are two antipodal points on the equator of the earth at which the temperatures are exactly the same.)

Proof Consider the functions $g : S^1 \longrightarrow \mathbb{R}$ and $h : [0, 1] \longrightarrow S^1$ defined as

$$g(x) := f(x) - f(-x) \text{ and } h(t) := (\cos(2\pi t), \sin(2\pi t)).$$

Since g and h are continuous, so is their composition $g \circ h : [0, 1] \longrightarrow \mathbb{R}$. Note that $h(0) = (1, 0) = -h(1/2)$. Therefore,

$$(g \circ h)(0) = f(h(0)) - f(-h(0)) = f(-h(1/2)) - f(h(1/2)) = -(g \circ h)(1/2).$$

If $(g \circ h)(0) = 0$, then take $z = h(0)$. Otherwise $(g \circ h)(0)$ and $(g \circ h)(1/2)$ are of opposite signs. By Intermediate Value Theorem, there exists $c \in (0, 1/2)$ such that $(g \circ h)(c) = 0$. Then $z = h(c)$ satisfies our requirements. □

Proposition 6.15 *[1, Theorem 2] Let $f : \mathbb{R} \longrightarrow \mathbb{R}$. Then f is continuous if and only if f maps intervals onto intervals, and compact sets onto compact sets.*

Proof The necessity follows by Theorems 5.31 and 6.12. For the converse, assume that f maps intervals onto intervals and compact sets onto compact sets, but is not continuous at some $x \in \mathbb{R}$. Then there exists some $\epsilon > 0$ and a sequence $\{y_n\}$ of real numbers such that

$$|y_n - x| < \frac{1}{n} \text{ and } |f(x) - f(y_n)| \geq \epsilon \text{ for all } n \in \mathbb{N}.$$

Then for each $n \in \mathbb{N}$, either $f(y_n) \geq f(x) + \epsilon$ or $f(y_n) \leq f(x) - \epsilon$. Since f maps intervals onto intervals, one can choose a real x_n between x and y_n such that

$$f(x_n) = f(x) + \epsilon\left(\frac{1}{2} + \frac{1}{n+1}\right) \text{ or } f(x) - \epsilon\left(\frac{1}{2} + \frac{1}{n+1}\right). \tag{6.1}$$

Write $X := \{x_n : n \in \mathbb{N}\} \cup \{x\}$. Since $x_n \longrightarrow x$, X is a closed and bounded subset of $\mathbb{R}$ and hence compact. By hypothesis, $f(X)$ is also compact. Also, (6.1) ensures that either $f(x) + \epsilon/2$ or $f(x) - \epsilon/2$ is an adherent point of $f(X)$, but not in $f(X)$, a contradiction. □

Alternative proofs of the Intermediate Value Theorem (6.12) will be suggested in Exercise 6.14.

History Notes 6.16 The first proof of the Intermediate Value Theorem appeared in a 60 pages book by Bolzano in 1817. In 1821, Cauchy provided its modern formulation (see [2, p. 847]).

Exercise 6.1 Show that every curve is a path connected compact set, given by a uniformly continuous map.

Exercise 6.2 Can a metric space have a path connected finite subset, with exactly two points?

Exercise 6.3 Characterize the set of continuous functions from $\mathbb{R}$ into $\mathbb{Z}$.

Exercise 6.4 Let $f : [0, 2] \longrightarrow \mathbb{R}$ be a continuous function with $f(0) = f(2)$. Prove that there exists some $x \in [0, 1]$ such that $f(x) = f(x + 1)$.

Exercise 6.5 Does there exist any normed linear space which is not path connected?

Exercise 6.6 Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be a continuous function such that $f(O)$ is an open set, for every open set O. Prove that f is strictly monotone.

Exercise 6.7 Does there exist any $p > 0$ such that the square root of the arithmetic mean of the p^{th} powers of 2, 3 and 4 is π?

Exercise 6.8 Does there exist a continuous bijection from $[0, 1]^2$ onto $[0, 1]$, or from $(0, 1)^2$ onto $(0, 1)$?

Exercise 6.9 Let $f : [0, 1) \cup [3, 4] \longrightarrow \mathbb{R}$ be a strictly increasing function such that the range of f is connected. Prove that f is a continuous function.

Exercise 6.10 If a, b, c are distinct reals, then what is the number of distinct real roots of the equation $(x - a)^3 + (x - b)^3 + (x - c)^3 = 0$?

Exercise 6.11 Let $n \in \mathbb{N}$, and $a_1, \ldots, a_{n-1} \in \mathbb{R}$ such that the polynomial given by $p(x) := x^n + a_{n-1}x^{n-1} + \cdots + a_1 x - 1$ has no roots in $\{z \in \mathbb{C} : |z| < 1\}$ and it satisfies $p(-1) = 0$. Prove or disprove:

(a) $p(1) = 0$.
(b) $p(2) > 0$.
(c) $\lim_{x\to\infty} p(x) = \infty$.
(d) $p(3) = 0$.

Exercise 6.12 Let $\{(X_i, \rho_i) : i = 1, \ldots n\}$ be a finite collection of path connected metric spaces and ρ be the metric on the Cartesian product $X := \prod_{i=1}^{n} X_i$ defined as in Exercise 2.24. Is (X, ρ) a path connected metric space?

Exercise 6.13 Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ such that f maps intervals onto intervals.

(a) Is it necessary that f is continuous?
(b) If $f^{-1}(\{r\})$ is closed for every $r \in \mathbb{Q}$, prove that f is continuous.

Exercise 6.14 Write alternate proofs for the Intermediate Value Theorem (6.12) using (a) the least upper bound property of $\mathbb{R}$ and (b) Heine-Borel Theorem (5.14) for $\mathbb{R}$.

Exercise 6.15 Write an alternate proof of Lemma 6.6, using relative open sets only.

6.2 Connected Sets

Motivated by Theorem 6.7, we now present the notion of connectedness.

Definition 6.17 A metric space X is said to be *connected* if it is not a union of two nonempty disjoint sets, closed in X.

A nonempty subset Y of a metric space X is said to be *connected* if Y is a connected subspace of X. Otherwise, Y is called *disconnected.*

It is evident from the definition that a metric space X is connected if and only if X is not a union of two nonempty disjoint sets, open in X.

Examples 6.18 (a) In any metric space, the singleton sets are connected.
(b) In $\mathbb{R}$, no finite set having more than one point is connected.
(c) By Lemma 6.6, every nonempty interval of reals is connected.
(d) By Theorem 6.7, every path connected metric space is connected.

In this chapter, we will provide several examples of connected metric spaces that are not path connected. However, for subspaces of $\mathbb{R}$, these two notions are equivalent.

Theorem 6.19 *Let $\emptyset \neq I \subset \mathbb{R}$. Then the following are equivalent:*

(a) *I is an interval.*
(b) *I is path connected.*
(c) *I is connected.*

Proof The implications $(a) \iff (b)$ and $(b) \Rightarrow (c)$ hold by Theorems 6.8 and 6.7, respectively. To prove that $(c) \Rightarrow (a)$, assume that I is not an interval. Then there are $a < c < b$ such that $a, b \in I$, but $c \notin I$. So I is a disjoint union of the nonempty sets $I \cap (-\infty, c)$ and $I \cap (c, +\infty)$, which are open in I. Hence, I is not connected. □

Just like path connected spaces, the continuous image of a connected space is connected.

Theorem 6.20 *Let X, Y be metric spaces such that X is connected and $f : X \longrightarrow Y$ be a continuous function. Then $f(X)$ is a connected subspace of Y.*

Proof Suppose that the result is not true. Then there exists continuous function f on a connected metric space X into another metric space Y such that $f(X)$ is not connected. Then $f(X) = O_1 \cup O_2$ for some nonempty disjoint sets O_1 and O_2, open in $f(X)$.

Then we obtain $X := f^{-1}(O_1) \cup f^{-1}(O_2)$, as a disjoint union. Since f is continuous, both $f^{-1}(O_1)$ and $f^{-1}(O_2)$ are open inside X. Since X is connected, either $f^{-1}(O_1) = \emptyset$ or $f^{-1}(O_2) = \emptyset$. So either $O_1 = f(\emptyset) = \emptyset$ or $O_2 = f(\emptyset) = \emptyset$, a contradiction. □

Corollary 6.21 *Let X be a connected metric space, $f : X \longrightarrow \mathbb{R}$ be a continuous function and $a, b \in X$ be such that $f(a) < f(b)$. Then for every $l \in (f(a), f(b))$, there exists some $c \in X$ such that $f(c) = l$.*

Proof Applying Theorems 6.20 and 6.19, $f(X)$ is a connected subset of $\mathbb{R}$ and hence an interval. Therefore, $l \in [f(a), f(b)] \subset f(X)$. Hence, $f(c) = l$, for some $c \in X$. □

Now we present a property of connected spaces which is not shared by path connected spaces, in general. Therefore, it serves as our main motivation to construct examples of connected sets which are not path connected.

Theorem 6.22 *If A is a connected subset of a metric space X, then any set B such that $A \subset B \subset \overline{A}$, is connected.*

Proof Let $B = B_1 \dot\cup B_2$, where B_1 and B_2 are disjoint sets, closed in B. Then

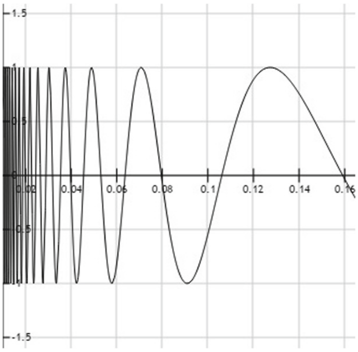
$$A = A \cap B = (A \cap B_1) \dot\cup (A \cap B_2)$$

is a union of disjoint sets, closed in A. Since A is connected, either $A \cap B_1 = \emptyset$ or $A \cap B_2 = \emptyset$. Without loss of generality, assume that $A \cap B_1 = \emptyset$. That is $A \subset B_2$.

For $E \subset B$, let $\widehat{E}$ denote the closure of E in the subspace B. Then $\widehat{A} = \overline{A} \cap B = B$, which implies $B = \widehat{A} \subset \widehat{B_2} = B_2$, as B_2 is closed in B. Therefore, $B_1 = \emptyset$. Hence, B is connected. □

Example 6.23 The *topologist's sine curve* T_s, defined as under, is connected but not path connected.

$$T_s := \left\{\left(x, \sin\frac{1}{x}\right) : x \in (0, 1]\right\} \bigcup \left\{(0, y) : |y| \leq 1\right\}.$$



Note that the set

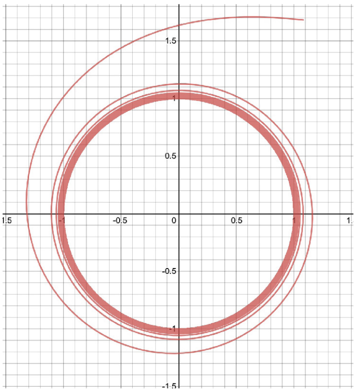
$$T := \{(x, \sin(1/x)) : x \in (0, 1]\}$$

is path connected and hence a connected subspace of $\mathbb{R}^2$.

Also, note that $\overline{T} = T_s$. Applying Theorem 6.22, the space T_s is connected.

It can also be verified that there is no path from $(0, 0)$ to $(1, \sin 1)$, in T_s.

Example 6.24 Let X denote the subspace of $\mathbb{R}^2$ given by the union of an infinite spiral and the unit circle as under:



$$\left\{\left(\left(1 + \frac{1}{t}\right)\cos t, \left(1 + \frac{1}{t}\right)\sin t\right); t \geq 1\right\}$$
$$\bigcup\{(x, y) : x^2 + y^2 = 1\}.$$

By Theorem 6.22, X is connected. However, X is not path connected.

It can be shown that there exists no path joining $(1, 0)$ and $(2\cos 1, 2\sin 1)$, inside this space.

A few other examples of connected but not path connected metric spaces will be presented Example 6.38 and Exercise 6.22.

Definition 6.25 Let X be a metric space and $A, B \subset X$. Then A and B are said to be *separated* if

$$A \cap \overline{B} = \emptyset \text{ and } \overline{A} \cap B = \emptyset.$$

Separated sets are always disjoint. However, the intervals $(0, 1)$ and $(1, 2)$ are separated in $\mathbb{R}$, while $[0, 1]$ and $(1, 2)$ are not. The converse holds under some additional hypotheses.

Proposition 6.26 *Let A and B be two disjoint subsets of a metric space.*

(a) If both A and B are closed, then these are separated.
(b) If both A and B are open, then these are separated.

Proof (a) This part is trivial, as then $A \cap \overline{B} = A \cap B = \emptyset$ and $\overline{A} \cap B = A \cap B = \emptyset$

(b) Assume that there are open disjoint sets A and B which are not separated. Then either $A \cap \overline{B} \neq \emptyset$ or $\overline{A} \cap B \neq \emptyset$. Without loss of generality, suppose that $A \cap \overline{B} \neq \emptyset$. Pick any $x \in A \cap \overline{B} \neq \emptyset$. Since A is open, there exists some $r > 0$ such that $B(x; r) \subset A$. Since $x \in \overline{B}$, we have $B(x; r) \cap B \neq \emptyset$. This implies that $A \cap B \neq \emptyset$, a contradiction. □

Proposition 6.27 *Let X be a metric space such that $X = A \dot{\bigcup} B$ for some separated sets A and B. Then both A and B are open (and hence closed) in X.*

Proof If either $A = \emptyset$ or $B = \emptyset$, the result is obvious. Assume otherwise. If $x \in A$, then $x \notin \overline{B}$. So there exists some $r > 0$ such that $B(x; r) \cap B = \emptyset$. Since $X = A \cup B$, we obtain $B(x; r) \subset A$. Hence, A is open. Similarly, B is open in X. □

Sets which are open as well as closed, are also known as *clopen sets*. Further, if X is a union of two nonempty disjoint clopen sets A and B, we say that $A|B$ is a *separation* of X or that $A \dot{\cup} B$ is a separation of X.

Now we present a few characterizations of disconnected spaces. Note that these characterizations correspond to characterizations of connected sets.

Theorem 6.28 *If X is a metric space, then the following are equivalent:*

(a) X is a union of two nonempty disjoint sets, open in X.
(b) X is a union of two nonempty disjoint sets, closed in X.
(c) X is a union of two nonempty separated sets.
(d) X contains a proper clopen subset.
(e) There exists a continuous surjective map $f : X \longrightarrow \{0, 1\}$.

Proof The equivalence $((a) \iff (b))$ is immediate from the definition of closed sets. The implications $((b) \Rightarrow (c))$ and $((a) \Rightarrow (c))$ follow from Proposition 6.26. Finally, $((c) \Rightarrow (b))$ and $((c) \Rightarrow (a))$ are evident from Proposition 6.27.

To prove $((b) \iff (d))$, assume that X is union of two nonempty disjoint closed sets A and B. Then $A = X \setminus B$ is an open subset of X, which is a proper subset of X as both A and B are nonempty. Conversely, if X contains a proper subset A which is both open as well as closed, then for $B := X \setminus A$, the second assertion holds.

To prove $((b) \iff (e))$, assume that X is union of two nonempty disjoint closed sets A and B. Define f to be 0 on A, and 1 on B. It can be shown that f is a continuous surjective map from X onto $\{0, 1\}$. Conversely, assume that there exists a continuous surjective map $f : X \longrightarrow \{0, 1\}$. Then the disjoint sets $A := f^{-1}(\{0\})$ and $B := f^{-1}(\{1\})$ serve our purpose. □

Theorem 6.29 *Let X be a compact and connected metric space such that $X \setminus \{x\}$ is disconnected, for some $x \in X$. Then there exist two different elements $x_1, x_2 \in X$ such that both $X \setminus \{x_1\}$ and $X \setminus \{x_2\}$ are connected.*

The above result is immediate when X is a subspace of $\mathbb{R}$. Its general proof is out of the scope of this textbook. Interested readers are referred to [3, p. 89, Theorem 6.6].

Metric spaces which are compact as well as connected are also known as *continuum* (see [3]). Further, a point x of a connected metric space X is known as a *cut point,* if $X \setminus \{x\}$ is disconnected. Hence, the above theorem concisely states the following:

If a continuum has a cut point, then it has at least two non-cut points.

Exercise 6.16 Prove that any curve in a metric space is a connected set.

Exercise 6.17 Does there exist a subset of $\mathbb{Q}$ which is both open and closed in $\mathbb{Q}$?

Exercise 6.18 Is finite intersection of connected subsets of a metric space always connected?

Exercise 6.19 Does there exist a connected subset whose interior is not connected?

Exercise 6.20 Prove or disprove:

(a) The interior of every path connected set is path connected.
(b) The interior of every connected set is path connected.

Exercise 6.21 Can you replace the word 'connected' with 'path connected' in Theorem 6.22?

Exercise 6.22 Show that *the infinite broom space,* given by

$$\left\{(x, y) \in \mathbb{R}^2 : x \in [0, 1], y = \frac{x}{n}; n \in \mathbb{N}\right\} \bigcup \{(1, 0)\}$$

is connected but not a path connected subspace of $\mathbb{R}^2$.

Exercise 6.23 If X is a metric space, prove that the following are equivalent:

(a) X is not a union of two nonempty disjoint sets, open in X.
(b) X is not a union of two nonempty disjoint sets, closed in X.
(c) X is not a union of two nonempty separated sets.
(d) X does not contain any proper clopen subset.
(e) There exists no continuous surjective map $f : X \longrightarrow \{0, 1\}$.

Exercise 6.24 Write a proof of Theorem 6.29 if X is a subspace of $\mathbb{R}$.

Exercise 6.25 In discrete metric spaces, prove that disjoint sets are separated.

Exercise 6.26 Let A be a subset of a metric space X. If A is connected, prove that so is $\overline{A}$. Is the converse true? Justify your answer.

Exercise 6.27 Let A, B be separated subsets of a metric space and $C \subset A$ and $D \subset B$. Prove that C and D are also separated.

Exercise 6.28 Let X be a metric space. Prove that

(a) singletons subsets of X are connected,
(b) the only finite connected subsets of X are singletons and
(c) if X is discrete, then only singletons are connected.

Exercise 6.29 Let X be a metric space and $A, B \subset X$ such that $dist(A, B) := \inf\{d(a, b) : a \in A, b \in B\} > 0$. Prove that A and B are separated sets. Is the converse true?

Exercise 6.30 Let X be a metric space with two nonempty subsets A and B satisfying $dist(A, B) > 0$ and $X = A \cup B$. Prove that both A and B are open as well as closed in X and thence conclude that X is not a connected metric space.

Exercise 6.31 Let $A \subset \mathbb{C}$ such that $A \notin \{\emptyset, \mathbb{C}\}$. Prove that A is not clopen in $\mathbb{C}$.

Exercise 6.32 Let A and B be closed subsets of a metric space such that both $A \cup B$ and $A \cap B$ are connected. Prove that both A and B are connected.

Exercise 6.33 Let $E \subset Y \subset X$. Prove or disprove: E is connected in X if and only if E is connected in Y.

Exercise 6.34 What difference does it make if we define empty sets to be connected or path connected?

Exercise 6.35 Let $A \subset B \subset C \subset X$. Prove or disprove: If A and C are connected, then so is B.

Exercise 6.36 Deduce Intermediate Value Theorem (6.12) from Theorem 6.20.

Exercise 6.37 Prove that a metric space X is disconnected if and only if there exists a continuous surjective function $f : X \longrightarrow \{0, 1\}$.

Exercise 6.38 Prove that a metric space X is disconnected if and only if there exists a continuous function $f : X \longrightarrow \mathbb{R}$ such that $f^{-1}(\{0\}) = \emptyset$, while both of the sets $f^{-1}(0, +\infty)$ and $f^{-1}(-\infty, 0)$ are nonempty.

Exercise 6.39 Let X denote the space of functions from $[0, 1]$ into itself, under uniform norm $\|.\|_\infty$ and let A be any connected subset of X. Prove that for every $x \in [0, 1]$, the set $\{a(x) : a \in A\}$ is either an interval or a singleton set.

Exercise 6.40 Show that the subspace $\left\{f \in C[0, 1] : \int_0^1 f \neq 0\right\}$ of $C[0, 1]$, under uniform norm $\|.\|_\infty$, is disconnected.

Exercise 6.41 Under uniform norm, is $C[0, 1]$ connected?

Exercise 6.42 Does there exist any $p \in [1, \infty]$ for which the sequence space ℓ^p is connected?

Exercise 6.43 Obtain the set of non-cut points of the Topologist's Sine Curve of Example 6.23.

Exercise 6.44 Does there exist a continuum having no non-cut point?

Exercise 6.45 Does there exist a continuum having

(a) no cut point?
(b) exactly one cut point?
(c) exactly n cut points, for every $n \in \mathbb{N} \setminus \{1\}$?
(d) infinitely many cut points?

Exercise 6.46 Does there exist a continuum having

(a) exactly one non-cut point?
(b) exactly n non-cut points, for every $n \in \mathbb{N} \setminus \{1\}$?
(c) infinitely many non-cut points?

6.3 Components

Definition 6.30 Let X be a metric space and $\emptyset \neq Y \subset X$. Then Y is said to be a *connected component* of X if Y is a maximal connected subset of X, that is

(a) Y is a connected subset of X and
(b) if $\widehat{Y}$ is a connected subset X with $\widehat{Y} \supset Y$, then $\widehat{Y} = Y$.

Analogously, we define *path components* or *path connected components* of a metric space, by replacing the term 'connected' with 'path connected', in the above definition.

Most of the significant fundamental results about (path) connected components emerge from the following theorem.

Theorem 6.31 *Let Ω be any nonempty collection of (path) connected subsets of a metric space X, containing a common point. Then $\bigcup_{E \in \Omega} E$ is also (path) connected.*

Proof Let $a \in X$ such that $a \in E$ for all $E \in \Omega$ and write $A := \bigcup_{E \in \Omega} E$.

(a) *Proof for path connectedness:* Pick any $x, y \in A$. Then there exist $E_x, E_y \in \Omega$ such that $x \in E_x$ and $y \in E_y$. Since E_x and E_y are path connected, there are continuous functions $f_x : [0, 1] \longrightarrow E_x$ and $f_y : [0, 1] \longrightarrow E_y$ such that $f_x(0) = x$, $f_x(1) = a = f_y(0)$ and $f_y(1) = y$. Define $f : [0, 1] \longrightarrow E_x \cup E_y$ such that

$$f(x) := \begin{cases} f_x(2t) & ; t \in [0, 1/2], \\ f_y(2t - 1) & ; t \in (1/2, 1]. \end{cases}$$

Then $f : [0, 1] \longrightarrow A$ is a continuous function such that $f(0) = x$ and $f(1) = y$. Hence, A is path connected.

(b) *Proof for connectedness:* Assume that A is a disjoint union of sets A_1 and A_2, closed in A. Since $a \in A$, without loss of generality we assume that $a \in A_1$. Pick any $E \in \Omega$. Then

$$E = E \cap A = (E \cap A_1) \,\dot{\cup}\, (E \cap A_2).$$

Note that $E \cap A_1$ and $E \cap A_2$ are closed in E. Since E is connected and $a \in E \cap A_1$, we obtain $E \cap A_2 = \emptyset$. Therefore, $E = E \cap A_1$, which implies that $E \subset A_1$. Since $E \in \Omega$ was arbitrary, we obtain $A \subset A_1$ and thus $A_2 = \emptyset$. Hence, A is connected. □

Theorem 6.32 *Let X be any metric space.*

(a) Any two connected components of X are either identical or disjoint.
(b) Every element of X belongs to a connected component of X.
(c) Every connected subset of X, is a subset of a connected component of X.
(d) X is a disjoint union of its connected components.

Each of the above assertions is true for path connected components.

Proof We establish the result only for connected components. The proofs for path connected components are analogous.

(a) Let A and B be two connected components of X. If $A \cap B \neq \emptyset$, by Theorem 6.31, $A \cup B$ is connected. Since $A \subset A \cup B$, we have $A \cup B = A$. Similarly, $A \cup B = B$. Hence, $A = B$.
(b) Let $a \in X$ and $\Omega := \{A : a \in A \subset X, A \text{ is connected}\}$. By Theorem 6.31, $T = \bigcup_{A \in \Omega} A$ is connected. It can be shown that T is the required component.
(c) Let S be a connected subset of X. Pick any $s \in S$. As in (b), for $\Omega := \{A : s \in A \subset X, A \text{ is connected}\}$, the set $T = \bigcup_{A \in \Omega} A$ is the connected component of X containing s. Since $S \in \Omega$, we obtain $S \subset T$.
(d) Follows from (a) and (b). □

Proposition 6.33 *Let $n \in \mathbb{N}$, U be an open subset of $\mathbb{R}^n$ and A be a connected component of the subspace U of $\mathbb{R}^n$. Then A is a clopen subset of U.*

Proof If $a \in A(\subset U)$, there exists some $\epsilon > 0$ such that $B := B(a; \epsilon) \subset U$. Since $a \in A \cap B$, we obtain $A \cap B \neq \emptyset$. By Theorem 6.31, $A \cup B$ is connected. So it is contained in a connected component of U. Since A is a connected component of U, we have $A = A \cup B$. Therefore, $B \subset A$. Hence, A is open.

Further by Theorem 6.22, $\overline{A} \cap U$ is a connected subset of the subspace U and it contains A. Consequently, $\overline{A} \cap U = A$. Hence, A is closed in U. □

Another important result on connected components will be discussed in Theorem 7.18. Below we provide one last consequence of Theorem 6.31.

Lemma 6.34 *If X and Y are (path) connected metric spaces, then so is* $X \times Y$.

Proof We prove the result for connectedness, as the case of path connectedness is analogous.

If X and Y are connected, then so are the 'horizontal' and 'vertical' slices in $X \times Y$, respectively, given by $X \times y := X \times \{y\}$ and $x \times Y := \{x\} \times Y$ for all $x \in X$ and $y \in Y$. Fix any $a \in X$ and $b \in Y$. Define

$$T_x := (X \times b) \cup (x \times Y) \text{ for all } x \in X.$$

Since both $X \times b$ and $x \times Y$ contain a common point (x, b), by Theorem 6.31, each T_x is connected. Further note that $\bigcup_{x \in X} T_x = X \times Y$ and each T_x contains a common point (a, b). Again by Theorem 6.31, the space $X \times Y$ is connected. □

Theorem 6.35 *If $X_1, \ldots, X_n$ are (path) connected metric spaces, then so is the product space is $\prod_{i=1}^{n} X_i$.*

Proof Apply induction on n. The result is trivial for $n = 1$ and true for $n = 2$ by Lemma 6.34. Assume the result for some $n \geq 2$. We establish it for $n + 1$.

Let $X_1, \ldots, X_{n+1}$ be (path) connected metric spaces. By induction hypothesis, so are the spaces $\prod_{i=1}^{n} X_i$ and $(\prod_{i=1}^{n} X_i) \times X_{n+1}$. Define $f : (\prod_{i=1}^{n} X_i) \times X_{n+1} \longrightarrow \prod_{i=1}^{n+1} X_i$ as

$$f\big((x_1, \ldots, x_n), x_{n+1}\big) := (x_1, \ldots, x_n, x_{n+1}).$$

It can be shown that f is a surjective map, continuous with respect to the usual product topologies. Hence, $\prod_{i=1}^{n+1} X_i = f\big((\prod_{i=1}^{n} X_i) \times X_{n+1}\big)$ is (path) connected. □

Let $GL_n(\mathbb{K})$ denote the collection of $n \times n$ invertible matrices over a field $\mathbb{K}$. Consider $GL_n(\mathbb{K})$ as a subspace of $\mathbb{K}^{n^2}$, equipped with the Euclidean metric in n^2 dimensions.

Example 6.36 $GL_n(\mathbb{C})$ is path connected, while its subspace $GL_n(\mathbb{R})$ is not even connected.

Proof For $M \in GL_n(\mathbb{C})$, let $det(M)$ denote the determinant of M. The space $GL_n(\mathbb{R})$ is not connected, as the image of $GL_n(\mathbb{R})$ under the continuous map $M \longmapsto det(M)$ is not connected.

Let A be an arbitrary element of $GL_n(\mathbb{C})$ and I denote the $n \times n$ identity matrix. It is enough to prove that there exists a path from A to I inside $GL_n(\mathbb{C})$.

Let $P(z) := det(A + z(I - A))$ for all $z \in \mathbb{C}$. Then $P(z)$ is a polynomial over $\mathbb{C}$ and hence has finitely many zeros. Thus, there exists a path from 0 to 1 in $\mathbb{C}$ which lies inside $\{z \in \mathbb{C} : P(z) \neq 0\}$, except possibly for the initial and terminating points 0 and 1. That is, there exists a continuous map h on $[0, 1]$ such that $h(0) = 0$, $h(1) = 1$ and $P(h(t)) \neq 0$ for all $t \in [0, 1]$.

If $H(t) := A + h(t)(I - A)$ for all $t \in [0, 1]$, then H is a continuous map from $[0, 1]$ into $GL_n(\mathbb{C})$ with $H(0) = A$ and $H(1) = I$. Hence the result. □

In Sect. 7.2, we shall come across some other interesting facts about connected components.

Exercise 6.47 Prove that every path component of a space is its connected component. Is the converse true?

Exercise 6.48 Let X be metric space with exactly two connected components. How many subsets of X are both open as well as closed?

Exercise 6.49 Let X be metric space with exactly n connected components. How many subsets of X are both open as well as closed?

Exercise 6.50 Let X be a metric space such that every two points of X are contained in some (path) connected subset of X. Prove that X is (path) connected.

Exercise 6.51 Let Ω be a collection of connected subsets of a metric space X such that $A \cap B \neq \emptyset$ for all $A, B \in \Omega$. Prove that $\bigcup_{A \in \Omega} A$ is a connected subset of X.

Exercise 6.52 Let $k \in \mathbb{N}$ and $\{A_1, \ldots, A_k\}$ be connected sets such that $A_n \cap A_{n+1} \neq \emptyset$ for all $n = 1, \ldots k - 1$. Prove that $\bigcup_{n=1}^{k} A_n$ is a connected set.

Exercise 6.53 Let $\{A_n\}$ be a sequence of connected sets such that $A_n \cap A_{n+1} \neq \emptyset$ for all $n \in \mathbb{N}$. Prove that $\bigcup_{n=1}^{\infty} A_n$ is also a connected set.

Exercise 6.54 If X has only finitely many connected components, prove that every connected component of X is clopen.

Exercise 6.55 If A and B are (path) connected subsets of $\mathbb{R}^2$, is $A + B$ also (path) connected?

Exercise 6.56 Let X be a metric space. Define $x \sim y \iff x$ and y lie in some connected subset of X. Prove that $\sim$ is an equivalence relation on X and the equivalence classes of this relation are precisely the connected components of X. State and prove analogous results for path components.

Exercise 6.57 Let X be the metric space of 2×2 invertible matrices over $\mathbb{R}$, equipped with the Euclidean metric in four dimensions. Which of the following spaces can be obtained as images of continuous maps on X?

(a) The usual space of real numbers $\mathbb{R}$?
(b) The subspace $\{(x, 1/x) : x \neq 0\}$ of $\mathbb{R}^2$?
(c) The subspace $\mathbb{R}^2 \setminus \{(x, 1/x) : x \neq 0\}$ of $\mathbb{R}^2$?
(d) The circle $\{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$?
(e) The closed disk $\{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\}$?

6.4 Miscellaneous

We wind up this chapter by discussing locally (path) connected sets, convex sets, and totally disconnected sets. We shall also explore their relationship with the notions presented earlier in this chapter.

6.4.1 Locally Connected and Locally Path Connected Spaces

A metric space X is said to be *locally (path) connected* if for all $x \in X$, every neighborhood of x contains a (path) connected neighborhood of x.

Since every path connected set is connected, it is immediate that every locally path connected space is also locally connected.

A natural question that arises here is whether local properties imply global properties or conversely. The answers are all negative.

There are locally path connected spaces, which are not path connected or even connected.

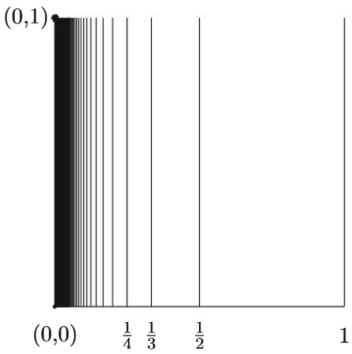
Examples 6.37 (a) $(0, 1) \cup (2, 3)$ is locally connected, but not connected.

(b) Let X be a discrete metric space with at least two elements. Then every singleton subset of X is path connected. Hence, X is locally path connected, but not connected.

The converse is also false. That is, there are connected spaces, which are not locally connected.

Example 6.38 The *comb space,* as under, is path connected but not locally connected.

$$Comb := \big(\{0\} \times [0, 1]\big) \bigcup \left(\left\{\frac{1}{n} : n \in \mathbb{N}\right\} \times [0, 1]\right) \bigcup \big([0, 1] \times \{0\}\big).$$



Further, the *deleted comb space* defined as

$$Comb_0 := Comb \setminus \big(\{0\} \times (0, 1)\big).$$

is not path connected, as there is no path from $(0, 0)$ to $(0, 1)$ inside $Comb_0$. Further, $C := Comb_0 \setminus \{(0, 0), (0, 1)\}$ is path connected and hence connected. Since $C \subset Comb_0 \subset \overline{C}$, by Theorem 6.22, $Comb_0$ is connected.

Example 6.39 The *Topologist's Sine Curve*, T_s of Example 6.23, is connected but not locally connected. Note that $(0, 1)$ has no connected neighborhood in T_s.

There are path connected spaces, which are not even locally connected.

Example 6.40 Let X denote the union of Topologist's Sine Curve T_s and a curve γ from $(0, 0)$ to $(1, \sin 1)$ such that γ does not meet T_s at any other point, except $(0, 0)$ and $(1, \sin 1)$. Then X is path connected but not locally path connected or even locally connected.

6.4.2 *Path Connectedness in Locally Path Connected Spaces*

Recall that every path connected space is connected and the converse holds for subspaces of $\mathbb{R}$. In Examples 6.23, 6.24 and 6.38, we have seen that the converses are false, in general.

However, for open subsets of locally connected spaces, the converse holds. In normed spaces, this holds with stronger consequences. Instead of general paths, we get polygonal paths, in that case. In case of finite-dimensional Euclidean spaces, the results become further stronger. First we present the notion of polygonal lines or polygonal paths in normed linear spaces.

Definitions 6.41 Let X be a normed linear space.

(a) If $x_1, x_2 \in X$, the *line segment* from x_1 to x_2 is defined as

$$[x_1, x_2] := \{(1-t)x_1 + tx_2 : 0 \leq t \leq 1\}.$$

(b) Let $x_1, \ldots, x_n$ be any finite number of elements in X, then the union of line segments $[x_1, x_2], \ldots, [x_{n-1}, x_n]$ is denoted by $[x_1, \ldots, x_n]$, and is called a *polygonal path*.
(c) A subset S of X is said to be *convex* if $[a, b] \subset S$ for all $a, b \in S$.

Examples 6.42 (a) Every interval is convex.
(b) The set $\mathbb{C} \setminus \{0\}$ is not convex.
(c) No finite subset of $\mathbb{R}^n$ is convex.
(d) Every rectangular (triangular) region in $\mathbb{R}^2$ is a convex set.

Proposition 6.43 *In normed linear spaces, every ball is a convex set.*

Proof Let $(X, \|.\|)$ be a normed linear space, $x \in X$ and $r > 0$. Write $B := B(x; r)$. Pick any $x_1, x_2 \in B$. To see that $[x_1, x_2] \subset B$, pick any $y \in [x_1, x_2]$. Then $y = tx_1 + (1-t)x_2$ for some $t \in [0, 1]$. Applying triangle inequality, we obtain

$$\|y - x\| \leq t\|y - x_1\| + (1-t)\|x_2 - y\| < r.$$

Therefore, $y \in B$. Hence, B is convex. □

Theorem 6.44 *Let O be an open connected subset of a locally path connected space X. Then O is path connected.*

Proof Let $x \in O$ be arbitrary. Let O_1 denote the collection of those $y \in O$ for which there exists a path from x to y, inside O. Write $O_2 := O \setminus O_1$. We claim that O_1 and O_2 are open in X, and hence in O.

Let $y_1 \in O_1$. Then $y_1 \in O$. Since O is open and X is locally path connected, there exists a path connected neighborhood U_1 of y_1 such that $U_1 \subset O$.

Since $y_1 \in O$, there exists a path P_1 inside O, from x to y_1. As P_1 and U contain a common point x, applying Theorem 6.31, $P_1 \cup U_1$ is path connected. Hence, for every $z_1 \in U_1$, there exists a path from x to z_1, inside O. Therefore, $U_1 \subset O_1$, which proves that O_1 is an open set.

Now, let $y_2 \in O_2$. Then $y_2 \in O$. As above there exists a path connected neighborhood U_2 of y_2 such that $U_2 \subset O$. We shall prove that $U_2 \subset O_2$, which will conclude that O_2 is open.

If possible, let $z_2 \in U_2 \setminus O_2$. Then $z_2 \in O_1$. Let P_2 be a path from x to z_2, inside O. As above $P_2 \cup U_2$ path connected. Thus, there exists a path from x to y_2, inside O. Hence, $y_2 \in O_1$, a contradiction.

Therefore, both O_1 and O_2 are open in X and hence also open in the open set O. The connectedness of O ensures that either $O_1 = \emptyset$ or $O_2 = \emptyset$. Since $x \in O_1$, we obtain $O_2 = \emptyset$. Hence, $O = O_1$ and the result follows. □

Note that Proposition 6.43 ensures that every normed linear space is locally path connected. Therefore, every open connected subset of a normed linear space is path connected. The following are stronger results, for particular cases.

Corollaries 6.45 *Let O be an open connected subset of a normed linear space X.*

(a) Then any two elements of O can be joined by a polygonal path, inside O.
(b) If $X = \mathbb{R}^m$ for some $m \in \mathbb{N}$, then any two elements of O can be joined by a polygonal path consisting of line segments parallel to the axes, inside O.

Proof (a) Let $x \in O$ be arbitrary and O_1 be the set of $y \in O$ which are connected to x through a polygonal path inside O.

The proof is analogous to Theorem 6.44. The only difference is that here we take U_1 and U_2 to be some open balls in X and use the fact that if P_1 is a polygonal path from x to y_1 inside O, then $P_1 \cup [y_1, z_1]$ is a polygonal path from x to z_1.

(b) Let $x \in O$ be arbitrary and O_1 be the set of $y \in O$ which are connected to x through a polygonal path, with line segments parallel to the axes, inside O.

The proof is analogous to part (a). Use the fact that any two points in a ball $B \subset \mathbb{R}^m$ can be joined by a polygonal path inside B, having line segments parallel to the axes. □

6.4.3 Quasi-components

Let X be a metric space and $x \in X$. The *quasi-component* of x in X is defined to be the intersection of all clopen subsets X, containing x. For this section, let C_x and Q_x denote the connected component and the quasi-component of x in X, respectively.

Theorem 6.46 *Let X be a metric space and $x \in X$. Then $C_x \subset Q_x$.*

Proof Suppose there exists some $y \in C_x \setminus Q_x$. Then there exists a clopen set A containing x such that $y \notin A$. Then $(C_x \cap A) \dot\cup (C_x \setminus A)$ is a separation of C_x, a contradiction. □

The opposite inclusion holds for compact metric spaces. That requires the following lemmas.

Lemma 6.47 *Let A and B be disjoint compact subsets of a metric space X. Then there exist disjoint open subsets U and V of X such that $U \supset A$ and $V \supset B$.*

Proof Let $\delta := \inf\{d(a, b) : a \in A, b \in B\}$. Since A and B are compact and disjoint, one can conclude that $\delta > 0$. Let

$$U := \bigcup_{a \in A} B(a; \delta/3) \text{ and } V := \bigcup_{b \in B} B(b; \delta/3).$$

It can be shown that U and V satisfy our requirements. □

Lemma 6.48 *Let X be a compact metric space, O be an open subset of X and $\mathcal{F}$ be a collection of closed subsets of X such that $\bigcap_{F \in \mathcal{F}} F \subset O$. Then there exist finitely many $F_1, \ldots, F_n \in \mathcal{F}$ such that $\bigcap_{i=1}^n F_i \subset O$.*

Proof Since $X \setminus O$ is a closed subset of the compact space X, it is a compact subset of X. The given hypothesis implies that $X \setminus O \subset \bigcup_{F \in \mathcal{F}} (X \setminus F)$. The compactness of $X \setminus O$ implies that there are finitely many $F_1, \ldots, F_n \in \mathcal{F}$ such that $X \setminus O \subset \bigcup_{i=1}^n (X \setminus F_i)$. □

Theorem 6.49 *Let X be a compact metric space and $x \in X$. Then $C_x = Q_x$.*

Proof By Theorem 6.46, we have $C_x \subset Q_x$. Since C_x is the largest connected set containing x and $C_x \subset Q_x$, to prove that $C_x \supset Q_x$, it is enough to prove that Q_x is a connected set.

If possible, let $Q_x = A \dot\bigcup B$ be a separation of Q_x. Without loss of generality, assume that $x \in A$. Since by definition Q_x is closed in X, the sets A and B are also closed in X. Since X is compact, A and B are also compact subsets of X.

Applying Lemma 6.47, there are disjoint open subsets U and V of X such that $U \supset A$ and $V \supset B$. Applying Lemma 6.48, there are finitely many clopen subsets $D_1, \ldots, D_n$ of X such that $x \in \bigcap_{i=1}^n D_i \subset U \dot\bigcup V$. Write $D := \bigcap_{i=1}^n D_i$. Then D is a clopen subset of X such that $x \in D \subset U \dot\bigcup V$.

Let $E := D \cap U$. Then E is open and $x \in E$, as $x \in A \subset U$. Also, $E := D \cap (X \setminus V)$ is closed. By definition $Q_x \subset E$. Consequently, $B = Q_x \cap V \subset E \cap V = \emptyset$. Hence the result. □

6.4.4 Totally Disconnected Sets

Definition 6.50 A subset A of a metric space is said to be *totally disconnected* if no two points of A lie in a connected subset of A.

In other words, A is totally disconnected if and only if singletons are its only connected subsets if and only if all singletons subsets are its connected components.

Examples 6.51 Finite metric spaces, all discrete metric spaces, the set of natural numbers, and the set of rational numbers are all totally disconnected spaces.

Definition 6.52 A complex number is said to be *algebraic* if it is a root of some polynomial over integers. Non-algebraic complex numbers are known as *transcendental* numbers.

The numbers e and π are two standard examples of transcendental numbers. In Example 7.15, we establish an abundance of transcendental numbers besides the algebraic ones.

Example 6.53 The set of real algebraic numbers is totally disconnected.

Proof Let $\mathbb{A}$ be the set of real algebraic numbers. To the contrary, assume that $\mathbb{A}$ is not totally disconnected. Then $\mathbb{A}$ has a connected component E containing two distinct reals, say $x < y$. Pick any $r \in \mathbb{Q}$ such that $\pi \in (x + r, y + r)$. Then we have $\pi - r \in (x, y)$.

By suitably modifying the polynomials satisfied by the algebraic numbers x and y, one can prove that $\pi - r$ is not an algebraic number.

Note that the sets $\{x \in E : x < \pi - r\}$ and $\{x \in E : x > \pi - r\}$ are nonempty and form a separation of E. This is impossible, as E is a connected set. Hence the result. □

Theorem 6.54 *Let X be a compact metric space. Then X is totally disconnected if and only if for every $x \in X$ and $r > 0$ there exists some clopen set A such that $x \in A \subset B(x; r)$.*

Proof The converse follows from the definition of totally disconnected sets. For the necessity part, let X be totally disconnected, $x \in X$, $r > 0$ and $B := B(x; r)$.

By Theorem 6.49, $Q_x = C_x = \{x\} \subset B$. Applying Lemma 6.48, there are finitely many clopen subsets $A_1, \ldots, A_n$ of X such that $x \in \bigcap_{i=1}^n A_i \subset B$. Write $A := \bigcap_{i=1}^n A_i$. Then A is a clopen subset of X and $x \in A \subset B$. Hence the result. □

Corollary 6.55 *Let X be a totally disconnectedcompact metric space, $K \subset O \subset X$ such that K is compact and O is open in X. Then X has a clopen subset A such that $K \subset A \subset O$.*

Proof For every $x \in K$, there exists some $r_x > 0$ such that $B(x; r_x) \subset O$. Applying Theorem 6.54, one can choose clopen sets A_x such that $x \in A_x \subset B(x; r_x)$ for all $x \in K$. Since K is compact and $K \subset \bigcup_{x \in K} A_x$, there are finitely many $x_1, \ldots, x_n \in K$ such that $K \subset \bigcup_{i=1}^n A_{x_1}$. Then $A := \bigcup_{i=1}^n A_{x_i}$ meets our requirements. □

Notes and Remarks 6.56 (a) It is possible to have three disjoint connected open sets in the plane which have the same boundary. For example, see Lakes of Wada [4, p. 138].

(b) In Sect. 10.5.3, we shall discuss a connected space, which is not path connected, known as the Cantor's leaky tent. It is such a connected metric space, which becomes totally disconnected after the removal of a particular point.

(c) Let X, Y be compact metric spaces and let $f : X \longrightarrow Y$. We state two characterizations of continuity. Readers interested in details are referred to [5].

  (i) f is continuous if and only if f maps compact sets onto compact sets, and $f^{-1}(y)$ is a closed set, for every $y \in Y$.
  (ii) If X is locally connected, then f is continuous if and only if f maps compact sets onto compact sets, and connected sets onto connected sets.

Exercise 6.58 Let $D := \{z \in \mathbb{C} : |z| < 1\}$ and $f : D \longrightarrow \mathbb{R}^3$ be a continuous function. How many subsets of $f(D)$ are both open as well as closed?

Exercise 6.59 Complete the details in the proof of Corollaries 6.45.

Exercise 6.60 Let X be a normed linear space. Prove the following:

(a) Every convex subset of a normed linear space is path connected.
(b) X is convex, and hence path connected.
(c) The converse of (a) does not hold.

Exercise 6.61 Prove that every discrete metric space is totally disconnected.

Exercise 6.62 Prove that both $\mathbb{Q}$ and $\mathbb{R} \setminus \mathbb{Q}$ are totally disconnected sets.

Exercise 6.63 After replacing $\sin \frac{1}{x}$ with $x \sin \frac{1}{x}$ in Example 6.23, does the resulting space become locally connected or path connected?

Exercise 6.64 Let A be a connected component of a locally connected metric space X. Prove that A is clopen in X.

Exercise 6.65 If E is a convex subset of a normed linear space, prove that so are the sets E^o and $\overline{E}$.

Exercise 6.66 If E is a convex subset of a normed linear space such that $E^o \neq \emptyset$, prove that $\overline{E} = \overline{E^o}$.

Exercise 6.67 Let X be a metric space in which every open ball is a closed subset of X. Prove that X is totally disconnected.

Exercise 6.68 Let X be a metric space. Prove that the following are equivalent:

(a) Quasi-components of X are singletons.
(b) For any $x, y \in X$, there exist disjoint clopen neighborhoods U_x and U_y of x and y, respectively, such that $U_x \bigcup\!\!\!\cdot\, U_y = X$. (Such an X is called a *totally separated space*.)

Exercise 6.69 Prove that every totally separated space is totally disconnected. Is the converse true?

Exercise 6.70 If d is an ultrametric on X, prove that (X, d) is totally separated.

Exercise 6.71 Let X be a totally disconnected compact metric space. Prove that for every $n \in \mathbb{N}$, there exists a finite open cover $\mathcal{U}_n$ of X with disjoint sets, each having diameter $< 2^{-n}$.

Exercise 6.72 Prove that a metric space X is locally connected if and only if for every open set O of X, each connected component of O is open in X.

Exercise 6.73 Prove that a metric space X is *locally path connected* if and only if for every open set O of X, each path component of O is open in X.

Exercise 6.74 Let X be a metric space. Prove that each path component of X lies in a connected component of X.

Exercise 6.75 If X is locally path connected, then prove that has same collection of path components and connected components.

Exercise 6.76 Let X be a metric space and $x \in X$. Let P_x, C_x and Q_x denote, respectively, the path component, connected component and the quasi-component of x. Prove that $P_x \subset C_x \subset Q_x$. If X is locally path connected, then prove that $P_x = C_x = Q_x$.

Exercise 6.77 A metric space X is said to be *weakly locally connected* at $x \in X$ if every neighborhood U of x contains a connected subspace S of X that contains a neighborhood of x. Prove that if X is weakly connected at each of its points, then X is locally connected. Is the converse true?

Exercise 6.78 State and prove the result analogous to Exercise 6.56 for quasi-components.

6.5 Hints and Solutions to Selected Exercises

6.2 No. Suppose that$a \neq b$ and $\{a, b\}$ be a path connected subset of a space X. Then there exists a continuous map $f : [0, 1] \longrightarrow \{a, b\}$ such that $f(0) = a$ and $f(1) = b$. Thus, $A := f^{-1}(a)$ and $B := f^{-1}(b)$ are nonempty disjoint closed subsets of $[0, 1]$ with union $[0, 1]$, a contradiction to Lemma 6.6.

6.4 Apply Intermediate Value Theorem (6.12) on $x \longmapsto f(x) - f(x + 1)$ on $[0, 1]$.

6.7 Yes. Apply Intermediate Value Theorem (6.12) on $f(p) := \sqrt{(2^p + 3^p + 4^p)/3}$.

6.8 No. Let $f : [0,1]^2 \longrightarrow [0,1]$ be a continuous bijection and x, y, z be any three elements from $[0,1]^2$. Since $[0,1]^2 \setminus \{x, y, z\}$ is path connected, so is its image under f. That is, $[0,1] \setminus \{f(x), f(y), f(z)\}$ is a path connected subset of $[0,1]$, a contradiction to Theorem 6.9. Similarly, the second part.

6.10 One. Let $f(x)$ denote the given cubic. Since f is a real polynomial of odd degree, it must have at least one real root. If it has two distinct roots, then f' must have a root between those. This is impossible as $f(x)$ is a sum of squares of reals and a, b, c are all distinct.

6.11 First three options are correct and the last one is false. Note that (c) is trivial. Let $-1, \alpha_1, \ldots, \alpha_{n-1}$ be all the roots of p, counting multiplicities. Equating coefficients from $p(x) = (x+1)\prod_{i=1}^{n-1}(x - \alpha_i)$, we obtain

$$\prod_{i=1}^{n-1} \alpha_i = (-1)^n, \text{ which implies } \prod_{i=1}^{n-1} |\alpha_i| = 1. \tag{6.2}$$

If α is a non-real root of p, then by hypothesis, $\overline{\alpha}$ is also a root of p and $|\alpha| \geq 1$. Thus, $\alpha\overline{\alpha} \geq 1$. Now (6.2) implies that $|\alpha| = 1$, that is $\alpha\overline{\alpha} = 1$, for every non-real root of p.

Similarly, since p has no real root in $(-1, 1)$, again (6.2) implies p has no real root in $\mathbb{R} \setminus [-1, 1]$. This concludes that (d) is false. Also, (b) is true, because if (b) is false then p will have a root in $[2, \infty)$, a contradiction.

By now we have established that all the roots of p lie on $\{z \in \mathbb{C} : |z| = 1\}$. If 1 is not a root of p, then the representation $(x+1)\prod_{i=1}^{n-1}(x - \alpha_i)$ of p will either have linear factors $(x+1)$ or pairs of complex conjugates $\alpha, \overline{\alpha}$ such that $\alpha\overline{\alpha} = 1$. Therefore, the constant term in this product must be 1, a contradiction. Hence, (a) is also true.

6.13 (a) No. For example, consider the function f as in Exercise 1.100.

(b) Assume that f is not continuous. Then there exists $x_n \longrightarrow x_0$ such that $f(x_n) \not\longrightarrow f(x_0)$. Without loss of generality, we can suppose that there is a rational number r and a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $f(x_0) > r > f(x_{n_k})$ for all $k \in \mathbb{N}$.

By hypothesis, one can choose t_k between x_0 and x_{n_k} such that $f(t_k) = r$ for all $k \in \mathbb{N}$. Then $t_k \longrightarrow x_0$. Since $\{t_k\}$ is a sequence in the closed set $f^{-1}(\{r\})$, we conclude that its limit x_0 also belongs to this set. Hence, $f(x_0) = r$, a contradiction.

6.14 Let f, a, b and l be as in Theorem 6.12.

(a) Write $S := \{t \in [a, b] : f(t) < l\}$. Then S is a nonempty subset of $[a, b]$, containing a, and bounded above by b. Let $c := \sup S$. Since the function f

is continuous at a and b, with $f(a) < l$ and $f(b) > l$, we obtain $a < c < b$. We claim that $f(c) = l$.

If $f(c) < l$, then the continuity of f at c implies that there exists some $\delta > 0$ such that $(c - \delta, c + \delta) \subset [a, b]$ and $f(x) < l$ for all $x \in (c - \delta, c + \delta)$. Therefore, $c + \delta/2 \in S$, a contradiction as $c = \sup S$. Similarly, if $f(c) > l$, there also exists $\delta > 0$ such that $(c - \delta, c + \delta) \subset [a, b]$ and $f(x) > l$ for all $x \in (c - \delta, c + \delta)$. Then $(c - \delta, c + \delta) \cap S = \emptyset$ which implies that $c \neq \sup S$, a contradiction.

(b) Assume that $f(t) \neq l$ for all $t \in [a, b]$. For $t \in [a, b]$, if $f(t) > l$, choose $\delta_t > 0$ such that $f(x) > l$ for all $x \in (t - \delta_t, t + \delta_t) \cap [a, b]$. Otherwise, choose $\delta_t > 0$ such that $f(x) < l$ for all $x \in (t - \delta_t, t + \delta_t) \cap [a, b]$.

By compactness of $[a, b]$, there are finitely many $t_1, \ldots, t_n \in [a, b]$ such that $[a, b] \subset \bigcup_{i=1}^{n}(t_i - \delta_{t_i}, t_i + \delta_{t_i})$. Since $f(a) < l$, it leads to $f(b) < l$, a contradiction.

6.15 Assume that $I = A \mathbin{\dot\cup} B$, for some disjoint nonempty sets A and B, open in I. Pick any $a \in A$ and $b \in B$. Without loss of generality, suppose $a < b$.

Write $S_1 := [a, b] \cap A$. Since S_1 contains a and is bounded above by b, it has a supremum. Let $s_1 := \sup S_1$. Then $s_1 \leq b$. We claim that $s_1 < b$.

If $s_1 = b$, then $I \cap (s_1 - \delta, s_1 + \delta) \subset B$, for some $\delta > 0$, as B is open in I. This implies that $(s_1 - \delta, s_1 + \delta) \cap A = \emptyset$. Therefore, $s_1 = \sup S_1 \leq s_1 - \delta$, a contradiction. Hence, $s_1 < b$.

Write $S_2 := [s_1, b] \cap B$ and $s_2 := \inf S_2$. Then $s_1 \leq s_2$. As above, one can show that $s_1 < s_2$. Let $s_3 \in (s_1, s_2)$. Since $s_2 = \inf S_2$, we have $s_3 \notin B$. Similarly, $s_1 = \sup S_1$ implies that $s_3 \notin A$. Therefore, I is not an interval, as $s_3 \in I \setminus A \cup B$.

6.18 No. Take the intersection of the x-axis and the unit circle, in $\mathbb{R}^2$. Then both A and B are connected, while $A \cap B = \{(0, 0), (0, 1)\}$ is not.

6.19 Yes. E.g. consider $\mathbb{R}^2 \setminus \{(x, 0) : x \in \mathbb{Q}\}$ in $\mathbb{R}^2$.

6.20 Both assertions are false. E.g., consider the subspace $\{(x, y) \in \mathbb{R}^2 : xy \geq 0\}$ of $\mathbb{R}^2$, which is a path connected set while the interior is neither connected nor path connected.

6.26 Apply Theorem 6.22. For the converse, take $A := [-1, 0) \cup (0, 1]$.

6.29 Let $\delta := dist(A, B)$. If $A \cap \overline{B} \neq \emptyset$, then there exists some $x \in A \cap \overline{B}$. By hypothesis $B(x; \delta/2) \cap B = \emptyset$. Therefore, $x \notin \overline{B}$, a contradiction. For the converse, take $A = (0, 1)$ and $B = (1, \infty)$ in reals with usual topology.

6.32 Let $A = A_1 \mathbin{\dot\bigcup} A_2$ be a separation of A. Then $A \cap B = (A_1 \cap B) \mathbin{\dot\bigcup} (A_2 \cap B)$ is a separation of $A \cap B$. Since $A \cap B$ is connected, one of these must be empty. Suppose $A_1 \cap B = \emptyset$. Then $A \cup B = A_1 \mathbin{\dot\cup} (A_2 \cup B)$. Since $A \cup B$ is connected, either $A_1 = \emptyset$ or $A_2 \cup B = \emptyset$, that is, either $A_1 = \emptyset$ or $A_2 = \emptyset$. Hence, A is connected. Similarly, B is connected.

6.39 Fix $x \in [0, 1]$. Define $\phi_x : A \longrightarrow [0, 1]$ as $\phi_x(a) := a(x)$ for all $a \in A$. Then ϕ_x is a continuous map, as $|\phi_x(a) - \phi_x(b)| = |a(x) - b(x)| \leq \|a - b\|_\infty$. Since A is connected, $\phi_x(A) = \{a(x) : a \in A\}$ is a connected subset of $[0, 1]$.

6.40 Define $\phi : C[0,1] \longrightarrow \mathbb{R}$ as $\phi(f) := \int_0^1 f$ for all $f \in C[0,1]$. Then ϕ is continuous, as $|\phi(f) - \phi(g)| \leq \int_0^1 \left|f - g\right| \leq \|f - g\|_\infty$. Hence,

$$\{f \in C[0,1] : \phi(f) > 0\} \,\dot{\cup}\, \{f \in C[0,1] : \phi(f) < 0\}$$

is a separation of $(C[0,1], \|.\|_\infty)$.

6.44 No. Apply Theorem 6.29.

6.50 Apply Theorem 6.31.

6.51 Let $\mathcal{A} := \bigcup_{A\in\Omega} A$ and fix any $A_0 \in \Omega$. Let T be the connected component of $\mathcal{A}$, containing A_0. For any $A \in \Omega$, we have $A_0 \cap A \neq \emptyset$ which implies $A_0 \cup A \subset T$ for all $\alpha \in \wedge$. Therefore, $\bigcup_{A\in\Omega} A \subset T$ which implies $T = \bigcup_{A\in\Omega} A$.

6.52 The result holds by induction on Theorem 6.31.

6.53 Write $A := \bigcup_{n=1}^{\infty} A_n$. Suppose that A is not connected. Then there exist nonempty disjoint sets E_1 and E_2, open in A such that $E_1 \dot{\bigcup} E_2 = A$. We claim that for every $n \in \mathbb{N}$, either $A \subset E_1$ or $A \subset E_2$. To show this, suppose that there exists some $n \in \mathbb{N}$ such that $A \cap E_1 \neq \emptyset$ and $A \cap E_2 \neq \emptyset$. Then A_n is not connected, as $A_n = (A_n \cap E_1) \dot{\bigcup} (A_n \cap E_2)$ and both $A_n \cap E_1$ and $A_n \cap E_2$ are open in A_n. This establishes our claim.

Hence, either $A_1 \subset E_1$ or $A_1 \subset E_2$. Without loss of generality, suppose that $A_1 \subset E_1$. Since $A_1 \cap A_2 \neq$, we have $A_2 \subset E_1$. Inducting like this, we obtain $A_n \subset E_1$ for all $n \in \mathbb{N}$. Hence, $A \subset E_1$ and thus $E_2 = \emptyset$, a contradiction. This proves the result.

6.54 Let $A_1, \ldots, A_n$ be the only connected components of X. By Theorem 6.22, each A_i is closed. By Theorem 6.32, X is a disjoint union of $A_1, \ldots, A_n$. Then $A_i = X \setminus \bigcup_{j\neq i} A_j$, for all i. Hence, each A_i is open.

6.55 Yes. Because $A \times B$ is a (path) connected subset of $\mathbb{R}^2 \times \mathbb{R}^2$ and the map $(a, b) \longrightarrow a + b$ from $A \times B$ onto $A + B$ is continuous.

6.64 Applying Theorem 6.22, A is closed. Let $a \in A$ and N_a be a connected neighborhood of a. By Theorem 6.31, $A \cup N_a$ is a connected subset of X. Since A is a maximal connected set, we obtain $A \cup N_a = A$ and hence $N_a \subset A$. Consequently, A is open.

6.65 Assume that E is convex. Let $t \in (0,1)$. Since E is convex, $tE^o + (1-y)E^o \subset E$. By previous part, $tE^o + (1-y)E^o$ is an open subset of E. Hence, $tE^o + (1-y)E^o \subset E^o$, that is, E^o is convex. To show that $\overline{E}$ is convex, let $x, y \in \overline{E}$. Then there are sequences $\{x_n\}$ and $\{y_n\}$ in E, convergent to x and y, respectively. Hence, $tx + (1-t)y = \lim_{n\to\infty}(tx_n + (1-t)y_n) \in \overline{E}$, as E is convex. Hence, $\overline{E}$ is convex.

6.66 Let X be the given normed linear space. Clearly, $\overline{E^o} \subset \overline{E}$. To prove that $\overline{E^o} \supset \overline{E}$, it is enough to show that $\overline{E^o} \supset E$. Since $E^o \neq \emptyset$, let $a \in E^o$. Then $B(a; r) \subset$

E for all $r > 0$. That is, $a + ry \in E$ for all $y \in X$ such that $\|y\| \leq 1$. Let $x \in E$ and $x_t := ta + (1 - t)x$. We claim that $B(x_t; tr) \subset E$.

Let $y \in X$ such that $\|y\| \leq 1$. Then $x_t + try = t(a + ry) + (1 - t)x \in E$, as E is convex. This proves our claim. Hence, $x_t \in E^o$. Hence, $x = \lim_{t\to 0} x_t \in E^o$.

6.67 Let $x \in X$ and A be the connected component of X containing x. We claim that $A = \{x\}$. If not, then pick any $y \in A \setminus \{x\}$. Let

$$\delta := d(x, y),\ B_x := B\left(x; \frac{\delta}{2}\right) \text{ and } B_y := B\left(x; \frac{\delta}{2}\right).$$

By hypothesis, both B_x and B_y are clopen in X. Therefore, $A \cap B_x$ and $A \cap B_y$ are clopen and nonempty in the connected subspace A, a contradiction.

6.69 Apply Theorem 6.46. A counter example will be provided in Exercise 10.81.

6.70 Let $x, y \in X$ such that $x \neq y$ and $r := d(x, y)$. By Exercise 3.65(a), both $B(x; r)$ and $X \setminus B(x; r)$ are disjoint clopen subsets of X containing x and y, respectively.

6.71 Fix $n \in \mathbb{N}$. Since X is totally disconnected and compact, by Theorem 6.54, it has a finite cover $\{O_1, \ldots, O_n\}$ consisting of clopen sets with diameter $< 2^{-n}$. Define $U_i := O_i \setminus \bigcup_{j<i} O_j$ for all i. Then each U_i is open and has diameter $< 2^{-n}$. The result follows, as we have $\bigcup_i U_i = \bigcup_i O_i$.

6.72 ($\Rightarrow$) Let O be an open subset of X and U be a connected component of O. Let $x \in U$. Since X is locally connected, there exists a connected neighborhood B_x of x such that $B_x \subset O$. Since U is a connected component of O, containing x, we obtain $B_x \subset U$. Hence, U is open in X.

($\Leftarrow$) Let $x \in X$ and U be any neighborhood of x. Then there exists an open set O such that $x \in O \subset U$. Let C_x be the connected component of x relative to O. By hypothesis, C_x is open in X. The result follows as $x \in C_x \subset O \subset U$.

6.73 Analogous to Exercise 6.72.

6.75 Let C be a connected component of X. Let $x \in C$ and P be the path component of x, inside X. Since P is connected, we obtain $P \subset C$. We claim that $P = C$. Suppose that $Q := C \setminus P \neq \emptyset$.

For each $q \in Q$, let P_q be the path component of q. Then each such P_q is connected and hence a subset of C. Consequently, Q is a union of path components of X. Applying Exercise 6.73, the union of all these path components, that is Q, is open in X. Similarly, the path component P is open in X. Since $C = P \,\dot\cup\, Q$ is connected, either $P = \emptyset$ or $Q = \emptyset$. But $x \in P$. Therefore, $Q = \emptyset$. Hence, $P = C$.

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Chapter 7
Cardinality

Can you write a given set as a sequence? In particular, can you write all real numbers together as a sequence? Such questions lead to the notions of countable and uncountable sets, and cardinality in general. This chapter discusses these ideas, along with their applications. It starts with countable and uncountable sets, followed by two special sections entitled 'Some Applications to Topology' and 'The Set of Discontinuities'. The latter includes the case of monotone functions along with the general case, which asserts that the set of discontinuities of a function between metric spaces is a countable union of closed sets. Finally, there is a section on cardinality which provides a glimpse into cardinal arithmetic.

7.1 Countable and Uncountable Sets

Definitions 7.1 Two sets A and B are said to be *equivalent* if there exists a bijection between A and B. In this case, we write $A \simeq B$.

Equivalent sets are also known as *equipotent sets* or sets having the same *cardinality.*

Examples 7.2 (a) The sets $\{a, b, c\}$ and $\{1, 2, 3\}$ are equivalent.
(b) The sets $\mathbb{N}$ and $\{3n : n \in \mathbb{N}\}$ are also equivalent.
(c) The sets $\{a, b, c, d, e\}$ and $\{1, 2, 3\}$ are not equivalent.

Definitions 7.3 A set E is said to be

(a) *finite* if either $E = \varnothing$ or $E \simeq \{1, 2, \dots, n\}$ for some $n \in \mathbb{N}$,
(b) *infinite* if E is not finite,
(c) *countably infinite* if $E \simeq \mathbb{N}$,
(d) *countable* if E is either finite or countably infinite and
(e) *uncountable* if E is not countable.

Note that every element of a countable set can be assigned a counting number. That is why the terms *enumerable* and *denumerable* are also used for countable sets.

Examples 7.4 (a) The set of English alphabet is finite and hence countable.

S. P. S. Kainth, *A Comprehensive Textbook on Metric Spaces*,
https://doi.org/10.1007/978-981-99-2738-8_7

(b) The set of odd natural numbers larger than 100 is countably infinite, as $n \longmapsto 100 + (2n - 1)$ is a bijection from $\mathbb{N}$ onto this set.
(c) The set of even natural numbers is countably infinite, as $f : \mathbb{N} \longrightarrow \{2, 4, \dots\}$ defined as $f(n) := 2n$, is a bijection.

Theorem 7.5 *A set is countably infinite if and only if it can be written as a sequence of distinct terms.*

Proof Let E be a countably infinite set. Then there exists a bijection $f : \mathbb{N} \longrightarrow E$. We claim that $\{f(n)\}$ is a sequence of distinct elements with $\{f(n) : n \in \mathbb{N}\} = E$. This sequence has distinct terms, as f is injective. Further the surjectivity of f implies that for every $x \in E$, there exists some $n \in N$ such that $x = f(n)$. Hence, $E \subset \{f(n) : n \in \mathbb{N}\}$. The other inclusion is immediate from the definition of f.

Conversely, let E be a set which can be written as a sequence of distinct terms, say $\{x_n\}$. Define $f : \mathbb{N} \longrightarrow E$ as $f(n) := x_n$ for all $n \in \mathbb{N}$. Since $\{x_n : n \in \mathbb{N}\} = E$, the function f is surjective. Further as the terms of the sequence are distinct, f is injective. Therefore, $E \simeq \mathbb{N}$. □

The following corollaries are immediate from Theorem 7.5.

Corollaries 7.6 *(a) Infinite subsets of countably infinite sets are countably infinite.*
(b) A nonempty set E is countable if and only if it can be written as a sequence.
(c) Subsets of countable sets are countable.
(d) Sets containing uncountable sets are uncountable.
(e) Countable sets having infinite subsets are countably infinite.

Example 7.7 The set of integers $\{0, -1, 1, \dots, -n, n, \dots\}$ is countably infinite.

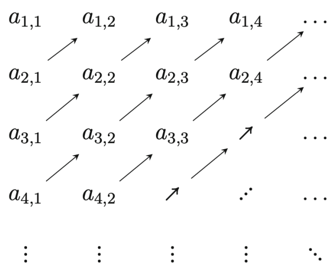
Theorem 7.8 *Countable union of countable sets is countable.*

Proof Let Ω be a countable collection of countable sets. Then Ω can be written as a sequence, say $\{A_n\}$.

Since each A_n is countable, one can write

$$A_n := \{a_{n,1}, \dots a_{n,k}, \dots\}.$$

Now we write $\bigcup_{n=1}^{\infty} A_n$ as an infinite array, as shown in the adjoining figure.



We choose to write this array diagonal-wise. Thus, $\bigcup_{n=1}^{\infty} A_n = \{a_{1,1}, a_{2,1}, a_{1,2}, a_{3,1}, a_{2,2}, a_{1,3}, \dots\}$. Hence, $\bigcup_{n=1}^{\infty} A_n$ is countable. □

Remarks 7.9 In the proof of Theorem 7.8, it can be shown that $a_{i,j}$ occurs as the $f(i,j)^{th}$ term of the sequence enumerating $\bigcup_{n=1}^{\infty} A_n$, where

$$f(i,j) := \sum_{k=1}^{i+j-1} k - (i-1) \text{ for all } i, j \in \mathbb{N}.$$

Further, one can enumerate the union $\bigcup_{n=1}^{\infty} A_n$ in various other ways. Two such methods have been demonstrated in the following figures.

$$\begin{array}{ccccccccc}
a_{1,1} & \rightarrow & a_{1,2} & & a_{1,3} & \rightarrow & a_{1,4} & & \cdots \\
 & & \downarrow & & \uparrow & & \downarrow & & \\
a_{2,1} & \leftarrow & a_{2,2} & & a_{2,3} & & a_{2,4} & & \cdots \\
\downarrow & & & & \uparrow & & \downarrow & & \\
a_{3,1} & \rightarrow & a_{3,2} & \rightarrow & a_{3,3} & & a_{3,4} & & \cdots \\
 & & & & & & \downarrow & & \\
a_{4,1} & \leftarrow & a_{4,2} & \leftarrow & a_{4,3} & \leftarrow & a_{4,3} & & \cdots \\
\downarrow & & & & & & & & \\
\vdots & & \vdots & & \vdots & & \vdots & & \ddots
\end{array}
\qquad
\begin{array}{ccccccccc}
a_{1,1} & & a_{1,2} & \rightarrow & a_{1,3} & & a_{1,4} & & \cdots \\
\downarrow & \nearrow & & \swarrow & & \nearrow & & & \\
a_{2,1} & & a_{2,2} & & a_{2,3} & & a_{2,4} & & \cdots \\
 & \swarrow & & \nearrow & & \swarrow & & & \\
a_{3,1} & & a_{3,2} & & a_{3,3} & & a_{3,4} & & \cdots \\
\downarrow & \nearrow & & \swarrow & & \nearrow & & & \\
a_{4,1} & & a_{4,2} & \rightarrow & a_{4,3} & & \ddots & & \cdots \\
 & & & & & & & & \\
\vdots & & \vdots & & \vdots & & \vdots & & \ddots
\end{array}$$

Example 7.10 The set of rational numbers is countable.

Proof Note that $\mathbb{Q} = \bigcup_{n=1}^{\infty} \{\frac{m}{n} : m \in \mathbb{Z}\}$. Also, the set $\{\frac{m}{n} : m \in \mathbb{Z}\}$ is countable, for each $n \in \mathbb{N}$. Applying Theorem 7.8, $\mathbb{Q}$ is countable. □

Proposition 7.11 *If A and B are countable, then so is their Cartesian product $A \times B$.*

Proof Since A and B are countable, these can be written as sequences, say $A = \{a_n : n \in \mathbb{N}\}$ and $B = \{b_n : n \in \mathbb{N}\}$. Imitating the proof of Theorem 7.8, we write

$$\begin{aligned} A \times B &= \{(a_m, b_n) : m, n \in \mathbb{N}\} \\ &= \{(a_1, b_1), (a_2, b_1), (a_1, b_2), (a_3, b_1), (a_2, b_2), (a_1, b_3), \ldots\}. \end{aligned}$$

Applying Corollary 7.6(b), we conclude that $A \times B$ is a countable set. □

Corollary 7.12 *The finite Cartesian product of countable sets is countable.*

Proof We prove the result by induction on the number of sets. By Proposition 7.11, it is already true for two sets. Assume it for n sets. Let $A_1, \ldots, A_{n+1}$ be countable sets. Note that the map $(a_1, \ldots, a_{n+1}) \longrightarrow \big((a_1, \ldots, a_n), a_{n+1}\big)$ is a bijection from $A_1 \times \cdots \times A_{n+1}$ onto $(A_1 \times \cdots \times A_n) \times A_{n+1}$. The induction hypothesis and Proposition 7.11 concludes the result. □

However, a countable Cartesian product of finite sets may not be countable.

Theorem 7.13 (Cantor) *The Cartesian product $\prod_{n=1}^{\infty} \{0, 1\}$ is uncountable.*

Proof Let $P := \prod_{n=1}^{\infty}\{0, 1\}$ be the set of sequences $\{x_n\}$ such that each $x_n = 0$ or 1. Suppose that P is uncountable. Therefore, we can write $P = \{a_n : n \in \mathbb{N}\}$. Note that each a_n is a sequence with terms either 0 or 1. Write $a_n := \{a_{n,m}\}_{m\in\mathbb{N}}$ for all n. Construct a sequence $x = \{x_n\}$ such that

$$x_n := \begin{cases} 1 & \text{if } a_{n,n} = 0 \\ 0 & \text{if } a_{n,n} = 1. \end{cases}$$

Then x is a sequence with terms either 0 or 1, while $x \neq a_n$ for any n. Hence, we obtain $x \in P \setminus \{a_n : n \in \mathbb{N}\} = \varnothing$, which is absurd. □

Example 7.14 $\mathbb{R}$ is uncountable.

Proof Let E denote the set of all those reals in [0, 1] whose decimal expansions consist of only digits 0 and 1. Then E is in a one-to-one correspondence with the set $\prod_{n=1}^{\infty}\{0, 1\}$. Applying Theorem 7.13, E is uncountable. The result follows, as $\mathbb{R}$ contains E. □

An Alternative Proof Since $\mathbb{R} \supset [0, 1]$, it is enough to prove that [0, 1] is uncountable. Suppose not. Then there exists a sequence $\{a_n\}$ such that $\{a_n : n \in \mathbb{N}\} = [0, 1]$. Write

$$[0, 1] = \left[0, \frac{1}{3}\right] \bigcup \left[\frac{1}{3}, \frac{2}{3}\right] \bigcup \left[\frac{2}{3}, 1\right].$$

Let I_1 be one of the above three closed intervals, on the right-hand side above, such that $a_1 \notin I_1$. Then we further partition I_1 into three closed intervals of length 1/9 and let I_2 be one of those intervals such that $a_2 \notin I_2$.

Inducting this way, we obtain a nested decreasing sequence of closed intervals $\{I_n\}$ such that

$$a_n \notin I_n,\ I_n \supset I_{n+1} \text{ and } l(I_n) = \frac{1}{3^n} \text{ for all } n \in \mathbb{N}.$$

Applying Nested Interval Property (1.23), we have $\bigcap_{n=1}^{\infty} I_n = \{x\}$ for some $x \in [0, 1]$. Therefore, $x = a_k$, for some $k \in \mathbb{N}$. Then $a_k = x \in I_k$, a contradiction. □

We already know that $\mathbb{Q}$ is countable. Hence, the set of irrational numbers is uncountable. In fact much 'larger supersets' of the set rational numbers, are countable. The set of algebraic numbers as defined in Definition 6.52, is also countable.

Examples 7.15 (a) For every $n \in \mathbb{N}$, there are only countably many polynomials with integral coefficients and degree n.
(b) The set of algebraic numbers is countable.
(c) The set of transcendental numbers is uncountable.

Proof (a) For every $n \in \mathbb{N}$, let $\mathbb{P}_n$ be the set of polynomials of degree n with integral coefficients. Let $p(x) := a_0 + a_1x + \cdots + a_nx^n$; $a_n \neq 0$, be any polynomial from $\mathbb{P}_n$. Then there are countably many choices for each a_i. Since finite Cartesian product of countable sets is countable, we obtain (a).

(b) Let $\mathbb{A}$ be the set of algebraic numbers. For each $n \in \mathbb{N}$, write

$$\mathbb{A}_n := \{a \in \mathbb{A} : p_n(a) = 0 \text{ for some } p_n \in \mathbb{P}_n\}.$$

Then $\mathbb{A} = \bigcup_{n=1}^{\infty} \mathbb{A}_n$. By (a), the set $\mathbb{P}_n$ is countable. Since each polynomial of degree n has at most n distinct roots, the set $\mathbb{A}_n$ is a countable union of finite sets. Therefore, each $\mathbb{A}_n$ is countable. Hence, $\mathbb{A} = \bigcup_{n=1}^{\infty} \mathbb{A}_n$ is countable.

(c) Since $\mathbb{R}$ is uncountable, so is its superset $\mathbb{C}$. Further (a) implies that the set of transcendental numbers $\mathbb{C} \setminus \mathbb{A}$ is uncountable. □

Notes and Remarks 7.16 In Exercise 7.23, we shall present a 'direct proof' of the uncountability of the set of transcendental numbers, as given in [1]. However, that 'direct proof' uses quite a 'tricky' idea that π is transcendental. Another proof for the uncountability of this set can be found in [2]. A popular proof for the countability of the set of algebraic numbers will be outlined in Exercise 7.21.

The first proof of Example 7.14 is the traditional one. We shall also provide its alternative proofs in Theorem 8.26 and in Corollary C.2. However, it must be noted that all these proofs depend upon some equivalent form of the completeness property of $\mathbb{R}$. For a thorough discussion on diverse proofs of the uncountability of $\mathbb{R}$, we refer the reader to [3].

Remarkably, there are bijections among the intervals $(0, 1)$, $(0, 1]$ and $[0, 1]$. These can be established in multiple ways (see Exercises 7.14-7.17). Two explicit formulas for this purpose appeared recently in [4]. We shall provide one of these in Exercise 7.22.

Every countably infinite set has an uncountable family of subsets, the intersection of any two of which is finite (see Exercise 7.24 or [5]).

Since there are four types of nitrogen bases of DNA, by Theorem 7.13, there are uncountably many DNA sequences!

Open Question 7.17 Is there any bijective polynomial from $\mathbb{Q} \times \mathbb{Q}$ onto $\mathbb{Q}$?

In particular, it is a celebrated *open problem* whether the polynomial $x^7 + 3y^7$ satisfies this requirement or not. A few partial results in this direction, through algebraic geometry, are documented in [6].

Exercise 7.1 Is countable intersection of countable sets always countable?

Exercise 7.2 Prove that every infinite set contains a countably infinite proper subset.

Exercise 7.3 Prove that every countable union of countably infinite sets is countably infinite.

Exercise 7.4 Which of the following subsets of $\mathbb{R}^2$ is/are countable:

$$\{(a, b) : a \le b\}, \{(a, b) : a + b \in \mathbb{Q}\}, \{(a, b) : ab \in \mathbb{Q}\}, \{(a, b) : a, b \in \mathbb{Q}\}?$$

Exercise 7.5 Prove that every finite Cartesian product of countably infinite sets is countably infinite.

Exercise 7.6 Let $\{A_n\}$ be a sequence of finite sets, each having at least two points. Which of the following sets is/are countable $\bigcup_{n=1}^{\infty} A_n$, $\prod_{n=1}^{\infty} A_n$, $\bigcup_{n=1}^{\infty} \prod_{k=1}^{n} A_k$?

Exercise 7.7 Prove that a set is infinite if and only if it is in one-to-one correspondence with a proper subset of itself.

Exercise 7.8 In $\mathbb{R}$, prove that every unbounded set is infinite. Is the converse true?

Exercise 7.9 Prove that the collection of polynomials over $\mathbb{Q}$ is countable.

Exercise 7.10 Prove that every unbounded subset of a metric space contains a countably infinite subset.

Exercise 7.11 Prove that any collection of disjoint open intervals is countable.

Exercise 7.12 Let E be any nonempty set. Prove that the set of sequences with terms from E, is either a singleton set or an uncountable set.

Exercise 7.13 Let A be any nonempty set. Show that there exists a bijection between the family of functions $f : A \longrightarrow \{0, 1\}$ and the power set of A.

Exercise 7.14 Let $\{x_n\}$ be any sequence of distinct terms in $(0, 1)$. Prove that the map $f : (0, 1) \longrightarrow [0, 1]$ defined as under is a bijection:

$$f(x) := \begin{cases} 0 & \text{if } x = x_1 \\ 1 & \text{if } x = x_2 \\ x_{n-2} & \text{if } x = x_n \text{ and } n > 2 \\ x & \text{if } x \in (0, 1) \setminus \{x_n : n \in \mathbb{N}\}. \end{cases}$$

Deduce that if $A \subset [0, 1]$ is countable, then there exists a bijection from $[0, 1]$ onto $[0, 1] \setminus A$.

Exercise 7.15 Establish a bijection between the following pairs of sets:

(a) $\mathbb{R}$ and $(0, \infty)$,
(b) $\mathbb{R}$ and $(100, \infty)$,
(c) $\mathbb{N}$ and $\{\frac{\sqrt{3}}{n} : n \in \mathbb{N}\}$,
(d) $\mathbb{N}$ and $\{0, 1\} \times \mathbb{Z}$.

Exercise 7.16 Obtain a bijection between

(a) $(0, 1)$ and $(0, \infty)$.
(b) $\mathbb{R}$ and $(0, \infty)$.
(c) $\mathbb{R}$ and $(0, 1)$.

Exercise 7.17 Let $a, b, c, d \in \mathbb{R}$ be such that $a < b$ and $c < d$. Establish a bijection between

(a) $[0, 1]$ and $[a, b]$,
(b) $[a, b]$ and $[c, d]$,

(c) $(0, 1)$ and $(1, \infty)$,
(d) $(0, 1)$ and $(-\infty, \infty)$,
(e) any two bounded intervals, having at least two points,
(f) any two intervals, having at least two points.

Conclude that all non-degenerate intervals are uncountable.

Exercise 7.18 Prove or disprove:

(a) The set $\mathbb{Q}^{100000}$ is countable.
(b) There is a surjective map from $\mathbb{Q}^{10^{1000}}$ onto $(0, 10^{-1000})$.

Exercise 7.19 If $x \in (0, \infty)$ and $\sum_{k=0}^{\infty} \frac{x_k}{10^k}$ is the (unique) infinite decimal expansion of x, write

$$f(x) := \left(\sum_{k=0}^{\infty} \frac{x_{3k}}{10^k}, \sum_{k=0}^{\infty} \frac{x_{3k+1}}{10^k}, \sum_{k=0}^{\infty} \frac{x_{3k+2}}{10^k} \right).$$

This defines a function $f : (0, \infty) \longrightarrow (0, \infty)^3$. Is f a bijection?

Exercise 7.20 Let E_5 be the set of those reals in $(0, 1)$ whose infinite decimal representations contain the digit 5. Prove that $E_5 \simeq (0, 1)$.

Exercise 7.21 Let $n \in \mathbb{N}$ and $\mathbb{A}(n)$ denote all the set of roots of all polynomials $a_0 + a_1 x + \cdots + a_k x^k$ with integral coefficients such that $k + |a_0| + \cdots + |a_k| = n$. Prove that $\mathbb{A}(n)$ is countable. Deduce that the set of algebraic numbers is countable.

Exercise 7.22 (Witkowski, 2020 [4]) If $x \longmapsto \lceil x \rceil$ defines the least integer next to x, prove that

$$f(x) := \frac{1}{\lceil x^{-1} \rceil} + \frac{1}{\lceil x^{-1} \rceil - 1} - x,$$

defines a bijection from $(0, 1)$ onto $(0, 1]$.

Exercise 7.23 (Jaime, 2014 [1]) Let $\mathbb{A}$ be the set of algebraic numbers and $f : (0, +\infty) \longrightarrow \mathbb{C} \setminus \mathbb{A}$ be defined as

$$f(x) := \begin{cases} \pi + x \ ; \pi - x \in \mathbb{A}, \\ \pi - x \ ; \pi + x \in \mathbb{A}. \end{cases}$$

Prove that f is well-defined and injective. Conclude that the set of transcendental numbers is uncountable.

Exercise 7.24 (Buddenhagen, 1971 [5]) Prove the following:

(a) Let $\mathcal{S}$ be the collection of sequences of rational numbers, convergent to some irrational number. Show that $\mathcal{S}$ is an uncountable family of subsets of $\mathbb{Q}$ such that $A \cap B$ is finite, for all $A, B \in \mathcal{S}$.

(b) For each $\theta \in [0, \pi)$, let S_θ be a strip in $\mathbb{R}^2$ with width > 1, inclined at an angle θ with the positive direction of x-axis and $T_\theta := S_\theta \cap \mathbb{Z}^2$. Prove that $\Omega := \{T_\theta : \theta \in [0, \pi)\}$ is an uncountable family of subsets of $\mathbb{Z}^2$ such that $A \cap B$ is finite, for all $A, B \in \Omega$.

(c) Let X be any countably infinite set. Prove that there exists an uncountable family Ω of subsets of X such that $A \cap B$ is finite, for all $A, B \in \Omega$.

7.2 Some Applications to Topology

In this section, we present some results about the basic topology, as applications of the notion of countable sets. It mainly comprises some consequences about connected subsets of $\mathbb{R}$, which use countability of the set of rational numbers.

Theorem 7.18 *Let O be an open subset of $\mathbb{R}^n$ for some $n \in \mathbb{N}$. Then O can have at most countably many connected components.*

Proof Let Ω be the set of connected components of O. By Theorem 6.32 and Proposition 6.33, Ω contains disjoint open subsets of $\mathbb{R}^n$. Define

$$X := \{(x_1, \ldots, x_n) : x_1, \ldots, x_n \in \mathbb{Q}\}.$$

Then X is a countable subset of $\mathbb{R}^n$. Also, note that X intersects with every nonempty open subset of $\mathbb{R}^n$. For every $A \in \Omega$, fix an element $x_A \in A \cap X$.

Therefore, the function $f : \Omega \longrightarrow X$ defined as $f(A) := x_A$ is injective. Thus, f is a bijection between Ω and $f(\Omega)$. Since X is countable, so is its subset $f(\Omega)$. Hence, Ω is countable. □

Corollary 7.19 *Every open subset of reals is a countable union of disjoint open intervals.*

Proof Let O be an open subset of reals. By Theorem 7.18, O is a countable union of its connected components, say $O := \dot{\bigcup}_{n=1}^{\infty} O_n$. Applying Theorem 6.19 and Proposition 6.33, each O_n is an open interval. Hence the result. □

Next, we show that every connected metric space with at least two elements is uncountable.

Theorem 7.20 *Let (X, d) be a connected metric space with $|X| > 1$. Then X is uncountable.*

Proof Let $a, b \in X$ such that $a \neq b$. Define a function $f : X \longrightarrow \mathbb{R}$ as follows:

$$f(x) := \frac{d(x, a)}{d(x, a) + d(x, b)} \text{ for all } x \in X.$$

Then f is continuous on X, as $x \longmapsto d(x, a)$ and $x \longmapsto d(x, b)$ are continuous maps on X and $a \neq b$. Since $f(a) = 0$ and $f(b) = 1$, by Intermediate Value Theorem (6.12), we conclude that $[0, 1] \subset f(X)$. Hence, X is uncountable. □

Corollary 7.21 *Every countable metric space is totally disconnected.*

Proposition 7.22 *No continuous real function switches rationals with irrationals. In other words, there exists no continuous function* $f : \mathbb{R} \longrightarrow \mathbb{R}$ *such that*

$$f(\mathbb{Q}) \subset \mathbb{R} \setminus \mathbb{Q} \text{ and } f(\mathbb{R} \setminus \mathbb{Q}) \subset \mathbb{Q}. \tag{7.1}$$

Proof Suppose that there exists a continuous function $f : \mathbb{R} \longrightarrow \mathbb{R}$ such that (7.1) holds true. Then $f(\mathbb{R}) = f(\mathbb{Q}) \cup f(\mathbb{R} \setminus \mathbb{Q}) \subset f(\mathbb{Q}) \cup \mathbb{Q}$, which is a countable set. Since f is continuous, by Theorem 6.20, $f(\mathbb{R})$ is connected. By Theorem 6.19, $f(\mathbb{R})$ is an interval. Since it is countable, $f(\mathbb{R})$ is a singleton. Hence, f is a constant.

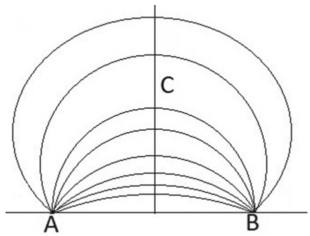
Therefore, there exists a constant $c \in \mathbb{R}$ such that $f(x) = c$ for all $x \in \mathbb{R}$. If $c \in \mathbb{Q}$, then $f(\mathbb{Q}) = \{c\} \not\subset \mathbb{R} \setminus \mathbb{Q}$. On the other hand, if $c \in \mathbb{R} \setminus \mathbb{Q}$, then $f(\mathbb{R} \setminus \mathbb{Q}) = \{c\} \not\subset \mathbb{Q}$. Therefore, we have a contradiction. Hence the result. □

Theorem 7.23 *Let S be any countable subset of $\mathbb{R}^2$. Then the subspace $\mathbb{R}^2 \setminus S$ of $\mathbb{R}^2$ is path connected. In particular, the set $\mathbb{R}^2 \setminus \mathbb{Q}^2$ is path connected.*

Proof Let $E := \mathbb{R}^2 \setminus S$ and $A, B \in E$. We shall show a circular arc joining A and B, inside E.

Let C be any point on the perpendicular bisector of segment AB except for the midpoint of AB. Then there exists a unique circle passing through A, C and B.

Since there are uncountably many such points C, there are uncountably many circles through A and B.



Since S is countable, it intersects with only countably many such circles through A and B. Hence, there are uncountably many circles in E, passing through both A and B. □

Remarks 7.24 The open interval $(0, 1)$ is not a countable union of pairwise disjoint closed subsets of $\mathbb{R}$. Readers interested in the proof are referred to [7, p. 51].

Another application of the notion of countability is that in the definition of compact metric spaces, we need not consider open covers consisting of arbitrary large number of sets. The open covers with only countably many sets are sufficient.

Definitions 7.25 A metric space X is said to be *countably compact* if every countable open cover of X has a finite subcover.

It is immediate that every compact metric space is countably compact. We shall establish the converse too. The following result can be established, analogous to Theorem 5.18. So we omit its proof.

Proposition 7.26 *A metric space X is countably compact if and only if every countable collection of closed subsets of X with the finite intersection property has a nonempty intersection.*

Theorem 7.27 *A metric space X is countably compact if and only if X is compact.*

Proof The converse follows from definitions. Assume that X is countably compact. Applying Theorem 5.27, it is enough to prove that every infinite subset of X has a limit point in X. Let A be an infinite subset of X. Choose a sequence $\{a_n\}$ of distinct terms from A. Write

$$A_n := \{a_k : k \geq n\} \text{ for all } n \in \mathbb{N}.$$

Then for $m < n$, we have $\overline{A_n} \cap \overline{A_m} = \overline{A_n} \neq \varnothing$. Hence, the collection $\{\overline{A_n} : n \in \mathbb{N}\}$ has finite intersection property. Applying Proposition 7.26, we obtain $\bigcap_{n=1}^{\infty} \overline{A_n} \neq \varnothing$.

Pick any $a \in \bigcap_{n=1}^{\infty} \overline{A_n}$. We claim that $a \in A'$. Let O be a neighborhood of a. Then $O \cap A_n \neq \varnothing$ for all $n \in \mathbb{N}$. Therefore, $O \cap A_n$ is an infinite set. Hence, $a \in A'$ and the result follows. □

In Exercise 7.31, we shall also establish that for a metric space to be complete, it is enough that its countable closed subspaces are complete.

Exercise 7.25 Is $\mathbb{Q}$ a union of countably many monotone sequences?

Exercise 7.26 Does every line segment in $\mathbb{C}$ contain an algebraic number?

Exercise 7.27 Let X be a connected metric space with at least two elements. What is the cardinality of X?

Exercise 7.28 Let F be a closed subset of $\mathbb{R}$ and $f : F \longrightarrow \mathbb{R}$ be a continuous function. Prove that there exists a continuous function $g : \mathbb{R} \longrightarrow \mathbb{R}$ such that $g(x) = f(x)$ for all $x \in F$. Is the same true if F is not closed in $\mathbb{R}$?

Exercise 7.29 Let X denote the set of sequences with terms 0 or 1, and Y denote the set of sequences from X which are eventually zero. Is Y a countable set?

Exercise 7.30 Which of the following sets are countable:

(a) the set of finite subsets of natural numbers?
(b) the set of sequences from $\mathbb{N}$, which are in some arithmetic progression?
(c) the set of strictly increasing sequences of natural numbers?

Exercise 7.31 Prove that a metric space X is complete if and only if every countable closed subset of X is a complete subspace.

Exercise 7.32 Does there exist a sequence having uncountably many subsequences, each convergent to a different limit?

Exercise 7.33 Does there exist a subset E of $\mathbb{R}$ having uncountably many isolated points?

Exercise 7.34 Does there exist an uncountable subset of reals having only countably many limit points?

Exercise 7.35 Let S be the collection of step functions on $[a, b]$, which assume values only in $\mathbb{Q}$ and are discontinuous only on $\mathbb{Q}$. Prove that S is countable.

Exercise 7.36 Prove or disprove: There exists a sequence of real numbers that has a monotone rearrangement.

Exercise 7.37 Let E be a countable subset of $\mathbb{R}$ having no limit point in $\mathbb{R}$. Prove that E can be written as a sequence $\{x_n\}$ such that $|x_n| \leq |x_{n+1}|$ for all $n \in \mathbb{N}$.

Exercise 7.38 Does there exist a subset of $\mathbb{C}$ with

(a) countably many connected components?
(b) uncountably many connected components?

Exercise 7.39 Does there exist an uncountable complete subspace of $\mathbb{R} \setminus \mathbb{Q}$?

Exercise 7.40 Does there exist a complete metric on the set $\mathbb{R} \setminus \mathbb{Q}$?

Exercise 7.41 Does there exist a continuum having only countably infinite number of cut points?

Exercise 7.42 Does there exist a continuum having only countably infinite number of non-cut points?

Exercise 7.43 Let $n \in \mathbb{N}$ and $S \subset \mathbb{R}^n$ be countable. Prove that the space $\mathbb{R}^n \setminus S$ is connected.

Exercise 7.44 Let $\varnothing \neq X \subset \mathbb{R}$ and O be an open subset of the subspace X. Prove that O is a countable union of clopen subsets of X.

Exercise 7.45 Prove that every interval can be written as a countable union of compact intervals.

Exercise 7.46 Does there exist a function $f : \mathbb{R} \longrightarrow \mathbb{R}$ continuous at every rational and discontinuous at uncountably many irrationals?

7.3 The Set of Discontinuities

In this section, we establish that the set of points of discontinuities of a monotone real function is at most countable. We also show that the same set, for any function between metric spaces, is at most a countable union of closed sets.

7.3.1 The Case of Monotone Functions

First, we establish that monotone real functions can have only jump discontinuities. Let us recall the notions of monotone functions and one-sided limits. For this subsection, let I denote an open interval, $d \in I$ and $f : I \longrightarrow \mathbb{R}$. Then

(a) f is said to be

  (i) *monotonically increasing* if $f(x) \leq f(y)$ for all $x, y \in I$ with $x < y$,
  (ii) *monotonically decreasing* if '-f' is monotonically increasing and
  (iii) *monotone* if f is either monotonically increasing or monotonically decreasing.

(b) f is said to have a

  (i) *left-handed limit* at d if there exists $l \in \mathbb{R}$ such that for every $\epsilon > 0$ there exists some $\delta > 0$ such that

$$|f(x) - l| < \epsilon \text{ for all } x \in (d - \delta, d).$$

  In this case, l is called the left-handed limit of f at d, and is denoted by $\lim_{x \to d^-} f(x)$ and $f(d^-)$. The *right-handed limit* $\lim_{x \to d^+} f(x)$ or $f(d^+)$. is defined analogously.
  (ii) *jump discontinuity* at d if both $f(d^-)$ and $f(d^+)$ exist but are not equal.

Lemma 7.28 *Let $f : I \longrightarrow \mathbb{R}$ be a monotonically increasing function and $d \in I$. Then both $f(d^-)$ and $f(d^+)$ exist with*

$$f(d^-) = \sup\{f(x) : x < d, x \in I\} \text{ and } f(d^+) = \inf\{f(x) : x > d, x \in I\}.$$

Proof Let $E := \{f(x) : x < d, x \in I\}$. Since $d \in I$, there are $a, b \in I$ such that $a < d < b$. Then $f(a) \in E$. Since f is monotonically increasing, E is a nonempty set bounded above by $f(b)$. Hence, E has a supremum, say s.

To prove that $s = f(d^-)$, let $\epsilon > 0$ be given. Since $s = \sup E$, there exists some $y \in I$ with $y < d$ such that $s - \epsilon < f(y)$. The monotonicity of f implies $s - \epsilon < f(y) \leq f(x) \leq s$ for all $d > x > y$. So $|f(x) - s| < \epsilon$ for all $x \in (y, d)$. Hence, we obtain $\lim_{x \to d^-} f(x) = s$. The second assertion is similar. □

Now we prove that monotone real functions can have only jump discontinuities.

Theorem 7.29 *Let $f : I \longrightarrow \mathbb{R}$ be a monotone function, discontinuous at $d \in I$. Then d is a jump discontinuities of f.*

Proof Suppose f is monotonically increasing. By Lemma 7.28, both $f(d^-)$ and $f(d^+)$ exist. Consequently, $f(d^-) \neq f(d^+)$ and therefore, d is a jump discontinuity of f.

If f is monotonically decreasing, then $-f$ is monotonically increasing. As above, $-f$ and thence f can have only jump discontinuities. Hence the result. □

Theorem 7.30 *The set of discontinuities of any monotone function $f : I \longrightarrow \mathbb{R}$ is at most countable.*

Proof First suppose that f is monotonically increasing. Being monotone, f can have only jump discontinuities. Let D_f be the set of discontinuities of f. For every $d \in D_f$, let $J_d := \big(f(d^-), f(d^+)\big)$. We claim that

$$J_{d_1} \cap J_{d_2} = \varnothing \text{ for all } d_1 \neq d_2. \tag{7.2}$$

To see this, let $d_1, d_2 \in D_f$ such that $d_1 < d_2$. Then there are reals a, b such that $d_1 < a < b < d_2$. Since f is monotonically increasing, we obtain

$$f(d_1^+) = \inf\{f(x) : x > d_1\} = \inf\{f(x) : d_1 < x < a\} \leq f(a).$$

Similarly, $f(d_2^-) \geq f(b)$. Hence, $f(d_1^+) \leq f(a) \leq f(b) \leq f(d_2^-)$. This establishes (7.2). For every $d \in D_f$, pick any $r_d \in J_d \cap \mathbb{Q}$. Define $\phi : D_f \longrightarrow \mathbb{Q}$ as follows:

$$\phi(d) := r_d \text{ for all } d \in D_f.$$

Note that (7.2) ensures that ϕ is an injective map. Hence, D_f is countable. In case f is monotonically decreasing, $-f$ is monotonically increasing. Therefore, D_{-f}, the set of discontinuities of $-f$, is countable. The result follows from the fact that $D_{-f} = D_f$. □

An alternative proof of Theorem 7.30 will be provided in Exercise 7.52.

Theorem 7.31 *Let $-\infty \leq a < b \leq +\infty$ and A be a countable subset of (a, b). Then there exists a monotonically increasing function $f : (a, b) \longrightarrow \mathbb{R}$ which is discontinuous precisely on A.*

Proof If A is a finite set, then take f as a suitable step function. Assume that A is countably infinite and write $A := \{d_n : n \in \mathbb{N}\}$. Define $f : (a, b) \longrightarrow \mathbb{R}$ as follows:

$$f(x) := \sum_{d_n \leq x} \frac{1}{2^n} \text{ for all } x \in (a, b).$$

Note that f is a monotonically increasing function, as for $a < x < y < b$, we have

$$f(y) - f(x) = \sum_{x<d_n \le y} \frac{1}{2^n} \ge 0 \text{ which implies } f(y) \ge f(x).$$

We claim that f is discontinuous only on A. Let $n \in \mathbb{N}$ and $x < d_n$. As earlier

$$f(d_n) - f(x) = \sum_{x<d_k \le d_n} \frac{1}{2^k} \ge \frac{1}{2^n} > 0.$$

Thus, f is discontinuous at each $d_n \in A$. Now pick any $x \in (a, b) \setminus A$. To prove that f is continuous at x, let $\epsilon > 0$ be given. Pick any $n \in \mathbb{N}$ such that $\frac{1}{2^n} < \epsilon$.

Let $\delta > 0$ be such that $(x - \delta, x + \delta) \subset (a, b)$ and $(x - \delta, x + \delta)$ contains none of the points $d_1, d_2, \ldots, d_n$. Then for any $y \in (x - \delta, x + \delta)$, we see that

$$|f(y) - f(x)| = \sum_{x<d_k \le y \text{ or } y<d_k \le x} \frac{1}{2^k} \le \sum_{k>n} \frac{1}{2^k} = \frac{1}{2^n} < \epsilon.$$

This ensures that f is continuous at each $x \in (a, b) \setminus A$. Hence the result. □

Corollary 7.32 *There exists a strictly increasing function* $f : \mathbb{R} \longrightarrow \mathbb{R}$ *such that* f *is continuous on* $\mathbb{R} \setminus \mathbb{Q}$ *and discontinuous on* $\mathbb{Q}$.

Proof The function obtained by Theorem 7.31 for $(a, b) = \mathbb{R}$ and $A = \mathbb{Q}$ is strictly increasing, as between any two reals there exists a rational number. □

Remarks 7.33 (a) In Corollary 8.39(b), we will show that there exists no $\mathbb{R} \longrightarrow \mathbb{R}$ function that is continuous on $\mathbb{Q}$ and discontinuous on $\mathbb{R} \setminus \mathbb{Q}$.

(b) For characterizations of monotone functions, see Exercise 7.62 and [10].

7.3.2 The General Case

Let X and Y be metric spaces, $f : X \longrightarrow Y$ be a function and

$$D_f := \{x \in X : f \text{ is discontinuous at } x\}.$$

We shall show that D_f is at most a countable union of closed subsets of X. Consequently, the set of continuities of f is at most a countable intersection of open subsets of X. There is a special terminology for such sets, which is given below.

Definitions 7.34 A subset E of a metric space is said to be

(a) an F_σ*-set,* if E is a countable union of closed sets.
(b) a G_δ*-set,* if E is a countable intersection of open sets.

Examples 7.35 (a) $\{0\} = \bigcap_{n=1}^{\infty} \big(-1/n, 1/n \big)$ is a G_δ-set.
(b) $\mathbb{N} = \bigcup_{n=1}^{\infty} \{n\}$ is an F_σ-set.
(c) Any countable subset of any metric space is an F_σ-set.
(d) The complement of any F_σ-set is a G_δ-set and vice versa.

To move ahead we need the notion of *oscillation* of a function f at a point $x \in X$, defined as

$$\omega(f; x) := \inf\{diam(f(B(x; \delta))) : \delta > 0\}.$$

It is easy to see that $\omega(f; x) = \lim_{\delta \to 0} diam\big(f(B(x; \delta))\big)$.

Proposition 7.36 *Let $x \in X$. Then f is continuous at x if and only if $\omega(f; x) = 0$.*

Proof Let $\epsilon > 0$ be given. If f is continuous at x, $f(B(x; \delta)) \subset B(f(x); \epsilon/2)$ for some $\delta > 0$. Then for any $y, z \in B(x; \delta)$, we have $d(f(y), f(z)) \leq d(f(y), f(x)) + d(f(x), f(z)) < \epsilon$. Hence, $diam(f(B(x; \delta))) \leq \epsilon$. Since $\epsilon > 0$ is arbitrary, we obtain $\omega(f; x) = 0$.

Conversely, if $\omega(f; x) = 0$, there exists some $\eta > 0$ such that $diam(f(B(x; \eta))) < \epsilon$. Hence, $d(f(y), f(x)) \leq diam(f(B(x; \eta))) < \epsilon$ for all $y \in B(x; \eta)$. Thus, f is continuous at x. □

Theorem 7.37 *Let X and Y be metric spaces, $f : X \longrightarrow Y$ and D_f is the set of discontinuities of f. Then D_f is an F_σ-set in X.*

Proof Applying Proposition 7.36, we note that

$$D_f = \{x \in X : \omega(f; x) > 0\} = \bigcup_{n=1}^{\infty} \left\{ x \in X : \omega(f; x) \geq \frac{1}{n} \right\}.$$

Hence, it is enough to prove that, for every $r > 0$, the set $\{x \in X : \omega(f; x) \geq r\}$ is closed or equivalently $E_r := \{x \in X : \omega(f; x) < r\}$ is an open subset of $\mathbb{R}$.

Pick any $r > 0$ and $x \in E_r$. Then $diam(f(B(x; \delta))) < r$ for some $\delta > 0$. Since $B(x; \delta)$ is an open set, for every $y \in B(x; \delta)$, there exists some $\eta > 0$ such that $B(y; \eta) \subset B(x; \delta)$. Therefore, $diam(B(y; \eta)) \leq diam(B(x; \delta)) < r$. This implies that $y \in E_r$. Consequently, $B(x; \delta) \subset E_r$ and hence the result. □

The converse of the above theorem is also true for the real case.

Theorem 7.38 *Let F be an F_σ subset of $\mathbb{R}$. Then there exists a bounded function $f : \mathbb{R} \longrightarrow \mathbb{R}$ such that $D_f = F$.*

Proof If F is a closed set, then it can be shown that for $f := \chi_{\mathbb{R} \setminus F} + \chi_{F^o \cap \mathbb{Q}}$, we have $D_f = F$.

In general, let $\{E_n\}$ be a sequence of closed sets such that $\bigcup_{n=1}^{\infty} E_n = F$. Write $F_0 = \varnothing$ and $F_n := \bigcup_{k \leq n} E_k$ for all $n \in \mathbb{N}$. Then each F_n is closed, $\bigcup_{n=0}^{\infty} F_n = F$ and $F_n \subset F_{n+1}$ for all $n \in \mathbb{N}$. Without loss of generality, suppose that $F_n \neq F_{n+1}$ for all $n \geq 0$.

Let $f_n := \chi_{\mathbb{R}\setminus F_n} + \chi_{F_n^o \cap \mathbb{Q}}$ for all $n \in \mathbb{N}$. Then each f_n is discontinuous only on F_n. Let $\{a_n\}$ be a sequence of positive reals such that $a_n > \sum_{k>n} a_k$ for all $n \in \mathbb{N}$. For example, $a_n := 2^{-n(n+1)}$ is one such sequence.

By Theorem 1.62, the series $\sum_{n=1}^{\infty} a_n f_n(x)$ converges uniformly on $\mathbb{R}$. Write $f(x) := \sum_{n=1}^{\infty} a_n f_n(x)$ for all $x \in \mathbb{R}$. By Theorem 3.33, f is continuous on $\mathbb{R} \setminus F$.

Let $x \in F = \bigcup_{n=1}^{\infty} F_n = \dot{\bigcup}_{n=1}^{\infty}(F_n - F_{n-1})$. Then $x \in F_n - F_{n-1}$ for some unique $n \in \mathbb{N}$. Note that

$$\begin{aligned}\omega(f; x) &= \lim_{\delta \to 0} \sup\{|f(t) - f(s)| : t, s \in (x - \delta, x + \delta)\} \\ &= \lim_{\delta \to 0} \sup \left\{ \sum_{k \geq n} a_k f_k(t) - \sum_{k \geq n} a_k f_k(s) : t, s \in (x - \delta, x + \delta) \right\}.\end{aligned}$$

If $x \in F_n^o$, there exist sequences $\{x_k\} \subset F_n^o \cap \mathbb{Q}$ and $\{y_k\} \subset F_n^o \setminus \mathbb{Q}$ both convergent to x. Therefore, $f_n(x_k) = 1$ and $f_n(y_k) = 0$ for all $k \in \mathbb{N}$. In case $x \in F_n \setminus F_n^o$, there exist sequences $\{x_k\} \subset \mathbb{R} \setminus F_n$ and $\{y_k\} \subset F_n \setminus F_n^o$ both convergent to x. Again $f_n(x_k) = 1$ and $f_n(y_k) = 0$ for all $k \in \mathbb{N}$.

Let $\delta > 0$ be given. Then there are $x_0, y_0 \in (x - \delta, x + \delta)$ such that $f_n(x_0) = 1$ and $f_n(y_0) = 0$. Thus, $\omega(f; (x - \delta, x + \delta)) \geq a_n - \sum_{k>n} a_k$. Therefore, $\omega(f; x) \geq a_n - \sum_{k>n} a_k > 0$. Consequently, f is discontinuous at x. Hence the result. □

It is natural to question that which subsets of $\mathbb{R}$ are F_σ or G_δ? In Corollary 8.39(a), we shall see that the set of rational numbers $\mathbb{Q}$, is not a G_δ. Hence, there exists no $\mathbb{R} \longrightarrow \mathbb{R}$ function, which is continuous precisely on $\mathbb{Q}$.

However, by the following proposition, every closed subset of a metric space is a G_δ-set and therefore, every open subset of a metric space is an F_σ-set.

Theorem 7.39 (Mazurkiewicz) *Let F be a closed subset of a metric space (X, d). Then F is a G_δ-set.*

Proof For each $n \in \mathbb{N}$, let $O_n := \bigcup_{x \in F} B\left(x; \frac{1}{n}\right)$. Since open balls open, each O_n is open in X. Hence, it is sufficient to prove that $F := \bigcap_{n=1}^{\infty} O_n$.

Clearly, for every $n \in \mathbb{N}$, we have $F \subset O_n$. Therefore, $F \subset \bigcap_{n=1}^{\infty} O_n$. For the opposite inclusion, let $y \in \bigcap_{n=1}^{\infty} O_n$. Then there exists a sequence $\{x_n\} \subset F$ such that $d(x_n, y) < \frac{1}{n}$ for all $n \in \mathbb{N}$. Therefore, $\{x_n\} \longrightarrow y$. Hence, $y \in \overline{F} = F$, as F is closed. □

Remarks 7.40 In F_σ, the letter F stands for *ferme (which means closed)* and σ stands for *summe (which means sum),* while in G_σ, the letter G stands for *geneit (which means region)* and δ stands for *durchschnitt (which means intersection).* Note that we need not to separately define F_δ and G_σ sets, as any union (intersection) of open (closed) sets is always open (closed).

Exercise 7.47 Let $f, g : I \longrightarrow \mathbb{R}$ be such that f is monotonically increasing and g is strictly increasing on I. Prove that $f + g$ is strictly increasing on I.

Exercise 7.48 Prove that the map $f : \mathbb{R} \longrightarrow \mathbb{R}$ defined as $f(x) := \sin x + x$ is strictly increasing.

Exercise 7.49 If f is monotonically increasing on (a, b) and $c \in (a, b)$, prove that

$$f(c^+) - f(c^-) = \inf\{f(c+h) - f(c-k) : h, k > 0\}.$$

Exercise 7.50 Let I be an interval and $f, g : I \longrightarrow \mathbb{R}$ be a given function.

(a) If f is monotone, prove that f is continuous if and only if $f(I)$ is an interval.
(b) If f is continuous, prove that f is injective if and only if f is strictly monotone.

Exercise 7.51 Let $f : [a, b] \longrightarrow \mathbb{R}$ be a monotonically increasing function and $a = x_0 < \cdots < x_n = b$. Prove that $\sum_{i=1}^{n-1} \big(f(x_i^+) - f(x_i^-)\big) \le f(b) - f(a)$.

Exercise 7.52 Let (a, b) be a bounded open interval and $f : (a, b) \longrightarrow \mathbb{R}$ be a monotone function. Prove that for every $m \in \mathbb{N}$, the set

$$\left\{d \in (a, b) : f(d^+) - f(d^-) > \frac{1}{m}\right\}$$

is finite. Conclude that the set of discontinuities of f is a countable set.

Exercise 7.53 If $f : \mathbb{R} \longrightarrow \mathbb{R}$ is monotonically increasing, prove that

$$\omega(f; c) = f(c+) - f(c-) \text{ for all } c \in \mathbb{R}.$$

Exercise 7.54 Does there exist a real function with uncountably many jump discontinuities?

Exercise 7.55 Let $f : [0, 1] \longrightarrow \mathbb{R}$ be an injective function. Prove that the range of f contains an irrational number.

Exercise 7.56 If $f : \mathbb{R} \longrightarrow \mathbb{R}$, prove that there exists a bounded function $g : \mathbb{R} \longrightarrow \mathbb{R}$ such that both f and g have same set of discontinuities.

Exercise 7.57 If Y is a complete subspace of a metric space X, prove that Y is a G_δ in X.

Exercise 7.58 Let $f : [0, 1] \longrightarrow \mathbb{R}$ and $\epsilon > 0$ be given. Show that there exists $n \in \mathbb{N}$ such that $\omega(f; [x_{k-1}, x_k]) < \epsilon$ for all $1 \le k \le n$, where $x_k := k/n$.

Exercise 7.59 For $E \subset \mathbb{R}$, show that the set of continuity of χ_E is $E^o \cup (\mathbb{R} \setminus \overline{E})$.

Exercise 7.60 Let F be a closed subset of $\mathbb{R}$ and $f = \chi_{\mathbb{R}\setminus F} + \chi_{F^o \cap \mathbb{Q}}$. Prove that $D_f = F$.

Exercise 7.61 Let X be a metric space with a dense set A such that $A^o = \varnothing$ and F be a countable union of closed subsets of X. Prove that there exists a bounded function $f : X \longrightarrow \mathbb{R}$, which is discontinuous precisely on F. Further show that the same is true, with the codomain of f as any given Banach space.

Exercise 7.62 Let I be any interval and $f : I \longrightarrow \mathbb{R}$. Prove that the following are equivalent:

(a) f is monotonically increasing on I.
(b) f is locally increasing at every point of I.
(c) For every $x \in I$, there exists some $\delta_x > 0$ such that

$$f(y) \leq f(x) \leq f(z) \text{ whenever } x - \delta_x < y < x < z < x + \delta_x.$$

Exercise 7.63 Let f be as in the proof of Theorem 7.31. Prove that f is right continuous on $\mathbb{R}$, that is $f(x+) = f(x)$ for all $x \in \mathbb{R}$.

Exercise 7.64 Let $f : [-1, 1] \longrightarrow \mathbb{R}$ be a differentiable function such that $f'(0) > 0$.

(a) Prove that there exists some $\delta > 0$ such that $f(x) < f(0) < f(y)$ for all $-\delta < x < 0 < y < \delta$.
(b) If f' is continuous at 0, prove that f is strictly increasing in the interval $(-\delta, \delta)$ for some $\delta > 0$.
(c) Show that f may not be strictly increasing in any neighborhood of 0.

Exercise 7.65 Let A, B be closed subsets of $\mathbb{R}$ and $f : A \longrightarrow B$ be a strictly increasing surjection. Prove that f is a continuous map.

7.4 Cardinality

As stated earlier, two given sets are said to have the same *cardinality,* if there exists a bijection between them. Now we discuss a weaker notion.

Definitions 7.41 For any sets X and Y, we write

(a) $X \preceq Y$ if and only if X is in one-to-one correspondence with a subset of Y.
(b) $X \prec Y$ if and only if $X \preceq Y$ and $X \not\simeq Y$.

Note that $X \preceq Y$ if and only if there exists an injective map from X into Y if and only if there exists a surjective map from Y onto X. Hence, X is countable if and only if $X \preceq \mathbb{N}$.

Theorem 7.42 (Cantor-Schröder-Bernstein) *If* $X \preceq Y$ *and* $Y \preceq X$, *then* $X \simeq Y$.

Proof Since $X \preceq Y$ and $Y \preceq X$, there exist injective maps $f : X \longrightarrow Y$ and $g : Y \longrightarrow X$. First we partition both X and Y into three sets each.

Pick any $x \in X$. If it has no preimage under g, stop. Otherwise, let $y_1 \in Y$ be that preimage. Then consider the preimage of y_1 under the map f. If it doesn't exist, stop. Otherwise, let x_2 be that preimage. Continuing this process, for every $x \in X$, either this process terminates somewhere or goes on indefinitely.

Let E_X and O_X, respectively, denote subsets of X with only even and odd number of successive preimages, respectively. The set E_X also contains elements of X having no preimage.

Let $I_X := X \setminus (E_X \cup O_X)$. Analogously, we obtain subsets E_Y, O_Y, and I_Y of Y, having even, odd, and infinitely many successive preimages of their elements, respectively. Define a map $H : X \longrightarrow Y$ as follows:

$$H(x) := \begin{cases} g^{-1}(x) & \text{if } x \in O_X \\ f(x) & \text{if } x \in X \setminus O_X. \end{cases}$$

We leave it for the readers to show that H is a bijection between the pairs O_X and E_Y; E_X and O_Y; and I_X and I_Y. Hence H is a bijection between X and Y. □

Notations 7.43 Let X and Y be any two nonempty sets. Then

(a) the set of functions from $X \longrightarrow Y$, will be denoted by Y^X and
(b) the set of subsets of X (the *power set* of X) will be denoted by $\mathcal{P}(X)$.

Due to the following result, the power set $\mathcal{P}(X)$ is also denoted by 2^X.

Proposition 7.44 *If X is a nonempty set, then there exists a one-to-one correspondence between $\mathcal{P}(X)$ and $\{0, 1\}^X$.*

Proof Consider a function $f : \mathcal{P}(X) \longrightarrow \{0, 1\}^X$ given by $f(A) := f_A$, where $f_A : X \longrightarrow \{0, 1\}$ is defined as

$$f_A(x) := \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A. \end{cases}$$

Note that $f(A) = f(B)$ that is $f_A = f_B$ implies $x \in A$ if and only if $x \in B$, that is $A = B$. Hence, f is injective. Clearly, f is surjective and hence a bijection. □

Example 7.45 $\mathcal{P}(\mathbb{N}) \simeq [0, 1]$.

Proof If $S \subset \mathbb{N}$, define $f(S) := \sum_{n=1}^{\infty} \frac{s_n}{10^n} = (0.s_1 s_2 \ldots s_n \ldots)_{10}$, where

$$s_n := \begin{cases} 1 \,; n \in S, \\ 2 \,; n \notin S. \end{cases}$$

This defines a function $f : \mathcal{P}(\mathbb{N}) \longrightarrow [0, 1]$. Note that, for each $S \subset \mathbb{N}$, $f(S)$ is a real inside $[0, 1]$ with a unique decimal representation. We show that f is an injective map. Let S and T be subsets of $\mathbb{N}$ such that $f(S) = f(T)$. Then for all $n \in \mathbb{N}$, $s_n = t_n$, which implies $n \in S$ if and only if $n \in T$. Hence, $S = T$.

Applying Theorem 7.42, it is enough to provide an injective function $g : [0, 1] \longrightarrow \mathcal{P}(\mathbb{N})$. Let $x \in [0, 1]$. Write its binary representation, say, $x = (0.x_1x_2 \ldots x_n \ldots)_2$. If x has two binary representations, then one of those must be finite and the other must be infinite. In this case, consider only the finite binary representation of x.

Define $g(x) := S$, where $S := \{n \in \mathbb{N} : x_n = 1\}$. If $x = (0.x_1x_2 \ldots x_n \ldots)_2$, $y = (0.y_1y_2 \ldots y_n \ldots)_2 \in [0, 1]$ such that $g(x) = g(y)$, then $n \in g(x)$ if and only if $n \in g(y)$, which implies $x_n = 1$ if and only if $y_n = 1$. Hence, $x = y$ and thus g is injective.

Applying Theorem 7.42, we conclude that $\mathcal{P}(\mathbb{N}) \simeq [0, 1]$. □

Theorem 7.46 (Cantor 1891) $X \prec \mathcal{P}(X)$, *for every set* X.

Proof Define $i : X \longrightarrow \mathcal{P}(X)$ as $i(x) := \{x\}$ for all $x \in X$. Then i is a bijection of X with the collection of singleton subsets of X. Hence, $X \preceq \mathcal{P}(X)$.

Suppose $X \simeq \mathcal{P}(X)$. Then there exists a bijection $f : X \longrightarrow \mathcal{P}(X)$. Let $Y := \{a \in X : a \notin f(a)\}$. Then $Y \subset X$. Since f is onto, there exists some $x \in X$ such that $Y = f(x)$. Now $x \in f(x)$ if and only if $x \in Y$ if and only if $x \notin f(x)$, a contradiction. Hence the result. □

The above result immediately implies that the collection of all sets, although well-defined, cannot be regarded as a set. Because if X is the collection of all sets, then $\mathcal{P}(X) \subset X$ and hence $\mathcal{P}(X) \preceq X \prec \mathcal{P}(X)$, a contradiction. This questioned the foundations of the Naive Set Theory, which lead to the development of the Axiomatic Set Theory.

The argument used in the proof of Theorem 7.46 leads to the following paradox, known as *Russell's paradox* or the *barber paradox:*

Suppose there exists a barber who shaves all those people who do not shave themselves.
Does this barber shave himself?

However, it is not a paradox since such a hypothetical barber cannot exist.

History Notes 7.47 It was the first congress of the German Mathematical Association in 1891 during which Georg Cantor presented and proved Theorem 7.46 (see [8, p. 77]).

The Cantor-Schröder-Bernstein theorem, also known as the *equivalence theorem,* was first published by Georg Cantor in 1886, but without proof. In 1897, a 19-year-old student Felix Bernstein proved this theorem in Cantor's seminar. Almost simultaneously, in 1897, Ernst Schröder independently discovered its proof. For the history and other proofs of the Cantor-Schröder-Bernstein theorem, the reader is referred to [9].

7.4.1 Cardinal Numbers

For any set X, applying Theorem 7.46, we obtain a strictly increasing sequence of cardinalities, given by

$$X \prec \mathcal{P}(X) \prec \mathcal{P}(\mathcal{P}(X)) \prec \mathcal{P}(\mathcal{P}(\mathcal{P}(X))) \prec \dots. \tag{7.3}$$

If X is a finite set, we define its *cardinal number* (or cardinality), as follows

$$card(X) := \text{ number of elements in } X.$$

Therefore, (7.3) leads to $n < 2^n < 2^{2^n} < \dots$. In case of infinite sets, defining cardinality is quite tricky, and out of scope of this textbook. However, we write

$$card(\mathbb{N}) := \aleph_0 \text{ and } card(\mathbb{R}) = c.$$

Recall that for any set X, $card(\mathcal{P}(X)) = 2^{card(X)}$. With these notations, we have

$$\aleph_0 < c < 2^c < 2^{2^c} < \dots$$

At this point, it is important to give a brief introduction to the Axiom of Choice and the Continuum Hypothesis. However, a more detailed presentation of this axiom along with the standard Zermelo-Fraenkel Axioms will be provided in Appendix A.

Continuum Hypothesis 7.48 (Georg Cantor, 1878) No set has cardinality strictly between $\aleph_0$ and c.

Axiom of Choice 7.49 For every nonempty family of nonempty sets, there exists a set containing one element from each set of the collection.

Let $\Omega := \{X_\alpha : \alpha \in \wedge\}$ be any nonempty collection of nonempty sets. The Cartesian product $\prod_{\alpha\in\wedge} X_\alpha$ is defined to be the collection of functions $f : \wedge \longrightarrow \bigcup_{\alpha\in\wedge} X_\alpha$ such that $f(\alpha) \in X_\alpha$ for all $\alpha \in \wedge$. Each such f is known as a *choice function* for $\{X_\alpha : \alpha \in \wedge\}$.

Therefore, the Axiom of Choice holds if and only if the Cartesian product of any nonempty collection Ω of nonempty sets is nonempty; that is, Ω admits a choice function.

Proposition 7.50 *If* $card(A) = c = card(B)$, *then* $card(A \cup B) = c$.

Proof Let $g_1 : A \longrightarrow (0, +\infty)$ and $g_2 : B \longrightarrow (-\infty, 0)$ be bijections. Define $g : A \cup B \longrightarrow \mathbb{R}$ as

$$g(x) := \begin{cases} g_1(x) \; ; x \in A, \\ g_2(x) \; ; x \notin A. \end{cases}$$

Then g is an injection. Hence, $card(A \cup B) \leq card(\mathbb{R}) = c$. Since $A \subset A \cup B$, we already have $c = card(A) \leq card(A \cup B)$. Hence, $card(A \cup B) = c$. □

Theorem 7.51 *If* $card(F) = c$, *then* $card(\prod_{n\in\mathbb{N}} F) = c$.

Proof Since $\mathbb{N}$ is bijective to $\mathbb{N} \times \mathbb{N}$, for any set A, we obtain

$$card\Big(\prod_{n\in\mathbb{N}} A\Big) = card\Big(\prod_{n,m\in\mathbb{N}} A\Big). \tag{7.4}$$

Since $card(F) = c$, we have $F \simeq [0, 1]$. By Proposition 7.44 and Example 7.45,

$$F \simeq [0, 1] \simeq \mathcal{P}(\mathbb{N}) \simeq \{0, 1\}^{\mathbb{N}}.$$

Recall that $\{0, 1\}^{\mathbb{N}}$ is the family of sequences with terms 0 and 1, that is $\{0, 1\}^{\mathbb{N}} = \prod_{n\in\mathbb{N}}\{0, 1\}$. Therefore, $card(\prod_{n\in\mathbb{N}}\{0, 1\}) = c$. With $A = \{0, 1\}$ in (7.4), we obtain

$$c = card\Big(\prod_{n\in\mathbb{N}}\{0, 1\}\Big) = card\Bigg(\prod_{n\in\mathbb{N}}\Bigg(\prod_{m\in\mathbb{N}}\{0, 1\}\Bigg)\Bigg) = card\Big(\prod_{n\in\mathbb{N}} F\Big).$$

This proves the result. □

Theorem 7.52 *If $card(X) \le c$ and $card(Y) \le c$, then $card(X \times Y) \le c$.*

Proof Without loss of generality, assume that $X = \mathcal{P}(\mathbb{N}) = Y$. Define $g, h : \mathbb{N} \longrightarrow \mathbb{N}$ by $g(n) := 2n$ and $h(n) := 2n - 1$ for all $n \in \mathbb{N}$. Define $f : X \times Y \longrightarrow \mathcal{P}(\mathbb{N})$ as $f(A, B) := g(A) \uplus h(B)$. It can be shown that f is a bijection. □

Theorem 7.53 *Let A be a nonempty set with $card(A) \le c$ and $\{F_x : x \in A\}$ be a collection of sets such that $card(F_x) = c$ for all $x \in A$. Then $card(\bigcup_{x\in A} F_x) = c$.*

Proof If $a_0 \in A$, then $F_{a_0} \subset \bigcup_{a\in A} F_a$ and hence $c = card(F_{a_0}) \le card(\bigcup_{a\in A} F_a)$.

To prove the opposite inequality, for each $a \in A$, consider a bijection $f_a : \mathbb{R} \longrightarrow F_a$. Define $f : \mathbb{R} \times A \longrightarrow \bigcup_{a\in A} F_a$ as $f(x, a) := f_a(x)$. Now it is a routine exercise to show that f is a surjective map. Applying Theorem 7.52, we obtain $card(\bigcup_{a\in A} F_a) \le card(\mathbb{R} \times A) \le c$. □

7.4.2 Notes and Remarks

The Axiom of Choice or its negation can be assumed along with the Zermelo-Fraenkel Axioms or ZF-axioms, without any penalty. In other words, if ZF-axioms are consistent, then so are $ZF + AC$ (Godel 1938) and $ZF + \neg AC$ (Cohen 1963). The system $ZF + AC$ is written as ZFC-axioms.

The validity of the Continuum Hypothesis was the first of Hilbert's 23 problems presented in the ICM, in 1900. In 1963, it was proved to be independent of the ZFC-set theory.

In 1940, Godel proved that the Continuum Hypothesis cannot be disproved using the ZFC-axioms. In 1963, Paul Cohen established that it cannot be proven using ZFC-axioms. This concluded that the Continuum Hypothesis is undecidable in ZFC-set theory. Due to this, Cohen was awarded the Fields Medal in 1966.

A brief history of the life of George Cantor can be found in [11, p. 112-113] and [12]. For the independence of the Continuum Hypothesis from ZFC, see [13–16]. More on cardinal numbers, and various natural models of set theory (along with the Zermelo-Fraenkel set theory) can be found in [17].

From an infinite number of pairs of socks, the Axiom of Choice is necessary to select a set having one from each pair. However, the same is not true if we have pairs of shoes instead of socks, Bertrand Russell.

Exercise 7.66 Prove that $\mathcal{P}(\mathbb{N}) \simeq \mathbb{R}$.

Exercise 7.67 Prove that $[0, 1] \simeq \{0, 1\}^{\mathbb{N}}$.

Exercise 7.68 Prove that the set of open subsets of $\mathbb{R}$ has cardinality c.

Exercise 7.69 Prove that the set of uncountable closed subsets of real numbers has cardinality c.

Exercise 7.70 If X is the collection of all sequences from $\mathbb{R}$, then what is $card(X)$?

Exercise 7.71 Does there exist a set whose power set is countably infinite? Justify.

Exercise 7.72 Prove that every connected metric space with at least two points has cardinality at least c.

Exercise 7.73 Let $n \in \mathbb{N}$ and S be any subset of $\mathbb{R}^n$ with $card(S) < c$. Prove that the subspace $\mathbb{R}^2 \setminus S$ is path connected.

Exercise 7.74 Assuming that every sentence in English language is of finite length, prove that there exists no longest sentence in English.

Exercise 7.75 Assuming that every sentence of English language is of finite length, prove that there are only countably many sentences in the English language.

Exercise 7.76 Prove that $card(\ell^p) = c$ for all $1 \leq p \leq \infty$.

Exercise 7.77 Let A be a set with $card(A) = c$ and Ω be the collection of all sequences from A. Prove that the $card(\Omega) = c$.

Exercise 7.78 Does there exist $A \subset \mathbb{R}$ such that $card(A) = c = card(\overline{A} \setminus A)$?

Exercise 7.79 Let $A \subset [0, 1]$ such that the decimal expansion of every $a \in A$ is eventually constant. Prove that A is countable.

Exercise 7.80 Let A be the collection of all $a \in [0, 1]$ which contain at least two consecutive identical terms, in their decimal expansions. Prove that $card(A) = c$.

Exercise 7.81 Let A and B be any two sets with the same cardinality. What is the cardinality of the set of bijections from A onto B?

7.5 Hints and Solutions to Selected Exercises

7.3 Let $\{A_n : n \in \mathbb{N}\}$ be a countable collection of countably infinite sets. By Theorem 7.8, $\bigcup_{n=1}^{\infty} A_n$ is countable. Since this union has an infinite subset A_1, it is countably infinite.

7.7 Let E be an infinite set and $x_0 \in E$. Pick a sequence $\{x_n\}$ of distinct terms from $E \setminus \{x_0\}$. Define $f : E \setminus \{x_0\} \longrightarrow E$ as follows:

$$f(x) := \begin{cases} x_0 & \text{if } x = x_1 \\ x_{n-1} & \text{if } x = x_n \text{ and } n > 1 \\ x & \text{if } x \in E \setminus \{x_n : n \in \mathbb{N}\}. \end{cases}$$

Then f is a bijection. The converse is trivial.

7.11 Between any two disjoint open intervals, there exists a rational number.

7.16 (a) Consider $f : (0, 1) \longrightarrow (0, \infty)$ defined as $f(x) := \frac{x}{x+1}$, for all $0 < x < \infty$.
(b) Consider $g : \mathbb{R} \longrightarrow (0, \infty)$ defined as $g(x) := 2^x$, for all $x \in \mathbb{R}$.
(c) Consider the map $g \circ f^{-1}$.

7.17 Note that the maps $f : [0, 1] \longrightarrow [a, b]$ defined as $f(t) := a + t(b - a)$ and $g : [a, b] \longrightarrow [c, d]$ defined as $g(t) := c + (t - a)\frac{d-c}{b-a}$ are bijections. Deal with the end points in semi open intervals, as in Exercise 7.15. Hence, all bounded closed intervals are uncountable.

Similarly, all bounded open intervals are bijective to (0, 1) and thus uncountable.

Since $x \longrightarrow 1/x$ is a bijection from $(0, 1)$ and $(1, \infty)$, every open bounded interval is bijective to intervals of the form (a, ∞). As earlier, we can conclude that every bounded interval is bijective to one-sided unbounded intervals.

Since $x \longrightarrow \tan x$ is a bijection from $(-\pi/2, \pi/2)$ to $\mathbb{R}$, any two intervals are bijective. Hence, all non-trivial intervals are uncountable.

7.22 Note that f maps $[\frac{1}{n+1}, \frac{1}{n})$ bijectively onto $(\frac{1}{n+1}, \frac{1}{n}]$ and use the fact that

$$(0, 1) = \dot{\bigcup_{n=1}^{\infty}} \left[\frac{1}{n+1}, \frac{1}{n}\right) \text{ and } (0, 1] = \dot{\bigcup_{n=1}^{\infty}} \left(\frac{1}{n+1}, \frac{1}{n}\right].$$

7.23 f is well-defined, because if there exists $x \in (0, +\infty)$ such that $\pi - x \in \mathbb{A}$ and $\pi + x \in \mathbb{A}$, then $\pi = \frac{1}{2}\big((\pi - x) + (\pi + x)\big) \in \mathbb{A}$, a contradiction.

Also, f is injective, as if $f(x) = f(y)$, then $x = |f(x) - \pi| = |f(y) - \pi| = y$. Therefore, $(0, +\infty)$ is bijective to a subset of $\mathbb{C} \setminus \mathbb{A}$.

7.24 (a) Apply Exercise 1.42 with S as the collection of sequences of rational numbers, convergent to some irrational number.
(b) For each $\theta \in [0, \pi)$, let S_θ be a strip in $\mathbb{R}^2$ with width > 1, inclined at an angle θ with the positive direction of x-axis and $T_\theta := S_\theta \cap \mathbb{Z}^2$. Then $\Omega := \{T_\theta : \theta \in [0, \pi)$ is the required collection.
(c) Take bijection with $\mathbb{Q}$ or $\mathbb{Z}^2$ and then apply (a).

7.28 Write $\mathbb{R} \setminus F = \bigcup_{n=1}^{\infty}(a_n, b_n)$, as a countable union of open intervals. On each interval $[a_n, b_n]$, define g to be a linear function whose graph is the line segment joining the points $(a_n, f(a_n))$ and $(b_n, f(b_n))$ in $\mathbb{R}^2$.

If F is not closed, the result is false. For example, consider the function $f(x) := 1/x$ for all $x \in (0, 1)$.

7.29 Yes. Note that $Y = \bigcup_k Y_k$, where

$$Y_k := \big\{\{x_n\} \in Y : \{x_n\} \text{ has exactly } k \text{ zeros}\big\} \text{ for all } k \in \mathbb{N}.$$

Then each Y_k is countable. Hence, so is $\bigcup_k Y_k = Y$.

7.30 (a) Countable. Because there are only countably many subsets of $\mathbb{N}$ with n elements, for every $n \in \mathbb{N}$.
(b) Countable. Because every A.P. of $\mathbb{N}$ is characterized by only two parameters: the initial term and the common difference. Both of these come from a countable set $\mathbb{N}$.
(c) Uncountable. Because the given set is equipotent to the set of infinite subsets of $\mathbb{N}$, which is the complement of the set in (a), inside $\mathcal{P}(\mathbb{N})$.

7.31 The necessity follows from Theorem 4.3. For sufficiency, let $\{x_n\}$ be a Cauchy sequence in X, which is not convergent in X. Then the set $Y := \{x_n : n \in \mathbb{N}\}$ has no limit point in X, otherwise a convergent subsequence will imply the convergence of the given sequence. Since $Y' = \varnothing$, the set Y is closed and countable. By hypothesis, Y is a complete subspace of X. Consequently, the Cauchy sequence $\{x_n\}$ must be convergent, a contradiction.

7.32 Yes. Consider any sequence of all rational numbers.

7.33 No. Let E be any subset of reals. We prove that $E \setminus E'$ is always countable. For every $x \in E \setminus E'$, choose an open interval I_x with rational end points such that $E \cap I_x = \{x\}$. Then for $x \neq y$, we have $I_x \neq I_y$. Since the collection of intervals $\{I_x : x \in E \setminus E'\}$ is countable, the set $E \setminus E'$ is countable.

7.34 No. Let E be any uncountable subset of reals. We prove that E' is an uncountable set. By Exercise 7.33, the set $E \setminus E'$ is countable. If E' is countable, the set $\overline{E} = E' \cup (E \setminus E')$ is countable. Therefore, $E(\subset \overline{E})$ is countable, a contradiction.

7.35 For each $n \in \mathbb{N}$, let S_n be the subset of S consisting of only those step functions, which take at most n values and have at most n discontinuities. Then $S = \bigcup_{n=1}^{\infty} S_n$. Since $\mathbb{Q}$ is countable, each S_n is countable. Hence, S is countable.

7.36 False. For example, take any enumeration of $\mathbb{Q}$. Then between any two terms of any such sequence, there exists another rational number.

7.38 (a) Yes. Consider the set $\bigcup_{n=1}^{\infty} B(3n; 1)$.
(b) Yes. Consider the set of irrationals.

7.41 Let $a_n := 2 - 3.2^{-n}$ for all $n \in \mathbb{N}$. Then $\{2\} \bigcup (\bigcup_{i=1}^{\infty}\{z \in \mathbb{C} : |z - a_i| \leq 2^{-i}\}$ has countably infinitely many cut points.

7.42 Consider the comb Space, already presented in Example 6.38.

7.50 (a) Apply Intermediate Value Theorem (6.12) and Theorem 7.29.

(b) Assume that f is not strictly monotone. Then there are $x < y$ and $z < w$ in I such that $f(x) < f(y)$ and $f(z) > f(w)$. Hence, there are $a < b < c$ in I such that either $f(b) > \max\{f(a), f(c)\}$ or $f(b) < \min\{f(a), f(c)\}$.

Suppose $f(b) > \max\{f(a), f(c)\}$ Since f is injective, $f(a) \neq f(c)$. If $f(a) < f(c)$, by Intermediate Value Theorem (6.12), $f(a) = f(d)$ for some $d \in (b, c)$. Otherwise, $f(c) = f(d)$ for some $d \in (a, b)$. This is a contradiction, as f is injective. Similarly, $f(b) < \min\{f(a), f(c)\}$ is also impossible. The converse is immediate.

7.51 If $y_i \in (x_i, x_{i+1})$ for all $i = 0, \ldots, n-1$, then $f(x_i^-) \leq f(y_i)$ and $f(y_{i-1}) \leq f(x_i^+)$, and therefore, $f(x_i^+) - f(x_i^-) \leq f(y_i) - f(y_{i-1})$. Hence

$$\sum_{i=1}^{n-1} \left(f(x_i^+) - f(x_i^-)\right) \leq \sum_{i=1}^{n-1} \left(f(y_i) - f(y_{i-1})\right) \leq f(y_{n-1}) - f(y_0) \leq f(b) - f(a).$$

7.52 It is enough to prove the result for monotonically increasing functions f. Being monotone, f has only jump discontinuities. Let D be the set of discontinuities of f inside (a, b). Note that

$$D = \bigcup_{m=1}^{\infty} \left\{d \in (a, b) : f(d^+) - f(d^-) > \frac{1}{m}\right\} = \bigcup_{m=1}^{\infty} D_m, \text{(say)}.$$

Therefore, it is enough to prove that each D_m is a finite set. Let $m \in \mathbb{N}$ and $x_1, \ldots, x_{n-1} \in D_m$ be such that $a < x_1 < \cdots < x_{n-1} < b$. Then we have

$$\sum_{i=1}^{n-1}\left(f(x_i^+) - f(x_i^-)\right) > \frac{n-1}{m}.$$

Along with Exercise 7.51, we obtain

$$\frac{n-1}{m} < \sum_{i=1}^{n-1} \left(f(x_i^+) - f(x_i^-)\right) \leq f(b) - f(a).$$

Thus, $|D_m| \leq m(f(b) - f(a)) < \infty$, that is D_m is finite. Hence the result.

7.56 Take $g(t) := \tan^{-1}\left(f(x)\right)$ for all $x \in \mathbb{R}$.

7.57 Apply Theorems 4.2 and 7.39.

7.59 Write $f := \chi_E$. If $x \in E^o$, then f is 1 in an open interval containing x and hence continuous at x. In case $x \in \mathbb{R} \setminus \overline{E}$, then there exists $\delta > 0$ such that $(x - \delta, x + \delta) \cap E = \varnothing$. Again f is the constant 0 on $(x - \delta, x + \delta)$ and so continuous at x.

Conversely, let $x \in \mathbb{R} \setminus \left(E^o \cup (\mathbb{R} \setminus \overline{E})\right) = (E^o) \cap \overline{E}$. Then there are sequences $\{x_n\} \subset E$ and $\{y_n\} \subset \mathbb{R} \setminus E$ such that both $x_n \longrightarrow x$ and $y_n \longrightarrow x$. But $1 = f(x_n) \longrightarrow 1$, while $0 = f(y_n) \longrightarrow 0$. Hence, f is discontinuous at x.

7.60 Note that f is continuous on $\mathbb{R} \setminus F$. Let $x \in F$. If $x \in F^o$, there exists $\delta > 0$ such that $F \supset (x - \delta, x + \delta)$. Choose sequences $\{r_n\} \subset (x - \delta, x + \delta) \cap \mathbb{Q}$ and $\{s_n\} \subset (x - \delta, x + \delta) \setminus \mathbb{Q}$ such that $r_n \longrightarrow x$ and $s_n \longrightarrow x$. Then $1 \equiv f(r_n) \longrightarrow 1$, while $0 \equiv f(s_n) \longrightarrow 0$. Hence, f is not continuous at x.

In case $x \in F \setminus F^o$, we have $(x - \eta, x + \eta) \cap (\mathbb{R} \setminus F) \neq \emptyset$ for all $\eta > 0$. Hence, there exists a sequence $\{x_n\} \subset \mathbb{R} \setminus F$ such that $x_n \longrightarrow x$. Then $1 \equiv f(x_n) \longrightarrow 1$, while $f(x) = 0$. Therefore, f is not continuous at x.

7.62 The implications $(a) \Rightarrow (b)$ and $(b) \Rightarrow (c)$ are trivial. For $(c) \Rightarrow (a)$, let $a, b \in I$ such that $a < b$. Then for every $x \in [a, b]$, pick any $\delta_x > 0$ as per the condition in (c) and write $I_x := (x - \delta_x, x + \delta_x)$.

Using compactness of $[a, b]$, there are finitely many $x_1, \ldots, x_n \in I$ such that $[a, b] \subset \bigcup_{i=1}^{n} I_{x_i}$. Without loss of generality, suppose $a = x_1 < \cdots < x_n = b$. For $1 \le i \le n - 1$, pick $y_i \in I_{x_i} \cap I_{x_{i+1}}$. Hence $f(a) \le f(y_1) \le f(x_2) \le f(y_2) \le \cdots \le f(b)$. This ensures (a).

7.64 Note that (a) and (b) follow by the definition of $f'(0)$. For (c), consider

$$f(x) = \begin{cases} x + 2x^2 \sin \frac{1}{x} & ; x \in [-1, 1] \setminus \{0\}, \\ 0 & ; x = 0. \end{cases}$$

Then f is differentiable on $[-1, 1]$ with $f'(0) = 1$ and

$$f'(x) = 1 + 4x \sin \frac{1}{x} - 2 \cos \frac{1}{x} \text{ for all } x \in [-1, 1] \setminus \{0\}.$$

Then f is not increasing in any neighborhood of 0 as $f'(\frac{1}{2n\pi}) = -1$ for all $n \in \mathbb{N}$.

7.65 Analogous to Lemma 7.28, one can prove that $f(d^-) = \sup\{f(x) : x < d, x \in A\}$, whenever d is a left limit point of A.

If $f(d^-) \neq f(d)$, then $f(d^-) < f(d)$, as f is strictly increasing. Since B is closed, we obtain $f(d-) \in B$. Then there exists $a \in A$ such that $f(a) = f(d-) < f(d)$. Hence, $a < d$. Since d is a left limit point of A, there exists a strictly increasing sequence $\{x_n\} \subset (a, d) \cap A$, convergent to d. Thus, $f(a) < f(x_1) \le \lim_{n\to\infty} f(x_n) = f(d-) = f(a)$, a contradiction.

Hence, $f(d^-) = f(d)$. Similarly, if f is a right limit point of A, then $f(d^+) = f(d)$. This ensures that $f : A \longrightarrow B$ is continuous.

7.66 By Exercise 7.17, $\mathbb{R} \simeq [0, 1]$. Further Example 7.45 implies that $\mathcal{P}(\mathbb{N}) \simeq \mathbb{R}$.

7.67 Define $f : \{0, 1\}^{\mathbb{N}} \longrightarrow [0, 1]$ and $g : [0, 1] \longrightarrow \{0, 1\}^{\mathbb{N}}$, as follows:

For any $\{s_n\} \in \{0, 1\}^{\mathbb{N}}$, define $f(\{s_n\}) := (0.s_1 s_2 \ldots s_n \ldots)_{10}$. Now pick any $x \in [0, 1]$. Write its binary representation, say, $x = (0.x_1 x_2 \ldots x_n \ldots)_2$. In case x has two binary representations, then one of those must be finite and the other must be infinite. In this case, write only the finite binary representation of x. Define $g(x) := \{x_n\}$.

Now show that both f and g are injective and apply Theorem 7.42.

7.68 Let Ω be the collection of open subsets of $\mathbb{R}$ and $\wedge := \{(r, s) : r, s \in \mathbb{Q}\}$. Then $\wedge$ is countable, say $\wedge = \{I_n : n \in \mathbb{N}\}$.

Since every $O \in \Omega$ is a union of intervals from $\wedge$, one can choose $S \subset \mathbb{N}$ such that $\bigcup_{n \in S} I_n = O$. Define $f(O) := S$. This defines a map $f : \Omega \longrightarrow \mathcal{P}(\mathbb{N})$. It can be shown that f is injective. Hence, $card(\Omega) \leq card(\mathcal{P}(\mathbb{N})) = c$. Also, $g : \mathbb{R} \longrightarrow \Omega$ defined as $g(x) := (x, +\infty)$ is injective, which implies that $c = card(\mathbb{R}) \leq card(\Omega)$. Now apply Theorem 7.42.

7.69 Let Ω denote the collection of uncountable closed subsets of $\mathbb{R}$. Since $[a, +\infty] \subset \Omega$ for all reals $a \in \mathbb{R}$, we obtain $card(\Omega) \geq card(\mathbb{R}) = c$.

Since there exists a one-to-one correspondence between closed sets and open sets given by set complements, by Exercise 7.68, the family of all closed sets has cardinality c. Hence, $card(\Omega) \leq c$. Finally Theorem 7.42 concludes the result.

7.70 $card(X) = card(\mathbb{R}) = c$.

7.71 No. Assume that the answer is affirmative. Let X be a set such that $\mathcal{P}(X)$ is countable. If $|X| < \infty$, say $|X| = n$ for some $n \in \mathbb{N}$, then $\mathcal{P}(X) = 2^n$, which is not true. So X must be infinite. Then it will contain a countably infinite subset, say $Y \subset X$. Then $\mathcal{P}(X) \supset \mathcal{P}(Y)$ and by Example 7.45, we have $\mathcal{P}(Y)$ is uncountable, a contradiction. Hence there exists no set whose power set is countably infinite.

7.72 Imitate the proof of Theorem 7.20.

7.73 Imitate the proof of Theorem 7.23.

7.74 Let S be any sentence. Then the following sentence is larger than S :

The sentence <include S here > is not largest.

7.75 Let $\wedge$ denote the collection of English Alphabet, along with all punctuation marks including space (between words). Note that $|\wedge| < \infty$ and a sentence in English is a finite (meaningful and grammatically correct) ordered set of terms from $\wedge$. For a sentence S, define its length $l(S)$ to be the number of elements of $\wedge$ in S, including their multiplicity. Let Ω be the set of sentences in English and

$$\Omega_k := \{S \in \Omega : n(S) = k\} \text{ for all } k \in \mathbb{N}.$$

Then $|\Omega_k| \leq k^{|\wedge|} < \infty$ for all $k \in \mathbb{N}$. Hence, $\Omega := \bigcup_{n=1}^{\infty} \Omega_n$ is countable.

7.76 Apply Theorem 7.51.

7.77 Apply Theorem 7.53.

7.78 Yes. Take $A := \big([0, 1] \cap \mathbb{Q}\big) \cup \big([1, 2] \setminus \mathbb{Q}\big)$.

7.80 For any $n \in \mathbb{N}$, we define A_n to be the collection of $a \in [0, 1]$ such that the decimal expansion of a contains n consecutive identical terms, starting at the n^{th} decimal place. Then $card(A_n) = c$ for all $n \in \mathbb{N}$. The result follows, as $A_2 \subset \bigcup_{n=1}^{\infty} A_n \subset \mathbb{R}$.

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Chapter 8
Denseness

Uncountable spaces are often difficult to handle, as one can't 'list up' all the elements. The situation is much better when a metric space contains a countable dense subset, as that can approximate all its elements and thus is a good representative of the space itself (e.g. $\mathbb{Q}$ in $\mathbb{R}$). Such spaces are known as separable spaces. In this chapter, we discuss some notions emanating from denseness such as separability, perfect sets, Baire category, and equicontinuity.

We start with a section on separable spaces, which also presents some standard Polish spaces, along with the relationship of separability with different types of bases such as the topological bases and the Schauder bases. Then we introduce perfect sets and discuss the Cantor-Bendixon theorem. It is followed by the Baire Category Theorem, along with a variety of applications. We wind up this chapter with equicontinuity and related results on the compactness of $C[a, b]$.

8.1 Separability

Recall that a subset E of a metric space X is said to be *dense* if $\overline{E} = X$.

Definition 8.1 A metric space X is said to be *separable* if it has a countable dense subset.

Further, a nonempty subset E of X will be called a *separable set* if it is a separable subspace of X. That is, if and only if there exists a countable set $A \subset E$ such that $E \subset \overline{A}$.

Examples 8.2 (a) Every countable metric space is separable.
(b) $\mathbb{Q}$, $\mathbb{R}$, all real open intervals are all separable subspaces of $\mathbb{R}$.
(c) For each $n \in \mathbb{N}$, the Euclidean space $\mathbb{R}^n$ is separable. It can be shown that the countable set $\{(r_1, \ldots, r_n) : r_i \in \mathbb{Q}, 1 \leq i \leq n\}$ is dense in $\mathbb{R}^n$.

Theorem 8.3 *Every totally bounded metric space is separable.*

S. P. S. Kainth, *A Comprehensive Textbook on Metric Spaces*,
https://doi.org/10.1007/978-981-99-2738-8_8

Proof Let X be a totally bounded metric space. For every $n \in \mathbb{N}$, choose a finite subset A_n of X such that $X \subset \bigcup_{y \in A_n} B(y; 1/n)$. Write $A := \bigcup_{n=1}^{\infty} A_n$. Then A is a countable set. Now it is enough to show that A is dense in X, that is $\overline{A} = X$.

Let $\epsilon > 0$ be given. Pick any $n \in \mathbb{N}$ such that $1/n < \epsilon$. If $x \in X$, there exists some $y \in A_n$ such that $x \in B(y; 1/n)$ which implies that $y \in B(x; \epsilon) \cap A$. Hence $x \in \overline{A}$ and the result follows. □

Corollary 8.4 *Compact metric spaces are separable.*

Proof Apply Proposition 5.3 and Theorem 8.3. □

Theorem 8.5 *Every separable metric space has cardinality at most c.*

Proof Let X be any separable metric space and A be any countable dense subset of X. Then every element of X is the limit of a sequence from A.

Hence, the cardinality of X is at most the cardinality of the collection of sequences from A. Since A is countable, we obtain $card(X) \leq c$. □

Separable complete metric spaces are also known as *Polish spaces*. In Theorem 10.18, we show that every Polish space is either countable or has cardinality c.

In this section, we discuss a few common examples of Polish spaces. Already each $\mathbb{R}^n$ is a Polish space. Another important Polish space is $C[a, b]$. The separability of this space can be ensured by Theorem B.1. However, we provide a direct proof. Without loss of generality, we assume that $a = 0$ and $b = 1$. First we discuss a particular subclass of $C[0, 1]$. It will also be required in establishing the existence of a 'large' class of continuous but nowhere differentiable functions (see Lemma B.9 and Theorem B.10).

Definition 8.6 A function $f : [0, 1] \longrightarrow \mathbb{R}$ is said to be

(a) *linear* on a subinterval $[a, b]$ of $[0, 1]$ if the graph of f on $[a, b]$ is a straight line, that is,

$$f(t) := \frac{b-t}{b-a} f(a) + \frac{t-a}{b-a} f(b) \text{ for all } t \in [a, b].$$

(b) *piecewise linear* if there exists a partition $0 = x_0 < x_1 < \cdots < x_n = 1$ of $[0, 1]$ such that f is linear on each subinterval $[x_{i-1}, x_i]$.

Theorem 8.7 $C[0, 1]$ *is separable.*

Proof Let $\mathcal{P}$ denote the collection of all piecewise linear continuous functions on $[0, 1]$, which are differentiable on $[0, 1] \setminus \mathbb{Q}$ and assume rational values at their points of non-differentiability. Then $\mathcal{P}$ is countable. We claim that $\mathcal{P}$ is dense in $C[0, 1]$.

Let $f \in C[0, 1]$ and $\epsilon > 0$ be given. Since $[0, 1]$ is compact, f is uniformly continuous on $[0, 1]$. Therefore, there exists some $\delta > 0$ such that

$$|f(x) - f(y)| < \frac{\epsilon}{6} \text{ for all } x, y \in [0, 1] \text{ such that } |x - y| < \delta.$$

Let $N \in \mathbb{N}$ such that $1/N < \delta$ and $x_i := i/N$ for all $i = 0, \ldots, N$. Let $p \in \mathcal{P}$ be such that p is linear on $[x_{i-1}, x_i]$ and $p(x_i) \in (f(x_i) - \frac{\epsilon}{6}, f(x_i) + \frac{\epsilon}{6}) \cap \mathbb{Q}$ for all $i = 0, \ldots, N$. It is enough to prove that $p \in B(f; \epsilon)$.

Let $x \in [0, 1]$ be arbitrary. Then there exists i such that $x \in [x_{i-1}, x_i]$. Note that

$$|p(x) - p(x_i)| \leq |p(x_{i-1}) - p(x_i)| \leq |f(x_{i-1}) - f(x_i)| + \frac{2\epsilon}{6} < \frac{\epsilon}{6} + \frac{\epsilon}{3} = \frac{\epsilon}{2}.$$

Therefore, we obtain

$$\begin{aligned} |p(x) - f(x)| &\leq |p(x) - p(x_i)| + |p(x_i) - f(x_i)| + |f(x_i) - f(x)| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{6} + \frac{\epsilon}{6} = \frac{5\epsilon}{6}. \end{aligned}$$

Hence, $\|p - f\|_\infty \leq \frac{5\epsilon}{6} < \epsilon$ and thus $p \in B(f; \epsilon)$. This proves the result. □

Next, we discuss the connection between separability and bases. There are mainly two kinds of bases, closely related to separability. Countable topological bases of metric spaces and Schauder bases of normed spaces. We discuss these one by one.

Definition 8.8 Let X be a metric space. A collection Ω of open subsets of X is said to be a *topological basis* for X if for every open set G of X and for every $x \in G$, there exists some $O \in \Omega$ such that $x \in O \subset G$.

Equivalently, a collection Ω of open subsets of X is said to be a topological base for X if for every $x \in X$ and $r > 0$, there exists some $O_{x,r} \in \Omega$ such that $x \in O_{x,r} \subset B(x; r)$. Elements of a basis are known as *basic open sets.*

Examples 8.9 (a) The set of open intervals is a topological base for $\mathbb{R}$.
(b) The set of open rectangles forms a topological base for $\mathbb{R}^2$.
(c) The set $\{(a, b) : a, b \in \mathbb{Q}\}$ is a topological base for $\mathbb{R}$.
(d) Let X be a metric space. Then $\{B(x; 1/n) : x \in X, n \in \mathbb{N}\}$ is a base for X.

Several properties of topologies can be reduced to statements about a base generating that topology (e.g. see Exercise 8.15). Often it is also convenient to consider a much smaller basis, instead of all open sets. In this context, countable basis serve better, if they exist. The existence of such a basis is equivalent to separability of the space.

Theorem 8.10 *A metric space has a countable topological basis if and only if it is separable.*

Proof Let X be a metric space with a countable topological basis $\{U_n : n \in \mathbb{N}\}$. Without loss of generality, suppose that each U_n is nonempty. Choose $u_n \in U_n$ for all $n \in \mathbb{N}$. It is enough to show that $U := \{u_n : n \in \mathbb{N}\}$ is dense in X. Let O be any nonempty open subset of X. Then it contains some U_m and hence u_m. Consequently, $O \cap U \neq \emptyset$. Hence, $\overline{U} = X$.

Conversely, let X be a separable metric space and A be a countable dense subset of X. Consider the collection of balls

$$\mathcal{B} := \{B(x; r) : x \in A, r \in \mathbb{Q} \cap (0, \infty)\}.$$

Then $\mathcal{B}$ is a countable set. So it is enough to prove that $\mathcal{B}$ is a basis for X.

Let O be an open neighborhood of y. Then $O \cap A \neq \emptyset$, as $\overline{A} = X$. Pick any $x \in O \cap A$. Since O is open, there exists some $\epsilon > 0$ such that $B(x; \epsilon) \subset O$. If $r \in (0, \epsilon) \cap \mathbb{Q}$, then $B(x; r) \subset O$. Hence, $\mathcal{B}$ is a topological basis for X and the result follows. □

Every subspace of a separable metric space is separable (also see Example D.9).

Corollary 8.11 *If X is separable and $\emptyset \neq Y \subset X$, then Y is also separable.*

Proof Applying Theorem 8.10, let $\{B_n : n \in \mathbb{N}\}$ be a countable basis of X. Then $\{B_n \cap Y : B_n \cap Y \neq \emptyset, n \in \mathbb{N}\}$ is a countable basis of Y. Again by Theorem 8.10, Y is separable. □

Let us now discuss the Schauder basis of a normed space and its relationship with separability. Such bases were first described by the Polish mathematician Juliusz Schauder, in 1927.

Definition 8.12 Let X be a normed space. A sequence $\{x_n\}$ from X is said to be a *Schauder basis* of X, if for every $x \in X$ there exists a unique sequence of scalars $\{k_n\}$ such that

$$x = \sum_{n=1}^{\infty} k_n x_n.$$

In other words, a sequence $\{x_n\}$ is called a Schauder basis of X if every element of X can be written as a *unique countable linear combination of* $\{x_n\}$.

Remark 8.13 It is pertinent to note that the order of x_n's in the above series is important, as it may not converge unconditionally. Therefore, we use a sequence, in defining Schauder basis, not a countable set.

Examples 8.14 For $n \in \mathbb{N}$, let e_n be the sequence with all terms zero but 1, as the n^{th} term. Then $\{e_n\}$ is a Schauder basis for the space ℓ^p for all $p \in [1, \infty)$.

Theorem 8.15 *If a normed space X has a Schauder basis, then X is separable.*

Proof We prove the result when the scalar field is $\mathbb{R}$. The case of complex scalars is similar. Let $\{x_n\}$ be a Schauder basis of X. By replacing x_n with $\frac{x_n}{\|x_n\|}$, if necessary, we can assume that $\|x_n\| = 1$ for all $n \in \mathbb{N}$. Write $A := \{\sum_{n=1}^{\infty} r_n x_n : r_n \in \mathbb{Q}\}$. Since $\mathbb{Q}$ is countable, so is the set A. Therefore, it is enough to prove that A is dense in X.

Let $x \in X$ and $\epsilon > 0$ be given. Choose a sequence of scalars $\{k_n\}$ such that $x = \sum_{n=1}^{\infty} k_n x_n$. For each $n \in \mathbb{N}$, choose $r_n \in \mathbb{Q}$ such that $|k_n - r_n| < \frac{\epsilon}{2^n}$. Then for $y := \sum_{n=1}^{\infty} r_n x_n \in A$, we obtain

$$\|y - x\| = \Big\| \sum_{n=1}^{\infty} (k_n - r_n) x_n \Big\| \le \sum_{n=1}^{\infty} |k_n - r_n| \|x_n\| < \sum_{n=1}^{\infty} \frac{\epsilon}{2^n} = \epsilon.$$

Therefore, $x \in \overline{A}$. Hence the result. □

Corollary 8.16 *For each* $1 \le p < \infty$, *the sequence space* ℓ^p *is separable.*

Proof Apply Example 8.14 and Theorem 8.15. □

Theorem 8.17 *The sequence space* ℓ^∞ *is not separable.*

Proof Let E denote the collection of all sequences with terms 0 or 1 only. Then E is uncountable and

$$\|s - s'\|_\infty = 1 \text{ for all } s, s' \in E \text{ such that } s \ne s'.$$

If ℓ^∞ is separable, then it will have a countable dense subset, say $A := \{x_n : n \in \mathbb{N}\}$. Then $E \subset \ell^\infty = \bigcup_{n=1}^\infty B(x_n; \epsilon)$ for all $\epsilon > 0$. In particular, $E \subset \bigcup_{n=1}^\infty B(x_n; 1/4)$.

Since E is uncountable, there exist $k \in \mathbb{N}$ and $s, s' \in E$ with $s \ne s'$ such that $s, s' \in B(x_n; 1/4)$. Therefore,

$$1 = \|s - s'\|_\infty \le \|s - x_n\|_\infty + \|x_n - s'\|_\infty < \frac{1}{4} + \frac{1}{4} < \frac{1}{2},$$

which is impossible. Hence the result. □

Corollary 8.18 *The sequence space* ℓ^∞ *does not have any Schauder basis.*

Proof Apply Theorems 8.15 and 8.17. □

For the sake of completion, we present the notion of an algebraic basis of a linear space, often termed as a *Hamel basis.*

Definition 8.19 Let X be a linear space over a field $\mathbb{F}$. A subset B of X is known as an *algebraic basis* of X, if every $x \in X$ has a unique representation $x = \sum_{i=1}^n k_i x_i$, up to rearrangements, for some finite sets $\{k_1, \ldots, k_n\} \subset \mathbb{F}$ and $\{x_1, \ldots, x_n\} \subset B$.

Examples 8.20 (a) Let X be a normed space having a countable algebraic basis $\{b_n\}$. Then $\{b_n\}$ is also a Schauder basis of X.

(b) For every $1 \le p < \infty$, the sequence $\{e_n\}$ is a Schauder basis for the space ℓ^p, but not an algebraic basis of ℓ^p.

Remark 8.21 If the context is clear, the term ‘basis’ is used instead of ‘algebraic basis’ or ‘topological basis’. For example, if a metric space is not a linear space, there is no harm in simply using the term ‘basis’. A natural question: *Do all linear spaces have an algebraic basis?* Indeed yes, the Axiom of Choice is equivalent to the statement ‘every linear space has a basis’ (see [1]).

History Notes 8.22 It had been an open question for a long time that whether every separable Banach space has a Schauder basis. In 1973, it was settled in the negative by Enflo [2] who found a closed subspace of c_0 having no Schauder basis.

Exercise 8.1 Let $E \subset X$. Prove that E is dense in X if and only if every nonempty open set in X intersects with E.

Exercise 8.2 Let $E \subset Y \subset X$. If E is dense in X, prove that E is dense in Y. Is the converse true?

Exercise 8.3 Let $Z \subset Y \subset X$ such that Y is dense in X and Z is dense in the subspace Y. Does it imply that Z is dense in X?

Exercise 8.4 If E is a separable subset of X, then prove that so is $\overline{E}$.

Exercise 8.5 Is the converse of Theorem 8.3 true?

Exercise 8.6 Prove that every countable union of separable sets is separable.

Exercise 8.7 Obtain a proper subset of $\mathbb{Q}$ which is dense in itself. Do all countable metric spaces have proper dense subsets?

Exercise 8.8 Prove that the set of real algebraic numbers as well as the set of real transcendental numbers are dense subsets of reals.

Exercise 8.9 Give an explicit example of a countable dense subset of the subspace of transcendental numbers.

Exercise 8.10 Is every collection of disjoint open subsets of a separable space is countable?

Exercise 8.11 Does there exist any separable uncountable discrete metric spaces?

Exercise 8.12 Prove that the following are bases for a metric space X :

(a) the collection of all open subsets of X.
(b) the collection of all open balls in X.
(c) the collection of all open balls with rational radii.

Exercise 8.13 Prove that the following are bases for the usual metric on $\mathbb{R}$:

(a) the collection of all open intervals with rational end points.
(b) the collection of all open intervals with rational end points and length ≤ 1.

Exercise 8.14 Let X be a metric space with a topological basis Ω. Prove that $\{x\} \in \Omega$, for every isolated point x of X.

Exercise 8.15 Let Ω be a topological base for a metric space X, $x \in X$ and $\{x_n\}$ be a sequence in X. Prove that sequence $\{x_n\} \longrightarrow x$ if and only if for every $B \in \Omega$ such that $x \in B$ there exists some $N \in \mathbb{N}$ such that $x_n \in B$ for all $n > N$.

Exercise 8.16 Prove that a collection Ω of open subsets of X is a base if and only if every open set in X is a union of a sub-collection of Ω.

Exercise 8.17 Prove that a metric space X is separable if and only if every open cover of X has a countable subcover.

Exercise 8.18 If A is dense in a metric space X, prove that $\bigcup_{a\in A} B(a;r) = X$ for all $r > 0$.

Exercise 8.19 Prove that open subsets of separable metric spaces are countable unions of (a) open balls (b) closed balls.

Exercise 8.20 Let O_1 and O_2 be dense subsets of a metric space X such that O_1 is open. Prove that $O_1 \cap O_2$ is also dense in X. Further, show that the result is not true if both O_1 as well as O_2 are not open.

Exercise 8.21 Let X be a metric space. Prove that the number of its dense subsets is either infinite or 2^n, for some $n \in \mathbb{N}$.

Exercise 8.22 Prove that a metric space having infinitely many dense subsets, always has at least c-many dense subsets.

Exercise 8.23 Let O be an open subset of a separable metric space. Prove that O can have at most countably many connected components.

Exercise 8.24 Let X denote the linear space of bounded real valued functions on $[0, 1]$. Show that the normed space $(X, \|.\|_\infty)$ is not separable.

Exercise 8.25 Prove that the set $\{f \in C[0, 1] : f(0) \neq 0\}$ is dense in $C[0, 1]$.

Exercise 8.26 Let $A := \{a_n : n \in \mathbb{N}\}$ be a dense subset of a metric space X, $r > 0$ and $\{r_n\}$ be a sequence of positive reals. Prove or disprove:

(a) $\bigcup_{n=1}^{\infty} B(a_n; r) = X$.
(b) $\bigcup_{n=1}^{\infty} B(a_n; r_n) = X$.
(c) $\bigcap_{n=1}^{\infty} \bigcup_{n=1}^{\infty} B\big(a_n; \frac{1}{n}\big) = A$.

Exercise 8.27 Prove that there exists a bijection between $\mathbb{R}$ and the space $C[0, 1]$.

Exercise 8.28 Does there exist any compact or totally bounded metric space with cardinality strictly greater than c?

Exercise 8.29 Let X be a metric space having a dense subset A with $card(A) = c$. Prove that $card(X) = c$.

Exercise 8.30 For each $1 \le p < \infty$, prove that c_{00} is dense in ℓ^p, but not a closed subset of ℓ^p.

Exercise 8.31 Blumberg's famous theorem asserts that any $\mathbb{R} \longrightarrow \mathbb{R}$ function is continuous when restricted on some dense subset S of $\mathbb{R}$. Find S for the function

$$f(x) := \begin{cases} 0 & \text{if } x \text{ is a nonnegative rational,} \\ 1 & \text{if } x \text{ is a negative rational,} \\ 2 & \text{if } x \text{ is a positive irrational,} \\ 3 & \text{if } x \text{ is a negative irrational.} \end{cases}$$

Exercise 8.32 (a) Let $a < b$ be real numbers. Is it possible to write $[a, b]$ as a disjoint union of two uncountable dense subsets of $[a, b]$?
(b) Is it possible to write any uncountable subset E of real numbers as a disjoint union of two uncountable dense subsets of E?

Exercise 8.33 Let $1 \leq p < \infty$ and for each n, let e_n denote the sequence with all terms zero but 1, as the n^{th} term.

(a) Prove that $\{e_n\}$ forms a basis for c_{00}.
(b) Does $\{e_n\}$ form a basis for ℓ^p?
(c) Prove that $\{e_n\}$ forms a Schauder basis for ℓ^p.

Exercise 8.34 Without using Theorem 8.15, prove that the sequence space ℓ^p is separable, for every $1 \leq p < \infty$.

Exercise 8.35 Let X be a normed linear space and $\{u_n : n \in \mathbb{N}\}$ be a countably infinite linearly independent subset of X such that $\|u_n\| = 1$ for all $n \in \mathbb{N}$. Is $\left\{ \sum_{n=1}^{\infty} \frac{u_{i_n}}{i_n^2} : 1_n < 2_n < \cdots < i_n < \ldots \right\}$ linearly independent?

Exercise 8.36 Let X be a normed linear space and $B := \{u_n\}$ be a countable subset of X. Prove that B is a Schauder basis of X if and only if $\overline{span(B)} = X$. (Here $span(B)$ denotes the subspace of X spanned by B.)

Exercise 8.37 Prove that $\{m + n\sqrt{2} : m, n \in \mathbb{Z}\}$ is dense in $\mathbb{R}$. In general, if $a \in \mathbb{N}$ is not a perfect square, prove that $\overline{\{m + n\sqrt{a} : m, n \in \mathbb{Z}\}} = \mathbb{R}$.

The next exercise is exclusive to readers familiar with groups. Others may skip.

Exercise 8.38 Show that subgroups of the additive group $(\mathbb{R}, +)$ are either cyclic or dense in $\mathbb{R}$, but can't be both. Conclude that the additive group $\{m + n\sqrt{2} : m, n \in \mathbb{Z}\}$ is not cyclic.

8.2 Perfect Sets

Definition 8.23 A subset E of a metric space X is said to be *perfect* if $E' = E$.

Thus, perfect sets are closed sets with no isolated points. Further, X is called a *perfect space* or *dense in itself* if X is a perfect subset of itself or equivalently, if X has no isolated point.

Examples 8.24 (a) The empty set, the set of reals, and all closed intervals having at least two points are all perfect sets.
(b) If I is a nonempty bounded open interval, then I is not a perfect subset of $\mathbb{R}$.
(c) $\mathbb{Q}$ is a perfect metric space, while not a perfect subset of $\mathbb{R}$.

Example 8.25 If X is a connected metric space with $|X| > 1$, then X is perfect.

Proof Assume that X is not perfect. Then X has an isolated point, say x. Therefore, $\{x\}$ is a non-trivial clopen subset of X. Hence, X is not connected. □

Now we provide sufficient conditions, under which perfect sets are either empty or uncountable.

Theorem 8.26 *Every complete perfect metric space is uncountable. Consequently, every non-degenerated interval is an uncountable set.*

Proof Suppose that there exists a countable complete perfect metric space X. Since every point of X is its limit point, it is an infinite set. So X can be written as a sequence of distinct terms, say $\{x_n : n \in \mathbb{N}\}$.

Let $B_1 := B(x_1; 1)$. If $n \in \mathbb{N}$ and a ball B_n is already chosen, choose a ball B_{n+1}, with a suitable radius and center such that

$$x_n \notin \overline{B_{n+1}}, \quad \overline{B_{n+1}} \subset B_n \text{ and } \quad diam(B_{n+1}) \leq \frac{1}{2^n}. \tag{8.1}$$

Let $y \in B_n \subset X'$. Since y is a limit point of X and B_n is a neighborhood of y, the set B_n is infinite. Thus, we can choose $z \in B_n$ such that $z \neq x_n$. Let

$$r := \min\left\{\frac{1}{2^{n+2}}, \frac{d(z, x_n)}{2}\right\}.$$

Then $B(z; r) \cap B_n$ is a neighborhood of z. Consequently, there exists some $s > 0$ such that $B(z; s) \subset B(z; r) \cap B_n$. Let $B_{n+1} := B(z; s/2)$. Then B_{n+1} satisfies the requirements of (8.1).

Therefore, by induction, we obtain a sequence $\{B_n\}$ of balls in X, satisfying the conditions of (8.1). Consequently, $\{\overline{B_n}\}$ is a nested decreasing sequence of nonempty closed subsets of X with $diam(\overline{B_n}) \longrightarrow 0$.

Since X is complete, by Cantor Intersection Property (4.15), $\bigcap_{n=1}^{\infty} \overline{B_n} = \{x\}$ for some $x \in X$. Then $x = x_i$ for some $i \in \mathbb{N}$. Hence, $x_i \in \overline{B_{i+1}}$, a contradiction. □

It must be noted that the completeness of a metric space is not necessary for its nonempty perfect subsets to be uncountable. We now provide a weaker condition.

Recall that a neighborhood of a point x in a metric space X, is defined to be a subset of X containing an open ball centered at x.

Corollary 8.27 *Let X be a metric space and P be any perfect subset of X. If there exists some $p \in P$ such that p has a complete neighborhood, then P is uncountable.*

Proof Let U be a complete neighborhood of p. Then there exists $\epsilon > 0$ such that $B(p; \epsilon) \subset U$. Write $B := B(x; \epsilon/2)$. Then $\overline{B} \subset U$. Also, note that $P \cap B$ has no isolated point, otherwise P is not perfect. Therefore, $\overline{P \cap B}$ is a perfect subset of $\overline{B}$ and hence of the complete subspace U. Applying Theorem 8.26, $\overline{P \cap B}$ is uncountable and hence so is its superset P. □

In Corollary 8.27, the hypothesis that 'some $p_1 \in P$ has a complete neighborhood' is redundant!

Example 8.28 Every nonempty perfect subset of the metric space $\mathbb{R} \setminus \mathbb{Q}$, under usual metric, is uncountable.

Proof Let P_0 be a nonempty perfect subset of $\mathbb{R} \setminus \mathbb{Q}$. Then P_0 has no isolated points. Considering P_0 as a subset of $\mathbb{R}$, its closure in reals, say $P := \overline{P_0}$ is a perfect subset of $\mathbb{R}$. Hence, P is uncountable. Since $P \setminus P_0$ is a subset of $\mathbb{Q}$, it is at most countable. Hence, $P_0 = P \setminus (P \setminus P_0)$ is be uncountable. □

Remarks 8.29 (a) The completeness of the metric space is not redundant in Theorem 8.26. For example, $\mathbb{Q}$ under usual metric is countable and perfect. In fact, there is 'no other' countable perfect metric space (see Theorem 9.22).

(b) In [3], the uncountability of reals as well as of perfect sets has been proved through an infinite game. We shall discuss the same in Sect. C.1.

(c) In Corollaries 8.36 and 10.17, we provide alternative proofs to Theorem 8.26.

(d) Two extensions of Example 8.28 will be presented in Examples 10.19.

It is interesting to note that every closed subset of a separable metric space can be written as a disjoint union of a countable set and a perfect set. This result has various applications in measure theory, particularly related to the notorious Bernstein sets.

Definition 8.30 Let S be a subset of a metric space X. A point $x \in X$ is said to be a *condensation point* of S, if every neighborhood of x contains uncountably many points of S.

The set of condensation points of S will be denoted by S_u. Also, note that the set $S \setminus S_u$ contains all isolated points of S in X.

Theorem 8.31 (Cantor-Bendixson) *Let F be a closed subset of a separable metric space X. Then F_u is perfect and $F \setminus F_u$ is countable. Consequently, F is a union of a perfect set and a countable set.*

Proof Applying Theorem 8.10, let $\{U_n\}$ be a countable basis for X. Let

$$M := \{n \in \mathbb{N} : F \cap U_n \text{ is countable}\}$$
$$\text{and } A := \bigcup_{n \in M} U_n = \{x \in X : x \in U_n \text{ for some } n \in M\}.$$

We claim that $F_u = A^c$. This will imply that $F \setminus F_u = F \cap F_u^c = F \cap A$ is countable.

To prove it, pick any $x \in A^c$. Let $\epsilon > 0$ be given. Then $x \in U_n \subset B(x;\epsilon)$ for some n. Since $x \notin A$, the set $F \cap U_n$ is uncountable. Therefore, $F \cap B(x;\epsilon)$ is uncountable. Hence, $x \in F_u$.

On the other hand, let $x \in F_u$. If $x \in A$, we have $x \in U_n$ such that $F \cap U_n$ is countable. Let $\eta > 0$ such that $U_n \supset B(x;\eta)$. Then $F \cap B(x;\eta)$ is countable. Hence, $x \notin F_u$, a contradiction. Therefore, $x \in A^c$. This establishes our claim that $F_u = A^c$.

Since A is a union of open subsets of X, it is open. Therefore, $F_u = A^c$ is a closed set. To show that F_u is perfect, pick any $x \in F_u$. Let O be a neighborhood of x. Then $F \cap O$ is uncountable. We need to prove that $F_u \cap (O \setminus \{x\}) \neq \emptyset$.

If $F_u \cap (O \setminus \{x\}) = \emptyset$, we have $O \setminus \{x\} \subset F_u^c = A$. This proves that $O \subset A \cup \{x\}$. Therefore, $F \cap O \subset (F \cap A) \cup \{x\}$. Since $F \cap A$ is countable, so is $F \cap O$, a contradiction.

Since F is closed, $F_u \subset F' \subset F$. Hence, $F = F_u \cup (F \setminus F_u)$ is the required decomposition. □

Remark 8.32 In the above discussion, we intentionally avoided the Cantor set, which is one of the most important perfect sets. It has so many marvelous properties that it deserves a special treatment. For that purpose, we have devoted a complete chapter at the end of this book, to the Cantor set only.

Exercise 8.39 In $\mathbb{R}^2$, which of the following sets are prefect:

closed balls, closed rectangular regions, finite union of closed balls,
open balls, open squares, arbitrary countable unions of closed balls?

Exercise 8.40 Prove that curves, surfaces, and solids in $\mathbb{R}^3$ are perfect sets.

Exercise 8.41 Characterize perfect subsets of discrete metric spaces.

Exercise 8.42 Do all nonempty open subsets of a perfect metric spaces form perfect subspaces?

Exercise 8.43 Does there exist any nonempty perfect subset of $\mathbb{R}$ containing only rationals?

Exercise 8.44 Does there exist a compact metric space having a countably infinite perfect subset?

Exercise 8.45 Prove that every finite union of perfect subspaces, of a metric space, is perfect. Is the same true for finite intersections or infinite unions?

Exercise 8.46 Let P be a perfect subset of reals and I be an open interval such that $I \cap P \neq \emptyset$. Prove that $I \cap P$ is an uncountable set.

Exercise 8.47 If $E \subset \mathbb{R}$ has no isolated points, prove that $\overline{E}$ is a perfect set.

Exercise 8.48 If $k \in \mathbb{N}$ and $O \subset \mathbb{R}^k$ is open, prove that $\overline{O}$ is a perfect set.

Exercise 8.49 Let X be a perfect space with a countable dense set A.

(a) Does there exist a convergent sequence $\{a_n\}$ in X such that $\{a_n : n \in \mathbb{N}\} = A$?
(b) Does there exist a Cauchy sequence $\{a_n\}$ in X such that $\{a_n : n \in \mathbb{N}\} = A$?

Exercise 8.50 Let $X := \mathbb{Q} \times \mathbb{N}$, d_2 be the usual metric on X and

$$\rho(x, y) := \frac{d_2(x, y)}{1 + d_2(x, y)} \text{ for all } x, y \in X.$$

Prove that (X, ρ) is a countable perfect metric space.

Exercise 8.51 Without using Theorem 8.26, show that the following set is not perfect:

$$\left\{\frac{1}{n_1} + \frac{2}{n_2} + \cdots + \frac{k}{n_k} : k, n_1, \ldots, n_k \in \mathbb{N} \text{ such that } n_1 < n_2 < \cdots < n_k\right\}.$$

Exercise 8.52 Does there exist a metric space X having an uncountable perfect set P such that no $p \in P$ has a complete neighborhood inside X?

Exercise 8.53 Let (X, d) be a complete metric space and Y be a subspace of X such that the set $X \setminus Y$ is countable. Prove that every nonempty perfect subset of (Y, d) is uncountable.

Exercise 8.54 Let O be an nonempty open subset of a totally disconnected perfect compact metric space X and $n \in \mathbb{N}$. Show that there are disjoint open subsets $O_1, \ldots, O_n$ of X such that $\dot{\bigcup}_{i=1}^{n} O_n = O$.

Exercise 8.55 Let $a \in [0, 1]$ be an irrational number and $(0.a_1 \ldots a_n \ldots)_2$ denote its infinite binary representation. Let

$$E := \{(0.b_1 \ldots b_n \ldots)_2 : b_{2n} := a_n \text{ for all } n \in \mathbb{N}\}.$$

Prove that E is a perfect set containing only irrationals.

Exercise 8.56 Let (X, ρ_r) denote the ultrametric space of Example 2.5. Prove that

(a) (X, ρ_r) is perfect.
(b) (X, ρ_r) is compact; and hence separable.
(c) For every $x \in X$, the set $S_x := \{\rho_r(x, y) : y \in X\}$ has no limit point in $(0, +\infty)$.
(d) For every $x \in X$, the above set S_x is countable and has a unique limit point 0.

Exercise 8.57 Let X be a normed linear space. Prove the following:

(a) X is perfect.
(b) If E is a closed and convex subset of X, then E is perfect.

8.3 Baire Category Theorem

Note that in general the assertion $\overline{A} \cap \overline{B} = \overline{A \cap B}$ is not valid. However, if both A and B are dense and open in a metric space X, then it can be shown that $A \cap B$ is also dense in X. The same is trivial for arbitrary finite intersections. The Baire Category Theorem generalizes it to countable intersections.

Theorem 8.33 *(Baire, 1899) In any complete metric space X, countable intersection of dense open sets is dense in X.*

Proof Let $\{O_n\}$ be a sequence of dense open subsets of X. Let $x_0 \in X$ and B_0 be a ball containing x_0. It is enough to prove that $B_0 \cap (\bigcap_{n=1}^{\infty} O_n)$ is a nonempty set.

Since O_1 is dense in X, we have $B_0 \cap O_1 \neq \emptyset$. Since $B_0 \cap O_1$ is open, there exists a ball $B_1 \subset B_0 \cap O_1$. By considering a smaller ball, if necessary, one can assume that $diam(\overline{B_1}) < 1$ and $\overline{B_1} \subset B_0 \cap O_1$.

Since O_2 is dense in X, $B_1 \cap O_2 \neq \emptyset$. Thus, there exists an open ball $B_2 \subset B_1 \cap O_2 \subset B_0 \cap O_1 \cap O_2$. As earlier, by considering a smaller ball, if necessary, one can further assume that

$$diam(\overline{B_2}) < \frac{1}{2} \text{ and } \overline{B_2} \subset B_0 \cap O_1 \cap O_2.$$

Inducting like this, we choose a sequence $\{B_n\}$ of open balls in X such that

$$\overline{B_n} \subset B_0 \cap \Big(\bigcap_{k=1}^{n} O_k\Big) \text{ and } diam(\overline{B_n}) < \frac{1}{n} \text{ for all } n \in \mathbb{N}.$$

Hence, $\bigcap_{n=1}^{\infty} \overline{B_n} \subset B_0 \cap (\bigcap_{n=1}^{\infty} O_n)$. Since X is complete, applying Cantor Intersection Property (4.15), $\bigcap_{n=1}^{\infty} \overline{B_n}$ is nonempty. Hence, $B_0 \cap (\bigcap_{n=1}^{\infty} O_n) \neq \emptyset$ and the result follows. □

Note that the set $\mathbb{Q} \cap [0, 1]$ is not dense in $\mathbb{R}$ but is dense in the subspace $[0, 1]$ of reals. So this set is 'somewhere dense'. Motivated by this, we define nowhere dense sets as follows.

Definition 8.34 A subset E of a metric space X is called *nowhere dense* if $\left(\overline{E}\right)^o = \emptyset$.

It can be shown that E nowhere dense if and only if $X \setminus \overline{E}$ is dense in X.

Corollary 8.35 *No complete metric space is a countable union of its nowhere dense subsets.*

Proof Let X be a metric space such that there exists a sequence $\{E_n\}$ of nowhere dense subsets of X with $\bigcup_{n=1}^{\infty} E_n = X$. Since $\left(\overline{E_n}\right)^o = \emptyset$, for each $n \in \mathbb{N}$, it follows that $\left(\overline{E_n}\right)^c$ is dense in X. From $X = \bigcup_{n=1}^{\infty} \overline{E_n}$, we conclude that $\bigcap_{n=1}^{\infty} \left(\overline{E_n}\right)^c = \emptyset$, a contradiction to Theorem 8.33. □

Corollary 8.36 *If X is a complete perfect metric space, then X is uncountable. Consequently, $\mathbb{R}$ is uncountable.*

Proof Assume that X is countable and write $X = \{x_n : n \in \mathbb{N}\}$. Since X has no isolated points, every singleton subset of X is nowhere dense. Therefore, $X = \bigcup_{n=1}^{\infty}\{x_n\}$ is a countable union of its nowhere dense subsets, a contradiction to Corollary 8.35. □

We shall generalize Corollary 8.36 in Theorem 8.38, and draw some interesting consequences. That requires the following elementary result, which we present here for the sake of completion.

Lemma 8.37 *Let O be an open dense subset of a perfect metric space X and $x \in O$. Then $O \setminus \{x\}$ is also an open dense subset of X.*

Proof It is immediate that $O \setminus \{x\}$ is an open subset of X. To see that it is dense in X, pick any $y \in X$. By hypothesis, y is not an isolated point of X. Since O is an open set containing y, we conclude that y is a limit point of O.

Consequently, one can choose a sequence $\{y_n\}$ of distinct terms from O such that $y_n \longrightarrow y$. By omitting at most one term from $\{y_n\}$, we obtain another sequence of distinct terms from $O \setminus \{x\}$, convergent to y. Therefore, $y \in (O \setminus \{x\})' \subset \overline{O \setminus \{x\}}$. Since $y \in X$ was arbitrary, we conclude that $X \subset \overline{O \setminus \{x\}}$. Hence the result. □

Theorem 8.38 *Let $\{O_n\}$ be a sequence of dense open subsets of a perfect complete metric space. Then $\bigcap_{n=1}^{\infty} O_n$ is uncountable.*

Proof Suppose that $\bigcap_{n=1}^{\infty} O_n$ is countable. Write $\bigcap_{n=1}^{\infty} O_n = \{x_n : n \in \mathbb{N}\}$. By Lemma 8.37, each $U_n := O_n \setminus \{x_n\}$ is a dense open subset of X. Now Theorem 8.33, ensures that $\bigcap_{n=1}^{\infty} U_n = \emptyset$ is a dense subset of X, a contradiction. □

Next, we show that $\mathbb{Q}$ is not a G_δ. Consequently, $\mathbb{R} \setminus \mathbb{Q}$ is not an F_σ-set.

Corollaries 8.39 *(a) The set of rational numbers $\mathbb{Q}$ is not a G_δ.*
(b) There exists no $\mathbb{R} \longrightarrow \mathbb{R}$ function, continuous precisely on $\mathbb{Q}$.
(c) There exists no sequence $\{f_n\}$ of continuous $\mathbb{R} \longrightarrow \mathbb{R}$ functions, pointwise convergent to the Dirichlet function $\chi_{\mathbb{Q}}$.

Proof (a) Suppose there exists a countable collection $\{O_n\}$ of open subsets of $\mathbb{R}$ such that $\mathbb{Q} = \bigcap_{n=1}^{\infty} O_n$. Since each O_n contains $\mathbb{Q}$, each O_n is dense in $\mathbb{R}$. By Theorem 8.38, $\mathbb{Q}$ is uncountable, a contradiction.
(b) Apply Theorem 7.37 along with (a).
(c) Suppose $\{f_n\}$ is one such sequence. Then $O_n := \{x \in \mathbb{R} : f_n(x) > 1/2\}$ is open, for every $n \in \mathbb{N}$. Therefore, each $G_n := \bigcup_{k \geq n} O_n$ is open. Note that

$$\bigcap_{n=1}^{\infty} G_n = \bigcap_{n=1}^{\infty} \bigcup_{k \geq n} O_k = \left\{ x \in \mathbb{R} : f_n(x) > \frac{1}{2}, \text{ for infinitely many } n \in \mathbb{N} \right\} = \mathbb{Q}.$$

Hence, $\mathbb{Q}$ is a G_δ, a contradiction to (a). □

Definition 8.40 A subset E of metric space is said to be of *first category* if it can be written as a countable union of nowhere dense sets. Sets which are not of first category are called sets of *second category.*

Therefore, the Baire Category Theorem (8.33) can be restated as

No complete metric space is of the first category.

A set of first category is also called a *meager set* or a *thin set* and a set of second category is also called a *non-meager set* or a *thick set.* The complement of a meager subset of a metric space is known as a *co-meager set* or a *residual set.*

Remark 8.41 The Baire Category Theorem (8.33) was first proved by a French mathematician René-Louis Baire in his 1899 doctoral thesis. The sets of first category relative to a given set and the corresponding version of the category theorem can be found in [4, Chap. 5].

Our last application of the category theorem requires the notions of negligible sets and the length function for intervals. The *length of an interval* I is defined as

$$l(I) := \begin{cases} b-a \ ; & \text{if } I \text{ is bounded with end points } a \leq b, \\ \infty \ ; & \text{if } I \text{ is unbounded.} \end{cases}$$

Definition 8.42 A subset E of real numbers is said to be a *negligible set* if

$$\inf \left\{ \sum_{n=1}^{\infty} l(I_n) : \{I_n\} \text{ is a sequence of open intervals with } E \subset \bigcup_{n=1}^{\infty} I_n \right\} = 0.$$

That is, E is negligible if for every $\epsilon > 0$ there exists a sequence $\{I_n\}$ of intervals such that

$$E \subset \bigcup_{n=1}^{\infty} I_n \text{ and } \sum_{n=1}^{\infty} l(I_n) < \epsilon.$$

It is immediate that every subset of a negligible set is negligible. Negligible sets are also known as *null sets* or *sets with Lebesgue outer measure zero.*

Example 8.43 Every countable subset of $\mathbb{R}$ is negligible.

Proof Let $E := \{x_1, \ldots, x_n, \ldots\} \subset \mathbb{R}$ be countable. Let $\epsilon > 0$ be given. Then

$$E \subset \bigcup_n \left(x_n - \frac{\epsilon}{2^{n+1}}, x_n + \frac{\epsilon}{2^{n+1}} \right) \text{ and } \sum_n l\left(\left(x_n - \frac{\epsilon}{2^{n+1}}, x_n + \frac{\epsilon}{2^{n+1}} \right) \right) = \sum_n \frac{\epsilon}{2^n} \leq \epsilon.$$

Since $\epsilon > 0$ is arbitrary, E is negligible. □

In Proposition 10.5, we shall show that the converse of Example 8.43 is not true.

Theorem 8.44 *$\mathbb{R}$ contains a negligible subset of second category. Consequently, $\mathbb{R}$ is a disjoint union of a negligible set and a set of first category.*

Proof Let $\{r_n\}$ be an enumeration of rational numbers and

$$U_m := \bigcup_{n=1}^{\infty}(r_n - 2^{-n-m}, r_n + 2^{-n-m}) \text{ for all } m \in \mathbb{N}.$$

Then each U_m is dense in $\mathbb{R}$ and its complement $U_m^c := \mathbb{R} \setminus U_m$ is nowhere dense. Consequently, $A := \bigcup_{n=1}^{\infty} U_m^c$ is a set of first category. Since $\mathbb{R}$ is of second category, $B := \mathbb{R} \setminus A = \bigcap_{n=1}^{\infty} U_n$ is of the second category.

Let $\epsilon > 0$ be given. Pick any $k \in \mathbb{N}$ such that $2^{-k+2} < \epsilon$. Since $B \subset U_k$, so B is contained in the union of intervals $\{(r_m - 2^{-k-m}, r_m + 2^{-k-m}) : m \in \mathbb{N}\}$. The sum of lengths of these intervals is at most 2^{-k+1}, which is less than ϵ. Hence, B is negligible. □

At this point, we can't resist stating the differentiation theorem for monotone real functions, although it is not much important from topological point of view.

Theorem 8.45 *(Lebesgue-Young) If $f : \mathbb{R} \longrightarrow \mathbb{R}$ is a monotone function, then there exists a negligible set E such that f is differentiable on $\mathbb{R} \setminus E$.*

On the other hand, for every negligible set $E \subset \mathbb{R}$, there exists a monotone function $f : \mathbb{R} \longrightarrow \mathbb{R}$ such that E is the set of points of non-differentiability of f (see [5, p. 114, Exercise 10]).

Further, every Lipschitz continuous $\mathbb{R} \longrightarrow \mathbb{R}$ function is a difference of two monotone functions, and hence differentiable almost everywhere. We omit the proofs of these results, as the tools required for these are out of scope for this textbook. Interested reader can find the proof from any standard textbook on measure theory (e.g. see [5, Chap. 6]). For a historical background of Theorem 8.45, we refer [6, p. 212].

Geometric and elementary proofs of Theorem 8.45 can be found in [7, 8], respectively. In the 29^{th} volume of Real Analysis Exchange, three consecutive articles appeared on the differentiation theorem (see [9–11]).

Exercise 8.58 If X is a metric space, prove that finite intersection of dense open subsets of X is dense in X.

Exercise 8.59 Let X be a metric space. Prove that

(a) every subset of a nowhere dense set is nowhere dense.
(b) every finite union of nowhere dense sets is nowhere dense.
(c) every finite set is nowhere dense, provided X has no isolated points.

Exercise 8.60 Let A be a nowhere dense subset of a metric space X and $B \subset \overline{A}$. Prove that B is nowhere dense.

Exercise 8.61 Prove that a subset E of a metric space X is nowhere dense if and only if $X \setminus E$ contains a dense open subset.

Exercise 8.62 Is there any closed subset of $\mathbb{R} \setminus \mathbb{Q}$ that is not nowhere dense?

Exercise 8.63 Let $E \subset \mathbb{R}$. If E is nowhere dense, prove that E is totally disconnected. Show that the converse is false. Is that true, assuming E is closed?

Exercise 8.64 Write a direct proof to show that $\mathbb{R}$ is of second category.

Exercise 8.65 Prove that every closed proper subspace of a normed space is nowhere dense.

Exercise 8.66 Let X be a metric space and $E \subset X$. Prove that E is nowhere dense if and only if $X \setminus \overline{E}$ is dense in X.

Exercise 8.67 Let X be a metric space. Prove that

(a) every subset of a meager set is meager.
(b) every countable union of meager sets is meager.
(c) every countable set is meager, provided X has no isolated points.

Exercise 8.68 Let X be a complete metric space. Prove that the following are equivalent:

(a) X is of second category.
(b) Every countable intersection of dense open sets is dense in X.
(c) Every nonempty open subset of X is of second category.
(d) Every countable intersection of dense open subsets of X is nonempty.

Exercise 8.69 Let X be a complete metric space. Prove that every open ball as well as every closed ball (with positive radius) is a non-meager set in X.

Exercise 8.70 Does the conclusion of Baire Category Theorem (8.33) hold for some non-complete metric space?

Exercise 8.71 Let X be a complete metric space. Is every intersection of dense open subsets of X nonempty?

Exercise 8.72 If $\mathbb{R} = \bigcup_{n=1}^{\infty} F_n$, where each F_n is closed, prove that some F_n contains a non-degenerate interval.

Exercise 8.73 Let E be a non-empty closed subset of a complete metric space X and $\{E_n\}$ be a sequence of subsets of X such that $E \subset \bigcup_{n=1}^{\infty} E_n$. Prove the following:

(a) If each E_n is closed, then there exists some $N \in \mathbb{N}$ and an open set O such that $\emptyset \neq E \cap O \subset E_N$.
(b) If each E_n is an F_σ-set, then there exists some $N \in \mathbb{N}$ and an open set O such that $\emptyset \neq E \cap O \subset E_N$.
(c) There exists some $N \in \mathbb{N}$ and an open set O such that E_N is dense in $E \cap O$.

Exercise 8.74 Does there exist $E \subset \mathbb{R}$ such that E is neither a G_δ nor an F_σ?

Exercise 8.75 Generalize Corollaries 8.39(a) and 8.39(b) for complete perfect metric spaces.

Exercise 8.76 Prove that the set of rational numbers as well as the set of algebraic numbers are both negligible sets.

Exercise 8.77 Prove that every countable union of negligible sets is also negligible.

Exercise 8.78 If I is any non-degenerated interval, prove that I is not negligible.

Exercise 8.79 Does there exist any nonempty perfect subset of reals consisting of only irrational numbers?

Exercise 8.80 Prove that every subset of $\mathbb{R}$ is a disjoint union of a negligible set and a set of first category.

8.4 Equicontinuity

This section presents the Arzelá-Ascoli Theorem, which characterizes compact subsets of $C(X)$, where X is a compact metric space. Before delving into this result, let us establish some notions and definitions.

Definitions 8.46 Let S be any set. A sequence $\{f_n\}$ of $S \longrightarrow \mathbb{R}$ functions is called

(a) *pointwise bounded,* if the sequence $\{f_n(x)\}$ is bounded for each $x \in S$.
(b) *uniformly bounded,* if there exists some $M > 0$ such that

$$|f_n(x)| < M, \text{ for every } x \in S \text{ and for every } n \in \mathbb{N}.$$

Definition 8.47 A family $\mathcal{F}$ of real valued functions defined on a metric space (X, d) is said to be *equicontinuous* on X if for every $\epsilon > 0$ there exists $\delta > 0$ such that

$$|f(y) - f(x)| < \epsilon \text{ for all } x, y \in X \text{ satisfying } d(x, y) < \delta \text{ and for all } f \in \mathcal{F}.$$

It is immediate that every function in an equicontinuous family on X is uniformly continuous on X.

Theorem 8.48 *Let $\{f_n\}$ be a sequence of continuous real valued functions on a compact metric space (X, d). If $\{f_n\}$ converges uniformly on X, then $\{f_n\}$ is equicontinuous on X.*

Proof Let $\epsilon > 0$ be given. Since $\{f_n\}$ converges uniformly on X, by Cauchy criterion for uniform convergence, there exists $N \in \mathbb{N}$ such that

$$|f_{n_2}(x) - f_{n_1}(x)| < \epsilon \text{ for all } x \in X \text{ and for all } n_2 > n_1 \geq N.$$

Since f_N is continuous on X, which is a compact space, it is uniformly continuous on X. Let $\delta > 0$ be such that $|f_N(x) - f_N(y)| < \epsilon$ for all $x, y \in X$ satisfying $d(x, y) < \delta$. Therefore for all $n \geq N$ and for all $x, y \in X$ satisfying $d(x, y) < \delta$, we have

$$|f_n(x) - f_n(y)| \leq |f_n(x) - f_N(x)| + |f_N(x) - f_N(y)| + |f_N(y) - f_n(y)| < \epsilon + \epsilon + \epsilon = 3\epsilon.$$

Since $\epsilon > 0$ is arbitrary, the result follows. □

Lemma 8.49 *Let X be a countable set and $\{f_n\}$ be a pointwise bounded sequence of real valued functions on X. Then $\{f_n\}$ has a subsequence that converges pointwise on X.*

Proof Write $X = \{x_n : n \in \mathbb{N}\}$. By hypothesis, the sequence $\{f_n(x_1)\}_n$ is bounded and hence has a convergent subsequence, say $\{f_{1_k}(x_1)\}_k$.

Again by hypothesis, $\{f_n(x_2)\}_n$ is bounded. So its subsequence $\{f_{1_k}(x_2)\}_k$ is also bounded. Consequently, it has a convergent subsequence, say $\{f_{2_k}(x_2)\}_k$.

Inducting like this, for all $n \in \mathbb{N} \setminus \{1\}$, we have a subsequence $\{f_{n_k}\}_k$ of $\{f_n\}_n$ such that $\{f_{n_k}(x_n)\}_k$ is convergent and that $\{f_{n_k}\}_k$ is a subsequence of $\{f_{(n-1)_k}\}_k$, for all $n > 1$.

If X is finite, then $X = \{x_1, \ldots, x_m\}$ for some $m \in \mathbb{N}$. In this case, the sequence $\{f_{m_k}(x_n)\}_k$ is convergent, for every $n \in \mathbb{N}$. If X is infinite, then $\{f_{k_k}(x_n)\}_{k \geq n}$ is a subsequence of the convergent sequence $\{f_{n_k}(x_n)\}_{k \geq 1}$ for all n. Hence, $\{f_{k_k}(x_n)\}_k$ is convergent, for all $n \in \mathbb{N}$. □

Theorem 8.50 *(Ascoli-Arzelá) Let (X, d) be a totally bounded metric space and $\{f_n\}$ be a sequence of pointwise bounded and equicontinuous real valued functions on X. Then*

(a) $\{f_n\}$ is uniformly bounded on X and
(b) $\{f_n\}$ contains a subsequence, uniformly convergent on X.

Proof Let $\epsilon > 0$ be given. Since $\{f_n\}$ is equicontinuous on X, there exists some $\delta > 0$ such that for every $n \in \mathbb{N}$, we have

$$|f_n(y) - f_n(x)| < \epsilon \text{ for all } x, y \in X \text{ satisfying } d(x, y) < \delta. \tag{8.2}$$

Since X is totally bounded, $X = \bigcup_{i=1}^{m} B(x_i; \delta)$ for finitely many $x_1, \ldots, x_m \in X$. Also, by Theorem 8.3, X is separable. By appending a countable set to $\{x_1, \ldots, x_m\}$, we can assume that $\{x_k : k \in \mathbb{N}\}$ is a countable dense subset of X.

(a) Since for each $k \in \{1, \ldots, m\}$, the sequence $\{f_n(x_k)\}_{n \in \mathbb{N}}$ is bounded, one can fix $M_k > 0$ such that $|f_n(x_k)| < M_k$ for each $n \in \mathbb{N}$. Write $M := \max\{M_1, \ldots, M_m\}$.

If $x \in X$, then $x \in B(x_i; \delta)$, for some $i \leq m$ and therefore

$$|f_n(x)| \leq |f_n(x) - f_n(x_i)| + |f_n(x_i)| < \epsilon + M \text{ for all } n \in \mathbb{N}.$$

This proves that $\{f_n\}$ is uniformly bounded by $\epsilon + M$, on X.

(b) By Lemma 8.49, $\{f_n\}$ has a subsequence $\{f_{n_k}\}$, pointwise convergent on $\{x_n : n \in \mathbb{N}\}$. Write $g_k := f_{n_k}$, to simplify the notation. Since $\{g_k\}$ is pointwise convergent on $\{x_1, \ldots, x_m\}$, there exists $N \in \mathbb{N}$ such that for every $k_2 > k_1 > N$, we have

$$|g_{k_2}(x_i) - g_{k_1}(x_i)| < \epsilon, \text{ for each } i = 1, \ldots, m. \tag{8.3}$$

Pick any $x \in X$. Then $x \in B(x_i; \delta)$, for some $i \leq m$. Then (8.2) and (8.3) imply that for all $k_2 > k_1 > N$, we have

$$\begin{aligned}|g_{k_2}(x) - g_{k_1}(x)| \leq |g_{k_2}(x) - g_{k_2}(x_i)| &+ |g_{k_2}(x_i) - g_{k_1}(x_i)| \\ &+ |g_{k_1}(x_i) - g_{k_1}(x)| < 3\epsilon.\end{aligned}$$

Therefore, $\{g_k\}$ is uniformly Cauchy, and hence uniformly convergent on X. □

Lemma 8.51 *Let (X, d) be a compact metric space and $\mathcal{F}$ be a totally bounded subset of $C(X)$. Then the collection of functions $\mathcal{F}$ is uniformly bounded and equicontinuous on X.*

Proof Being totally bounded, $\mathcal{F}$ is a bounded subset of $C(X)$ and hence uniformly bounded. To prove the equicontinuity of $\mathcal{F}$, let $\epsilon > 0$ be given.

Since $\mathcal{F}$ is totally bounded, there exist $f_1, \ldots, f_n \in \mathcal{F}$ such that $\mathcal{F} \subset \bigcup_{i=1}^n B(f_i; \epsilon/3)$. Since X is a compact space, the finite collection $\{f_1, \ldots, f_n\}$ is equicontinuous. Then there exist $\delta > 0$ such that for every $i = 1, \ldots, m$ and for all $x, y \in X$ such that $d(x, y) < \delta$, we have $|f_i(x) - f_i(y)| < \frac{\epsilon}{3}$.

Pick any $f \in \mathcal{F}$. Then there exists i such that $\|f - f_i\|_\infty < \epsilon/3$. Let $x, y \in X$ such that $d(x, y) < \delta$. Then we obtain

$$\begin{aligned}|f(x) - f(y)| &\leq |f(x) - f_i(x)| + |f_i(x) - f_i(y)| + |f_i(y) - f(y)| \\ &< \frac{\epsilon}{3} + 2\|f - f_i\|_\infty < \epsilon.\end{aligned}$$

Hence, $\mathcal{F}$ is equicontinuous on X. □

Theorem 8.52 *Let X be a compact metric space and $\mathcal{F} \subset C(X)$. Then $\mathcal{F}$ is compact if and only if $\mathcal{F}$ is closed, bounded, and equicontinuous.*

Proof If $\mathcal{F}$ is compact, it is closed, bounded and totally bounded. Applying Lemma 8.51, $\mathcal{F}$ is equicontinuous on X.

Conversely, assume that $\mathcal{F}$ is closed, bounded and equicontinuous. Since $C(X)$ is complete, its closed subspace $\mathcal{F}$ is also complete (see Theorem 4.3). To prove compactness, it is enough to show that $\mathcal{F}$ is totally bounded, that is every sequence in $\mathcal{F}$ has a Cauchy subsequence. That follows by Theorem 8.50. □

Remarks 8.53 The Ascoli-Arzelá Theorem is an important tool in Analysis. It is used in the standard proofs of the Peano Existence Theorem and Riemann Mapping Theorem. Another application of this theorem to the Isoperimetric Theorem is provided in [12, p. 30].

In most textbooks, Theorem 8.50 is presented with an additional hypothesis that X is compact (see [13, Theorem 7.25, p. 158]). However, the above proof ensures that X is not required to be complete. Moreover, the results of this section hold for functions taking values in finite-dimensional normed spaces.

Exercise 8.81 Let $\mathcal{F}$ be a set of real valued functions on a metric space X.

(a) If $\mathcal{F}$ is a finite collection of uniformly continuous functions on X, prove that $\mathcal{F}$ is equicontinuous on X.
(b) Prove that $\mathcal{F}$ is equicontinuous on X if and only if $\mathcal{F} \setminus F$ is equicontinuous on X for every finite set $F \subset \mathcal{F}$.

Exercise 8.82 Let $\mathcal{F}$ be a family of Lipschitz continuous real valued functions on $[0, 1]$ with Lipschitz constant 1. Prove that $\mathcal{F}$ is equicontinuous.

Exercise 8.83 Let $\mathcal{F}$ be a bounded subset of $C[0, 1]$ such that every $f \in \mathcal{F}$ is differentiable and $|f'| < 1$ for all $f \in \mathcal{F}$. Prove that $\mathcal{F}$ is equicontinuous.

Exercise 8.84 Let $\{f_n\}$ be a sequence of real valued differentiable functions on $[0, 1]$ such that $|f_n'| < 1$ for all $n \in \mathbb{N}$. Prove that $\{f_n\}$ has a uniformly convergent subsequence.

Exercise 8.85 Let X be a compact metric space and $f : X \times X \longrightarrow \mathbb{R}$ be any continuous function. Define $f_y(x) := F(x, y)$ for all $x, y \in X$. Prove that $\{f_y : y \in X\}$ is a family of equicontinuous functions on X.

Exercise 8.86 Give an example of a countably infinite collection of real valued equicontinuous functions on $[0, 1]$.

Exercise 8.87 Does there exist any uncountable family of real valued equicontinuous functions on $[0, 1]$?

Exercise 8.88 Let $\mathcal{F}$ be the set of continuous functions $f : (0, \infty) \longrightarrow \mathbb{R}$ such that $f(2x) = f(x)$ for all $x > 0$. Which of the following statements is/are true?

(a) Every $f \in \mathcal{F}$ is bounded.
(b) Every $f \in \mathcal{F}$ is uniformly continuous.
(c) Every $f \in \mathcal{F}$ is differentiable.
(d) Every uniformly bounded sequence of functions from $\mathcal{F}$ has a uniformly convergent subsequence.

Exercise 8.89 Let X be a compact metric space and $\{f_n\}$ be a convergent sequence in $C(X)$. Prove that $\{f_n\}$ is uniformly bounded and equicontinuous.

Exercise 8.90 Prove that every uniformly convergent sequence of bounded functions is uniformly bounded.

Exercise 8.91 Let $\{f_n\}$ be a sequence of uniformly continuous functions on a metric space X. If $\{f_n\}$ converges uniformly on X, prove that $\{f_n\}$ is equicontinuous on X.

Exercise 8.92 Let $f_n(x) := \sin nx$ for all $x \in [0, 2\pi], n \in \mathbb{N}$. Prove that $\{f_n\}$ is uniformly bounded on $[0, 2\pi]$, and not even pointwise convergent for any $x \in [0, 2\pi]$. In fact no subsequence of $\{f_n\}$ is uniformly convergent on $[0, 2\pi]$.

Exercise 8.93 Let $f_n(x) := \frac{x^2}{x^2+(1-nx)^2}$ for all $x \in [0, 1], n \in \mathbb{N}$. Prove that $\{f_n\}$ is uniformly bounded on $[0, 1]$, but not uniformly convergent. Also, prove that no subsequence of $\{f_n\}$ is uniformly convergent on $[0, 1]$. Conclude that $\{f_n\}$ is not equicontinuous on $[0, 1]$.

8.5 Hints and Solutions to Selected Exercises

8.2 The converse is false. For example, take $X := \mathbb{R}$ under usual metric, $E := (0, 1)$, and $Y := [0, 1]$.

8.5 No, for example take $X := \mathbb{R}$ or a countably infinite discrete metric space.

8.7 Note that $\mathbb{Q} \setminus \{0\}$ is dense in $\mathbb{Q}$. Further countably infinite discrete spaces have no proper dense subsets.

8.9 $\{r + \pi : r \in \mathbb{Q}\}$. To see this, note that for any transcendental numbers $a < b$, there exists a rational number r such that $a - \pi < r < b - \pi$. Hence, $a < r + \pi < b$.

8.17 Apply Theorem 8.10.

8.20 Let O be a nonempty open subset of X. Since O_1 is open and dense in X, $O \cap O_1$ is a nonempty and open subset of X. Since O_2 is dense in X, $O \cap O_1 \cap O_2 \neq \emptyset$. Hence, $O_1 \cap O_2$ is dense in X. The result is false if both O_1 and O_2 are not open. For example, take $X := \mathbb{R}$, $O_1 := \mathbb{Q}$ and $O_2 := \mathbb{R} \setminus \mathbb{Q}$.

8.21 Suppose X has only finitely many dense subsets. Then X will have only finitely many limit points $x_1, \ldots, x_n$. Note that E is dense in X if and only if $X \setminus E \subset \{x_1, \ldots, x_n\}$.

8.23 Imitate Theorem 7.18.

8.24 For every $c \in [0, 1]$, let $p_c \in X$ be defined as $p_c := \chi_{[0,c]}$. Let $\mathcal{F}$ be any dense subset of $C[0, 1]$. Then there exists some $f_c \in \mathcal{F}$ such that $\|p_c - f_c\|_\infty < 1/2$. Write $B_c := B(p_c; 1/2)$ for all $c \in [0, 1]$.

Let $c, d \in [0, 1]$ such that $c \neq d$ be arbitrary. Then $\|p_c - p_d\|_\infty = 1$, which implies $B_c \cap B_d = \emptyset$ and hence $f_c \neq f_d$. Therefore, $\mathcal{F}$ is uncountable.

8.25 Let $f \in C[0, 1]$. Then $f_n := f + \frac{1}{n}$ is a sequence in the given set, uniformly convergent to f on $[0, 1]$.

8.26 (a) True, as A is dense in X.
(b) False. Take $X := \mathbb{R}$, $A := \mathbb{Q}$ and $r_n := 2^{-n}$ for all $n \in \mathbb{N}$.
(c) True, by (a).

8.27 Since for every $a \in [0, 1]$, the constant map $x \longmapsto a$ on $[0, 1]$ is continuous, we have $card(C[0, 1]) \geq card([0, 1]) = c$. The fact that $card(C[0, 1]) \leq c$ holds by Theorems 8.5 and 8.7.

8.28 No. Apply Theorems 8.3 and 8.5.

8.29 Note that $card(X)$ is same as the cardinality of the set of convergent sequences from A, which is $\leq$ the cardinality of the set of sequences from A, that is c (see Exercise 7.77). Hence, $card(X) \leq c$. On the other hand, $c = card(A) \leq card(X)$, as $A \subset X$.

8.30 The sequence $(1, \ldots, 1/n, 0, 0, \ldots) \in c_{00}$ converges to $(1, \ldots, 1/n, \ldots) \in \ell^p \setminus c_{00}$. Hence, c_{00} is not closed in ℓ^p. Now let $x = \{x_n\} \in \ell^p$ be arbitrary and $y_k \in \ell^p$ be the restriction of $\{x_n\}$ till k terms, followed by all zeros. Then each $y_k \in c_{00}$ and $y_k \longrightarrow x$, in ℓ^p. Hence, c_{00} is dense in ℓ^p.

8.32 (a) Yes. Let Q_1 denote the set of rational numbers p/q inside $[a, b]$ such that p, q are co-prime integers and q is an even natural number. Write $Q_2 := [a, b] \cap (\mathbb{Q} \setminus Q_1)$. Note that both Q_1 and Q_2 are disjoint sets of rational numbers, dense in $[a, b]$. Fix any $c \in [a, b)$ and let

$$E_1 := ([a, c] \cup Q_1) \setminus Q_2 \text{ and } E_2 := ((c, b] \cup Q_2) \setminus Q_1.$$

Then E_1 and E_2 are the required sets.

(b) Yes. Let E_0 denote the set of isolated points of E. Then $E \setminus E_0 = E'$. Since $\mathbb{R}$ is separable and E is uncountable, we conclude that $E' \neq \emptyset$.

Let $\{I_n : n \in \mathbb{N}\}$ be the sequence of intervals with rational end points such that for all $n \in \mathbb{N}$, $I_n \cap E' \neq \emptyset$ and hence the set $I_n \cap E$ is infinite. Applying induction, we choose two sequences $\{x_n\}$ and $\{y_n\}$ from E, as follows:

Let $x_1, y_1 \in I_1 \cap E$ such that $x_1 \neq y_1$. Let $n \in \mathbb{N}$ and assume that $x_1, \ldots, x_n$ and $y_1, \ldots, y_n$ have been chosen. Choose $x_{n+1}, y_{n+1} \in I_{n+1} \cap E \setminus \{x_i, y_i : i = 1, \ldots, n\}$ such that $x_{n+1} \neq y_{n+1}$. This completes the induction step.

Let $Q_1 := \{x_n : n \in \mathbb{N}\}$ and $Q_2 := \{y_n : n \in \mathbb{N}\}$. Note that both Q_1 and Q_2 are disjoint sets, dense in E. Fix any $c \in E$ such that both $E \cap (-\infty, c)$ and $E \setminus (-\infty, c)$ are uncountable.

$$E_1 := ((E \cap (-\infty, c)) \cup Q_1) \setminus Q_2 \text{ and } E_2 := ((E \setminus (-\infty, c)) \cup Q_2) \setminus Q_1.$$

Then E_1 and E_2 are the required sets.

8.33 (b) No. Let x denote the sequence with $n^{-\frac{2}{p}}$ as its n^{th}-term. Then x is not a finite linear combination of elements from the set $\{e_n : n \in \mathbb{N}\}$. Also, compare it with Theorem 9.55.

8.34 If $1 \le p < \infty$, let A be the family of sequences $(r_1, r_2, \ldots, r_n, 0, 0, \ldots)$, where $r_1, \ldots r_n$ are all rational numbers and n is any natural number. Then A is countable. So it is enough to show that A is dense in ℓ^p.

Let $x = \{x_k\}$ be any sequence in ℓ^p and $\epsilon \in (0, 1)$. Then we have $\sum_{n=N+1}^{\infty} |x_n|^p < \frac{\epsilon}{2}$ for some $N \in \mathbb{N}$. For every $n \in \{1, \ldots, N\}$ there exists $r_n \in \mathbb{Q}$ such that $|x_n - r_n|^p < \frac{\epsilon}{2N}$. Note that $r := (r_1, \ldots, r_N, 0, 0, \ldots) \in A$. By Minkowsky's inequality, we have $\|r - x\|_p^p < \frac{\epsilon}{2} + \sum_{n=1}^{N} \frac{\epsilon}{2N} = \epsilon$. Hence, $\|r - x\|_p < \epsilon^p < \epsilon$.

8.35 No. For example, $\sum_{n=1}^{\infty} \frac{u_n}{n^2}$, $\sum_{n=1}^{\infty} \frac{u_{2n}}{(2n)^2}$ and $\sum_{n=1}^{\infty} \frac{u_{2n-1}}{(2n-1)^2}$ are linearly dependent.

8.36 Suppose that B is a Schauder basis of X and pick any $x \in X$. Choose a unique sequence of scalars $\{k_n\}$ such that $x = \sum_{n=1}^{\infty} k_n u_n$ and write $x_n := \sum_{i=1}^{n} k_i u_i$ for all $n \in \mathbb{N}$. Then $\{x_n\}$ is a sequence in $span(B)$ and $x_n \longrightarrow x$. Hence, $x \in \overline{span(B)}$.

Conversely, suppose $\overline{span(B)} = X$ and let $x \in X$. Then there exists a sequence $\{b_n\} \subset span(B)$ such that $b_n \longrightarrow x$. Write $a_1 := 1$ and $a_n := b_n - b_{n-1}$ for all $n > 1$. Then $\sum_{n=1}^{\infty} a_n = x$. Since each $b_n \in span(B)$, we have $a_n \in span(B)$ for all $n \in \mathbb{N}$.

8.37 Let $S := \{m + n\sqrt{2} : m, n \in \mathbb{Z}\}$. Note that $m + n\sqrt{2} = m' + n'\sqrt{2} \iff m = m'$ and $n = n'$. For $m \in \mathbb{Z}$, let $t_m := m\sqrt{2} - [m\sqrt{2}]$, where $[x]$ denotes the greatest integer less than or equal to x. Note that $t_m = t_{m'} \iff m = m'$. Hence, $T := \{t_m : m \in \mathbb{Z}\}$ is an infinite subset of $[0, 1] \cap S$.

Let a and b be any reals such that $a < b$ and $\epsilon \in (0, b - a)$. Pick any $n \in \mathbb{N}$ such that $1/n < \epsilon$ and write $x_i := i/n$ for all $i = 0, 1, \ldots, n$. Since T is an infinite subset of $[0, 1]$, there exists some $i \in \{0, 1, \ldots, n\}$ such that $[x_{i-1}, x_i]$ contains two different $t_m, t_{m'} \in T$. Hence, $t := |t_m - t_{m'}| \le 1/n < \epsilon$. Thus, $0 < t < \epsilon$ and $t \in T \subset S$.

Since $a \in \mathbb{R} \subset \dot{\bigcup}_{m \in \mathbb{Z}} [(m-1)t, mt)$, we have $a \in [(m-1)t, mt)$, for a unique $m \in \mathbb{Z}$. We claim that $mt < b$. If not, then $b - a \le mt - (m-1)t = t < \epsilon$, a contradiction. Therefore, there exists $nt \in S$ such that $nt \in (a, b)$. The generalization is analogous.

8.38 The conclusion follows from Exercise 8.37. To prove the main part, let G be any subgroup of $(\mathbb{R}, +)$. Define $a := \inf\{|x| : x \in G \setminus \{0\}\}$.

In case $a > 0$, we claim that $G = \{ma : m \in \mathbb{Z}\}$. To establish this, let $g \in G$. Then $ka \le |g| < (k+1)a$ for some $k \in \mathbb{Z}$. Therefore, $\big||g| - ka\big| = |g| - ka < a$ and $|g| - ka \in G$. Hence, by the definition of a, we have $|g| - ak = 0$ which implies $g = \pm ka$. Consequently, $G \subset \{ma : m \in \mathbb{Z}\}$. The opposite inclusion is obvious.

Now suppose that $a = 0$. Let $x \in \mathbb{R}$ and $\epsilon > 0$ be given. Since $a = 0$, there exists some $y \in (0, \epsilon) \cap G$. Then $ky \le x < (k+1)y$ for some $k \in \mathbb{Z}$. Note that $ky \in G$ and $0 \le x - ky < (k+1)y - ky = y < \epsilon$. Hence, $x \in \overline{G}$.

8.44 No, as compact metric spaces are complete.

8.45 Apply $(A \cup B)' = A' \cup B'$ for the first part. For the second part, note that $\{1\} = [0, 1] \cap [1, 2]$ is not perfect. The result is false for infinite unions, e.g. $\bigcup_{n=1}^{\infty} [1/n, 1]$ is not even closed.

8.46 Apply Corollary 8.27.

8.47 By hypothesis, we have $E \subset E'$. Since E' is closed, we have $(E')' \subset E'$. Hence

$$(\overline{E})' = (E \cup E')' = E' \cup E'' = E' = E' \cup E = \overline{E}.$$

8.48 Apply Exercise 8.47.

8.49 The answer is negative for both questions. Below we refute (a) only.

Assume that $a_n \longrightarrow a$ in X and $a_1 \neq a$. Then $B(a; d(a, a_1)/2)$ contains all but finitely many terms of $\{a_n\}$. Hence, $B(a_1; d(a, a_1)/2)$ contains only finitely terms of $\{a_n\}$. Consequently, a_1 is an isolated point of X, a contradiction.

8.51 Let E denote the given set, as in Exercise 3.54, we obtain $\overline{E} = [0, 1] = E'$. Since E is countable and E' is uncountable, we have $E \neq E'$.

8.52 Yes. Let $X := \mathbb{R} \setminus \mathbb{Q}$ under usual metric and $P := (0, 1) \setminus \mathbb{Q}$.

8.54 Using induction, it is enough to prove the result for $n = 2$. Since $O \neq \emptyset$, it can't be a single point, as X is perfect. Let $x, y \in O$ such that $x \neq y$. Let d be the distance between x and y. By Theorem 6.54, there exists some clopen set A such that $x \in A \subset B(x; d)$. Then $O_1 := O \cap A$ and $O_2 := O \setminus A$ satisfy our requirement.

8.56 We prove the results for $r > 0$. The case of $r = 0$ is similar.

(a) Let $a = \{a_n\} \in X$ be arbitrary. Write $x_n = \{x_n(m)\}_m$; where

$$x_n(m) := \begin{cases} a_m & ; m \neq n, \\ 1 - a_m & ; m = n. \end{cases}$$

Then $\{x_n\}$ is a sequence from the set $X \setminus \{a\}$, convergent to a. Hence, a is a limit point of X.

(b) We will prove that (X, ρ_r) is sequentially compact. Let $\{x_n\}$ be any sequence in X. Write $x_n = \{x_n(m)\}_m$ for all n.

Let $a_1 \in \{0, 1\}$ such that a_1 repeats infinitely many times in the sequence $\{x_n(1)\}_n$. Assume that $a_1, \ldots, a_n$ have been chosen. Let $a_{n+1} \in \{0, 1\}$ such that a_{n+1} repeats infinitely many times in the set $\{x_k(n+1) : k \in \mathbb{N}, x_k(i) = a_i, i = 1, \ldots, n\}$.

Therefore, by induction we have chosen a sequence $a := \{a_n\} \in X$. For each $k \in \mathbb{N}$, let $n_k = \min\{m : x_m(k) = a_k\}$. It can be shown that $x_{n_k} \longrightarrow a$ in X. By Corollary 8.4, X is separable.

(c) Suppose that $s \in (0, \infty)$ is a limit point of S_x. One can choose a sequence $\{s_n\}$, from $S_x \setminus \{s\}$, convergent to s. Also, each $s_n = \rho_r(y_n, x)$ for some $y_n \in X$. By (b), X is compact. So there exists $y_0 \in X$ and a subsequence $\{y_{n_k}\}$ of $\{y_n\}$ such that $y_{n_k} \longrightarrow y_0$. Therefore,

$$\rho_r(y_0, x) = \lim_{k\to\infty} \rho_r(y_{n_k}, x) = \lim_{k\to\infty} s_{n_k} = s.$$

Since $s > 0$ and $\rho_r(y_{n_k}, y_0) \longrightarrow 0$, for all sufficiently large k we obtain

$$s = \rho_r(y_0, x) \leq \max\{\rho_r(y_0, y_{n_k}), \rho_r(y_{n_k}, x)\} \leq s_{n_k}$$
$$\text{and } s_{n_k} = \rho_r(y_{n_k}, x) \leq \max\{\rho_r(y_{n_k}, y_0), \rho_r(y_0, x)\} \leq s.$$

So $s_{n_k} = s$, a contradiction to our choice of the sequence $\{s_n\}$. Hence, $S'_x \cap (0, \infty) \neq \emptyset$.

(d) By (a), and (c), $S'_x = \{0\}$. Assume that S_x is uncountable. Since $S_x = \bigcup_{n=1}^{\infty}(S_x \cap [1/n, n])$, there exists $k \in \mathbb{N}$ such that $S_x \cap [1/k, k]$ is uncountable and hence is an infinite subset of $[1/k, k]$. Thus, S_x has a limit point in $[1/k, k] \subset (0, \infty)$, a contradiction to (c).

8.57 (a) Apply Example 8.25 along with the fact that every normed linear space is path connected and hence connected.

(b) Use (a) and that every nonempty convex set is path connected and hence connected.

8.63 $\mathbb{Q}$ is not nowhere dense, but totally disconnected.

8.64 Suppose not. Then $\mathbb{R} = \bigcup_{n=1}^{\infty} A_n$, for a sequence $\{A_n\}$ of nowhere dense subsets of $\mathbb{R}$. In this proof, all the intervals are non-degenerate ones.

Let I_1 be any closed interval disjoint from A_1. Then A_2 is not dense in I_1. So there exists a closed subinterval I_2 of I_1, disjoint from A_2. Continuing like this, choose a nested decreasing sequence $\{I_n\}$ of closed intervals such that $I_n \cap A_n = \emptyset$.

By Nested Interval Property (1.23), there exists some $x \in \bigcap_{n=1}^{\infty} I_n$. Since $x \in \mathbb{R} = \bigcup_{n=1}^{\infty} A_n$, we obtain $x \in A_k$ for some k. Hence, $x \in I_k \cap A_k$, a contradiction.

8.66 We need to prove that $\left(\overline{E}\right)^o = \emptyset$ if and only if $\overline{\left(\overline{E}\right)^c} = X$.

Assume that $\left(\overline{E}\right)^c$ is not dense in X. Let $x \in X \setminus \overline{(\overline{E})^c}$. Then there exists some $\epsilon > 0$ such that $B(x; \epsilon) \cap \left(\overline{E}\right)^c = \emptyset$. Therefore, $B(x; \epsilon) \subset \overline{E}$. Hence, $x \in \left(\overline{E}\right)^o$ which implies $\left(\overline{E}\right)^o \neq \emptyset$.

Conversely, suppose $\left(\overline{E}\right)^o \neq \emptyset$. Let $x \in \left(\overline{E}\right)^o$. Then there exists $\epsilon > 0$ such that $B(x; \epsilon) \subset \overline{E}$. Hence, $B(x; \epsilon) \cap \left(\overline{E}\right)^c = \emptyset$ which implies that $\left(\overline{E}\right)^c$ is not dense in the space X.

8.68 The result is a repeated use of Exercise 8.66.

$(a) \Rightarrow (b)$ Assume that there are dense open subsets U_n of X such that $\bigcap_{n=1}^{\infty} U_n$ is not dense in X. Then there exists a nonempty open subset U of X such that $\bigcap_{n=0}^{\infty} U_n = \emptyset$, where $U_0 = U$. A contradiction can be obtained by imitating the proof of Theorem 8.33.

$(b) \Rightarrow (c)$ Let O be a nonempty open subset of X such that $O = \bigcup_{n=1}^{\infty} E_n$, where each E_n is nowhere dense. Since $\overline{E_n}$ is nowhere-dense, each $\left(\overline{E_n}\right)^c$ is dense in X. Since $O \subset \bigcup_{n=1}^{\infty} \overline{E_n}$, we have $O \cap \left(\bigcap_{n=1}^{\infty} \left(\overline{E_n}\right)^c\right) = \emptyset$. Hence a countable intersection of dense open sets is not dense in X.

$(c) \Rightarrow (d)$ Assume that there are dense open subsets U_n of X such that $\bigcap_{n=1}^{\infty} U_n = \emptyset$. Hence, $U_1 = \bigcup_{n>1}(U_n)^c$ is of the first category. and each U_n^c is nowhere dense.

$(d) \Rightarrow (a)$ Suppose that $X = \bigcup_{n=1}^{\infty}(U_n)^c$, where each U_n^c is nowhere dense. Therefore, $\bigcap_{n=1}^{\infty} U_n = \emptyset$ and each U_n is dense in X.

8.70 Yes. The interval (0, 1) is also of second category, by Exercise 8.68.

8.71 No. For example, take $X := \mathbb{R}$ and $O_x := \mathbb{R} \setminus \{x\}$ for all $x \in X$.

8.73 (a) Since $E \subset \bigcup_{n=1}^{\infty} E_n$, we have $E = \bigcup_{n=1}^{\infty}(E \cap E_n)$. Applying Baire Category Theorem (8.33), there exists some $N \in \mathbb{N}$ such that $(E \cap E_N)^o \neq \emptyset$. So there exists a non-empty open set O such that $O \subset E \cap E_N$. Hence, $\emptyset \neq E \cap O \subset E_N$.

(b) Follows from (a) and the definition of an F_σ set.

(c) Since $E \subset \bigcup_{n=1}^{\infty} E_n$, we have $E = \bigcup_{n=1}^{\infty}(E \cap \overline{E_n})$. Applying Baire Category Theorem (8.33), there exists some $N \in \mathbb{N}$ and a non-empty open set O such that $O \subset (E \cap \overline{E_N})$. Hence, $O \cap E \subset \overline{E_N}$. So E_N is dense in $E \cap O$.

8.74 Yes. Consider $E := [(-\infty, 0) \setminus \mathbb{Q}] \cup [\mathbb{Q} \cap [0, +\infty)]$. Apply Corollary 8.39(a).

8.78 Let $a < b$ be any two points from I. If I is negligible, then so is $[a, b]$. Hence, there exists a sequence $\{I_n\}$ of open intervals such that $[a, b] \subset \bigcup_{n=1}^{\infty} I_n$ and $\sum_{n=1}^{\infty} l(I_n) < (b - a)/2$.

Since $[a, b]$ is compact, $[a, b] \subset \bigcup_{k=1}^{N} I_k$, for some $N \in \mathbb{N}$. For each k, write $I_k := (a_k, b_k)$. Without loss of generality, we can assume that

$$a_1 < a \leq a_2 < b_1 < a_3 < b_2 < \cdots < a_N < b_{N-1} \leq b < b_N.$$

Therefore, we obtain the following contradiction

$$\begin{aligned} b - a < b_N - a_1 &= \sum_{i=1}^{N}(b_i - a_i) - \sum_{i=1}^{N-1}(b_i - a_{i+1}) \\ &< \sum_{i=1}^{N}(b_i - a_i) \leq \sum_{n=1}^{\infty} l(I_n) \leq \frac{b-a}{2}. \end{aligned}$$

8.79 Yes. If $\{r_n\}$ is an enumeration of $\mathbb{Q}$, then $\mathbb{R} \setminus \bigcup_{n=1}^{\infty}(r_n - 2^{-n-1}, r_n + 2^{-n-1})$ is a closed subset of $\mathbb{R}$ with infinite measure. Hence, it is a complete subspace $\mathbb{R}$. Applying Theorem 8.31, we obtain the required set perfect set.

8.86 Apply Theorem 8.48.

8.87 Yes. Define $f_a(x) := ax$ for all $x \in [0, 1]$, $a \in (0, 1)$. Let $\mathcal{F} := \{f_a : a \in (0, 1)\}$. It can be shown that $\mathcal{F}$ is an uncountable equicontinuous subset of $C[0, 1]$.

8.88 The given condition ensures that f is completely characterized by its values on the interval $[1, 2)$. Note that $(0, \infty) = \dot{\bigcup}_{m=-\infty}^{\infty}[2^m, 2^{m+1})$.

(a) True. Being continuous on $[1, 2]$, we conclude that f is bounded on this interval and hence on $(0, \infty)$.

(b) False. For a counter example, take $f(x) := \sin\big((x+1)\pi\big)$ for all $x \in [1, 2)$ and extend it for all positive reals, using the given condition.
(c) False. Consider the example, as in (b).
(d) False. Let $n \in \mathbb{N}$ and f_n be defined as the piecewise linear function on $[1, 2]$, whose graph is given by joining the points

$$(1, 0), (1 + 2^{-n}, 0), (1 + 3.2^{-n-1}, 1), (1 + 2^{-n+1}, 0) \text{ and } (2, 0).$$

Now extend f_n to $(0, \infty)$ using the given condition. It can be shown that $\{f_n\}$ is a uniformly bounded sequence from $\mathcal{F}$ which has no uniformly convergent subsequence.

8.89 Apply Lemma 8.51 and use the fact that convergence sequences are Cauchy and hence form a totally bounded set.

8.90 Let $\{f_n\}$ be a sequence of bounded functions uniformly convergent to f on a metric space X. For $\epsilon = 1$, there exists $m \in \mathbb{N}$ such that

$$|f_n(x) - f(x)| < 1 \text{ for all } n \geq m \text{ and for all } x \in X.$$

Since f_m is bounded, let $M_m := \sup\{|f_m(x)| : x \in E\}$. Then for all $x \in X$,

$$|f(x)| \leq |f(x) - f_m(x)| + |f_m(x)| < 1 + M_m.$$

Hence, for all $x \in X$, we have

$$|f_n(x)| \leq |f_n(x) - f(x)| + |f(x)| < 2 + M_m \text{ for all } n \geq m.$$

Write $M := \max\{M_1, \ldots, M_m, 2 + M_m\}$, where $M_i := \sup\{|f_i(x)| : x \in E\}$, for each $i = 1, \ldots m$. Then $|f_n| < M$ on X for all $n \in \mathbb{N}$.

8.91 Analogous to the proof of Theorem 8.48.

8.93 The uniform bound is 1, pointwise limit is 0 and $f_n(1/n) = 1$ ensures that no subsequence of $\{f_n\}$ converges uniformly on $[0, 1]$. Now apply Theorem 8.50 to conclude the result.

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Chapter 9
Homeomorphisms

Homeomorphisms are essentially topological isomorphisms. In other words, homeomorphic spaces are same from the topological viewpoint. In Sect. 4.4, we have already discussed isometries, as a particular class of homeomorphisms.

A general discussion on homeomorphisms begins in Sect. 9.16. Until that we strive on some of its particular cases given by topologically, uniformly, and Lipschitz equivalent metrics. The third section of this chapter provides several extension theorems, including the results by Tietze, Kuratowski, and Lavrentiev. Finally, we present the case of normed spaces, particularly the equivalence of all norms on finite-dimensional spaces.

9.1 Equivalent Metrics

In this section, we address the question that if two different metrics are defined on the same space, under what conditions they generate the same class of open sets?

Definition 9.1 Two metrics ρ_1 and ρ_2 on a space X are said to be *equivalent* (or *topologically equivalent*) if they generate the same class of open subsets of X.

We adopt the notation $B_d(x; r)$ for balls in (X, d), instead of $B(x; r)$, whenever we shall be dealing with multiple metrics on the same space.

Examples 9.2 (a) As presented in Exercise 3.12, open balls in any of the metric spaces $(\mathbb{R}^2, d_1)$, $(\mathbb{R}^2, d_2)$ and $(\mathbb{R}^2, d_\infty)$ are open sets in the remaining two metric spaces. Therefore, all these metric spaces have the same the class of open sets. Hence, d_1, d_2 and d_∞ are equivalent metrics on $\mathbb{R}^2$.

(b) In $\mathbb{R}^n$, the discrete metric d_c induces a bigger collection of open sets than any of d_1, d_2 or d_∞. Hence, d_c is not equivalent to any of these metrics on $\mathbb{R}^n$.

Definition 9.3 A metric ρ_1 on a space X is said to be *topologically stronger* than another metric ρ_2 on X, if every open set in (X, ρ_2) is open in (X, ρ_1). In this case, we also say that ρ_2 is *topologically weaker* than ρ_1.

S. P. S. Kainth, *A Comprehensive Textbook on Metric Spaces*,
https://doi.org/10.1007/978-981-99-2738-8_9

Example 9.4 The discrete metric on any space X is topologically stronger than any other metric on X.

Now we present some characterizations of topologically stronger metrics. These lead to characterizations of topologically equivalent metrics. A few other such characterizations will be provided in Exercise 9.9.

Theorem 9.5 *Let ρ_1 and ρ_2 be metrics on X. Then the following are equivalent:*

(a) Every open subset of (X, ρ_2) is open in (X, ρ_1).
(b) Every open ball of (X, ρ_2) contains an open ball of (X, ρ_1) with same center.
(c) Every convergent sequence in (X, ρ_1) is convergent in (X, ρ_2).

Proof $((a) \Rightarrow (b))$ Let $x \in X$ and $r > 0$ be arbitrary. Since $B_{\rho_2}(x; r)$ is open in (X, ρ_2), it is open in (X, ρ_1). Therefore, x is its interior point in (X, ρ_1). Hence, there exists some $r' > 0$ such that $B_{\rho_1}(x; r') \subset B_{\rho_2}(x; r)$.

$((b) \Rightarrow (c))$ Consider a sequence $\{x_n\}$ such that $x_n \longrightarrow x$ in (X, ρ_1). To prove that $x_n \longrightarrow x$ in (X, ρ_2), pick any $\epsilon > 0$. By (b), let $\epsilon_0 > 0$ be such that $B_{\rho_1}(x; \epsilon_0) \subset B_{\rho_2}(x; \epsilon)$. Since $x_n \longrightarrow x$ in (X, ρ_1), there exists some $N \in \mathbb{N}$ such that $x_n \in B_{\rho_1}(x; \epsilon_0) \subset B_{\rho_2}(x; \epsilon)$ for all $n \geq N$. Hence, $x_n \longrightarrow x$ in (X, ρ_2).

$((c) \Rightarrow (a))$ Let O be an open subset of (X, ρ_2). Then $F := X \setminus O$ is closed in (X, ρ_2). Let x be a limit point of F in (X, ρ_1). Choose a sequence $\{x_n\}$ from F such that $x_n \longrightarrow x$ in (X, ρ_1). By (c), $x_n \longrightarrow x$ in (X, ρ_2). Since F is closed in (X, ρ_2), we have $x \in F$. Therefore, $F = X \setminus O$ is closed in (X, ρ_1). Hence, O is open in (X, ρ_1). □

Remark 9.6 Since equivalent metrics admit the same class of open sets, they admit same convergent sequences. However, they do not always admit same Cauchy sequences or bounded sets. The completeness property is also not shared by equivalent metrics, in general.

Example 9.7 Let $X := (0, 1]$. Then

$$d(x, y) := \left|\frac{1}{x} - \frac{1}{y}\right| \text{ for all } x, y \in X$$

defines a metric on X, equivalent to the usual metric on X. The sequence $\{\frac{1}{n}\}$ is Cauchy under the usual metric but is neither Cauchy nor bounded in (X, d). Further X is complete under d, but not under the usual metric.

Proof Let ρ denote the usual metric on X. It is trivial to see that d is a metric on X and $\{\frac{1}{n}\}$ is a Cauchy sequence under the usual metric but neither Cauchy nor bounded in (X, d). To prove the equivalence of the metrics, pick any $x > 0$ and $r \in \left(0, \frac{1}{x}\right)$. Then

$$B_d(x; r) = \left\{y \in X : \left|\frac{1}{y} - \frac{1}{x}\right| < r\right\} = \left(\frac{x}{1 + xr}, \frac{x}{1 - xr}\right).$$

Hence, for every $x > 0$ and $r \in \left(0, \frac{1}{x}\right)$, it can be shown that

$$B_\rho\left(x; \frac{x^2 r}{1+xr}\right) \subset B_d(x;r) \subset B_\rho\left(x; \frac{x^2 r}{1-xr}\right).$$

By Theorem 9.5, we conclude that both d and ρ are topologically stronger than each other and hence equivalent. Further, it is clear that X is not complete under the usual metric, as $\{\frac{1}{n}\}$ is Cauchy in (X, ρ) but not convergent. Next, we show that (X, d) is complete.

Let $\{x_n\}$ be a Cauchy sequence in (X, d). Then $\{\frac{1}{x_n}\}$ is Cauchy under the usual metric and hence convergent to some real number, say $y \in \mathbb{R}$. If $y < 1$, then $\frac{1}{x_m} < 1$ for some $m \in \mathbb{N}$, that is $x_m > 1$. This is impossible, as each $x_n \in (0, 1]$. Hence, $y \geq 1$ which implies $\frac{1}{y} \in (0, 1] = X$. Consequently, $x_n \longrightarrow \frac{1}{y}$ in (X, d). □

Uniform continuity is also not preserved by equivalent metrics, in general.

Example 9.8 Let $X := (-\pi/2, \pi/2)$ and for every $x, y \in X$, define

$$\rho_1(x, y) := |x - y| \text{ and } \rho_2(x, y) := |\tan x - \tan y|.$$

Let $f(x) := \tan x$ for all $x \in X$. Then $f : (X, \rho_2) \longrightarrow \mathbb{R}$ is a uniformly continuous function.

If $f : (X, \rho_1) \longrightarrow \mathbb{R}$ is uniformly continuous, then Theorem 5.42 implies that f is a bounded function, a contradiction. Therefore, $f : (X, \rho_1) \longrightarrow \mathbb{R}$ is not uniformly continuous.

Definition 9.9 Let ρ_1 and ρ_2 on X and $f : (X, \rho_1) \longrightarrow (X, \rho_2)$ denote the identity map. Then ρ_1 and ρ_2 are said to be

(a) *uniformly equivalent* if both f and f^{-1} are uniformly continuous.
(b) *Lipschitz equivalent* if both f and f^{-1} are Lipschitz continuous.

Note that $f^{-1} : (X, \rho_2) \longrightarrow (X, \rho_1)$ is also an identity map. Further ρ_1 and ρ_2 are *Lipschitz equivalent* if and only if there are positive constants α and β such that

$$\alpha\rho_2(x, y) \leq \rho_1(x, y) \leq \beta\rho_2(x, y) \text{ for all } x, y \in X.$$

Since Lipschitz continuity implies uniform continuity, which in turn implies continuity, it follows that Lipschitz equivalent metrics are uniformly equivalent, and uniformly equivalent metrics are equivalent.

Therefore, Lipschitz equivalent metrics share all the properties of uniformly equivalent metrics, and uniformly equivalent metrics share all the properties of topologically equivalent metrics. However there are some additional properties in both of these cases.

Theorem 9.10 *Uniformly equivalent metrics share the same classes of Cauchy sequences, totally bounded sets, and complete subspaces.*

Proof Let ρ_1 and ρ_2 be two uniformly equivalent metrics on a space X. By Theorem 5.42, it follows that ρ_1 and ρ_2 share the same class of Cauchy sequences and totally bounded sets.

Let Y be a complete subspace of (X, ρ_1). To see that (Y, ρ_2) is complete, let $\{x_n\}$ be a Cauchy sequence in (Y, ρ_2). By Theorem 5.42, $\{x_n\}$ is also a Cauchy sequence in (Y, ρ_1), which is complete. Let $x \in Y$ be such that $x_n \longrightarrow x$ in (Y, ρ_1). Since ρ_1 and ρ_2 are equivalent, by Theorem 9.5, $x_n \longrightarrow x$ in (Y, ρ_2). Hence the result. □

Uniformly equivalent metrics may have different classes of bounded sets (see Theorem 9.15).

Theorem 9.11 *Lipschitz equivalent metrics share the same class of bounded subsets.*

Proof Let ρ_1 and ρ_2 be Lipschitz equivalent metrics on a space X and E be a bounded subset of (X, ρ_1). Let $M > 0$ such that $\rho_2(x, y) \leq M\rho_1(x, y)$ for all $x, y \in X$.

Let $D := \sup\{\rho_1(x, y) : x, y \in E\}$. Since E is bounded in (X, ρ_1), we have $D < \infty$. Also, $\rho_2(x, y) \leq MD$ for all $x, y \in E$. Consequently, E is bounded in (X, ρ_2). Similarly, every bounded subset of (X, ρ_1) is bounded in (X, ρ_2). □

Example 9.12 In $\mathbb{R}^2$, the metrics d_1, d_2 and d_∞ are Lipschitz equivalent. Hence, these metrics admit the same class of open sets.

Proof Let $x = (x_1, x_2), y = (y_1, y_2) \in \mathbb{R}^2$. Then we have

$$d_1(x, y) \leq 2d_\infty(x, y) \text{ and } d_\infty(x, y) \leq d_1(x, y).$$

A bit of squaring and algebraic manipulations imply that $d_2(x, y) \leq d_1(x, y)$. Applying Cauchy-Schwarz inequality, we conclude that

$$d_1(x, y) = |x_1 - y_1|.1 + |x_2 - y_2|.1 \leq \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2}\sqrt{1^2 + 1^2} = \sqrt{2}d_2(x, y).$$

Consequently,

$$\begin{aligned} d_2(x, y) &\leq d_1(x, y) \leq 2d_\infty(x, y) \\ \text{and } d_\infty(x, y) &\leq d_1(x, y) \leq \sqrt{2}d_2(x, y). \end{aligned}$$

Hence, d_1, d_2 and d_∞ are Lipschitz equivalent. □

Proposition 9.13 *Let (X, d) be any metric space, $f : X \longrightarrow X$ be a bijection and*

$$\rho(x, y) := d(f(x), f(y)) \text{ for all } x, y \in X.$$

Then ρ is a metric on X. Further the metrics d and ρ are

(a) *equivalent on X if and only if both f and f^{-1} are continuous.*
(b) *uniformly equivalent if and only if both f and f^{-1} are uniformly continuous.*

(c) Lipschitz equivalent if and only if both f and f^{-1} are Lipschitz continuous.

Proof Since $f : X \longrightarrow X$ is a bijection, ρ is a metric on X. Further note that the continuity, uniform continuity or the Lipschitz continuity of the identity maps $(X, \rho) \longrightarrow (X, d)$ and $(X, d) \longrightarrow (X, \rho)$ are equivalent to the continuity, uniform continuity or the Lipschitz continuity of f^{-1} and f, respectively. □

There are topologically equivalent metrics that are not uniformly equivalent.

Example 9.14 Let d be the usual metric on $\mathbb{R}$ and $\rho(x, y) := |x^3 - y^3|$ for all $x, y \in \mathbb{R}$. Then d and ρ are equivalent, but not uniformly.

Proof Note that $x \longrightarrow x^3$ is a continuous bijection with continuous inverse. However, it is not uniformly continuous. So, the result follows by Proposition 9.13.

To see that $x \longrightarrow x^3$ is not uniformly continuous, let $\epsilon = 3$. Suppose there exists some $\delta > 0$ such that $|x^3 - y^3| < 3$ for all $|x - y| < \delta$. Let $n \in \mathbb{N}$ such that $1/n < \delta$. Then $(n + 1/n) - n < \delta$ but $(n + 1/n)^3 - n^3 > 3$, a contradiction. □

Our next proposition produces an abundance of examples of uniformly equivalent metrics, which are not Lipschitz equivalent. For example, let $X = \mathbb{R}$ and d be the usual metric in the next proposition. A similar result is provided in Exercise 9.13.

Theorem 9.15 *Let d be a metric on X and*

$$\rho(x, y) := \frac{d(x, y)}{1 + d(x, y)} \text{ for all } x, y \in X.$$

Then

(a) ρ is a metric on X.
(b) ρ is uniformly equivalent to d.
(c) If d is unbounded, then ρ and d are not Lipschitz equivalent.

Proof The function ρ clearly is positive definite and symmetric. To prove the triangle inequality, after usual simplifications we conclude that $\rho(x, y) \le \rho(x, z) + \rho(z, y)$ if and only if

$$d(x, y) \le d(x, z) + d(z, y) + 2d(x, z)d(z, y) + d(x, y)d(x, z)d(z, y).$$

This holds by the triangle inequality for d. Since $\rho \le d$, the identity map $(X, d) \longrightarrow (X, \rho)$ is uniformly continuous. To see that the identity map $(X, \rho) \longrightarrow (X, d)$ is uniformly continuous, let $\epsilon > 0$. Then for $x, y \in X$ such that $\rho(x, y) < \frac{\epsilon}{1+\epsilon}$, we obtain $d(x, y) < \epsilon$. This proves (b).

To prove (c), note that $\rho \le d$ and $\rho \le 1$. If there exists some $M > 0$ such that $d \le M\rho$, then d is bounded by M, a contradiction if d is given to be unbounded. Hence the result. □

In Exercises 10.39-10.42, we shall discuss two uniformly equivalent metrics on countable product of metric spaces, which are not Lipschitz equivalent. Also, see Exercise 9.14.

Exercise 9.1 Let (X, d) be a metric space. Is $\sqrt{d}$ a metric equivalent to d?

Exercise 9.2 Prove that every metric on a finite set is equivalent to the discrete metric.

Exercise 9.3 Let $\{x_k\}$ be a sequence in $\mathbb{R}^n$ and $x \in \mathbb{R}^n$ be such that

$$x_k = (x_k[1], \ldots, x_k[n]) \text{ and } x = (x[1], \ldots, x[n]).$$

Prove that the following are equivalent:

(a) $x_k \longrightarrow x$, in $(\mathbb{R}^n, d_\infty)$,
(b) $x_k \longrightarrow x$, in $(\mathbb{R}^n, d_1)$ and
(c) $x_k[i] \longrightarrow x[i]$ for all $1 \le i \le n$.

Exercise 9.4 Let (X, d) be a metric space. Prove that the following are equivalent.

(a) X is equivalent to a discrete metric space.
(b) Every convergent sequence in X is eventually constant.
(c) Every subspace of X is a complete metric space.

Exercise 9.5 Generalize Example 9.12 for $\mathbb{R}^n$, for all $n \in \mathbb{N}$.

Exercise 9.6 If X is a metric space, prove that the topological, uniform, and Lipschitz equivalences are equivalence relations on the family of metrics on X.

Exercise 9.7 Let $f : (X, d_X) \longrightarrow (Y, d_Y)$ be continuous. Does there exist a metric ρ on X, equivalent to d_X, such that $f : (X, \rho) \longrightarrow (Y, d_Y)$ is Lipschitz continuous?

Exercise 9.8 Prove or disprove:

(a) $(0, 1)$ with usual topology admits a complete metric.
(b) $[0, 1]$ with usual topology admits a metric that is not complete.

Exercise 9.9 Let ρ, ρ_1 and ρ_2 be metrics on a space X. Prove that the following are equivalent:

(a) Every open subset of (X, ρ_2) is open in (X, ρ_1).
(b) Every closed subset of (X, ρ_2) is closed in (X, ρ_1).
(c) Every open ball of (X, ρ_2) contains an open ball of (X, ρ_1) with same center.
(d) The identity map $(X, \rho_1) \longrightarrow (X, \rho_2)$ is continuous.
(e) Every convergent sequence in (X, ρ_1) is convergent in (X, ρ_2).
(f) If $f : (X, \rho_2) \longrightarrow (Y, \rho)$ is continuous, then so is $f : (X, \rho_1) \longrightarrow (Y, \rho)$.
(g) If $f : (Y, \rho) \longrightarrow (X, \rho_1)$ is continuous, then so is $f : (Y, \rho) \longrightarrow (X, \rho_2)$.

Deduce characterizations of equivalent metrics, implied by of the above statements.

Exercise 9.10 Let ρ_1 and ρ_2 be two topologically equivalent metrics on a space X. Prove that the spaces (X, ρ_1) and (X, ρ_2) admit the same classes of open sets, closed sets, dense sets, compact sets, connected sets, convergent sequences and their limits, continuous functions with domain X, and continuous functions with X as codomain.

Exercise 9.11 If d and ρ are metrics on X such that d is topologically stronger than ρ. If (X, d) is compact, then d and ρ are equivalent.

Exercise 9.12 Let ρ_1 and ρ_2 be uniformly equivalent metrics on X. Prove that the spaces (X, ρ_1) and (X, ρ_2) admit the same

(a) uniformly continuous functions with domain X,
(b) uniformly continuous functions with X as codomain.

Exercise 9.13 If d is a metric on X and $\eta(x, y) := \min\{1, d(x, y)\}$ for all $x, y \in X$, prove that

(a) η is a metric on X,
(b) η is a uniformly equivalent to d, and
(c) η and d are not Lipschitz equivalent, provided d is unbounded.

Exercise 9.14 Let d denote the usual metric on $\mathbb{R}$ and

$$\rho(x, y) := d\left(\frac{x}{1+|x|}, \frac{y}{1+|y|}\right) \text{ for all } x, y \in \mathbb{R}.$$

Prove that ρ is a metric on $\mathbb{R}$, equivalent to d but not uniformly equivalent to d.

Exercise 9.15 Let (X, d) be a metric space, η and ρ be the metrics on X, as defined in Exercise 9.13 and Theorem 9.15, respectively. Write explicit formulas, in terms of center and radii, to show that

(a) every ball of (X, η) contains a ball of (X, d) with same center and vice versa.
(b) every ball of (X, ρ) contains a ball of (X, d) with same center and vice versa.

Exercise 9.16 Let (X, d) be a metric space, $y \notin X$ and $Y := X \cup \{y\}$. Define

$$\rho(a, b) := \begin{cases} \min\{1, d(a, b)\} & ; a, b \in X, \\ 0 & ; a = b = y, \\ 1 & ; \text{otherwise}. \end{cases}$$

Prove that ρ is a metric on Y and is equivalent to d, on X.

Exercise 9.17 Let d and d' be Lipschitz equivalent metrics on a space X. Prove that (X, d) is complete if and only if (X, d') is complete. Conclude that $(\mathbb{R}^n, d_1)$ and $(\mathbb{R}^n, d_\infty)$ are also complete metric spaces.

Exercise 9.18 Let ρ_1 and ρ_2 be two Lipschitz equivalent metrics on a space X. Prove that the spaces (X, ρ_1) and (X, ρ_2) admit the same class of

(a) Lipschitz continuous functions with domain X.
(b) Lipschitz continuous functions with X as codomain.

Exercise 9.19 Let (X, d) be any complete metric space and $r > 0$. Prove that there exists a complete metric ρ on X, equivalent to d and bounded by r.

Exercise 9.20 Let A and B be subsets of a metric space (X, d) such that

$$\inf\{d(a, b) : a \in A, b \in B\} > 0.$$

If ρ is a metric on X equivalent to d, then is $\inf\{\rho(a, b) : a \in A, b \in B\} > 0$?

Exercise 9.21 Does there exist an incomplete space having an equivalent complete metric?

Exercise 9.22 Does there exist a metric d on $\mathbb{R}$ such that $(\mathbb{R}, d)$ is incomplete?

Exercise 9.23 Prove that two metrics admit same class of Cauchy sequences if and only if they admit same class of totally bounded sets.

Exercise 9.24 If $a, b \in \mathbb{R}^* = [-\infty, +\infty]$ such that either of a and b is finite, define

$$\rho_1(a, b) := \min\{1, |a - b|\} \text{ and } \rho_2(a, b) := \frac{|a - b|}{1 + |a - b|}.$$

If $a, b \in \{-\infty, +\infty\}$, let $\rho_1(a, b) := \rho_2(a, b) = 0$ if $a = b$; and 1 if $a \neq b$. For $i = 1, 2$, prove that

(a) ρ_i is a metric on $\mathbb{R}^*$ and equivalent to the usual metric on the induced space $\mathbb{R}$.
(b) the singletons $\{-\infty\}$ and $\{+\infty\}$ are open as well as closed in $(\mathbb{R}^*, \rho_i)$.
(c) The collection of open subsets of $(\mathbb{R}^*, \rho_i)$ is given by $\big\{O \cup A : A \subset \{-\infty, +\infty\},$ O is open in $\mathbb{R}\big\}$.
(d) $(\mathbb{R}^*, \rho_i)$ is neither compact nor connected.
(e) If $\{x_n\}$ converges to either $-\infty$ or $+\infty$ in $(\mathbb{R}^*, \rho_i)$, then it is eventually constant.
(f) $(\mathbb{R}^*, \rho_i)$ is complete.
(g) $(\mathbb{R}^*, \rho_i)$ is separable.
(h) $(\mathbb{R}^*, \rho_i)$ is not totally bounded.

Exercise 9.25 Let ρ_1 and ρ_2 be metric on X and $\alpha, \alpha_1, \alpha_2 \in (0, +\infty)$ such that for all $x, y \in X$ satisfying $\rho_2(x, y) < \alpha$, we have

$$\alpha_1\rho_1(x, y) < \rho_2(x, y) < \alpha_2\rho_1(x, y).$$

Prove that ρ_1 and ρ_2 are uniformly equivalent and share the same collection of bounded sets.

Exercise 9.26 Let x, y be arbitrary reals with infinite decimal expansions $x = x_0.x_1 \ldots x_n \ldots$ and $y = y_0.y_1 \ldots y_n \ldots$. Define

$$d(x, y) := \begin{cases} 10^{-m} & ; m := \min\{k : x_k \neq y_k\}, x \neq y, \\ 0 & ; x = y. \end{cases}$$

Is d a metric on $\mathbb{R}$? Is it Lipschitz or uniformly equivalent to the usual metric on $\mathbb{R}$?

Exercise 9.27 Let (X, ρ_r) be the ultrametric space of Example 2.5 and $0 \leq r < s < 1$ be arbitrary. Prove that ρ_r and ρ_s are uniformly equivalent, but not Lipschitz equivalent.

Exercise 9.28 If two metrics on a space have same class of compact sets, are they equivalent?

Exercise 9.29 If two metrics d and d' on a space X have same class of Cauchy sequences and complete subspaces, are they equivalent?

Exercise 9.30 If a metric space is an open subset of its completion, prove that it has an equivalent complete metric.

9.2 Homeomorphisms

Definition 9.16 Let X and Y be metric spaces. Then X, Y are said to be *homeomorphic* if there exists a continuous bijection $f : X \longrightarrow Y$ such that f^{-1} is also continuous. In this case, f is known as a *homeomorphism* from X onto Y.

Functions which carry open sets onto open sets are also known as *open maps*. Therefore, a homeomorphism is a bijection which is continuous as well as open.

Examples 9.17 (a) If $a < b$ and $c < d$ are reals, then the linear mapping $x \longmapsto c + (x - a)\frac{d-c}{b-a}$ is a homeomorphism from $[a, b]$ onto $[c, d]$.

(b) The unit interval $[0, 1]$ is not homeomorphic to $\mathbb{R}$. To see this, suppose $f : [0, 1] \longrightarrow \mathbb{R}$ is a homeomorphism. Then f is a continuous surjective function. By Theorem 5.31, $\mathbb{R} = f([0, 1])$ is compact, a contradiction.

(c) If d_1 and d_2 are topologically equivalent metrics on a space X, then (X, d_1) and (X, d_2) are homeomorphic, with the identity map as a homeomorphism.

(d) By Theorem 5.34, every continuous bijection from a compact metric space onto a metric space is a homeomorphism.

Example 9.18 Every isometry is a homeomorphism, while the converse is not true. For For example, the mapping $x \longmapsto 2x$ from $[0, 1]$ onto $[0, 2]$ is a homeomorphism, but not an isometry.

Proposition 9.19 (Stereographic Projection) *Let* $n \in \mathbb{N}$ *and* $S^n := \{x \in \mathbb{R}^{n+1} : \|x\|_2 = 1\}$ *be the unit sphere in* $\mathbb{R}^{n+1}$. *Let* $N := (0, \ldots, 0, 1)$ *and* $S := S^n \setminus \{N\}$. *Then the spaces* S *and* $\mathbb{R}^n$ *are homeomorphic.*

Proof Let $f : S \longrightarrow \mathbb{R}^n$ be defined as

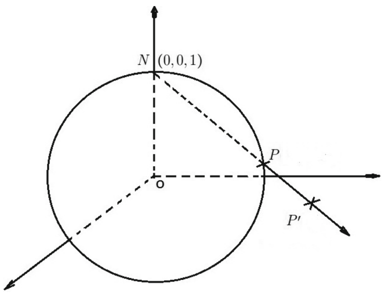
$$f(x) := \left(\frac{x_1}{1 - x_{n+1}}, \dots, \frac{x_n}{1 - x_{n+1}} \right) \text{ for all } x = (x_1, \dots, x_{n+1}) \in S.$$

Note that f is a bijection and its inverse $g : \mathbb{R}^n \longrightarrow S$ is given by

$$g(y) := \left(\frac{2y_1}{\|y\|_2^2 + 1}, \dots, \frac{2y_n}{\|y\|_2^2 + 1}, \frac{\|y\|_2^2 - 1}{\|y\|_2^2 + 1} \right) \text{ for all } y = (y_1, \dots, y_n) \in \mathbb{R}^n.$$

It can be shown that both f and g are continuous maps. Hence the result. □

Remark 9.20 For $n = 2$, the above homeomorphism has the following geometric interpretation for the one-to-one correspondence between $\mathbb{R}^2$ and S (in $\mathbb{R}^3$).



Let P be any point on S and P' be the point of intersection of the line passing through N and P with the plane $\mathbb{R}^2$.

Similarly, given any point Q' on the plane $\mathbb{R}^2$, let Q be the point of intersection of the line passing through N and Q' with S.

We leave it for the reader to verify that $f(P) := P'$ and $g(Q') := Q$.

For $n > 2$, analogous geometric interpretation of the stereographic projection may be given.

Proposition 9.21 *Any bijection between two metric spaces, which also induces a bijection between their bases, is a homeomorphism.*

Proof Let X, Y be metric spaces with basis B_X and B_Y, respectively. Suppose that $f : X \longrightarrow Y$ is a bijection which induces a bijection from B_X onto B_Y.

Let $x \in X$ be arbitrary, and $V \in B_Y$ be an open neighborhood of $f(x)$. Then $x \in f^{-1}(B_Y) \in B_X$, and hence $f^{-1}(B_Y)$ is an open neighborhood of x. Thus, f is continuous at x, and hence f is continuous. Similarly, f^{-1} is also continuous. □

Our next result provides a topological characterization of the space $\mathbb{Q}$ of rational numbers.

Theorem 9.22 (Sierpiński, 1920) *Any two countable perfect metric spaces are homeomorphic.*

Proof (Dashiell, [5]) Let (X, d_X) and (Y, d_Y) be countable perfect metric spaces. Enumerate X and Y as sequences of distinct terms, say $X = \{x_n : n \in \mathbb{N}\}$ and $Y = \{y_n : n \in \mathbb{N}\}$. Let

$$D := \{d_X(x_n, x_m) : n \neq m\} \cup \{d_Y(y_n, y_m) : n \neq m\}.$$

Then D is a countable subset of positive reals. Let S be an arbitrary nonempty open subset of X. Define $x(S) := x_k$, where $k = \min\{i : x_i \in S\}$. For $r > 0$, write

$$B(S; r) := B_X\big(x(S); r\big) \text{ and } G(S; r) := S \setminus B(S; r).$$

We *claim* that there exists an arbitrary small $r > 0$ such that $B(S; r)$ and $G(S; r)$ split S into two nonempty disjoint open sets.

To see this, write $x := x(S)$. Since x is not isolated, there exists $x' \in S$ such that $x' \neq x$. Pick any $r \in (0, d_X(x, x')) \setminus D$ such that $B(x; r) \subset S$. Then $x' \in S \setminus B(x; r) = G(S; r)$. Since $r \notin D$, we have $G(S; r) = \{s \in S : d_X(s, x) > r\}$ which is open and nonempty. This proves our claim.

Analogously we treat nonempty open subsets of Y, and write $y(S)$ in place of $x(S)$.

We will use such a splitting to generate sequences of partitions of X and Y indexed by W_G. Here W_G is the set of words consisting of letters B and G whose first letter is always G; that is $W_G := \{G, GG, GB, GGG, GGB, GBG, GBB, \dots\}$. For $t \in W_G$, its *length* $|t|$ is the number of letters in t, and t is called *blue* (respectively, *green*) if its last letter is B (respectively, G); tB and tG extend t by letters B and G, respectively.

Let $U_G := X$ and $V_G := Y$. Suppose that open sets U_t and V_t are defined for all $t \in W_G$ with $|t| = n$, for some $n \in \mathbb{N}$. Note that there are 2^{n-1} words $t \in W_G$ with $|t| = n$. By our previous claim, one can choose a single $r_{n+1} \in \big(0, \frac{1}{n+1}\big)$ to split U_t and V_t for all such t. For all $t \in W_G$ with $|t| = n$, define

$$U_{tB} := B(U_t; r_{n+1}) \text{ and } U_{tG} := G(U_t; r_{n+1})$$
$$V_{tB} := B(V_t; r_{n+1}) \text{ and } V_{tG} := G(V_t; r_{n+1}).$$

Note that if $x \in X$ is equal to $x(U_t)$ for some green t, then it becomes the center of the balls U_s, for all $s \in \{tB, tBB, tBBB, \dots\}$; and these form a decreasing sequence of balls with center x and radii $\longrightarrow 0$. Also, each $x \in X$ is equal to x_{U_t} for some green t. Hence, each $x \in X$ is equal to $x(U_t)$ for a unique green t. It also ensures that $\{U_t : t \in W_G\}$ is a base for open subsets of X. Similar results hold for $\{V_t : t \in W_G\}$.

Next, we *claim* that the desired homeomorphism $f : X \longrightarrow Y$ is given by

$$f(x(U_t)) := y(V_t) \text{ for all green } t \in W_G.$$

From our previous observation, it is immediate that f is a bijection from X onto Y. Applying Proposition 9.21, it is enough to show that $f(U_t) = V_t$ for every $t \in W_G$. To see this, let $y \in V_t$. Then $y = y(V_{tt'})$, for some $tt' \in W_G$. Hence, $y = y(V_{tt'}) = f(x(U_{tt'})) \subset f(U_t)$. Therefore, $V_t \subset f(U_t)$. Similarly, $f(U_t) \subset V_t$. This proves the result. □

Corollary 9.23 *Every countable metric space is homeomorphic to a subspace of the space of rational numbers.*

Proof Let X be a countable metric space. If $X' \neq \emptyset$, then X' is a perfect countable metric space, and hence homeomorphic to $\mathbb{Q} \cap (0, 1/2)$. Also, the set of isolated points of X, that is $X \setminus X'$, is homeomorphic to a subset of $\mathbb{N}$. Consequently, $X = X' \cup (X \setminus X')$ is homeomorphic to a subspace of $(\mathbb{Q} \cap (0, 1/2)) \cup \mathbb{N}$, and hence to a subspace of $\mathbb{Q}$. □

History Notes 9.24 The Sierpiński's theorem (9.22) was published by Wacław Sierpiński in 1920 in the first volume of Fundamenta Mathematicae (see [1]). Since then various proofs of this result appeared in the literature, e.g. [2, p. 370], [3, p. 318] and [4]. However, there had been no complete self-contained metric space proof of this theorem, which can be given in a course in real analysis until 2021. For more details, the reader is referred to [5].

Definition 9.25 A property P of metric spaces is said to be a *topological property* if it is invariant under homeomorphisms. That is, if a space X has property P, then every space homeomorphic to X also possesses the property P.

None of completeness, boundedness, or total boundedness is a topological property.

Examples 9.26 The mapping $x \longrightarrow \tan \frac{\pi x}{2}$ is a homeomorphism from $(-1, 1)$ onto $\mathbb{R}$. Therefore, $(-1, 1)$ is homeomorphic to $\mathbb{R}$.

(a) Note that $(-1, 1)$ is bounded and totally bounded, while $\mathbb{R}$ is not even bounded. Therefore, neither boundedness nor total boundedness is a topological property.
(b) Further $(-1, 1)$ is incomplete, while $\mathbb{R}$ is complete. Hence, completeness is also not a topological property.

Whether two given topological spaces are homeomorphic, is a fundamental problem in topology. To establish that spaces are not homeomorphic, it is enough to find a topological property which is not shared by them.

Examples 9.27 Compactness is a topological property, since continuous image of every compact set is compact (see Theorem 5.31). Similarly, by Theorems 6.9 and 6.20, path connectedness and connectedness are also topological properties. Hence,

(a) $\mathbb{R}$ is not homeomorphic to $[0, 1]$, since $[0, 1]$ is compact while $\mathbb{R}$ is not.
(b) $(0, 1)$ is not homeomorphic to $\mathbb{R} \setminus \{0\}$, as $(0, 1)$ is connected while $\mathbb{R} \setminus \{0\}$ is not connected.

Next, we show that real normed spaces which are isometric as metric spaces (via an isometry fixing the origin) are actually isometric as normed spaces. It is an important result which lead to the development of Lipschitz geometry.

Theorem 9.28 (Mazur-Ulam, 1932) *Let X and Y be normed spaces over $\mathbb{R}$ and $T : X \longrightarrow Y$ be a surjective isometry such that $T(0) = 0$. Then T is linear.*

Proof ***(Nica, [6])*** Since T is continuous, it is enough to show that

$$T\left(\frac{x+y}{2}\right) = \frac{T(x)+T(y)}{2} \text{ for all } x, y \in X.$$

Fix $x, y \in X$. For every surjective isometry $F : X \longrightarrow Y$ with $F(0) = 0$, define its defect as

$$def(F) := \left\| F\left(\frac{x+y}{2}\right) - \frac{F(x)+F(y)}{2} \right\|.$$

Since F is an isometry, it follows that

$$def(F) \leq \left\| F\left(\frac{x+y}{2}\right) - F(x) \right\| + \left\| F\left(\frac{x+y}{2}\right) - F(y) \right\| = \left\| \frac{x-y}{2} \right\|. \tag{9.1}$$

For each $z \in X$, define $S(z) := T(x) + T(y) - z$ and $T_1(z) := (T^{-1} \circ S \circ T)(z)$. Then S and T_1 are surjective isometries from X onto Y, fixing origin. Further, it can be shown that $T_1(x) = y$, $T_1(y) = x$, and $def(T_1) = 2def(T)$. If $def(T) \neq 0$, iterating like this, we obtain a surjective isometry $T : X \longrightarrow Y$ for which $def(T)$ exceeds $\|\frac{x-y}{2}\|$, a contradiction to (9.1). So $def(T) = 0$, and hence the result. □

Definition 9.29 Two metric spaces X and Y are said to be

(a) *uniformly equivalent* if there exists a uniformly continuous bijection $f : X \longrightarrow Y$ such that f^{-1} is also uniformly continuous.
(b) *Lipschitz equivalent* or *lipeomorphic* if there exists a Lipschitz continuous bijection $f : X \longrightarrow Y$ such that f^{-1} is also Lipschitz continuous. In this case, f is said to be a *lipeomorphism* from X onto Y.

Homeomorphic spaces are also known as *topologically equivalent spaces.* The relationship among these three types of equivalences and the properties preserved by them can be discussed analogous to the case of equivalent metrics of the previous section.

Notes and Remarks 9.30 We conclude this section with a few important results about homeomorphisms. We are omitting their proofs as they are beyond the scope of this textbook.

(a) If $m \neq n$, then $\mathbb{R}^n$ and $\mathbb{R}^m$ are not homeomorphic. The result is trivial if either of m or n is 1 but the general case is not so quick. For details we refer [7].
(b) Recall that compact and connected metric spaces are known as *continuum.* A point of a connected metric space whose complement is disconnected is termed as a *cut point.* The following results can be found in [8]:

   (i) If X is a continuum, then X has exactly two non-cut points if and only if it is homeomorphic to $[0, 1]$.
   (ii) Let X be a continuum such that $X \setminus \{a, b\}$ is not connected, for all distinct $a, b \in X$. Then X is homeomorphic to the circle $\{(x, y) : x^2 + y^2 = 1\}$.

(c) Every ultrametric space can be embedded isometrically in a real Hilbert space (see [9]).

(d) Normed linear spaces X and Y are said to be *linearly homeomorphic (isometric)* if there exists a linear homeomorphism (isometry) from X onto Y. Every separable infinite-dimensional Hilbert space is linearly isometric to the sequence space ℓ^2 (see [10, p. 397, Theorem 22.9]).
Any two separable infinite-dimensional Banach spaces are (topologically) homeomorphic (see [11]). If a separable Banach space X embeds isometrically into another Banach space Y, then X embeds linearly isometrically into Y (see [12, p. 131, Corollary 3.3]). Further, it remains an open problem that whether any two (infinite-dimensional) lipeomorphic separable Banach spaces are linearly homeomorphic? A thorough survey in this direction can be found in [13].

Exercise 9.31 Let $S := \{0\} \cup \{1/n : n \in \mathbb{N} \setminus \{1\}\}$ and $f : \mathbb{N} \longrightarrow S$ be defined as

$$f(1) := 0 \text{ and } f(n) := \frac{1}{n} \text{ for all } n \in \mathbb{N} \setminus \{1\}.$$

Prove that f is continuous, but not a homeomorphism.

Exercise 9.32 Let X and Y be metric spaces and $f : X \longrightarrow Y$ be a bijective function. Prove the following assertions:

(a) f maps open sets onto open sets $\iff x_n \longrightarrow x$, whenever $f(x_n) \longrightarrow f(x)$.
(b) f is a homeomorphism $\iff$ both f and f^{-1} preserve convergent sequences.

Exercise 9.33 Prove that the mapping $x \longmapsto x/(1+|x|)$ from $\mathbb{R}$ onto $(-1, 1)$ is a homeomorphism, but not an isometry.

Exercise 9.34 Prove that separability is a topological property.

Exercise 9.35 Prove that cut points are preserved under homeomorphisms.

Exercise 9.36 Prove that a metric space X is not totally bounded if and only if X has an infinite subspace, homeomorphic to a discrete metric space.

Exercise 9.37 Is any of the following intervals homeomorphic to $\mathbb{R}$:

(a) $(-1, 1)$, (b) $(0, +\infty)$, (c) $[0, 1)$, (d) $[0, +\infty)$?

Exercise 9.38 For the proof of Proposition 9.19, show that f is bijective with inverse g and that both f and g are continuous functions.

Exercise 9.39 Prove that $\{z \in \mathbb{C} \setminus \{1\} : |z| = 1\}$ is homeomorphic to $\mathbb{R}$.

Exercise 9.40 Prove that every open interval is homeomorphic to $\mathbb{R}$.

Exercise 9.41 Prove that any two open balls in $\mathbb{R}^n$ are homeomorphic.

Exercise 9.42 Let $n \in \mathbb{N}$. Prove that $\mathbb{R}$ is homeomorphic to $\mathbb{R}^n$ if and only if $n = 1$.

Exercise 9.43 Prove that circles and ellipses in $\mathbb{R}^2$ are homeomorphic.

Exercise 9.44 Is (0, 1) homeomorphic to $\{(t, t^2) \in \mathbb{R}^2 : t \in \mathbb{R}\}$?

Exercise 9.45 Let $f : X \longrightarrow Y$ be a continuous function. Define

$$G_f := \{(x, f(x)) : x \in X\}.$$

Does there exist a homeomorphism between X and the subspace G_f of $X \times Y$?

Exercise 9.46 Let $f : X \longrightarrow Y$ be a homeomorphism and $A \subset X$. Does it imply that A and $f(A)$ are also homeomorphic?

Exercise 9.47 Let X and Y be metric spaces and $f : X \longrightarrow Y$ be a bijection. Prove that f is a homeomorphism if and only if $f(\overline{E}) = \overline{f(E)}$ for every $E \subset X$.

Exercise 9.48 Prove that every homeomorphism maps connected components onto connected components.

Exercise 9.49 Let $\mathcal{C}$ denote the collection of subspaces of $\mathbb{R}^2$, given by the graphs of the capital letters $\{A, B, C, D, E, H, P, X\}$ of the English alphabet. Prove that no two elements of $\mathcal{C}$ are homeomorphic.

Exercise 9.50 Classify all English alphabets into equivalence classes, given by the homeomorphism relation on their shape as capital letters in $\mathbb{R}^2$.

Exercise 9.51 Let $f : \mathbb{C} \longrightarrow \mathbb{C}$ be a homeomorphism which maps $\{z \in \mathbb{C} : |z| = 1\}$ onto x-axis. Prove that f maps $\{z \in \mathbb{C} : |z| < 1\}$ onto the upper half plane $\{x + iy : y > 0\}$ if and only if the imaginary part of $f(0)$ is positive.

Exercise 9.52 (Pythagorean Triplets via Stereographic Projection) Consider the stereographic projection of the real line over the unit circle in $\mathbb{R}^2$. Show that under this transformation, the point $m/n \in \mathbb{Q}$ corresponds to

$$\left(\frac{2mn}{n^2 + m^2}, \frac{n^2 - m^2}{n^2 + m^2}\right),$$

which leads to the Euclid's formula for Pythagorean triples.

Exercise 9.53 Let $f : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be an isometry which fixes three non-collinear points. Prove that $f(x) = x$ for all $x \in \mathbb{R}^2$.

Exercise 9.54 Let $f, g : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ be two isometries which agree on $n + 1$ non-collinear points. Prove that $f \equiv g$ on $\mathbb{R}^n$.

Exercise 9.55 Let C be an arbitrary circle in $\mathbb{R}^3$ and L be any line disjoint from C. The surface obtained by revolving C around L is known as a *torus* in $\mathbb{R}^3$. In $\mathbb{R}^3$, prove that no sphere is homeomorphic to any torus.

Exercise 9.56 If $d(a, b) := |\tan^{-1} a - \tan^{-1} b|$ for all $a, b \in [-\infty, +\infty]$, prove that

(a) d is a metric on $[-\infty, +\infty]$.
(b) d is equivalent to the usual metric on the induced space $\mathbb{R}$.
(c) $([-\infty, +\infty], d)$ is homeomorphic to $[-\pi/2, +\pi/2]$.
(d) $([-\infty, +\infty], d)$ is compact.

Exercise 9.57 Let X_n be the union of n unit segments in $\mathbb{R}^n$, emanating from origin. Prove that X_n is homeomorphic to $X_m \iff m = n$ or $(m, n) \in \{(1, 2), (2, 1)\}$.

Exercise 9.58 Let X_1 and Y_1 be metric spaces, homeomorphic to X_2 and Y_2, respectively. Let f_x and f_y be homeomorphisms from X_1 onto X_2 and from Y_1 onto Y_2, respectively. Let $g : X_1 \longrightarrow Y_1$ be a uniformly continuous function. Is it necessary that $f_y \circ \left(g \circ (f_x^{-1})\right) : X_2 \longrightarrow Y_2$ is uniformly continuous?

Exercise 9.59 Up to homeomorphisms, characterize metric spaces having a unique dense subset.

Exercise 9.60 Does there exist any metric d on $\mathbb{Q}$, such that $(\mathbb{Q}, d)$ is (a) complete, (b) compact, or (c) connected? Justify your answers.

Exercise 9.61 Let X be any countably infinite set. Does there exist any metric d on X, such that the metric space (X, d) is (a) perfect, (b) perfect as well as complete?

Exercise 9.62 Does there exist a metric space X, not homeomorphic to a discrete space, such that every open ball in X is a closed subset of X?

Exercise 9.63 Let X be the vector space of all sequences $\{x_n\}$ of complex numbers such that $\sum_{n=1}^{\infty} 2^n |x_n| < \infty$. Define $\|\{x_n\}\| := \sum_{n=1}^{\infty} 2^n |x_n|$ for all $\{x_n\} \in X$. Is X a complete normed space? What is the dimension of X?

Exercise 9.64 Let X and Y be normed spaces over $\mathbb{R}$ and $f : X \longrightarrow Y$ be a continuous map. Prove that the following are equivalent:

(a) $f\left(\frac{a+b}{2}\right) = \frac{f(a)}{2} + \frac{f(b)}{2}$ for all $a, b \in X$.
(b) $f(ta + (1 - t)b) = tf(a) + (1 - t)f(b)$ for all $a, b \in X$ and $t \in [0, 1]$.
(c) $f(ta + (1 - t)b) = tf(a) + (1 - t)f(b)$ for all $a, b \in X$ and $t \in \mathbb{R}$.
(d) The mapping $x \longmapsto f(x) - f(0)$ is linear.

Exercise 9.65 Let O be a nonempty open subset of a complete metric space (X, d). Prove that O is homeomorphic to a complete metric space.

Exercise 9.66 Prove Baire Category Theorem (8.33) for nonempty open subsets of complete metric spaces.

9.3 Extension Theorems for Continuous Functions

Consider the question of extending a continuous function from a subset of metric space to the whole space continuously. One such extension theorem has already been presented in Corollary 5.39. A natural question is to ask whether continuous extensions always exist? The answer is in the negative, in general.

Example 9.31 The function $x \longmapsto \frac{1}{x}$ is continuous on $(0, 1)$ but cannot be extended to $[0, 1]$ as a continuous function (see Corollary 5.32(b)).

In some particular situations, it is possible. We shall discuss some results in this direction. First we present the Tietze Extension Theorem, which requires the Urysohn's Lemma.

Lemma 9.32 (Urysohn's lemma, 1925) *Let F_1 and F_2 be disjoint nonempty closed subsets of a metric space X. Then there exists a continuous function $f : X \longrightarrow [0, 1]$ such that*

$$f|_{F_1} \equiv 0 \text{ and } f|_{F_2} \equiv 1.$$

Proof Consider a function $f : X \longrightarrow [0, 1]$ defined as

$$f(x) := \frac{dist(x; F_1)}{dist(x; F_1) + dist(x; F_2)} \text{ for all } x \in X.$$

We leave it for the reader to show that f satisfies all our requirements □

Remark 9.33 In fact the above function f is Lipschitz continuous, and there are several other functions with these properties (see Exercises 9.68-9.69).

Theorem 9.34 (Tietze Extension Theorem, 1915) *Let F be a closed subset of metric space X and $g : F \longrightarrow \mathbb{C}$ be a continuous function. Then there exists a continuous function $f : X \longrightarrow \mathbb{C}$ such that $f|_F \equiv g$. Moreover if $|g| \leq 1$ on F, then f can be chosen so that $|f| \leq 1$ on X.*

Proof Let $g(F)$ denote the range of g. If $g(F) \subset [1, 2]$, then one can show that the function f defined as under has the desired properties.

$$f(x) := \begin{cases} g(x) & ; x \in F, \\ \inf\{g(a)d(x, a) : a \in F\}/dist(x; F) & ; x \in X \setminus F. \end{cases}$$

Consider the case when $g(F) \subset [-1, 1]$. Since the interval $[-1, 1]$ is homeomorphic to $[1, 2]$, there exists a continuous function $f : X \longrightarrow [-1, 1]$ satisfying our requirement.

Suppose $g(F) \subset \mathbb{R}$. Let $h : \mathbb{R} \longrightarrow (-1, 1)$ be a homeomorphism and write $g_1 := h \circ g$. Since $g_1 : F \longrightarrow (-1, 1) \subset [-1, 1]$ is a continuous function, there exists a continuous function $f_1 : X \longrightarrow [-1, 1]$ such that $f_1|_F \equiv g_1$. Let $F_1 := \{x \in X :$

$|f_1(x)| = 1\}$. If $F_1 = \emptyset$, then $f := h^{-1} \circ f_1$ is the required function. Otherwise, we proceed as follows:

Applying Lemma 9.32, consider a continuous function $p : X \longrightarrow [0, 1]$ such that $p|_{F_1} \equiv 0$ and $p|_F \equiv 1$. Let $P := p \times f_1$. Then $P : X \longrightarrow (-1, 1)$ is a continuous function and $P|_F = g_1$. Write $f := h^{-1} \circ P$. Then $f : X \longrightarrow \mathbb{R}$ is a continuous function such that $f|_F = g$, as required.

Finally, let $g : F \longrightarrow \mathbb{C}$. Then its real and imaginary parts, respectively denoted by $Re(g)$ and $Im(g)$, are continuous functions from F to $\mathbb{R}$. Applying previous calculations, there are continuous functions $H_1, H_2 : X \longrightarrow \mathbb{R}$ such that $H_1|_F = Re(g)$ and $H_2|_F = Im(g)$. Then $f := H_1 + iH_2$ satisfies our requirements.

Suppose that $|g| \leq 1$. Let G be a homeomorphism from $\{z \in \mathbb{C} : |z| \leq 1\}$ onto $\{x + iy \in \mathbb{C} : -1 \leq x, y \leq 1\}$. Since $Re(G \circ g)$ and $Im(G \circ g)$ are continuous functions from F into $[-1, 1]$, there are continuous functions h_1 and h_2 from X onto $[-1, 1]$ such that $h_1|_F = Re(G \circ g)$ and $h_2|_F = Im(G \circ g)$. It can be shown that $f := G^{-1} \circ (h_1 + ih_2) : X \longrightarrow \mathbb{C}$ satisfies our requirements. □

Corollary 9.35 (Beer, [14]) *Let (X, d) be a metric space. Then the following are equivalent:*

(a) X is complete.
(b) Every continuous map from X onto another metric space Y maps Cauchy sequences onto Cauchy sequences.
(c) Every continuous map from X onto another metric space Y maps totally bounded sets sequences onto totally bounded sets.

Proof We prove the result by showing that (a) ⇒ (b) ⇒ (c) ⇒ (a).

(a) ⇒ (b): This holds as every Cauchy sequence in X is convergent.

(b) ⇒ (c): Let Y be a metric space, $f : X \longrightarrow Y$ be continuous, and let A be a totally bounded subset of X. To prove that $f(A)$ is totally bounded, let $\{y_n\}$ be a Cauchy sequence in $f(A)$. Then there exists a sequence $\{x_n\} \subset A$ such that $y_n = f(x_n)$ for all $n \in \mathbb{N}$. Since A is totally bounded, $\{x_n\}$ has a Cauchy subsequence,. Without loss of generality, suppose that $\{x_n\}$ itself is Cauchy. By (b), $\{f(x_n)\}$ is Cauchy.

(c) ⇒ (a): Suppose there exists a Cauchy sequence $\{x_n\}$ in X which does not converge in X. Without loss of generality, suppose that all terms of $\{x_n\}$ are distinct. By Tietze Extension Theorem (9.34), there exists a continuous function $f : X \longrightarrow \mathbb{R}$ such that $f(x_n) = n$ for all $n \in \mathbb{N}$. So $\{f(x_n) : n \in \mathbb{N}\}$ is not totally bounded, while the set $\{x_n : n \in \mathbb{N}\}$ is totally bounded. □

In general, homeomorphic extensions are not possible even in some of the simplest cases.

Examples 9.36 (a) Let $f : \{0, 1\} \longrightarrow \{0, 1\}$ be defined as $f(1) := 0$ and $f(0) := 1$. Then there exists no homeomorphic extension of f from $[-1, 1]$ onto itself.

(b) Let $f : \{-1, 0, 1\} \longrightarrow \{-1, 0, 1\}$ be defined as $f(-1) := 0$, $f(0) := -1$ and $f(1) := 1$. Then there exists no homeomorphic extension of f from $\mathbb{R}$ onto itself.

Proof (a) Suppose there exists such a homeomorphic extension of f. Since 0 is a cut point of $[-1, 1]$ and cut points are preserved under homeomorphisms, we obtain $f(0) = 1$ as a cut point of $[-1, 1]$, a contradiction.

(b) Suppose g is a homeomorphic extension of f. By Intermediate Value Theorem (6.12), $g([0, 1])$ contains the interval $[-1, 1]$. Hence, $g(-1) = 0 \in [-1, 1] \subset g([0, 1])$, which implies that $-1 \in [0, 1]$, a contradiction. □

However, homeomorphisms between compact subsets of $\mathbb{R}$ extend to self-homeo morphisms on $\mathbb{R}^2$.

Theorem 9.37 *If $f : A \longrightarrow B$ is a homeomorphism between compact subsets A and B of $\mathbb{R}$, then it has a homeomorphic extension $g : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$, identifying $\mathbb{R}$ as the subspace $\mathbb{R} \times \{0\}$ of the plane $\mathbb{R}^2$.*

Proof Identifying $\mathbb{R}$ with $I := (-1, 1)$, we assume that $A, B \subset (-1, 1)$. Note that it is enough to obtain a homeomorphic extension of f from I^2 onto itself.

Let K be a compact set, containing both A and B, inside I. For $t \in K$, let $H_t : \overline{I} \longrightarrow \overline{I}$ be the (unique) homeomorphism which maps linearly $[-1, t]$ onto $[-1, 0]$ and $[t, 1]$ onto $[0, 1]$.

Define $\phi : B \longrightarrow K$ by $\phi(x) := f^{-1}(x)$ By Tietze Extension Theorem (9.34), there exists a continuous extension $\eta : I \longrightarrow K$ of ϕ. Define $h_1 : I^2 \longrightarrow I^2$ as

$$h_1(x, y) := (H_{\eta(y)}(x), y) \text{ for all } x, y \in I.$$

It can be proved that h_1 is a homeomorphism. Further for $a \in A$, we have

$$h_1(a, f(a)) = (H_{\eta(f(a))}(a), f(a)) = (H_a(a), f(a)) = (0, f(a)).$$

If $G := \{(a, f(a)) : a \in A\} \subset I^2$, then $h_1(G) = \{0\} \times B$. Define $h_2 : I^2 \longrightarrow I^2$ as

$$h_2(x, y) := (x, H_{\eta(y)}(x)) \text{ for all } x, y \in I.$$

As above, h_2 is a homeomorphism and $h_2(a, f(a)) = (a, 0)$ for all $a \in A$. Let $\xi : I^2 \longrightarrow I^2$ be the homeomorphism given by $(x, y) \longmapsto (y, x)$ and $g := \xi \circ h_1 \circ h_2^{-1}$. It can be shown that g is a self-homeomorphism on I^2 with

$$\begin{aligned} g(a, 0) &= (\xi \circ h_1 \circ h_2^{-1})(a, 0) = \xi(h_1(h_2^{-1}(a, 0))) \\ &= \xi(h_1(a, f(a))) = \xi(0, f(a)) = (f(a), a). \end{aligned}$$

This completes the proof. □

Remark 9.38 The above result is taken from [15, p. 260]. In [15, Chap. 6], three methods to construct new homeomorphisms from the old are presented. For more on homeomorphic extensions, we refer [16, Sects. 6.9 and 6.11], and [17]. The latter paper provides conditions under which homeomorphisms between compact subsets of spaces X and Y extend to a homeomorphisms between X and Y. This paper ends with the following open question:

Open Question 9.39 Is there an integer $n > 1$ such that every homeomorphism between compact subsets of $\mathbb{R}^n$ is extendible to a self-homeomorphism of $\mathbb{R}^n$?

Next, we extend the notion of oscillations, already discussed in Sect. 7.3.2, to functions on an arbitrary subset of a given space. Let X and Y be metric spaces, $A \subset X$ and $f : A \longrightarrow Y$ be a given function. The *oscillation* of a function f at an element $x \in X$ is defined as

$$\omega(f; x) := \inf\{diam(f(B(x; \delta) \cap A)) : \delta > 0\}.$$

Theorem 9.40 (Kuratowski) *Let X and Y be metric spaces such that Y is complete. Let $A \subset X$ and $f : A \longrightarrow Y$ be continuous. Then there exists a G_δ-set G with $A \subset G \subset \overline{A}$ and a continuous extension $g : G \longrightarrow Y$ of f.*

Proof Let $G := \overline{A} \cap \{x \in X : \omega(f; x) = 0\}$. By Theorems 7.37 and 7.39, and Proposition 7.36, G is a G_δ-subset of X with $A \subset G \subset \overline{A}$. Let $x \in G$ and choose a sequence $\{x_n\}$ from A, convergent to x. Then

$$\lim_{n\to\infty} diam(\{f(x_m) : m \geq n\}) = 0.$$

Therefore, $\{f(x_n)\}$ is a Cauchy sequence in Y and hence convergent in Y. Define $g(x) := \lim_{n\to\infty} f(x_n)$. We leave it for the reader to show that g is independent of the choice of sequence $\{x_n\}$ and extends f.

To show that g is continuous on G, let $x \in G$ and $\delta > 0$. Then the definition of g implies that $g(B(x; \delta)) \subset \overline{f(B(x; \delta))}$. So, $diam(g(B(x; \delta))) \leq diam(\overline{f(B(x; \delta))}) = diam\big(f(B(x; \delta))\big)$, which implies that $0 \leq \omega(g; x) \leq \omega(f; x) = 0$, as f is continuous at x. Hence, $\omega(g; x) = 0$ and therefore, g is continuous at x. □

Theorem 9.41 (Lavrentiev) *Let X and Y be complete metric spaces, $A \subset X$, $B \subset Y$ and $f : A \longrightarrow B$ be a homeomorphism. Then f can be extended to a homeomorphism $h : G \longrightarrow H$, where both G, H are G_δ-sets satisfying $A \subset G \subset \overline{A}$ and $B \subset H \subset \overline{B}$.*

Proof By Theorem 9.40, obtain G_δ-sets G_1, H_1 satisfying $A \subset G_1 \subset \overline{A}$, and $B \subset H_1 \subset \overline{B}$ and continuous extensions $f_1 : G_1 \longrightarrow Y$ and $g_1 : H_1 \longrightarrow X$ of f and f^{-1}, respectively. Let

$$G := \{x \in G_1 : (g_1 \circ f_1)(x) = x\} \text{ and } H := \{y \in H_1 : (f_1 \circ g_1)(y) = y\}.$$

Then $A \subset G \subset G_1$ and $B \subset H \subset H_1$. Define $h : G \longrightarrow H$ as $h := f_1|_G$. Then h is a homeomorphism. Therefore, it is enough to show that G and H are G_δ-sets.

Let $p : G_1 \longrightarrow X \times Y$ denote the continuous mapping $x \longmapsto (x, f_1(x))$. Then $G = p^{-1}(S)$, where $S := \{(x, y) : x = g_1(y)\}$. Since g_1 is continuous, S is a closed subset of $X \times H_1$. By Theorem 7.39, S is a G_δ in $X \times Y$. Since $G = p^{-1}(S)$ and p is continuous, G is a G_δ in X. Similarly, H is also a G_δ in Y. Hence the result. □

Corollary 9.42 *Let (X, d) be a complete metric space and $Y \subset X$ such that (Y, d') is complete, for some metric d' on Y equivalent to d. Then Y is a G_δ-subset of (X, d).*

Proof Let $f : Y \longrightarrow Y$ be the identity map. Since it is continuous, by Theorem 9.40, there exists a G_δ-set G with $Y \subset G \subset \overline{Y}$ and a continuous extension $g : G \longrightarrow Y$ of f. Then Y is dense in G. Thus, g is also an identity map on G which implies $G \subset Y$. Consequently, $Y = G$. Hence the result. □

Next, we generalize Corollary 5.39. For a variant, see Exercise 4.19.

Theorem 9.43 *Let X, Y be metric spaces such that Y is complete. Let A be a dense subset of X and $f : A \longrightarrow Y$ be a uniformly continuous map. Then there exists a unique uniformly continuous function $F : X \longrightarrow Y$ such that $F|_A = f$.*

Proof *Uniqueness:* Let f_1 and f_2 be continuous extensions of f to X. Pick any $x \in X = \overline{A}$. Then there exists a sequence $\{x_n\} \subset A$ such that $x_n \longrightarrow x$. Since f_1, f_2 are continuous at x,

$$f_1(x) = \lim_{n\to\infty} f_1(x_n) = \lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} f_2(x_n) = f_2(x).$$

Existence: Let $x \in X = \overline{A}$. Choose a sequence $\{a_n\}$ in A such that $a_n \longrightarrow x$. Being convergent, $\{a_n\}$ is Cauchy and hence $\{f(a_n)\}$ is a Cauchy sequence in the complete metric space Y. Define $F(x) := \lim_{n\to\infty} f(a_n)$. We claim that $F : X \longrightarrow Y$ is the required extension. Let d_X and d_Y denote the metrics on X and Y, respectively.

To see that F is well-defined, let $\{a_n\}$ and $\{b_n\}$ be sequences from A, convergent to some x. Since $d_X(a_n, b_n) \le d_X(a_n, x) + d_X(x, b_n)$, we obtain $d_X(a_n, b_n) \longrightarrow 0$. As f is uniformly continuous on A, $d_Y(f(a_n), f(b_n)) \longrightarrow 0$. Hence, both $\{f(a_n)\}$ and $\{f(b_n)\}$ converge to the same limit.

It is immediate that $F(a) = f(a)$ for all $a \in A$. To prove that F is uniformly continuous on X, let $\epsilon > 0$ be given. Then there exists some $\delta > 0$ such that

$$d_Y(f(a), f(b)) < \frac{\epsilon}{3} \text{ for all } a, b \in A \text{ with } d_X(a, b) < \delta.$$

Let $x, y \in X$ such that $d_X(x, y) < \frac{\delta}{3}$. The definition of F ensures $a, b \in A$ such that

$$d_X(x, a) < \frac{\delta}{3}, d_X(y, b) < \frac{\delta}{3}, d_Y(F(x), f(a)) < \frac{\epsilon}{3} \text{ and } d_Y(F(y), f(b)) < \frac{\epsilon}{3}.$$

Then

$$d_X(a, b) \le d_X(a, x) + d_X(x, y) + d_X(y, b) < \frac{\delta}{3} + \frac{\delta}{3} + \frac{\delta}{3} = \delta.$$

Therefore, $d_Y(f(a), f(b)) < \frac{\epsilon}{3}$ and consequently, we obtain

$$\begin{aligned} d_Y(F(x), F(y)) &\le d_Y(F(x), f(a)) + d_Y(f(a), f(b)) + d_Y(f(b), F(y)) \\ &< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon. \end{aligned}$$

This establishes that F is uniformly continuous. Hence the result. □

Note that every complete subspace of a metric space is its closed subset and hence a G_δ (see Theorems 4.2 and 7.39). The converse is not true, as $(0, 1)$ is a G_δ but not a complete subspace of $\mathbb{R}$. However, the following result can be treated as a weaker converse.

Theorem 9.44 (Alexandroff) *Let (X, d) be a complete metric space and Y be a nonempty G_δ-subset of X. Then there exists a metric d' on Y such that*

(a) d' and d are topologically equivalent on Y.
(b) (Y, d') is a complete metric space.

Proof Since Y is a G_δ in X, we have $Y = \bigcap_{n=1}^{\infty} O_n$, for a sequence $\{O_n\}$ of open subsets of X. Let $F_n := X \setminus O_n$ for all $n \in \mathbb{N}$. For each $x, y \in Y$, define

$$d'(x, y) := d(x, y) + \sum_{n=0}^{\infty} \min\left\{\frac{1}{2^{n+1}}, \left|\frac{1}{d(x, F_n)} - \frac{1}{d(y, F_n)}\right|\right\},$$

here $d(t, F_n) := \min\{d(t, s) : s \in F_n\}$. It can be shown that d' is a metric on Y. We claim that d' is the required candidate.

Let $\{y_k\}$ be a Cauchy sequence in (Y, d'). Then it is also Cauchy sequence in (Y, d). Since (X, d) is complete there exists some $y \in X$ such that $y_k \longrightarrow y$ in (X, d). Fix $n \in \mathbb{N}$. Note that $\left\{\frac{1}{d(y_k, F_n)}\right\}_k$ is a Cauchy sequence of reals and hence convergent. Consequently, $\{d(y_k, F_n)\}_k$ is convergent in $\mathbb{R} \setminus \{0\}$. Since $\{d(y_k, F_n)\}_k \longrightarrow d(y, F_n)$, as $k \longrightarrow \infty$, we conclude that $d(y, F_n) > 0$. Hence, $y \in X \setminus F_n = O_n$. Therefore, $y \in \bigcap_{n=1}^{\infty} O_n = Y$. Clearly, $y_k \longrightarrow y$ in (Y, d').

This shows that (Y, d') is a complete metric space. We leave it for the readers to show that d and d' are equivalent on X, which establishes the result. □

Remarks 9.45 A separable metric space X is called *countably dense homogeneous (CDH)* if for any two countably dense subsets A and B of X, there exists a homeomorphism $f : X \longrightarrow X$ such that $f(A) = B$ and $f(B) = A$.

Finite-dimensional Euclidean spaces $\mathbb{R}^n$ are CDH (see [18, p. 46]). Very little is known about CDH spaces. A collection of open problems in this direction can be found in [19].

History Notes 9.46 A special case of Tietze Extension Theorem (9.34) was first proved by L. E. J. Brouwer and Henri Lebesgue, where X is a finite-dimensional real vector space. It was extended to metric spaces by Heinrich Tietze and then to normal topological spaces by Paul Urysohn.

Exercise 9.67 Is the hypothesis that 'F is a closed set', redundant in the Tietze Extension Theorem (9.34)?

Exercise 9.68 Let F_1, F_2 and (X, d) be as in Urysohn's lemma (9.32), and f be the continuous function presented in the proof of this lemma. Prove the following:

(a) Let $\delta > 0$ and $S := \{(a, b) \in \mathbb{R}^2 : a \geq 0, b \geq 0, a + b \geq \delta\}$. Then the map $g : S \longrightarrow \mathbb{R}$ defined as $g(a, b) := \frac{a}{a+b}$ is Lipschitz continuous on S with Lipschitz constant $1/\delta$.
(b) The maps $x \longmapsto dist(x, F_i)$; $i = 1, 2$, are Lipschitz continuous with Lipschitz constant 1.
(c) If $\delta := dist(F_1, F_2)$, then $\delta > 0$ and f is Lipschitz continuous with Lipschitz constant $1/\delta$.

Exercise 9.69 Let F_1, F_2 and (X, d) be as in Urysohn's lemma (9.32). For every $x \in X$, we define

$$h_1(x) := \min\left\{1, \frac{dist(x, F_1)}{dist(F_1, F_2)}\right\} \text{ and } h_2(x) := \max\left\{0, 1 - \frac{dist(x, F_1)}{dist(F_1, F_2)}\right\}.$$

Prove that both h_1 and h_2 are Lipschitz continuous on X with Lipschitz constant $1/dist(F_1, F_2)$; and satisfy the requirements of the Urysohn's lemma (9.32).

Exercise 9.70 Define $f : \{-1, 0, 1\} \longrightarrow \{-1, 0, 1\}$ as

$$f(-1) := 0,\ f(0) := 1 \text{ and } f(1) := -1.$$

Does there exists a homeomorphic extension of f from $[-1, 1]$ onto itself?

Exercise 9.71 Does there exist any homeomorphism $f : (0, 4) \longrightarrow (0, 4)$ such that $f(1) = 2$, $f(2) = 1$ and $f(3) = 3$?

Exercise 9.72 Let $a < c < d < b$ be reals. Does there exist a homeomorphism $f : (a, b) \longrightarrow (a, b)$ which swaps c and d, that is, $f(c) = d$ and $f(d) = c$?

Exercise 9.73 Let I be an interval and $a < b < c < d$ be reals inside I. Does there exist a homeomorphism $f : I \longrightarrow I$ which swaps a with b; and c with d?

Exercise 9.74 Let A and B be finite subsets of reals with same number of elements. Prove that there exists a homeomorphism $f : \mathbb{R} \longrightarrow \mathbb{R}$ such that $f(A) = B$.

Exercise 9.75 Let $n \in \mathbb{N}$ and $\alpha, \beta \in \mathbb{R}^n$. Give an example of a homeomorphism from $\mathbb{R}^n$ onto itself, which swaps α and β.

Exercise 9.76 Prove that the function h_1, as in the proof of Theorem 9.37 is a homeomorphism.

Exercise 9.77 Complete all routine verifications in the proof of Tietze Extension Theorem.

Exercise 9.78 In the proof of Theorem 9.44, show that the metrics d and d' are equivalent on X.

Exercise 9.79 In the proof of Theorem 9.40, show that the function g is independent of the choice of the sequence $\{x_n\}$ and extends f.

Exercise 9.80 Let X be a metric space, $A \subset X$ and $f : A \longrightarrow \mathbb{R}$ be a uniformly continuous function. Prove that there exists a continuous function $F : X \longrightarrow Y$ such that $F|_A = f$.

Exercise 9.81 Let X, Y be metric spaces, A be a dense subset of X, $\widetilde{Y}$ be the completion of Y and $f : A \longrightarrow Y$ be a uniformly continuous map. Then there exists a unique uniformly continuous function $F : X \longrightarrow \widetilde{Y}$ such that $F|_A = f$.

Exercise 9.82 Can the uniform continuity in Theorem 9.43 be replaced with continuity?

Exercise 9.83 Let X be a metric space and $C_0(X)$ denote the set of $f \in C(X)$ such that for every $\epsilon > 0$ there exists a compact set $K \subset X$ such that $|f(x)| < \epsilon$ for all $x \in X \setminus K$.

(a) Prove that $C_0(X)$ is a Banach space under uniform norm.
(b) If X is compact and there exists $x_0 \in X$ such that $X_0 := X \setminus \{x_0\}$ is not compact, prove that every $f \in C_0(X_0)$ can be extended to a function in $C(X)$.

Exercise 9.84 Let X and Y be complete metric spaces. Prove that homeomorphisms between dense subsets of X and Y can be extended to a homeomorphism between G_δ subsets of X and Y, containing the given dense subsets.

Exercise 9.85 Let F be a closed subset of metric space X and $g : F \longrightarrow \mathbb{R}$ be a bounded continuous function. Prove that there exists a continuous function $f : X \longrightarrow \mathbb{R}$ such that

$$f|_F \equiv g, \inf f(X) = \inf g(F) \text{ and } \sup f(X) = \sup g(F).$$

9.4 Finite-Dimensional Normed Linear Spaces

In this section, we shall conclude that all norms on a finite-dimensional linear space are equivalent. Further, on such spaces, every closed and bounded set is compact.

Theorem 9.47 *Let X be a linear space with two norms $\|.\|_1$ and $\|.\|_2$. Let ρ_i be the metric induced by $\|.\|_i$; $i = 1, 2$. Then the following are equivalent.*

(a) ρ_1 and ρ_2 are topologically equivalent
(b) ρ_1 and ρ_2 are uniformly equivalent, and
(c) ρ_1 and ρ_2 are Lipschitz equivalent.

Proof The implications $(c) \Rightarrow (b) \Rightarrow (a)$ are immediate from the definitions. To prove $(a) \Rightarrow (c)$, assume that ρ_1 and ρ_2 are topologically equivalent. Then the identity maps $(X, \rho_1) \longrightarrow (X, \rho_2)$ and $(X, \rho_2) \longrightarrow (X, \rho_1)$ are both continuous. Since the identity map on a normed space is linear, by Theorem 5.50, both of the above maps are Lipschitz continuous. □

Hence, for a vector space X, two norms $\|.\|$ and $\|.\|'$ on X are said to be *equivalent* if there exist positive constants α and β such that

$$\alpha\|x\| \leq \|x\|' \leq \beta\|x\| \text{ for all } x \in X.$$

Lemma 9.48 *Let X be a finite-dimensional linear space with basis $\{b_1, \ldots, b_m\}$. For $x = k_1 b_1 + \cdots + k_m b_m \in X$, define*

$$\|x\|_\infty := \max\{|k_1|, \ldots, |k_m|\}.$$

Then $\|.\|_\infty$ is a norm on X and $S := \{x \in X : \|x\|_\infty = 1\}$ is a compact subset of $(X, \|.\|_\infty)$.

Proof It can be shown that $\|.\|_\infty$ defines a norm on X. Since every normed space is a metric space, by Theorem 5.27, it is enough to prove that every sequence in S has a subsequence, convergent in S.

Let $\{x_n\}$ be any sequence in S. Write $x_n := \sum_{i=1}^{m} k_{n,i} b_i$ for all $n \in \mathbb{N}$. Then for every $i \in \{1, \ldots, m\}$, the sequence $\{k_{n,i}\}_n$ is a bounded sequence of scalars, as

$$|k_{n,i}| \leq \|x_n\|_\infty = 1 \text{ for all } n \in \mathbb{N}.$$

Applying the Bolzano-Weierstrass Property of the scalar field and the diagonal procedure finitely many times, as in Theorem 2.27, we conclude that $\{x_n\}$ has a convergent subsequence, say $\{x_{n_j}\} \longrightarrow x$ in $(X, \|.\|_\infty)$. The continuity of the map $x \longmapsto \|x\|_\infty$ implies that $\|x\|_\infty = \lim_{j\to\infty} \|x_{n_j}\|_\infty = 1$, that is, $x \in S$. Hence, S is compact in $(X, \|.\|_\infty)$. □

Theorem 9.49 *On any finite-dimensional linear space, all norms are equivalent*

Proof Let X be a finite-dimensional linear space, $\{b_1, \ldots, b_m\}$ and $\|.\|_\infty$ be as in Lemma 9.48, and $\|.\|$ be any norm on X. Then for $x = \sum_{i=1}^{m} k_i b_i \in X$, we have

$$\|x\| \leq \sum_{i=1}^{m} |k_i|\|b_i\| \leq \|x\|_\infty \sum_{i=1}^{m} \|b_i\| = \beta\|x\|_\infty, (say). \tag{9.2}$$

By Lemma 9.48, $S := \{x \in X : \|x\|_\infty = 1\}$ is a compact subset of $(X, \|.\|_\infty)$. For $x \in X$, define $T(x) := \|x\|$. From (9.2), it follows that $T : (X, \|.\|_\infty) \longrightarrow (X, \|.\|)$ is a continuous function which maps the compact set S into $(0, \infty)$. By Theorem 5.33, there exists $\alpha > 0$ such that $\|x\| \geq \alpha$ for all $x \in S$. If $x \neq 0$, then $\frac{x}{\|x\|_\infty} \in S$ and thus

$$\left\| \frac{x}{\|x\|_\infty} \right\| \geq \alpha \text{ which implies } \|x\| \geq \alpha\|x\|_\infty.$$

Hence, $\|.\|$ and $\|.\|_\infty$ are equivalent norms on X. Since the equivalence of norms on a linear space X is an equivalence relation, the result follows. □

Recall that given normed spaces X and Y are called *linearly homeomorphic* if there exists a linear homeomorphism $f : X \longrightarrow Y$.

Theorem 9.50 *If $(X, \|.\|)$ is a finite-dimensional normed linear space, then X is linearly homeomorphic to $\mathbb{K}^m$ for some $m \in \mathbb{N}$. Consequently, X is a Banach space.*

Proof Let $\{b_1, \ldots, b_m\}$ and $\|.\|_\infty$ be as in Lemma 9.48. Define $f : X \longrightarrow \mathbb{K}^m$ as

$$f(x) := (k_1, \ldots, k_m) \text{ for all } x = \sum_{i=1}^{m} k_i b_i \in X.$$

It is immediate that f is a linear bijection with $\|f(x)\|_\infty = \|x\|_\infty$ for all $x \in X$. By Theorem 5.50, we conclude that $f : (X, \|.\|_\infty) \longrightarrow (\mathbb{K}^m, \|.\|_\infty)$ is a homeomorphism. By Theorem 9.49, $(X, \|.\|)$ is linearly homeomorphic to the complete normed space $(\mathbb{K}^m, \|.\|_\infty)$. Applying Theorem 9.10, it follows that X is a Banach space. □

Corollary 9.51 *Every finite-dimensional subspace of a normed linear space is its closed subspace.*

Proof Apply Theorems 9.50 and 4.2. □

Next, we provide a generalization of the Heine-Borel Theorem (5.14).

Theorem 9.52 *Let $(X, \|.\|)$ be a finite-dimensional normed linear space and $E \subset X$. Then E is compact if and only if E is closed and bounded in X.*

Proof The necessity is immediate by Theorem 5.8 and Corollary 5.4. For the converse, let E be a closed and bounded subset of X. Applying Theorem 9.50, X is Banach, there exists $n \in \mathbb{N}$ and a linear homemorphism $f : (X, \|.\|) \longrightarrow (\mathbb{K}^n, \|.\|_2)$.

Being a closed subset of a complete space X, E is complete. By Theorem 5.50, f is Lipschitz continuous. Therefore $f(E)$ is a complete and bounded subset of the finite-dimensional space $(\mathbb{K}^n, \|.\|_2)$ and hence compact. Finally, the continuity of f^{-1} ensures that $E = f^{-1}(f(E))$ is compact in X. □

Note that Theorem 9.52 is not true in arbitrary normed linear spaces.

Example 9.53 Let $X = c_{00}$ under $\|.\|_2$ norm and $\{e_n\}$ be as in Example 4.43. Let $E := \{e_n : n \in \mathbb{N}\}$. Then E is a closed and bounded $(X, \|.\|_2)$, but not compact.

Proof The set E is bounded, as for all $m, n \in \mathbb{N}$ such that $m \neq n$, we have

$$\|e_n - e_m\|_2 = \sqrt{2}. \tag{9.3}$$

To see that E is closed, let x be a limit point of E in X. Then there exists a sequence $\{x_n\} \subset E$ such that $x_n \longrightarrow x$ in X. Hence, $\{x_n\}$ is Cauchy. By (9.3), we conclude that $\{x_n\}$ is eventually constant. Therefore, $x \in E$ and hence E is closed. The above arguments also conclude that $\{e_n\}$ has no subsequence, convergent in X. Hence, E is not sequentially compact. Applying Theorem 5.27, we conclude that E is not compact. □

Theorem 9.54 *If all norms on a linear space X are equivalent, then X is finite-dimensional.*

Proof Assume that X is an infinite-dimensional space on which all norms are equivalent. Let $\|.\|$ be any norm on X. Choose a sequence $\{u_n\}$ of linearly independent elements of X such that $\|u_n\| = 1$ for all n. Let B be a basis of X containing $\{u_n : n \in \mathbb{N}\}$.

Let Y denote the subspace of X, spanned by $\{u_n : n \in \mathbb{N}\}$. For every $y = \sum_{n=1}^{\infty} k_n u_n \in Y$, define $\phi(y) := \sum_{n=1}^{\infty} n k_n$. Note that all but finitely many terms of the sequence $\{k_n\}$ are zero. Extend ϕ linearly to X by setting $\phi(b) = 0$ for all $b \in B \setminus \{u_n : n \in \mathbb{N}\}$. Define

$$\|x\|_0 := \|x\| + |\phi(x)| \text{ for all } x \in X.$$

It can be shown that $\|.\|_0$ is a norm on X. By hypothesis, there exist some $\alpha > 0$ such that $\|x\|_0 \leq \alpha\|x\|$ for all $x \in X$. Therefore,

$$\sqrt{n} = \left|\phi\left(\frac{u_n}{\sqrt{n}}\right)\right| \leq \left\|\frac{u_n}{\sqrt{n}}\right\|_0 \leq \alpha\left\|\frac{u_n}{\sqrt{n}}\right\| = \frac{\alpha}{\sqrt{n}} \longrightarrow 0,$$

as $n \longrightarrow \infty$, which is absurd. Hence the result □

Theorem 9.55 *If X is an infinite-dimensional Banach space, then X does not have a countable basis.*

Proof If possible, let $\{u_n\}$ be a countable basis of X. Since X is infinite-dimensional, this basis is countably infinite. For each $n \in \mathbb{N}$, let Y_n be the subspace of X spanned by $\{u_1, \ldots, u_n\}$. By Corollary 9.51, each Y_n is a closed subspace of X. Applying Theorem 3.35, each Y_n is nowhere dense. Since $\{u_n\}$ is a basis of X, we have $X = \bigcup_{n=1}^{\infty} Y_n$, a contradiction to the Baire Category Theorem (8.33). □

Remark 9.56 It is well known that any two algebraic bases of a linear space are in one-to-one correspondence, which allows us to define the dimension of a linear space as the cardinality of its basis. In 1945, it was proved that if X is an infinite-dimensional Banach space, then its dimension is at least c. In 1973, a short proof (without the continuum hypothesis) proving that the dimension of any infinite-dimensional separable Banach space is c, appeared in AMS monthly. For details, see [20] and [21, Theorem I-1, p. 158].

All non-trivial (non-zero) normed linear spaces are unbounded and hence non-compact. Therefore, it is natural to look out for normed spaces satisfying some weaker property.

Definition 9.57 A metric space X is said to be *locally compact* if every $x \in X$ has a compact neighborhood.

Examples 9.58 The spaces $(0, 1)$ and $\mathbb{R}$ are locally compact, while $\mathbb{Q}$ is not.

Theorem 9.59 *A normed linear space X is locally compact if and only if its closed unit ball $B[0; 1]$ is compact.*

Proof The converse is immediate from $B[x; 1] = x + B[0; 1]$. Assume that X is locally compact and consider a compact neighborhood V of 0. Pick any $r > 0$ such that $B(0; r) \subset V$. Applying Theorem 3.38, $B[0; r] = \overline{B(0; r)}$ is a closed subset of $\overline{V} = V$. Therefore, $B[0; r]$ is compact. Hence, so is the closed unit ball $B[0; 1] = \frac{1}{r} B[0; r]$. Hence the result. □

The next result is popularly known as the *Riesz's lemma.* It can be easily proved for finite-dimensional Euclidean spaces. We here present the general case.

Lemma 9.60 (F. Riesz, 1918) *Let Y be a closed proper subspace of a normed linear space X and $0 < r < 1$. Then there exists $x_r \in X$ with $\|x_r\| = 1$ such that*

$$\|x_r - y\| > r \text{ for all } y \in Y.$$

Proof Since $Y \neq X$, there exists some $x \in X \setminus Y$. Let $d := \inf\{\|x - y\| : y \in Y\}$. Since Y is closed, $d > 0$. Let $y_r \in Y$ be such that $\|x - y_r\| < \frac{d}{r}$. Set

$$x_r := \frac{x - y_r}{\|x - y_r\|}.$$

We show that x_r is the required candidate. Clearly, $\|x_r\| = 1$. Pick any $y \in Y$. Since Y is a subspace, we have $y_r + \|x - y_r\| y \in Y$. Hence,

$$\|x_r - y\| = \frac{\|x - (y_r + \|x - y_r\|)y\|}{\|x - y_r\|} > \frac{d}{d/r} = r.$$

This proves the lemma. □

Theorem 9.61 *Let $(X, \|.\|)$ be a normed linear space. Then X is locally compact if and only if X is finite-dimensional.*

Proof The converse following from the fact that X is linearly homeomorphic to $\mathbb{K}^n$, for some n; and that $\mathbb{K}^n$ is locally compact. To prove the necessity, write $S := \{x \in X : \|x\| = 1\}$, and assume that X is locally compact.

By Theorem 9.59, $B[0; 1]$ is compact, and hence so is its closed subset S. Therefore, $S \subset \bigcup_{i=1}^n B(x_i; 1/2)$, for some finitely many $x_1, \ldots, x_n \in X$. Let Y be the subspace of X spanned by $x_1, \ldots, x_n$. By Corollary 9.51, Y is closed.

If X is infinite-dimensional, then $Y \neq X$. Applying Lemma 9.60 with $r = 1/2$, there exists some $x \in X$ such that $\|x\| = 1$ and $\|x - x_i\| \geq 1/2$ for all $i = 1, \ldots, n$. Therefore, we obtain $x \in S \setminus \bigcup_{i=1}^n B(x_i; 1/2)$, a contradiction. Hence the result. □

Applying Theorems 5.10, 9.49, 9.54, 9.52, 9.59, and 9.61, we conclude the following:

Corollary 9.62 *If X is a normed space, then the following are equivalent:*

(a) *X is finite-dimensional,*
(b) *X is locally compact,*
(c) *all norms on X are equivalent,*
(d) *the unit sphere* $\{x \in X : \|x\| = 1\}$ *is compact,*
(e) *the closed unit ball* $\{x \in X : \|x\| \leq 1\}$ *is compact, and*
(f) *every closed and bounded subset of X is compact.*

Remark 9.63 Several characterizations of finite-dimensional Banach spaces in terms of Lipschitz functions can be found in [22].

Exercise 9.86 Show that the map $x \longmapsto \|x\|_\infty$, as in Lemma 9.48 is a norm on X.

Exercise 9.87 In the proof of Theorem 9.55, why $\bigcup_{n=1}^{\infty} Y_n = X$?

Exercise 9.88 Let Y and Z be linear subspaces of a normed linear space X such that Y is closed and Z is finite dimensional. Prove that $Y + Z$ is a closed subspace of X.

Exercise 9.89 Let X be any infinite-dimensional normed linear space, $x \in X$ and $r \geq 0$. Prove that the set $\{y \in X : \|y - x\| \leq r\}$ is compact if and only if $r = 0$.

Exercise 9.90 Let X be the vector space of all polynomials over reals. Does there exist a norm on X which makes it a Banach space?

Exercise 9.91 Let Y be a finite-dimensional subspace of a normed linear space X. Prove that Y is closed in X.

Exercise 9.92 Does there exist an infinite-dimensional normed linear space X such that every subspace of X is complete?

Exercise 9.93 Prove that equivalent metrics admit the same class of locally compact subsets.

Exercise 9.94 Let X be a normed linear space. Prove that the Riesz lemma with $r = 1$ holds if and only if for every closed proper subspace Y of X, there are $x \in X$ and $y \in Y$ such that $\|x - y\| = dist(x; Y)$.

Exercise 9.95 Let Y be a proper subspace of a finite-dimensional normed linear space X. Prove that there exists some $x \in X$ such that $\|x\| = 1$ and $dist(x; Y) = 1$.

Exercise 9.96 Let $X := \{f \in C[0, 1] : f(0) = 0\}$ equipped with the supremum norm and $Y := \{f \in X : \int_0^1 f = 0\}$. Show that Y is a proper closed subspace of X but there exists no $x \in X$ such that $\|x\| = 1$ and $dist(x; Y) = 1$.

9.5 Hints and Solutions to Selected Exercises

9.7 Yes. Take $\rho(x, y) := d_X(x, y) + d_Y(f(x), f(y))$ for all $x, y \in X$.

9.8 (a) True. By scaling and translating, it is enough to prove the result for the interval $X := (-\pi/2, \pi/2)$. Define $d(x, y) := |\tan x - \tan y|$ for all $x, y \in X$. Then d is a metric on X. Let $\{x_n\}$ be a Cauchy sequence in (X, d). Then $\{\tan x_n\}$ is a Cauchy sequence of reals and hence convergent, say $\tan x_n \longrightarrow y_0$. Since $tan : X \longrightarrow \mathbb{R}$ is surjective, $y_0 = \tan x_0$ for some $x_0 \in X$. Hence, $x_n \longrightarrow x_0$ in (X, d).

(b) False. Suppose d is a metric on $[0, 1]$ such that $([0, 1], d)$ has the usual topology. Then it is compact and hence complete.

9.11 Let F be closed in (X, d). Then F is compact in (X, d). Since d is stronger than ρ, F is also compact in (X, ρ). Hence, F be closed in (X, ρ).

9.13 Both (a) and (b) follow from the basic definitions, as in Theorem 9.15. For (c), note that $\eta \le d$ and $\eta \le 1$. If there exists $M > 0$ such that $d \le M\eta$, then d must be bounded by M. It is impossible, if d is unbounded. This proves (c).

9.14 Note that

$$\rho(x, y) = 0 \iff \frac{x}{1 + |x|} = \frac{y}{1 + |y|} \iff x + x|y| = y + y|x|$$

and x, y have same sign. Usual componendo dividendo gives definiteness of ρ. Further,

$$|x_n - x| \longrightarrow 0 \iff \frac{x_n}{1 + |x_n|} \longrightarrow \frac{x}{1 + |x|} \iff \rho(x_n, x) \longrightarrow 0.$$

Hence, d and ρ are equivalent. Assume that the identity map $(X, \rho) \longrightarrow (X, d)$ is uniformly continuous. Then for $\epsilon = 1$, there exists $\delta > 0$ such that

$$\left|\frac{x}{1 + |x|} - \frac{y}{1 + |y|}\right| < \delta \Rightarrow |x - y| < 1.$$

Since $\frac{n}{1+n} \longrightarrow 1$, for x and y in a suitable tail of this sequence, the above relation is not satisfied, a contradiction.

9.15 (a) Pick any $x \in X$ and $r > 0$. Then for $r' := \min\{r, 1/2\}$,

$$B_d(x; r) \subset B_\eta(x; r) \text{ and } B_\eta(x; r') \subset B_d(x; r).$$

To prove the second inclusion, let $y \in B_\eta(x; r')$. Then $\eta(y, x) < r' \le 1/2$, which implies that $\eta(y, x) = d(y, x)$. Hence, $d(y, x) < r' \le r$.

(b) For any $x \in X$ and $r > 0$, we have

$$B_d(x; r) \subset B_\rho(x; r) \text{ and } B_\rho\left(x; \frac{r}{r + 1}\right) \subset B_d(x; r).$$

9.20 Yes. Let

$$\delta := \inf\{d(a, b) : a \in A, b \in B\},$$
$$A_1 := \{x \in X : d(x, a) < \delta/3 \text{ for some } a \in A\}$$
$$\text{and } B_1 := \{x \in X : d(x, b) < \delta/3 \text{ for some } b \in B\}.$$

Then $\overline{A_1}$ and $\overline{B_1}$ are disjoint closed subsets of (X, d). Since ρ is equivalent to d, these disjoint sets are closed in (X, ρ) too. Hence,

$$\inf\{\rho(a, b) : a \in A, b \in B\} \geq \inf\{\rho(a, b) : a \in \overline{A_1}, b \in \overline{B_1}\} > 0.$$

9.21 Yes. See Example 9.7.

9.24 (a) Apply Exercise 9.16 twice.
(b) Note that $B_{\rho_i}(-\infty; 1/2) = \{-\infty\}$ and $B_{\rho_i}(\infty; 1/2) = \{\infty\}$ for $i \in \{1, 2\}$.
(c) Use (b).
(d) $(\mathbb{R}^*, \rho_i)$ is not compact as $\{(-n, n) : n \in \mathbb{N}\} \cup \{\pm\infty\}$ has no finite subcover. It is not connected as it has non-trivial clopen sets $\{-\infty\}$ and $\{+\infty\}$.
(e) Apply the definition of convergence with $\epsilon = 1/2$.
(f) Let $\{x_n\}$ be a Cauchy sequence in $(\mathbb{R}^*, \rho_i)$. If it is bounded, it is a sequence of reals so convergent. Otherwise it is eventually constant and in $\{-\infty, +\infty\}$.
(g) $\mathbb{Q} \cup \{-\infty, +\infty\}$ is a countable dense subset of the space.
(h) $(\mathbb{R}^*, \rho_i)$ is not totally bounded, as the subspace $\mathbb{R}$ is not totally bounded.

9.25 The second assertion is immediate from the given inequality. Let $\epsilon > 0$ be arbitrary and $\eta := \min\{\alpha, \epsilon\alpha_1\}$. Then $\rho_2(x, y) < \eta$, we have $\rho_1(x, y) < \eta\alpha^{-1} \leq \epsilon$. Hence, the identity map $(X, \rho_2) \longrightarrow (X, \rho_1)$ is uniformly continuous. Similarly, the identity map $(X, \rho_1) \longrightarrow (X, \rho_2)$ is uniformly continuous.

9.26 Yes, d is a metric on $\mathbb{R}$. Since d is bounded, d is not Lipschitz equivalent to the usual metric on $\mathbb{R}$. By Exercise 9.25, d is uniformly equivalent to the usual metric on $\mathbb{R}$.

9.27 Let $f : (X, \rho_r) \longrightarrow (X, \rho_s)$ be the identity map, $g := f^{-1}$, and $\epsilon > 0$.

Case I: $r, s \in (0, 1)$. Then there exists some $k \in \mathbb{N}$ such that $s^k < \epsilon$. If $x, y \in X$ such that $\rho_r(x, y) < r^k$, then at least first k terms of x and y are identical, and thus $\rho_s(x, y) < s^k < \epsilon$. Hence, f is uniformly continuous. Interchanging r and s, we conclude that $g = f^{-1}$ is also uniformly continuous. Hence, ρ_r and ρ_s are uniformly equivalent.

Now assume that f is Lipschitz continuous, for some $0 < r < s < 1$. Then there exists some $M > 0$ such that $\rho_s(x, y) \leq M\rho_r(x, y)$ for all $x, y \in X$. Therefore, $s^k \leq Mr^k$, that is, $(\frac{s}{r})^k \leq M$ for all $k \in \mathbb{N}$. Passing limit $k \longrightarrow \infty$ we obtain $M = \infty$, a contradiction. Hence, ρ_r and ρ_s are not Lipschitz equivalent.

Case II: $r = 0, s \in (0, 1)$. Then there exists $k \in \mathbb{N}$ such that $\max\{s^k, 1/k\} < \epsilon$. If $x, y \in X$ satisfy $\rho_0(x, y) < 1/k$, then at least first k terms of x and y are identical; and thus $\rho_s(x, y) < s^k < \epsilon$. Similarly, if $x, y \in X$ satisfy $\rho_s(x, y) <$

s^k, then at least first k terms of x and y are identical; and so $\rho_0(x, y) < 1/k < \epsilon$. Hence, ρ_0 and ρ_s are uniformly equivalent.

Now assume that there exists some $s \in (0, 1)$ and $M > 0$ such that $\rho_0(x, y) \leq M\rho_s(x, y)$ for all $x, y \in X$. Therefore, $1/k \leq Ms^k$, that is, $1 \leq Mks^k$ for all $k \in \mathbb{N}$. Passing limit $k \longrightarrow \infty$ we obtain $1 \leq 0$, a contradiction. Hence, ρ_0 and ρ_s are not Lipschitz equivalent.

9.28 Yes. Let d and d' be any two metrics on a space X with same compact sets. It is enough to show that they have same convergent sequences. Let $x_n \longrightarrow x_0$ in (X, d). Write $A := \{x_n : n \in \mathbb{N} \cup \{0\}\}$. If A is a finite set, then this sequence is eventually constant and thence convergent in (X, d') too.

Without loss of generality, suppose that $x_n \neq x_m$ for all $n \neq m$. Note that A is a compact subset of (X, d) and hence so in (X, d'). Therefore, the infinite set A has a limit point, say y, in (A, d'). If $y = x_k$ for some $k > 0$, then $A \setminus \{y\}$ is compact in (X, d) but not in (X, d'), a contradiction. Hence, $y = x_0$.

If $\{x_{n_k}\}$ is any subsequence of $\{x_n\}$, then $A_0 := \{x_{n_k} : k \in \mathbb{N}\}$ is an infinite subset of the compact set A and hence has a limit point in (A, d'). Applying Lemma 4.21, there exists a subsequence of $\{x_{n_k}\}$, convergent to x_0. Hence, by Exercise 2.48, we conclude that $x_n \longrightarrow x_0$ in (X, d'). Similarly, it can be shown that every convergent sequence in (X, d') is also convergent in (X, d).

9.29 Yes. It is enough to show that they have same convergent sequences. Let $x_n \longrightarrow x_0$ in (X, d). Write $A := \{x_n : n \in \mathbb{N} \cup \{0\}\}$. If A is a finite set, then this sequence is eventually constant and thence convergent in (X, d') too.

Without loss of generality, suppose that $x_n \neq x_m$ for all $n \neq m$. Note that A is a complete subspace of (X, d) and hence (A, d') is also complete. Since $\{x_n\}$ is also Cauchy in (X, d'), there exists some $y \in A$ such that $x_n \longrightarrow y$, in (A, d'). If $y = x_k$ for some $k > 0$, then $\{x_{n+k}\}$ is Cauchy in d and d' but $\{x_{n+k} : n \in \mathbb{N}\} \cup \{x_0\}$ is complete in d but not in d'. Consequently, $x_n \longrightarrow x_0$ in (X, d').

9.30 Without loss of generality, let X be a complete metric space and O be any nonempty subset of X. Write $F := X \setminus O$ and for $x, y \in O$, we define

$$\rho(x, y) := d(x, y) + \left| \frac{1}{dist(x; F)} - \frac{1}{dist(y; F)} \right|,$$

where $dist(x; F) := \inf\{d(x, t) : t \in F\}$. It is easy to see that ρ is a metric on O and that has same convergent sequences as the metric d on O. Hence, ρ and d are equivalent on O. We claim that (O, ρ) is a complete metric space.

Let $\{x_n\}$ be any Cauchy sequence in (X, ρ). Then $\{x_n\}$ is Cauchy in (X, d) and $\{1/dist(x_n; F)\}$ is Cauchy in $\mathbb{R}$. Consequently, there exist $x_0 \in X$ and $r \in \mathbb{R}$ such that $\{x_n\} \longrightarrow x_0$ in (X, d) and $\{1/dist(x_n; F)\} \longrightarrow r$ in $\mathbb{R}$.

Note that $\{x_n\} \longrightarrow x_0$ in (X, d) implies that $\{dist(x_n; F)\} \longrightarrow dist(x_0; F)$. If $x_0 \notin O$, then $x \in F$ and hence $\{1/dist(x_n; F)\}$ is an unbounded sequence, a contradiction. Hence, $x_0 \in O$. Therefore, every Cauchy sequence in (O, ρ) is convergent in O.

9.31 $\{1\}$ is an open subset of $\mathbb{N}$, while $f(1) = \{0\}$ is not an open subset of Y.

9.37 (a) Yes. The map $x \longmapsto \tan \frac{\pi x}{2}$ is a homeomorphism from $(-1, 1)$ onto $\mathbb{R}$
(b) Yes. The map $x \longmapsto e^x$ is a homeomorphism from $\mathbb{R}$ onto $(0, +\infty)$.
(c) No. Because if $f : [0, 1) \longrightarrow \mathbb{R}$ is a homeomorphism, then $f((0, 1)) = \mathbb{R} \setminus \{f(0)\}$ would be connected, which is not true.
(d) No. Because if $f : [0, +\infty) \longrightarrow \mathbb{R}$ is a homeomorphism, then $f((0, +\infty)) = \mathbb{R} \setminus \{f(0)\}$ would be connected, which is not true.

9.47 Let f be a homeomorphism and $E \subset X$. Since f^{-1} is continuous, $f(\overline{E})$ is a closed subset of Y. Therefore, $f(\overline{E}) \supset \overline{f(E)}$. The opposite inclusion holds by Exercise 3.91.

Conversely, f is continuous by Exercise 3.91. Let F be a closed subset of X. Then $f(F) = \overline{f(F)}$ which implies that $f(F)$ is closed. Hence, f^{-1} is also continuous.

9.50 The homeomorphic classes are

$$\{A, R\}, \{B\}, \{C, G, I, J, L, M, N, S, U, V, W, Z\},$$
$$\{D, O\}, \{E, F, T, Y\}, \{H, K\}, \{P, Q\}, \{X\}.$$

9.51 Apply Exercise 9.48.

9.53 Let a, b, c be three non-collinear points, fixed by f, and $\|.\|$ denote the usual norm on $\mathbb{R}^2$. Then for $x \in \mathbb{R}^2$, we have $\|f(x) - a\| = \|f(x) - f(a)\| = \|x - a\|$, which implies $f(x).a = x.a$. Similarly, $f(x).b = x.b$ and $f(x).c = x.c$. Hence $(f(x) - x)(b - a) = 0 = (f(x) - x)(c - a)$, that is, $f(x) - x$ is orthogonal to two linearly independent vectors $b - a$ and $c - a$. Consequently, $f(x) = x$ for all $x \in \mathbb{R}^2$. The generalization is analogous.

9.54 Apply Exercise 9.53 on $g^{-1} \circ f$.

9.55 Removing the original circle from the torus is a connected space, while removing any proper closed curve from any sphere will make it disconnected.

9.57 Note that X_1 has two cut points and X_n has exactly n cut points, for each $n > 1$. Use the fact that cut points are preserved under homeomorphisms.

9.58 No. See Example 9.8.

9.59 Answer: Spaces homeomorphic to discrete spaces. Clearly, every discrete space X has a unique dense subset X. Conversely, let X be such a metric space and $x \in X$. Then the set $X \setminus \{x\}$ is not dense in X. Hence, $\{x\}$ is open in X.

9.60 Let $X := \mathbb{Q}$ and $Y := \{\frac{1}{n} : n \in \mathbb{N}\} \cup \{0\}$ and $f : X \longrightarrow Y$ be any bijection. Define $d(x, y) := |f(x) - f(y)|$ for all $x, y \in X$. Then f is a homeomorphism from (X, d) onto the subspace Y of $\mathbb{R}$. Hence, (X, d) is compact and therefore, complete. By Theorem 7.20, no metric d makes (X, d) connected.

9.61 Let $Y := \mathbb{Q}$ and $f : X \longrightarrow Y$ be any bijection. Define $d(x, y) := |f(x) - f(y)|$ for all $x, y \in X$. Then f is a homeomorphism from (X, d) to the subspace $\mathbb{Q}$ of $\mathbb{R}$. Hence, (X, d) is perfect. By Theorem 8.26, no metric d makes (X, d) perfect and complete.

9.63 Let $f : X \longrightarrow \ell^1$ be defined as $f(\{x_n\}) := \sum_{n=1}^{\infty} 2^n x_n$. Then $\|f(x)\|_1 = \|x\|$ for all $x \in X$. Hence, f is an isometry between X and ℓ^1. Since isometries are Lipschitz continuous, X is also complete and has dimension c.

9.65 Apply Exercise 9.30.

9.66 Apply Exercise 9.65.

9.68 We prove part (a) only. The other parts follow immediately. Note that for every $(a, b), (c, d) \in S$, we have

$$|g(a,b)-g(c,d)| = \left|\frac{ad-bc}{(a+b)(c+d)}\right| = \left|\frac{(a-c)d-(b-d)c}{(a+b)(c+d)}\right|$$
$$\leq \frac{|a-c|}{a+b} \times \frac{d}{c+d} + \frac{|b-d|}{a+b} \times \frac{c}{c+d} \leq \frac{1}{\delta}\big(|a-c|+|b-d|\big).$$

9.69 It is immediate that $h_i|_{F_1} \equiv 0$, $h_i|_{F_2} \equiv 1$ and $0 \leq h_i \leq 1$ on X. Define $p : X \longrightarrow [0, 1]$ as $p(x) := \min\{dist(F_1, F_2), dist(x, F_1)\}$. Let $x, y \in X$. It is enough to prove that

$$|p(x) - p(y)| \leq d(x, y).$$

This inequality is immediate for the case $p(x) = dist(F_1, F_2) = p(y)$. If $p(x) = dist(x, F_1)$ and $p(y) = dist(y, F_1)$, then it follows from Exercise 9.68(b). Now assume that $p(x) = dist(x, F_1) < dist(F_1, F_2)$ and $p(y) = dist(F_1, F_2)$. Then

$$|p(x) - p(y)| = dist(F_1, F_2) - dist(x, F_1) \leq dist(y, F_1) - dist(x, F_1) \leq d(x, y).$$

9.72 Yes. Consider

$$f(x) := \begin{cases} b - \frac{b-d}{c-a}(x-a) & ; a \leq x \leq c, \\ c + d - x & ; c \leq x \leq d, \\ c - \frac{c-a}{b-d}(x-d) & ; d \leq x \leq b. \end{cases}$$

9.75 Consider the map $x \longmapsto \alpha + \beta - x$.

9.80 Apply Theorems 9.34 and 9.43.

9.81 Apply Theorem 9.43, as $f : A \longrightarrow \widetilde{Y}$.

9.82 No. For example, take $A := (0, 1]$, $X := [0, 1]$ and $f(x) := \frac{1}{x}$ for all $0 < x \leq 1$.

9.85 If $g \equiv 0$, the result is trivial. Suppose that $g \not\equiv 0$. Since g is bounded, there exists some $r \in \mathbb{R}$ such that for $g_1 \equiv g - r$, we have $\sup g_1(F) = -\inf g_1(F) = M$, (say). Applying Tietze Extension Theorem (9.34), let f_1 denote a continuous function on X such that $f_1|_F \equiv g_1/M$ and $|f_1| \leq 1$ on X. Then $f \equiv Mf_1 + r$ satisfies our requirements.

9.90 No. Apply Corollary 9.55.

9.92 No. Let X be a infinite-dimensional normed linear space. Then X has a countably infinite linearly independent subset, say Y. By Theorem 9.55, $span(Y)$ is not complete.

9.95 Let $x \in X \setminus Y$. Since Y is complete, $t := dist(x; Y) > 0$. Take $z := x/t$. It can be verified that $dist(z; Y) = 1$, and hence there exists a sequence $\{y_n\} \subset Y$ such that $\|y_n - z\| \longrightarrow 1$. Therefore, $\{y_n\}$ is a bounded sequence in the finite-dimensional space Y. So it has a convergent subsequence, say $\{y_{n_k}\} \longrightarrow y$ in Y. Then $\|y - z\| = \lim_{k\to\infty} \|y_{n_k} - z\| = 1$. For $a := y - z$, we obtain $\|a\| = 1$ and $dist(a; Y) = dist(y - z; Y) = dist(z; Y) = 1$.

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Chapter 10
The Cantor Set

This chapter is a treatise on the Cantor set. It starts with a thorough discussion on the basic properties of this set. Then we present a weaker version of Tychonoff's theorem, which leads to an infinite product representation of the Cantor set. In the next section, we discuss a result of Alexandroff and Hausdorff which states that every complete perfect metric space contains a copy of the Cantor discontinuum.

We provide various characterizations of the Cantor space and their applications; including the Brouwer's theorem which states that every totally disconnected, compact, and perfect metric space is homeomorphic to the Cantor set. We also present a continuous real function that interpolates every bounded sequence of real numbers. This chapter winds up with some miscellaneous topics such as the Cantor function, homeomorphic permutations, and Cantor's leaky tent.

10.1 Introduction

The Cantor set is defined inductively by removing middle third open intervals from the unit interval [0, 1]. Consider the sequence of sets, of this inductive process:

$$
\begin{aligned}
C_0 &:= [0, 1],\\
C_1 &:= C_0 \setminus (1/3, 2/3) = [0, 1/3] \cup [2/3, 1],\\
C_2 &:= C_1 \setminus (1/9, 2/9) \cup (7/9, 8/9) = [0, 1/9] \cup [2/9, 1/3] \cup [2/3, 7/9] \cup [8/9, 1]\\
&\vdots
\end{aligned}
$$

In general, for $n \in \mathbb{N}$, the set C_n is a disjoint union of 2^n closed intervals, each of length $1/3^n$. The *Cantor set,* denoted by C, is defined as

$$C := \bigcap_{n=1}^{\infty} C_n.$$

In terms of ternary expansions, the sequence of sets $\{C_n\}$ can be expressed as

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$$C_0 = [0, (0.\overline{2})_3],$$
$$C_1 = [0, (0.0\overline{2})_3] \cup [(0.2)_3, (0.\overline{2})_3],$$
$$C_2 = [0, (0.00\overline{2})_3] \cup [(0.02)_3, (0.0\overline{2})_3] \cup [(0.2)_3, (0.20\overline{2})_3] \cup [(0.22)_3, (0.\overline{2})_3]$$
$$\vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots$$

Let $a \in C = \bigcap_{n=1}^{\infty} C_n$ and write a ternary representation of a, say

$$a = \sum_{n=1}^{\infty} \frac{a_n}{3^n} = (0.a_1a_2\ldots)_3; \text{ where each } a_n \in \{0, 1, 2\}.$$

If a has the finite ternary expansion $(0.1)_3$, then we also have $a = (0.0\overline{2})_3$. Note that the open interval $\big((0.1)_3, (0.2)_3\big)$ is removed initially from C_0, during the construction of C. Therefore, we have $a_1 \neq 1$. Similarly, one can observe that $a_n \neq 1$ for all $n \in \mathbb{N}$.

We also leave it to the readers to show that each real number with a ternary expansion of the form $(0.a_1a_2\ldots)_3$; $a_n \in \{0, 2\}$ belongs to the Cantor set. Hence

$$C = \big\{(0.a_1a_2\ldots)_3 : a_n \in \{0, 2\}\big\} = \left\{\sum_{k=1}^{\infty} \frac{a_k}{3^k} : a_k \in \{0, 2\}, k \in \mathbb{N}\right\}. \tag{10.1}$$

In this chapter, all the expansions of reals will be their ternary expansions, C and C_n will denote the sets defined above. We first establish that the Cantor set is perfect, uncountable, and negligible.

Theorem 10.1 *The Cantor set is a perfect set.*

Proof Being an intersection of closed sets C_n, it is immediate that C is closed. Let $a \in C$ and $\epsilon > 0$ be given. Then there exists $k \in \mathbb{N}$ such that $2/3^k < \epsilon$. Write $a = (0.a_1a_2\ldots)_3$, where each $a_n \in \{0, 2\}$. Let $b := (0.b_1b_2\ldots)_3$, where

$$b_n := \begin{cases} a_n & \text{if } n \neq k \\ 2 - a_k & \text{if } n = k. \end{cases}$$

Then $b \in C$ and $0 < |b - a| = \frac{2}{3^k} < \epsilon$. Consequently, a is a limit point of C. □

Since nonempty perfect subsets of $\mathbb{R}$ are uncountable (Theorem 8.26), the Cantor set is uncountable. Another proof for this fact will be presented in Proposition 10.34. Two alternate proofs are provided as under.

Theorem 10.2 *The Cantor set is uncountable.*

Proof Let X be the set of sequences with terms from $\{0, 2\}$. By (10.1), we have

$$C := \big\{(0.a_1a_2\ldots)_3 : a_n \in \{0, 2\}\big\}.$$

Define $f : C \longrightarrow X$ as $f\big((0.a_1a_2\ldots)_3\big) := \{a_n\}$, where each $a_n \in \{0, 2\}$. Then f is a bijection. By Theorem 7.13, X is uncountable. Hence, C is also uncountable. □

As a consequence of the above theorem, we conclude that the Cantor set contains uncountably many irrationals and uncountably many transcendental numbers.

Remark 10.3 It is well known that for any α such that $0 < |\alpha| < 1$, the numbers $\sum_{n=0}^{\infty} \alpha^{2^n}$ and $\sum_{n=0}^{\infty} \alpha^{n!}$ are transcendental. Hence, the reals given by

$$\sum_{n=0}^{\infty} \frac{2}{3^{2^n}} \text{ and } \sum_{n=0}^{\infty} \frac{2}{3^{n!}}$$

are transcendental numbers, which belong to the Cantor set.

It is immediate that the Cantor set contains infinitely many rational numbers, especially the endpoints of the middle third intervals that are removed during its construction.

In [1], Mahler poses eight open problems, including the following one:

Open Question 10.4 Does the Cantor set contain any irrational algebraic number?

Now we show that the converse of Example 8.43 is not true.

Proposition 10.5 *The Cantor set is an uncountable negligible set.*

Proof The uncountability of C is already proved in Theorem 10.2. As per our notations, $C \subset C_n$ and C_n is a union of 2^n intervals, each having length $1/3^n$. Since $n \in \mathbb{N}$ is arbitrary, by Definition 8.42, C is a negligible set. □

The sum or difference of two nowhere dense subsets of reals can be a set of positive measure. These can even be the whole space.

Theorem 10.6 $C + C = [0, 2]$, *that is* $\{a + b : a, b \in C\} = [0, 2]$.

Proof It is immediate that $C + C \subset [0, 2]$. For the opposite inclusion, pick any $x \in [0, 2]$ and write $\frac{x}{2}$ in its ternary expansion, say $\sum_{n=1}^{\infty} \frac{x_n}{3^n}$. For each $n \in \mathbb{N}$, define a pair (y_n, z_n) as follows:

$$(y_n, z_n) := \begin{cases} (0, 0)\ ; x_n = 0, \\ (2, 0)\ ; x_n = 1, \\ (2, 2)\ ; x_n = 2. \end{cases}$$

Define $y := \sum_{n=1}^{\infty} \frac{y_n}{3^n}$ and $z := \sum_{n=1}^{\infty} \frac{z_n}{3^n}$. Then $y, z \in C$ and $y_n + z_n = 2x_n$ for all $n \in \mathbb{N}$. Therefore, $y + z = 2 \times \frac{x}{2} = x$. Hence, $x \in C + C$ and the result follows. □

Corollary 10.7 $C - C = [-1, 1]$, *that is* $\{a - b : a, b \in C\} = [-1, 1]$.

Proof As usual, the inclusion $C - C \subset [-1, 1]$ is trivial. For the opposite inclusion, let $x \in [-1, 1]$. By Theorem 10.6, $x + 1 = y + z$ for some $y, z \in C$. Then $x = y - (1 - z) \in C - C$ Hence the result. □

The Cantor set C is totally disconnected. Hence, it is also known as *the Cantor discontinuum.* In Theorem 10.28, we shall establish that, up to homeomorphisms, C is the only metric space that is compact, perfect and totally disconnected.

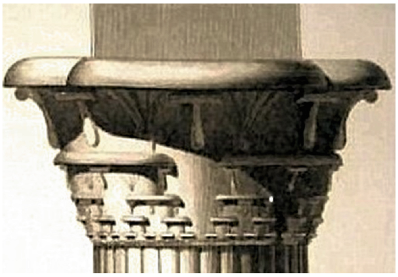
Let us wind up this section, with the notion of end points of the Cantor set, which will be used throughout this chapter.

Definition 10.8 A real number x is said to be an *end point* of the Cantor set if there exists $n \in \mathbb{N}$ and an interval $I \subset C_n$ with $l(I) = 3^{-n}$ such that x is an end point of I.

In other words, an end point of C is an end point of some connected component of some C_n. All other points of C are known as the *internal points* of C.

History Notes 10.9 It seems that the Cantor set is inspired by ancient Egyptian architecture.

As per [2, p. 17], 'Napoleon's Expedition brought this picture to Europe in their report, Description de L'Egypte. Notice the startling resemblance to the Cantor set diagram. Did George Cantor see the Egyptian columns before he conceived the set? We don't know, but it is a possibility because Cantor's cousin was a student of Egyptology.'



A brief history of the Cantor set can be found in [3]. For more on Theorem 10.6, see [4].

Exercise 10.1 Prove that $a \in C$ if and only if a can be expressed as a ternary expansion of the form

$$a = (0.a_1a_2 \ldots a_n \ldots)_3 = \sum_{k=1}^{\infty} \frac{a_k}{3^k};\ a_n \in \{0, 2\}.$$

Exercise 10.2 Prove Theorem 10.1, without using ternary expansions.

Exercise 10.3 Give an explicit example of an irrational number in the Cantor set.

Exercise 10.4 As a subset of $\mathbb{R}$, prove that the Cantor set has no interior point and it is nowhere dense.

Exercise 10.5 Prove that $1/4 \in C$. Obtain three points in C, which are not its end points.

Exercise 10.6 For each $n \in \mathbb{N}$, let L_n denote the set of left end points of the largest possible closed intervals of C_n, or equivalently the set of left end points of the connected components of C_n. Prove the following:

(a) Each $l \in L_n$ has a finite ternary expansion, which is zero after the first n-digits.
(b) $\bigcup_{n=1}^{\infty} L_n \subset C$.
(c) $\overline{\bigcup_{n=1}^{\infty} L_n} = C$.

Exercise 10.7 Does there exist a subspace of C, homeomorphic to $\mathbb{Q}$?

Exercise 10.8 Let L_n denote the subset of C, as in Exercise 10.6. Is $\bigcup_{n=1}^{\infty} L_n$ a complete subspace of $\mathbb{R}$?

Exercise 10.9 Let $\mathcal{I}$ denote the countable collection of closed intervals, inductively obtained while removing middle thirds in the construction of the Cantor set C. Let Ω denote the collection of nested decreasing sequences $\{I_n\}$ of distinct intervals from $\mathcal{I}$. Prove that

$$C = \left\{ \bigcap_{I \in \mathcal{M}} I : \mathcal{M} \in \Omega \right\} = \bigcup_{\mathcal{M} \in \Omega} \bigcap_{I \in \mathcal{M}} I.$$

Exercise 10.10 Is the Cantor set C totally disconnected?

Exercise 10.11 Prove that the family of connected components of the Cantor set is uncountable.

Exercise 10.12 Prove that the Cantor set is compact, hence conclude that it is totally bounded and separable.

Exercise 10.13 Show that the Cantor set is separable by constructing a countable dense subset of it.

Exercise 10.14 (Fat Cantor Set) Let $0 < \alpha < 1$. Analogous to the Cantor set, construct the fat Cantor set C_α by successively removing 2^n middle open intervals each of length $\alpha/3^n$ at n^{th} inductive step. Prove that C_α is a nonempty perfect non-negligible set with an empty interior.

Exercise 10.15 (Smith-Volterra-Cantor set) Analogous to the Cantor set, consider a set V obtained by removing the middle $\frac{8}{10}^{th}$ open interval of the available closed intervals at each stage, starting with $[0, 1]$. Prove that A is the set of reals in $[0, 1]$ having decimal expansion consisting only digits 0 and 9. Is V also negligible?

Exercise 10.16 Analogous to the Cantor set, construct a set $A \subset [0, 1]$ by inductively removing the second and fourth open intervals, after partitioning all the available intervals into five equal parts. E.g. at the first stage, we have $[0, 1/5] \cup [2/5, 3/5] \cup [4/5, 1]$ and so on. Which properties of A are analogous to the Cantor set?

Exercise 10.17 Does there exist a sequence of continuous functions on $[0, 1]$, pointwise convergent to a function discontinuous at uncountably many points?

Exercise 10.18 Let A be a subset of $\mathbb{R}$ such that A is not negligible. Is it necessary that A contains an interval?

Exercise 10.19 Prove that the Cantor set is a G_δ-set as well as an F_σ-set.

Exercise 10.20 Does there exist a nonempty countable set A such that $C \setminus A$ is a perfect set.

Exercise 10.21 Let C denote the Cantor set and $\epsilon > 0$ be given. Does there exist a sequence $\{r_n\}$ of rationals in $[0, 1]$ such that $C \setminus \bigcup_{n=1}^{\infty} \left(r_n - \frac{\epsilon}{2^n}, r_n + \frac{\epsilon}{2^n}\right) = \emptyset$?

Exercise 10.22 Let C denote the Cantor set and $f : \mathbb{R} \longrightarrow \mathbb{R}$ be defined as

$$f(x) := \begin{cases} 1 \,;\, x \in C, \\ 0 \,;\, x \in \mathbb{R} \setminus C. \end{cases}$$

Prove that f is discontinuous at every point of C.

Exercise 10.23 Is $[0, 1] \setminus C$ dense in $[0, 1]$?

Exercise 10.24 Does there exist any uncountable closed subset of the Cantor set consisting of only irrationals numbers?

Exercise 10.25 Let P be a perfect subset of reals and I be an open interval such that $I \cap P \neq \emptyset$. Does it imply that $\overline{I} \cap P$ is a perfect set?

Exercise 10.26 Let $f : C \longrightarrow C$ be a homeomorphism. Prove that there exists $m \in \mathbb{N}$ such that for $x, y \in C$ with $x - y < 3^{-m}$, both $f(x)$ and $f(y)$ have the same digit at the first place of their infinite ternary expansions.

Exercise 10.27 Let A be any countably infinite subset of the Cantor set. Prove that $C \setminus A$ is non-compact. Also, do the same if $0 < card(A) < c$.

Exercise 10.28 Prove that there exists $x \in \mathbb{R}$ such that $C + x \subset \mathbb{R} \setminus \mathbb{Q}$.

Exercise 10.29 Let A be any dense subset of C and $\{L_n\}$ be the sequence of sets as in Exercise 10.6. Prove that there exists a dense subset A_0 of A and a homeomorphism $f : A_0 \longrightarrow \bigcup_{n=1}^{\infty} L_n$.

Exercise 10.30 A number is chosen at random from the Cantor set. If all possibilities are equally likely, what is the probability that it is a rational number?

10.2 An Infinite Product Representation

In this section, we shall establish that the Cantor set is homeomorphic to a countable infinite product of discrete metric spaces. First we present a metric on such a product of metric spaces, which makes the product compact if and only if each component is compact.

Let $\{(X_n, d_n)\}$ be a sequence of metric spaces. Without loss of generality, suppose that each $d_n \leq 1$, otherwise replace the corresponding d_n with the equivalent metric $d_n/(1 + d_n)$ (see Theorem 9.15). Let $X := \prod_{n=1}^{\infty} X_n$, that is, the collection of sequences $\{x_n\}$ such that $x_n \in X_n$ for all $n \in \mathbb{N}$. For every $x = \{x_n\}$, $y = \{y_n\} \in X$, define

$$d(x, y) := \sup\left\{\frac{d_n(x_n, y_n)}{n} : n \in \mathbb{N}\right\}.$$

It can be shown that d is a metric on X, which we leave as an exercise for the reader. This metric, referred to as the *product metric,* will be considered as the default metric on the *product space* $\prod_{n=1}^{\infty} X_n$. The topology of (X, d) will be known as the *product topology* on $\prod_{n=1}^{\infty} X_n$.

Let $B(\cdot, \cdot)$ and $B_n(\cdot, \cdot)$ denote open balls in X and X_n, respectively. Note that for every $x = \{x_n\} \in X$ and $r > 0$, we have

$$\begin{aligned} B(x; r) &= \{y \in X : d(y, x) < r\} \\ &\subset \prod_{n=1}^{\infty}\{y_n \in X_n : d_n(x_n, y_n) < nr\} = \prod_{n=1}^{\infty} B_n(x_n; nr). \end{aligned}$$

Let $k \in \mathbb{N}$ such that $1/k < r$. Then

$$B(x; r) \subset \prod_{n=1}^{k-1} B_n(x_n; nr) \times \prod_{n=k}^{\infty} X_n \subset \prod_{n=1}^{k-1} B_n(x_n; kr) \times \prod_{n=k}^{\infty} X_n.$$

This motivates us to consider the following type of sets:

$$P_m(x; r) := \prod_{n=1}^{m-1} B_n(x_n; r) \times \prod_{n=m}^{\infty} X_n, \text{ where } m \in \mathbb{N}, x \in X \text{ and } r > 0.$$

Let

$$\beta := \{P_m(x; r) : m \in \mathbb{N}, x \in X, r > 0\}.$$

Throughout this section, the notations (X, d), (X_n, d_n), $B_n(\cdot, \cdot)$, $B(\cdot, \cdot)$, $P_m(\cdot, \cdot)$ and β will be reserved for the notions, discussed above.

Proposition 10.10 *β is a topological basis for the metric space (X, d).*

Proof We need to establish the following:

(a) For every $x \in X$ and $r > 0$ there exists some $P \in \beta$ such that $x \in P \subset B(x;r)$.
(b) Every $P \in \beta$ is an open subset of (X, d).

If $x \in X$ and $r > 0$, then $mr > 1$, for some $m \in \mathbb{N}$. Hence, $P_m(x;r) \subset B(x;r)$. This proves (a).

For (b), consider any $P_k(y;s) \in \beta$, where $y \in X, k \in \mathbb{N}$ and $s > 0$. Let $z \in P_k(y;s)$. Then $d_n(z_n, y_n) < s$ for all $n < k$. Let

$$\delta := \min\left\{\frac{s - d_n(z_n, y_n)}{n} : n < k\right\}.$$

We claim that $B(z;\delta) = \prod_{n=1}^{\infty} B_n(z_n; n\delta) \subset P_k(y;s)$. To see this, let $w = \{w_n\} \in B(z;\delta)$. Then for all $n < k$, we have

$$d_n(w_n, z_n) < n\delta \leq s - d_n(z_n, y_n) \text{ which implies } d_n(w_n, y_n) < s.$$

Consequently, $w \in P_k(y;s)$. This proves our claim and hence (b). □

Theorem 10.11 *Let $x \in X$ and $\{x^{(k)}\}_k$ be a sequence in X. Write*

$$x = \{x_n\}_n \text{ and } x^{(k)} = \{x_n^{(k)}\}_n \text{ for all } k \in \mathbb{N}.$$

Then $\lim_{k\to\infty} x^{(k)} = x$, *in X if and only if* $\lim_{k\to\infty} x_n^{(k)} = x_n$, *in X_n for all $n \in \mathbb{N}$.*

Proof The necessity follows from the inequality $d_n \leq nd$ for all $n \in \mathbb{N}$. For the converse, assume that $\lim_{k\to\infty} x_n^{(k)} = x_n$ in X_n for all $n \in \mathbb{N}$. Let $r > 0$ be given. Then $mr > 1$, for some $m \in \mathbb{N}$. As in Proposition 10.10, we obtain $P_m(x;r) \subset B(x;r)$. Since $\lim_{k\to\infty} x_n^{(k)} = x_n$ in X_n for all $n \leq m$, there exists $N_1 \in \mathbb{N}$ such that $x_n^{(k)} \in B_n(x_n;r)$ for all $k \geq N_1$ and for all $n \leq m$. Thus $x^{(k)} \in P_m(x;r) \subset B(x;r)$ for all $n \geq N_1$. Hence, $\lim_{k\to\infty} x^{(k)} = x$ in X. □

For every $n \in \mathbb{N}$, the *projection map* $\pi_n : (X, d) \longrightarrow (X_n, d_n)$ is defined as

$$\pi_n(x) := x_n \text{ for all } x = \{x_n\} \in X.$$

Then each π_n is Lipschitz continuous, as $d_n(x_n, y_n) \leq nd(x, y)$ for all $x = \{x_n\}, y = \{y_n\} \in X$.

Theorem 10.12 (Tychonoff) $\prod_{n=1}^{\infty} X_n$ *is compact if and only if each X_n is compact.*

Proof If $\prod_{n=1}^{\infty} X_n$ is compact, by Theorem 5.31, each $X_n = \pi_n(X)$ is compact. Conversely, assume that each X_n is compact. By Theorem 5.27, it is enough to show that every sequence in the metric space (X, d) has a convergent subsequence.

Let $\{x^{(k)}\}_k$ be a sequence in X. Write $x^{(k)} := \{x_n^{(k)}\}_n$ for all $k \in \mathbb{N}$. Since $\{x_1^{(k)}\}_k$ is a sequence in the compact metric space X_1, it has a convergent subsequence, say $\{x_1^{(1_i)}\}_i$. Now $\{x_2^{(1_i)}\}_i$ is a sequence in the compact space X_2 and hence has a convergent subsequence, say $\{x_2^{(2_i)}\}_i$.

Inducting this way, for all $n \in \mathbb{N} \setminus \{1\}$, we obtain a sequence $\{n_i\}_i$ of $\mathbb{N}$ such that $\{n_i\}_i$ is a subsequence of $\{(n-1)_i\}_i$ and $\{x_n^{(n_i)}\}_i$ is convergent in X_n.

Then for all $n \in \mathbb{N}$, $\{i_i\}_i$ is a subsequence of $\{n_i\}_i$ and thus $\{x_n^{(i_i)}\}_i$ is convergent in X_n. Applying Theorem 10.11, we conclude that $\{x^{(i_i)}\}_i$ is convergent in X. Hence the result. □

Theorem 10.13 *Let $X_n := \{0, 1\}$ be the discrete space, for every $n \in \mathbb{N}$. Then the Cantor set is homeomorphic to the product space $\prod_{n=1}^{\infty} X_n$.*

Proof Write $X := \prod_{n=1}^{\infty} X_n$. Consider a function $f : X \longrightarrow C$ defined as

$$f(\{x_n\}) := \sum_{n=1}^{\infty} \frac{2x_n}{3^n} \text{ for all } \{x_n\} \in X.$$

We claim that f is a homeomorphism. It is a routine exercise to prove that f is a bijection, which we leave to the reader. By Theorem 10.12, X is a compact metric space. Applying Theorem 5.34, it is enough to show that f is a continuous function.

Let $x = \{x_n\} \in X$ and $\epsilon > 0$ be given. We shall find an open subset O of X containing x such that $|f(y) - f(x)| < \epsilon$ for all $y \in O$. Let $N \in \mathbb{N}$ such that $\sum_{n>N} \frac{2}{3^n} < \epsilon$ and

$$O := \{x_1\} \times \{x_2\} \times \cdots \times \{x_N\} \times \prod_{n>N} X_n.$$

Due to the discrete topology on each X_n, the set O is open in $\prod_{n=1}^{\infty} X_n$. If $y \in O$, then $y = (x_1, \ldots, x_N, y_{N+1}, y_{N+2}, \ldots)$ for some $y_{N+k} \in X_{N+k}$ for all $k \geq 1$. Note that

$$|f(y) - f(x)| = \left| \sum_{n>N} 2\frac{y_n - x_n}{3^n} \right| \leq \sum_{n>N} \frac{2}{3^n} < \epsilon.$$

Therefore, f is continuous at x. Hence the result. □

Finally, we establish $[0, 1]^{\mathbb{N}}$ as a universal compact metric space.

Theorem 10.14 *Every compact metric space embeds homeomorphically into* $[0, 1]^{\mathbb{N}}$.

Proof If (X, d) is a compact metric space, then it is separable. Let $S = \{s_n : n \in \mathbb{N}\}$ be any countable dense subset of X. By replacing d with an equivalent metric $d/(1 + d)$, if required, without loss of generality, we assume that $0 \leq d \leq 1$. Define $f : X \longrightarrow [0, 1]^{\mathbb{N}}$ as

$$f(x) := \{d(x, s_n)\} \text{ for all } x \in X.$$

We claim that $f : X \longrightarrow f(X)$ is the required homeomorphism. Since $x \longmapsto d(x, s_n)$ is a continuous map from X into $[0, 1]$ for all n, analogous to Theorem 10.11, it can be shown that $f : X \longrightarrow [0, 1]^{\mathbb{N}}$ is a continuous map.

Suppose $f(x) = f(y)$ for some $x, y \in X$. Then $d(x, s) = d(y, s)$ for all $s \in S$. Since $x \in X = \overline{S}$, there exists a sequence $\{x_n\} \subset S$ such that $x_n \longrightarrow x$. Therefore, $d(y, x_n) = d(x, x_n) \longrightarrow 0$ which implies $y = x$. Hence, f is injective.

If F is any closed subset of X, then F is compact, and hence so is $f(F)$ in $f(X)$. Therefore, $f(F)$ is a closed subset of $f(X)$. This establishes that $f^{-1} : f(X) \longrightarrow X$ is a continuous map. Hence the result. □

Remarks 10.15 It is pertinent to mention that Theorem 10.12 is a particular case of one of the most non-trivial results in topology (see [5, p. 232]). For more on the product metric, the reader is referred to [6, Chapter 6].

Exercise 10.31 Prove that the function f in the proof of Theorem 10.13 is a bijection.

Exercise 10.32 Prove that the function f in the proof of Theorem 10.14 is continuous.

Exercise 10.33 Prove that every totally bounded space embeds homeomorphically into $[0, 1]^{\mathbb{N}}$.

Exercise 10.34 Let X be a discrete metric space. Prove that X is countable if and only if X is homeomorphic to a subspace of $[0, 1]^{\mathbb{N}}$.

Exercise 10.35 Prove that $[0, 1]^{\mathbb{N}}$ is a continuous image of the Cantor set $\{0, 1\}^{\mathbb{N}}$.

Exercise 10.36 Let $\{x^{(k)}\}_k$ be a sequence in X and $x^{(k)} = \{x_n^{(k)}\}_n$ for all $k \in \mathbb{N}$. Prove that $\{x^{(k)}\}_k$ is Cauchy in $X \iff \{x_n^{(k)}\}_k$ is Cauchy in X_n for all $n \in \mathbb{N}$.

Exercise 10.37 Let $\{(X_n, d_n) : n \in \mathbb{N}\}$ be any countable collection of metric spaces. Define $X := \prod_{n=1}^{\infty} X_n$. For any $x = \{x_n\}, y = \{y_n\} \in X$, define

$$\overline{d}(x, y) := \sum_{n=1}^{\infty} \frac{1}{2^n} \cdot \frac{d_n(x_n, y_n)}{1 + d_n(x_n, y_n)}.$$

Prove that $(X, \overline{d})$ is compact if and only if each (X_n, d_n) is a compact.

Exercise 10.38 As per the notations of this section, prove that

(a) $\prod_{n=1}^{\infty} X_n$ is complete if and only if each X_n is a complete metric space.
(b) $\prod_{n=1}^{\infty} X_n$ is totally bounded if and only if each X_n is totally bounded.

Exercise 10.39 As per the notations of this section, define

$$\rho(x, y) := \sum_{n=1}^{\infty} \frac{d_n(x_n, y_n)}{2^n} \text{ for all } x = \{x_n\}, y = \{y_n\} \in X.$$

Prove that ρ defines a metric on X, β is also a base for ρ and that ρ and d are equivalent.

Exercise 10.40 Let ρ be as in Exercise 10.39. Without using the fact that ρ and d are equivalent, prove the following:

(a) (X, ρ) is complete if and only if each (X_n, d_n) is a complete metric space.
(b) (X, ρ) s totally bounded if and only if each (X_n, d_n) is totally bounded.
(c) (X, ρ) s compact if and only if each (X_n, d_n) is compact.

Exercise 10.41 Let ρ be as in Exercise 10.39. Prove that there exists some $M < \infty$ such that

$$\rho(x, y) \leq Md(x, y) \text{ for all } x, y \in X.$$

Also, show that the metrics ρ and d are not Lipschitz equivalent.

Exercise 10.42 Are the metrics ρ and d, as in Exercise 10.39, uniformly equivalent?

Exercise 10.43 Prove that $\prod_{n=1}^{\infty} X_n$ is path connected if and only if each X_n is path connected.

Exercise 10.44 Let $\{X_n\}$ be a sequence of connected spaces, $X = \prod_{i=1}^{\infty} X_n$ and $a = \{a_n\}_{n \in \mathbb{N}}$ be a fixed point of X. Given any $n \in \mathbb{N}$, let $Y_n := \{\{x_n\} \in X : x_k = a_k \text{ for all } k > n\}$. Prove that

(a) For every $n \in \mathbb{N}$, the space Y_n is homeomorphic to $\prod_{i=1}^{n} X_i$ and hence connected.
(b) $Y = \bigcup_{n=1}^{\infty} Y_n$ is connected.
(c) Y is a dense subset of X.
(d) X is connected.

Exercise 10.45 Prove that $\prod_{n=1}^{\infty} X_n$ is connected if and only if each X_n is connected.

Exercise 10.46 Show that each projection map π_n is uniformly continuous and maps open sets onto open sets.

Exercise 10.47 Prove that $\prod_{n=1}^{\infty} A_n$ is dense in $\prod_{n=1}^{\infty} X_n$ if and only if A_n is dense in X_n for all $n \in \mathbb{N}$.

Exercise 10.48 If $X := \prod_{n=1}^{\infty}[0, 1]$, and $d(x, y) := \sum_{n=1}^{\infty} \frac{1}{2^n}|x_n - y_n|$ for all $x = \{x_n\}, y = \{y_n\} \in X$, does there exist any countably infinite nonempty perfect subset of (X, d)?

Exercise 10.49 Prove that the Cartesian product of any finite collection of Cantor sets is also homeomorphic to the Cantor set.

10.3 Embedding Cantor Set Inside Metric Spaces

In this section, we discuss a classical result of Alexandroff and Hausdorff that every complete perfect metric space contains a copy of the Cantor discontinuum.

Theorem 10.16 *Every perfect complete metric space contains a subspace that is homeomorphic to the Cantor set.*

Proof Let X be a perfect complete metric space. Applying induction, we shall construct a copy of the Cantor set inside X. Pick any two points $x_0, x_1 \in X$ and write $r := d(x_0, x_1)$. Let

$$K_1 := \overline{I_0} \dot{\bigcup} \overline{I_1}, \text{ where } I_i := B\left(x_i; \frac{r}{4}\right) \text{ for all } i = 0, 1.$$

Then for each $i \in \{0, 1\}$, pick two different points $x_{i,0}, x_{i,1} \in I_i$. This is possible, as X is perfect and I_i is a neighborhood of x_i, so I_i contains infinitely many points of X. Applying triangle inequality, we obtain $d(x_{i,0}, x_{i,1}) < \frac{r}{2}$, for $i = 0, 1$. Define

$$J_{i_1,i_2} := B\left(x_{i_1,i_2}; \frac{d(x_{i_1,0}, x_{i_1,1})}{4}\right) \text{ and } I_{i_1,i_2} := K_1 \bigcap J_{i_1,i_2} \text{ for every } i_1, i_2 \in \{0, 1\}.$$

Write $K_2 := \overline{I_{0,0}} \dot{\bigcup} \overline{I_{0,1}} \dot{\bigcup} \overline{I_{1,0}} \dot{\bigcup} \overline{I_{1,1}}$. Again by triangle inequality, $diam(I_{i_1,i_2}) < \frac{r}{4}$ for every $i_1, i_2 \in \{0, 1\}$. Let

$$K_n := \dot{\bigcup}_{i_1,\ldots,i_n \in \{0,1\}} \overline{I_{i_1,\ldots,i_n}}$$

has been constructed such that $diam(I_{i_1,\ldots,i_n}) \le \frac{r}{2^n}$, for every $i_1, \ldots, i_n \in \{0, 1\}$. For each tuple $t = (i_1, \ldots, i_n) \in \{0, 1\}^n$, pick two different points $x_{i_1,\ldots,i_n,0}$ and $x_{i_1,\ldots,i_n,1}$ from $I_{i_1,\ldots,i_n}$. Define

$$\begin{aligned} J_{i_1,\ldots,i_{n+1}} &:= B\left(x_{i_1,\ldots,i_{n+1}}; \frac{d(x_{i_1,\ldots,i_n,0}, x_{i_1,\ldots,i_n,1})}{4}\right) \\ I_{i_1,\ldots,i_{n+1}} &:= K_n \bigcap J_{i_1,\ldots,i_{n+1}} \text{ for every } i_1, \ldots, i_{n+1} \in \{0, 1\} \\ \text{and } K_{n+1} &:= \dot{\bigcup}_{i_1,\ldots,i_{n+1} \in \{0,1\}} \overline{I_{i_1,\ldots,i_{n+1}}} \end{aligned}$$

Again by the triangle inequality, $diam(I_{i_1,\ldots,i_{n+1}}) < \frac{r}{2^{n+1}}$ for every $i_1, \ldots, i_{n+1} \in \{0, 1\}$. This completes the induction process, and finally, we define $K := \bigcap_{n=1}^{\infty} K_n$.

Let C denote the standard Cantor set inside $[0, 1]$. Define $f : K \longrightarrow C$ as

$$f\left(\bigcap_{i_1,\ldots,i_n,\ldots} \overline{I_{i_1,\ldots,i_n,\ldots}}\right) = \sum_{n=1}^{\infty} \frac{2i_n}{3^n}.$$

It can be proven that f is a homeomorphism from K onto C. □

Corollaries 10.17 *(a) Every nonempty perfect subset of* $\mathbb{R}$ *has cardinality* c.
(b) Every uncountable closed subset of $\mathbb{R}$ *has cardinality* c.
(c) Every perfect complete metric space is uncountable.

Proof (a) Apply Theorem 10.16.
(b) Apply (a), along with Theorem 8.31.
(c) Apply Theorem 10.16, along with Theorem 10.2. □

Since every closed subset of $\mathbb{R}$ is either countable or has cardinality c, such subsets of reals satisfy the continuum hypothesis. The same holds in complete separable metric spaces.

Theorem 10.18 *Let E be a closed subset of a complete separable metric space X. Then E is either countable or has cardinality c.*

Proof Assume that E is uncountable. Let E_0 denote the set of isolated points of E. Since X is separable, E_0 is countable. Hence, $E \setminus E_0$ is a complete and perfect metric space. Applying Theorem 10.16, $E \setminus E_0$ contains a subspace homeomorphic to the Cantor set. Hence, $card(E) \geq card(E \setminus E_0) \geq c$. From Theorem 8.5, we have $card(E) \leq c$. Hence, $card(E) = c$. □

Now we present a generalization to Example 8.28.

Examples 10.19 (a) Every nonempty perfect subset of the space $\mathbb{R} \setminus \mathbb{Q}$, under usual metric, has cardinality c.
(b) Let X be a complete metric space and Y be a subspace of X such that $card(X \setminus Y) < c$. Then every nonempty perfect subset of Y has cardinality c.

Proof (a) Consider any nonempty perfect subset P_0 of $\mathbb{R} \setminus \mathbb{Q}$. Then P_0 has no isolated points. Therefore the closure of P_0 in reals, say P is a perfect subset of $\mathbb{R}$. Hence, $card(P) = c$. Since $P \setminus P_0$ is a subset of $\mathbb{Q}$, it is at most countable, which implies that $card(P_0) = c$.
(b) Note that Theorem 10.16 ensures that every perfect subset of X contains a homeomorphic copy of C and hence has cardinality at least c. The result follows by imitating the arguments in part (a). □

Remarks 10.20 A natural question is that whether there exists a perfect metric space X with $card(X) > c$? The answer is in the affirmative. In fact, for any cardinal number α, there exists a perfect metric space with cardinality $> \alpha$ (see [7]).

Let Ω denote the collection of cardinal numbers α such that there exists a perfect metric space with cardinality α. Then, by [7], Ω contains each of the cardinal numbers $c, 2^c, 2^{2^c}, \ldots$. It is also easy to see that Ω contains all α such that $\aleph_0 \leq \alpha \leq c$.

The question remains open whether Ω contains any other cardinal number, apart from these.

We also refer to [8] for Classical Descriptive Set Theory by Kechris, which contains a more detailed treatment to Polish spaces.

Exercise 10.50 Is there any perfect complete metric space with cardinality $< c$?

Exercise 10.51 Is there any complete separable metric space with cardinality $< c$?

Exercise 10.52 Show that no hypothesis of Theorem 10.18 is redundant.

Exercise 10.53 If f is the mapping as defined in the proof of Theorem 10.16, prove that $f : K \longrightarrow C$ is a homeomorphism.

Exercise 10.54 Prove that every uncountable separable complete metric space contains a subspace homeomorphic to the Cantor set.

Exercise 10.55 Prove that every complete metric space without isolated points contains a perfect set, homeomorphic to the Cantor set.

Exercise 10.56 Let $E \subset \mathbb{R}$ be homeomorphic to the Cantor set. Prove that E is perfect, bounded, and nowhere dense.

Exercise 10.57 Let X be a complete separable metric space. Prove that the collection of closed subsets of X satisfies the continuum hypothesis. That is, show that every closed subset of X is either countable or has cardinality c.

Exercise 10.58 Let X be an uncountable complete separable metric space, and X_0 be its set of isolated points. If X_0 is countable, prove that X contains a homeomorphic copy of the Cantor set.

Exercise 10.59 Let $A \subset \mathbb{R}$ be any uncountable set. Prove that there exists a perfect metric space having cardinality same as A.

10.4 Characterizations in Terms of the Cantor Set

This section is devoted to some topological notions which can be characterized in terms of the Cantor set. First consider the case of total boundedness and compactness.

10.4.1 Cantor Set and Compact Metric Spaces

Theorem 10.21 *A metric space X is totally bounded if and only if there exists a uniformly continuous bijection from a subset of the Cantor set onto X.*

Proof The converse is easy. Let C_0 be a subset of the Cantor set C and $f : C_0 \longrightarrow X$ be a uniformly continuous bijection. Being a compact set, C is totally bounded and thus so is its subset C_0. By Theorem 5.42, X is totally bounded.

To prove the necessity, for each $m \in \mathbb{N}$, let $\mathcal{B}_m$ be a finite collection of open balls with radius $\frac{1}{m}$ such that $X = \bigcup_{B \in \mathcal{B}_m} B$. Then the collection of all such open balls, i.e. $\bigcup_{m \in \mathbb{N}} \mathcal{B}_m$, is countable. We write

$$\bigcup_{m \in \mathbb{N}} \mathcal{B}_m = \{U_n : n \in \mathbb{N}\}.$$

Without loss of generality, assume that the balls in $\mathcal{B}_m$ precede the balls in $\mathcal{B}_{m+1}$ in this enumeration. Then $diam(U_n) \longrightarrow 0$, as $n \longrightarrow \infty$. For every $x \in X$, define

$$h(x) := \sum_{n=1}^{\infty} \frac{\alpha_n(x)}{3^n}, \text{ where } \alpha_n(x) := \begin{cases} 2 \,; x \in U_n, \\ 0 \,; x \notin U_n. \end{cases}$$

Then $h(x) \in C$ for all $x \in C$. Also, note that every $x \in X$ belongs to infinitely many U_n. So $\alpha_n(x) = 2$, for infinitely many $n \in \mathbb{N}$.

We *claim* that $h : X \longrightarrow C$ is injective. Pick any $x, y \in X$ such that $x \neq y$. Pick any $N_1 \in \mathbb{N}$ such that $\frac{2}{N_1} < d(x, y)$. Let $N_2 := |\mathcal{B}_1| + \cdots + |\mathcal{B}_{N_1}|$, here $|A|$ denotes the number elements in the set A. Let $n > N_2$ such that $x \in U_n$. Then $\alpha_n(x) = 2$ and $diam(U_n) < \frac{2}{N_1} < d(x, y)$. Therefore, $\alpha_n(y) = 0$. Consequently, $h(x) \neq h(y)$.

This proves that $h : X \longrightarrow C$ is an injective function. Write $C_0 := h(X)$ and $f := h^{-1}$. Then $f : C_0 \longrightarrow X$ is a bijection. Now we prove that it is uniformly continuous.

Let $\epsilon > 0$ be given. Then there exists $N_3 \in \mathbb{N}$ such that $\frac{2}{N_3} < \epsilon$ and write $N_4 := |\mathcal{B}_1| + \cdots + |\mathcal{B}_{N_3}|$. Then for all $n > N_4$, $diam(U_n) \leq \frac{2}{N_3} < \epsilon$. Pick any $a, b \in C_0$ such that $|a - b| < 3^{-N_4}$. Write $x := f(a)$ and $y := f(b)$. Then $a = h(x)$ and $b = h(y)$.

Since $|a - b| < 3^{-N_4}$, we have $\alpha_n(x) = \alpha_n(y)$ for all $n \leq N_4$. That is, for all $n \leq N_4$, $x \in U_n$ if and only if $y \in U_n$. Pick $q \leq N_4$ such that $x \in U_q \in \mathcal{B}_{N_3}$. Then $y \in U_q$ and hence

$$d\big(f(a), f(b)\big) = d(x, y) \leq diam(U_q) \leq \frac{2}{N_3} < \epsilon.$$

Consequently, $f : C_0 \longrightarrow X$ is uniformly continuous. Hence the result. □

Now we prove that compact metric spaces are precisely the continuous images of the Cantor set. It is also known as the *universal surjectivity of the Cantor Set.*

Theorem 10.22 (Alexandroff-Hausdorff) *A metric space X is compact if and only if there exists a surjective continuous map from the Cantor set onto X.*

Proof Let C denote the Cantor set. The converse is an immediate consequence of Theorem 5.31, as the Cantor is compact.

To prove the necessity, apply Theorem 10.21 to obtain subset $C_0 \subset C$ and a uniformly continuous bijection $f : C_0 \longrightarrow X$. Since X is complete, by Theorem 9.43, f has a unique uniformly continuous surjective extension $f_1 : \overline{C_0} \longrightarrow X$. Now, extend f_1 to C using Tietze Extension Theorem (9.34). □

Corollary 10.23 *For every positive integer n, there exists a continuous surjective function from* $[0, 1]$ *onto* $[0, 1]^n$.

Proof By Theorem 10.22, there exists a surjective continuous map f from the Cantor set C onto $[0, 1]^n$. Apply Tietze Extension Theorem (9.34) to extend f as a continuous function from $[0, 1]$ onto $[0, 1]^n$. Hence the result. □

Since continuous maps on [0, 1] are known as curves, the maps given by Corollary 10.23 are known as *space filling curves.* Such maps were first discovered by Peano in 1890, and are hence known as *Peano curves.*

Our next application of Theorem 10.22 provides a 'universal continuous function' that interpolates every bounded sequence. First consider the following result.

Theorem 10.24 *There is a continuous map* $f : \mathbb{R} \longrightarrow [-1, 1]$ *such that for every doubly infinite sequence* $\{x_m\}_{m=-\infty}^{+\infty} \in [-1, 1]^{\mathbb{Z}}$, *there exists some* $t \in \mathbb{R}$ *satisfying*

$$f(t+m) = x_m \text{ for all } m \in \mathbb{Z}.$$

Proof As in Sect. 10.2, the countable product $[-1, 1]^{\mathbb{Z}}$ is a compact metric space. Applying Alexandroff-Hausdorff Theorem (10.22), Consider a continuous surjective map $\phi_0 : C \longrightarrow [-1, 1]^{\mathbb{Z}}$. Define $\phi(t) := \phi_0(2t)$ for all $t \in \frac{C}{2} := \{x : 2x \in C\}$. Then the mapping $\phi : \frac{C}{2} \longrightarrow [-1, 1]^{\mathbb{Z}}$ is also continuous and surjective.

For every $m \in \mathbb{Z}$, the component map $t \longmapsto (\phi(t))_m$ is the composition of the projection map π_m with ϕ, and hence continuous. Note that $(\frac{C}{2} + m) \cap (\frac{C}{2} + n) = \emptyset$ for all integers $m \neq n$. Let $A := \bigcup_{m\in\mathbb{Z}}(\frac{C}{2} + m)$ and define

$$f_0(t+m) := (\phi(t))_m \text{ for all } t \in \frac{C}{2}, m \in \mathbb{Z}.$$

Then $f_0 : A \longrightarrow [-1, 1]$ is continuous on A. Let $f : \mathbb{R} \longrightarrow [-1, 1]$ denote a continuous extension of f_0 by Tietze Extension Theorem (9.34). If $\{x_m\}_{m=-\infty}^{+\infty} \in [-1, 1]^{\mathbb{Z}}$, then there exists some $t_0 \in \frac{C}{2}$ such that $\phi(t_0) = \{x_m\}_{m=-\infty}^{+\infty}$. Hence, $x_m = (\phi(t_0))_m = f(t_0 + m)$ for all $m \in \mathbb{Z}$. □

Corollary 10.25 *If* $\{M_m\}_{m\in\mathbb{Z}}$ *is a doubly infinite sequence of positive reals, then there exists a continuous function* $f : \mathbb{R} \longrightarrow \mathbb{R}$ *such that for every* $\{x_m\}_{m=-\infty}^{+\infty} \in [-M_m, M_m]^{\mathbb{Z}}$, *there exists some* $t \in \mathbb{R}$ *satisfying*

$$f(t+m) = x_m \text{ for all } m \in \mathbb{Z}.$$

Proof Replace the product $[-1, 1]^{\mathbb{Z}}$ with $[-M_m, M_m]^{\mathbb{Z}}$ in Theorem 10.24. □

Corollary 10.26 *There exists a continuous function* $f : \mathbb{R} \longrightarrow \mathbb{R}$ *such that for every bounded sequence* $\{x_n\}$ *of real numbers, there exists some* $t \in \mathbb{R}$ *satisfying*

$$f(t+n) = x_n \text{ for all } n \in \mathbb{N}.$$

Proof Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be the map given by Corollary 10.25 with $M_m = m$ for all $m \in \mathbb{Z}$. Let $\{x_n\}_{n\in\mathbb{N}}$ be a bounded sequence of real numbers. Then there exists $N \in \mathbb{N}$ such that $|x_n| < N$ for all $n \in \mathbb{N}$. Then there exists $s \in \mathbb{R}$ such that

$$f(s+m) = \begin{cases} 0 & ; m \leq N, \\ x_{m-N} & ; m > N. \end{cases}$$

Then for $t := s + N$, we obtain $f(t+n) = f(s+N+n) = x_n$ for all $n \in \mathbb{N}$. □

Various other applications of Alexandroff-Hausdorff Theorem (10.22) can be found in [9, 10]. We now state the Hahn–Mazurkiewicz theorem. Readers interested in the proof are referred to [11, p. 221-222, Theorem 31.5].

Theorem 10.27 (Hahn–Mazurkiewicz) *A space X is a continuous image of* $[0, 1]$ *if and only if X is compact, connected and locally connected.*

10.4.2 Cantor Set and Totally Disconnected Metric Spaces

Theorem 10.28 (Brouwer, 1910) *Every totally disconnected compact perfect metric space is homeomorphic to the Cantor set.*

As an immediate consequence, we conclude that any two totally disconnected perfect compact metric spaces are homeomorphic. Before we prove the result in general, we present a simple proof for subspaces of $\mathbb{R}$. An eager reader may skip it and move onto the most general case.

Proof of Theorem 10.28 for Reals Let P be any compact perfect totally disconnected subset of $\mathbb{R}$. Since P is compact, it is bounded and closed. Write $m := \inf P$ and $M := \sup P$. We outline the construction of a strictly increasing function $f : [0, 1] \longrightarrow [m, M]$ with $f(C) = P$.

Since P is closed, by Theorem 7.19, $[m, M] \setminus P$ is a countable union of disjoint open intervals. Since P is has no isolated points, no two intervals in this collection have common end points.

Let I_0 be an interval, from this collection, having maximal length. Note that there are at most finitely many such intervals with (same) maximal length. Define f to be a strictly increasing linear map on I_0 with image $(1/3, 2/3)$.

Let I_1 and I_2 be intervals with maximal length on the left and right of I_0, respectively. Define f as strictly increasing and linear on these such that $f(I_1) = (1/9, 2/9)$ and $f(I_2) = (7/9, 8/9)$.

Inducting like this, we obtain a strictly increasing function $f : [m, M] \setminus P \longrightarrow [0, 1] \setminus C$. Extend f to $[m, M]$, as a continuous function. Then $f : P \longrightarrow C$ is the required homeomorphism.

To see this, let $p_0, p_1, p_2, \ldots$ denote the left end points of the open intervals $I_0, I_1, I_2, \ldots$ Due to the continuity of f, we have $f(p_0) = \frac{1}{3}$, $f(p_1) = \frac{1}{9}$ and $f(p_2) = \frac{7}{9}$ and so on. Since the set of points $\{\frac{1}{3}, \frac{1}{9}, \frac{7}{9}, \ldots\}$ is dense in C, the set $\{p_0, p_1, p_2, \ldots\}$ is dense in P. Hence, f can be extended to P, using its values on $\{p_0, p_1, p_2, \ldots\}$. It can be shown that $x_n \longrightarrow x$ if and only if $f(x_n) \longrightarrow f(x)$. Consequently, $f : P \longrightarrow C$ is a homeomorphism. Hence the result. □

***General Proof of Theorem* 10.28** Let X be a totally disconnected, perfect, compact metric space. Then X is separable and hence has a countable basis, say $\{U_n\}$, see Corollary 8.4 and Theorem 8.10. For every $n \in \mathbb{N}$, let

$$J_n := \{m \in \mathbb{N} : \overline{U_m} \subset U_n\}.$$

Pick any $n \in \mathbb{N}$ and $m \in J_n$. Applying Corollary 6.55, choose some clopen set $A_{n,m}$ such that $\overline{U_m} \subset A_{n,m} \subset U_n$. Then $\mathcal{B} := \{A_{n,m} : m, n \in \mathbb{N}\}$ is a countable basis of X, consisting of only clopen sets. Write $\mathcal{B} = \{W_n : n \in \mathbb{N}\}$.

Now we shall construct nonempty clopen subsets $C_{i_1,\ldots,i_n}$, $n \in \mathbb{N}$, $i_k \in \{0, 1\}$, of X satisfying the following conditions:

(a) $X = C_0 \dot{\bigcup} C_1$,
(b) $C_{i_1,\ldots,i_n} = C_{i_1,\ldots,i_n,0} \dot{\bigcup} C_{i_1,\ldots,i_n,1}$ for all $n \in \mathbb{N}$ and $i_k \in \{0, 1\}$,
(c) each W_n is the union of some of the sets $C_{i_1,\ldots,i_n}$; $i_k \in \{0, 1\}$.

For $n = 1$, if W_1 is a proper subset of X, take $C_0 := W_1$. Otherwise, take any other proper clopen subset of X as C_0. Let $C_1 := X \setminus C_0$. Let $n \in \mathbb{N}$ and assume that the sets $C_{i_1,\ldots,i_m}$; $i_k \in \{0, 1\}$ have been defined for every $m \leq n$.

If $W_{n+1} \cap C_{i_1,\ldots,i_n}$ is a proper subset of $C_{i_1,\ldots,i_n}$, we define $C_{i_1,\ldots,i_n,0} := W_{n+1} \cap C_{i_1,\ldots,i_n}$. Otherwise, take any other proper clopen subset of $C_{i_1,\ldots,i_n}$ as $C_{i_1,\ldots,i_n,0}$. Define

$$C_{i_1,\ldots,i_n,1} := C_{i_1,\ldots,i_n} \setminus C_{i_1,\ldots,i_n,0}.$$

Note that for every $I = \{i_n\} \in \{0, 1\}^{\mathbb{N}}$, the sets $C_{i_1,\ldots,i_n}$ form a nested decreasing sequence of nonempty closed subsets of X and hence have a non-trivial intersection (see Theorem 5.18). Let x_I be a point in this intersection. We claim that this intersection contains no other point.

If possible, let $x \in \bigcap_{n=1}^{\infty} C_{i_1,\ldots,i_n} \setminus \{x_I\}$. Since $\mathcal{B}$ is a basis, there exists some $n \in \mathbb{N}$ such that $x_I \in W_n$ but $x \notin W_n$. Since $C_{i_1,\ldots,i_n} \subset W_n$, we have $x \notin C_{i_1,\ldots,i_n}$. Hence, our claim is established.

Similarly, for every $x \in X$, there exists a unique $I = \{i_n\} \in \{0, 1\}^{\mathbb{N}}$ such that $x = x_I$. Hence, the function $f : \{0, 1\}^{\mathbb{N}} \longrightarrow X$ defined as $f(I) := x_I$ is a bijection. We shall show that f is a homeomorphism. By our construction, we have

$$f^{-1}(C_{i_1,\ldots,i_n}) = \{\{j_k\} : j_l = i_l \text{ for all } 1 \leq l \leq n\}.$$

Note that the sets $\{C_{i_1,\ldots,i_n} : i_j \in \{0, 1\}, n \in \mathbb{N}\}$ form a basis for the topology on X. Also, the collection of sets of the form $\{\{j_k\} : j_l \text{ is fixed, for all } 1 \leq l \leq n\}$ form a basis for the topology on $\{0, 1\}^{\mathbb{N}}$. Hence, f is a bijection between these bases of X and Y. Applying Proposition 9.21, we conclude that f is a homeomorphism. The result follows by Theorem 10.13. □

Corollary 10.29 *Every totally disconnected compact metric space is homeomorphic to a subspace of the Cantor set.*

Proof Let X be a totally disconnected compact metric space and X_0 be its set of isolated points. Since X is compact, by Theorem 5.27, X_0 cannot be infinite. Hence, there exists a finite set X_0 such that $X \setminus X_0$ is perfect.

Since X_0 is a discrete subspace of X, it is homeomorphic to a finite subspace of $C \cap [2/3, 1]$. If $X \neq X_0$, by Theorem 10.28, $X \setminus X_0$ is homeomorphic to C and hence to $C \cap [0, 1/3]$. Hence, $X = (X \setminus X_0) \cup X_0$ is homeomorphic to a subspace of the Cantor space. □

For alternative proofs of Theorem 10.28, the reader is referred to [8, Theorem 7.8] and [11, p. 216, Theorem 30.3].

10.4.3 Open Subsets of the Cantor Set

Based upon [12], we now present another characterization of the Cantor set.

Lemma 10.30 *Let O be a nonempty open subset of the Cantor set C. Then there are countably many disjoint nonempty clopen subsets $\{O_n\}$ of C such that $\dot{\bigcup}_n O_n = O$.*

Proof Let U be an open subset of $\mathbb{R}$ such that $U \cap C = O$. Applying Corollary 7.19, there are countably many disjoint open intervals $\{I_n : n = 1, 2, \dots\}$ such that $\dot{\bigcup}_n I_n = U$. Then for every $n = 1, 2, \dots$, we have $C \setminus I_n = C \cap (\dot{\bigcup}_{m \neq n} I_m)$, which is open in C. Hence, each $I_n \cap C$ is clopen in C. Consequently, $O = U \cap C = \dot{\bigcup}_n (I_n \cap C)$. □

Theorem 10.31 *Let O be a nonempty open subset of the Cantor set C. Then O is either homeomorphic to C or homeomorphic to $C \setminus \{0\}$.*

Proof By Lemma 10.30, there exists a countable collection $\{O_n : n = 1, 2, \dots\}$ of disjoint non-empty clopen subsets of C such that $\dot{\bigcup}_n O_n = O$. If this collection is finite, then O is closed in C. Since O is open and C is perfect, O has no isolated point. Hence, O is a compact perfect totally disconnected non-empty subset of reals. Applying Theorem 10.28, we conclude that O is homeomorphic to C.

Now consider the case when the collection $\{O_n : n = 1, 2, \dots\}$ is infinite. Note that $C \setminus \{0\}$ is an open subset of C, which is not closed. Applying Lemma 10.30, there exists a countably infinite union of disjoint non-empty clopen subsets $\{V_n\}$ of C such that $\dot{\bigcup}_{n=1}^{\infty} V_n = C \setminus \{0\}$. As above, each V_n as well as each O_n is a compact perfect totally disconnected subspace of reals. Again by Theorem 10.28, V_n and O_n are homeomorphic, for all $n \in \mathbb{N}$. Hence, $O = \dot{\bigcup}_{n=1}^{\infty} O_n$ is homeomorphic to $C \setminus \{0\} = \dot{\bigcup}_{n=1}^{\infty} V_n$. □

Definition 10.32 A metric space X is said to have *property* $\mathcal{C}$ if

(a) X has a nonempty compact open subset and a non-compact open subset.
(b) Any two nonempty compact open subsets of X are homeomorphic.
(c) Any two non-compact open subsets of X are homeomorphic.

Note that $\mathcal{C}$ is a topological property, that is it is preserved by homeomorphisms.

Theorem 10.33 *Let X be a compact metric space. Then X satisfies property $\mathcal{C}$ if and only if X is homeomorphic to C.*

Proof The converse follows from Theorem 10.31 and the fact that the property $\mathcal{C}$ is a topological property. Assume that X satisfies property $\mathcal{C}$. By Theorem 10.28, it is enough to show that X is a perfect and totally disconnected metric space.

Assume that X is not perfect. Then X has an isolated point, say x. By (b) above, $\{x\}$ is homeomorphic to X. Then X does not satisfy (a), a contradiction.

We first show that X is disconnected. Let $x, y \in X$ such that $x \neq y$ and let d be the distance between x and y. Then $U := B(x; d/3) \biguplus B(y; d/3)$ is a disconnected open subset of X. If U is homeomorphic to X, then X is disconnected. Otherwise, U is not compact. Since X is perfect, $X \setminus \{x\}$ is non-compact, for every $x \in X$. By (c), U is homeomorphic to $X \setminus \{x\}$, for every $x \in X$. Hence, for every $x \in X$, the subspace $X \setminus \{x\}$ is disconnected. By Theorem 6.29, X is disconnected.

Finally, let $x \in X$ be arbitrary and C_x be the connected component of x, in X. By Theorem 6.49, C_x is also the quasi-component of x. Consequently, C_x closed in X and hence compact. If C_x is open in X, then by (b), C_x is homeomorphic to X and hence X is connected, a contradiction. Therefore, $X \setminus C_x$ is not a closed subset of X and hence non-compact. Consequently, $X \setminus C_x$ is homeomorphic to $X \setminus \{x\}$.

Let Ω be the collection of all clopen subsets of X, containing x. Then $\bigcap_{O\in\Omega} O = C_x$ and thus $X \setminus C_x = \bigcup_{O\in\Omega}(X \setminus O)$. Hence, for each $y \in X \setminus C_x$, there exists a clopen neighborhood U of x such that $y \notin U$. Then $(X \setminus C_x) \setminus U$ is a clopen neighborhood of y.

Since $X \setminus C_x$ and $X \setminus \{x\}$ are homeomorphic, every $z \in X \setminus \{x\}$, has a clopen neighborhood in $X \setminus \{x\}$. Thus, X is totally disconnected. Hence the result. □

Exercise 10.60 Let $\{X_n\}$ be a sequence of finite metric spaces with $|X_n| > 1$ for all $n \in \mathbb{N}$. Prove that $\prod_{n=1}^{\infty} X_n$ is homeomorphic to the Cantor set.

Exercise 10.61 Prove that the Cartesian product of countably many Cantor sets is homeomorphic to the Cantor set.

Exercise 10.62 In Theorem 10.24, can you replace the bound 1 on doubly infinite sequences $\{x_m\}_{m=-\infty}^{+\infty}$ with any other fixed positive real?

Exercise 10.63 Does there exist a continuous function $f : \mathbb{R} \longrightarrow \mathbb{R}$ such that

(a) for every doubly infinite constant sequence $\{x_m\}_{m=-\infty}^{+\infty}$ of reals, there exists $t \in \mathbb{R}$ satisfying $f(t+m) = x_m$ for all $m \in \mathbb{Z}$?
(b) for every doubly infinite sequence $\{x_m\}_{m=-\infty}^{+\infty}$ of reals, there exists some $t \in \mathbb{R}$ satisfying $f(t+m) = x_m$ for all $m \in \mathbb{Z}$?

Exercise 10.64 Let E be a subspace of $\mathbb{R}$ which is closed, bounded, and nowhere dense. Prove that E is homeomorphic to a subspace the Cantor set.

Exercise 10.65 Let E be a subspace of $\mathbb{R}$. Prove that E is homeomorphic to the Cantor set if and only if E is bounded, perfect, and nowhere dense.

Exercise 10.66 Let X be a non-compact metric space. Prove that X satisfies property $\mathcal{C}$ if and only if X is homeomorphic to $C \setminus \{0\}$.

Exercise 10.67 Let X be a locally compact metric space. Prove that X is totally disconnected if and only if it has a basis of topology consisting of compact open sets.

Exercise 10.68 Let A and B be any countably infinite subsets of the Cantor set. Prove that $C \setminus A$ and $C \setminus B$ are homeomorphic. Also, do the same if both A and B are nonempty sets with cardinality strictly less than c.

10.5 Miscellaneous

In this section, we present three major themes associated with the Cantor set, say C. The first one is the Cantor function, which is an important tool in measure theory. The second one provides homeomorphisms $C \longrightarrow C$ which do not map a specific countable set onto any end point of C. The third one is one of the trickiest examples of a connected space, which becomes totally disconnected after the removal of a particular point.

10.5.1 The Cantor Function

The Cantor function is a function $[0, 1] \longrightarrow [0, 1]$, denoted by f_c, defined as follows:

Pick any $a \in [0, 1]$. If $a \in C$, we write $a = (0.a_1a_2 \dots a_n \dots)_3$, where each $a_n \in \{0, 2\}$. Define

$$f_c\big((0.a_1a_2 \dots a_n \dots)_3\big) := \left(0.\frac{a_1}{2}\frac{a_2}{2} \cdots \frac{a_n}{2} \cdots\right)_2.$$

In other words, we define

$$f_c\left(\sum_{k=1}^{\infty} \frac{a_k}{3^k}\right) := \left(\sum_{k=1}^{\infty} \frac{a_k/2}{2^k}\right) \text{ for every } a_k \in \{0, 2\}.$$

It is then extended to $[0, 1]$ as a suitable constant, on every connected component of $[0, 1] \setminus C$. Note that these components are precisely the open intervals, inductively removed from $[0, 1]$, in the process of constructing the Cantor set.

If $a \in [0, 1] \setminus C$, we write its infinite ternary expansion $a = \sum_{k=1}^{\infty} \frac{a_k}{3^k}$, where $a_k \in \{0, 1, 2\}$. Let $n \in \mathbb{N}$ be the least number such that $a_n = 1$. Then we define

$$f_c(a) := \sum_{k=1}^{n-1} \frac{a_k/2}{2^k} + \frac{1}{2^n}.$$

Proposition 10.34 *The Cantor function f_c is a surjective map from C onto* $[0, 1]$. *Consequently, C is uncountable.*

Proof Let $a \in [0, 1]$. Write $a = (0.a_1a_2 \dots a_n \dots)_2$, in its binary representation. Then

$$a = f_c\big((0.(2a_1)(2a_2)\dots(2a_n)\dots)_3\big).$$

Hence, $f_c : C \longrightarrow [0, 1]$ is surjective. Therefore, $[0, 1]$ is in bijection with a subset of C. Hence, C is uncountable. □

Proposition 10.35 *The Cantor function is uniformly continuous on* $[0, 1]$.

Proof Since continuous functions on closed bounded intervals are uniformly continuous, it is enough to prove that f_c is continuous on $[0, 1]$.

Pick any $a \in [0, 1]$. If $a \in [0, 1] \setminus C$, then f_c is constant in a neighborhood of a and hence continuous at a. If $a \in C$, then we can write $a = (0.a_1a_2 \dots a_n \dots)_3$, where each $a_k \in \{0, 2\}$. Let $\epsilon > 0$ be given. Then there exists some $n \in \mathbb{N}$ such that $2^{-n} < \epsilon$ and $\delta = 3^{-n}$.

Let $b \in [0, 1]$ be such that $|b - a| < 3^{-n}$ and write $b := (0.b_1b_2 \dots b_n \dots)$, where each $b_k \in \{0, 1, 2\}$. Since $|b - a| < 3^{-n}$, the first n-digits in the ternary representations of a and b must be same. Therefore, by the definition of f_c, we have $|f_c(b) - f_c(a)| < 2^{-n} < \epsilon$. Hence the result. □

Remarks 10.36 Readers interested in the differentiability of the Cantor function, are referred to [13–16]. It is also known that if $p > 3$ is any prime, then the Cantor set contains only finitely many fractions of the form a/p^k; $k \geq 1$ (see [17]).

History Notes 10.37 The Cantor function has been introduced by George Cantor in 1884, as a counter example to an extension of the fundamental theorem of calculus claimed by Harnack. A thorough survey of the properties of the Cantor function is provided in [18].

10.5.2 Homeomorphic Permutations of the Cantor Set

Let X be a metric space. By a *homeomorphic permutation* of X, we shall refer to a self homeomorphism $f : X \longrightarrow X$. As always, C will denote the Cantor set and for $c \in C$, the representation $c = 0.c_1 \dots c_n \dots$ will always refer to the ternary expansion of c consisting of 0 and 2 only.

Fix any $a = 0.a_1 \dots a_n \dots \in C$. For all $x = 0.x_1 \dots x_n \dots \in C$, define $g_a(x) := 0.y_1 \dots y_n \dots$, where

$$y_n := \begin{cases} x_n & ; a_n = 0, \\ 2 - x_n & ; a_n = 2. \end{cases}$$

This defines a function $g_a : C \longrightarrow C$. In particular, if $a = 0.2$, then g_a translates $C \cap [0, 0.0\overline{2}]$ onto $C \cap [0.2, 0.\overline{2}]$ and vice versa. Similarly, with $a = 0.02$, the permutation g_a translates $C \cap [0, 0.00\overline{2}]$ onto $C \cap [0.02, 0.0\overline{2}]$, and $C \cap [0.2, 0.20\overline{2}]$ onto $C \cap [0.22, 0.\overline{2}]$, and vice versa.

Theorem 10.38 *The function g_a is a homeomorphic permutation of the Cantor set.*

Proof It is immediate that $g_a : C \longrightarrow C$ is a bijection with self inverse, that is $g_a^{-1} = g_a$. Therefore, it is enough to establish that g_a is a continuous map.

Pick any $x_0 = 0.x_{0,1} \dots x_{0,n} \dots \in C$. Let $\{x_k\}$ be a sequence in C convergent to x_0. Write $x_k = 0.x_{k,1} \dots x_{k,n} \dots$ for all $k \in \mathbb{N}$. For every $n \in \mathbb{N}$, $|x_{k,n} - x_{0,n}| \leq 3^n |x_k - x_0| \longrightarrow 0$, as $k \longrightarrow \infty$. Hence, $x_{k,n} \longrightarrow x_{0,n}$ for all $n \in \mathbb{N}$.

Consequently, $g_a(x_k) \longrightarrow g_a(x_0)$, as $k \longrightarrow \infty$. Therefore, g_a is continuous at x_0. Since $x_0 \in C$ is arbitrary, g_a is a continuous on C. Hence the result. □

In the following, let E denote the collection of end points of C. It can be proven that a point $c = 0.c_1 \dots c_n \dots \in E$ if and only if the sequence $\{c_n\}$ is eventually constant.

More precisely, c is a *left end point of* C, (i.e. the set of left end point some connected component of some C_n) if and only if $\{c_n\}$ is eventually 0. Analogously, c is a *right end point of* C if and only if $\{c_n\}$ is eventually 2.

Theorem 10.39 *Let X be a countable subset of C. Then there exists a homeomorphic permutation $f : C \longrightarrow C$ such that $f(X)$ contains no end point of C.*

Proof Since $\mathbb{N} \times \mathbb{N}$ is countable, there exists a bijection $K : \mathbb{N} \times \mathbb{N} \longrightarrow \mathbb{N}$. Write $X := \{x^{(1)}, \dots, x^{(n)}, \dots\}$. Let $x^{(n)}(k)$ denote the digit at k^{th} place of the ternary expansion of $x^{(n)}$ consisting of 0 and 2 only. Define a subset A of $\mathbb{N}$ as follows:

$$A := \left\{ K(m, 2n-1) : x^{(m)}(2n-1) = 0 \right\} \bigcup \left\{ K(m, 2n) : x^{(m)}(2n) = 2 \right\}.$$

Then A is a countable set. Consider $a \in C$ such that $a = 0.a_1 \dots a_n \dots$, where

$$a_n := \begin{cases} 2 \,; n \in A, \\ 0 \,; n \in \mathbb{N} \setminus A. \end{cases}$$

Let $f := g_a$, where g_a is as in Theorem 10.38. We show that f satisfies our requirements.

Applying Theorem 10.38, f is a homeomorphic permutation on C. Pick any $m \in \mathbb{N}$ and write $y^{(m)} = f(x^{(m)})$. Then $y^{(m)}(K(m, 2n)) = 0$ and $y^{(m)}(K(m, 2n-1)) = 2$ for all $n \in \mathbb{N}$. Since K is injective, we conclude that $y^{(m)}(j) = 0$ for infinitely many j and 2 for infinitely many j. Therefore, $y^{(m)}$ is not an end point of C. □

The above result is taken from [19, p. 317], where it is used in the proof of Serpiński's Theorem (9.22).

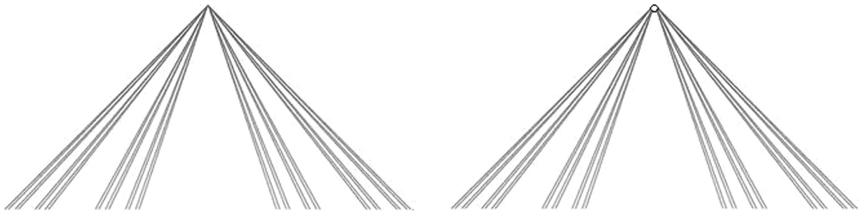
10.5.3 Cantor's Leaky Tent

Let p denote the point $(1/2, 1/2)$ in $\mathbb{R}^2$. Join p, via a line segment, to every point of the Cantor set C. Remove points with irrational ordinate, on the lines joining p to the end points of C and remove points with rational ordinate, on the lines joining p to the internal points (non-end points) of C. The remaining set is known as the *Cantor's leaky tent.*

Formally, for every $c \in C$, let L_c denote the line segment joining p and c. Define

$$X_c := \begin{cases} \{(x, y) \in L_c : y \in \mathbb{Q}\} & ; \text{if } c \text{ is an end point of } C, \\ \{(x, y) \in L_c : y \in \mathbb{R} \setminus \mathbb{Q}\} & ; \text{if } c \text{ is an internal point of } C. \end{cases}$$

The *Cantor's leaky tent* is defined as $T := \bigcup_{c \in C} X_c$, as the induced subspace of $\mathbb{R}^2$. We claim that T is connected, while $T \setminus \{p\}$ is totally disconnected.



Theorem 10.40 *The subspace $T \setminus \{p\}$ is totally disconnected.*

Proof Write $T_0 := T \setminus \{p\}$. Note that every point of T_0 lies on unique L_c. Define $f : T_0 \longrightarrow C$ as $f(x, y) := c$, if $(x, y) \in L_c$. It can be shown that f is continuous.

Let A be any connected subset of T_0. Then so will be $f(A)$. Since $f(A)$ is a subset of the totally disconnected set C, it must be a singleton. Hence, $A \subset L_c$ for some $c \in C$. Since each L_c is totally disconnected, A is also a singleton. Hence, T_0 is totally disconnected. □

Theorem 10.41 *The Cantor's leaky tent T is a connected metric space.*

Proof Assume that $T := A \dot{\bigcup} B$ for some disjoint clopen subsets A and B of T. Without loss of generality, suppose that $p \in A$. We shall establish that $B = \emptyset$.

Let $c \in C$ be arbitrary. For $(x_1, y_1), (x_2, y_2) \in L_c$, define $(x_1, y_1) < (x_2, y_2)$ if and only if $y_1 < y_2$. Under the analogous notions of supremum of a nonempty subset of L_c, we define

$$l_c := \begin{cases} \sup B \cap L_c & ; B \cap L_c \neq \emptyset, \\ c & ; B \cap L_c = \emptyset. \end{cases}$$

Observe that the above supremums will certainly exist in the complete metric space $\mathbb{R}^2$. We claim that exactly one of the following cases holds: (a) $l_c = c$ (b) $l_c \notin T$.

To see this, assume that $l_c \neq c$. If $l_c \in A$, then A contains a ball around l_c and thus an 'interval' of the subspace L_c, a contradiction to the definition of l_c. In case $l_c \in B$, again B contains an 'interval' of the subspace L_c centered around l_c, a contradiction. Hence, $l_c \notin T$.

Let $\{r_k\}$ be an enumeration of rational numbers in $(0, 1/2]$. For every $k \in \mathbb{N}$, we define

$$H_k := \left\{(x, r_k) : (x, r_k) = l_c \text{ for some } c \in C \setminus E\right\}.$$

Then each H_k is a bounded subset of the horizontal line $y = r_k$. Hence, $\overline{H_k}$ is a compact subset of $\mathbb{R}^2$. By our previous observation, each $H_k \cap T = \emptyset$.

Pick any $k \in \mathbb{N}$ such that $H_k \neq \emptyset$. Then for each $z \in \overline{H_k}$, there exists a unique $c \in C$ such that $\overline{H_k} \cap L_c = \{z\}$. Define $P_k : \overline{H_k} \longrightarrow C$ as $P_k(z) := c$, as the projection map from p onto C. It can be shown that each P_k is a continuous map. Therefore, $F_k := P_k(\overline{H_k})$ is compact and hence a closed subset of C.

Being a subset of C, each F_k is nowhere dense. Then $F := \bigcup_{k=1}^{\infty} F_k$ is a set of first category and hence so is the set $E \cup F$. Applying Baire Category Theorem (8.33), $S := C \setminus E \cup F$ is dense in C.

If possible, let $z \in B$. Then $B \supset B(z; \delta)$ for some $\delta > 0$. Since S is dense in C, the ball $B(z; \delta)$ contains a portion of L_c for some $c \in S = C \setminus E \cup F$. From the definition of S, we obtain $l_c = c$ and $L_c \setminus \{c\} \subset A$, a contradiction. Hence, $B = \emptyset$. This concludes that T is connected. □

Other popular terms for the *Cantor's leaky tent* are *Cantor teepee* and *Knaster-Kuratowski fan,* named after Polish mathematicians Bronislaw Knaster and Kazimierz Kuratowski.

Remarks 10.42 Let $\sum_{n=1}^{\infty} x_n$ be an absolutely convergent series of real numbers such that $|x_n| > \sum_{i>n} |x_i|$ for all $n \in \mathbb{N}$. Then its set of *subsums* $\{\sum_{n=1}^{\infty} \epsilon_n x_n : \epsilon_n \in \{0, 1\}, n \in \mathbb{N}\}$ is homeomorphic to the Cantor set.

In fact, the set of subsums of an absolutely convergent series is either a finite set, a finite union of compact intervals, homeomorphic to the Cantor set, or a *Cantorval* (a particular kind of Cantor type set due to Guthrie-Nymann). We omit the proofs of these results. Interested readers are referred to [20, 21].

Exercise 10.69 Obtain $C_0 \subset C$ such that the Cantor function $f_c : C_0 \longrightarrow [0, 1]$ is a bijection.

Exercise 10.70 Is the Cantor function differentiable at $1/3$?

Exercise 10.71 Prove that the Cantor function is not Lipschitz continuous.

Exercise 10.72 Show that the Cantor function is monotonically increasing.

Exercise 10.73 Let C and f_c, respectively, denote the Cantor set and the Cantor set function. Prove that $[0, 1] \setminus \mathbb{Q} \subset f_c(C)$.

Exercise 10.74 Prove that the Cantor function is the *unique monotone real valued function* f *on* $[0, 1]$ satisfying $f(0) = 0$, $f\left(\frac{x}{3}\right) = \frac{f(x)}{2}$ and $f(1 - x) = 1 - f(x)$ for all $x \in [0, 1]$.

Exercise 10.75 Establish the continuity of the Cantor function by considering a suitable uniformly convergent sequence of piecewise linear continuous functions on the interval $[0, 1]$.

Exercise 10.76 Prove that the Cantor function is not differentiable at $1/3$; and is differentiable outside a negligible subset of $[0, 1]$.

Exercise 10.77 Is the Cantor's leaky tent compact, complete or perfect?

Exercise 10.78 Does there exist any proper non-degenerate compact connected subspace of the Cantor's leaky tent?

Exercise 10.79 Show that f, in the proof of Theorem 10.40, is a continuous map.

Exercise 10.80 Prove that each P_k, in the proof of Theorem 10.41, is a continuous map.

Exercise 10.81 In the Cantor's leaky tent T of the last subsection, show that $X_c \setminus \{p\}$ is a quasi-component of $T \setminus \{p\}$ for every $c \in C$. Conclude that $T \setminus \{p\}$ is not totally separated, but totally disconnected.

Exercise 10.82 If $f : C \longrightarrow C$ is a homeomorphism, can you conclude $f \equiv f_a$ for some $a \in C$?

10.6 Hints and Solutions to Selected Exercises

10.3 The number $(0.0200220002220000222200000222222\ldots)_3$ can't be rational, as this ternary representation doesn't repeat anywhere.

10.7 Note that the countable subset set $\bigcup_{n=1}^{\infty} L_n$ of C, as in Exercise 10.6, has no isolated points. Applying Theorem 9.22, $\bigcup_{n=1}^{\infty} L_n$ is homeomorphic to $\mathbb{Q}$.

10.8 No, since $\left\{ \sum_{k=1}^{n} \frac{2}{3^{2^{2k}}} \right\}_n (\subset \bigcup_{n=1}^{\infty} L_n)$ converges to $\frac{1}{4} \in C \setminus \bigcup_{n=1}^{\infty} L_n$.

10.13 Use Exercise 10.6.

10.17 Yes. For each $n \in \mathbb{N}$, define $f_n : [0, 1] \longrightarrow \mathbb{R}$ as

$$f_n(x) = max\{0, 1 - n \times dist(x; C)\} \text{ for all } x \in [0, 1].$$

Then $|dist(x; C) - dist(y; C)| = |x - y|$ for all $x, y \in [0, 1]$. So the map $x \longmapsto dist(x; C)$ is continuous. Hence, each f_n is continuous on $[0, 1]$. Note that

$$\lim_{n\to\infty} f_n(x) := \begin{cases} 1 \,;\, x \in C, \\ 0 \,;\, x \in [0, 1] \setminus C. \end{cases}$$

Hence, the sequence of functions $\{f_n\}$ is continuous on $[0, 1]$ and pointwise convergent to χ_C, which is discontinuous on the uncountable set C.

10.19 Apply Theorem 7.39 and the fact that C is a closed subset of $\mathbb{R}$..

10.20 No. Suppose that such a set A exists and let $a \in A$. Then for every $\epsilon > 0$, the set $(a-\epsilon, a+\epsilon) \cap C \setminus \{a\}$ is uncountable. Therefore, a is a limit point of $C \setminus (A \setminus \{a\})$.

10.21 Yes. In fact, for every $\epsilon > 0$, the Cantor set is contained in finitely many intervals, each having length at most ϵ.

10.22 Applying Exercise 7.59, the required set is $\mathbb{R} \setminus C$.

10.23 Yes, as every end point of C is a limit point of $[0, 1] \setminus C$.

10.25 No. E.g., let C be the Cantor set in $[0, 1]$, $P := C \cup (C + 2)$ and $I := (0, 2)$.

10.26 Since f is uniformly continuous, there exists $\delta > 0$ such that $|f(x) - f(y)| < \frac{1}{3}$ for all $|x - y| \le \delta$. Let $m \in \mathbb{N}$ such that $3^{-m} < \delta$. The result holds as $|f(x) - f(y)| < \frac{1}{3}$ for all $|x - y| \le \frac{1}{3^m}$.

10.27 Let $a \in A$. Then $a \in C = C'$. So every neighborhood of a contains c-many points of C and thus it contains c-many points of $C \setminus A$. Hence, we obtain $a \in (C \setminus A)' \setminus (C \setminus A)$. Consequently, $C \setminus A$ is not closed.

10.28 Suppose not. Then $(C + x) \cap \mathbb{Q} \neq \emptyset$ for all $x \in \mathbb{R}$. Therefore, $\mathbb{R} = \bigcup_{q \in \mathbb{Q}} (q - C)$. Since C is nowhere dense, so are $q - C$ for all $q \in \mathbb{Q}$. This contradicts Corollary 8.35.

10.29 Write $B := \bigcup_{n=1}^{\infty} L_n$. Since A is a subset of $\mathbb{R}$, it is a separable space. Thus, there exists a countable dense subset A_0 of the space A. By Exercise 8.3, A_0 is dense in C. Since C has no isolated point, A_0 is a perfect metric space. Note that B is also a countably infinite perfect metric space. By Theorem 9.22, we conclude that A_0 is homeomorphic to B.

10.33 Apply Theorem 10.14 to the completion of the space.

10.34 Assume that X is countable, say $X = \{x_n : n \in \mathbb{N}\}$. Define $f : X \longrightarrow [0, 1]^{\mathbb{N}}$ as $f(x_n) := (1, 2, \ldots, n, 0, \ldots)$ for all $n \in \mathbb{N}$. Then f is injective, continuous, and open.

Conversely, assume that X is homeomorphic to a subspace of $[0, 1]^{\mathbb{N}}$. Since $[0, 1]^{\mathbb{N}}$ is compact, it is separable and hence so is X. Being discrete and separable, X is countable.

10.35 Define $f_0 : \{0, 1\}^{\mathbb{N}} \longrightarrow [0, 1]$ and $f : \{0, 1\}^{\mathbb{N}} \longrightarrow [0, 1]^{\mathbb{N}}$ as

$$f_0(\{x_i\}) := \sum_{n=1}^{\infty} \frac{x_i}{2^n} \text{ and } f(\{x_i\}) := \{f_0(\{x_i\})\} \text{ for all } \{x_i\} \in \{0, 1\}^{\mathbb{N}}.$$

Then f_0 and f are surjective continuous functions.

10.42 Yes. Let $\epsilon > 0$ be given. Then there exists some $m \in \mathbb{N}$ such that $1/m < \epsilon$. Let $\delta := \frac{1}{m2^m}$. Pick any $x = \{x_n\}$, $y = \{y_n\} \in X$ such that $\rho(x, y) < \delta$. If $n > m$, we have $d_n(x_n, y_n)/n \le 1 < 1/m < \epsilon$. If $n \le m$, then we obtain

$$\frac{d_n(x_n, y_n)}{2^n} < \frac{1}{m2^m} \text{ which implies } \frac{d_n(x_n, y_n)}{n} < \frac{1}{nm2^{m-n}} < \frac{1}{m} < \epsilon.$$

Consequently, $d(x, y) < \epsilon$. Hence, ρ and d uniformly equivalent.

10.43 Write $X := \prod_{n=1}^{\infty} X_n$. The necessity follows from the continuity of projection maps and Theorem 6.9. Towards the converse, assume that each X_n is path connected. Let $x = \{x_n\}$, $y = \{y_n\} \in X$ be arbitrary. For each $n \in \mathbb{N}$, choose a continuous map $f_n : [0, 1] \longrightarrow X_n$ such that $f_n(0) = x_n$ and $f_n(1) = y_n$. Define $f : [0, 1] \longrightarrow X$ such that

$$f(a) := \{f_n(a_n)\}_n \text{ for all } a = \{a_n\} \in X.$$

It can be shown that f is a path in X from x to y.

10.48 No. Use Theorem 8.26.

10.51 Yes. Consider $\mathbb{N}$ with usual topology of $\mathbb{R}$.

10.54 Apply Theorems 10.16 and 8.31.

10.55 Apply Theorem 10.16.

10.59 Try $X := A \cup \mathbb{Q}$ under usual metric.

10.63 (a) No. For example, if f is such a function, then for every real r, there there exists $t_r \in \mathbb{R}$ satisfying $f(t_r + m) = r$ for all $m \in \mathbb{Z}$. Therefore, $\mathbb{R} \subset f([0, 1])$, which is impossible as f continuous and hence bounded on the compact interval $[0, 1]$.
(b) No. Follows from (a).

10.64 Being closed and bounded, E is compact. Being nowhere dense, E is totally disconnected. Now apply Corollary 10.29.

10.67 Proceed as in Theorem 6.54.

10.68 Apply Exercise 10.27 and Theorem 10.33.

10.70 No. As for $x_n := \frac{1}{3} - \frac{1}{3^n}$, we have $f_c(x_n) := \frac{1}{2} - \frac{1}{2^n}$. Hence

$$\frac{f_c(x_n) - f_c(1/3)}{x_n - 1/3} = \frac{3^n}{2^n} \longrightarrow \infty.$$

10.73 Use the fact that $f([0, 1] \setminus C) \subset \mathbb{Q}$.

10.82 No. For example, let I_1, I_2, I_3 and I_4 be the four consecutive compact intervals of length 3^{-2} such that $C \subset \bigcup_{i=1}^{4} I_i$ and $x < y$ for all $x \in I_i$, $y \in I_j$ with $i < j$. Let g be a suitable shift operator which maps I_1 onto I_4. Then $g : I_1 \longrightarrow I_4$ is a homeomorphism. Define $f : C \longrightarrow C$ as

$$f(x) := \begin{cases} g(x) & ; x \in I_1, \\ g^{-1}(x) & ; x \in I_4, \\ x & ; x \in I_2 \cup I_3. \end{cases}$$

It can be shown that f is a homeomorphism. Also, the first ternary digits of $f(0)$ and 0 are different, while the same for $f(1/3)$ and $1/3$ are the same. Hence, $f \not\equiv f_a$ for all $a \in C$.

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Appendix A
Axiomatic Set Theory

This appendix provides a glimpse into the axiomatic set theory. We start with an introduction to the formal language of set theory and a standard set of Zermelo-Fraenkel axioms. Then we use these axioms to deduce the set of non-negative integers and the well-ordering principle. After that, we discuss the axiom of choice and observe some proofs which intrinsically assume it. Most of the times, the choice axiom can be bypassed by a minor change in the proof. Sometimes a completely different proof is required to avoid this axiom. We discuss both of these situations.

A.1 The Language of Set Theory

In Naive Set Theory, a set is defined as a well-defined collection of distinct objects. This leads to some paradoxes. For example if X is the collection of all sets, then X is well-defined and $\mathcal{P}(X) \subset X$. Along with Theorem 7.46 this leads to $X \prec \mathcal{P}(X) \preceq X$, a contradiction.

To sort out this kind of paradoxes and for a further rigorous foundation of mathematics, the Axiomatic Set Theory came into the picture. This includes a formal Language of Set Theory followed by Zermelo-Fraenkel Axioms. These axioms require a *'Language of Set Theory,'* comprising of the following:

(a) Names: for sets as well for the members of sets
(b) Symbols: $\in$ (membership), = (equality) and $\neg$ (negation)
(c) Logical connectives: $\vee$ (or), $\wedge$ (and)
(d) Quantifier symbols: $\forall$ (for all) and $\exists$ (there exist)
(e) Brackets (,).

Some authors leave the notion of a set undefined for a while. For example, as per [1], '...as any mathematical theory begins with undefined concepts. Therefore, the notion of a set and the notion of $\in$ (is an element of) are better left undefined.' However, the notion of sets will be intrinsically present in the Zermelo-Fraenkel

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Axioms, given in the next section. Any expression using either of (a)-(e), is called a *formula* of the Language of Set Theory (LAST).

Exercise A.1 If f and g are formulas of LAST, prove that so are $f \vee g$ and $f \wedge g$.

Exercise A.2 Express union of two sets A and B as a formula of LAST.

A.2 Zermelo-Fraenkel Axioms

There are various equivalent versions of the *Zermelo-Fraenkel axioms* or *ZF-axioms.* Here we present the one given in [2].

(ZF1) *Null Set Axiom:* There exists a set $\emptyset$, having no element.
(ZF2) *Power Set Axiom:* If X is a set, then there exists a set $\mathcal{P}(X)$ consisting of all and only subsets of X.
(ZF3) *Axiom of Infinity:* There exists a set X such that $\emptyset \in X$ and $\{x\} \in X$ for all $x \in X$.
(ZF4) *Axiom of Extensionality:* If two sets have same elements, then they are equal.
(ZF5) *Axiom of Union:* If X is a set, there exists a set $\cup X$, consisting of all elements of all elements of X.
(ZF6) *Axiom of Subset Selection:* Let X be a set and $\phi(x)$ be a formula of LAST. Then there exists a set consisting of only those $x \in X$ such that $\phi(x)$ holds true.
(ZF7) *Axiom of Replacement:* Let $\phi(x, y)$ be any formula of LAST such that for every a there exists a unique b such that $\phi(a, b)$ holds true. Let X be a set. Then there exists a set Y consisting of only those b such that $\phi(a, b)$ is true for some $a \in X$.
(ZF8) *Axiom of Foundation:* $\in$ is a well-founded relation, that is, if X is a nonempty set, then there exists some $x \in X$ such that $x \cap X = \emptyset$.

Remarks A.1 (a) The above eight axioms are not all independent. For example, ZF3 and ZF6 together imply ZF1. To see this, let X be the infinite set given by ZF3. Then its subset $\{x \in X : x \neq x\}$ is empty.

(b) The following axiom (Axiom of Pairing) is often stated in ZF-axioms:

If x and y are two sets, then there exists a set having elements x and y.

It holds by ZF7, as by ZF1 and ZF2, $\{\emptyset, \{\emptyset\}\}$ is a set.

(c) The Axiom of Foundation, also known as the *Axiom of Regularity,* ensures that no set can be a member of itself. Without this axiom, there is a set X containing all sets. Then $\mathcal{P}(X) \subset X$ and we have the contradiction $X \prec X$, as discussed in the beginning of this appendix. The Axiom of Foundation avoids these types of paradoxes, such as the Burali-Forti Paradox and the Russel's Paradox.

Exercise A.3 Express the intersection of a given family of sets in terms of ZF-axioms. Conclude that there is no need for a separate axiom of intersection in ZF.

A.3 The Set of Non-Negative Integers

'God made the natural numbers; all else is the work of man,' Kronecker.

In 1889, Peano presented a set of axioms for the set of natural numbers, known as Peano axioms. Here we deduce these numbers from the ZF-axioms. In ZF, using $ZF1$, $ZF2$, $ZF3$, $ZF6$, and $ZF7$, we consider an infinite set $\mathbb{Z}_{\geq 0}$, defined as under

$$\mathbb{Z}_{\geq 0} := \{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}, \dots\} = \{0, 1, 2, \dots\} \quad \text{(say)},$$

where 0 denotes $\emptyset$, 1 denotes $\{0\}$, ..., n denotes $\{0, 1, \dots, n-1\}$, in general. For every $k \in \mathbb{Z}_{\geq 0}$, we define $k + 1 := \{0, 1, \dots, k\}$.

Note that for every $n, m \in \mathbb{Z}_{\geq 0}$, either $n \subset m$ or $m \subset n$. We shall write $n \leq m$ if $n \subset m$ and $m \leq n$ otherwise. Further if $n \leq m$ but $n \neq m$, we write $n < m$. Now we establish that the set $\mathbb{Z}_{\geq 0}$ is well ordered under the relation $<$.

Theorem A.2 *Every nonempty subset of $\mathbb{Z}_{\geq 0}$ contains its smallest element.*

Proof Let S be any nonempty subset of $\mathbb{Z}_{\geq 0}$. Consider the set

$$E := \{x : x \in s \text{ for all } s \in S\} = \bigcap_{s \in S} s,$$

here this intersection is ensured by the axioms of union and subset selection. By definitions of E and $\mathbb{Z}_{\geq 0}$, we obtain $E \subset \mathbb{Z}_{\geq 0}$ and $E \subset s$ for every $s \in S$. Therefore, $E \leq s$ for every $s \in S$. Hence, E is a lower bound of S. Next, we claim that E is the smallest element of S.

First we show that $E \in \mathbb{Z}_{\geq 0}$. If not, then there will exist some $k_1, k_2 \in E$ and $k_3 \in \mathbb{Z}_{\geq 0}$ such that $k_1 < k_3 < k_2$. This will ensure some $s_0 \in S$ such that $k_3 \notin s_0$, while $k_1, k_2 \in s_0$. This is not possible, as $s_0 \in \mathbb{Z}_{\geq 0}$. Hence, $E \in \mathbb{Z}_{\geq 0}$.

If $E \notin S$, then $E + 1 \leq s$ for all $s \in S$. So $E + 1 \subset s$ for all $s \in S$. The definition of E implies that $E + 1 \subset E$, a contradiction. Hence the result. □

It must be emphasized here that the above theorem is a basis for the *principle of mathematical induction,* which is in fact equivalent to this.

Theorem A.3 *Let $P(n)$ be a statement for every $n \in \mathbb{Z}_{\geq 0}$ such that $P(0)$ is true and $P(n+1)$ is true whenever $P(n)$ is true. Then $P(n)$ is true, for all $n \in \mathbb{Z}_{\geq 0}$.*

Proof Let $E := \{n \in \mathbb{Z}_{\geq 0} : P(n) \text{ is not true}\}$. If possible, suppose $E \neq \emptyset$. Applying Theorem A.2, let m be the smallest element of E. Since $P(0)$ is true, $0 \notin E$. Therefore, $m > 0$. So there exists some $k \in \mathbb{Z}_{\geq 0}$ such that $k + 1 = m$. Since $P(m)$ is not true, by hypothesis, $P(k)$ is also not true. Thus, $k \in E$, contradicting the choice of m. Hence, $E = \emptyset$. □

Write $m + 2 := (m + 1) + 1$ and define

$$m + (n + 1) := (m + n) + 1 \text{ for all } m, n \in \mathbb{Z}_{\geq 0}.$$

This defines a binary operation $+$ on $\mathbb{Z}_{\geq 0}$. For $m, n \in \mathbb{Z}_{\geq 0}$, define $m \times n$ inductively by adding m to itself n times. Hence, we obtain the usual binary operations $+$ and $\times$ on $\{1, 2, \dots\}$.

The inverses of these binary operations lead to the negative and the rational numbers. Further we obtain reals, as on page 105. For further details, we refer [3, 4].

Exercise A.4 Prove that the principle of mathematical induction implies that every nonempty subset of $\mathbb{Z}_{\geq 0}$ contains a smallest element.

A.4 Axiom of Choice

The *Axiom of Choice,* also known as the *Choice Axiom,* has already been presented in Sect. 7.4.1. It has various equivalent forms, some of which are equivalent in a highly non-trivial manner. The following is a popular version of this axiom:

For every nonempty family Ω of nonempty sets, there exists a set $C \subset \bigcup_{A\in\Omega} A$ such that C contains at least one element from each $A \in \Omega$.

Restricting Ω to countably infinite collections defines the *Axiom of Countable Choice.*

The Zermelo-Fraenkel Axioms along with the Axiom of Choice are known as *ZFC-axioms.* Unless otherwise specified, we always work within the framework of ZFC-axioms.

Several proofs of this textbook use Choice Axiom, without explicitly mentioning it. It is a good exercise to locate those proofs. Such proofs are known as *ZFC-proofs.* The proofs using ZF-axioms but independent of the choice axiom are called *ZF-proofs* or *choice-free proofs.*

Most of the times, the ZFC-proofs can be modified to ZF-proofs, with a minor change. But there are cases when drastically different ZF-proofs are required. In some cases, the use of Axiom of Choice cannot be avoided. In this section, we present the first two cases.

First we discuss some examples, where the Axiom of Choice can be avoided, with a little change in the ZFC-proof. Few other cases will be presented in the exercises.

Example A.4 In the alternative proof of Theorem 5.37, the Choice Axiom is used to choose δ_x for each $x \in X$. However, the same proof can be made choice free by taking $\delta_x := \frac{1}{n_x}$, where

$$n_x := \min\left\{n \in \mathbb{N} : f\left(B_X\left(x; \frac{1}{n}\right)\right) \subset B_Y\left(f(x); \frac{\epsilon}{2}\right)\right\}.$$

Since in ZF, every subset of $\mathbb{N}$ has its least element, the above choice of δ_x is independent of the Axiom of Choice.

Example A.5 The proof of Theorem 5.28 uses the Axiom of Choice while choosing open set O_x for every $x \in X$. However, the following little modification makes it choice free.

ZF-Proof of Theorem 5.28 Let X be a compact metric space. If possible, let E be an infinite subset of X with $E' = \emptyset$. Then for every $x \in X$, there exists an open set O_x containing x such that O_x contains at most one point of E, namely x. Let

$$\Omega := \{O : O \text{ is an open set such that } O \cap E \text{ is at most a singleton}\}.$$

Note that Ω is an open cover of X. Since X is compact, there exists $O_1, \ldots, O_n \in \Omega$ such that $X \subset \bigcup_{i=1}^{n} O_i$. So $E = E \cap X \subset \bigcup_{i=1}^{n}(E \cap O_i)$, which is set having at most n elements. This is a contradiction, as E is an infinite set. $\square$

We now present a case, where the Axiom of Choice can be avoided, but with a completely different proof. Note that the converse part of the proof of Corollary 1.47 uses the Axiom of Choice while choosing a suitable sequence $\{x_n\}$. So that proof is invalid in ZF. Here we provide a related result.

Theorem A.6 *In ZF, a function $f : \mathbb{R} \longrightarrow \mathbb{R}$ is continuous, if $f(x_n) \longrightarrow f(x)$ whenever $x_n \longrightarrow x$.*

Proof Let $\{r_n : n \in \mathbb{N}\}$ be an enumeration of $\mathbb{Q}$. For every $x \in \mathbb{R}$, let $f_x := f|_{\mathbb{Q}\cup\{x\}}$ be the restriction of f to the set $\mathbb{Q} \cup \{x\}$. First we claim that f_x is continuous at x, for every $x \in \mathbb{R}$.

Suppose some f_x is not continuous at x. Then there exists $\epsilon > 0$ such that for all $n \in \mathbb{N}$, there exists $y_n \in [x - \frac{1}{n}, x + \frac{1}{n}] \cap \mathbb{Q}$ such that $|f_x(y_n) - f_x(x)| \geq \epsilon$. For each $n \in \mathbb{N}$, let $y_n := r_{k_n}$, where

$$k_n := \min\left\{m : |f_x(r_m) - f_x(x)| \geq \epsilon \text{ and } r_m \in \left[x - \frac{1}{n}, x + \frac{1}{n}\right] \cap \mathbb{Q}\right\}.$$

Since every subset of $\mathbb{N}$ has the smallest element, the above choice of y_n is independent of the Choice Axiom. Note that $y_n \longrightarrow x$, while $f(y_n) \not\longrightarrow f(x)$, a contradiction. This proves our claim.

Fix any $x \in \mathbb{R}$ and $\eta > 0$. Then there exists $\delta > 0$ such that $|f(y) - f(x)| < \frac{\eta}{2}$, for every $y \in (x - \delta, x + \delta) \cap \mathbb{Q}$. Let $z \in (x - \delta, x + \delta)$. Since f_z is continuous at z, $|f(z) - f(w)| < \frac{\eta}{2}$ for some $w \in (x - \delta, x + \delta) \cap \mathbb{Q}$. Hence, $|f(x) - f(z)| \leq |f(x) - f(w)| + |f(w) - f(z)| < \eta$. Therefore, f is continuous at x. Hence the result. $\square$

The proofs dependent upon Axiom of Choice establish only the existence of the required object and are non-constructive. However, a choice-free proof is not always non-constructive.

Theorem A.7 *There are irrationals a and b such that a^b is a rational number.*

Proof Let $x = \sqrt{2}^{\sqrt{2}}$. If x is a rational number, take $a := \sqrt{2}$ and $b := \sqrt{2}$. Otherwise, take $a := \sqrt{2}^{\sqrt{2}}$ and $b := \sqrt{2}$, as $(\sqrt{2}^{\sqrt{2}})^{\sqrt{2}} = \sqrt{2}^{2} = 2 \in \mathbb{Q}$. □

The above proof does not explicitly provides a and b, so it is not a constructive proof.
Constructive Proof of Theorem A.7 Take $a = \sqrt{2}$ and $b := \log_2 9$. Then $a^b = 3 \in \mathbb{Q}$. Clearly a is an irrational. Suppose $\log_2 9 = m/n$, for some positive integers m, n. Then $9^n = 2^m$. This is impossible, as the former is odd and the latter is even. □

Notes and Remarks A.8 (a) In ZF, the following pointwise analogue of Theorem A.6 does not hold (see [5, p. 74, Theorem 4.54]).

A function $f : \mathbb{R} \longrightarrow \mathbb{R}$ *is continuous at* x, *if* $f(x_n) \longrightarrow f(x)$, *whenever* $x_n \longrightarrow x$.

(b) It is known that the five axioms; axioms of extensionality, replacement, power set, union, and choice are consistent. Further, each of the axioms of extensionality, replacement, and power set is independent of the remaining four axioms (see [6, 7]). A thorough treatise on the Axiom of Choice is available in [5].
(c) In ZF, the cardinalities of any two sets are comparable if and only if the Axiom of Choice holds (see [5, p. 52, Theorem 4.20]).
(d) The Axiom of Choice leads to several paradoxes. The Banach-Tarski Paradox is on outstanding one, which defies the common intuition. It is often loosely stated that in $\mathbb{R}^3$ we can break a ball into a finite number of pieces, and with these pieces, build two balls having the same size as the initial ball. Interested readers are referred to [8–10].
(e) Recently, in a paper entitled 'A paradox arising from the elimination of a paradox,' Taylor and Wagon presented the *division paradox*. It states that in ZF without the Choice Axiom, the set of real numbers can be written into disjoint classes, so that there are more classes than real numbers (see [11]).

Exercise A.5 Prove that the Axiom of Choice implies that for every nonempty family Ω of nonempty disjoint sets, there exists a set $C \subset \bigcup_{A \in \Omega} A$ such that C contains exactly one element of each set $A \in \Omega$. Is it still true if the sets in Ω are not disjoint?

Exercise A.6 In ZF, prove that the Axiom of Choice holds if and only if for every nonempty family of nonempty sets Ω, the Cartesian product $\prod_{E \in \Omega} E$ is nonempty.

Exercise A.7 Is the proof of Theorem 5.13 (as on p. 126) dependent upon Axiom of Choice? If yes, can you provide a proof which does not depend upon it?

Exercise A.8 Let n be a positive integer. In ZF, prove that every bounded infinite subset of $\mathbb{R}^n$ has a limit point in $\mathbb{R}^n$.

A.5 Hints and Solutions to Selected Exercises

A.5 The result holds, as sets in Ω are disjoint. Further, this hypothesis is necessary. Otherwise, for $\Omega := \{\{1, 2\}, \{2, 3\}, \{3, 1\}\}$, there exists no set $C \subset \{1, 2\} \cup \{2, 3\} \cup \{3, 1\} = \{1, 2, 3\}$ such that $C \cap A$ is a singleton, for every $A \in \Omega$.

A.7 Yes. The choice of j_0 at each step of induction, in Theorem 5.13, is not made by any fixed rule. So, the sequence $\{I_n\}$ is chosen using the Axiom of Choice. For a choice-free proof, define a suitable order on k-cells, and at n^{th} step, choose the 'smallest legal' k-cell.

A.8 Apply Exercise A.7, along with the ZF-proof of Theorem 5.28.

Appendix B
More on Continuous Functions

In this appendix, we provide a further discussion on continuous functions. It starts with the Weierstrass approximation theorem, which asserts the density of polynomials on $[a, b]$ in the space $C[a, b]$ under supremum norm. Then we provide a standard example of a continuous but nowhere differentiable function, along with the Banach-Mazurkiewicz theorem; which states that 'most' continuous functions are nowhere differentiable.

B.1 Weierstrass Approximation Theorem

At the age of 70, Karl Weierstrass (1815-1897) presented the following approximation theorem for continuous functions. We shall prove it in several steps. The technical verifications have been worked out separately in three lemmas. Throughout this section, let $[a, b]$ denote an arbitrary closed and bounded interval.

Theorem B.1 (Weierstrass, 1885) *If $f : [a, b] \longrightarrow \mathbb{C}$ is a continuous function, there exists a sequence of polynomials $\{P_n\}$, uniformly convergent to f on $[a, b]$. Further, if f is a real valued function, each P_n can be taken as a real polynomial.*

Lemma B.2 *If Theorem B.1 holds for*

(a) all functions on interval $[0, 1]$, then it holds for $[a, b]$.
(b) all functions f on interval $[0, 1]$ such that $f(0) = 0 = f(1)$, then it holds for any complex valued continuous function f on $[a, b]$.

Proof (a) Let $f : [a, b] \longrightarrow \mathbb{C}$ be a continuous function. Note that

$$[a, b] = \{a(1 - t) + bt : t \in [0, 1]\} = \{a + (b - a)t : t \in [0, 1]\}.$$

Define a function $g : [0, 1] \longrightarrow \mathbb{C}$ as

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$$g(x) := f(a + (b - a)x) \text{ for all } x \in [0, 1].$$

Then g is continuous with $g(0) = f(a)$ and $g(1) = f(b)$. By hypothesis, let $\{R_n\}$ be a sequence of polynomials, uniformly convergent to g on $[0, 1]$. Define

$$P_n(x) := R_n\left(\frac{x - a}{b - a}\right) \text{ for all } x \in [a, b].$$

Then $\{P_n\}$ is a sequence of polynomials, uniformly convergent to f on $[a, b]$.

(b) By (a), without loss of generality, we assume that $[a, b] = [0, 1]$. Let $f : [0, 1] \longrightarrow \mathbb{C}$ be a continuous function. Define $g : [0, 1] \longrightarrow \mathbb{C}$ as

$$g(x) := f(x) - f(0) - x\big(f(1) - f(0)\big) \text{ for all } x \in [0, 1].$$

Then g is continuous and $g(0) = 0 = g(1)$. By hypothesis, let $\{S_n\}$ be a sequence of polynomials, uniformly convergent to g on $[0, 1]$. Define

$$P_n(x) := S_n(x) + f(0) + x\big(f(1) - f(0)\big) \text{ for all } x \in [0, 1].$$

Then each $P_n(x)$ is a polynomial in x and $P_n \longrightarrow f$, uniformly on $[0, 1]$. □

Lemma B.3 *Let* $n \in \mathbb{N} \setminus \{1\}$ *and* $x \in (0, 1)$. *Then* $(1 - x^2)^n > 1 - nx^2$ *and*

$$\int_{-1}^{1} (1 - x^2)^n dx > \frac{1}{\sqrt{n}}.$$

Proof The second inequality follows from the first, as

$$\begin{aligned}\int_{-1}^{1} (1 - x^2)^n dx &\geq 2\int_{0}^{1/\sqrt{n}} (1 - x^2)^n dx \\ &\geq 2\int_{0}^{1/\sqrt{n}} (1 - nx^2) dx = \frac{4}{3\sqrt{n}} > \frac{1}{\sqrt{n}}.\end{aligned}$$

The first inequality can be proven by induction on n. □

Lemma B.4 *If* $\delta \in (0, 1)$, *then* $\lim_{n\to\infty} \sqrt{n}(1 - \delta^2)^n = 0$.

Proof Fix $\delta \in (0, 1)$. Then, we obtain

$$\begin{aligned}\lim_{n\to\infty} \sqrt{n}(1 - \delta^2)^n &= \lim_{n\to\infty} \frac{\sqrt{n}}{(1 - \delta^2)^{-n}} \\ &= \lim_{n\to\infty} \frac{1}{2\sqrt{n}(-n)(1 - \delta^2)^{-n-1}} = \lim_{n\to\infty} \frac{-(1 - \delta^2)^{n+1}}{2n^{\frac{3}{2}}} = 0.\end{aligned}$$

Note that above, we have made use of L'Hôpital's Rule. □

***Proof of Theorem* B.1** Applying Lemma B.2, without loss of generality, we assume that $a = 0, b = 1$ and $f(0) = 0 = f(1)$. Further, define $f(x) := 0$ for all $x \in \mathbb{R} \setminus [0, 1]$.

For each $n \in \mathbb{N}$, let $Q_n(x) := c_n(1 - x^2)^n$, where c_n is a constant such that $\int_{-1}^{1} Q_n(x)dx = 1$. Then $c_n \in \mathbb{R}$ and by Lemma B.3, $c_n \in (0, \sqrt{n})$, for each $n \in \mathbb{N}$. Fix any $\delta \in (0, 1)$. Then

$$Q_n(x) \leq \sqrt{n}(1 - \delta^2)^n, \text{ for all } \delta \leq |x| \leq 1 \text{ and for all } n \in \mathbb{N}. \tag{B.1}$$

Define P_n as follows:

$$P_n(x) := \int_{-1}^{1} f(x + t)Q_n(t)dt \text{ for all } x \in [0, 1].$$

Since f is zero outside $[0, 1]$, by a change of variable, we have

$$P_n(x) = \int_{-x}^{1-x} f(x + t)Q_n(t)dt = \int_{0}^{1} f(t)Q_n(t - x)dt \text{ for all } x \in [0, 1].$$

Since each Q_n is a polynomial, the last integral above implies that, each $P_n(x)$ is a polynomial in x. If f is real valued, then so is each polynomial P_n.

Let $\epsilon > 0$ be given. Since f is continuous on $[0, 1]$, it is uniformly continuous on $[0, 1]$ and hence on $\mathbb{R}$. Pick any $\delta \in (0, 1)$ such that

$$|f(s) - f(t)| < \frac{\epsilon}{2} \text{ whenever } |s - t| < \delta.$$

Let $M := \sup\{|f(x)| : x \in [0, 1]\}$. By Lemma B.4, let $m \in \mathbb{N}$ such that

$$\sqrt{n}(1 - \delta^2)^n < \frac{\epsilon}{2(4M + 1)} \text{ for all } n > m.$$

Since $Q_n \geq 0$ and $\int_{-1}^{1} Q_n = 1$. Applying (B.1), for all $x \in [0, 1]$ and $n > m$, we obtain

$$\begin{aligned}
|P_n(x) - f(x)| &= \left| \int_{-1}^{1} f(x + t)Q_n(t)dt - f(x) \right| \leq \int_{-1}^{1} |f(x + t) - f(x)|Q_n(t)dt \\
&\leq 2M \int_{-1}^{-\delta} Q_n(t)dt + \frac{\epsilon}{2} \int_{-\delta}^{\delta} Q_n(t)dt + 2M \int_{\delta}^{1} Q_n(t)dt \\
&\leq 4M\sqrt{n}(1 - \delta^2)^n + \frac{\epsilon}{2} < \epsilon.
\end{aligned}$$

Hence the result. □

Exercise B.1 For any $a > 0$, prove that there exists a sequence of real polynomials $\{P_n\}$, uniformly convergent to $|x|$ on $[-a, a]$ such that $P_n(0) = 0$, for every $n \in \mathbb{N}$.

Exercise B.2 Let $\phi : [c, d] \longrightarrow [a, b]$ be a bijection and $f_n \longrightarrow f$, uniformly on $[a, b]$. Prove that $f_n \circ \phi \longrightarrow f \circ \phi$, uniformly on $[c, d]$.

Exercise B.3 Is it possible to generalize Theorem B.1 for continuous functions $f : \mathbb{R} \longrightarrow \mathbb{C}$?

Exercise B.4 Using Theorem B.1, prove that $C[0, 1]$ is separable.

B.2 A Continuous but Nowhere Differentiable Function

Theorem B.5 *There exists a real valued function which is continuous on* $\mathbb{R}$, *but differentiable nowhere.*

Proof Define a function $f_0 : \mathbb{R} \longrightarrow \mathbb{R}$ with period 2 such that $f_0(x) := |x|$ for all $x \in [-1, 1]$. Then f_0 is continuous on $\mathbb{R}$ such that $f_0(x + 2) = f_0(x)$ and $|f_0(x)| \leq 1$ for all $x \in \mathbb{R}$. Also

$$|f_0(y) - f_0(x)| \leq |y - x| \text{ for all } x, y \in \mathbb{R}. \tag{B.2}$$

Note that f_0 is a 'sawtooth' function on $\mathbb{R}$, which is linear on the intervals $[k, k + 1]$ for every $k \in \mathbb{Z}$. For each $n \in \mathbb{N}$, let

$$f_n(x) := \left(\frac{3}{4}\right)^n f_0(4^n x) \text{ for all for all } x \in \mathbb{R}.$$

Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be defined as $f(x) := \sum_{n=0}^{\infty} f_n(x)$ for all $x \in \mathbb{R}$. By Weierstrass M-test, the series for f is uniformly convergent on $\mathbb{R}$. Since each f_n is continuous, f is a continuous function. Now we prove that f is not differentiable at any $x \in \mathbb{R}$. Fix any $x \in \mathbb{R}$ and $m \in \mathbb{N}$.

Let $\delta_m = \pm \frac{4^{-m}}{2}$, where the sign is chosen such that no integer lies strictly between $4^m x$ and $4^m(x + \delta_m)$. This is possible, as $4^m |\delta_m| = \frac{1}{2}$. Let

$$y_n := \frac{f_n(x + \delta_m) - f_n(x)}{\delta_m}.$$

If $n > m$, then $4^n(x + \delta_m) - 4^n x = 4^n \delta_m = \pm \frac{4^{n-m}}{2}$ is an even integer, and hence $y_n = 0$. If $n < m$, then (B.2) implies

$$|y_n| = \left| \frac{f_n(x + \delta_m) - f_n(x)}{\delta_m} \right| = 3^n \left| \frac{f_0(4^n(x + \delta_m)) - f_0(4^n x)}{4^n \delta_m} \right| \leq 3^n.$$

For $n = m$, we have $4^m(x + \delta_m) - 4^m x = 4^m \delta_m = \pm \frac{1}{2}$. Since no integer lies between $4^m x$ and $4^m(x + \delta_m)$, the images of both of $4^m x$ and $4^m(x + \delta_m)$ under f_0, lie on single a straight line with slope ± 1. Therefore,

$$|y_m| = \left| \frac{f_m(x+\delta_m) - f_m(x)}{\delta_m} \right| = 3^m \left| \frac{f_0(4^m(x+\delta_m)) - f_0(4^m x)}{4^m \delta_m} \right| = 3^m.$$

Hence, by triangle inequality, we obtain

$$\begin{aligned} \left| \frac{f(x+\delta_m) - f(x)}{\delta_m} \right| &= \left| \sum_{n=0}^{\infty} \frac{f_n(x+\delta_m) - f_n(x)}{\delta_m} \right| \\ &= \left| \sum_{n=0}^{m} y_n \right| \geq 3^m - \sum_{n=0}^{m-1} 3^n = \frac{1}{2}(3^m + 1). \end{aligned}$$

Therefore, f is not differentiable at x, otherwise we would have

$$|f'(x)| = \lim_{\delta \to 0} \left| \frac{f(x+\delta) - f(x)}{\delta} \right| = \lim_{m \to \infty} \left| \frac{f(x+\delta_m) - f(x)}{\delta_m} \right| = +\infty,$$

which is absurd, as the derivative is defined to be a finite number. □

Remarks B.6 (a) Along with Theorem 8.45 we conclude that there exists a continuous $\mathbb{R} \longrightarrow \mathbb{R}$ function that is not monotone on any open interval.
(b) A continuous function, differentiable only on rationals is provided in [12].

Exercise B.5 Let f be the function given by Theorem B.5 and $g(x) := x^2 f(x)$, for all $x \in \mathbb{R}$. Show that g is a continuous function, differentiable only at 0.

B.3 Most Continuous Functions are Nowhere Differentiable

The function presented in the proof of Theorem B.5 is far from being an isolated example of such a continuous everywhere and nowhere differentiable function. In the sense of Baire Category, 'almost all' functions in $C[0, 1]$ nowhere differentiable. In other words, it is 'exceptional' for a continuous function to have a derivative at some point.

Let $C[0, 1]$ be the Banach space of continuous real valued functions on $[0, 1]$ under uniform norm and $ND[0, 1]$ be the collection of nowhere differentiable functions from $C[0, 1]$. Write $D[0, 1] := C[0, 1] \setminus ND[0, 1]$. We shall prove that $D[0, 1]$ is a meager subset of $C[0, 1]$. For $m, n \in \mathbb{N}$, define

$$A_{m,n} := \Bigg\{ f \in C[0,1] : \text{ there exists some } x \in [0,1] \text{ such that } \\ \left| \frac{f(t) - f(x)}{t - x} \right| \leq m \text{ whenever } 0 < |t - x| < \frac{1}{n} \Bigg\}.$$

Lemma B.7 $D[0, 1] \subset \bigcup_{m=1}^{\infty} \bigcup_{n=1}^{\infty} A_{m,n}$.

Proof Let $f \in D[0, 1]$. Then there exists some $x \in [0, 1]$ such that $f'(x)$ exists. Pick any $m \in \mathbb{N}$ such that $|f'(x)| < m$. Then there exists some $\delta > 0$ such that

$$\left|\frac{f(t) - f(x)}{t - x}\right| \leq m \text{ whenever } 0 < |t - x| < \delta.$$

Let $n \in \mathbb{N}$ such that $1/n < \delta$. Then $f \in A_{m,n}$. Hence the result. □

Lemma B.8 *Each $A_{m,n}$ is a closed subset of the Banach space $C[0, 1]$.*

Proof Let f be a limit point of $A_{m,n}$ in $C[0, 1]$. Then there exists a sequence $\{f_k\}$ in $A_{m,n}$ convergent to f. Therefore, $f_k \longrightarrow f$, uniformly on $[0, 1]$. For every $k \in \mathbb{N}$, $f_k \in A_{m,n}$ implies that there exists $x_k \in [0, 1]$ such that

$$\left|\frac{f_k(t) - f_k(x_k)}{t - x_k}\right| \leq m \text{ whenever } 0 < |t - x_k| < \frac{1}{n}.$$

By Theorem 1.22, there exists a subsequence of $\{x_k\}$ convergent to some $x \in [0, 1]$. Without loss of generality, suppose that $x_k \longrightarrow x$. Then for $0 < |x - t| < 1/n$,

$$\left|\frac{f(t) - f(x)}{t - x}\right| = \lim_{k\to\infty} \left|\frac{f_k(t) - f_k(x_k)}{t - x_k}\right| \leq m.$$

Therefore, $f \in A_{m,n}$. Hence the result. □

Lemma B.9 *Each $A_{m,n}$ is nowhere dense.*

Proof Let $PL[0, 1]$ denote the collection of piecewise linear functions on $[0, 1]$ (see Definition 8.6). Pick any $m, n \in \mathbb{N}$. By Lemma B.8, it is enough to prove that $A_{m,n}$ has no interior point. Let $f \in A_{m,n}$ and $\epsilon > 0$ be given. It is enough to find some $g \in B(f; \epsilon) \setminus A_{m,n}$.

As in the proof of Theorem 8.7, there exists some $p \in B(f; \epsilon/2) \cap PL[0, 1]$. Since p is piecewise linear, it is differentiable on $[a, b]$ except possibly at a finite number of points. Therefore,there exists some $M > 0$ such that $|p'(x)| < M$ for every x, wherever this derivative exists. Pick any integer $k > 2(M + m)/\epsilon$.

Let $\phi_k \in PL[0, 1]$ such that $|\phi_k| \leq 1$ and $\phi_k'(x) = \pm k$, whenever it exists. Define $g(x) := p(x) + \frac{\epsilon}{2}\phi_k(x)$ for all $x \in [0, 1]$. Then

$$\|f - g\| \leq \|f - p\| + \|p - g\| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

To prove that $g \notin A_{m,n}$, let $x \in [0, 1]$. If p and ϕ_k are differentiable at x, then

$$|g'(x)| = |p'(x) \pm \frac{\epsilon}{2}k| \geq \frac{k\epsilon}{2} - |p'(x)| > (M + m) - M = m.$$

Thus, there exists some $t \in [0, 1]$ such that $0 < |x - t| < \frac{1}{n}$ and $\left|\frac{g(t)-g(x)}{t-x}\right| > m$. Hence, $g \notin A_{m,n}$.

In general, we find $l > n$ such that the restriction functions $g|_{[x,x+1/l]}$ and $g|_{[x-1/l,x]}$ are linear. As above, the absolute value of the slopes of these two restriction functions is greater than m. That is,

$$\left|\frac{g(t) - g(x)}{t - x}\right| > m \text{ for all } 0 < |x - t| < \frac{1}{l} < \frac{1}{n}.$$

Therefore, $g \notin A_{m,n}$. Hence the result. □

Theorem B.10 (Banach-Mazurkiewicz, 1931) *$D[0, 1]$ is a meager subset of the space $C[0, 1]$. Consequently, $ND[0, 1]$ is a non-meager set.*

Proof By Lemma B.7, we have $D[0, 1] \subset \bigcup_{m=1}^{\infty} \bigcup_{n=1}^{\infty} A_{m,n}$. By Lemma B.9, each $A_{m,n}$ is nowhere dense. Therefore, $D[0, 1]$ is a meager set. By Baire Category Theorem (8.33), $C[0, 1]$ is a non-meager set. Hence, $ND[0, 1]$ is non-meager. □

Notes and Remarks B.11 (a) A diverse range of particular examples of functions which are continuous but nowhere differentiable, has been given by several mathematicians starting from the Bolzano function (∼1830) to the recent Wen function (2002). A thorough discussion of all such functions is provided in the masters' thesis [13].

(b) In the first volume of Studia Mathematica in 1929, H. Steinhaus posed the following question, 'of what category is the set of continuous nowhere differentiable functions in the space of all continuous functions...'. The answer was published in the third volume of the same journal in 1931, simultaneously by Banach and Mazurkiewicz, who established special versions of Theorem B.10 (see [14–16]).

(c) The space $C[0, 1]$ is universal in the sense that every separable Banach space is linearly isometric to a subspace of $C[0, 1]$, known as the Banach-Mazur Theorem (see [17, p. 18]). In [18], it has been shown that every separable Banach space is linearly isometrically embeddable into $ND[0, 1]$. Further, $ND[0, 1]$ has positive Wiener measure (see [19]).

Appendix C
Proofs Through Games

This appendix offers some proofs in terms of simple two-player games. First, we establish the uncountability of reals and perfect sets through such an infinite two-player game. Then we present the Banach-Mazur game to prove Baire Category Theorem (8.33).

C.1 An Infinite Game and Uncountable Sets

To establish the uncountability of certain subsets of $\mathbb{R}$, we consider a game of numbers. There are two players A and B, and the game is defined as follows:

A subset E of real numbers is fixed in the beginning. First A chooses a real number a_1. Then B chooses another real $b_1 > a_1$. Then A chooses $a_2 \in (a_1, b_1)$ and B chooses $b_2 \in (a_2, b_1)$. Inducting this way, both A and B choose a number, alternatively, strictly between the previously chosen two numbers. Therefore, we obtain two sequences $\{a_n\}$ and $\{b_n\}$ such that

$$a_n < a_{n+1} < b_{n+1} < b_n \text{ for all } n \in \mathbb{N}.$$

Since $\{a_n\}$ is a monotonically increasing sequence of reals bounded above by b_1, it is convergent in $\mathbb{R}$. Write $a := \lim_{n\to\infty} a_n$ and declare that

$$A \text{ wins the game if } a \in E, \text{ and } B \text{ wins the game if } a \notin E.$$

One can easily see that if E is a finite set, then B has a winning strategy. In fact, the same holds for all countable sets.

Theorem C.1 *If E is countable, then B has a winning strategy.*

Proof If $E = \emptyset$, then the result is obvious. Otherwise, enumerate E as a sequence, say $E = \{x_n : n \in \mathbb{N}\}$. Consider the following strategy for B.

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For $n \in \mathbb{N}$, B chooses x_n, if it is a legal move, that is if $a_n < x_n < b_{n-1}$. Otherwise, B chooses any other number in (a_n, b_{n-1}).

If $a \in E$, then $a = x_m$ for some $m \in \mathbb{N}$. Hence, $a_n < a = x_m < b_n$ for all $n \in \mathbb{N}$. In particular, $a_m < a = x_m < b_m$, a contradiction. Hence the result. □

Corollary C.2 $\mathbb{R}$ *is uncountable.*

Proof If $E = \mathbb{R}$, then $a \in E$, no matter what B does. Hence, $\mathbb{R}$ is not countable. □

Next, we will establish the uncountability of non-empty perfect sets through our game. Before that, let us discuss the notion of *one-sided limit points.*

Definition C.3 Let $E \subset \mathbb{R}$. A real number x is called a *right limit point* of E, if

$$(x, x + \epsilon) \cap E \neq \emptyset \text{ for all } \epsilon > 0.$$

The set of the right limit points of E will be denoted by E^+.

Analogously, we define *left limit points* of E and denote that set by E^-.

Lemma C.4 *Let* $E \subset \mathbb{R}$. *Then*

(a) $E' = E^+ \cup E^-$ *and*
(b) $\inf E \in E^+$, *if E is bounded below closed set.*

Proof (a) This part is trivial from the definitions.
(b) Since E is bounded below and closed, $\inf E$ exists and $\inf E \in E = E' = E^+ \cup E^-$. The result holds as, by definition we have $\inf E \notin E^-$. □

Theorem C.5 *Let E be a perfect subset of reals, $a \in E^+$ and $\epsilon > 0$. Then the interval $(a, a + \epsilon)$ contains an element of E^+. Consequently, $E^+ \subset (E^+)^+$.*

Proof Repeatedly using $a \in E^+$, we obtain numbers $z, y, x \in E$ such that $a < x < y < z < a + \epsilon$. Since $E \cap (x, z)$ contains y, it is a nonempty subset of reals bounded below by x.

If $x \in E^+$, we are done. Suppose $x \notin E^+$ and let $b := \inf(E \cap (x, z))$. As in Lemma C.4, $b \in E^+$. Therefore $b \neq x$. So $x < b < z < a + \epsilon$. Hence the result. □

Lemma C.6 *Let E be a nonempty perfect subset of $\mathbb{R}$ which is bounded below. Then E uncountable.*

Proof Applying Lemma C.4, A first chooses any $a_1 \in E^+$. Using Theorem C.5, A can choose a sequence $\{a_n\}$ from E^+. Therefore, $a = \sup_n a_n \in E' = E$ and A will win. Hence, A has the winning strategy. By Theorem C.1, E is uncountable. □

Theorem C.7 *If E is a nonempty perfect subset of reals, then E uncountable.*

Proof Without loss of generality, assume that E is bounded. Note that $E = E' = E^- \cup E^+$. If $E^+ \neq \emptyset$, A picks any $a_1 \in E^+$ and proceed as in Lemma C.6. In case $E^+ = \emptyset$, then $E^- = E' = E \neq \emptyset$. Write

$$E_1 := -E := \{-x : x \in E\}.$$

Note that $E_1^+ = -(E^-) = -E \neq \emptyset$. Applying Lemma C.6 for E_1, we conclude that $E_1 = -E$ is uncountable. Hence, E is uncountable. □

The above infinite game is based upon Cantor's original proof of the uncountability of the reals, see [20]. It is taken from [21], where the following natural questions have been raised.

Open Questions C.8 Do there exist uncountable subsets of $\mathbb{R}$, for which

(a) B has a winning strategy or
(b) A does not have a winning strategy or
(c) neither A nor B has a winning strategy?

Exercise C.1 Let $a < b$ be reals. Define an infinite game to prove that $[a, b]$ is uncountable. (Hint: Restrict the choice of a_n, b_n to (a, b) in our game).

Exercise C.2 Show that the assumption that 'E is bounded', does not violate any generality in the proof of Theorem C.7.

Exercise C.3 Give an example of an infinite game to extend Theorem C.7 to complete metric spaces.

C.2 The Banach-Mazur Game

Two players A and B decide to play a game. A subset E of $\mathbb{R}$ is fixed in the beginning. First, A chooses a closed interval I_1 of X. Then B chooses a closed subinterval I_2 of O_1. Again A chooses a closed subinterval I_3 of O_2.

Inducting this way, both A and B choose a closed interval, alternatively, as a subset of the previously chosen interval. Therefore, we obtain a nested decreasing sequence $\{I_n\}$ of closed intervals of $\mathbb{R}$. Declare that

$$A \text{ wins the game if } \left(\bigcap_{n=1}^{\infty} I_n\right) \cap E \neq \emptyset. \text{ Otherwise, } B \text{ wins the game.}$$

Theorem C.9 *If E is of first category, then the player B has a winning strategy.*

Proof Assume that E is of first category. Then $E = \cup E_n$ for some nowhere dense sets E_n. Then for every n, there exists an open interval disjoint from E_n.

Note that player B chooses the subsequence $\{I_{2n}\}$, in this game. Suppose that B chooses I_{2n} to be a closed interval, disjoint from E_n. Then $\bigcap_{n=1}^{\infty} I_n = \bigcap_{n=1}^{\infty} I_{2n}$ will contain no point of $\bigcup_{n=1}^{\infty} E_n = E$. Hence, this is a winning strategy for B. □

Corollary C.10 $\mathbb{R}$ *is not of first category.*

Proof Suppose A chooses I_1 to be a bounded closed interval. Then $\bigcap_{n=1}^{\infty} I_n \in \mathbb{R}$, no matter what B does. The result follows by Theorem C.9. □

The converse of Theorem C.9 also holds (see [22, p. 28, Theorem 6.1]).

Exercise C.4 (Choquet Game) Let X be a metric space. Two players A and B, alternatively, choose a nested decreasing sequence $\{O_n\}$ of nonempty open subsets of X. Declare that

$$A \text{ wins the game if } \bigcap_{n=1}^{\infty} O_n \neq \emptyset. \text{ Otherwise, } B \text{ wins the game.}$$

Prove that if X is of first category, then the player B has a winning strategy. Further conclude the Baire Category Theorem (8.33) for complete metric spaces.

Appendix D
A Glimpse into General Topology

With this last appendix, we strive to provide a quick overview of general topological spaces. After introducing basic notions, we show that various properties of metric spaces are not shared by general topological spaces. This provides a few examples of non-metrizable topological spaces. A collection of standard topological spaces is provided in the exercises.

D.1 Introduction to Topological Spaces

As observed in the chapter on homeomorphisms, the notions like convergence, continuity, compactness, and connectedness are topological properties (see Sect. 9.2). In particular, for a metric space (X, d), these are unaltered even if d is replaced with some topologically equivalent metric. So these depend only upon the open subsets of the metric space.

Recall that each of these notions can be defined purely in terms of open subsets of the space (see Exercise 3.19, Theorem 3.28, Exercise 3.93, Definition 5.1, and Definition 6.17). So why don't we discuss convergence, continuity, compactness, and connectedness for spaces when only open sets are given? But such open sets should be 'trustable'. That is, these should satisfy the requirements of Theorem 3.2.

Motivated by this, we define the notion of open sets in arbitrary spaces, which leads to the notion of topological spaces. Throughout this appendix, X will denote a nonempty set.

Definition D.1 A *topology* on X is a family $\mathcal{T}$ of subsets of X such that

(a) $\emptyset, X \in \mathcal{T}$,
(b) $\mathcal{T}$ is closed under arbitrary unions, and
(c) $\mathcal{T}$ is closed under finite intersections.

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If $\mathcal{T}$ is a topology on X, we say that $(X, \mathcal{T})$ is a *topological space.* If the underlying topology is evident, we simply write 'X is a topological space'. The sets in $\mathcal{T}$ are known as *open sets* of $(X, \mathcal{T})$; their complements are called *closed sets*.

Example D.2 The collection of usual open subsets of $\mathbb{R}^n$, equipped with the Euclidean metric, is a topology on $\mathbb{R}^n$. It is commonly known as the *usual topology, standard topology* or the *Euclidean topology on* $\mathbb{R}^n$.

Examples D.3 Let X be any nonempty set.

(a) If $X = \{a, b, c\}$ and $\mathcal{T} := \{\emptyset, \{a\}, \{b, c\}, X\}$, then $\mathcal{T}$ is a topology on X.
(b) If d is a metric on X, the family $\mathcal{T}_d := \{O : O \text{ is open in } (X, d)\}$ is a topology on X. Therefore,every metric space is a topological space with same open sets.
(c) The collection $\mathcal{T}_1$ of all subsets of X is a topology on X. It is known as the *discrete topology* on X, and is given by the discrete metric on X.
(d) The collection $\mathcal{T}_0 := \{\emptyset, X\}$ is a topology on X, known as the *indiscrete topology* on X. It is also known as the *trivial topology.*
(e) If $\mathcal{T}$ is a topology on X and $\emptyset \neq Y \subset X$, then $\mathcal{T}_Y := \{O \cap Y : O \in \mathcal{T}\}$ is a topology on Y. It is known as the *subspace topology* on Y and $(Y, \mathcal{T}_Y)$ is called a *subspace* of $(X, \mathcal{T})$.
(f) Apart from the discrete and indiscrete spaces, there are topological spaces in which a set is open if and only closed. E.g. let $X := \{1, 2, 3\}$ and $\mathcal{T} := \{\emptyset, \{1\}, \{2, 3\}, X\}$.

Several other examples of topological spaces will be presented in the exercises.

Definition D.4 Topologies induced by metrics are called *metrizable.* In other words, a topological space $(X, \mathcal{T})$ is said to be a *metrizable space,* if there exists a metric d on X such that $\mathcal{T} = \mathcal{T}_d$.

Examples D.5 Let X be any nonempty set.

(a) Every discrete topological space $(X, \mathcal{T}_1)$ is metrizable. Note that $\mathcal{T}_1 = \mathcal{T}_{d_c}$, where d_c denotes a discrete metric on X (see Example 2.3(b)).
(b) If $|X| > 1$, then the indiscrete topology $\mathcal{T}_0 := \{\emptyset, X\}$ on X is not metrizable. To see this, assume that there exists a metric d on X which induces the topology $\mathcal{T}_0$. Let $a, b \in X$ such that $a \neq b$. If $r := d(a, b)/2$, then $B(a; r)$ and $B(b; r)$ are disjoint nonempty open subsets of X. This is a contradiction, as $\mathcal{T}_0$ has only one nonempty open set X.

Exercise D.1 (A-exclusive topology) If $A \subset X$, then prove that $\mathcal{E}_A := \{X\} \cup \{O \subset X : O \cap A = \emptyset\}$ is a topology on X. What is this topology when $A = \emptyset$ or $A = X$?

Exercise D.2 (A-inclusive topology) If $A \subset X$, then prove that $\mathcal{I}_A := \{\emptyset\} \cup \{B : A \subset B \subset X\}$ is a topology on X. What is this topology when $A = \emptyset$ or $A = X$?

Exercise D.3 (Co-finite topology) If X is a nonempty set, prove that the collection of sets $\{A : X \setminus A \text{ is a finite set or } A = \emptyset\}$ is a topology on X.

Exercise D.4 (Co-countable topology) If X is a nonempty set, prove that the collection of sets $\{A : X \setminus A$ is a countable set or $A = \emptyset\}$ is a topology on X.

Exercise D.5 (a) If X is a finite set, prove that the co-finite topology on X is discrete.
(b) If X is a countable set, prove that the co-countable topology on X is discrete.

Exercise D.6 (Fort's space) Let X be any infinite set and ∞ be any fixed point of X. Prove that the following collection defines a topology on X.

$$\mathcal{T} := \{G \subset X : \infty \notin G\} \cup \{G : \infty \in G \text{ and } X \setminus G \text{ is a finite set}\}.$$

Exercise D.7 Let C denote a collection of subsets of X containing $\emptyset$ and X such that C is closed under arbitrary intersections and under finite unions. Prove that there exists a unique topology on X, in which C is the collection of all closed sets.

Exercise D.8 (a) Let X be an infinite set with co-finite topology. Prove that

(i) no two (nonempty) open subsets of X are disjoint,
(ii) the co-finite topology on X is not metrizable.

(b) Let X be an uncountable set with co-countable topology. Prove that

(i) no two (nonempty) open subsets of X are disjoint,
(ii) the co-countable topology on X is not metrizable.

Exercise D.9 (Lower limit topology) Prove that the collection $\{\emptyset\} \cup \{A : A$ is a union of intervals of the form $[a, b)\}$ forms a topology on $\mathbb{R}$.

Exercise D.10 (Left ray topology) Prove that $\{\emptyset, \mathbb{R}\} \cup \{(-\infty, a) : a \in \mathbb{R}\}$ is a topology on $\mathbb{R}$.

Exercise D.11 Is $\{\emptyset, \mathbb{R}\} \cup \{[a, +\infty) : a \in \mathbb{R}\}$ a topology on $\mathbb{R}$?

Exercise D.12 Is $\{\emptyset\} \cup \{A \subset \mathbb{R} : x \in A \iff -x \in A\}$ a topology on $\mathbb{R}$?

Exercise D.13 Prove that the collections of sets $\{\emptyset\} \cup \{\{n, n+1, \dots\} : n \in \mathbb{N}\}$ as well as $\{\emptyset, \mathbb{N}\} \cup \{\{1, \dots, n\} : n \in \mathbb{N}\}$ are topologies on $\mathbb{N}$.

Exercise D.14 Let $\mathbb{Z}_e$ and $\mathbb{Z}_o$ denote the collections of all even and odd integers, respectively. Prove that the collection $\{\emptyset, \mathbb{Z}, \mathbb{Z}_e, \mathbb{Z}_o\}$ is a topology on $\mathbb{Z}$.

Exercise D.15 Let $\mathcal{T}_1$ and $\mathcal{T}_2$ be two topologies on X.

(a) Is $\mathcal{T}_1 \cup \mathcal{T}_2$ a topology on X?
(b) Prove that there exists a smallest topology on X which contains both $\mathcal{T}_1$ and $\mathcal{T}_2$.

Exercise D.16 If $\emptyset \neq E \subset \mathbb{R}$, what is the topology in $\mathbb{R}^2$, in which every line with slope in E is an open set?

Exercise D.17 (Line with two 'origins') Consider two lines $L_i := \{(x, i) : x \in \mathbb{R}\}$; $i = 1, 2$. Let $X = L_1 \cup L_2$ and identify $(x, 1)$ with $(x, 2)$, unless $x = 0$. Consider a topology $\mathcal{T}$ on X, generated by the subspace topologies of L_1 and L_2. Show that there are no disjoint open sets in $\mathcal{T}$, separating the points $(0, 1)$ with $(0, 2)$.

D.2 Analogies and Contrasts

This section examines the similarities and differences between metric spaces and general topological spaces. The following are a few notions in the general setting.

Definition D.6 Let $(X, \mathcal{T})$ be a topological space.

(a) A sequence $\{x_n\}$ in X is said to be *convergent* to some $x \in X$ if for every $O \in \mathcal{T}$ such that $x \in O$, there exists some $m \in \mathbb{N}$ such that

$$x_n \in O \text{ for all } n \geq m.$$

(b) A *neighborhood* of a point $x \in X$ is defined to be a set $N_x \subset X$ such that

$$x \in O \subset N_x \text{ for some } O \in \mathcal{T}.$$

Similarly, the notions of interior points, limit points, isolated points, adherent points, continuity, continuity at a point, compactness, and connectedness can also be defined in terms of open sets, in general topological spaces.

These topological properties can also be equivalently defined only in terms of neighborhoods. Here we present a few of such definitions.

Definitions D.7 Let X, Y be topological spaces, $E \subset X$ and $x \in X$. For every $a \in X$ (or $a \in Y$), let $\mathcal{N}_a$ denote the family of all neighborhoods of a in X (or in Y).

(a) A sequence $\{x_n\}$ in X is said to be *convergent* to $x \in X$ if for every $O \in \mathcal{N}_x$, there exists some $m \in \mathbb{N}$ such that $x_n \in O$ for all $n \geq m$.
(b) A function $f : X \longrightarrow Y$ is said to be *continuous at* $x \in X$ if $f^{-1}(O) \in \mathcal{N}_x$ for all $O \in \mathcal{N}_{f(x)}$. Further, f is called continuous on E, if f is *continuous* at every $x \in E$.
(c) An element $x \in X$ is said to be
   (i) an *interior point* of E if $O \subset E$ for some $O \in \mathcal{N}_x$.
   (ii) an *isolated point* of E if $O \cap E = \{x\}$ for some $O \in \mathcal{N}_x$.
   (iii) an *adherent point* of E if $O \cap E \neq \emptyset$ for every $O \in \mathcal{N}_x$.
   (iv) a *limit point* of E if $O \cap E \setminus \{x\} \neq \emptyset$ for every $O \in \mathcal{N}_x$.
   (v) a *boundary point* of E if $O \cap E \neq \emptyset$ and $O \cap (X \setminus E) \neq \emptyset$ for all $O \in \mathcal{N}_x$.
(d) A set $E \subset X$ is said to be
   (i) *perfect* if E is closed in X and every point of E is its limit point.
   (ii) *connected* if E is not a union of two nonempty disjoint sets, clopen in E.
   (iii) *compact* if every open cover of E has a finite subcover; i.e. if Ω is a collection open sets such that $E \subset \bigcup_{O\in\Omega} O$, then $E \subset \bigcup_{i=1}^{n} O_i$, for some finitely many $O_1, \ldots, O_n \in \Omega$.

The sets of interior points, limit points, adherent points, and boundary points of E will be again denoted by E^o, E', $\overline{E}$ and ∂E, respectively. These are known as *interior, derived set, closure* and *boundary* of E, respectively.

The reader may verify that analogues of Theorems 3.5, 3.10, 3.19, 5.10, 5.18, 5.31, 5.33, 6.9, 6.20, 6.22, 6.28, 6.31, 6.32, 6.35, and 6.44 hold for arbitrary topological spaces. However, the same is not true for several other properties of metric spaces.

Examples D.8 Let $X := \{0, 1\}$ with the indiscrete topology. Then

(a) 0 is a limit point of X but no neighborhood of 0 contains infinitely many points of X. Also, there exists no sequence of distinct elements in X convergent to 0.
(b) X is a finite set with limit points.
(c) Every sequence in X is convergent and has two limits 0 as well as 1.
(d) $\{0\}$ is a compact subset of X, while it is not closed in X.

Therefore, in general topological spaces, compact sets need not be closed. Further, closure of a compact set may not be compact. In contrast to Corollary 8.11, a subspace of a separable topological space may not be separable. Doesn't it appear paradoxical because separability is a measure of smallness of a set?

Example D.9 Let $\mathcal{T} := \{S \subset \mathbb{R} : 0 \in S\} \cup \{\emptyset\}$. Then

(a) $(\mathbb{R}, \mathcal{T})$ is a topological space.
(b) The singleton set $\{0\}$ is dense in $(\mathbb{R}, \mathcal{T})$, and hence $(\mathbb{R}, \mathcal{T})$ is separable.
(c) The subspace $\mathbb{R} \setminus \{0\}$ is not separable.
(d) The set $\{0\}$ is compact, but not its closure.

Proof (a) Left to the reader.
(b) To prove that $\overline{\{0\}} = \mathbb{R}$, let $x \in \mathbb{R}$ and O be an open subset of $\mathbb{R}$ containing x. Then $O \neq \emptyset$ and thus $0 \in O$. Therefore, every neighborhood of x intersects $\{0\}$. Hence, $x \in \overline{\{0\}}$.
(c) The subspace $\mathbb{R} \setminus \{0\}$ is uncountable and has discrete topology. Therefore, it is not separable.
(d) Being a singleton set, $\{0\}$ is compact. Further $\overline{\{0\}} = \mathbb{R}$ is not compact in $(\mathbb{R}, \mathcal{T})$, as $\{\{0, x\} : x \in \mathbb{R}\}$ is an open cover of $\mathbb{R}$, having no finite subcover. □

Similarly, various characterizations of compactness, as in Theorem 5.27, are not all equivalent in general topological spaces. It is a good exercise to look out for the results on metric spaces, which hold (or do not hold) for general topological spaces. That will also be an amusing way to revise this textbook.

There are topological spaces, known as *uniform spaces,* in which the notions like Cauchy sequences, completeness, total boundedness, and uniform continuity can be discussed. For such spaces, analogues of Theorems 5.26 and 5.37 hold true (see [23, Chapter 9] or [24]). A characterization of complete uniform spaces can be found in [25, Proposition 4.10].

The readers in search of an expository book on topology are referred to [26]. For a smooth take off from metric spaces to topological spaces, we refer [27]. A vast collection of counterexamples in topology can be found in [28].

Exercise D.18 Establish analogues of Theorems 3.5, 3.10, 3.19, 5.10, 5.18, 5.31, 5.33, 6.9, 6.20, 6.22, 6.28, 6.31, 6.32, 6.35, and 6.44 for arbitrary topological spaces.

Exercise D.19 Do any of the Exercises 3.18, 3.31, 3.58 or 8.15 hold true for all topological spaces?

Exercise D.20 Let X and Y be topological spaces and $f : X \longrightarrow Y$. Prove that f is continuous, if either X is a discrete space or Y is an indiscrete space.

Exercise D.21 Let X be an indiscrete topological space with $|X| > 1$. Prove that every subset of X is compact, perfect as well as connected.

Exercise D.22 Let X be a nonempty set and $\mathcal{T}_p := \{S \subset X : p \in S\} \cup \{\emptyset\}$ for all $p \in X$. Prove that $\mathcal{T}_p$ and $\mathcal{T}_q$ are non-comparable homeomorphic topologies on X, for all distinct $p, q \in X$. (Two topologies are called *comparable* if one of them is contained in the other.)

Exercise D.23 Characterize the collections of compact, connected, and perfect subsets in a discrete topological space.

Exercise D.24 Let X be a topological space. Prove the following:

(a) If X is indiscrete, then every sequence in X converge to every point of X.
(b) If X is discrete, then only eventually constant sequences converge in X.
(c) If X is co-countable, then only eventually constant sequences converge in X.

Exercise D.25 Let $\{x_n\}$ be a sequence in a co-finite space X. Prove the following:

(a) If no term of $\{x_n\}$ repeats infinitely many times, then it converges to every $x \in X$.
(b) If exactly one term, say x', of $\{x_n\}$ repeats infinitely many times, then $\{x_n\}$ has a unique limit x' in X.
(c) If there are two terms of $\{x_n\}$ which repeat infinitely many times, then $\{x_n\}$ is not convergent in X.

Exercise D.26 Prove that every convergent sequence in topological space X has a unique limit if and only if any two distinct points of X are contained in disjoint open sets. (Such a topological space is called a *Hausdorff space*.)

Exercise D.27 Prove that every compact subset of a Hausdorff space is closed.

Exercise D.28 Let $\mathcal{T}$ be a topology on X and $\mathcal{B} \subset \mathcal{T}$. Prove that the following statements are equivalent:

(a) Every open set in X is a union of members of $\mathcal{B}$.
(b) For every set $O \in \mathcal{T}$ and $x \in O$, there is some $B \in \mathcal{B}$ such that $x \in B \subset O$.

(A collection $\mathcal{B}$ satisfying (a) or (b) is called a *base* or *basis* for $\mathcal{T}$.)

Exercise D.29 Let X be any nonempty set and $\mathcal{B}$ be a collection of subsets of X. Prove that $\mathcal{B}$ is a base for some topology on X if and only if

(a) X is a union of members of $\mathcal{B}$.
(b) For any $B_1, B_2 \in \mathcal{B}$, the intersection $B_1 \cap B_2$ is a union of members of $\mathcal{B}$.

Further, show that $\left\{ \bigcup_{B \in \mathcal{B}_0} B : \mathcal{B}_0 \subset \mathcal{B} \right\}$ is a topology on X. (It is known as *the topology generated by base* $\mathcal{B}$. If $\mathcal{B}_0 = \emptyset$, then $\bigcup_{B \in \mathcal{B}_0} B$ is taken as the empty set.)

Exercise D.30 Let $\{(X_\alpha, \mathcal{T}_\alpha) : \alpha \in \wedge\}$ and $\prod_{\alpha \in \wedge} X_\alpha$ be as on page 203. Let $\mathcal{B}$ denote the collection of sets of the form $\prod_{\alpha \in \wedge} U_\alpha$ such that there exists a finite set $\{a_1, \ldots, \alpha_n\} \subset \wedge$ such that U_{α_i} is open in X_{α_i} for all $i = 1, \ldots, n$, and $U_\alpha = X_\alpha$ for all $\alpha \in \wedge \setminus \{a_1, \ldots, \alpha_n\}$. Prove that $\mathcal{B}$ is a base for some topology on X. (That topology is called the *product topology* on X).

Exercise D.31 Let $\{(X_\alpha, \mathcal{T}_\alpha) : \alpha \in \wedge\}$ be a nonempty collection of topological spaces and consider the product topology on $X := \prod_{\alpha \in \wedge} X_\alpha$. Let $x \in X$ and $\{x_n\}$ be a sequence in X. Prove that $x_n \longrightarrow x$ in X if and only if $x_n(\alpha) \longrightarrow x(\alpha)$ in X_α for all $\alpha \in \wedge$.

Exercise D.32 Prove that every topological space with a countable basis is separable. Is the converse true? (A space with a countable basis is called *second countable.*)

Exercise D.33 Let $X = \mathbb{R}$, with the lower limit topology. Prove that X is separable but not second countable. Conclude that X is not metrizable.

Exercise D.34 Let X be a Hausdorff space such that every $x \in X$ has a compact neighborhood. Prove that every nonempty perfect subset of X is uncountable.

Exercise D.35 (Brouwer) Prove that every totally disconnected, compact, perfect, Hausdorff, and second countable topological space is homeomorphic to the Cantor set.

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