Chapter 9
Design of Quarter Car Model for Active Suspension System and Control Optimization

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Abstract Quarter car model can be used to approximate a response of the suspension systems to obtain a behavioral relationship between the suspension and the body. The major aim of the quarter is to obtain a stable working control system for achieving three major control states. The three states include the passenger's comfort design, the road handling design, and a balanced design. The objective is to obtain the three control systems by using the H-infinity synthesis and designing a controller based on the defined states and control inputs of the car. The designed controller can therefore be optimized for the account of uncertainty. The use of μ-synthesis for the optimization of the designed controller and the balanced design is considered for the optimization. For the analysis of the control system, initially the system is given a disturbance of 7 cm. For the optimization, the input was increased and a bump of 10 cm was considered for achieving a greater disturbance and gains in the measurements. The bode plots recorded for the calculation would verify the performance of the control system which would take the quarter car model as a state space and the controller for the feedback module for the control of the suspension system.

Keywords Quarter car · Active suspension system · Bode plots · MATLAB · Control optimization

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I. R. Kale and A. Sadollah (eds.), *Optimization Methods for Structural Engineering*, Engineering Optimization: Methods and Applications,
https://doi.org/10.1007/978-981-99-2378-6_9

9.1 Introduction

Mathematical modeling deals with representing a system or a component of system mathematically. The key for representing a system mathematically is behavior study of the system and its components. The obtained mathematical model shall behave the same as the actual model under the usual circumstances. The behavior of the components in the system serves as the building blocks for the response of the system. The behavior of each components their connections with other system and their control of gains with respect to the system objective lead to a successful mathematical model. The mathematical model of a component is usually determined by the behavior of the component; i.e., if the components work with the displacement gain (spring), it generally works linearly. The dampers on the other hand works with the velocity gains, and thus the dampers work in a parabolic response. The components' performance may be different from the desired output since the practical circumstance and conditions play a vital role for the actual response. Mathematical models are used for representing systems and its corresponding components in mathematical format. The models are a great way in determining the system definition and obtain a more calculative approach. The system in the simulation system can be edited for a change in any of the component, and the simulation results can provide a similar result of the actual change of response in the system.

The mathematical modeling of a system usually starts with components and their respective behavior; usually, an equation or a state space is used for the definition of the components. The components involved in the suspension system are spring which responds linearly, the damper which responds parabolically, and the masses which vibrates with the road disturbance. As we solve the mathematical model of the system, all the corresponding components shall be tested for achieving the response of the suspension system.

The mathematical model of a **spring** can be determined as a **linear function**.

$$\boldsymbol{F} = \boldsymbol{k}\boldsymbol{x}$$

The mathematical model of a **damper** can be defined as **first-order differential**.

$$\boldsymbol{F} = \boldsymbol{c}\frac{\partial \boldsymbol{x}}{\partial \boldsymbol{t}}$$

The mathematical model of the **masses** falls under the **second-order differential**.

$$\boldsymbol{F} = \boldsymbol{m}\frac{\partial^2 \boldsymbol{x}}{\partial \boldsymbol{t}^2}$$

As we sum up all the terms we obtain an equation for the suspension system which can be used for the analysis.

$$\boldsymbol{m}\frac{\partial^2 \boldsymbol{x}}{\partial \boldsymbol{t}^2} + \boldsymbol{c}\frac{\partial \boldsymbol{x}}{\partial \boldsymbol{t}} + \boldsymbol{kx} = \boldsymbol{F}$$

The actuator can be represented in the cumulative force in the ***rhs*** of the equation.

Thite et al. (2012) developed a refined model for quarter car analysis. The research focused around the discomfort caused due to the vibration. Sharaf et al. (2013) presented quarter car analytical model and depicted the effect of vehicle speeds on ride comfort. Yu et al. (2019) illustrated experiments on series active variable geometry suspension system and modeled using H-infinity control scheme. Ahmed et al. (2021) employed linear quadratic regulator and fuzzy PID controllers to get optimized active suspension system of a quarter car. Hassaan et al. (2015) developed a passive suspension-based quarter model car and analyzed using MATLAB. Had et al. (1992) developed a semi-active suspension system based on 2 DoF with actual road conditions simulation. Paliwal et al. (2020) presented effect of different road profiles on nonlinear quarter car model. Allamraju et al. (2016) depicted the numerical modeling of quarter car model suspension system to get modal parameters. Ebrahimi-Nejad et al. (2020) utilized sports car to develop a modal equations using Lagrange's equations. Alvarez-Sánchez et al. (2013) used linear mathematical model to have a robust control scheme. Based on the literature review, it is observed that design of car with active suspension system and control optimization is at nascent stage. This study aims to obtain the three control systems by using the H-infinity synthesis and designing a controller based on the defined states and control inputs of the car.

9.2 Methodology

The simulation of the active suspension system in the simulation system requires defining all the components in the system and defining the design setup. A quarter car model is generally designed by defining the individual components and setting up gains from all the components according to their connection to the car and suspension system. The normally used suspensions usually have a spring and a damper (Zeng 2019). These are usually connected to the car body on one end and to the wheel on the other end. Under the components, the specific characteristics like the spring constant and the damping coefficient of the damper can be changed to achieve the desired results. Under the modern suspension system, the same configuration observes an increase in the installation of the actuator which is normally hydraulic and it is controlled by a feedback controller. The components need to be individually defined in the MATLAB for the generation of the state space of the quarter car model, as depicted in Fig. 9.1. The mass of the car body is defined in terms of **mb** (in kg). The mass of the wheel assembly is denoted by **mw** (in kg). The spring and damper are defined by **ks** and **bs**, respectively. The compressibility of the tire is accounted as **kt**. The linear first-order measurements and the disturbances are defined as **xb** (body travel), **xw** (wheel travel), and **r** (road disturbance). All the linear measurements are in meters. The hydraulic actuator discussed earlier is denoted as the force that is used

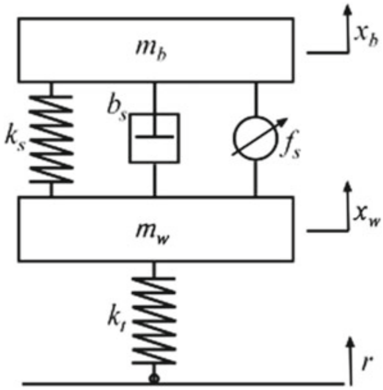

Fig. 9.1 Mathematical model of active suspension system

to control the system as **fs** (in kN). The actuator serves as the active component of the suspension system.

The values of the components can be referred as given below

$$(\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{x}_3, \boldsymbol{x}_4) = (\boldsymbol{x}_b, \dot{\boldsymbol{x}}_b, \boldsymbol{x}_w, \dot{\boldsymbol{x}}_w)$$

The state-space equations for the quarter car model are defined as listed below

$$\dot{\boldsymbol{x}}_1 = \boldsymbol{x}_2$$

$$\dot{\boldsymbol{x}}_2 = -\left(\frac{1}{\boldsymbol{m}_b}\right)\left[\boldsymbol{k}_s(\boldsymbol{x}_1 - \boldsymbol{x}_3) + \boldsymbol{b}_s(\boldsymbol{x}_2 - \boldsymbol{x}_4) - 10^3 \boldsymbol{f}_s\right]$$

$$\dot{\boldsymbol{x}}_1 = \boldsymbol{x}_2$$

$$\dot{\boldsymbol{x}}_4 = -\left(\frac{1}{\boldsymbol{m}_w}\right)\left[\boldsymbol{k}_s(\boldsymbol{x}_1 - \boldsymbol{x}_3) + \boldsymbol{b}_s(\boldsymbol{x}_2 - \boldsymbol{x}_4) - \boldsymbol{k}_t(\boldsymbol{x}_3 - \boldsymbol{r}) - 10^3 \boldsymbol{f}_s\right]$$

Further, defining all the elements in a state matrices which would aid in providing with the control of the quarter car model. After defining the matrices of the state space, we would define the state space of the quarter car model and define the inputs and outputs of the system. The transfer function related to actuator and chassis travel makes zero with the imaginary axis with natural frequency, i.e., **tire-hop frequency**. Similarly, actuator and the suspension travel give a zero with the imaginary axis at the natural frequency known as the **rattle space frequency**. The road disturbance is one of the factors that affect the motion of a vehicle, the suspension primarily targets the road disturbance, and the required control is used for achieving the specific design target. The road disturbance usually provides an input to the car state space with the deflection in the suspension, and the control is achieved by the force applied by the actuator. There are zeros of the imaginary axes that affect the feedback control and

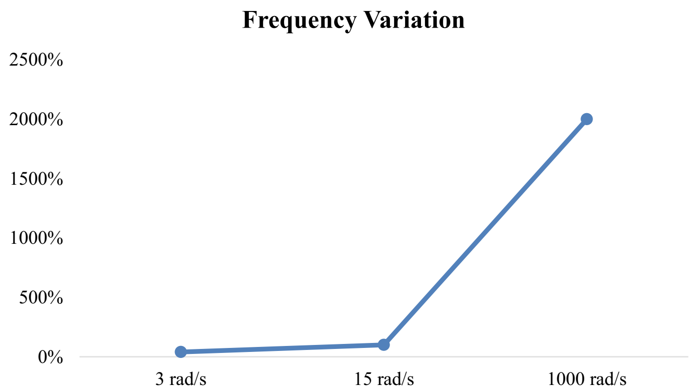

Fig. 9.2 Variations in the actuator with respect to the frequency

would cause not to improve the response from road disturbance to the acceleration of the body that is obtained at the tire-hop frequency. The similar would be the case of achieving the control of the suspension travel at the rattle space frequency. There is a relation that defines that the wheel position follows the road surface $x_w = x_b - \mathrm{sd}$. Under the conditions of lower frequencies, i.e., less than 5 rad/s, there is observed a trade-off between passenger comfort and suspension travel. The actuator is used for the force application to the car body, and the wheel is usually a hydraulic actuator. The actuator is placed between car body $\boldsymbol{m_b}$ and the wheel $\boldsymbol{m_w}$. The hydraulic actuator dynamics can be best represented by the first-order transfer function $1/((1+s)/60)$ with a maximum allowable displacement of **0.05 m**. The defined actuator model, but the actuator is named as a nominal model of a true actuator. This is due to the results that may be obtained would be only an approximation of the true actuator. This family of actuators would consist of a nominal model, the uncertainty would be frequency dependent. Figure 9.2 shows the result variations in the actuator with respect to the frequency. The weighting function $\boldsymbol{W}_{\mathbf{unc}}$ can be used to interpret the amount of uncertainty with frequency.

The two major goals of the suspension system are the passenger's comfort and the vehicle handling. The goals refer to the control of the body acceleration and the suspension travel. There are other factors that account for the control design which includes the actual surface of the road, data/measurements received from the sensors, noise of the sensor, and the actual working constraints of the actuator. The H-infinity synthesis algorithms can be used to express the objectives with the help of a single cost function and the target would be to minimize the cost function for achieving the control of the system. The $\boldsymbol{H}_\infty$ controller would use the measurements $\boldsymbol{y}_1$, $\boldsymbol{y}_2$ of the travel of suspension $\boldsymbol{sd}$ and chassis acceleration $\boldsymbol{ab}$ this will be used for obtaining the signal **u**, which will then be used to drive the actuator. There are three external sources of disturbance:

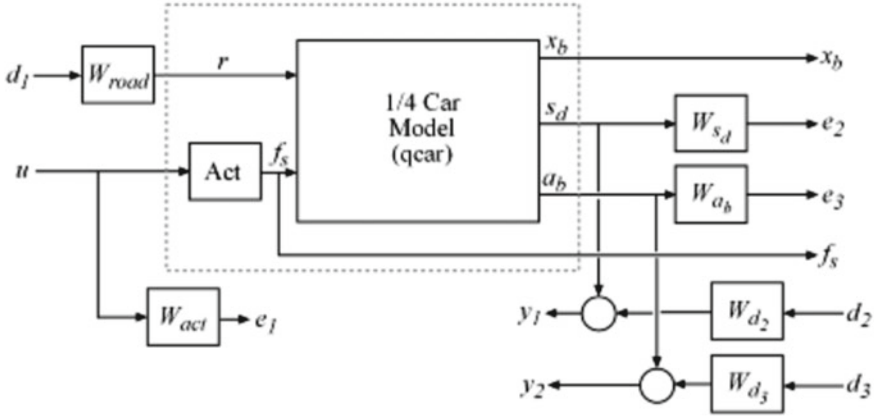

Fig. 9.3 Design setup for active suspension system

- The surface of the road $\boldsymbol{r}$ is returned as a normalized signal $\boldsymbol{d}_1$ which is further classified under the weighting function $\boldsymbol{W_{road}}$. To account for the road bump of height 7 cm, the weight is constant $\boldsymbol{W_{road}} = 0.07$
- The sensors used for the measurement of the current disturbance results with the noise, and the noise of the sensors is accounted as $\boldsymbol{W}_{d2}$ and $\boldsymbol{W}_{d3}$. The weighting functions are used, $\boldsymbol{W}_{d2} = 0.01$ and $\boldsymbol{W}_{d3} = 0.5$ to account for the noise of the sensor. The noise intensity is kept constant as **0.01** and **0.5**. The constant intensity locks the scope of the analysis and a result in a nominal result under the fixed intensity of noise. The analysis can be made realistic by adding a frequency-dependent sensor noise relationship and a weight function that corresponds to the noise dependency can be defined. Figure 9.3 depicts the quarter car model design setup with active suspension system.

After the definition, all the components and their respective signal models corresponding to the weight functions. The road surface signal is accounted for as $\boldsymbol{d}_1, \boldsymbol{d}_2, \boldsymbol{d}_3$ on a combined weight function for the control signal $\boldsymbol{u}$, suspension travel $\boldsymbol{sd}$, and body acceleration $\boldsymbol{ab}$. The result obtained of the simulation would be in terms of H∞ norm, the greater the system the lesser the control authority. Thus, the H∞ norm can be represented as the impact of the road surface to the car. The result can be summarized as the design of the controller accounting the inputs $\boldsymbol{d}_1, \boldsymbol{d}_2, \boldsymbol{d}_3$ to error signals $\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3$. To proceed further, we need to define the weight functions of the design and denote the input and output channels for accounting the interconnection of the different weights defined. We need to use a high-pass filter for the weight of the actuator $\boldsymbol{W_{act}}$ to account for the high-frequency content of the control signal. This would aid in limiting the range of operation of the controller. The need arises for specifically defining the closed-loop gain targets for the road disturbance r to the suspension deflection sd and the body acceleration ab (Gawad 2021; Shao 2020; Elattar et al. 2016; Liu 2019). Due to the actuator uncertainty, the system shall seek the control only below 10 rad/s. The weights W_{sd}, W_{ab} can be defined as the reciprocals of the comfort and handling goals. To observe the variation between passenger comfort and road handling, we would develop three sets of weights (βW_{sd}, $(1-\beta)W_{ab}$) corresponding to three different variations: comfort ($\beta = 0.01$), balanced ($\beta = 0.5$), and handling ($\beta = 0.99$). The use of connect function for constructing a model `qcaric` of the block diagram. `qcaric` is an amalgamation of three models, one for each design point β. `qcaric` can be interpreted as an uncertain model. The reason for the uncertainty is the presence of the actuator which we earlier termed as an uncertain model and derived only the nominal values for the simulation. The use of `hinfsyn` for computing an H_∞ controller for every value of the blending factor β. Now we need to define closed-loop systems for observing the different results of the road disturbance impact to x_b, s_d, a_b for the passive and active type suspensions. The results obtained reveal that the controllers reduce the suspension deflection and the other control variable below the rattle space frequency, i.e., 21.52 rad/s.

9.3 Results

The frequency of the wheel assembly that is attained due to the road disturbance is called tire-hop frequency and the obtained value after the simulation is 56.27 rad/s. Similarly, the frequency of the car body that is transferred from the suspension to the body is called rattle space frequency. The frequency obtained after the simulation is 21.52 rad/s. Figure 9.4 shows the bode plots for suspension system for open-loop gain from disturbance and actuator force to the acceleration of body and travel of suspension.

Figure 9.5 depicts the comparison of actuator response values with nominal values. Response of the hydraulic actuator with total 20 samples for its comparison with the nominal value.

Figure 9.6 depicts the comparison of open-loop to closed-loop targets obtaining results through bode plots for achieving open-loop targets and closed-loop targets.

Figure 9.7 depicts closed-loop model tests according to design points. The closed-loop target represents the active suspension system, and the reaction of the suspension system to various parameters aids in understanding the working of the suspension system. Figure 9.8 depicts the results for response of suspension system. Figure 9.9 depicts the comparison of design objectives with suspension deflection and control force.

The controller that has been designed in the earlier stages has been working for only three different conditions according to the control required. The major three conditions included the control for comfort, handling, and balanced. Under the balanced condition, both the comfort and handling were observed to be compromised in a considerable manner. The error in this condition is observed due to the actuator model used for the analysis. The actuator model used for the quarter car analysis was only accounting the nominal values, and it served only as an approximation to the true actuator. The model didn't encounter the model errors and uncertainty that may

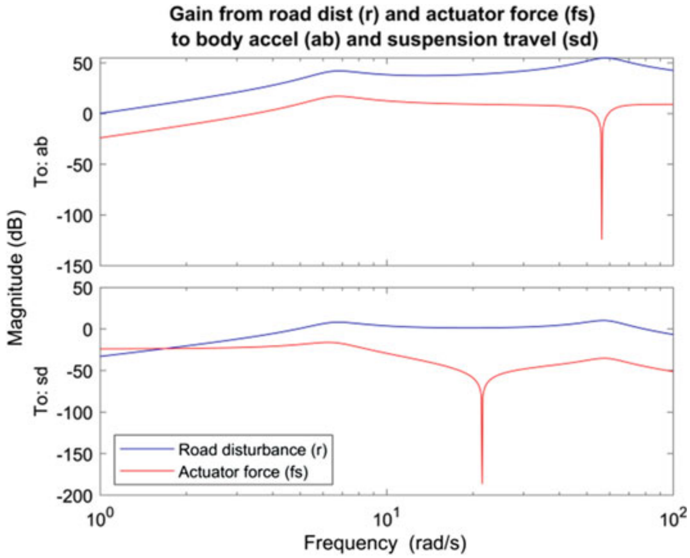

Fig. 9.4 Behavior of passive suspension system with actuator

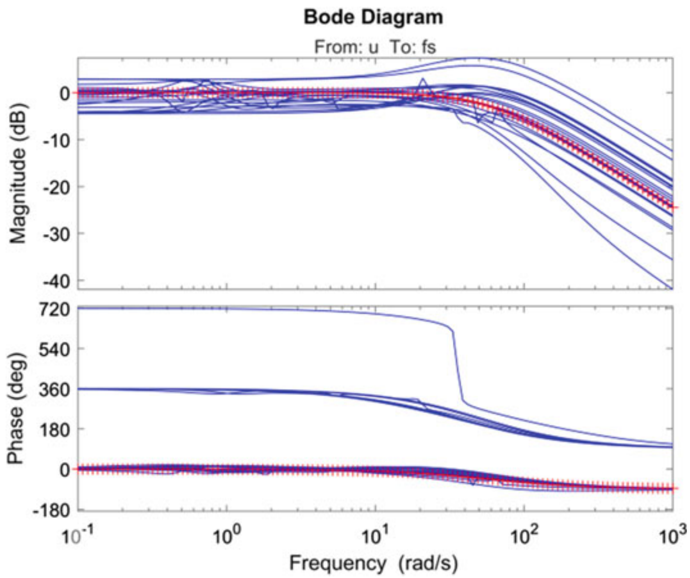

Fig. 9.5 Actuator model comparison with nominal actuator

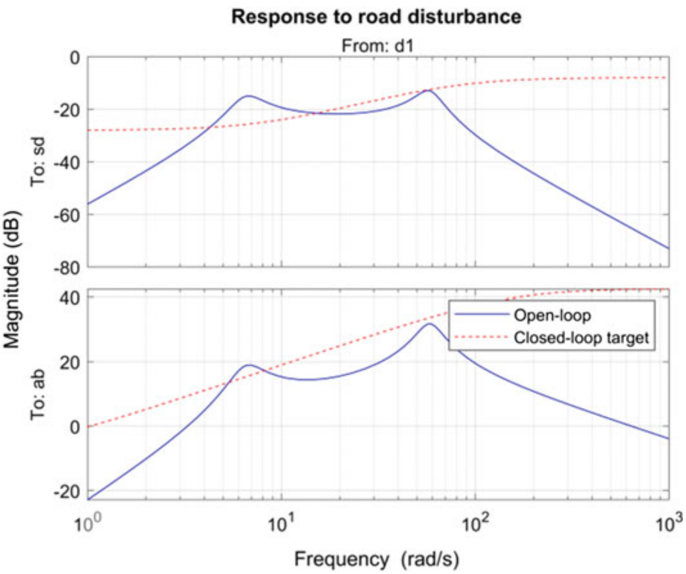

Fig. 9.6 Comparison of active suspension system with passive suspension system

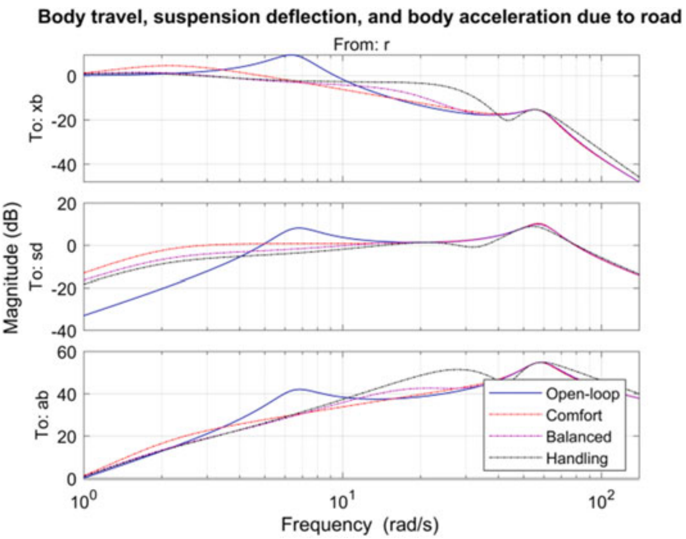

Fig. 9.7 Comparison of parameters with different design objectives

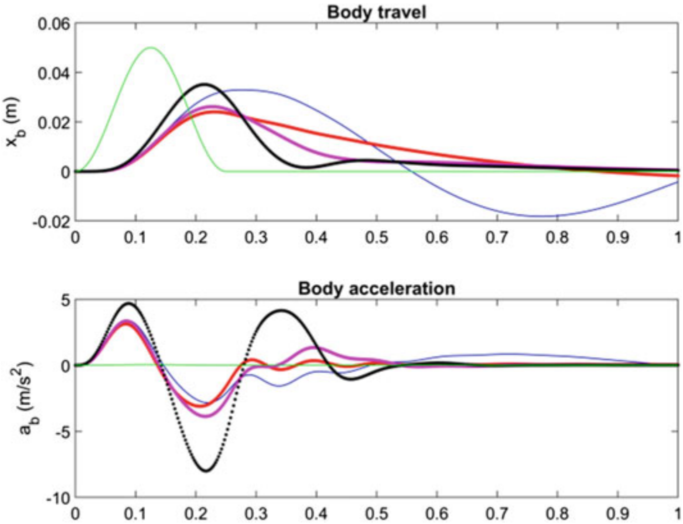

Fig. 9.8 Comparison of design objectives with body travel and body acceleration

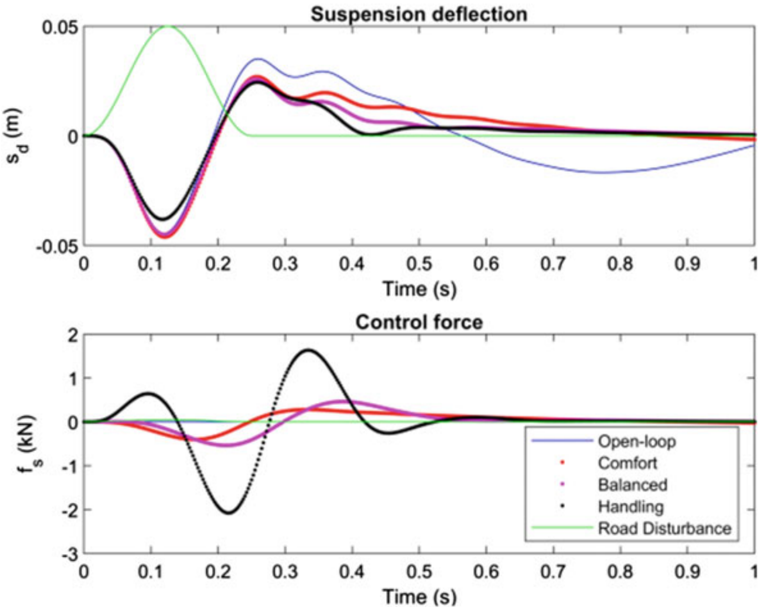

Fig. 9.9 Comparison of design objectives with suspension deflection and control

be observed in the true actuator. In the earlier stage of analysis, during the definition of the actuator model, we only used the nominal value of the actuator but have defined the actuator family including of total 20 sample actuators. The 20 samples generated for the actuator models would aid in accounting for the uncertainty. For obtaining the desired response of the suspension to a bump, we need to add extra feedback to the existing control system which would in turn act as the reduction in the uncertainty in the response and would provide a more robust response from the controller. The aim is to achieve a robust performance according to the road conditions, and accordingly, the control authority can be achieved for the desired system. The μ synthesis is used for obtaining the robust performance in reducing the uncertainty.

The term robust states "the ability of the system to resist change without adapting the initial stable configuration." In terms of control systems, robustness can be defined as "Approach for a Controller Design that explicitly deals with Uncertainty." μ-synthesis falls under the class of the sliding mode control criterion. The control criterion states that "it's a nonlinear control method that alters the dynamics of a nonlinear system by applying a discontinuous control signal that forces a system to 'slide' along a cross section of a system's normal behavior." The current suspension refers to the system as we have defined the state space of the suspension system. Under our analysis, the synthesis can change the feedback control from one continuous structure to another based on the current position of the suspension system. Thus, our control system can be termed as a hybrid dynamical system which contains a continuous state space and discrete control modes. The state space in the synthesis can be referred to the state space of the car, and the uncertain and discrete control modes can be referred to the family of actuator models which were generated earlier. The use of the `musyn` function of MATLAB is used for the analysis. The balanced performance is selected for the analysis, and a robust performance is to be obtained for the balanced design point. The output results in a *D-K* Iteration which provides us with the robust performance achievement.

The results show that the robust performance was achieved best as 1.09 for a bump of 10 cm. Further, we need to obtain the performance of the closed-loop control with the help of the **Krob** controller currently obtained. The result of the **`Krob`** controller performance is shown in Fig. 9.10. The controller performance of the H_∞ controller is to be calculated for the family of actuators that were generated earlier. The result of the H_∞ controller with the different discrete actuator models is shown in Fig. 9.11. To obtain robust performance, only some of the discrete actuator shall be used and the range of the currently obtained H_∞ controller needs to be reduced. The results with the robust **Krob** controller are shown in Fig. 9.12.

9.4 Discussion

The results we have obtained till now are the performance of two controllers, and the initially obtained results were from the nominal H_∞ controller. The results were not optimal since the actuator model used was possessing only the nominal values

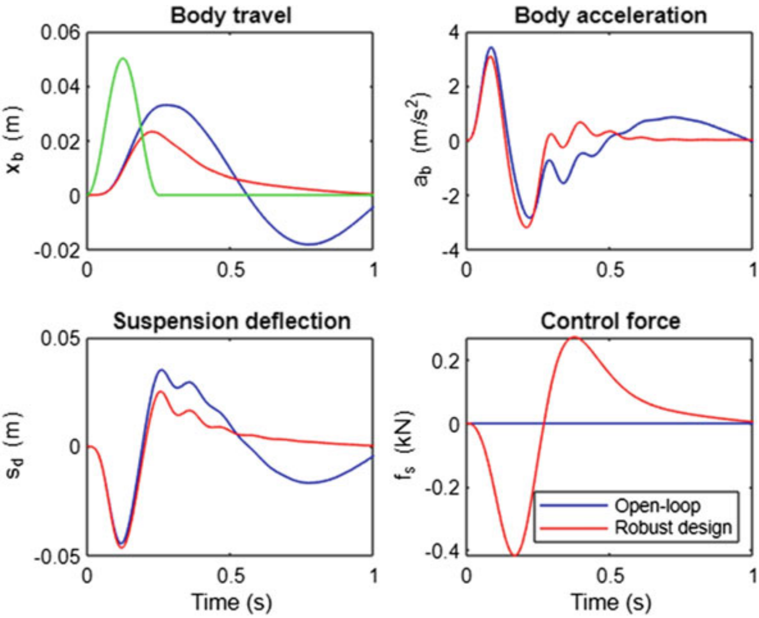

Fig. 9.10 Performance of robust controller

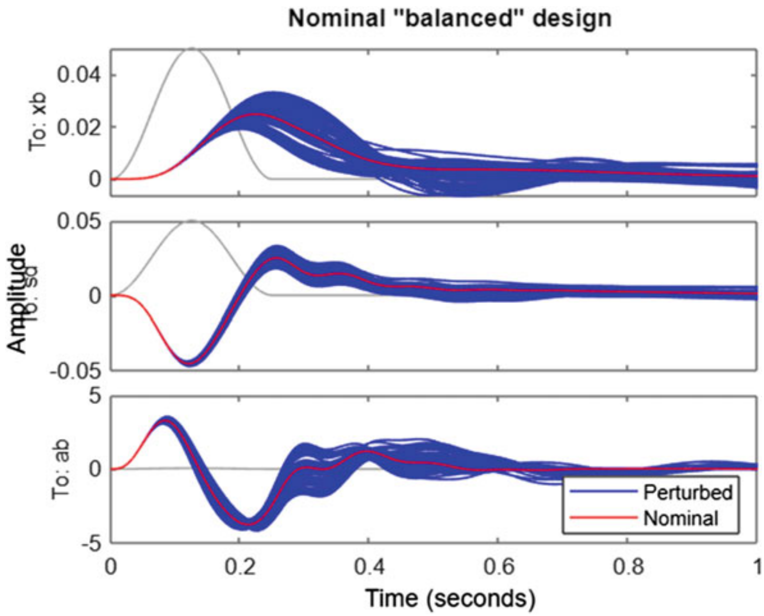

Fig. 9.11 Performance of initial controller with 20 actuator models

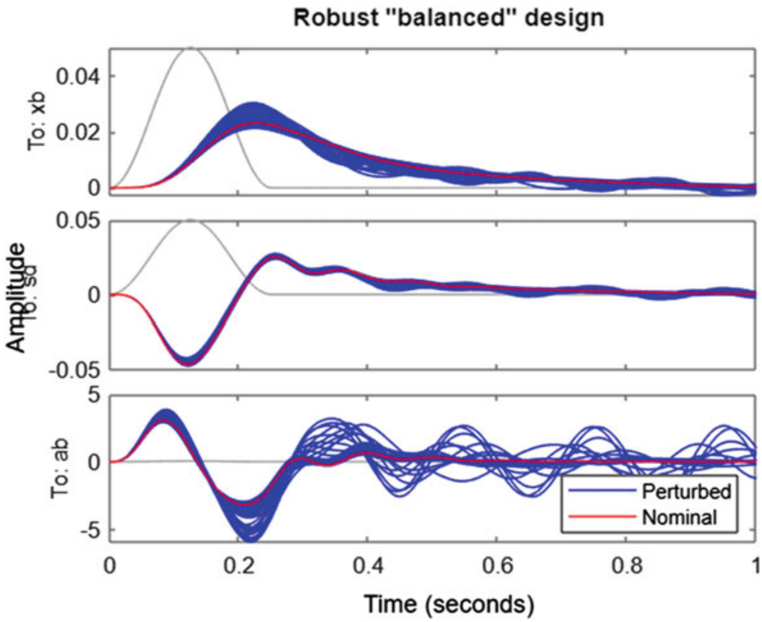

Fig. 9.12 Performance of robust controller amplitude versus time

of the true actuator. Thus, the results obtained were only an approximation of the true response of the actuator, if the controller were to be in actual practice. The second major result obtained was from the family of actuators (discrete generated actuators). The results obtained have been plotted with respect to the response of the suspension to body acceleration, suspension deflection, and body travel. The results show the range in which the results may variate according to the situation. The last results obtained were the performance of the robust controller which was designed for the optimizing the H_∞ controller feedback and obtaining a robust performance. The results obtained after the μ-synthesis works mainly for reducing the variation in the peak MU obtained, i.e., the peak gain in the μ-synthesis. As soon as the variation stops and a consistent performance is achieved, the said performance is called robust performance. Under the use of Krob controller, the results were considerably narrowed down and the desired robust control was achieved. Further, the results were compared to the passive suspension system. The results clearly depict that the robust performance of the controller aided in obtaining a balanced control between the passenger comfort and road handling. The same can be achieved for prioritizing the other design points, the design would vary accordingly, and the gains and their control would change accordingly.

9.5 Future Scope

The results obtained were obtained by the optimization of the initially used H_∞ controller by the μ-synthesis method. The process used for the optimization results in the controller with an auto-defined order controller. Thus, the order of the controller may be more comparatively and there is a chance that the same or similar performance may be achieved by a relatively lower-order controller. The obtained controller can further be reduced in terms of the operational order. The fine-tune of the controller order can be done to reduce the complexity of the operation and can be optimized in that domain. Alternatively, we can use a fixed order controller, obtain results for it, and compare it to the robust performance of results. As the desired results are obtained, the controller obtaining the nearby performance can be used for application. The currently used model was defined and analyzed for only a quarter of a car. The same can be done for the half car model which includes two independent quarter car models. Their respective dependencies and relations can be defined in a state space and the inputs need to change accordingly. The controllers we have been using are the nominal H∞ controller, and the optimization can be carried out with the μ-synthesis as we carried out in the quarter car model. The entire analysis can be done on a full car model, which would be a true analysis for a car since the quarter car only accounts for the response to the road disturbance. But under practical circumstances, the car's movements may result under some deflection to the suspension system as well and counting only the road disturbance for the analysis would be only an approximation. The use of dependency equations for the four wheels and their effects on one another need to be studied for the relative response to the road disturbance. The full car model analysis would result in a much complex result and achieving the desired design points would be more complex since there would be more variables to be optimized.

9.6 Conclusions

The simulation of the quarter car model has been carried out by defining the system and their entire models. The system was defined in terms of the state-space equations and further was converted into state-space matrices. The state-space matrices have been then used as the main systems. The suspension system is then considered for completing the entire system. The suspension system consists of various components like spring, damper, tire, the car body, and the actuator. All the components need to be individually defined and then all the components are to be connected to form a system of suspension. As we defined the system and its components, we now have the system ready for simulation and the next step was to define the control criteria and the disturbances to be given as input. All the values of the system are defined as different weight functions. The weight functions include the control inputs, road disturbance, error signals, and the measured output. While defining the weight function, we have been keeping the error signal intensity and the road disturbance to be constant. Thus, the dependency of variables is reduced. The entire system control is defined as the active suspension system. The controller that provides the command for maintenance of the design goal is designed using the H- infinity synthesis. The controller designed works on the reduction on the H norm achieved at the end of every output measured from the system. Thus, a gain reduction principle is followed, and for testing the system, a bump was set to 7 cm. The design of the controller was initially designed by the H-infinity synthesis, the results obtained achieved the basic goals of achieving the control authority. The controller designed only achieved the performance with only the nominal actuator in the suspension system. The practical conditions would have an uncertain actuator model, and to model the practical/true actuator, the family of actuators are considered and 20 different actuator models were generated and simultaneously tested. The current controller was also optimized using the μ- synthesis approach. The same controller was tested for the 20 different actuator models and provided with a range of control with the aid of different actuators.

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