Input Design Analysis for the Capacity of Finite Impulse Response Actuator

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Abstract. Mutual information (MI) is usually a computational heavy work when used as optimal metric in control system. In this paper, the modeling of Finite Impulse Response (FIR) system with entropy-rate framework is shown. A practical signal sampling algorithm to calculate the MI in the case of probability density function (PDF) with regenerative property is given. The MI calculation method is based on random sampling, which can reduce the computational complexity. The simulation results show our entropy-rate framework can be employed in analysis for the capacity of FIR actuator.

Keywords: FIR Actuator · Information Theory · Entropy

1 Introduction

Information theory has been adapted to control problems in various aspects including optimal metric, bound constraint and system explanation. It is well known that information theory and cybernetics share in a common theoretical and engineering background, both of which are concerned with signals and dynamic systems. Most traditional control system analysis assumes that the observed signal is available to controller in its entire decision period [1]. In this paper, the information theory modeling of FIR actuator is a commonly used convolutional structure.

There exists numerous work using information theory to explain and resolve problems in optimal control [2,3]. Touchette and Lloyd [4] proved that the one-step reduction in entropy of the final state is upper bounded by MI between control variables and current state of the system. Minimum of the one-step reduction in entropy can be found by dynamic programming, but the computational complexity of Dynamic Programming (DP) grows exponentially with the number of time steps and control variables. In fact, a quantization problem arises when information is exchanged between the controller and the controlled dynamic object [7–9]. An actuator channel can only transmit a finite number of

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states at any given time [10]. There is a trade-off between the fineness of the control process and the time delay required to send the data over the channel. J.Baillieul demonstrated in [12] that adequate data quantization necessitates a bigger channel capacity or more data transmission time. In [13], directed information quantity is adopted as a measure of information transmission in feedback control system. However, the specific expressions for distortion measures other than quadratic distortion is rarely studied and researchers paid little attention to signal distribution in non-Gaussian PDF cases. Recently, authors in [14] have conducted some fundamental limit research in control field using information theory. Information theory is adopted as the main mathematical tool to obtain generic bounds on the variance of estimation errors in time series analysis in [15,16].

In this paper, we will demonstrate how to derive the entropy-rate function about a given actuator with noise, which is based on traditional rate-distortion theory. Then we define the conditional entropy as a distortion metric in an actuator system. Finally, we derive and simulate the MI between system state and output.

This paper is organized as follows. Two preliminaries of our analysis process, entropy-rate function and Sampling A Posterior (SAP) method are introduced in Sect. 2. The modeling of FIR actuator and derivation of the posterior estimation are given in Sect. 3. We derive and calculate the MI between system state and output in Sect. 4. Simulation of a non causal track system is given in Sect. 5.

2 Preliminaries

Throughout the paper, we consider real-valued continuous random variables and random vectors, as well as discrete time stochastic processes. All random variables, random vectors, and stochastic processes are zero mean. Given a stochastic process $\{x_k\}$, we assume the excitation of the control system be a set of finite-length K-dimensional independent and identically distributed Gaussian vectors, the system is linear and has memory. The output signal is an N-dimensional Gaussian vector $\mathbf{Y}$.

While classical linear system theory uses frequency response theory to characterize noise and system transfer functions, modern control theory uses linear algebra and state space to establish dynamic equations. Specifically, we consider the basic system with state-space model given by

$$\begin{cases} \hat{\mathbf{x}}_k = \mathbf{x}_k + \mathbf{w}_k \\ \hat{\mathbf{y}}_k = \mathbf{A}\hat{\mathbf{x}}_k \end{cases} \tag{1}$$

where $\mathbf{A}$ is the FIR actuator matrix, $\hat{y}_k$ is real output at step k, x_k is system state at step k. The impluse response sequence h(d) has the length of d. The sequence is then filled with zero to the length of $N_S = N_{cp} + d$, where N_{cp} is the length of circulant prefix. The finite implues response $\mathbf{A}$ during the i^{th} symbol, which is N $\times$ N toeplitz [17],

$$\mathbf{A} = \mathbf{R}_{cp}\mathbf{H}\mathbf{T}_{cp}$$
$$= \begin{bmatrix} h(0) & 0 & \dots & 0 & h(d-1) & \dots & h(1) \\ h(1) & h(0) & 0 & \dots & \dots & \dots & \dots \\ \dots & \dots & \ddots & \dots & \dots & \dots & h(d-1) \\ h(d-1) & \dots & \dots & \ddots & \dots & \dots & 0 \\ 0 & \dots & \dots & \dots & \ddots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \ddots & 0 \\ 0 & \dots & 0 & h(d-1) & \dots & \dots & h(0) \end{bmatrix}, \tag{2}$$

where $\mathbf{R}_{cp} = \begin{bmatrix} 0_{N\times(N_S-N)} & I_N \end{bmatrix}$ is $N \times N_S$, $\mathbf{T}_{cp} = \begin{bmatrix} 0_{N-Ncp} & I_{Ncp} \\ & I_N \end{bmatrix}$ is $N_S \times N$. $\mathbf{y}_k$ is defined as a sequence of designed inputs over some number of timesteps k, $\hat{\mathbf{y}}_k$ is the real actuator output. It is reasonable to assume that direct control over designed input $\mathbf{y}_k$ exists, but not over $\hat{\mathbf{y}}_k$. This is a common noise assumption for many robotic systems in which command input must pass through a lower-level controller. A prototypical example is the steering and throttle inputs for a car which are then used as set-point targets for low level servomotor controllers.

The MI calculation method we used in this paper is called SAP method. It is based on a random sampling of the posterior probability generated by each received signal. The statistical distribution of this estimate will naturally approximate the theoretical posterior probability distribution of the parameter to be estimated. The main constraint is that the PDFs must be regenerative, eg. Gaussian, Cauchy, Gamma, Poisson, Exponentional distribution. The sampling posterior probability estimation method is described in Algorithm 1. Firstly, generate the received signal vector, then match conditional PDF of received signal vector. The key step is generating an estimate by random sampling of the posterior PDF. Through this way the empirical entropy and the average estimation performance will be calculated.

3 FIR Actuator Model and Posteriori Estimation

Assuming that the channel is constant (slow fading) during one symbol period, its finite impulse response sequence in the i_{th} symbol period sequence is

$$\begin{aligned} y(n) &= \hat{x}(n) * h(n) = \sum_{l}^{n} \hat{x}(l)h(n-l) \\ &= \hat{x}(n)h(0) + \sum_{l}^{n-1} \hat{x}(l)h(n-l), \end{aligned} \tag{3}$$

where $l = n - d + 1$.

Algorithm 1: Sampling from A Posterior Method

Data: received signal vector, posterior PDF,
prior entropy H_prior
Result: posterior entropy H, mutual information

initialization;
for $j = 1; j \leq sampling\_iteration\_num$ **do**
    Using random sampling method to get sample points from posterior PDF, eg. randsample() in matlab ;
    while *sampling_num not end* **do**
        H = H - log(*sampling_point*);
    end
    H = H / sampling_num;
    I = H_prior - H;
end
return I;

The FIR system state Eq. (1) can be extended as

$$\begin{cases} \mathbf{x}_k = A\mathbf{x}_{k-1} + B\mathbf{e}_{k-1} + \mathbf{v}_k \\ \mathbf{y}_k = C\mathbf{x}_k + D\mathbf{e}_k + \mathbf{w}_k \end{cases}, \tag{4}$$

where Gaussian noise v_k, w_k are independent. Combine (4) with (1)(2), we give the whole system model, the notation e_k denotes the input of FIR system, y_k denotes the referred signal, $\hat{y}_k$ denotes the output signal. Based on (3), we write the state equation as

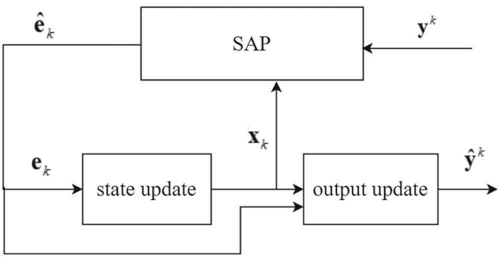

Fig. 1. System Input Design with SAP method

$$\begin{cases} p\left(\mathbf{y}_k \mid \mathbf{e}_k, \mathbf{x}_k\right) = f_w\left(\mathbf{y}_k - C\mathbf{x}_k - D\mathbf{e}_k\right) \\ p\left(\mathbf{x}_k \mid \mathbf{x}_{k-1}, \mathbf{e}_{k-1}\right) = f_v\left(\mathbf{x}_k - A\mathbf{x}_{k-1} - B\mathbf{e}_{k-1}\right) \end{cases}, \tag{5}$$

$$\begin{aligned} & p\left(e_k \mid y_k, y^{k-1}, e^{k-1}, x^{k-1}, x_k\right) \\ & = \frac{p\left(e_k, y_k, y^{k-1}, e^{k-1}, x^{k-1}, x_k\right)}{p\left(y_k, y^{k-1}, e^{k-1}, x^{k-1}, x_k\right)}. \end{aligned} \tag{6}$$

Based on Bayes' Rule, we continue to derive (6) and get the step estimation equation

$$\begin{aligned} & p\left(e_k \mid y_k, y^{k-1}, e^{k-1}, x^{k-1}, x_k\right) \\ & \quad = \frac{f_w f_v p\left(y^{k-1}, e^{k-1}, x^{k-1}\right) p\left(e_k\right)}{E_{p(e_k)}\left[f_w f_v p\left(y^{k-1}, e^{k-1}, x^{k-1}\right)\right]}. \end{aligned} \tag{7}$$

The next stages are derived in the same way. Other PDF with property of regenerativity can also do this SAP algorithm.

4 Mutual Information of FIR Actuator

This section will examine the MI-distortion relation of FIR Actuator. The main problem is the processing of MI in this model. SAP algorithm based on sampling method is given to quantitatively figure out the MI between the system output and state.

Based on the actuator model shown in (4), it is reasonable to assume that the stochastic process in both sides of (4) share the same statistical characteristic, thus the auto correlation of a stochastic process $\{\hat{x}_k\}$ is

$$\begin{aligned} R(d) &= E\left[\hat{\mathbf{x}}h(0)[\hat{\mathbf{x}}h(0)]^T\right] \\ &= \begin{bmatrix} R_{xx}(0) & R_{xx}(-1) & \cdots & R_{xx}(-d+1) \\ R_{xx}(1) & R_{xx}(0) & \cdots & R_{xx}(-d+2) \\ \vdots & \vdots & \ddots & \vdots \\ R_{xx}(d-1) & R_{xx}(d-2) & \cdots & R_{xx}(0) \end{bmatrix}, \end{aligned} \tag{8}$$

designed output $y \sim N(0, \Sigma_Y)$, distortion $(y - \hat{y}) \sim N(0, \mathbf{D})$. It is known that circulation matrix $\mathbf{A}$ in (2) can be diagonalized by Fourier matrices. Hence we have $\hat{\mathbf{Y}} = \Sigma\hat{\mathbf{X}}$. To the same effect, the conditional probability distribution representing the relationship between finite precision output and ideal output is

$$p(\hat{\boldsymbol{y}} \mid \boldsymbol{y}) = \frac{1}{\sqrt{(2\pi)^N \left|R_\delta\right|}} \exp\left\{-\frac{1}{2}(\hat{y} - \boldsymbol{y}) R_\delta^{-1} (\hat{\boldsymbol{y}} - \boldsymbol{y})^T\right\}, \tag{9}$$

based on the rate-distortion, we write the conditional entropy as the distortion metric constraint for minimization

$$\frac{1}{2}\log(2\pi e)^N D = H(\mathbf{Y} \mid \hat{\mathbf{Y}}), \tag{10}$$

we have mutual information which indicates the information capacity of the FIR actuator,

$$\begin{aligned} I(\mathbf{y};\mathbf{x}) &= \iint p(\mathbf{x},\mathbf{y}) \ln \frac{p(\mathbf{y} \mid \mathbf{x})}{p(\mathbf{y})} d\mathbf{x} d\mathbf{y} = h(\mathbf{y}) - h(\mathbf{y} \mid \mathbf{x}) \\ &= \frac{1}{2}\ln(2\pi)^N \left|\boldsymbol{\Sigma}_{\mathbf{Y}}\right| + \frac{\sqrt{(2\pi)^N \left(\left|\boldsymbol{\Sigma}_{\mathbf{Y}}\right| + |\mathbf{D}|\right)}}{|\mathbf{D}|}\left[\ln \frac{\sqrt{(2\pi)^N \left(\left|\boldsymbol{\Sigma}_{\mathbf{Y}}\right| + |\mathbf{D}|\right)}}{(2\pi)^N \left|\boldsymbol{\Sigma}_{\mathbf{Y}}\right| |\mathbf{D}|} - 1\right]. \end{aligned} \tag{11}$$

By using SAP algorithm described in Sect. 3 to calculate MI, the results are shown in Fig. 3. Other PDFs with regenerativity such as Cauchy, Gamma, Poisson, Exponentional distribution also apply to this deduction.

5 Simulation of a Non Causal Track System

We carry out the system simulation based on (5). The FIR system has 1 Hz bandwidth and it functions as a track system with reference input y.

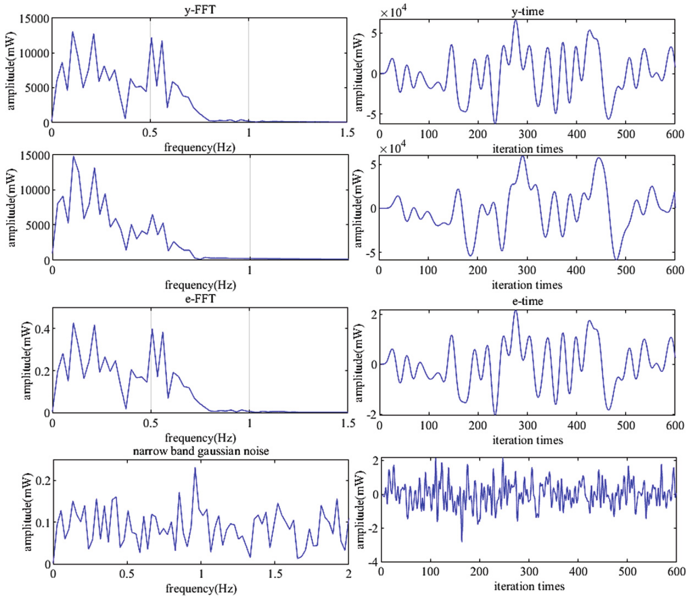

Fig. 2. Simulation of the track system

As can be shown in Fig. 2, the left column is the FFT of time series and the right column is the time domain series. The first row is the designed output y, and the second row is the actual output $\hat{y}$. The third row is designed input signal e based on estimation Eq. (6) and the forth row is the noise w in system equations. This track system is a lowpass filter 1 Hz bandwidth. The MI between the first row y and the second row $\hat{y}$ is shown in Fig. 3. It can be observed in Fig. 3 that as the tracked frequency y getting closer to the system's cutoff 1 Hz, tracking process performs worse. The worse performance is reflected by the decrease of MI $I(y;\hat{y})$ from 0.3 to 0.2.

MI result in Fig. 3 reflects the capacity of this track system. $I(y;\hat{y})$ tells us the information from y to $\hat{y}$ in Fig. 1. As the input frequency exceeds the allowed entropy-rate relation, the signal tracking process will not keep up. When the

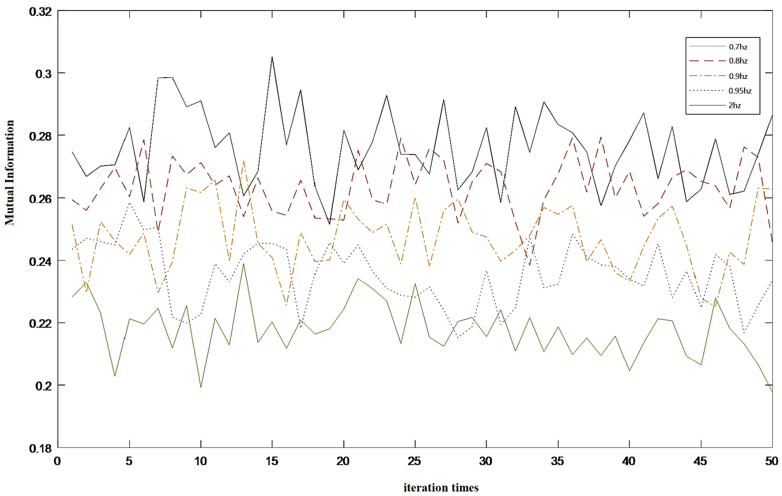

Fig. 3. Mutual Information $I(y;\hat{y})$ with different frequenciesies from 0.7 Hz 2 Hz

tracked frequency gradually 1 Hz, the MI decreases at the same time. The same phenomenon also exists in Mean Square Error (MSE), as input frequency getting closer to cutoff frequency, MSE between y and $\hat{y}$ also decreases.

6 Conclusion

In this paper, we proposed an entropy-rate framework for the analysis of a FIR actuator model and gave the input estimation equation and posterior PDF recursive deduction. The simulation result about the MI of system is based on SAP method, and the calculation complexity of SAP method is reduced The MI metric can reflect the capacity of FIR actuator based on simulation results. Our entropy-rate relation can be a feasible framework to explain control system's capacity. Other PDFs with regenerativity such as Cauchy, Gamma, Poisson, Exponentional distribution also apply to this analysis framework.

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