Optimum LQR Controller for Inverted Pendulum Using Whale Optimization Algorithm

Bharti Panjwani, Vipul Kumar, Jyoti Yadav, and Vijay Mohan

Abstract In this work, a Linear-Quadratic Regulator (LQR)-based control scheme is designed for a highly nonlinear and unstable inverted pendulum system. The system is linearised about its vertical position based on certain assumptions. Initially, weight matrices of the LQR controller are selected based on a trial and error method. These matrices are then optimised using a multi-objective genetic algorithm (GA) and whale optimization algorithm (WOA). The robustness of designed controllers is tested by reference tracking and parametric uncertainty analysis. The results reveal that optimisation of LQR by WOA provides superior performance compared to GA.

Keywords LQR · GA · WOA · Inverted pendulum · Robustness analysis

1 Introduction

The inverted pendulum is a platform for testing several control algorithms because of its nonlinear and unstable behaviour [1]. An inverted pendulum has its centre of mass above its pivot point, due to which it falls when released from a slight angle about its vertical position. It is a classical control theory problem for verifying different control techniques [2, 3]. The aim is to move the cart to prevent the pendulum from falling [4]. It is done with the help of a DC motor and a control technique [5–7]. Modern and advanced control techniques are available to control such systems [8–11].

LQR designed by trial and error method gives a satisfactory response close to the desired response but not optimal. The optimum values of weight matrices are

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required for an efficient response, which is achieved using GA and PSO [4, 12]. A genetic algorithm works on the reactions from its environment and can solve multiple dimensions optimisation problems [13, 14].

In this paper, the whale optimisation algorithm [15] is also utilised to obtain the optimal feedback gain matrix '*K*' of LQR controller. WOA copies the social behaviour of Humpback whales and follows their hunting method known as a bubble net method. Simulation results for inverted pendulum show that the optimal solution obtained from WOA leads to superior performance compared to GA and trial and error method in overshoot, settling time and Integral Absolute Error (IAE). Furthermore, uncertainty analysis is performed to show the robustness of the WOA optimised LQR controller over others.

The paper is organised as follows. Section 2 describes the mathematical modelling of the inverted pendulum. Controller design and its optimisation are presented in Sects. 3 and 4, respectively. Section 5 summarises the simulation results. Lastly, Sect. 6 concludes the research work.

2 Mathematical Modelling of Inverted Pendulum

The structure of the inverted pendulum is shown in Fig. 1. It consists of a pendulum connected to a movable cart that can move left and right on a rail to prevent the pendulum from falling [16]. The system's parameters, their nominal values, and SI units are given in Table 1.

The dynamic behaviour of angle and position of the system varies proportionally to the control force '*F*' [2]. Differential equations relate the kinetic and potential energy of the system with control force [5]. The state space model of the system can be derived from 'LaGrange mechanics' as,

$$\frac{d}{dt}\left[\frac{\partial}{\partial q^{\circ}}L\right]-\frac{\partial}{\partial q}L=\tau \tag{1}$$

where Lagrange Function $L = K - V$. K is the kinetic energy and V is potential energy of the system given as

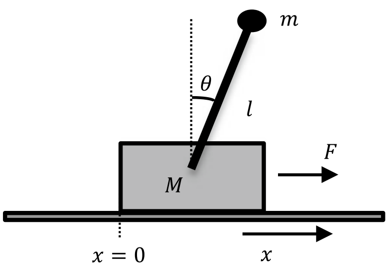

Fig. 1 Schematic diagram of inverted pendulum

Table 1 Parameters of inverted pendulum [17]

Symbol	Quantity	Value/SI unit
M	Mass of the cart	2.4 kg
m	Mass of the pendulum	0.23 kg
l	Length of pendulum	0.36 m
f	Coefficient of friction	0.1 N/m/s
g	Acceleration due to gravity	9.8 m/s^2
F	Control force applied to the cart	Newton
θ	Angle between pole and vertical direction	Radian
x	Position of the cart	Metre

$$K = \frac{1}{2}M\dot{x}^2 + \frac{1}{2}m[(\dot{x} + l\dot{\theta}\cos\theta)^2 + \left(l\dot{\theta}\sin\theta)^2\right] \tag{2}$$

$$V = mgl\cos\dot{\theta} \tag{3}$$

Solving Eq. (1) by substituting Eqs. (2) and (3) yields

$$(M + m)\ddot{x} + ml\ddot{\theta}\cos\theta - ml\dot{\theta}^2\sin\theta = F \tag{4}$$

$$l\ddot{\theta} + \ddot{x}\cos\theta - g\sin\theta = -f\dot{\theta} \tag{5}$$

where $f\dot{\theta}$ denotes the friction in the rotational link of system. The state obtained are $X(t) = [x\dot{x}\theta\dot{\theta}]^{\mathrm{T}}$. After eliminating algebraic loops, the state model is

$$\begin{cases} \dot{x}_1 = \ddot{x} = x_2 \\ \dot{x}_2 = \ddot{x} = \frac{-mg\sin x_3\cos x_3 + mlx_4^2\sin x_3 + fmx_4\cos x_3 + F}{M + \left(1 - \cos^2 x_3\right)m} \\ \dot{x}_3 = \dot{\theta} = x_4 \\ \dot{x}_4 = \frac{(M+m)(g\sin x_3 - fx_4) - \left(lmx_4^2\sin x_3 + F\right)\cos x_3}{l\left(M + \left(1 - \cos^2 x_3\right)m\right)} \end{cases} \tag{6}$$

The state space obtained above is nonlinear model that needs to be linearised in order to introduce a modern control scheme. Thus, system is linearised around its equilibrium point $\left[x\dot{x}\theta\dot{\theta}\right] = [0000]$. The linear model can be obtained by approximation of certain terms. These approximations are $\sin x_3 = x_3$, $\cos x_3 = 1$, $x_4^2 = 0$, $x_3x_4 = 0$.

$$\begin{cases} \begin{bmatrix} \dot{x} \\ \ddot{x} \\ \dot{\theta} \\ \ddot{\theta} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & -0.9392 & 0.0096 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 311.2802 & -0.3044 \end{bmatrix} \begin{bmatrix} x \\ \dot{x} \\ \theta \\ \dot{\theta} \end{bmatrix} + \begin{bmatrix} 0 \\ 0.4167 \\ 0 \\ -1.1574 \end{bmatrix} u(t) \\ y(t) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ \dot{x} \\ \theta \\ \dot{\theta} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix} u(t) \end{cases} \tag{7}$$

3 Controller Design

The control objective is to move the cart so that it keeps the pendulum in an upright position. This means that at $\theta = \dot{\theta} = 0$ [4]. An optimal LQR controller is designed to stabilise the system state feedback. The control input $u(t)$ is obtained by reducing the cost function J.

$$J = \int_0^\infty \left(x^T Q x + u^T R u\right) \mathrm{d}t \tag{8}$$

Here Q is a positive semi-definite matrix known as state variable weighting matrix, and R is positive definite matrix as input variable weighting matrix [12]. Hence, input vector u is designed which reduces the cost function J [1]. Therefore, control signal $u(t)$ is called optimal control [2]:

$$u = -K*x = -R^{-1}B^{\mathrm{T}}P*x \tag{9}$$

where P is obtained by solving Riccati equation and K is the feedback gain matrix. Now, solving Riccati equation:

$$A^{\mathrm{T}}P + PA - PBR^{-1}B^{\mathrm{T}}P + Q = 0 \tag{10}$$

where [13] $Q = \mathrm{diag}\left(\left[\,100\ 1\ 200\ 1\,\right]\right)$ and $R = 1$. Then,

$$\begin{aligned} K &= -R^{-1}B^T P = \left[\,K1\ K2\ K3\ K4\,\right] \\ &= \left[\,-10.0\ -8.10\ -633.86\ -35.64\,\right] \end{aligned} \tag{11}$$

This feedback gain matrix is then used in the system model of the inverted pendulum to get the results. The system model of inverted pendulum stabilisation using LQR controller is given in Fig. 2.

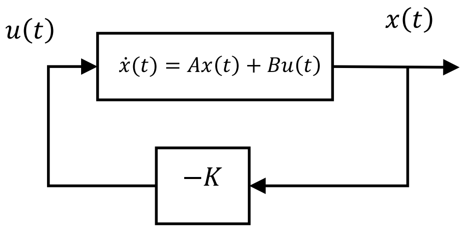

Fig. 2 LQR control structure

4 Optimisation of Controller

The feedback gain matrix is obtained using two weighting matrices Q and R as defined above. The elements of Q and R matrices emphasis on the two main state variables–angle and position. These values are varied to get desired response. To get an optimal feedback gain matrix, we have used two optimisation algorithms: GA and WOA. Aim is to obtain the optimal values of matrices Q and R, which minimises the settling time and overshoot of the response.

4.1 Genetic Algorithm

The genetic algorithm (GA) is one of the evolutionary algorithms that work on the principle of laws of natural selection and evolution. This algorithm creates a virtual environment where better responses are emphasised, whereas other responses are disregarded. The basis of the algorithm is the three operations of evolution: reproduction, crossover, and mutation [1]. The design steps for GA are given in literature [1]. The two objectives of minimisation in the fitness function are: settling time and overshoot. This algorithm is carried out by using optimisation toolbox in MATLAB. Convergence plot is shown in Fig. 3 and the optimum value of K obtained is:

$$K = [-12.0458 - 14.0369 - 707.8372 - 39.8890]$$

4.2 Whale Optimization Algorithm

This algorithm is a nature inspired algorithm which follows a biological phenomenon of impersonating the behaviour of Humpback whales. It is found that they are intelligent animals and have emotions as well. However, they are predators and preys on small fishes near the surface of the ocean. Their method of hunting is known as bubble net feeding method. Humpback whale first goes deep inside the water and

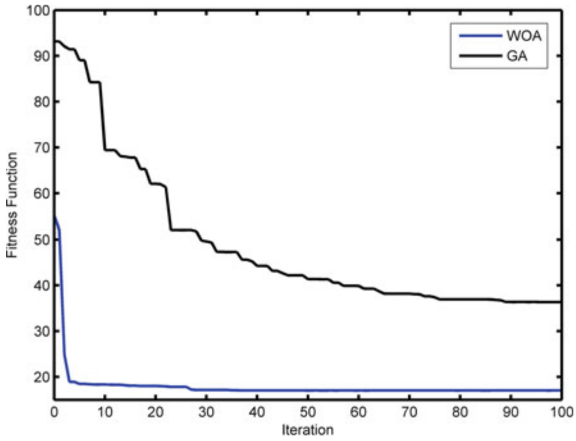

Fig. 3 Convergence curve for GA and WOA

form bubbles in the shape of spirals. These spirals look like small circles encircling the group of small fishes present near the surface of water. Whale then swims up to the surface to kill its prey. The key mathematical formulation and design procedure of the WOA algorithm is presented in literature [15]. The convergence curve obtained is shown in Fig. 3 and the value of K obtained using WOA is:

$$K = [-1.2015 - 19.3238 - 839.1297 - 58.2497]$$

5 Simulation and Results

The system model of the inverted pendulum is designed in MATLAB, and the results are analysed for reference tracking and parametric uncertainties. First, LQR is developed and then optimal feedback gain matrix is then obtained using GA and WOA. Comparative analysis of LQR, GA-LQR, and WOA-LQR controller is shown in Table 2. Figure 4 shows the open loop response of the pendulum on the cart (Fig. 5).

From Table 2, it is clear that the best response is achieved by WOA-LQR, as it reduces the settling time by 99% and 53.9%, respectively. The use of WOA-LQR

Table 2 Time-domain specifications for LQR, GA-LQR, and WOA-LQR

Time-domain specifications	Angle of the pendulum			Position of the cart		
	LQR	GA	WOA	LQR	GA	WOA
Settling time (s)	17.2	9.54	0.16	57.18	33.88	26.32
Overshoot	0.012	0.0114	0.0025	1.52	1.39	1.02
IAE	0.149	0.149	0.104	8.42	5.42	4.39

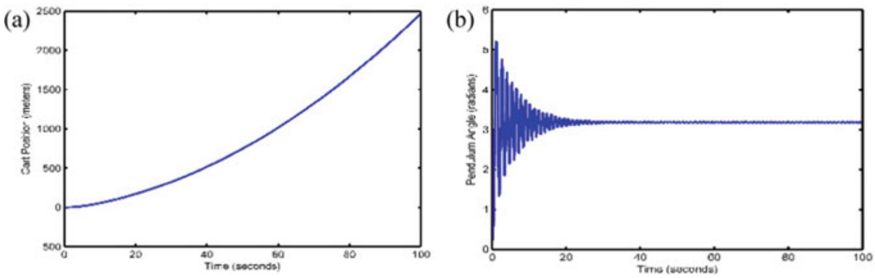

Fig. 4 Open loop response **a** cart position **b** pendulum angle

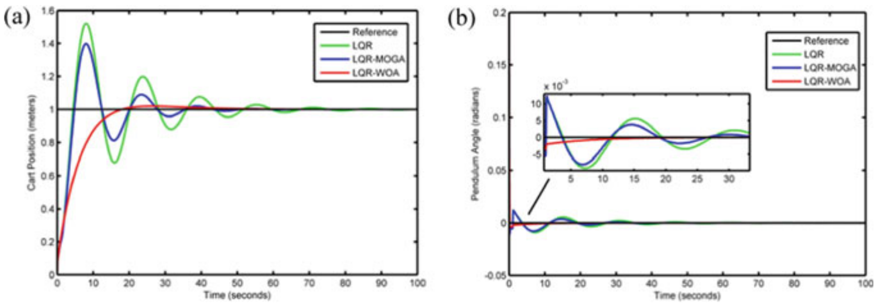

Fig. 5 Reference tracking graph **a** cart position **b** pendulum angle

significantly improves the overshoot of the cart and pendulum as compared to GA-LQR and LQR. Also, Integral Absolute Error (IAE) is minimum for WOA-LQR optimised controller. Results are also compared with the literature [13].

5.1 Uncertainty Analysis

Uncertainty analysis is carried out for the mass of cart alone, mass of the pendulum alone, both the masses, length of pendulum alone, and both masses and length together. IAE of different controllers are obtained for parametric uncertainty. Controller performance under parameter uncertainties is shown in Table 3 and Fig. 6.

6 Conclusion

In this paper, the dynamics of a nonlinear inverted pendulum is derived using the Euler–Lagrange formulation and linearised about its operating point. Further, an LQR controller is designed using GA and WOA optimisation. The performance of

Table 3 IAE for ±50% parametric uncertainty

Uncertainty in parameter	IAE of pendulum					
	LQR		GA-LQR		WOA-LQR	
	+50%	−50%	+50%	−50%	+50%	−50%
M	0.152	0.146	0.107	0.101	0.030	0.029
m	0.148	0.149	0.104	0.104	0.029	0.028
L	0.145	0.154	0.102	0.107	0.030	0.029
M & *m*	0.151	0.146	0.106	0.102	0.030	0.028
M, *m* & *l*	0.149	0.153	0.107	0.106	0.033	0.028

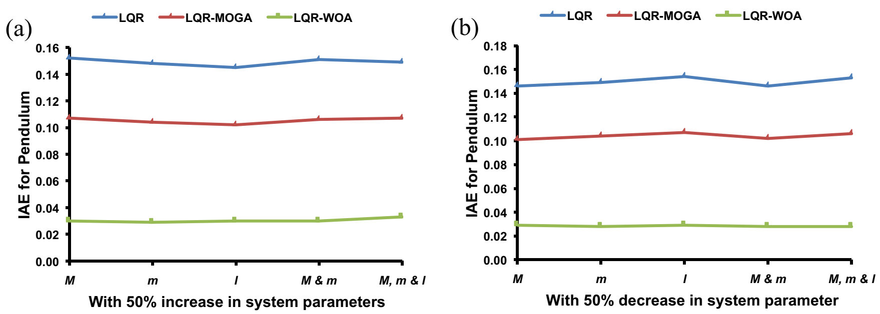

Fig. 6 IAE variations for model parameter uncertainties **a** 50% increase **b** 50% decrease

WOA is superior to GA in reference tracking. Furthermore, uncertainty analysis is successfully carried out, showing that WOA-LQR is more robust than GA-LQR controller.

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