Estimations of the Bounds for the Zeros of Polynomials Using Matrices

Ahmad Al-Swaftah, Aliaa Burqan, and Mona Khandaqji

Abstract Let $p(z) = z^n + \alpha_n z^{n-1} + \alpha_{n-2} z^{n-2} + \cdots + \alpha_2 z + \alpha_1$ be a monic polynomial of degree $n \geq 7$ with complex coefficients $\alpha_n, \alpha_{n-1}, \ldots .\alpha_1$, where $\alpha_1 \neq 0$. This paper investigates and estimates the upper bounds for the moduli of the zeros of p depending on the spectral norms, spectral radii, and the fifth power of the Frobenius companion. These upper bounds allow us to locate all the zeros of p in smaller annuli in the complex plane.

Keywords Bounds for the zeros of polynomials · Companion matrix · Spectral radius

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1 Introduction

Locating the zeros of polynomials is essential in many fields of study, including signal processing, control theory, communication theory, coding theory, and cryptography. Beginning with Cauchy, this classic problem drew a large number of mathematicians across time. Recently, several famous classical upper bounds for the moduli of the zeros of the monic complex polynomials have been established using the Frobenius companion matrix, which is a key connection between matrix theory and polynomial geometry. These bounds include Cauchy's bound [2], Carmichael and Mason's bound, Montel's bound [2] and Fujii and Kubo's bound [3]. In this paper, we will give a new estimate for the zeros of polynomials using the spectral norm and the spectral radius for the fifth power of the Frobenius companion matrix.

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Let $M_n(\mathbb{C})$ stands for the algebra of all $n \times n$ complex matrices. For $A \in M_n(\mathbb{C})$, the eigenvalues of A are denoted by $\lambda_i(A)$, for $i = 1, 2, \ldots, n$, arranged in such a way that

$$|\lambda_1(A)| \geq |\lambda_2(A)| \geq \cdots \geq |\lambda_n(A)| .$$

The singular values of A, (the eigenvalues of $|A| = (A^*A)^{\frac{1}{2}}$) are denoted by $s_i(A)$, $(1 \leq i \leq n)$, where they are arranged in such a way that

$$s_1(A) \geq s_2(A) \geq \cdots \geq s_n(A) .$$

Recall that $s_i^2(A) = \lambda_j(A^*A) = \lambda_j(AA^*)$, for $j = 1, 2, \ldots, n$ and $s_1(A) = \|A\|$, where $\|A\|$ represent the spectral norm of A. For $A \in M_n(\mathbb{C})$, if λ is the eigenvalue of A and $r(A)$ represents the spectral radius of A, then for any matrix norm $|\|\cdot\||$, we have

$$|\lambda| \leq r(A) \leq |\|A\|| .$$

Let $p(z) = z^n + \alpha_n z^{n-1} + \alpha_{n-2} z^{n-2} + \cdots + \alpha_2 z + \alpha_1$ be a monic polynomial of degree $n \geq 7$ with complex coefficients $\alpha_n, \alpha_{n-1}, \ldots . \alpha_1$, where $\alpha_1 \neq 0$. The following matrix

$$C = \begin{bmatrix} -\alpha_n & -\alpha_{n-1} & \cdots & -\alpha_2 & -\alpha_1 \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix}_{n \times n}$$

is called the Frobenius companion matrix for p. It is well known that the characteristic polynomial of C is p itself, and so the eigenvalues of C are the zeros of p, see [4]. Using the fact that the eigenvalues of C are the roots of $p(z) = 0$, then for any matrix norm $|\|\cdot\||$, $|z| \leq |\|C\||$, where z is the zero of the monic polynomial p. Many mathematicians in the area have used Frobenius companion matrix C to derive bounds for the moduli of the zeros of the polynomial p; we list some of them below. Let z be any zero of p, then we note that some bounds are obtained by the classical approach.

Cauchy [2], proved that

$$|z| \leq 1 + \max\{|\alpha_1|, |\alpha_1|, \ldots, |\alpha_n|\},$$

Montal [2], proved that

$$|z| \leq 1 + |\alpha_1| + |\alpha_1| + \ldots + |\alpha_n|,$$

Cramichael and Mason [2], proved that

$$|z| \leq \left(1 + |\alpha_1|^2 + |\alpha_2|^2 + \ldots + |\alpha_n|^2\right)^{\frac{1}{2}} .$$

Others have provided bounds for zeros of polynomials based on matrix inequalities using the Frobenius companion matrix, such as

Fujii and Kubi [3], proved that

$$|z| \leq \cos\left(\frac{\pi}{n+1}\right) + \frac{1}{2}\left(|\alpha_n| + \left(\sum_{j=1}^{n} |\alpha_j|^2\right)^{\frac{1}{2}}\right),$$

Linden [7], proved that

$$|z| \leq \frac{|\alpha_n|}{2} + \left(\frac{n-1}{n}\left(n-1+\left|\sum_{j=1}^{n} |\alpha_j|^2 - \frac{|\alpha_n|^2}{2}\right|\right)\right)^{\frac{1}{2}},$$

Kittaneh [5], proved that

$$|z| \leq \frac{1}{2}\left(|\alpha_n| + 1 + \sqrt{(|\alpha_n|-1)^2 + 4\sqrt{\sum_{j=1}^{n-1} |\alpha_j|^2}}\right).$$

Based on certain estimates for spectral norms and spectral radii of the square of the Frobenius companion matrices Kittaneh and Shebrawi, [6] obtained new bounds for the zeros of p as follows:

$$|z| \leq \left(1 + \left(\sum_{j=1}^{n} |\alpha_j|^2 + |b_j|^2\right)\right)^{\frac{1}{4}}, \text{ where } b_j = \alpha_n \alpha_j - \alpha_{j-1}.$$

Also

$$|z| \leq \left(\frac{1}{2}\left(|b_n| + \beta + \sqrt{(|b_n| - \beta)^2 + 4\gamma\sqrt{1+|\alpha_n|^2}}\right)\right)^{\frac{1}{2}},$$

where

$$\gamma = \left(\sum_{j=1}^{n-1} |b_j|^2\right)^{\frac{1}{2}}$$

and

$$\beta = \sqrt{\frac{1}{2}\left(1 + \sum_{j=1}^{n-1} |\alpha_j|^2 + \sqrt{1 + \sum_{j=1}^{n-1} |\alpha_j|^2 - 4\left(|\alpha_1|^2 + |\alpha_2|^2\right)}\right)}.$$

They also obtained new bounds based on the spectral norms and the spectral radii of the cube of the Frobenius companion matrix.

Recently, Al Sawaftah and Burqan [1] have given another bound for the zeros of polynomials depending on the spectral norm of fourth of the Frobenius companion matrix. In this paper we will present more accurate bounds depending on the spectral norm and the spectral radii of C^5. In this paper let $N = C^5$. Thus,

$$N = \begin{bmatrix} e_n & e_{n-1} & \cdots & e_6 & e_5 & \cdots & e_1 \\ d_n & d_{n-1} & \cdots & d_6 & d_5 & \cdots & d_1 \\ c_n & c_{n-1} & \cdots & c_6 & c_5 & \cdots & c_1 \\ b_n & b_{n-1} & \cdots & b_6 & b_5 & \cdots & b_1 \\ -\alpha_n & -\alpha_{n-1} & \cdots & -\alpha_6 & -\alpha_5 & \cdots & -\alpha_1 \\ 1 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 & \cdots & 0 \end{bmatrix},$$

where

$$\begin{aligned} b_j &= \alpha_n \alpha_j - \alpha_{j-1}, c_j = -\alpha_n b_j + \alpha_{n-1}\alpha_j - \alpha_{j-2}, \\ d_j &= -\alpha_n c_j - \alpha_{n-1} b_j + \alpha_{n-2}\alpha_j - \alpha_{j-3}, \\ e_j &= -\alpha_n d_j - \alpha_{n-2} c_j - \alpha_{n-2} b_j + \alpha_{n-3}\alpha_j - \alpha_{n-4,} \end{aligned}$$

for $j = 1, 2, \ldots, n$.

2 Main Results

In this section, we obtain bounds for the spectral norm and the spectral radius of the matrix N, which we use it to estimate the zeros of polynomials.

Theorem 1 *Let z be a zero of $p(z) = z^n + \alpha_n z^{n-1} + \alpha_{n-2} z^{n-2} + \cdots + \alpha_2 z + \alpha_1$, with degree $n \geqslant 7$, then*

$$|z| \leq \left(1 + \sum_{j=1}^{n} |\alpha_j|^2 + |b_j|^2 + |c_j|^2 + |d_j|^2 + |e_j|^2\right)^{\frac{1}{10}}.$$

Proof Consider the following matrices

$$G_1 = \begin{bmatrix} e_n & e_{n-1} & \cdots & e_2 & e_1 \\ 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 \end{bmatrix}_{n\times n}, \quad G_2 = \begin{bmatrix} 0 & 0 & \cdots & 0 & 0 \\ d_n & d_{n-1} & \cdots & d_2 & d_1 \\ 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 \end{bmatrix}_{n\times n},$$

$$G_3 = \begin{bmatrix} 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \\ c_n & c_{n-1} & \cdots & c_2 & c_1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 \end{bmatrix}_{n\times n}, \quad G_4 = \begin{bmatrix} 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \\ b_n & b_{n-1} & \cdots & b_2 & b_1 \\ 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 \end{bmatrix}_{n\times n},$$

$$G_5 = \begin{bmatrix} 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \\ -\alpha_n & -\alpha_{n-1} & \cdots & -\alpha_2 & -\alpha_1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 \end{bmatrix}_{n\times n}$$

and the block matrix $G_6 = \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ I_{n-5} & \mathbf{0} \end{bmatrix}_{n\times n}$, where I_{n-5} is the identity of order $n-5$. Then $\sum_{l=1}^{6} G_l = N$ with $G_l^* G_m = 0,\ 1 \le l, m \le 6,\quad l \ne m$. Thus by the triangle inequality, and using the fact that $\|A\|^2 = \|A^*A\|$, for any matrix $A \in M_n(\mathbb{C})$, we get

$$\begin{aligned} \|N\|^2 = \|N^*N\| &= \left\| \sum_{l=1}^{6} G_l^* G_l \right\| \\ &\le \sum_{l=1}^{6} \|G_l^* G_l\| = \sum_{l=1}^{6} \|G_l\|^2 \\ &= \sum_{j=1}^{n} \left(|e_j|^2 + |d_j|^2 + |c_j|^2 + |b_j|^2 + |\alpha_j|^2 \right) + 1. \end{aligned}$$

Since

$$\|G_1\|^2 = \max\left\{\lambda : \lambda \in \sigma\left(G_1^* G_1\right)\right\} = \sum_{j=1}^{n} \left|e_j\right|^2,$$

Also

$$\|G_2\|^2 = \sum_{j=1}^{n} \left|d_j\right|^2, \|G_3\|^2 = \sum_{j=1}^{n} \left|c_j\right|^2, \|G_4\|^2 = \sum_{j=1}^{n} \left|b_j\right|^2, \|G_5\|^2 = \sum_{j=1}^{n} \left|\alpha_j\right|^2$$

$\left\|G_6^* G_6\right\| = 1.$

Therefore,

$$\left\|C^5\right\| = \|N\| \leq \left(1 + \sum_{j=1}^{n} \left|e_j\right|^2 + \left|d_j\right|^2 + \left|c_j\right|^2 + \left|b_j\right|^2 + \left|\alpha_j\right|^2\right)^{\frac{1}{2}}.$$

Using the fact that $|z| \leq \left\|C^5\right\|^{\frac{1}{5}}$, we get

$$|z| \leq \left(1 + \sum_{j=1}^{n} \left|e_j\right|^2 + \left|d_j\right|^2 + \left|c_j\right|^2 + \left|b_j\right|^2 + \left|\alpha_j\right|^2\right)^{\frac{1}{10}}.$$

■

Let us recall some important Lemmas which are essential to establish our next results in this paper. These Lemmas can be found in [4].

Lemma 2 *If* $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, *then the spectral radius of* A,

$$r(A) = \frac{1}{2}\left(a + d + \sqrt{(a-d)^2 + 4bc}\right).$$

Lemma 3 *Let* $A \in M_n(\mathbb{C})$ *be partitioned as* $A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$, *where* A_{ij} *is an* $n_i \times n_j$ *matrix for* $i, j = 1, 2$ *with* $n_1 + n_2 = n$. *If* $\tilde{A} = \begin{bmatrix} \|A_{11}\| & \|A_{12}\| \\ \|A_{21}\| & \|A_{22}\| \end{bmatrix}$, *then* $r(A) \leq r\left(\tilde{A}\right)$ *and* $\|A\| \leq \left\|\tilde{A}\right\|$.

Lemma 4 *Let* $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, *then the spectral norm of* A *is*

$$\|A\| = \left(\frac{1}{2}\left(|a|^2 + |b|^2 + |c|^2 + |d|^2 + \gamma\right)\right)^{\frac{1}{2}},$$

where $\gamma = \sqrt{\left(|a|^2 + |c|^2 - |b|^2 - |d|^2\right)^2 + 4\left|a\overline{b} + c\overline{d}\right|^2}$.

Lemma 5 *Let*

$$B = \begin{bmatrix} -\alpha_{n-1} & -\alpha_{n-2} & \cdots & -\alpha_6 & -\alpha_5 & \cdots & -\alpha_1 \\ 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 1 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 & \cdots & 0 \end{bmatrix}_{n\times n},$$

with $n \geq 7$, *then*

$$\|B\| = \frac{1}{2}\left(1 + \mu + \sqrt{(1+\mu)^2 - 4\sum_{j=1}^{5}\left|\alpha_j\right|^2}\right),$$

where $\mu = \sum_{j=1}^{n-4}\left|\alpha_j\right|^2$.

The following partition matrix is needed to obtain the next result. For the matrix

$$N = \begin{bmatrix} e_n & e_{n-1} & \cdots & e_6 & e_5 & \cdots & e_1 \\ d_n & d_{n-1} & \cdots & d_6 & d_5 & \cdots & d_1 \\ c_n & c_{n-1} & \cdots & c_6 & c_5 & \cdots & c_1 \\ b_n & b_{n-1} & \cdots & b_6 & b_5 & \cdots & b_1 \\ -\alpha_n & -\alpha_{n-1} & \cdots & -\alpha_6 & -\alpha_5 & \cdots & -\alpha_1 \\ 1 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 & \cdots & 0 \end{bmatrix},$$

partition the matrix as $N = \begin{bmatrix} N_{11} & N_{12} \\ N_{21} & N_{22} \end{bmatrix}$, where

$$N_{11} = \begin{bmatrix} e_n & e_{n-1} & e_{n-2} & e_{n-3} \\ d_n & d_{n-1} & d_{n-2} & d_{n-3} \\ c_n & c_{n-1} & c_{n-2} & c_{n-3} \\ b_n & b_{n-1} & b_{n-2} & b_{n-3} \end{bmatrix}_{4\times 4},$$

$$N_{12} = \begin{bmatrix} e_{n-4} & \cdots & e_6 & e_5 & \cdots & e_1 \\ d_{n-4} & \cdots & d_6 & d_5 & \cdots & d_1 \\ c_{n-4} & \cdots & c_6 & c_5 & \cdots & c_1 \\ b_{n-4} & \cdots & b_6 & b_5 & \cdots & b_1 \end{bmatrix}_{4\times (n-4)},$$

$$N_{21} = \begin{bmatrix} -\alpha_n & -\alpha_{n-1} & -\alpha_{n-2} & -\alpha_{n-3} \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 \end{bmatrix}_{(n-4)\times 4},$$

$$N_{22} = \begin{bmatrix} -\alpha_{n-4} & -\alpha_{n-5} & \cdots & -\alpha_6 & -\alpha_5 & -\alpha_4 & -\alpha_3 & -\alpha_2 & -\alpha_1 \\ 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}_{(n-4)\times (n-4)}.$$

Now, as a result, we get the following:

Theorem 6 *Let z be a zero of $p(z) = z^n + \alpha_n z^{n-1} + \alpha_{n-2} z^{n-2} + \cdots + \alpha_2 z + \alpha_1$, with degree $n \geqslant 7$, then*

$$|z| \le \left[\|N_{11}\| + \|N_{22}\| + \sqrt{(\|N_{11}\| - \|N_{22}\|)^2 + 4\,\|N_{12}\|\,\|N_{21}\|} \right]^{\frac{1}{5}}.$$

Proof Since N is partitioned as $N = \begin{bmatrix} N_{11} & N_{12} \\ N_{21} & N_{22} \end{bmatrix}$, applying Lemma 3, we have

$$r(N) \le r\left(\begin{bmatrix} \|N_{11}\| & \|N_{12}\| \\ \|N_{21}\| & \|N_{22}\| \end{bmatrix} \right).$$

To find $\|N_{11}\|$, we partition N_{11} as $N_{11} = \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix}$, where

$$S_{11} = \begin{bmatrix} e_n & e_{n-1} \\ d_n & d_{n-1} \end{bmatrix}, \ S_{12} = \begin{bmatrix} e_{n-2} & e_{n-3} \\ d_{n-2} & d_{n-3} \end{bmatrix}, \ S_{21} = \begin{bmatrix} c_n & c_{n-1} \\ b_n & b_{n-1} \end{bmatrix}$$

and

$$S_{22} = \begin{bmatrix} c_{n-2} & c_{n-3} \\ b_{n-2} & b_{n-3} \end{bmatrix}.$$

Now, find the spectral norm for each S_{ij}, $i, j = 1, 2$, by using Lemma 4 as follows:

$$\alpha = \|S_{11}\| = \left(\frac{1}{2} \left(\sum_{j=n-1}^{n} \left|e_j\right|^2 + \left|d_j\right|^2 + \sqrt{\left(|e_n|^2 + |d_n|^2 - |e_{n-1}|^2 - |d_{n-1}|^2\right)^2 + 4\left|e_n\overline{e_{n-1}} + d_n\overline{d_{n-1}}\right|^2} \right) \right)^{\frac{1}{2}},$$

$$\beta = \|S_{12}\| = \left(\frac{1}{2} \left(\sum_{j=n-3}^{n} \left|e_j\right|^2 + \left|d_j\right|^2 + \sqrt{\left(|e_{n-2}|^2 + |d_{n-2}|^2 - |e_{n-3}|^2 - |d_{n-3}|^2\right)^2 + 4\left|e_{n-2}\overline{e_{n-3}} + d_{n-2}\overline{d_{n-3}}\right|^2} \right) \right)^{\frac{1}{2}},$$

$$\gamma = \|S_{21}\| = \left(\frac{1}{2} \left(\sum_{j=n-1}^{n} \left|c_j\right|^2 + \left|bd_j\right|^2 + \sqrt{\left(|c_n|^2 + |b_n|^2 - |c_{n-1}|^2 - |b_{n-1}|^2\right)^2 + 4\left|c_n\overline{c_{n-1}} + b_n\overline{b_{n-1}}\right|^2} \right) \right)^{\frac{1}{2}},$$

and

$$\delta = \|S_{22}\| = \left(\frac{1}{2} \left(\sum_{j=n-3}^{n-2} \left|c_j\right|^2 + \left|b_j\right|^2 + \sqrt{\left(|c_{n-2}|^2 + |b_{n-2}|^2 - |c_{n-3}|^2 - |b_{n-3}|^2\right)^2 + 4\left|c_{n-2}\overline{c_{n-3}} + b_{n-2}\overline{b_{n-3}}\right|^2} \right) \right)^{\frac{1}{2}},$$

Also, by Lemma 3, we have

$$\|N_{11}\| \leq \left\| \begin{bmatrix} \|S_{11}\| & \|S_{12}\| \\ \|S_{21}\| & \|S_{22}\| \end{bmatrix} \right\| = \left\| \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} \right\|.$$

Again using Lemma 3 to get

$$\|N_{11}\| \leq \left(\frac{1}{2} \left(\alpha^2 + \beta^2 + \gamma^2 + \delta^2 + \sqrt{\left(\alpha^2 + \beta^2 - \gamma^2 - \delta^2\right)^2 + 4\,|\alpha\beta + \gamma\delta|^2} \right) \right)^{\frac{1}{2}}.$$

Now, $\|N_{12}\| = \left(r\left(N_{12}N_{12}^*\right)\right)^{\frac{1}{2}}$, where

$$N_{12}N_{12}^* = \begin{bmatrix} \sum_{j=1}^{n-4} |e_j|^2 & \sum_{j=1}^{n-4} e_j\overline{d_j} & \sum_{j=1}^{n-4} e_j\overline{c_j} & \sum_{j=1}^{n-4} e_j\overline{b_j} \\ \sum_{j=1}^{n-4} d_j\overline{e_j} & \sum_{j=1}^{n-4} |d_j|^2 & \sum_{j=1}^{n-4} d_j\overline{c_j} & \sum_{j=1}^{n-4} d_j\overline{b_j} \\ \sum_{j=1}^{n-4} c_j\overline{e_j} & \sum_{j=1}^{n-4} c_j\overline{d_j} & \sum_{j=1}^{n-4} |c_j|^2 & \sum_{j=1}^{n-4} c_j\overline{b_j} \\ \sum_{j=1}^{n-4} b_j\overline{e_j} & \sum_{j=1}^{n-4} b_j\overline{d_j} & \sum_{j=1}^{n-4} b_j\overline{c_j} & \sum_{j=1}^{n-4} |b_j|^2 \end{bmatrix}.$$

To find $\|N_{12}\|$, we partition $N_{12}N_{12}^*$ as $\begin{bmatrix} W_{11} & W_{12} \\ W_{21} & W_{22} \end{bmatrix}$, where

$$W_{11} = \begin{bmatrix} \sum_{j=1}^{n-4} |e_j|^2 & \sum_{j=1}^{n-4} e_j\overline{d_j} \\ \sum_{j=1}^{n-4} d_j\overline{e_j} & \sum_{j=1}^{n-4} |d_j|^2 \end{bmatrix}, \quad W_{12} = \begin{bmatrix} \sum_{j=1}^{n-4} e_j\overline{c_j} & \sum_{j=1}^{n-4} e_j\overline{b_j} \\ \sum_{j=1}^{n-4} d_j\overline{c_j} & \sum_{j=1}^{n-4} d_j\overline{b_j} \end{bmatrix}, .$$

$$W_{21} = \begin{bmatrix} \sum_{j=1}^{n-4} c_j\overline{e_j} & \sum_{j=1}^{n-4} c_j\overline{d_j} \\ \sum_{j=1}^{n-4} b_j\overline{e_j} & \sum_{j=1}^{n-4} b_j\overline{d_j} \end{bmatrix}, \quad W_{22} = \begin{bmatrix} \sum_{j=1}^{n-4} |c_j|^2 & \sum_{j=1}^{n-4} c_j\overline{b_j} \\ \sum_{j=1}^{n-4} b_j\overline{c_j} & \sum_{j=1}^{n-4} |b_j|^2 \end{bmatrix}.$$

Using Lemma 4 to get the spectral norm for each $W_{ij},\ i, j = 1, 2$ as follows:

$$\|W_{11}\| = \left(\frac{1}{2} \left(\left| \sum_{j=1}^{n-4} |e_j|^2 \right|^2 + \left| \sum_{j=1}^{n-4} |d_j|^2 \right|^2 + \left| \sum_{j=1}^{n-4} e_j\overline{d_j} \right|^2 + \left| \sum_{j=1}^{n-4} d_j\overline{e_j} \right|^2 + \sqrt{a+b} \right) \right)^{\frac{1}{2}},$$

$$\|W_{12}\| = \left(\frac{1}{2}\left(\left|\sum_{j=1}^{n-4} e_j\overline{c_j}\right|^2 + \left|\sum_{j=1}^{n-4} e_j\overline{b_j}\right|^2 + \left|\sum_{j=1}^{n-4} d_j\overline{c_j}\right|^2 + \left|\sum_{j=1}^{n-4} d_j\overline{b_j}\right|^2 + \sqrt{c+d}\right)\right)^{\frac{1}{2}},$$

$$\|W_{21}\| = \left(\frac{1}{2}\left(\left|\sum_{j=1}^{n-4} c_j\overline{e_j}\right|^2 + \left|\sum_{j=1}^{n-4} c_j\overline{d_j}\right|^2 + \left|\sum_{j=1}^{n-4} b_j\overline{e_j}\right|^2 + \left|\sum_{j=1}^{n-4} b_j\overline{d_j}\right|^2 + \sqrt{e+f}\right)\right)^{\frac{1}{2}},$$

$$\|W_{22}\| = \left(\frac{1}{2}\left(\left|\sum_{j=1}^{n-4} |c_j|^2\right|^2 + \left|\sum_{j=1}^{n-4} |b_j|^2\right|^2 + \left|\sum_{j=1}^{n-4} b_j\overline{c_j}\right|^2 + \left|\sum_{j=1}^{n-4} c_j\overline{b_j}\right|^2 + \sqrt{g+h}\right)\right)^{\frac{1}{2}},$$

where

$$a = \left(\left|\sum_{j=1}^{n-4} |e_j|^2\right|^2 + \left|\sum_{j=1}^{n-4} d_j\overline{e_j}\right|^2 - \left|\sum_{j=1}^{n-4} e_j\overline{d_j}\right|^2 - \left|\sum_{j=1}^{n-4} |d_j|^2\right|^2\right)^2,$$

$$b = 4\left|\left(\sum_{j=1}^{n-4} |e_j|^2\right)\left(\sum_{j=1}^{n-4} e_j\overline{d_j}\right) + \left(\sum_{j=1}^{n-4} d_j\overline{e_j}\right)\left(\sum_{j=1}^{n-4} |d_j|^2\right)\right|^2,$$

$$c = \left(\left|\sum_{j=1}^{n-4} e_j\overline{c_j}\right|^2 + \left|\sum_{j=1}^{n-4} d_j\overline{c_j}\right|^2 - \left|\sum_{j=1}^{n-4} e_j\overline{b_j}\right|^2 - \left|\sum_{j=1}^{n-4} d_j\overline{b_j}\right|^2\right)^2,$$

$$d = 4\left|\left(\sum_{j=1}^{n-4} e_j\overline{c_j}\right)\left(\sum_{j=1}^{n-4} e_j\overline{b_j}\right) + \left(\sum_{j=1}^{n-4} d_j\overline{c_j}\right)\left(\sum_{j=1}^{n-4} d_j\overline{b_j}\right)\right|^2,$$

$$e = \left(\left|\sum_{j=1}^{n-4} c_j\overline{e_j}\right|^2 + \left|\sum_{j=1}^{n-4} b_j\overline{e_j}\right|^2 - \left|\sum_{j=1}^{n-4} c_j\overline{d_j}\right|^2 - \left|\sum_{j=1}^{n-4} b_j\overline{d_j}\right|^2\right)^2,$$

$$f = 4\left|\left(\sum_{j=1}^{n-4} c_j\overline{e_j}\right)\left(\sum_{j=1}^{n-4} c_j\overline{d_j}\right) + \left(\sum_{j=1}^{n-4} b_j\overline{d_j}\right)\left(\sum_{j=1}^{n-4} d_j\overline{b_j}\right)\right|^2,$$

$$g = \left(\left|\sum_{j=1}^{n-4} |c_j|^2\right|^2 + \left|\sum_{j=1}^{n-4} b_j\overline{c_j}\right|^2 - \left|\sum_{j=1}^{n-4} c_j\overline{b_j}\right|^2 - \left|\sum_{j=1}^{n-4} |b_j|^2\right|^2\right)^2$$

$$h = 4\left|\left(\sum_{j=1}^{n-4} |c_j|^2\right)\left(\sum_{j=1}^{n-4} c_j\overline{b_j}\right) + \left(\sum_{j=1}^{n-4} b_j\overline{c_j}\right)\left(\sum_{j=1}^{n-4} b_j\overline{c_j}\right)\right|^2.$$

By Lemmas 2 and 3, we have

$$
\begin{aligned}
\|N_{12}\| &= \left(r\left(N_{12}N_{12}^{*}\right)\right)^{\frac{1}{2}} \leq \left(r\left(\begin{bmatrix} \|w_{11}\| & \|w_{12}\| \\ \|w_{21}\| & \|w_{22}\| \end{bmatrix}\right)\right)^{\frac{1}{2}} \\
&= \left(\frac{1}{2}\left(\|w_{11}\| + \|w_{11}\| + \sqrt{(\|w_{11}\| - \|w_{22}\|)^2 + 4\,\|w_{12}\| \ \ \|w_{21}\|}\right)\right)^{\frac{1}{2}}.
\end{aligned}
$$

Now,

$$
\|N_{21}\| = \sqrt{|\alpha_n|^2 + |\alpha_{n-1}|^2 + |\alpha_{n-2}|^2 + |\alpha_{n-3}|^2 + 1},
$$

and Lemma 5 yields

$$
\|N_{22}\| = \left(\frac{1}{2}\left(1 + \mu + \sqrt{(1+\mu)^2 - 4\left(|\alpha_1|^2 + |\alpha_2|^2 + |\alpha_3|^2 + |\alpha_4|^2 + |\alpha_5|^2\right)}\right)\right)^{\frac{1}{2}},
$$

where $\mu = \sum_{j=1}^{n-4} \left|\alpha_j\right|^2$ .Thus,

$$
\begin{aligned}
r\,(N) &\leq r\left(\begin{bmatrix} \|N_{11}\| & \|N_{12}\| \\ \|N_{21}\| & \|N_{22}\| \end{bmatrix}\right) \\
&= \frac{1}{2}\left(\|N_{11}\| + \|N_{22}\| + \sqrt{(\|N_{11}\| - \|N_{22}\|)^2 + 4\,\|N_{12}\|\,\|N_{21}\|}\right).
\end{aligned}
$$

Since $|z| \leq r\,(C) = \left(r\left(C^5\right)\right)^{\frac{1}{5}} = (r\,(N))^{\frac{1}{5}}$, we have

$$
|z| \leq \left(\frac{1}{2}\left(\|N_{11}\| + \|N_{22}\| + \sqrt{(\|N_{11}\| - \|N_{22}\|)^2 + 4\,\|N_{12}\|\,\|N_{21}\|}\right)\right)^{\frac{1}{5}}.
$$

This completes the proof. ■

References

1. Al Sawafteh, A., Burqan, A.: Bounds for the Zeros of Polynomials, Master thesis, Zarqa University (2021)
2. Fujii, M., Kubo, F.: Operator norms as bounds for roots of algebraic equations. Proc. Jpn. Acad. **49**(10), 805–808 (1973)
3. Fujii, M., Kubo, F.: Buzano's inequality and bounds for roots of algebraic equations. Proc. Am. Math. Soc. **117**(2), 359–361 (1993)
4. Horn, R. A., Johnson, C. R.: Matrix Analysis. Cambridge University Press (2012)

5. Kittaneh, F.: Bounds for the zeros of polynomials from matrix inequalities. Archiv der Mathematik **81**(5), 601–608 (2003)
6. Kittaneh, F., Shebrawi, K.: Bounds for the zeros of polynomials from matrix inequalities - II. Linear Multilinear Algebra **55**(2), 147–158 (2007)
7. Linden, H.: Bounds for zeros of polynomials using traces and determinants. Seminarberichte Fachbereich Mathematik FeU Hagen **69**, 127–146 (2000)