Hamiltonian-Jacobi Equation for Torsional Problems of Corrugated Steel Web Box Girders

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Abstract. Due to the use of corrugated steel plate in the web, the torsional performance of the beam body will be weakened to a certain extent. In this paper, the Hamiltonian-Jacobian equation is used to analyze the torsion of the corrugated steel web composite box girder. The Lagrangian function of the system is obtained by taking the warping function and the relative torsion angle of the cross-section as the two generalized coordinates, and the Hamiltonian-Jacobian equation for the torsion problem is established by using the regular transformation. The accuracy of the proposed method is verified by comparing the obtained results with the literature values, and the fact that the torsional performance of the beam is weakened by using corrugated steel plate is verified by using the results of this paper.

Keywords: Corrugated Steel Web · Torsional Properties · Warpage Function; Relative Torsion Angle · Hamiltonian-Jacobian Equations

1 The Hamiltonian-Jacobian Equation for the Torsion Problem

The warping function of the cross-section and the relative torsion angle are selected as the two generalized coordinates of the torsion problem [5], they are both functions of the coordinates z, $\beta = \beta(z)$ and $\varphi = \varphi(z)$, and the time variable t in Hamiltonian mechanics is replaced by the coordinate z, and the Lagrangian function of the torsion problem is [3, 4]:

$$L = \frac{1}{2}\left(\sum_{i=1}^{4} E_i I_{\omega i}\right)\dot{\beta}^2 + \frac{1}{2}\left(\sum_{i=1}^{4} G_i J_i\right)\dot{\varphi}^2 + \frac{1}{2}\left(\sum_{i=1}^{4} G_i (I_{Pi} - J_{Bi})\right)(\dot{\varphi} - \beta)^2 - m\varphi \tag{1}$$

where $I_{\omega i}$ is the main sector moment of inertia of the cross-section, J_i is the torsional moment of inertia of the cross-section, I_{Pi} is the polar moment of inertia of the cross-section, and J_{Bi} is the Brett torsional moment of inertia of the cross-section [6], which are all cross-section constants that can be found and calculated by the engineering manual. m is the moment of the external force acting on the beam per unit length. Let $A = \frac{1}{2}\left(\sum_{i=1}^{4} E_i I_{\omega i}\right) B = \frac{1}{2}\left(\sum_{i=1}^{4} G_i J_i\right), C = \frac{1}{2}\left(\sum_{i=1}^{4} G_i (I_{Pi} - J_{Bi})\right)$

G. Feng (Ed.): ICCE 2023, LNCE 526, pp. 714–719, 2024.
https://doi.org/10.1007/978-981-97-4355-1_71

Then the generalized momentum corresponding to the generalized coordinates is [1]:

$$\begin{aligned} P_\beta &= \tfrac{\partial L}{\partial \dot{\beta}} = 2A\dot{\beta} \\ P_\varphi &= \tfrac{\partial L}{\partial \dot{\varphi}} = 2B\dot{\varphi} + 2C(\dot{\varphi} - \beta) \end{aligned} \tag{2}$$

Substituting (2) into the Lagrangian function yields the Hamiltonian function of the system:

$$H = 0.25A^{-1}P_\beta^2 + 0.25(B+C)^{-1}P_\varphi^2 - BC(B+C)^{-1}\beta^2 + C(B+C)^{-1}\beta P_\varphi + m\varphi \tag{3}$$

The Hamiltonian function H does not contain coordinates z, so there is a generalized energy integral h (h is a constant), and the parent function of the canonical transformation can be selected as follows:

$$S(\beta, \varphi; z) = -hz + W(\beta, \varphi) + K \tag{4}$$

where K is the integral constant of the addition, and the regular transformation relation is as follows:

$$\begin{aligned} P_\beta &= \frac{\partial W}{\partial \beta} \\ P_\varphi &= \frac{\partial W}{\partial \varphi} \end{aligned} \tag{5}$$

Substituting (2) and (5) into (3), we get the Hamiltonian-Jacobian equation for the torsion problem as:

$$0.25A^{-1}\left(\frac{\partial W}{\partial \beta}\right)^2 + 0.25(B+C)^{-1}\left(\frac{\partial W}{\partial \varphi}\right)^2 - BC(B+C)^{-1}\beta^2 + C(B+C)^{-1}\beta\frac{\partial W}{\partial \varphi} + m\varphi = h \tag{6}$$

2 The Solution to the Torsion Problem

2.1 Simplification of Equations

(6) is a first-order nonlinear partial differential equation, which has been shown to be an important factor affecting the torsional characteristics of the corrugated steel plate, while the coefficient of the nonlinear term is a dimensionless coefficient, which no longer includes the stiffness of the corrugated steel, so the influence of the nonlinear term is negligible, and the equation can be simplified as:

$$\left[0.25A^{-1}\left(\frac{\partial W}{\partial \beta}\right)^2 - BC(B+C)^{-1}\beta^2\right] + \left[0.25(B+C)^{-1}\left(\frac{\partial W}{\partial \varphi}\right)^2 + m\varphi\right] = h \tag{7}$$

2.2 Separation of Variables

(7) It can be solved by using the separation variable method [2], let $W(\beta, \varphi) = W_1(\beta) + W_2(\varphi)$ Substituting (7) gives us two ordinary differential equations:

$$\begin{aligned} 0.25A^{-1}\left(\frac{dW_1}{d\beta}\right)^2 - BC(B+C)^{-1}\beta^2 = \eta \\ 0.25(B+C)^{-1}\left(\frac{dW_2}{d\varphi}\right)^2 + m\varphi = h - \eta \end{aligned} \tag{8}$$

In this case, the parent function of the regular transformation is:

$$\begin{aligned} S(\beta, \varphi; z) = -hz + W(\beta, \varphi) + K = \eta\left(A\tfrac{B+C}{BC}\right)^{\frac{1}{2}} \ln\left[\left(\tfrac{ABC}{B+C}\right)^{\frac{1}{2}}\beta + \left(A\eta + \tfrac{ABC}{B+C}\beta^2\right)^{\frac{1}{2}}\right] \\ +\beta\left(A\eta + \tfrac{ABC}{B+C}\beta^2\right)^{\frac{1}{2}} - \tfrac{4(B+C)^{\frac{1}{2}}(h-\eta-m\varphi)^{\frac{3}{2}}}{3m} - hz + K \end{aligned} \tag{9}$$

where η, h, K are the three integration constants, which are determined by the boundary conditions.

2.3 Analytic Expressions for $\varphi(z)$ and $\beta(z)$

The regular transformation relation expressed in terms of the parent function is as follows:

$$\begin{aligned} \frac{\partial S}{\partial \eta} = \gamma_1 \\ \frac{\partial S}{\partial h} = \gamma_2 \end{aligned} \tag{10}$$

Substituting the expression (10) of S into (11) gives the expression of $\varphi(z)$ as follows:

$$\varphi(z) = \frac{h-\eta}{m} - \frac{m}{4(B+C)}(\mathrm{z} + \gamma_2)^2 \tag{11}$$

The expression for $\beta(z)$ is as follows:

$$\begin{aligned} \left(\tfrac{A}{B+C}\right)^{\frac{1}{2}} \ln\left[(BC)^{\frac{1}{2}}\beta + \left(B\eta + C\eta + BC\beta^2\right)^{\frac{1}{2}}\right] + \tfrac{\beta}{2}\left(B\eta + C\eta + BC\beta^2\right)^{-\frac{1}{2}} \\ +\eta(B+C)\left(\tfrac{A}{BC}\right)^{\frac{1}{2}}\left[(BC)^{\frac{1}{2}}\beta + \left(B\eta + C\eta + BC\beta^2\right)^{\frac{1}{2}}\right]^{-1}\left(B\eta + C\eta + BC\beta^2\right)^{-\frac{1}{2}} = \gamma_1 - \tfrac{z+\gamma_2}{(B+C)^{\frac{1}{2}}} \end{aligned} \tag{12}$$

where h, η, γ_1, γ_2 are the 4 integration constants, which are determined by the boundary conditions.

3 Analysis and Discussion

3.1 Case Analysis

In order to verify the effectiveness of the proposed algorithm, a specific example Ref. [9] is selected for calculation.

The beam bears the mid-span torque, which is solved by the algorithm in this paper. The results of the warpage function are shown in Fig. 1.

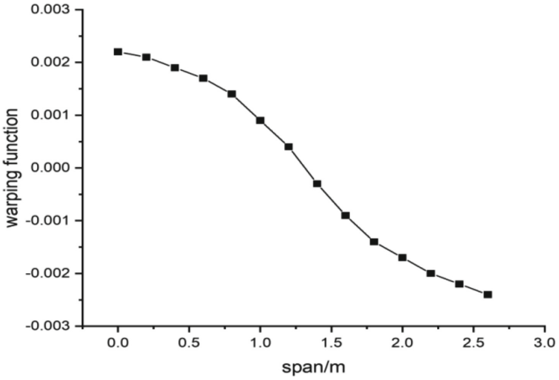

Fig. 1. Calculation of the warpage function

Then, according to the warping function [7], the values of the normal stresses at each point on the cross-section can be calculated, and the expression of the normal stresses is as follows:

$$\sigma_z = -E_i \dot{\beta} \omega \tag{13}$$

where ω is called the main sector coordinate of the box section, and the specific calculation method can be referred to Ref. [10], which will not be repeated in this article. The following Table 1 shows the normal stress values of some measurement points at a span of 0.975 m, where point A is the edge point of the roof flange, point B is the intersection point of the roof and the web, and point C is the edge of the bottom plate. It can be seen that the error between the normal stress value of the obtained measurement point and the literature value is not more than 3%, and the correctness and accuracy of the proposed algorithm are verified by the comparison of stress values.

Table 1. Comparison of normal stress values of cross-sections(MPa)

Measuring points	Measuring points A	Measuring points B	Measuring points C
Calculated values in this article	0.973	−0.206	−10.39
Calculated values from Ref. [9]	0.950	−0.200	−10.13

3.2 Torsional Performance Analysis

The torsional performance of a beam can be measured using the torsional angle per unit length. Deriving the expression (11) yields the absolute value of the torsion angle per unit length as follows:

$$|\dot{\varphi}(z)| = (B + C)^{-1}\left|0.25(z + \gamma_2)^2\dot{m}(z) + 0.5(z + \gamma_2)m(z)\right| \tag{14}$$

When the boundary conditions and the external force couple moment are constant, the magnitude of the torsion angle per unit length is only related to the magnitude of (B + C). It can be seen from the above that when the corrugated steel plate is used instead of the concrete slab, the shear stiffness of the cross-section will decrease [8], and the value of (B + C)will increase, so the torsional angle per unit length will also be larger, so the torsional performance of the beam will be weakened, which is consistent with the existing research conclusions.

4 Conclusion

In this paper, the Hamiltonian-Jacobian equation for the torsion problem of the corrugated steel web is given, and an approximate solution to the torsion problem is obtained. The accuracy of the proposed algorithm is verified by comparing with the calculated results of the existing literature. The results of this paper can well meet the needs of the project, and its engineering significance is as follows:

Although the use of corrugated steel plate has the advantages of light weight and good crack resistance, the torsional performance of the beam will be weakened to a certain extent, so special attention should be paid to the torsion-related strength and stiffness calculation in the engineering design to ensure the safe operation of the beam.

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