Chapter 3
Phenomenology and Scaling Theories

Abstract The statistics of the velocity and temperature differences, between measurements taken at two points separated by a distance l, can reveal the structure of turbulence. These structure functions often exhibit power laws or scaling laws in l. We introduce the important concept of energy cascade in turbulent flows and the different theories for the scaling behavior of the velocity and temperature fluctuations. We start with the scaling theory for non-buoyant turbulent flows and then discuss how the presence of buoyancy would affect and modify the scaling behavior. A crossover between the two types of scaling behavior is expected to occur at a length scale, the Bolgiano length, above which buoyancy is significant. Furthermore, there are corrections to these scaling theories due to the intermittent nature of turbulent fluctuations, and we discuss the idea of refined similarity hypothesis used to account for these corrections.

Keywords Energy cascade · Kolmogorov scaling · Four-fifth law · Obukhov–Corrsin scaling · Bolgiano–Obukhov scaling · Bolgiano length

3.1 Richardson's Energy Cascade

One important concept of fluid turbulence is the energy cascade introduced by Richardson [1]. Turbulent flows are dissipative thus energy input by external forces is required to maintain a turbulent fluid flow. The characteristic scale of this energy input is typically of the size of the system, known as the integral scale, denoted by l_0. On the other hand, the characteristic scale of energy dissipation by viscous effects, known as the dissipative scale and, denoted by l_d, is much smaller than the integral scale. As a result, there must be a transfer of energy from large to small scales. A transfer of energy between scales indicates an interaction between the Fourier modes of velocity of different wave numbers or scales. This is possible because of the nonlinear advection term $\vec{U} \cdot \vec{\nabla} \vec{U}$ in the Navier–Stokes equation. Richardson pictured this

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energy transfer as a cascade process. Large eddies of the integral scale are produced by the external forces. They are unstable due to the nonlinearity of the dynamics and break up into eddies of smaller scale. These smaller eddies are themselves unstable and break up into eddies of even smaller scale. This process continues until dissipative effects due to viscosity are significant, and the turbulent kinetic energy is then dissipated into heat. This picture of energy cascade is succinctly summarized in Richardson's famous rhyme [1]:

Big whorls have little whorls
Which feed on their velocity
And little whorls have lesser whorls
And so on to viscosity
(in the molecular sense)

An inherent feature of the cascade picture is that the energy transfer among scales is local, that is, the effective energy exchange between modes of different wave numbers decreases as the ratio of the wave numbers increases. It is expected that the statistics at the integral scale are determined by the mechanism of energy input and would vary from flow to flow. Because of the locality of the energy transfer, the statistics at small scales, scales further down the cascade and far from the integral scale, are not directly influenced by the mechanism of energy input. Thus this locality feature of the energy cascade allows for the possibility of universal characteristics for the statistics of small scales.

3.2 The Kolmogorov 1941 Theory

Based on Richardson's energy cascade, Kolmogorov developed in 1941 a phenomenological theory (K41) [2] for the statistics of velocity difference,

$$\delta\vec{U}(\vec{r},\vec{l}) \equiv \vec{U}(\vec{r}+\vec{l},t) - \vec{U}(\vec{r},t) \tag{3.1}$$

measured at the same time and at two positions separated by a displacement vector $\vec{l}$. There are several hypotheses in the K41 theory. We shall focus on two of them. The first one is an assumption of statistical homogeneity and isotropy of the small-scale turbulent motion when, the Reynolds number (Re) is sufficiently high and far from the boundaries. For statistically homogeneous fluctuations, $\delta U(\vec{r},\vec{l}) = \delta u(\vec{r},\vec{l})$ as $\langle\vec{U}(\vec{r},t)\rangle = \langle\vec{U}(\vec{r}+\vec{l},t)\rangle$. Thus under this hypothesis, the statistics of $\delta\vec{u}(\vec{r},\vec{l})$ do not depend on $\vec{r}$ nor the direction of $\vec{l}$ but depend only on $l = |\vec{l}|$ for $l \ll l_0$. The second assumption is that under the same conditions stated in the first assumption, there exists a range of intermediate length scales in which the statistics of $\delta\vec{u}(l)$ are uniquely and universally determined by the mean energy transfer rate and l. This range of intermediate length scales, $l_d \ll l \ll l_0$, is known as the inertial range. The locality of energy cascade makes it possible that the statistics in the inertial range to be universal. The mean energy transfer rate is equal to the mean energy dissipation rate as well as to the mean energy input rate. The mean energy dissipation rate is given by $\langle\epsilon\rangle$, where $\epsilon(\vec{r}, t)$ is defined by Eq. (1.29), and the ensemble average, can be taken as the spatial average in statistically homogeneous turbulent flows.

Using these two hypotheses and dimensional analysis, one therefore obtains

$$\langle\delta\vec{u}(l)\cdot\delta\vec{u}(l)\rangle = C\langle\epsilon\rangle^{2/3} l^{2/3} \tag{3.2}$$

where C is a universal constant. Since

$$\langle\delta\vec{u}(l)\cdot\delta\vec{u}(l)\rangle = 2[\langle\vec{u}\cdot\vec{u}\rangle - \langle\vec{u}(\vec{r}+\vec{l}, t)\cdot\vec{u}(\vec{r}, t)\rangle] \tag{3.3}$$

we obtain

$$\langle\delta\vec{u}(l)\cdot\delta\vec{u}(l)\rangle = 4\int_0^\infty E(k)\left[1 - \left(\frac{\sin kl}{kl}\right)\right] dk \tag{3.4}$$

using Eq. (2.55). As a result, Eq. (3.2) is equivalent to the result that the spatial energy spectrum $E(k)$ follows a $k^{-5/3}$ law over a suitable range of wave number k. There is good experimental support for the $-5/3$ power-law in the energy frequency spectrum $E(f)$ (see [5] for details). Similar arguments can be applied to give higher-order statistics of $\delta\vec{u}(l)$. The longitudinal velocity difference along the direction of the separation, denoted by $\delta u_{\|}(l)$, is given by

$$\delta u_{\|}(l) \equiv \delta\vec{u}(l)\cdot\frac{\vec{l}}{l} \tag{3.5}$$

Then we have

$$S_p(l) \equiv \langle[\delta u_{\|}(l)]^p\rangle = C_p\langle\epsilon\rangle^{p/3} l^{p/3} \tag{3.6}$$

for arbitrary $p > 0$. Here, $S_p(l)$ is known as the longitudinal velocity structure function of order p. A power-law dependence of $S_p(l)$ on l indicates that the inertial-range turbulent statistics are scale-invariant and Eq. (3.6) is a statement of the K41 scaling. For $p = 3$, an exact result can be derived from the Navier–Stokes equation for statistically homogeneous and isotropic flows:

$$S_3(l) = -\frac{4}{5}\langle\epsilon\rangle l \tag{3.7}$$

This exact result is known as the four-fifth law and was derived by Kolmogorov also in 1941 [3]. It is one of the very few exact results for turbulent flows, and will be discussed in details in the next section.

In the above discussion, the K41 scaling is obtained by dimensional analysis based on the requirement that in the inertial range $S_p(l)$ depends only on $\langle\epsilon\rangle$ and the scale l. Thus we have the same scaling behavior for the velocity structure function using $\delta u(l) = |\delta\vec{u}(l)|$:

$$S_p^u(l) \equiv \langle [\delta u(l)]^p \rangle \sim \langle \epsilon \rangle^{p/3} l^{p/3} \tag{3.8}$$

The K41 scaling can also be obtained by the requirement that the rate of energy transfer (per unit mass) is scale-independent in the inertial range. To see this, think of $\delta u(l)$ as the velocity of a turbulent eddy of scale l. The energy transfer rate (per unit mass) at scale l can be estimated as $[\delta u(l)]^2/t_l$. Here, t_l, known as the eddy turnover time, is the typical time for the eddy of size l to deform or change in energy, and can be estimated as $t_l = l/\delta u(l)$. Requiring the rate of energy transfer to be independent of scale l implies $[\delta u(l)]^3/l \sim const$ but the mean energy transfer rate has to be equal to the mean energy dissipation rate thus $const = \langle \epsilon \rangle$, hence

$$\frac{[\delta u(l)]^3}{l} \sim \langle \epsilon \rangle \tag{3.9}$$

This gives

$$\delta u(l) \sim \langle \epsilon \rangle^{1/3} l^{1/3} \tag{3.10}$$

which further implies Eq. (3.8).

The dissipative scale l_d can be estimated as the scale at which the rate of dissipation due to viscosity is comparable to $\langle \epsilon \rangle$:

$$\nu \left[\frac{\delta u(l_d)}{l_d} \right]^2 \sim \langle \epsilon \rangle \tag{3.11}$$

Then take l_d to be at the edge of the inertial range such that Eq. (3.10) holds at $l = l_d$. Eliminating $\delta u(l_d)$ from Eqs. (3.10) and (3.11) gives $l_d \sim (\nu^3/\langle \epsilon \rangle)^{1/4}$. The Kolmogorov dissipative scale η_K is defined as:

$$\eta_K \equiv \left(\frac{\nu^3}{\langle \epsilon \rangle} \right)^{1/4} \tag{3.12}$$

and typically l_d is of the order of $10\eta_K$.

Denote the normalized velocity difference by Y_l:

$$Y_l \equiv \frac{\delta u(l)}{\langle [\delta u(l)]^2 \rangle^{1/2}} \tag{3.13}$$

An important consequence of Eq. (3.8) is that all the moments of Y_l are independent of l. This l-independence follows directly from the proportionality of the scaling exponents $p/3$ of $S_p^u(l)$ to p, and further implies that the PDF of Y_l is independent of l. Thus the K41 theory predicts that the statistics of inertial-range turbulent velocity fluctuations are scale-independent and thus self-similar. Experiments confirmed the power-law dependence or scaling of $S_p^u(l)$ but show that the scaling exponents, defined by $S_p^u(l) \sim l^{\zeta(p)}$, depend on p in a nonlinear fashion. This deviation of the scaling behavior from the K41 prediction is known as anomalous scaling. Furthermore, the discrepancy between the observed scaling exponents $\zeta(p)$ and the predicted values of $p/3$ is known as intermittency corrections as the origin of the correction is believed to be due to the intermittent nature of turbulent fluctuations. The problem of anomalous scaling is a longstanding problem of turbulence and is remained to be solved. We shall discuss one particular idea, the refined similarity hypothesis, proposed by Kolmogorov and Obukhov in Sect. 3.7.

3.3 The Four-Fifth Law

We show in detail the derivation of the exact four-fifth law. We shall follow the treatment in [4] and [5]. In this subsection, we denote $\partial/\partial t$ and $\partial/\partial r_i$ by ∂_t and ∂_i, and adopt the Einstein notation of summation over repeated indices. We write the Navier–Stokes equation Eq. (1.1) with an external force in component form:

$$\partial_t U_i + U_k \partial_k U_i = -\frac{1}{\rho}\partial_i p + \nu \partial_k^2 U_i + f_i \tag{3.14}$$

Here, $\rho \vec{f}$ is the external force per unit volume. Denote quantities evaluated at $\vec{r}\,' = \vec{r} + \vec{l}$ by the same notations with a prime, e.g., $U_i' \equiv U_i(\vec{r}\,', t)$, and $\partial/\partial r_i'$ by ∂_i'. Then taking the ensemble average of the product of U_j' with Eq. (3.14) and U_i with Eq. (3.14) for U_j', we have

$$\begin{aligned} \partial_t \langle U_i U_j' \rangle = &-\partial_k \langle U_k U_i U_j' \rangle - \partial_k' \langle U_k' U_j' U_i \rangle - \frac{1}{\rho}\partial_i \langle U_j' p \rangle - \frac{1}{\rho}\partial_j' \langle U_i p' \rangle \\ &+ \nu(\partial_k^2 + \partial_k'^2)\langle U_i U_j' \rangle + \langle U_j' f_i \rangle + \langle U_i f_j' \rangle \end{aligned} \tag{3.15}$$

Here, we have used the interchangeability of taking derivative and ensemble average, incompressibility, and the derivative of primed quantities with respect to unprimed coordinates vanishes.

Consider turbulent flows that are statistically homogeneous and isotropic. Because of homogeneity and isotropy, the averages of the product of primed and unprimed quantities depend only on $l = |\vec{r}\,' - \vec{r}|$. Therefore,

$$\partial_i \langle \cdot \rangle = -\partial_i' \langle \cdot \rangle = -\partial_{l_i} \langle \cdot \rangle \tag{3.16}$$

Thus

$$\langle U_j' p \rangle = g(l) n_j \tag{3.17}$$

for some function $g(l)$ and $n_j \equiv l_j/l$. Incompressibility implies $0 = \partial_j' \langle U_j' p \rangle = \partial_{l_j} \langle U_j' p \rangle$, and using

$$\partial_{l_i} = \frac{\partial l}{\partial l_i}\partial_l = n_i \partial_l, \quad \partial_{l_i} = \frac{\partial n_k}{\partial l_i}\partial_{n_k} = \frac{1}{l}(\delta_{ik} - n_i n_k)\partial_{n_k} \tag{3.18}$$

we get

$$\frac{dg(l)}{dl} + \frac{2}{l}g(l) = 0 \Rightarrow g(l) = \frac{const}{l^2} \tag{3.19}$$

But $g(l)$ has to be finite at $l = 0$ thus $const = 0$ giving $g(l) = 0$ or $\langle U'_j p \rangle = 0$. Similarly, $\langle U_i p' \rangle = 0$. Define the velocity correlation and structure functions as follows.

$$b_{i,j} = \langle U_i U'_j \rangle \tag{3.20}$$

$$B_{ij} = \langle (U'_i - U_i)(U'_j - U_j) \rangle \tag{3.21}$$

$$b_{ij,m} = \langle U_i U_j U'_m \rangle \tag{3.22}$$

$$B_{ijm} = \langle (U'_i - U_i)(U'_j - U_j)(U'_m - U_m) \rangle \tag{3.23}$$

Statistical homogeneity and isotropy imply that these functions depend only on l. Moreover, $\langle U_i(\vec{r}, t) U_j(\vec{r} + \vec{l}, t) \rangle = \langle U_i(\vec{r} - \vec{l}, t) U_j(\vec{r}, t) \rangle = \langle U_i(\vec{r} + \vec{l}, t) U_j(\vec{r}, t) \rangle$, thus $b_{i,j} = b_{j,i}$ is symmetric in the indices i and j. Therefore, the most general forms for $b_{i,j}$ and $b_{ij,m}$ are:

$$b_{i,j} = A(l)\delta_{ij} + B(l) n_i n_j \tag{3.24}$$

$$b_{ij,m} = C(l)\delta_{ij} n_m + D(l)\left(\delta_{im} n_j + \delta_{jm} n_i\right) + F(l) n_i n_j n_m \tag{3.25}$$

The form in Eq. (3.25) takes into account the symmetry in the indices i and j. Homogeneity implies $\langle U'_j U'_k U_i \rangle = \langle U_j(\vec{r}, t) U_k(\vec{r}, t) U_i(\vec{r} - \vec{l}, t)$, which is equal to $-b_{jk,i}$ using Eq. (3.25). We write Eq. (3.15) for statistically homogeneous and isotropic turbulent flows:

$$\partial_t b_{i,j} = -\partial_k (b_{ki,j} + b_{kj,i}) + 2\nu \partial_k^2 b_{i,j} + \langle U'_j f_i \rangle + \langle U_i f'_j \rangle \tag{3.26}$$

Using Eq. (3.26), a relation between the second- and third-order longitudinal velocity structure functions can be derived and from this relation the four-fifth law follows.

As we are interested in the longitudinal structure functions, we let the x-axis be along the direction of $\vec{l}$. Then we take $i = j = 1$ in Eq. (3.26) and obtain

$$\partial_t b_{\parallel,\parallel} = -2\partial_k b_{k\parallel,\parallel} + 2\nu \partial_k^2 b_{\parallel,\parallel} + \frac{2}{3}\langle \vec{f} \cdot \vec{U} \rangle - \langle \delta f_\parallel \delta u_\parallel \rangle \tag{3.27}$$

where $\delta f_i = f'_i - f_i$. The subscript $\parallel$ denotes the component along the longitudinal direction along $\vec{l}$ and there is no summation over this direction. Next, we relate $\partial_k^2 b_{\parallel,\parallel}$ to $S_2(l)$. Now $S_2(l)$ can be written as

$$S_2(l) = \langle \delta u_\parallel^2 \rangle = B_{ij} n_i n_j = \frac{2}{3}\langle \vec{U} \cdot \vec{U} \rangle - 2b_{\parallel,\parallel} = \frac{2}{3}\langle \vec{U} \cdot \vec{U} \rangle - 2(A+B) \quad (3.28)$$

where we have used

$$\langle U_i' U_j' \rangle = \langle U_i U_j \rangle = \frac{1}{3}\langle \vec{U} \cdot \vec{U} \rangle \delta_{ij} \quad (3.29)$$

Using Eqs. (3.16) and (3.18), we get

$$\partial_k^2 b_{i,j} = \left(\frac{d^2 A}{dl^2} + \frac{2}{l}\frac{dA}{dl} + \frac{2}{l^2} B \right) \delta_{ij} + \left(\frac{d^2 B}{dl^2} + \frac{2}{l}\frac{dB}{dl} - \frac{6}{l^2} B \right) n_i n_j \quad (3.30)$$

The incompressibility condition gives $0 = \partial_j' b_{i,j}$, which implies

$$\frac{l}{2}\frac{d}{dl}(A+B) + B = 0 \quad (3.31)$$

Thus

$$\frac{dS_2}{dl} = -2\frac{d}{dl}(A+B) = \frac{4}{l}B(l) \quad (3.32)$$

and

$$\begin{aligned} \partial_k^2 b_{i,j} &= -\left(\frac{d^2 B}{dl^2} + \frac{4}{l}\frac{dB}{dl} \right) \delta_{ij} + \left(\frac{d^2 B}{dl^2} + \frac{2}{l}\frac{dB}{dl} - \frac{6}{l^2} B \right) n_i n_j \\ \Rightarrow \quad \partial_k^2 b_{\parallel,\parallel} &= -\frac{2}{l^4}\frac{d(l^3 B)}{dl} = -\frac{1}{2l^4}\frac{d}{dl}\left[l^4 \frac{dS_2}{dl} \right] \end{aligned} \quad (3.33)$$

Then we relate $\partial_k b_{k\parallel,\parallel}$ to $S_3(l)$. The incompressibility condition gives $0 = \partial_m' b_{ij,m}$. Using again Eqs. (3.16) and (3.18), we get

$$\left[\frac{dC}{dl} + \frac{2}{l}(C+D) \right] \delta_{ij} + \left[2\frac{dD}{dl} + \frac{dF}{dl} + \frac{2}{l}(F-D) \right] n_i n_j = 0 \quad (3.34)$$

Thus

$$\frac{dC}{dl} + \frac{2}{l}(C+D) = 0 \quad (3.35)$$

$$\frac{d(3C+2D+F)}{dl} + \frac{2}{l}(3C+2D+F) = 0 \quad (3.36)$$

Equation (3.36) is obtained by taking the trace of Eq. (3.34). The functions C, D, and F have to be finite at $l = 0$, thus

$$3C + 2D + F = 0 \quad (3.37)$$

We can then express D and F in terms of C and dC/dl:

$$D = -C - \frac{l}{2}\frac{dC}{dl} \tag{3.38}$$

$$F = l\frac{dC}{dl} - C \tag{3.39}$$

and obtain

$$b_{ij,m} = C\delta_{ij}n_m - \left(C + \frac{l}{2}\frac{dC}{dl}\right)(\delta_{im}n_j + \delta_{jm}n_i) + \left(l\frac{dC}{dl} - C\right)n_i n_j n_m \tag{3.40}$$

Thus

$$S_3(l) = \langle \delta u_{\parallel}^3 \rangle = B_{ijm}n_i n_j n_m \tag{3.41}$$

$$= 2(b_{ij,m} + b_{im,j} + b_{jm,i})n_i n_j n_m = -12C(l) \tag{3.42}$$

where we have used $\langle U_i' U_j' U_m' \rangle = \langle U_i U_j U_m \rangle$. Moreover,

$$-\partial_k b_{ki,j} = \left(-\frac{2}{l}C + 2\frac{dC}{dl} + \frac{l}{2}\frac{d^2C}{dl^2}\right)n_i n_j - \left(\frac{2}{l}C + 3\frac{dC}{dl} + \frac{l}{2}\frac{d^2C}{dl^2}\right)\delta_{ij}$$

$$\Rightarrow \quad -\partial_k b_{k\parallel,\parallel} = -\frac{1}{l^4}\frac{d}{dl}(l^4 C) = -\frac{1}{12l^4}\frac{d}{dl}(l^4 S_3) \tag{3.43}$$

Putting all the results together, we finally obtain

$$\frac{1}{6l^4}\frac{d}{dl}(l^4 S_3) - \frac{\nu}{l^4}\frac{d}{dl}\left(l^4\frac{dS_2}{dl}\right) = -\frac{2}{3}\langle\epsilon\rangle - \frac{1}{2}\partial_t S_2 + \langle \delta f_{\parallel}\delta u_{\parallel}\rangle \tag{3.44}$$

Here, we have used

$$\frac{1}{2}\partial_t\langle \vec{U}\cdot\vec{U}\rangle - \langle \vec{f}\cdot\vec{U}\rangle = -\langle\epsilon\rangle \tag{3.45}$$

which follows from Eq. (3.14). For decaying turbulence, $\vec{f} = 0$ and $\partial_t S_2 \approx 0$ for $l \ll l_0$. For stationary turbulence forced by $\vec{f}$ that acts only at the largest scales, $\partial_t S_2 = 0$ and $\langle \delta f_{\parallel}\delta u_{\parallel}\rangle \approx 0$ for $l \ll l_0$. Thus for both cases, we have

$$\frac{1}{6l^4}\frac{d}{dl}(l^4 S_3) - \frac{\nu}{l^4}\frac{d}{dl}\left(l^4\frac{dS_2}{dl}\right) = -\frac{2}{3}\langle\epsilon\rangle \tag{3.46}$$

for $l \ll l_0$. In the limit of $\nu \to 0$, the viscous term is negligible. On the other hand, it is assumed that $\langle\epsilon\rangle$ remains finite in this limit. This implies that the velocity gradients $\partial u_i/\partial r_j$ become unlimited as $\nu \to 0$ or as Re $\to \infty$, which further implies that vorticity is generated in turbulent flows and increases with Re. The result that the dissipation remains finite as Re $\to \infty$ is generally referred to as the "dissipative anomaly", and is well supported by experimental and numerical results. Thus in the limit of $\nu \to 0$, integrating Eq. (3.46) gives the four-fifth law Eq. (3.7). We note that for stationary homogeneous and isotropic turbulence with $\vec{f}$ acting over all scales, we have the more general result [6]

$$S_3(l) = -\frac{4}{5}\langle\epsilon\rangle l + \frac{6}{l^4}\int_0^l l'^4 \langle \delta f_\parallel(l')\delta u_\parallel(l')\rangle dl' \tag{3.47}$$

for $l \ll l_0$.

3.4 The Obukhov–Corrsin Theory for Passive Scalar

Obukhov [7] and Corrsin [8] extended Kolmogorov's 1941 theory to study temperature fluctuations in weakly-heated incompressible turbulent flows. The heating is so weak that the resulted temperature variations have no dynamical effect on the turbulent flow itself. As a result, the velocity field is still governed by the Navier–Stokes equation. In this case, the temperature is known as a passive scalar. The equations of motion are thus Eqs. (1.1) and (1.8). The Obukhov–Corrsin theory gives the statistics of the temperature difference, defined by

$$\delta T(\vec{r}, \vec{l}) \equiv T(\vec{r} + \vec{l}, t) - T(\vec{r}, t) \tag{3.48}$$

which is taken to be statistically homogeneous and isotropic. Besides the cascade of turbulent energy, there is also a cascade of temperature variance from large to small scales. The mean temperature dissipation rate is given by $\langle\chi\rangle$, where $\chi(\vec{r}, t)$ is defined in Eq. (1.30). In analogy to the K41 theory, the temperature variance transfer rate, estimated by $[\delta T(l)]^2/t_l$, is scale-independent and thus equals to $\langle\chi\rangle$ in the intermediate inertial-convective range, the range of scales within the inertial range where buoyancy is insignificant. That is,

$$\frac{[\delta T(l)]^2 \delta u(l)}{l} \sim \langle\chi\rangle \tag{3.49}$$

Together with Eq. (3.10) for $\delta u(l)$, we obtain

$$\delta T(l) \sim \langle\epsilon\rangle^{-1/6}\langle\chi\rangle^{1/2} l^{1/3} \tag{3.50}$$

and the Obukhov–Corrsin (OC) scaling for passive temperature fluctuations:

$$S_p^\theta(l) \equiv \langle[\delta T(l)]^p\rangle \sim \langle\epsilon\rangle^{-p/6}\langle\chi\rangle^{p/2} l^{p/3} \tag{3.51}$$

Here, S_p^θ is known as the pth order temperature structure functions. Experiments again confirm the power-law dependence but show that there are intermittency corrections to the OC scaling such that $S_p^\theta(l) \sim l^{\xi(p)}$ and $\xi(p)$ deviates from $p/3$ [9].

3.5 The Bolgiano–Obukhov Scaling

In turbulent convection, temperature variations result in a buoyancy force that drives the fluid motion, and temperature is now an active scalar. The presence of buoyancy could affect and modify the scaling behavior. In several theoretical studies [6, 10–13], arguments were given that buoyancy would give rise to a different scaling behavior:

$$S_p^u(l)\rangle \sim (\alpha g)^{2p/5} \langle \chi \rangle^{p/5} l^{3p/5} \tag{3.52}$$

$$S_p^\theta(l) \sim (\alpha g)^{-p/5} \langle \chi \rangle^{2p/5} l^{p/5} \tag{3.53}$$

This type of scaling behavior, which is known as the Bolgaino–Obukhov (BO) scaling, was originally proposed by Bolgiano [14] and Obukhov [15] for stably stratified flows (see also discussions in [16]) based on dimensional analysis and the argument that the velocity and temperature structure functions would depend only on αg, $\langle \chi \rangle$ and l. Here, αg is the additional parameter that describes the strength of buoyant coupling when buoyancy is significant. In turbulent Rayleigh–Bénard convection, the BO scaling can be obtained based on a cascade of temperature variance (Eq. (3.49)) or a cascade of entropy flux [12] (for $\theta \ll T_0$, $\int \theta^2 d^3x$ describes the entropy increase per unit mass and volume due to the temperature fluctuations [13]) together with the argument that the buoyant term dominates the dynamics and balances the nonlinear advection term:

$$\alpha g \delta T(l) \sim \frac{[\delta u(l)]^2}{l} \tag{3.54}$$

Equations (3.49) and (3.54) imply

$$\delta u(l) \sim (\alpha g)^{2/5} \langle \chi \rangle^{1/5} l^{3/5} \tag{3.55}$$

$$\delta T(l) \sim (\alpha g)^{-1/5} \langle \chi \rangle^{2/5} l^{1/5} \tag{3.56}$$

Then Eqs. (3.52) and (3.53) follow directly.

3.6 Crossover in Scaling

The BO scaling would hold only when buoyancy is significant. When buoyancy is negligible, temperature behaves as a passive scalar and K41 and OC scaling would hold. The buoyant term, estimated by $\alpha g \delta T(l) \delta u(l)$, increases with l. Thus one expects a crossover from the K41-OC scaling to the BO scaling to occur at the crossover scale l_c when

$$\langle \delta u(l_c)^{BO} \rangle = \langle \delta u(l_c)^{K41} \rangle \tag{3.57}$$

Using Eqs. (3.10) and (3.55), we get

$$l_c = \frac{\langle \epsilon \rangle^{5/4}}{(\alpha g)^{3/2} \langle \chi \rangle^{3/4}} \equiv L_B \tag{3.58}$$

Thus the crossover scale is given by L_B, which is known as the Bolgiano length and is the length scale above which buoyancy is important. The Bolgiano length was first defined in terms of αg, $\langle \epsilon \rangle$, and $\langle \chi \rangle$ using dimensional analysis [16], Furthermore, we have

$$\alpha g \langle \delta u(l) \delta T(l) \rangle \geq \langle \epsilon \rangle \qquad \text{for } l \geq L_B \tag{3.59}$$

therefore L_B is also the scale at which the power injected into the flow due to buoyancy is equal to the mean energy dissipation rate [17]. Using the exact relations Eqs. (1.41) and (1.42), L_B can be related to Nu and Ra:

$$L_B = \frac{\text{Nu}^{1/2}}{(\text{PrRa})^{1/4}} H \tag{3.60}$$

Hence, the picture emerging from these scaling theories is that the BO scaling is expected to hold in the buoyancy subrange, $l_0 \gg l > L_B$, while the K41-OC scaling is expected to hold in the inertial-convective subrange, $l_d \ll l < L_B$. If L_B is of the order of $l_0 \approx H$ or even larger, then only K41-OC scaling will be observed. On the other hand, if L_B is of the order of l_d or even smaller, then only the BO scaling would be observed [13]. However, there are two complications. The first complication is that turbulent Rayleigh–Bénard convection is inhomogeneous. Thus it is more appropriate to define a local crossover or Bolgiano length using the energy and thermal dissipation rates averaged over the local region of interest. As a result, it is possible that different scaling behavior is observed in different regions of the cell. This will be discussed in Chap. 4 when we examine the scaling behavior observed in experiments and numerical calculations. The second complication is the existence of intermittency corrections to the scaling behavior. In the next Section, we shall discuss one particular idea, the refined similarity hypothesis, which was proposed to account for the intermittency corrections.

3.7 Refined Similarity Hypothesis

To account for the intermittency corrections of velocity fluctuations, Kolmogorov proposed in 1962 [18] to refine his second hypothesis by replacing the mean energy dissipation rate $\langle \epsilon \rangle$ with a locally-averaged energy dissipation rate over a scale l, defined as

$$\epsilon_l(\vec{r}, t) \equiv \frac{3}{4\pi l^3} \int\limits_{|\vec{y}| \le l} \epsilon(\vec{x} + \vec{y}, t) d\vec{y} \tag{3.61}$$

Similar ideas were also proposed independently by Obukhov [19]. As a result of this refinement, which is known as the refined similarity hypothesis (RSH), Eq. (3.10) is modified to

$$\delta u(l)^{K41} \sim \epsilon_l^{1/3} l^{1/3} \tag{3.62}$$

$$\Rightarrow \qquad S_p^u(l) \sim \langle \epsilon_l^{p/3} \rangle l^{p/3} \tag{3.63}$$

Corrections to the K41 scaling can thus be resulted from the l-dependence of the moments of ϵ_l. In particular, let

$$\langle \epsilon_l^q \rangle \sim l^{\tau(q)} \tag{3.64}$$

then

$$\zeta(p) = \tau\left(\frac{p}{3}\right) + \frac{p}{3} \tag{3.65}$$

Different intermittency models have been proposed which give different results for $\tau(q)$.

A direct implication of Eq. (3.62) is

$$\langle [\delta u(l)]^p \, \big| \, \epsilon_l = x \rangle \sim x^{p/3} l^{p/3} \tag{3.66}$$

where $\langle [\delta u(l)]^p \, \big| \, \epsilon_l = x \rangle$ is the conditional velocity structure function of order p when the value of ϵ_l is fixed at a small range about x. Thus $\langle [\delta u(l)]^p \, \big| \, \epsilon_l = x \rangle \sim l^{p/3}$ exhibits the K41 scaling. Support for Eq. (3.66) has been found in both experiments [20] as well as in direct numerical simulations and large-eddy simulations [21].

The refined similarity hypothesis has been extended to temperature fluctuations by replacing also χ by the locally averaged $\chi_l(\vec{r}, t)$, which is similarly defined:

$$\chi_l(\vec{r}, t) = \frac{3}{4\pi l^3} \int\limits_{|\vec{y}| \le l} \chi(\vec{x} + \vec{y}, t) d^3 y \tag{3.67}$$

For passive temperature fluctuations, Eq. (3.50) becomes [22, 23]:

$$\delta T(l)^{OC} \sim \epsilon_l^{-1/6} \chi_l^{1/2} l^{1/3} \tag{3.68}$$

and for the BO scaling, Eqs. (3.55) and (3.56) become [24]:

$$\delta u(l)^{BO} \sim (\alpha g)^{2/5} \chi_l^{1/5} l^{3/5} \tag{3.69}$$

$$\delta T(l)^{BO} \sim (\alpha g)^{-1/5} \chi_l^{2/5} l^{1/5} \tag{3.70}$$

3.8 Conditional Structure Functions

We note the interesting observation that the dependence on χ_l is different for the two scaling behaviors, K41-OC and BO, as shown in Eqs. (3.62), (3.68), (3.69) and (3.70). This difference can be clearly spelled out by studying the conditional velocity and temperature structure functions evaluated at fixed values of χ_l:

$$\tilde{S}_p^u(l,x) \equiv \langle [\delta u(l)]^p \mid \chi_l = x \rangle \tag{3.71}$$

$$\tilde{S}_p^\theta(l,x) \equiv \langle [\delta T(l)]^p \mid \chi_l = x \rangle \tag{3.72}$$

We have used these conditional structure functions [24] and similar conditional structure functions evaluated at given values of local temperature variance transfer rate [25] to examine the validity of refined similarity hypothesis in turbulent Rayleigh–Bénard convection.

To evaluate $\tilde{S}_p^u(l,x)$ and $\tilde{S}_p^\theta(l,x)$ from Eqs. (3.62) and (3.68) in the case of the K41-OC scaling, we need to evaluate the conditional average $\langle \epsilon_l^q \mid \chi_l = x \rangle$ for various values of q. In this case, temperature is a passive scalar so we make use of the measured approximate statistical independence of ϵ_l and χ_l for passive scalar fluctuations [26] to approximate:

$$\langle \epsilon_l^q \mid \chi_l = x \rangle \approx \langle \epsilon_l^q \rangle \qquad \text{K41} - \text{OC} \tag{3.73}$$

As a result, we obtain

$$\tilde{S}_p^u(l,x) \sim \begin{cases} \langle \epsilon_l^{p/3} \rangle l^{p/3} & \text{K41} \\ (\alpha g)^{2p/5} x^{p/5} l^{3p/5} & \text{BO} \end{cases} \tag{3.74}$$

$$\tilde{S}_p^\theta(l,x) \sim \begin{cases} \langle \epsilon_l^{-p/6} \rangle x^{p/2} l^{p/3} & \text{OC} \\ (\alpha g)^{-p/5} x^{2p/5} l^{p/5} & \text{BO} \end{cases} \tag{3.75}$$

From Eqs. (3.74) and (3.75), we see the different x-dependence of $\tilde{S}_p^u$ and $\tilde{S}_p^\theta$ for the two different scaling behaviors: $\tilde{S}_p^u$ is independent of x for the K41 scaling but has a power-law dependence of $x^{p/5}$ for the BO scaling. Similarly, $\tilde{S}_p^\theta$ has the power-law dependence of $x^{p/2}$ for the OC scaling but a different dependence of $x^{2p/5}$ for the BO scaling. Hence it is possible to reveal the two different scaling behaviors by studying the x-dependence of $\tilde{S}_p^u(l,x)$ and $\tilde{S}_p^\theta(l,x)$. This method is particularly useful because the unknown intermittency corrections might hinder direct revelation of the scaling behavior. Details about this method will be discussed in Chap. 4.

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