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Arijit Chaudhuri
Sat N. Gupta
Rajkumar Roychoudhury *Editors*

A Tribute to the Legend of Professor C. R. Rao

The Centenary Volume

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A Tribute to the Legend of Professor C. R. Rao

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ISSN 2523-3114 ISSN 2523-3122 (electronic)
Indian Statistical Institute Series
ISBN 978-981-33-6990-0 ISBN 978-981-33-6991-7 (eBook)
https://doi.org/10.1007/978-981-33-6991-7

This Springer imprint is published by the registered company Springer Nature Singapore Pte Ltd.
The registered company address is: 152 Beach Road, #21-01/04 Gateway East, Singapore 189721, Singapore

Prof. C. R. Rao
b.10 Sept. 1920

Preface

In order to appropriately observe the centenary celebrations of an illustrious academic like Prof. C. R. Rao, preparations had to start in a timely manner. Since other important organizations and institutions were also expected to organize parallel celebrations, we felt it was not easy to find celebrated speakers and writers to agree to collaborate with us. But happily for us, Prof. Rao himself remained enthusiastic throughout and his advice and guidance were very critical. He gave us a short article to encourage us to go ahead with our project. He gave us an account of his bright career and emailed us a good many photographs, including his own.

We have included in this volume a copy of one of his early papers published by the Calcutta Mathematical Society which is well known among the statisticians around the world. We have annexed a copy of a permission for this insertion gathered from the present authorities of this publishing society.

The present President of Indian Statistical Institute (ISI) has also kindly blessed us with an article on Prof. C. R. Rao, and our present Director also has kindly done the same. Two former Directors of ISI have also contributed their write-ups. An article by one of them is a reproduction from a publication by him in a different source. Two documents authorizing us to implement this reproduction are also cited in this volume.

We were very delighted when the Abel Prize winner and the National Medal of Science (USA) winner Prof. S. R. Varadhan kindly came to Kolkata to give a talk on this occasion in our ISI Auditorium. He further gave us a write-up based on his speech which we have gratefully incorporated in this volume. This is a precious publication by our association.

We feel profound honor in putting together this edited book volume commemorating the centenary year of a legend, Prof. C. R. Rao. We are deeply indebted to all of the distinguished academics who contributed to this volume. We are also grateful to Prof. C. R. Rao himself for blessing this project. Academics like Prof. C. R. Rao appear only once in a generation, and we are extremely honored to play a small role in this commemoration.

Sincerely,

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Acknowledgements

The editors and the publishers gratefully acknowledge the help received from the Director, Indian Statistical Institute and the Reprography and Photography Unit of the Library, for permitting the insertion of the photographs in several articles by the contributing authors.

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<u>To Whom It May Concern</u>

Gathering that many contributing authors of articles being considered to be included in the forthcoming Springer publication of a Professor CR Rao Centenary Volume have inserted quite a number of photographs for illustrative purposes, I authorise hereby Professor Arijit Chaudhuri, one of its three editors, to negotiate with the Publishers telling them that I, as the Director of Indian Statistical Institute (ISI) , have no objection to insertion in the Volume of the photographs all gathered from the Photography & Reprography unit of our ISI.

Sanghamitra Bandyopadhyay

13 June 2020

(Sanghamitra Bandyopadhyay)
Director

About This Book

This centenary volume includes major presentations given during the five seminar sessions held at the Platinum Jubilee Building Seminar Hall of ISI on December 13, 2019, and January 17, 2020, and one more to be held in September 2020, but its content is included in the present volume in honor of Prof. Calyampudi Radhakrishna Rao, more popularly known as C. R. Rao, commemorating his centenary year, by the members of the Indian Statistical Institute Retired Employees Association. This volume also contains several contributions reflecting on the diverse aspects of the work of this august personality. The chapters pay tribute to C. R. Rao, the most decorated academic of our time. Two of the many honors conferred on C. R. Rao include the Padma Vibhushan Award by the Government of India (2001) and the National Medal of Science Award (2002) by the US President George W Bush. The book includes a chapter by Prof. S. R. Vardhan a former student of C. R. Rao and the Abel Prize winner (2007) and the National Medal of Science recipient (2010) from US President Barack Obama.

A Message from C. R. Rao

I am honored and touched that you are celebrating my centenary year. I appreciate your thoughtfulness and kindness.

I started my academic career with a master's degree in mathematics. In the early 1940s, the jobs in the field of mathematics were limited. I then stumbled into statistics by chance. Statistics was a relatively new subject in India then. I was one of the first five students to get a master's degree in statistics from Calcutta University. My career in statistics progressed as I got a Ph.D. in statistics from Cambridge University and I continued to work at the Indian Statistical Institute (ISI) in Calcutta. At the ISI, there was a rich, supportive environment to pursue research and to teach, in addition to my administrative responsibilities. After retirement from the ISI, I worked at several universities in the USA including the University of Pittsburgh and the Pennsylvania State University, which further enriched my experiences.

I was fortunate to have made some fundamental contributions to the field of statistics and to see the impact of my work in furthering research. I have some intellectual satisfaction for the esteem I earned from my peers in my field who introduced technical terms in statistical inference, attaching my name to them. My greatest contribution is the encouragement I provided to my Ph.D. students, 51 of them, some of whom have made outstanding contributions to statistics. They have in turn produced 649 Ph.D.'s to date. This is a matter of pride to me.

In my lifetime, I have seen statistics grow into a strong independent field of study based on mathematical, and more recently computational, tools. Its importance has spread across numerous areas such as business, economics, health and medicine, banking, management, physical, natural, and social sciences.

Statistics is the science of learning from data. Today is the age of data revolution. There is therefore a heightened need for statistics—both in terms of training in statistics to help analyze and interpret the data and in terms of research to answer new questions arising from the data. The demand for statisticians is growing worldwide. For instance, the US Bureau of Labor lists statistics as one of the fastest growing career fields—and predicts it will grow by 33% between 2016 and 2026.

Therefore, we have our work cut out for us as researchers and as teachers. This is the challenge for all of you.

Thank you very much for your efforts in celebrating my centenary year.

My best wishes and blessings to all of you—my friends and well-wishers.

C. R. Rao

CV of C. R. Rao with Photographs

Developer of Statistics as an Independent Discipline
C. R. Rao, Ph.D., Sc.D. (Cantab), FRS, Padma Vibhushan
Recipient of Indian and American Prestigious National Medals of Science
Honorary Life Member of International Biometric Society
Life Fellow of Kings College, Cambridge
39 Honorary Doctorate Degrees from Universities in Six Continents
Author of 15 Textbooks and 477 Research Papers
Date of Birth: September 10, 1920
Address: 29 Old Orchard St., Williamsville, NY 14221
Email: crr1@psu.edu, **Phone**: 716-817-6469

> *We study physics to solve problems in physics, chemistry to solve problems in chemistry, and botany to solve problems in botany. There are no statistical problems which we solve using statistics. We use statistics to provide a course of action with minimum risk in all areas of human endeavor under available evidence.*
>
> —C. R. Rao

Education

Sc.D.	Cambridge University
Ph.D.	Cambridge University (1945)
MA	Calcutta University (1943, first class, first)
MA	Andhra University (1940, first class, first)

Rao's school and college education was in India. He began his education joining the 3rd class in Gudur at the age of 5 and continued his school education in Nuzvid, Nandigama, and Visakhapatnam, all towns in Andhra Pradesh. In 1940, he received his MA degree in Mathematics with a first class and first rank from Andhra University in Waltair.

Rao joined the Indian Statistical Institute (ISI) in Calcutta on January 1, 1941, to continue his education. In a few months of being at ISI, he joined Calcutta University as a student in the newly started MA course in statistics. He obtained an MA degree in statistics in 1943 with a first class, first rank, gold medal, and record marks unbeaten for the last 85 years. Rao has the distinction of standing first in the final examination of all classes where he studied from primary school to university education.

Rao submitted a thesis to Cambridge University (CU) describing the new methods for analyzing multivariate data he developed during his 2 years stay (1946–1948) for the Ph.D. degree. He received the Ph.D. degree of Cambridge University in 1948.

A few years later, Cambridge University awarded Rao the higher doctorate degree Sc.D. based on peer review of his contributions to statistics. Kings College made him a life fellow of King's College, a rare honour.

Honorary Doctorate Degrees

Rao was awarded 39 honorary doctorate degrees from universities in 19 countries spanning six continents:

Asia—3 countries: **India** 14 degrees, **Philippines** 1 degree, **Sri Lanka** 1 degree
Australia—1 country, 1 degree
Africa—1 country, 1 degree
Europe—10 countries: **Russia**, **Greece**, **Poland**, **Spain**, **Cyprus**, **Germany**, **Finland**, **Portugal**, **Slovakia**, **Switzerland**, 11 degrees
North America—2 countries: **USA**, 6 degrees, and **Canada**, 2 degrees
South America—2 countries: **Brazil**, 1 degree and **Peru**, 1 degree

Research Contributions by Rao

Rao is the author of **475** research papers published in prestigious journals and **11** books, one of which, **Statistics and Truth**, was translated into French, German, Japanese, Main Land and Taiwan Chinese, Turkish, and Korean and another book **Linear Statistical Inference**, used as a text book, has been in the market for over 50 years and is available in Russian, German, Czech, Polish, Chinese, and Japanese languages.

He has edited 39 volumes of **Handbook of Statistics** dealing with the latest methodologies in statistics.

Breakthrough Papers in Statistics

Two of Rao's research papers published when he was 25 and 27 years of age have been reproduced in the book *Statistics Published in the Last Century*. One of the papers, "Information and Accuracy Attainable in the Estimation of Statistical Parameters," *Bull. Cal. Math. Soc.* 37, 81–91, 1945, published at the age of 25, generated the technical terms **Cramer–Rao inequality, Rao–Blackwellization, and Rao metric** and became the topic of research by mathematicians and quantum physicists.

Development of Multivariate Analysis (1946–1948)

At the invitation of Royal Statistical Society, Rao presented a discussion paper describing the new multivariate methods he developed in solving the *Jebel Moya* problem. The paper with discussion was published in the *Journal of Royal Statistical Society*, "Utilization of Multiple Measurements in Problems of Biological Classification." *J. Roy. Statist. Soc.* 10, 159–203, 1948. Some new multivariate tests Rao used in analyzing Jebel Moya data such as multivariate analysis of variance (MANOVA) are described in his paper, "Tests of Significance in Multivariate Analysis," *Biometrika*, 35, 58–59, 1948. These two papers provide the **foundations of multivariate analysis**. They are discussed in Rao's book, *Advanced Statistical Methods*, John Wiley, 1952.

Rao's Quadratic Entropy

Rao introduced quadratic entropy (QE) as a general measure of diversity or dissimilarity within a population. In a series of papers, he laid out mathematical foundations, statistical analysis, and practical applications and usefulness of QE. His new and unified approach is useful for performing analysis of variance (ANOVA) type of analysis of both **qualitative and quantitative data**. He also developed QE-based cross-entropy to measure and analyze differences between two distributions and established their important properties.

Rao's work on quadratic entropy and cross-entropy has made extraordinary impact on theoretical and applied research in many fields. Quadratic entropy has been used widely to measure and analyze diversity in biology, ecology, economics, sociology, bioinformatics, linguistic, and other fields. Discussions and applications of QE have appeared in many books and articles in a very wide range of journals.

His work on quadratic and cross-entropy has also stimulated significant theoretical research in information theory, differential geometry, asymptotic inference, quantum chemistry, econometrics, image processing, and other areas. His work has been often cited in articles in top-tier journals in such fields.

As far as diversity measurement is concerned, quadratic entropy is most relevant when applied to ultra-metric dissimilarities. The key paper on diversity measures is as follows.

Technical Terms Appearing in Text Books on Statistics and Engineering

Rao published about 55 research papers during the 10-year period 1940-1950 which gave rise to technical terms in his name:

Cramer–Rao inequality, Rao–Blackwellization, Rao metric, Rao's U test, multivariate analysis of variance (MANOVA), Orthogonal **arrays (described by Ford Magazine as a new manthra for American industries) used in designing products and producing goods of high quality),** ***Rao's paradox in multivariate analysis and sample surveys*****, and** ***generalized inverse of matrices***.

Additional Technical Terms Bearing Rao's Name in Specialized Books

The 1945 paper of Rao received attention from mathematicians, and they published a number of papers related to the results of Rao's papers. Their contributions resulted in the technical terms:

Quantum Cramer–Rao inequality, Cramer–Rao functional, Fisher–Rao theorem, Rao's theorem on second-order efficiency, and stereological Rao–Blackwell theorem.

Subsequent research by Rao gave rise to technical terms:

Burbea–Rao phi-entropy and divergence measures, Rao's quadratic entropy, Rao–Rubin, Lau–Rao, Lau–Rao–Shanbhag, Kagan–Linnik–Rao theorems on characterization of probability distributions, Hamming–Rao bound and Hamming–Rao construction, Geary–Rao theorem, and integrated Cauchy functional equation.

Technical terms bearing Rao's name arising out of papers by other authors:

Global Bayseian Cramer–Rao bound, Quantum Cramer–Rao bound, Cramer–Rao functional, stereological Rao–Blackwell theorem, Rao–

Blackwellized particle filters, quantum Cramer–Rao bound, and Rao measure.

Positions Held

Indian Statistical Institute, in various capacities (1941–1979)
Visiting Professor at University of Illinois (1951–1952)
National Professor of India (1987–1992)
University Professor, University of Pittsburgh (1979–1988)
Eberly Professor of Statistics, The Pennsylvania State University (PSU), PA, USA (1988–2001)
Eberly Professor Emeritus (PSU) (2001–)
Director, Centre for Multivariate Analysis, PSU (2001–2010)
Research Professor, University at Buffalo, SUNY (2010–)

He retired from active service at the age of 80 from The Pennsylvania State University, PA, USA, and continues to hold honorary Professorship at The Pennsylvania State University, PA, USA, and at the University at Buffalo, SUNY, USA.

Prestigious Awards Received by Rao

Rao received numerous prestigious awards in India, USA, and UK

1. **Shanti Swarup Bhatnagar Award**, a prestigious award for a scientist in India, **1963**
2. **Wilks Medal**, the highest award for a statistician in USA, **1989**
3. **Mahalanobis Birth Centenary Medal**, **1996**
4. **Army Wilks Medal**, **2000**
5. **National Medal of Science awarded by the President of USA**, as a prophet of new age “is the highest award given to a scientist in USA” **2002** (National Medal of Science Laureate)
6. **International Mahalanobis Prize**, **2003**
7. **India Science Award (Gold Medal)**, the highest award given to a scientist in India, **2010**
8. **Guy Medal in Gold** awarded by the president of Royal Statistical Society, the highest award given to a statistician in UK, **2010**: **C. R. Rao is first non-European and non-American to receive the award**.
9. **Neyman Medal** from the Polish Statistical Society, **2014** for outstanding contributions to statistics
10. **Megnad Saha Medal**, **1969**
11. **Honorary Fellowship of Kings College, Cambridge**, **1975**

12. **Jagadish Chandra Bose Medal** of Bose Institute, **1979**
13. **Ramanujan Medal from Indian National Science Association**, **2003**
14. **Army Wilks Medal, American Statistical Association**, **2000**
15. **Gold Medal from Calcutta University**, **1943**
16. **Rao was elected as Fellow of Royal Society (FRS), UK**, **1967**
17. **Fellow of National Academy of Science, USA**, **1995**
18. **Fellow of Third World Academy of Science**, **1983**
19. **Fellow of Lithuanian Academy of Science**, **1997**
20. **Member of Prometheus Society**, **1996**
21. **International Mahalanobis Prize**, **2003**
22. **Fellow of European Academy of Science**, **2009**
23. **Sirdar Patel Lifetime Achievement Award**, **2015**
24. **Fellow of Indian National Science Academy**
25. **The Institute of Combinatorial Mathematics and its Applications elected C. R. Rao as an Honorary Member with the citation**: "*as the world's leading expert in statistical design theory*."
26. "**The Department of Statistics and Programme Implementation (DOS&PI) of India instituted a National Award in honor of C. R. Rao, the renowned statistician of the country. The award given once in two years carries an amount of one lakh rupees and is reserved for young statisticians for the work done during the preceding three years in any field of statistics**."

Presidentship of Statistical Societies

1971–1976: The Indian Econometric Society (TIES), India
1973–1975: International Biometric Society (IBS)
1976–1977: Institute of Mathematical Statistics, (IMS), USA
1977–1979: International Statistical Institute (ISI)
1982–1984: Forum for Interdisciplinary Mathematics, USA

A Place in the History of Statistics as One of the Founders of Statistics (Information from Websites)

"Rao's early contributions to statistical theory earned for him a place in the history of statistics." (Website: **Founders of Statistics**, Pathak/Founders/.)

C. R. Rao is the only Asian included in the following accounts of History of Statistics, with photographs and brief description of the contributions.

1. Figures from the history of Probability and Statistics: John Aldrich, University of Southampton, UK, describes the work of 35 major contributors to the development of statistics since 1650.
2. Statisticians in History produced by the American Statistical Association has a list of 40 probabilists and statisticians.
3. Chronology of probabilists and statisticians listed by the University of Texas has a galaxy of 57 famous scientists (Newton, Kepler, Gauss, Galilio, Fisher, and Erdos to name some) during the period sixteenth and twentieth centuries arranged in the order of their birth years starting from 1501. (Photograph of main contributors is listed with other photographs issued separately)
4. 215 influential developments in statistics by Anirban Dasgupta, IMS Bulletin, Vol 42, issue 8, 10–11. Seven papers by Rao are quoted.

Career Path

Long Career at the Indian Statistical Institute (ISI)

Professor of Statistics at the Age of 29

Rao (with 2 MA first-class degrees) was offered and accepted the post of "Technical Apprenticeship" in ISI in 1943, at Rs. 75 a month. He worked in ISI starting at the age of 23 until the compulsory retirement age of 60. While at ISI, Rao worked in various capacities, as Technical Apprentice (1943–1944), Acting Statistician (1944–1948), Superintending Statistician (1948–1949), **Professor at the age of 29**, Head of Research and Training School (1949–1972), Secretary and Director (1972–1976), Jawaharlal Nehru Professor (1976–1984), and National Professor (1987–1992).

It is unusual for a person of 29 years to be appointed as a full professor in India. The following government notification refers to it:

> *Government has appointed C. R. Rao, an eminent statistician, as National Professor. Professor Rao is an outstanding and creative thinker in the field. He was appointed by Prof. Mahalanobis as full-fledged professor at the early age of 29 in recognition of his creativity.*
>
> —Government of India notification, 1987

Developer of Statistics as an Independent Discipline

When Rao started doing research, statistics was not considered as an independent discipline. Rao has the distinction along with some distinguished statisticians in Europe and USA as the developer of statistics as an independent discipline. (Photographs of developers of statistics are appended to this CV)

Indian School of Statistics

As the head of Research and Training School in ISI, C. R. Rao developed a number of courses in statistics leading to BSTAT and MSTAT degrees. He trained several students with specializations in different areas of theoretical and applied statistics.

In 1948, Rao started the Ph.D. program in statistical theory at the ISI and provided opportunities for research work of 51 students over a course 25 years for Ph.D. degree, who in turn produced over 576 Ph.D.'s up to the year 2016. (Website: **Mathematical Genealogy of CRRao**).

D. Basu, the first doctoral student (1948–1950), was the first Ph.D. produced in India in theoretical statistics. S. R. S. Varadhan, a Ph.D. from ISI, is the winner of the prestigious Abel Prize in Mathematics. Another Ph.D., V. S. Varadarajan, made valuable contributions to mathematics. Most of the other Ph.D.'s have become renowned researchers holding distinguished academic positions.

Rao developed the International Statistical Educational Center (ISEC) in ISI.

Under the guidance of International Statistical Education Center for training statisticians from developing countries. He was chairman of ISEC for a number of years.

Invitation from Cambridge University, UK

The Anthropology Department (AD) of Cambridge University (CU), UK, sent an expedition to *Jebel Moya* in Africa to dig out ancient graves and bring the skeletons of people buried there for study. The department wanted to analyze the measurements taken on the skeletons to determine the relationship of the people who lived there with people currently living there or in nearby areas. There were no multivariate statistical methods developed at that time to treat such problems, involving multiple measurements. Professor Mahalanobis, Director of Indian Statistical Institute (ISI), had suggested a method, called Mahalanobis Distance, to treat such issues. In July 1946, Dr. J. C. Trevor, Professor of Anthropology Department, Cambridge University (CU), sent a cable to Professor Mahalanobis to send someone from ISI to analyze the measurements on *Jebel Moya* skeletons using the methodology developed at the ISI.

Rao had developed some new multivariate methods for analyzing such multivariate data at the ISI and had the experience of solving such problems. Rao was selected along with R. K. Mukherji, an anthropologist, to go to Cambridge to take measurements on skeletons and analyze the data. They went to Cambridge and worked in the Anthropology Museum in Duckworth Laboratory for two years (1946–1948) as visiting scholars with pay. Rao had to develop some new multivariate methods to analyze the data. The results based on the analysis of measurements were reported in the book, *Ancient Inhabitants of Jebel Moya*, published by the

Cambridge University press in **1954** under the joint authorship of Rao and two anthropologists J. C. Trevor and R. K. Mukherji.

Work in USA

After taking compulsory retirement in India at the age of 60 with 40 years of service at ISI, Rao tried to get a suitable job in India to continue his research activities without administrative responsibilities. As opportunities to work in India were denied, he accepted unsolicited offers of distinguished professorships in USA.

He worked for another 25 years as University Professor at the University of Pittsburgh and Eberly (Chair) Professor of Statistics at The Pennsylvania State University continuing his research in diverse areas of statistics. He retired from active service at the age of 81 but continued his research activities as Eberly Professor Emeritus and Director of the Center for Multivariate Analysis at The Pennsylvania State University. In 2010, he moved to Buffalo accepting Research Professorship at the University at Buffalo. He published 201 research papers during his 40 years stay in India and 272 research papers during his 30 years of stay in USA.

C. R. Rao Advanced Institute of Mathematics Statistics and Computer Science (CRRAO AIMSCS)

C. R. Rao wrote a letter in December 2000 to Chandrababu Naidu Garu, the Chief Minister (CM) of Andhra Pradesh (AP) at that time and well known for making Hyderabad the technological capital of India, quoting statistics to show that research publications by scientists in AP were not at the same level as in Tamil Nadu and Mumbai, which may be due to the existence of research institutes IMS in Chennai and TIFR in Mumbai. He sought his help to establish a research institute in Hyderabad similar to IMS and TIFR. Rao did not meet him personally but the CM replied within a month sanctioning Rs. 50 Lakhs to construct a building for a research institute. Rao approached Dr. Hasnain, the Vice Chancellor of University of Hyderabad (UoH), and requested him to give 5 acres of land in the campus of UoH to construct the building. Rao had not known Dr. Hasnain earlier and so was surprised that he readily agreed to give 5 acres of land in the campus of UoH. The university engineer thought Rs. 50 lakhs was not sufficient to construct a building. Mrs. C. R. Rao (Bhargavi Rao) donated Rs. 25 lakhs from her personal earnings. With Rs.75 lakhs, a small building, named as *Aryabhatta*, was erected in the campus of HU. The Chairman of the Governing Body of the institute registered the Institute as CRRAO Advanced Institute of Mathematics Statistics and Computer Science.

The institute started functioning in *Aryabhatta* in early 2009. A few years later, Rao approached Dr. Manmohan Singh, the prime minister at that time, for help. He kindly sanctioned eight crores of rupees. With this money, the main building named as Ramanujan was constructed. The institute started functioning in Ramanujan from 2012. It took 10 years of hard work by Rao to have a research institute established in Hyderabad. (Photograph of the building Ramanujan is attached to this CV). In *Ramanujan*, C. R. Rao Gallery, designed by Rao's daughter, Dr. Tejaswini Rao, was inaugurated by Nobel Laureate Sir. Venkataraman (Venki). The gallery is open to visitors and students.

The Institute has qualified staff to do research in cryptology, statistics, and biometry. The institute conducts courses and organizes conferences on various aspects of statistics

Family

Calyampudi Radhakrishna Rao (C. R. Rao) was born on September 10, 1920, in Hadagali, a small town in Madras Presidency, during the British rule. He is the eighth child among 10 siblings (6 brothers and 4 sisters) to his parents, C. D. Naidu, a detective inspector of police, and Mrs. Naidu (Laxmikantamma). One of Rao's sisters, Sanjeevamma, with no school or college education, was a Telugu poetess and one of his brothers, C. V. Rao known for his enormous memory, was a gold medalist in anatomy from Andhra University.

Peer Reviews

1. **Bharat Ratna Abdul Kalam**, former president of India, in a speech delivered at the foundation stone laying ceremony of Sankhya, Museum of Statistics:

   "*One could say if Europe is the mother of differential calculus based on deterministic analysis, India could be called the mother of statistics. When I think of modern statistics, Dr. C. R. Rao features on the top of the list. He once said that "statistics is the technology of finding the invisible and measuring the immeasurable.*"

2. **Julian Champkin**, Editor of *Significance* (the official publication of The Royal Statistical Society, UK, and American Statistical Association, USA), Vol 18, No. 4, Nov 2011, 175–178, 2011 in the article, "*C. R. Rao, A Life in Statistics*", states:

   "*C. R. Rao is a great name from the golden age of statistics. His work was done in India and his intellect shaped statistics worldwide.*"

3. **S. Amari**, world renowned mathematician from Japan states:

   "***Dr. Rao** is a great scholar beyond the framework of statistics, which he himself has founded. It is a big surprise to see that his influence has been effective and* *has played a great role for more than half a century to produce fruitful developments in several scientific disciplines. The idea of connecting statistics and differential geometry was too early at that time (1945). However, after nearly half a century,* ***Rao's*** *idea has been developed to become one of most active and important topics in information sciences, connecting statistics, information theory, control and statistical physics.*"

4. **Anuj Srivastava**, Professor, Department of Statistics, Florida State University:

   "*I find the fact that under the Fisher-Rao metric, the eparameterization group acts on the function space by isometries, makes this metric one of* ***the most important tools in functional analysis***."

5. In an article entitled **The Statistical Century** published in the *Royal Statistical Society News* (Vol 22, Jan 1995), the distinguished American statistician **Bradley Efron**, states:

   "*Karl Pearson's famous chi-square paper appeared in the spring of 1900, an auspicious beginning to a wonderful century for the field of statistics. The first half of the century was the golden age of statistical theory, during which our field grew from ad hoc origins, similar to the current state of computer science, into a firmly grounded mathematical science. Men of the intellectual calibre of Fisher, Neyman, Pearson, Hotelling, Wald, Cramer and Rao were needed to bring statistical theory to maturity.*"

6. **Terry Speed**, Professor, University of Berkeley, writes in the Institute of mathematical Statistics Bulletin, Jan–Feb, 2010 issue:

   "*The 1940's were ungrudgingly C. R. Rao's. His 1945 paper, which contains the Cramer-Rao Inequality, Rao-Blackwell Theorem, and the beginning of differential geometry of parameter space will guarantee that, even had he done nothing else-but there was much else.*"

7. **Frank Nielsen**, Professor in Computer Science, Ecole Polytechnique, France, in his paper on *Cramer-Rao Lower Bound and Information Geometry* published in *Connected at Infinity*, 2, 18–37:

   "*The article focuses on an important piece of work of the world renowned Indian statistician, Calyampudi Radhakrishna Rao. In 1945, C. R. Rao (25 years old then) published a path breaking paper, which had a profound impact on subsequent statistical research. It opens up a novel paradigm by introducing differential geometric modeling ideas to the field of statistics. In recent years, this contribution has led to the birth of a flourishing field of Information Geometry.*"

8. In his book on "*Reminiscences of a Statistician*" (2008, Springer), **Prof. Erich L. Lehmann**, Professor, University of Berkeley and member of US National Academy of Sciences says:

"*… several of my early papers grew out of Rao's paper of 1945, and at the time of writing this section I am in fact also working on a paper that has its origin in Rao's work in the 1960s on efficiency.*"

9. A review of C. R. Rao's book, **Linear Statistical Inference and its Applications**, by the famous statistician, **W. G. Cochran** in the *Journal of the Franklin Institute* states the following:

   "*C. R. Rao would be found in almost any statistician's list of the five outstanding workers in the world of Mathematical Statistics today. His book represents a comprehensive account of the main body of results that comprise modern statistics theory.*"

10. **B. Efron**, president of the American Statistical Association mentions in the issue 327, September 2004, of AMSTAT while introducing the article by C. R. Rao on *Reflections on the past and visions for the future*:

    "*C. R. Rao, Eberly Professor Emeritus in the Stat. Dept. at Penn State is a towering figure in the postwar development of statistical theory. Among his great many honors, he was recently awarded the National Medal of Science, the (US) government's highest scientific prize.*"

11. **S. Karlin**, mathematician who won US National Medal of Science medal:

    "*C. R. Rao is among the world wide leaders in statistical science over the last five decades. His research, scholarship, and professional service have had a profound influence in the theory and applications of statistics and are incorporated into standard references for statistical study and practice. C. R. Rao is not only a highly creative theoretician but was attracted and labored with many data sets in health, biology, psychology and social sciences.*"

12. **P. Armitage**, Professor of Statistics, Oxford University, UK, writes in a review of *Statistical Analysis and Inference* (Ed. Y. Dodge):

    "*C. Radhakrishna Rao is a polymath amongst statisticians. …Rao's research interests include social, industrial and economic applications. He has been (and still is) an influential teacher especially in third world countries. …The group of papers are interspersed with quatrains from Rubaiyat of Omar Khayyam of Naishapur, whose hedonistic nihilism seems to accord ill with C. R. Rao's outlook:" Myself when young did eagerly frequent/ Doctor and Saint, and heard great Argument/ About it and about: but evermore/ came out by the same door as I went". Those of us who have frequented C.R.'s company have invariably found new doors open.*"

13. **Sir David Cox**, Nuffield College, Oxford

    "*Your pivotal ranging contributions to our field have had a major impact on my own and others work.*"

14. Press release by **Government of India** on appointment as **National Professor** (limited to 12 at any time)

    Government has appointed C. R. Rao, an eminent statistician, as National Professor. Professor Rao is an outstanding and creative thinker in the field. He was appointed by Professor Mahalanobis as full-fledged professor of the Indian Statistical Institute at the early age of 29 in recognition of his creativity.

15. **Times of India** dated 31 December 1988 chose C. R. Rao as one of the 10 top scientists of India; the list includes the outstanding scientists, J. C. Bose, S. N. Bose, S. Ramanujan, Harishchandra, H. Khurana (NL), C. V. Raman (NL), S. Chandrasekhar (NL), Salim Ali, and G. N. Ramachandran.

16. The introduction to *Glimpses of India's Statistical Heritage* (Wiley Eastern, edited by **J. K. Ghosh**, **S. K. Mitra**, and **K. R. Parthasarthy**) notes about Rao:

    "*As the Head of the Research and Training School of the Indian Statistical Institute during 1949–1963, he single-handedly created an international renowned team of young statisticians, economists and mathematicians who are currently holding prominent positions in several universities all over the world. Thanks to the well-known Cramer-Rao Bound and Rao-Blackwellization, he is a household name in the family of statisticians and now electrical engineers too.*"

17. **M. L. Straff**, President of American Statistical Association, in his presidential address *The Next Generation* published in JASA, 98, 461, 2003 said:

    "*This year, C. R. Rao was honored by our President with the National Medal of Science. We are all proud that a statistician was so recognized. Rao's citation refers to the impact of his work on the physical, biological, economics, and engineering sciences, showing the importance given to statistics for enriching other sciences.*"

18. The booklet on some famous scientists of modern India brought out on the occasion of the Open House for school students at the **Tata Institute of Fundamental Research**, on 2 November 2003 features C. R. Rao as one of those scientists "*who have been instrumental in building up the vast and rich scientific culture of modern India. All those chosen here have contributed immensely to science. Almost all have been great institution builders.*

    The compelling factor for the choice of these few, from among the many in the country, is the time and circumstances under which they worked. Their achievements are nothing short of heroic. With no infrastructure and with little support from the Government of the day, they have built up world class scientific institutions and a scientific heritage we can be proud of. The institutions they built still stand proud and are rated highly by the scientific community here and abroad."

19. **The Journal of Quantitative Economics** published a special issue in Rao's honor in 1991. The preface gives the following tribute:

 "*Dr. Rao is a very distinguished scientist and a highly eminent statistician of our time. His contributions to statistical theory and applications are well known, and many of his results, which bear his name, are included in the curriculum of courses in statistics at bachelor's and master's level all over the world. He is an inspiring teacher and has guided the research work of numerous students in all areas of statistics. His early work had greatly influenced the course of statistical research during the last four decades. One of the purposes of this special issue is to recognize Dr Rao's own contributions to econometrics and acknowledge his major role in the development of econometric research in India.*"

20. **The Institute of Combinatorial Mathematics and Its Applications** elected C. R. Rao as an Honorary Member with the citation:

 "*as the world's leading expert in statistical design theory.*"

21. **Czeslaw Domanski**, President of Polish Statistical Society writes in a letter dated 8-07-2014:

 "*By sending you the Jerzy Splawa-Neyman Medal, we are sending our best words of appreciation and admiration of your extraordinary scientific work and your invaluable commitment to the statistical education all over the world.*"

22. **D. Irini Moustaki and D.S. Deman, 2011**

 "*Professor C. R. Rao is one of country's foremost statisticians, entered statistics by chance and went on to become a household name in the field. He has achieved the golden age of 91 and today, he is a living legend who has transcended not just statistics, but has profound influence on other disciplines which have far reaching implications for fields like varying from economics, econometrics, genetics, anthropology, agronomy, geology, national planning, demography, biometry, biology to medicine … Rao is an institution unto himself and unique intellectual tradition leaving the lasting impact on generations. We are fortunate in having him midst us.*"

23. **Hon. D.Sc. by the Ohio State University, Honorary Doctorate citation**:

 "*Among the international community of scholars, you are widely acknowledged as one of the world's foremost statisticians. In the complex realms of statistics and higher mathematics, your research and scholarly writing have opened new doors of understanding. The statistical theories and applications, which bear your name, attest to the fundamental contributions you have made to your field and to the larger body of man's knowledge. Numerous honors and awards have followed, in tribute to an unusually distinguished and productive life of inquiry. You have earned the highest accolade of all, the esteem of your peers throughout the world of scholarship.*"

24. **R. A. Fisher**, the father of modern statistics said in his speech at the Indian Statistical Institute (ISI) on 12 February 1963:

"*Mathematicians of many nations have contributed to our knowledge of this very intriguing field, but I do not exaggerate if I say that Indian names are as numerous as all other together, and this must particularly be ascribed to the fact that early in the institute's history several of the young mathematicians were brought into intellectual contact by the institute's activity, found in this subject a type of problem ideally suited to their gifts.*"
"*The implementation of this broad educational policy seems to be extremely difficult. For its educational programs the institute needs not only leaders in mathematical thought like **Prof. Rao** who can uphold and maintain the high place in world opinion that Indians have already won....*"

Statistics in India

Development of Statistics in India: C. R. Rao worked at the Indian Statistical Institute for a period of 40 years in various capacities as Professor, Director, Jawaharlal Nehru Professor, and National Professor. During most of this period, there was no separate minister for statistics. All problems relating to the development of statistics in India were under direct administrative control of the Prime Minister Nehru as he was greatly interested in the development of statistics in India. He visited the ISI a number of times at the invitation of Prof. Mahalanobis, the founder of ISI. Rao had the opportunity to discuss the national statistical system and training of statisticians to work in statistical bureaus with Pandit Nehru. His successor Prime Minister Smt. Indira Gandhi was also interested in the development of statistics. Rao was made a member of Committee on Science and Technology (COST) in 1962 and chairman of National Committee on Statistics in 1969. He had several opportunities to discuss the development of research activities in India with Smt. Indira Gandhi.

Groundbreaking Statistical Concepts

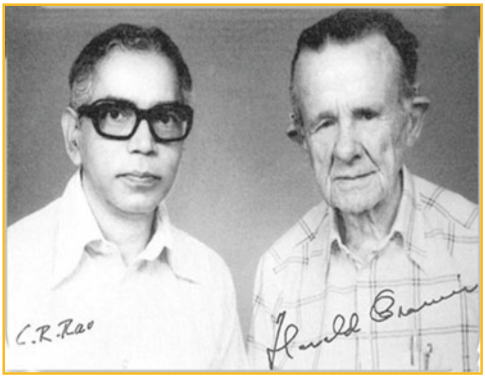
Information and accuracy attainable in the estimation of statistical parameters.
—Bull. Cal. Math. Soc., 37, 81–91 (1945)

The paper was reproduced as a separate chapter in the book, *Breakthroughs in Statistics*: 1890–1990, vol. 1, editors—S. Kotz and N. Johnson, John Wiley. The paper contains numerous results that bear Rao's name (as shown below) and includes groundbreaking concepts.

Rao with R. A. Fisher
Fisher-Rao Metric Named by Brody and Houghston in 2001

R.A. Fisher
(17 February, 1890 - 29 July, 1962)

R.A. Fisher, the most celebrated statistician of the 20th century, was a genius who almost single-handedly created the foundations of modern statistics. Rao was one of his few research scholars in statistics between 1946-1948 in Cambridge, UK



Rao with Harald Cramér
Cramér-Rao Bound
Named by J. Neyam and
E. Scott in 1948

Harald Cramér
(25 September, 1893 –
5 October, 1985)

Rao with David H. Blackwell
Rao-Blackwellization
Named by E. Lehman and
H. Scheffe in 1950

David H. Blackwell
(24 April, 1919 - 10 July, 2010)

Place in the History of Statistics

Rao's contributions to the development of statistics as an independent discipline earned for him a place in the history of statistics as one of the founders of statistics. Rao is the only Asian mentioned in all reports (websites on history of statistics) on main contributors to the development of statistics.

- Rao's major contributions to statistics leading to several technical terms incorporated into textbooks on statistics were made in the 1940s while working at the Indian Statistical Institute.
- Figures from the history of probability and statistics by Professor John Aldrich, University of Southampton, UK, describing the work of 35 major contributors to probability and statistics since 1650 includes Rao.
- Statisticians in History by American Statistical Association has Rao's name mentioned in a list of 50 contributors.
- Chronology of Probabilists and Statisticians: A list of 57 major contributors to probability and statistics from sixteenth to twentieth century prepared by Statistics Department, University of Texas at Elpasso, USA, has Rao's name.

- Rao is one of the 77 contemporary scientists from all over the world in all areas of science, selected by Gerard Piel, editor of Scientific American, covered in the book *Faces of Science* by Mariana Cook. The book contains portraits of scientists taken by the author with a short description of contributions made by each. The portraits were exhibited at the Gallery of Arts and Science, New York Academy of Sciences in New York and the National Academy of Sciences in Washington.

Prestigious Awards

Pictures

1. ***Shanti Swarup Bhatnagar Award**, Council of Scientific and Industrial Research, India, for "notable and outstanding research in statistics" from Pandit Jawaharlal Nehru—1963.*
2. ***Samuel S. Wilks Medal**, awarded by the American Statistical Association for the great influence he has had on the application of statistical thinking in different disciplines, embodying over a career of more than 40 years in the spirit and ideals of Samuel S. Wilks—1989.*
3. ***Padma Vibhushan**, second highest civilian award, from the Government of India for "outstanding contributions to Science and Engineering/Statistics"—2001.*
4. ***National Science Medal**, USA, honored as "prophet of a better age" from President Bush—2002.*
5. ***Mahalanobis Prize**, awarded by International Statistical Institute at the 54th Session held in Berlin for "lifetime achievement"—2003.*
6. ***India Science Award**, "for major contributions of a path-breaking nature based on work done in India" from Prime Minister Manmohan Singh—2009.*
7. ***Guy Medal in Gold**of the Royal Statistical Society, UK ,"for those who are judged to have merited a significant mark of distinction by reason of their innovative contribution to theory or application of statistics" from the president of the Royal Statistical Society—2011.*
8. *Honored as a "statistician of international repute" by the President of India, Pranab Mukherjee, at the World Telugu Conference in Tirupathi, India—2012.*
9. ***Jerzy Splawa-Neyman Medal**, from the Polish Statistical Association in recognition of "his outstanding contributions to the theory, applications, and teaching of statistics"—2014.*
10. *Rao and Bhargavi Rao with Professor Mahalanobis (founder of ISI) and Rani Mahalanobis at the convocation of ISI in 1967 where the election of Rao to Royal Society was announced.*
11. *James D. Watson (Nobel Laureate) and Rao in front of portrait of Rao exhibited at the "Faces of Science" display at the Gallery of Arts and Sciences, New York Academy of Sciences—2005.*
12. *C. R. Rao Advanced Institute of Mathematics, Statistics, and Computer Science, in University of Hyderabad Campus, Prof. C. R. Rao Road, established to promote basic research in science and technology—2011.*
13. *The C. R. Rao Gallery capturing Rao's life in statistics over 65 years was inaugurated by Nobel Laureate V. Ramakrishnan on December 22, 2013, at*

the C. R. Rao Advanced Institute of Mathematics, Statistics, and Computer Science, Hyderabad. The gallery setup by Dr. Tejaswini is open to the public.

Picture 1

Picture 2

Picture 3

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About the Editors

Arijit Chaudhuri is Honorary Visiting Professor at the Applied Statistics Unit (ASU) at the Indian Statistical Institute (ISI), Kolkata, India, since 1 September 2005. Professor Chaudhuri holds a Ph.D. in Statistics in the area of sample surveys from the University of Calcutta, Kolkata, India, from where he also graduated. He worked as a postdoctoral researcher for two years at the University of Sydney (1973–1975), Australia. He retired as a professor from the ISI, Kolkata, India, on 31 August 2002, where he then continued to work as CSIR Emeritus Scientist for three years up to 31st August 2005. His areas of research include mean square error estimation in multi-stage sampling, analytical study of complex surveys, randomized response surveys, and small area estimation. In the year 2000, he was elected as President of the Section of Statistics for the Indian Science Congress and worked for the Government of West Bengal for 12 years as Committee Chairman for the improvement of crop statistics. He has also worked with the Government of India to apply sophisticated methods of small area estimation in improving state and union territory level estimates of various parameters of national importance. He has worked on various global assignments upon invitation, including visiting professorships at universities/statistical offices in the USA, Canada, England, Germany, the Netherlands, Sweden, Israel, Cyprus, Turkey, Cuba, and South Africa, from 1979 to 2009. He has successfully mentored 10 Ph.D. students and published more than 150 research papers in peer-reviewed journals, a few of them jointly with his students and colleagues. He is the author of 11 books on survey sampling.

Sat N. Gupta is Professor of Statistics and Head of the Department of Mathematics and Statistics at the University of North Carolina at Greensboro, USA. He holds a Ph.D. in Statistics from Colorado State University, USA (1983), and a Ph.D. in Mathematics from the University of Delhi (1977). Earlier, he served as Professor of Statistics at the University of Southern Maine, USA. He has published over 130 journal articles and has several edited book volumes to his credit including one with Professor C. R. Rao. He has provided research guidance to students at all levels from undergraduate to doctorate, including 20 graduate students, 8 undergraduate students, and 10 Ph.D. students. His research has been funded by major US funding agencies like the National Science Foundation, National Institute of Health, Mathematical Association of America, American Statistical Association, and Robert Wood Johnson Foundation. Fellow of the American Statistical Association, Prof. Gupta has bagged several awards in his illustrious career, including the Senior University-wide Research Excellence Award; Senior College of Arts and Sciences Teaching Excellence Award at the University of North Carolina, Greensboro, USA; Outstanding Faculty Award and the Outstanding Teacher/Scholar Award from the University of Southern Maine, USA; Distinguished Service Award for the Cause of Statistics given by the North Carolina Chapter of the American Statistical Association; and the Sankhyiki Bhushan Award from the Indian Society of Agricultural Statistics. He is Founding Editor-in-Chief of the prestigious Journal of Statistical Theory and Practice (Springer) since 2007.

Rajkumar Roychoudhury is former Professor at the Physics and Applied Mathematics Unit of the Indian Statistical Institute, Kolkata. He holds a Ph.D. in Physics from the University of Durham, England (1970). He completed his M.Sc. in Applied Mathematics from the University of Calcutta, Kolkata, India, in 1965. He was awarded the President's Gold Medal in 1963 for obtaining the highest marks among all B.Sc. students and went to England on a state scholarship for a higher education in 1966. His areas of research are quantum mechanics, plasma physics, nonlinear dynamics, and quantitative linguistics. After a stint with the West Bengal Education Service, Prof. Roychoudhury joined Indian Statistical Institute (ISI), Kolkata, in 1977. He retired from ISI in 2006 as Professor and Head at the Physics and Applied Mathematics Unit. He has visited universities and institutions around the world, including in Canada, Taiwan, China, and Europe, as a visiting scientist and an invited speaker. He has more than 250 publications in international journals of repute to his credit.

Information and the Accuracy Attainable in the Estimation of Statistical Parameters

C. Radhakrishna Rao

Introduction

The earliest method of estimation of statistical parameters is the method of least squares due to Markoff. A set of observations whose expectations are linear functions of a number of unknown parameters being given, the problem which Markoff posed for solution is to find out a linear function of observations whose expectation is an assigned linear function of the unknown parameters and whose variance is a minimum. There is no assumption about the distribution of the observations except that each has a finite variance.

A significant advance in the theory of estimation is due to Fisher (1921) who introduced the concepts of *consistency*, *efficiency* and *sufficiency* of estimating functions and advocated the use of the maximum likelihood method. The principle accepts as the estimate of an unknown parameter θ, in a probability function $\phi(\theta)$ of an assigned type, that function $t(x_1, \ldots, x_n)$ of the sampled observations which makes the probability density a maximum. The validity of this principle arises from the fact that out of a large class of unbiassed estimating functions following the normal distribution the function given by maximising the probability density has the least variance. Even when the distribution of t is not normal the property of minimum variance tends to hold as the size of the sample is increased.

Taking the analogue of Markoff's set up Aitken (1941) proceeded to find a function $t(x_1, \ldots, x_n)$ such that

$$\int t\phi(\theta)\pi \, dx_i = \theta$$

and

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A. Chaudhuri et al. (eds.), *A Tribute to the Legend of Professor C. R. Rao*,
Indian Statistical Institute Series,
https://doi.org/10.1007/978-981-33-6991-7_1

$$\int (t-\theta)^2 \phi(\theta)\pi \, dx_i \qquad \text{is minimum.}$$

Estimation by this method was possible only for a class of distribution functions which admit sufficient statistics. Some simple conditions under which the maximum likelihood provides an estimate accurately possessing the minimum variance, even though the sample is finite and the distribution of the estimating function is not normal, have emerged.

The object of the paper is to derive certain inequality relations connecting the elements of the *Information Matrix* as defined by Fisher (1921) and the variances and covariances of the estimating functions. A class of distribution functions which admit estimation of parameters with the minimum possible variance has been discussed.

The concept of distance between populations of a given type has been developed starting from a quadratic differential metric defining the element of length.

Estimation by Minimising Variance

Let the probability density $\phi(x_1, \ldots, x_n; \theta)$ for a sample of n observations contain a parameter θ which is to be estimated by a function $t = f(x_1, \ldots, x_n)$ of the observations. This estimate may be considered to be the best, if with respect to any other function t', independent of θ, the probabilities satisfy the inequality

$$P(\theta - \lambda_1 < t < \theta + \lambda_2) \nless P(\theta - \lambda_1 < t' < \theta + \lambda_2) \tag{2.1}$$

for all positive λ_1 and λ_2 in an interval $(0, \lambda)$. The choice of the interval may be fixed by other considerations depending on the frequency and magnitude of the departure of t from θ. If we replace the condition (2.1) by a less stringent one that (2.1) should be satisfied for all λ we get as a necessary condition that

$$E(t-\theta)^2 \ngtr E(t'-\theta)^2, \tag{2.2}$$

where E stands for the mathematical expectation. We may further assume the property of unbiassedness of the estimating functions *viz.*, $E(t) = \theta$, in which case the function t has to be determined subject to the conditions $E(t) = \theta$ and $E(t-\theta)^2$ is minimum.

As no simple solution exists satisfying the postulate (2.1) the inevitable arbitrariness of these postulates of unbiassedness and minimum variance needs no emphasis. The only justification for selecting an estimate with minimum variance from a class of unbiassed estimates is that a necessary condition for (2.1) with the further requirement that $E(t) = \theta$ is ensured. The condition of unbiassedness is particularly defective in that many biassed estimates with smaller variances lose their claims as estimating functions when com-condition is that $\phi(x;\theta)$ the probability density of the sample observations satisfies the equality

$$\phi(x;\theta) = \Phi(T,\theta)\psi(x_1,\ldots,x_n), \tag{3.6}$$

where ψ does not involve θ and $\Phi(T,\theta)$ is the probability density of T. If t is an unbiassed estimate of θ then

$$\theta = \int t\phi\pi\, dx_i = \int f(T)\Phi(T,\theta)\, dT \tag{3.7}$$

which shows that there exists a function $f(T)$ of T, independent of θ and is an unbiassed estimate of θ. Also

$$\int (t-\theta^2)\phi\pi\, dx_i = \int [t-f(T)]^2\phi\pi\, dx_i + \int [f(T)-\theta]^2\Phi(T,\theta)\, dT$$
$$\geq \int [f(T)-\theta]^2\Phi(T,\theta)\, dT, \tag{3.8}$$

which shows that

$$V[f(T)] \ngtr V(t) \tag{3.9}$$

and hence we get the result that *if a sufficient statistic and an unbiassed estimate exist for θ, then the best unbiassed estimate of θ is an explicit function of the sufficient statistic*. It usually happens that instead of θ, a certain function of θ can be estimated by this method for a function of θ may admit an unbiassed estimate.

It also follows that if T is a sufficient statistic for θ and $E(T) = f(\theta)$, then there exists no other statistic whose expectation is $f(\theta)$ with the property that its variance is smaller than that of T.

It has been shown by Koopman (1936) that under certain conditions, the distribution function $\phi(x,\theta)$ admitting a sufficient statistic can be expressed as

$$\phi(x,\theta) = \exp(\Theta_1 X_1 + \Theta_2 + X_2), \tag{3.10}$$

where X_1 and X_2 are functions of $x_1, x_2, \ldots, x_n$ only and Θ_1 and Θ_2 are functions of θ only. Making use of the relation

$$\int \exp(\Theta_1 X_1 + \Theta_2 + X_2)\pi\, dx_i = 1, \tag{3.11}$$

we get

$$E(X_1) = -\frac{d\Theta_2}{d\Theta_1} \quad \text{and} \quad V(X_1) = -\frac{d^2\Theta_2}{d\Theta_1^2}. \tag{3.12}$$

If we choose $-\dfrac{d\Theta_2}{d\Theta_1}$ as the parameter to be estimated we get the minimum variance attainable is by (3.5) pared with unbiassed estimates with greater variances. There are, however, numerous examples where a slightly biassed estimate is preferred to an unbiassed estimate with a greater variance. Until a unified solution of the problem of estimation is set forth we have to subject the estimating functions to a critical examination as to its bias, variance and the frequency of a given amount of departure of the estimating function from the parameter before utilising it.

Single Parameter and the Efficiency Attainable

Let $\phi(x_1, \ldots, x_n)$ be the probability density of the observations $x_1, x_2, \ldots, x_n$, and $t(x_1, \ldots, x_n)$ be an unbiassed estimate of θ. Then

$$\int \cdots \int t\phi\pi \, dx_i = \theta. \tag{3.1}$$

Differentiating with respect to θ under the integral sign, we get

$$\int \cdots \int t\frac{d\phi}{d\theta}\pi \, dx_i = 1 \tag{3.2}$$

if the integral exists, which shows that the covariance of t and $\frac{1}{\phi}\frac{d\phi}{d\theta}$ is unity. Since the square of the covariance of two variates is not greater than the product of the variances of the variates we get using V and C for variance and covariance

$$V(t)V\left(\frac{1}{\phi}\frac{d\phi}{d\theta}\right) \nless \left\{C\left(t, \frac{1}{\phi}\frac{d\phi}{d\theta}\right)\right\}^2 \tag{3.3}$$

which gives that

$$V(t) \nless 1/I$$

where

$$I = V\left(\frac{1}{\phi}\frac{d\phi}{d\theta}\right) = E\left\{-\frac{d^2 \log \phi}{d\theta^2}\right\} \tag{3.4}$$

is the intrinsic accuracy defined by Fisher (1921). This shows that *the variance of any unbiassed estimate of θ is greater than the inverse of I which is defined independently of any method of estimation.* The assumption of the normality of the distribution function of the estimate is not necessary.

If instead of θ we are estimating $f(\theta)$, a function of θ, then

$$V(t) \nless \{f'(\theta)\}^2/I. \tag{3.5}$$

If there exists a sufficient statistic T for θ then the necessary and sufficient

$$\left\{\frac{d}{d\theta}\frac{d\Theta_2}{d\Theta_1}\right\}^2 \Big/ \left\{\frac{d^2\Theta_2}{d\Theta_1^2}\frac{d\Theta_1}{d\theta}\right\} = -\frac{d^2\Theta_2}{d\Theta_1^2} = V(X_1). \tag{3.13}$$

Hence X_1 is the best unbiassed estimate of $-\dfrac{d\Theta_2}{d\Theta_1}$. Thus for *the distributions of the type* (3.10), *there exists a function of the observations which has the maximum precision as an estimate of a function of* θ.

Case of Several Parameters

Let $\theta_1, \theta_2, \ldots, \theta_q$ be q unknown parameters occurring in the probability density $\phi(x_1, \ldots, x_n; \theta_1, \theta_2, \ldots, \theta_q)$ and $t_1, t_2, \ldots, t_q$ be q functions independent of $\theta_1, \theta_2, \ldots, \theta_q$ such that

$$\int \cdots \int t_i \phi \pi \, dx_j = \theta_i. \tag{4.1}$$

Differentiating under the integral sign with respect to θ_i and θ_j, we get, if the following integrals exist,

$$\int \cdots \int t_i \frac{\partial \phi}{\partial \theta_i} \pi \, dx_k = 1, \tag{4.2}$$

and

$$\int \cdots \int t_i \frac{\partial \phi}{\partial \theta_j} \pi \, dx_k = 0. \tag{4.4}$$

Defining

$$E\left[-\frac{\partial^2 \log \phi}{\partial \theta_i \partial \theta_j}\right] = I_{ij}, \tag{4.4}$$

and

$$E(t_i - \theta_i)(t_j - \theta_j) = V_{ij}, \tag{4.5}$$

we get the result that the matrix of the determinant

$$\begin{vmatrix} V_{ii} & 0 & \ldots & 1 & \ldots & 0 \\ 0 & I_{11} & \ldots & I_{1i} & \ldots & I_{1q} \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 1 & I_{i1} & \ldots & I_{ii} & \ldots & I_{iq} \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & I_{q1} & \ldots & I_{qi} & \ldots & I_{qq} \end{vmatrix} \tag{4.6}$$

being the dispersion matrix of the stochastic variates t_i and $\dfrac{1}{\phi}\dfrac{\partial \phi}{\partial \theta_j}$ ($j = 1,$

$2, \ldots, q$) is positive definite or semi-definite. If we assume that there is no linear relationship of the type

$$\sum \lambda_j \frac{1}{\phi} \frac{\partial \phi}{\partial \theta_j} = 0 \tag{4.7}$$

among the variables $\frac{1}{\phi} \frac{\partial \phi}{\partial \theta_j}$ ($j = 1, 2, \ldots, q$) then the matrix $\|I_{ij}\|$, which is known as the information matrix due to $\theta_1, \theta_2, \ldots, \theta_q$, is positive definite in which case there exists a matrix $\|I^{ij}\|$ inverse to $\|I_{ij}\|$. From (4.6) we derive that

$$V_{ii} - I^{ii} \geq 0 \tag{4.8}$$

which shows *that minimum variance attainable for the estimating function of θ_i when $\theta_1, \theta_2, \ldots, \theta_q$ are not known is I^{ii}, the element in the i-th row and the i-th column of the matrix $\|I^{ij}\|$ inverse to the information matrix $\|I_{ij}\|$*.

The equality is attained when

$$t_i - \theta_i = \sum \mu_j \frac{1}{\phi} \frac{\partial \phi}{\partial \theta_j}. \tag{4.9}$$

We can obtain a generalisation of (4.8) by considering the dispersion matrix of $t_1, t_2, \ldots, t_i$ and $\frac{1}{\phi} \frac{\partial \phi}{\partial \theta_r}$ ($r = 1, 2, \ldots, q$)

$$\begin{vmatrix} V_{11} & \ldots & V_{1i} & 1 & 0 & \ldots & 0 & \ldots & 0 \\ V_{21} & \ldots & V_{2i} & 0 & 1 & \ldots & 0 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ V_{i1} & \ldots & V_{ii} & 0 & 0 & \ldots & 1 & \ldots & 0 \\ 1 & \ldots & 0 & I_{11} & I_{12} & \ldots & I_{1i} & \ldots & I_{1q} \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & \ldots & 0 & I_{q1} & I_{q2} & \ldots & I_{qi} & \ldots & I_{qq} \end{vmatrix} \tag{4.10}$$

This being positive definite or semi-definite we get the result that the determinant

$$|V_{rs} - I^{rs}| \geq 0, \quad (r, s = 1, 2, \ldots, i) \tag{4.11}$$

for $i = 1, 2, \ldots, q$. The above inequality is evidently independent of the order of the elements so that, in particular, we get that the determinant

$$\begin{vmatrix} V_{ii} - I^{ii}, & V_{ij} - I^{ij} \\ V_{ji} - I^{ji}, & V_{jj} - I^{jj} \end{vmatrix} \geq 0, \tag{4.12}$$

which gives the result that if $V_{ii} = I^{ii}$, so that maximum precision is attainable for the estimation of θ_i, then $V_{ij} = I^{ij}$ for ($j = 1, 2, \ldots, q$).

In the case of the normal distribution

$$\phi(x; m, \sigma) = \text{const. exp} - \tfrac{1}{2}\{\sum (x_i - m)^2/\sigma^2\}, \tag{4.13}$$

we have

$$I_{mm} = n/\sigma^2, \qquad I_{m\sigma} = 0, \qquad I_{\sigma\sigma} = 2n/\sigma^2. \tag{4.14}$$

Since the mean of observations $(x_1 + x_2 + \cdots + x_n)/n$ is the best unbiassed estimate of the parameter m and the maximum precision is attainable *viz.*, $V_{mm} = I^{mm}$, it follows that any unbiassed estimate of the parameter σ is uncorrelated with the mean of observations for $V_{m\sigma} = I^{m\sigma} = 0$. *Thus in the case of the univariate normal distribution any function of the observations whose expectation is a function of σ and independent of m is uncorrelated with the mean of the observations.* This can be extended to the case of multivariate normal populations where any unbiassed estimates of the variances and covariances are uncorrelated with the means of the observations for the several variates.

If there exists no functional relationships among the estimating functions $t_1, t_2, \ldots, t_q$ then $\|V^{ij}\|$ the inverse of the matrix $\|V_{ij}\|$ exists in which case we get that the determinant

$$|V^{rs} - I_{rs}|, \quad (r, s = 1, 2, \ldots, i) \tag{4.15}$$

is greater than or equal to zero for $i = 1, 2, \ldots, q$, which is analogous to (4.11).

If a sufficient set of statistics $T_1, T_2, \ldots, T_q$ exist for $\theta_1, \theta_2, \ldots, \theta_q$ then we can show as in the case of a single parameter that the best estimating functions of the parameters or functions of parameters are explicit functions of the sufficient set of statistics.

Koopman (1936) has shown that under some conditions the distribution function $\phi(x_1, x_2, \ldots, x_n; \theta_1, \theta_2, \ldots, \theta_q)$ admitting a set of statistics $T_1, T_2, \ldots, T_q$ sufficient for $\theta_1, \theta_2, \ldots, \theta_q$ can be expressed in the form

$$\phi = \exp(\Theta_1 X_1 + \Theta_2 X_2 + \cdots + \Theta_q X_q + \Theta + X) \tag{4.16}$$

where X's are independent of θ's and Θ's are independent of x's. Making use of the relation

$$\int \phi \, dv = 1, \tag{4.17}$$

we get

$$\left.\begin{aligned} E(X_i) &= -\frac{\partial\Theta}{\partial\Theta_i}, \\ V(X_i) &= -\frac{\partial^2\Theta}{\partial\Theta_i^2}, \\ \operatorname{cov}(X_i X_j) &= -\frac{\partial^2\Theta}{\partial\Theta_i \partial\Theta_j}. \end{aligned}\right\} \tag{4.18}$$

This being the maximum precision available we get that for this class of distribution laws there exist functions of observations which are the best possible estimates of functions of parameters.

Loss of Information

If $t_1, t_2, \ldots, t_q$, the estimates of $\theta_1, \theta_2, \ldots, \theta_q$, have the joint distribution $\Phi(t_1, t_2, \ldots, t_q; \theta_1, \theta_2, \ldots, \theta_q)$ then the information matrix on $\theta_1, \theta_2, \ldots, \theta_q$ due to $t_1, t_2, \ldots, t_q$ is $\|F_{ij}\|$ where

$$F_{ij} = E\left\{-\frac{\partial^2 \log \Phi}{\partial\theta_i \partial\theta_j}\right\}. \tag{5.1}$$

The equality

$$I_{ij} = (I_{ij} - F_{ij}) + F_{ij} \tag{5.2}$$

effects a partition of the covariance between $\frac{1}{\phi}\frac{\partial\phi}{\partial\theta_i}$ and $\frac{1}{\phi}\frac{\partial\phi}{\partial\theta_j}$ as within and between the regions formed by the intersection of the surfaces for constant values of $t_1, t_2, \ldots, t_q$. Hence we get that the matrices

$$\|I_{ij} - F_{ij}\| \qquad \text{and} \qquad \|F_{ij}\| \tag{5.3}$$

which may be defined as the dispersion matrices of the quantities $\frac{1}{\phi}\frac{\partial\phi}{\partial\theta_i}$ $(i = 1, 2, \ldots, q)$ within and between the meshes formed by the surfaces of constant values of $t_1, t_2, \ldots, t_q$, is positive definite or semidefinite. This may be considered as a generalisation of Fisher's inequality $I_{ii} \geq F_{ii}$ in the case of a single parameter.

If $I_{ii} = F_{ii}$, then it follows that $I_{ij} = F_{ij}$ for all j for otherwise the determinant

$$\begin{vmatrix} I_{ii} - F_{ii} & I_{ij} - F_{ij} \\ I_{ij} - F_{ij} & I_{jj} - F_{jj} \end{vmatrix} < 0. \tag{5.4}$$

If in the determinant

$$|I_{ij} - F_{ij}|, \quad (i, j = 1, 2, \ldots, q), \tag{5.5}$$

the zero rows and columns are omitted, the resulting determinant will be positive and less than the determinant obtained by omitting the corresponding rows and columns in $|I_{ij}|$. If we represent the resulting determinants by dashes, we may define the loss of information in using the statistics $t_1, t_2, \ldots, t_q$ as

$$|I_{ij} - F_{ij}|'/|I_{ij}|'. \tag{5.6}$$

If Φ is the joint distribution of $t_1, t_2, \ldots, t_q$ the estimates of $\theta_1, \theta_2, \ldots, \theta_q$ with the dispersion matrix $\|V_{ij}\|$ then we have the relations analogous to (4.11) and (4.15) connecting the elements of $\|V_{ij}\|$ and $\|F_{ij}\|$ defined above. Proceeding as before we get that the determinants

$$|V_{rs} - F^{rs}| \qquad \text{and} \qquad |F_{rs} - V^{rs}|, \quad (r, s = 1, 2, \ldots, i), \tag{5.7}$$

are greater than or equal to zero for all $i = 1, 2, \ldots, q$.

The Population Space

Let the distribution of a certain number of characters in a population be characterised by the probability differential

$$\phi(x, \theta_1, \ldots, \theta_q)\, dv. \tag{6.1}$$

The quantities $\theta_1, \theta_2, \ldots, \theta_q$ are called population parameters. Given the functional form in x's as in (6.1) which determines the type of the distribution function, we can generate different populations by varying $\theta_1, \theta_2, \ldots, \theta_q$. If these quantities are represented in a space of q dimensions, then a population may be identified by a point in this space which may be defined as the population space (P.S).

Let $\theta_1, \theta_2, \ldots, \theta_q$ and $\theta_1 + d\theta_1, \theta_2 + d\theta_2, \ldots, \theta_q + d\theta_q$ be two contiguous points in (P.S). At any assigned value of the characters of the populations corresponding to these contiguous points, the probability densities differ by

$$d\phi(\theta_1, \theta_2, \ldots, \theta_q) \tag{6.2}$$

retaining only first order differentials. It is a matter of importance to consider the relative discrepancy $d\phi/\phi$ rather than the actual discrepancy. The distribution of this quantity over the x's summarises the consequences of replacing $\theta_1, \theta_2, \ldots, \theta_q$ by $\theta_1 + d\theta_1, \ldots, \theta_q + d\theta_q$. The variance of this distribution or the expectation of the square of this relative discrepancy comes out as the positive definite quadratic differential form

$$ds^2 = \sum \sum g_{ij}\, d\theta_i\, d\theta_j, \tag{6.3}$$

where

$$g_{ij} = E\left(\frac{1}{\phi}\frac{\partial \phi}{\partial \theta_i}\right)\left(\frac{1}{\phi}\frac{\partial \phi}{\partial \theta_j}\right). \tag{6.4}$$

Since the quadratic form is invariant for transformations in (P.S) it follows that g_{ij} form the components of a covariant tensor of the second order and is also symmetric for $g_{ij} = g_{ji}$ by definition. This quadratic differential form with its *fundamental tensor* as the elements of the *Information matrix* may be used as a suitable measure of divergence between two populations defined by two contiguous points. The properties of (P.S) may be studied with this as the *quadratic differential metric* defining the element of length. The space based on such a metric is called the Riemannian space and the geometry associated with this is the Riemannian geometry with its definitions of distances and angles.

The Distance Between Two Populations

If two populations are represented by two points A and B in (P.S) then we can find the distance between A and B by integrating along a geodesic using the element of length

$$ds^2 = \sum \sum g_{ij}\, d\theta_i\, d\theta_j. \tag{7.1}$$

If the equations to the geodesic are

$$\theta_i = f_i(t), \tag{7.2}$$

where t is a parameter, then the functions f_i are derivable from the set of differential equations

$$\sum_1^q {}_j\, g_{jk} \frac{d^2\theta_j}{dt^2} + \sum_1^q {}_{jl}\, [jl, k] \frac{d\theta_j}{dt}\frac{d\theta_l}{dt} = 0, \tag{7.3}$$

where $[jl, k]$ is the Christoffel symbol defined by

$$[jl, k] = \frac{1}{2}\left[\frac{\partial g_{jk}}{\partial \theta_l} + \frac{\partial g_{lk}}{\partial \theta_j} + \frac{\partial g_{jl}}{\partial \theta_k}\right]. \tag{7.4}$$

The estimation of distance, however, present some difficulty. If the two samples from two populations are large then the best estimate of distance can be found by substituting the maximum likelihood estimates of the parameters in the above expression for distance. In the case of small samples we can get the fiducial limits only in a limited number of cases.

We apply the metric (7.1) to find the distance between two normal populations defined by (m_1, σ_1) and (m_2, σ_2) the distribution being of the type

$$\phi(x, m, \sigma) = \frac{1}{\sqrt{(2\pi\sigma^2)}} \exp. -\frac{1}{2}\frac{(x-m)^2}{\sigma^2}. \tag{7.5}$$

The quantities g_{ij} defined above have the values

$$g_{11} = 1/\sigma^2, \qquad g_{12} = 0, \qquad g_{22} = 2/\sigma^2, \tag{7.6}$$

so that the element of length is obtained from

$$ds^2 = \frac{(dm)^2}{\sigma^2} + \frac{2}{\sigma^2}(d\sigma)^2. \tag{7.7}$$

If $m_1 \neq m_2$ and $\sigma_1 \neq \sigma_2$ then the distance comes out as

$$D_{AB} = \sqrt{2} \log \frac{\tan \theta_1/2}{\tan \theta_2/2} \tag{7.8}$$

where

$$\theta_i = \sin^{-1} \sigma_i/\beta \quad \text{and} \quad \beta = \sigma_1^2 + [(m_1 - m_2)^2 - 2(\sigma_2^2 - \sigma_1^2)]/8(m_1 - m_2)^2. \tag{7.9}$$

If $m_1 = m_2$ and $\sigma_1 \neq \sigma_2$

$$D_{AB} = \sqrt{2} \log(\sigma_2/\sigma_1). \tag{7.10}$$

If $m_1 \neq m_2$ and $\sigma_1 = \sigma_2$

$$D_{AB} = \frac{m_1 - m_2}{\sigma}. \tag{7.11}$$

Distance in Tests of Significance and Classification

The necessity for the introduction of a suitable measure of distance between two populations arises when the position of a population with respect to an assigned set of characteristics of a given population or with respect to a number of populations has to be studied. The first problem leads to tests of significance and the second to the problem of classification. Thus if the assigned values of parameters which define some characteristics in a population are $\bar{\theta}_1$, $\bar{\theta}_2, \ldots, \bar{\theta}_q$ represented by the point O, and the true values are $\theta_1, \theta_2, \ldots, \theta_q$ represented by the point A, then we can define the divergence from the assigned sets of parameters by D_{AO}, the distance defined before in the (P.S.). The testing of the hypothesis

$$\bar{\theta}_i = \theta_i, \quad (i = 1, 2, \ldots, q), \tag{8.1}$$

may be made equivalent to the test for the significance of the estimated distance D_{AO} on the large sample assumption. If $D_{AO} = \psi(\theta_1, \ldots, \theta_q; \bar{\theta}_1, \ldots, \bar{\theta}_q)$ and the maximum likelihood estimates of $\theta_1, \theta_2, \ldots, \theta_q$ are $\hat{\theta}_1, \hat{\theta}_2, \ldots, \hat{\theta}_q$, then the estimate of D_{AO} is given by

$$\hat{D}_{AO} = \psi(\hat{\theta}_1, \ldots, \hat{\theta}_q; \bar{\theta}_1, \ldots, \bar{\theta}_q). \tag{8.2}$$

The covariances between the maximum likelihood estimates being given by the elements of the information matrix, we can calculate the large sample approximation to the variance of the estimate of D_{AO} by the following formula

$$V(\hat{D}_{AO}) = \sum\sum \frac{\partial\psi}{\partial\theta_i}\frac{\partial\psi}{\partial\theta_j}\operatorname{cov}(\hat{\theta}_i\hat{\theta}_j) \tag{8.3}$$

We can substitute the maximum likelihood estimates of $\theta_1, \theta_2, \ldots, \theta_q$ in the expression for variance. The statistic

$$w = \frac{\hat{D}_{AO}}{[V(\hat{D}_{AO})]^{1/2}} \tag{8.4}$$

can be used as a normal variate with zero mean and unit variance to test the hypothesis (8.1).

If the hypothesis is that two populations have the same set of parameters then the statistic

$$w = \frac{\hat{D}_{AB}}{[V(\hat{D}_{AB})]^{1/2}}, \tag{8.5}$$

where $\hat{D}_{AB}$ is the estimate of the distance between two populations defined by two points A and B in (P.S) can be used as (8.4). The expression for variance has to be calculated by the usual large sample assumption.

If the sample is small the appropriate test will be to find out a suitable region in the sample space which affords the greatest average power over the surfaces in the (P.S) defined by constant values of distances. The appropriate methods for this purpose are under consideration and will be dealt with in a future communication.

The estimated distances can also be used in the problem of classification. It usually becomes necessary to know whether a certain population is closer to one of a number of given populations when it is known that populations are all different from one another. In this case the distances among the populations taken two by two settle the question. We take that population whose distance from a given population is significantly the least as the one closest to the given population.

This general concept of distance between two statistical populations (as different from tests of significance) was first developed by Prof. P.C. Mahalanobis. The generalised distance defined by him (Mahalanobis, 1936) has become a powerful tool in biological and anthropological research. A perfectly general measure of divergence has been developed by Bhattacharya (1942) who defines the distance between populations as the angular distance between two points representing the populations on a unit sphere. If $\pi_1, \pi_2, \ldots, \pi_k$ are the proportions in a population consisting of k classes then the population can be represented by a point with coordinates $\sqrt{\pi_1}, \sqrt{\pi_2}, \ldots, \sqrt{\pi_k}$ on a unit sphere in a space of k dimensions. If two populations have the proportions $\pi_1, \pi_2, \ldots, \pi_k$ and $\pi'_1, \pi'_2, \ldots, \pi'_k$ the points representing them have the co-ordinates $\sqrt{\pi_1}, \sqrt{\pi_2}, \ldots, \sqrt{\pi_k}$ and $\sqrt{\pi'_1}, \sqrt{\pi'_2}, \ldots, \sqrt{\pi'_k}$. The distance between them is given by

$$\cos^{-1}\{\sqrt{(\pi_1\pi'_1)} + \sqrt{(\pi_2\pi'_2)} + \cdots + \sqrt{(\pi_k\pi'_k)}\}. \tag{8.6}$$

If the populations are continuous with probability densities $\phi(x)$ and $\psi(x)$ the distance is given by

$$\cos^{-1} \int \sqrt{\{\phi(x)\psi(x)\}}\, dx. \tag{8.7}$$

The representation of a population as a point on a unit sphere as given by Bhattacharya (1942) throws the quadratic differential metric (7.1) in an interesting light. By changing $\theta_1, \theta_2, \ldots, \theta_q$ the parameters occurring in the probability density, the points representing the corresponding populations describe a surface on the unit sphere. It is easy to verify that the element of length ds connecting two points corresponding to $\theta_1, \theta_2, \ldots, \theta_q$ and $\theta_1 + d\theta_1, \ldots, \theta_q + d\theta_q$ on this is given by

$$ds^2 = \sum (d\phi)^2/\phi = \sum\sum g_{ij}\, d\theta_i\, d\theta_j, \tag{8.8}$$

where g_{ij} are the same as the elements of the quadratic differential metric defined in (7.1).

Further aspects of the problems of distance will be dealt with in an extensive paper to be published shortly.

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A Few Thoughts on India's Statistical System

Bibek Debroy

Introduction

I possess no expertise to write an essay on Professor C. R. Rao. I am neither a mathematician, nor a statistician. Economists need to use statistics, as do several others. Therefore, as a student of economics, I am familiar with his work on estimation, which one studied as part of course work. Subsequently, one can only marvel at his numerous contributions in mathematics, statistics and econometrics, recognized through a host of awards and honours. Therefore, though this is in a volume in honour of Professor C. R. Rao, I will stay from areas in which I possess no expertise. Instead, this essay will focus on India's statistical system.

Pre-Independence, there was quite a bit of statistical material for British India, though it tended to be fragmentary. For instance, there were district gazetteers, authored in the second half of the nineteenth century and later. Several decades later, they make for interesting reading. As an example, here is a quote from the Howrah District Gazetteer for 1909.[1] The quote describes the state of vital statistics, the system having been introduced in 1892. "Under the present system, compulsory registration is in force in the towns, i.e. parents, guardians or the persons directly concerned are required to report births and deaths to the town police. In rural circles each village watchman is provided with a pocket book, in which he is required to have all births and deaths that may occur within his jurisdiction recorded by the village panchayat; these are reported on parade days at the police stations and outposts, which are the registering centres. The statistics thus obtained are compiled and classified

[1] *Bengal District Gazetteers, Howrah*, L. S. S. O'Malley and Monmohan Chakravarti, Bengal Secretariat Book Depot, 1909.

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A. Chaudhuri et al. (eds.), *A Tribute to the Legend of Professor C. R. Rao*,
Indian Statistical Institute Series,
https://doi.org/10.1007/978-981-33-6991-7_2

by the police, and submitted monthly to the Civil Surgeon, who prepares the figures for the whole district for inclusion in the annual report of the Sanitary Commissioner. The statistics are checked from time to time by superior police officers and by Inspectors and Sub-Inspectors of Vaccination. In the towns, the higher level of intelligence and the fear of legal penalties tend to make registration and the classification of diseases more accurate than in the rural tracts. In the latter the reporting chaukidar is generally illiterate, and vital registration is less correct, the chief defects being that still-births are very often omitted, while births of females and births in outlying parts, and among the lowest castes, are overlooked. Deaths are more carefully recorded, but the causes of death, except cholera and small-pox, are hopelessly confused, the bulk being classified under the general head of fever." For British India as a whole, one should be amazed at the kind of statistical information available in something like the 1908 Imperial Gazetteer.[2]

Post-Independence, with the integration of the Princely States, the statistical system was still not robust enough, and its improvement was intricately linked to the evolution of the planning process. Minoo Masani (1905–98) is all but forgotten now. In his day, he was an influential thinker, politician and Parliamentarian. In 1940, he published a slim book directed at children.[3] It was socialist in tone and had views Minoo Masani would change in his later years. In the Preface to this book, he said, "Statistics of Indian life are so scanty and scrappy that reliance on them is bound to endanger one's conclusions." Indeed, data and statistics were scanty and scrappy in 1940. Proper statistical systems started to evolve in the 1950s. These days, if one wants official data, one often resorts to Economic Survey. There was nothing quite akin to Economic Survey then, apart from data being non-existent. Even after Independence in 1947 and the enactment of the Constitution in 1950, for more than a decade, there was no Economic Survey in the sense we understand it now. Publishing Economic Survey is not a Constitutional requirement. It is published because of an executive decision and because it has now become established precedence. As the seed of what would eventually become Economic Survey, a "White Paper" started to be included in the Budget papers from 1950 to 51. If one reads the first White Paper, included in the Budget papers for 1950–51, there is nothing on what is known by the name of human development indicators today. A proper Economic Survey started to be separately presented in 1957–58.

A trigger was provided by the formation of the National Planning Committee in 1938.[4] The National Planning Committee recommended that an all-India National Planning Commission should be established. With this interest in planning, it should not be surprising that plans should be developed, independent of the National Planning Committee. In histories of Indian economic policy-making, it is odd that M.

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[2] *The Imperial Gazetteer of India*, Clarendon Press, Oxford, 1908.

[3] *Our India*, Minoo Masani, Oxford University Press, 1940.

[4] This National Planning Committee never published any reports. But several details about its meetings are available in *National Planning Committee, Being an abstract of Proceedings and other particulars relating to the National Planning Committee*, K. T. Shah (Honorary General Secretary, National Planning Committee), Bombay, 1948.

Visvesvaraya's (1861–1962) book is rarely mentioned, though we remember him for other reasons.[5] This was almost certainly, the first "plan" developed for India. The book collated a wealth of statistical data and made cross-country comparisons between India and the rest of the world. Unlike Visvesvaraya's book, the Bombay Plan has consistently received a lot of attention.[6] Published in 1944, the People's Plan has often been described as a polar opposite of the Bombay Plan.[7] 1944 also saw the publication of a Gandhian Plan, with a Foreword written by Mahatma Gandhi.[8] This was the context in which the Planning Commission was set up through a Cabinet Resolution dated 15 March 1950 and attention also focused on improving India's statistical system.

In April 1951, the first report of the National Income Committee highlighted problems with the statistical system.[9] Parts of this report should be quoted, because some of the problems cited in 1951 have not substantively changed. "To begin with, when calculating the value of output, one normally proceeds on the assumption that the bulk of the commodities and services produced in the country are exchanged for money. In the case of India, however, a considerable proportion of output does not come into the market at all, being consumed by the producers themselves or bartered for other commodities and services. The problem of imputation of value thus arises and takes on significantly large proportions in some sectors of the economy…The problem of measurability is further complicated in India by the fact that many producers have nothing but the remotest notion of either the quantity or value of their output….Even if he could and did maintain accounts, the producer in India will find it hard put even to know the gross value of his output, let alone the net output; in the absence of accounts, we cannot even know, much less impart information on his income and expenditure." There was more in the same vein. The Chairman of the National Income Committee was Professor Prasanta Chandra Mahalanobis, instrumental in the setting up of the Indian Statistical Institute (ISI), with which Professor C. R. Rao was also associated. In collaboration with Gokhale Institute of Politics and Economics (Pune), the first round of the National Sample Survey (NSS) was started by ISI in 1950. At that time, NSS was still a Directorate. It became the National Sample Survey Organization (NSSO) in 1970 and is now the National Sample Survey Office. There have been more than 70 rounds of NSSO surveys so far, covering consumer expenditure, employment/unemployment, manufacturing, services and investment. More or less at the same time as NSSO, the Central Statistical Institute was set up in 1951. The Central Statistical Institute first became the Central Statistical Organization (CSO)

[5] *Planned Economy for India*. M. Visvesvaraya, Bangalore Press, Bangalore 1934.

[6] It has been edited and republished recently. *The Bombay Plan: Blueprint for Economic Resurgence* edited by Sanjaya Baru and Meghnad Desai, Rupa, 2018.

[7] *People's Plan for Economic Development of India*, Post-war Reconstruction Committee of Indian Federation of Labour, Brajendra Nath Banerjee, Govardhan Dhanaraj Parikh and V. M. Tarkunde, 1944. This initiative was driven by M. N. Roy.

[8] *The Gandhian Plan of Economic Development for India*, Shriman Narayan Agarwal, Padma Publications, 1944.

[9] *First Report of the National Income Committee*, Department of Economic Affairs, Ministry of Finance, April 1951.

and is now the Central Statistical Office. Very recently, CSO and NSSO have been brought together under the umbrella of the National Statistical Office (NSO). But to go back to the edifice that was being built in the 1950s, the now repealed Collection of Statistics Act was passed in 1953. The 1953 legislation was replaced by the Collection of Statistics Act of 2008, later amended in 2017.

There are several reasons why the collection of statistics is bound to be complicated in a country like India. The Seventh Schedule of the Constitution divides responsibilities between the Union government and State governments. Entry 69 in the Union List mentions the Census, while Entry 94 in the Union List mentions "inquiries, surveys and statistics for the purpose of any of the matters" in the Union List. Entry 30 in the Concurrent List is on vital statistics, while Entry 45 in the Concurrent List mentions "inquiries, surveys and statistics for the purpose of any of the matters" in the State List or the Concurrent List. The Seventh Schedule evolved at a certain point in time, through the Government of India Act of 1919 and then the Government of India Act of 1935, and there can be valid arguments for taking a relook at the Seventh Schedule. But that is a digression. For our purposes, the complexities are best described in the words of the Ministry of Statistics and Programme Implementation (MoSPI) itself.[10] This is a longish quote, but is necessary to explain the complexities. "The Indian Statistical System presently functions within the overall administrative framework of the country. The Indian federal structure has influenced the organization of the statistical system as well. The division of administrative functions between the Government of India and the State Governments is on the basis of the subject classifications under the Union, State and Concurrent Lists as detailed in the Constitution of India. At the Centre, the responsibilities are further divided amongst the various ministries and departments, according to the Allocation of Business Rules, 1961 that are amended from time to time. The collection of statistics on any subject generally vests in the authority (Central Ministry or Department or State Government Department) that is responsible for that subject according to its status in the Union, State or Concurrent Lists. By and large, the flow of statistical information emanates from the States to the Centre except in cases where the State-level operations are an integral part of Centrally- sponsored schemes or data are collected through national sample surveys…The collection of statistics for different subject-specific areas, like agriculture, labour, commerce, industry, etc. vests with the corresponding administrative ministries. More often than not, the statistical information is collected as a by-product of administration or for monitoring the progress of specific programmes…The Statistical System in the States is similar to that at the Centre. It is generally decentralised laterally over the Departments of the State Government, with major Departments, such as, agriculture or health, having large statistical divisions for the work of departmental statistics."

As it evolved in the 1950s, the Indian statistical system was among the best in countries that were then known as developing economies. Despite the warts and blemishes, it continues to be among the best. The warts and blemishes were extensively documented in the report of the National Statistical Commission, chaired by Dr. C.

[10]http://mospi.nic.in/142-present-indian-statistical-system-organisation.

Rangarajan.[11] This report was submitted in 2001, and there was a specific section on restructuring of the Indian statistical system. For example, "As official statistics play a major role in assessing the performance of Government, it is important that such statistics are not only accurate, but are also trusted as such by the layman as well as by its principal users. This is ensured if a high-level policy-making body that has commensurate authority and obligations, oversees the statistical system. This is not a new concept. For instance, in the Nehruvian era, Professor P. C. Mahalanobis, an independent non-official was the Honorary Statistical Adviser to the Cabinet, had this kind of authority, but his authority was entirely personal with no formal institutional arrangement. The Governing Council of the NSSO also had complete autonomy in respect of sample surveys conducted by the NSSO. The NABS was intended to be an institutional arrangement for the statistical system as a whole, but it did not succeed because it lacked legal backing. The Commission recommends creation of a permanent and statutory apex body—the National Commission on Statistics (NCS)—independent of the Government and responsible to the Parliament in respect of policy-making, coordination and certification of quality of Core Statistics." Subsequently, the National Statistical Commission (NSC) was set up in 2005, though not legislatively. There is now a draft Bill to provide statutory backing to NSC. It is available on MoSPI's Website, and public comments have been sought.[12] The Collection of Statistics Act was amended in 2008 and 2017, as I have stated earlier. There was also a 2011 N. R. Madhava Menon Committee on legislative changes that would be required to improve India's statistical system.[13]

In 2016, India adopted UN's "Fundamental Principles of Official Statistics." Some of the problems with the statistical system highlighted by the National Income Committee in 1951 still remain. The Indian economy still remains largely unorganized/informal.[14] Data gathering in relatively more formal economies is relatively simpler. For example, in an economy that is largely self-employed, even outside of agriculture, numbers on employment cannot be gleaned through enterprise surveys. They will have to be obtained through household surveys. Surveys, based on traditional systems, often suffer from lack of a sufficient number of enumerators, and methodologies vary over time, rendering inter-temporal comparisons difficult. Time-lags render surveys redundant for immediate policy-making, and there are incompatibilities between data gathered through surveys and those through Censuses, the latter gathered at intervals of ten years. Are there more modern and real-time methods one can tap? There was a time when statistics were gathered fundamentally through government sources. A tremendous amount of data is now generated outside the government system. Can one use data generated through private sources? How does

[11] http://www.mospi.gov.in/report-dr-rangarajan-commission.

[12] http://www.mospi.gov.in/sites/default/files/announcements/Draft%20NSC%20Bill%202019%20for%20seeking%20comments%20or%20suggestions.pdf?download=1.

[13] *Committee on Legislative measures In statistical matters*, Chaired by N. R. Madhava Menon, National Statistical Commission Secretariat, October 2011, http://www.mospi.gov.in/sites/default/files/committee_reports/legislative_measure_stat_matter_18jan12.pdf.

[14] There is a technical difference between unorganized and informal, but we are using the terms synonymously.

one validate it? There is a tremendous amount of data that is now generated through administrative Ministries and Departments, often virtually real-time. The dashboards of data.gov.in are illustrative. How can one use this? Most government programmes, Union as well as State, now use Socio-Economic Caste Census (SECC) as a base to identify beneficiaries. SECC has had multiple benefits, but is household-based. How does one correlate and match this with Aadhar or Ayushman Bharat, which are individual-based? Can one give some form of legal identity to MSME enterprises that are not necessarily legally incorporated? What about information on agriculture, transcending crop output, or even more narrowly, food-grains, or rice and wheat? The focus, since the Millennium Development Goals (MDGs) and even more so, since the Sustainable Development Goals (SDGs), is increasingly on sub-national and State-level data, indeed, on sub-State data.[15] Consider for instance the national indicators adopted by India to measure progress towards SDGs.[16] The effort required to obtain data on these indicators is nothing short of formidable.

Last year's Economic Survey had an excellent chapter on data as a public good.[17] This also listed the various kinds of data that are already collected by government. "The Government of India collects four distinct sets of data about people—administrative, survey, institutional and transactions data. While the latter two databases are in a fledgling state, the first two are comprehensive and robustly maintained….Governments hold *administrative data* for mainly non-statistical purposes. Administrative datasets include birth and death records, crime reports, land and property registrations, vehicle registrations, movement of people across national borders, tax records etc. Governments also gather data to evaluate welfare schemes; for example, the Ministry of Drinking Water and Sanitation gathers data on toilet usage to assess the efficacy of the Swachh Bharat Mission. *Survey data*, on the other hand, is data gathered predominantly for statistical purposes through systematic, periodic surveys. For example, the National Sample Survey Office conducts large-scale sample surveys across India on indicators of employment, education, nutrition, literacy etc. Because these data are gathered for statistical analyses, the identity of participants is irrelevant and unreported, although these identities may be securely stored at the back-end without violating any legal guidelines on privacy. *Institutional data* refers to data held by public institutions about people. For example, a government-run district hospital maintains medical records of all its patients. A government-run school maintains personal information about all its pupils. State-run universities maintain records of students' educational attainment and degrees awarded to them. Most such data are held locally, predominantly in paper-based form. This data can be digitized to enable aggregation at the regional or national level. *Transactions data* are data on an individual's transactions such as those executed on

[15] State-level, and even district-level, human development reports (HDRs) were a good source of information. But their publication has now been suspended.

[16] *Sustainable Development Goals, National Indicator Framework*, Baseline Report 2015–15, Ministry of Statistics and Programme.

[17] *Economic Survey 2018–19*, Department of Economic Affairs, Volume 1, Chap. 4.

the United Payment Interface (UPI) or BHIM Aadhaar Pay. This is a nascent category of data but is likely to grow as more people transition to cashless payment services." While addressing privacy concerns, the challenge is to integrate all this into a reinvention of India's statistical system. Because of the advent of new forms of technology, the marginal costs of collecting data have decline, while the marginal benefits have increased.

MoSPI has recently brought out a five year vision document for 2019–24.[18] This sets out what is contemplated, in terms of transforming the Indian statistical system. Those details should not be repeated. With the new Committee on Economic Statistics (headed by Pronab Sen) and the new NSC, that process of transformation has started.

[18] *Five Year Vision, 2019–2024*, Ministry of Statistics and Programme Implementation.

A Tribute to Prof. C R Rao

Sanghamitra Bandyopadhyay

A world leader in statistical science and a living legend, Professor C R Rao holds a special place in the Indian Statistical Institute as also for me personally. Having met Prof. Rao a couple of times as a young faculty, known about his work for many years and as the Director of the Indian Statistical Institute (ISI), I am familiar with his achievements, immense contribution to statistics and its applications in newer areas, and his service to ISI. His work and research results are relevant even today, and read by students of statistics in universities around the world, as it gives us Rao–Cramer inequality, Rao–Blackwell theorem relating to sufficient statistics, and Fisher–Rao metric and Rao distance in the space of probability distributions. He made fundamental contributions in multivariate analysis which fetched him his Ph.D. degree.

C R Rao came to ISI in 1941 for training for a year, left half way through for a Master's degree in Statistics in Calcutta University, then rejoined ISI in 1943 as a research scholar. He worked with Prof. P C Mahalanobis, the founder of ISI, on real-life problems with a salary of Rs. 75 per month. He then left for the UK and obtained his Ph.D. there. Later, be came back to ISI as a Professor at an young age of 28. He was the Director of the Institute from 1972 to 1976. He superannuated from the Institute at the age of 60, and thereafter continued working in USA for many more years till recently when age became a confining factor. However, intellectually he continues to be active. I remember that even about five years back, after he came to know about my selection as the Director of the Institute in April 2015, he had written to me expressing his happiness. He had also informed me about his interest in big data, and the books he was editing in this area. To me, this was a demonstration of his extreme intellectual energy and youth, as well as great adaptability to the demands of the changing times. He had once said, “Statistics is not a discipline like physics, chemistry or biology where we study a subject to solve problems in the same subject.

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A. Chaudhuri et al. (eds.), *A Tribute to the Legend of Professor C. R. Rao*,
Indian Statistical Institute Series,
https://doi.org/10.1007/978-981-33-6991-7_3

We study statistics with the main aim of solving problems in other disciplines." Today, this is all the more evident. In this era of data sciences and machine learning, the importance of statistics for solving problems of national importance and for the welfare of humankind is being felt all the more. C R Rao therefore continues to inspire.

Many organizations including our own ISI are celebrating Prof. C R Rao's birth centenary. The ISI Retired Employee's Association (ISIREA) is one of them. In 2008, at the initiative of Prof. Arijit Chaudhuri, ISIREA decided to bring out a book to show their respect and gratitude to Prof. P C Mahalanobis, the founding father of ISI. He contacted Prof. C R Rao to invoke his blessings before venturing on this mission. Within 48 hours Prof. C R Rao responded to his email volunteering to contribute one article. Later ISIREA published three more books and in two of them he again contributed. Prof. Chaudhuri narrated to me that in spite of his advancing age, Prof. Rao once wrote to Prof. Chaudhuri that he read one of these books from cover to cover including even those chapters written in Bengali. When they conveyed to him their intention and plan to celebrate his centenary, his favorable reaction was immediate and enthusiastic. I am sure this volume being brought out by ISIREA will be well appreciated by all. I take this opportunity to express by sincere gratitude to Prof. C R Rao who has dedicated his life for the cause of Statistics. Wishing you a healthy and productive life ahead, Sir.

The Polaron Problem

S. R. S. Varadhan

I want to thank ISIREA for giving me this opportunity to participate in the year long celebration of Dr. Rao's centennial. I remember fondly my years at ISI and the encouragement and support Dr. Rao provided during that period. He has always remained in touch. I take this occasion to wish him good health.

I want to report on a problem I have been working in recnet years with Chiranjib Mukherjee from Münster. My interest on this probelm originated when Elliot Lieb, some time around 1983, visited Donsker and me and outlined a problem that he thought was amenable to the large deviation tools that we had developed. The so-called Polaron problem has to do with the behavior of an electron in a crystal. The electron polarizes the crystal and as a rsult the elcetron looks heavier. The problem was studied by Pekar who had a conjecture, which can be stated in purely probabilistic terms.

$$Z(\alpha, T) = E^P\left[\exp\left[\alpha \iint\limits_{-T\le s\le t\le T} \frac{e^{-|t-s|}}{|x(t)-x(s)|}\right] \mathrm{d}t\,\mathrm{d}s\right] \qquad (1)$$

where $x(t)$ is the three-dimensional Brownian Motion and $\alpha > 0$ is the coupling constant. the free energy $F(\alpha)$ is defuned as

$$F(\alpha) = \lim_{T\to\infty} \frac{1}{2T} \log Z(\alpha, T) \qquad (2)$$

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The conjecture of Pekar [GP] is that

$$\lim_{n\to\infty} \frac{F(\alpha)}{\alpha^2} = c_0 \tag{3}$$

exists and equals

$$c_0 = \sup_{\|\phi\|_2=1}\left[\iint \frac{\phi^2(x)\phi^2(y)}{|x-y|}\mathrm{d}x\mathrm{d}y - \frac{1}{2}\int \|\nabla\phi\|^2\mathrm{d}x\right] \tag{4}$$

Large deviation theory [DV1] can be used to prove that the limit in (2) exists and get the following variational formula for the limit.

$$F(\alpha) = \sup_{Q\in S}\left[E^Q\left[\alpha\int_0^\infty \frac{e^{-t}}{|x(t)-x(0)}\right] - H(Q)\right] \tag{5}$$

Browninan motion is a process with stationary increments and given a realization of its increments in some interval $[-T, T]$, one can construct an empirical process out of it, which is similar to empirical distribution but contains joint empirical distributions of increments over intervals of different lengths. There is a law of large numbers as well as a large deviation principle. For any process Q with stationary increments, the entropy of the conditional distribution of increments of Q on $[0, T]$ given its increments over $(-\infty, 0]$ relative to white noise or increments of Brownian motion, is linear in T and equals $TH(Q)$ for some constant $H(Q)$. The probability that in $[-T, T]$ the empirical process is close to some other process Q with stationary increments is $\exp[-2TH(Q) + o(T)]$. One can then expect (5) to hold.

We can use Brownian scaling $x(\alpha^2.) \simeq \alpha x(.\cdot)$ and (5) reduces to

$$\frac{F(\alpha)}{\alpha^2} = \sup_{Q\in S}\left[E^Q\left[\int_0^\infty \frac{\epsilon\, e^{-\epsilon t}\mathrm{d}t}{|x(t)-x(0)}\right] - H(Q)\right]$$

where $\in = \alpha^{-2} \to 0$. For $t \gg 1$ and $x(t)$ and $x(0)$ are nearly independent with marginal distribution $\mu = qQ$.

$$\inf_{Q:q_Q=\mu} H(Q) = I(\mu) = \frac{1}{2}\int \|\nabla\phi\|^2\mathrm{d}x$$

where $\mathrm{d}\mu = \phi^2\mathrm{d}x$.

$$\lim_{\alpha\to\infty} \frac{F(\alpha)}{\alpha^2} = \sup_{Q\in S}\left[\int_0^\infty \frac{\mu(\mathrm{d}x)\mu(\mathrm{d}y)}{|x-y|} - I(\mu)\right]$$

There are some issues. Lack of compactness and coercivity. Replace R^3 by the three torus of size ℓ and let ℓ go to infinity. Usually, lower bound is not a problem and it is not here. Upper bound can be an issue. Fortunately, there is a monotonicity that makes the issue go away. The problem was solved in the 80s [DV2].

But there is another layer to the Polaron problem. If we define $Q_{\alpha,T}$ as

$$dQ_{\alpha,T} = [Z(\alpha,T]^{-1}\exp\left[\alpha \int\limits_{-T\leq s\leq t\leq T} \frac{e^{-|t-s|}}{|x(t)-x(s)|}\mathrm{d}t\mathrm{d}s\right]\mathrm{d}P$$

the following questions arise.

1. Does $Q_\alpha = \lim_{T\to\infty} Q_{\alpha,T}$ exist? What is it?
2. Is there a central limit theorem for $\frac{x(T)-x(-T)}{\sqrt{2T}}$ under $Q_{\alpha,T}$ or Q_α?
3. Is there a scaling limit for Q_α as $\alpha\to\infty$ under Brownian scaling? What is the limit?
4. What is the variance $\sigma^2(\alpha)I$ in the CLT of 2.
5. How does $\sigma^2(\alpha)$ behave as $\alpha\to\infty$? Conjecture $\simeq c\alpha^{-4}$

Chiranjib Mukherjee and I have made some progress over the last six years [MV1, MV2] and [MV3]. The space of probability distributions $M(R^3)$ needs to be compactified. The translation group $G = \{T_a\}$ acts on $M(R^3)$ and we need to compactify the orbit space $M(R^3)/G$. The compactification is a countable (can be finite or even empty) collection of orbits $\tilde{\mu}_{\mathrm{j}} = \{\mu_j, a\}$ of measures with masses $m_j = \mu_j, a(R^3) = \tilde{\mu}_{\mathrm{j}}(R^3)$ adding up to $\Sigma_j\, m_j \leq 1$. Now large deviation theory applies directly and lack of coercivity is no longer a problem.

The space of stationary processes can be completed to the space of processes with stationary increments on which the problem is coercive.

$$F(\alpha) = \sup_{Q\in S}\left[E^Q\left[\alpha\int_0^\infty \frac{e^{-t}}{|x(t)-x(0)|}\right] - H(Q)\right]$$

$$\lim_{\alpha\to\infty}\frac{F(\alpha)}{\alpha^2} = \sup_{\left\{\tilde{\mu}_j\,:\,\sum_j m_j\leq 1\right\}}\left[\sum_j \iint \frac{\tilde{\mu}_j(\mathrm{d}x)\tilde{\mu}_j(\mathrm{d}y)}{|x-y|} - I\left(\tilde{\mu}_j\right)\right]$$

The sup is attained at a single orbit $\{\mu_a\} = \{\phi^2\,(x-a)\,\mathrm{d}x\}$ that optimizes

$$\sup_{\|\phi\|_2=1}\left[\iint \frac{\phi^2(x)\phi^2(y)}{|x-y|}\mathrm{d}x\mathrm{d}y - \frac{1}{2}\int \|\nabla\phi\|^2\mathrm{d}x\right]$$

which is unique modulo translation and is spherically symmetric around its center. It is now clear that we can expect Q_α to be a process with stationary increments that optimizes (1). Is it unique? What is it? It was shown in [MV] that it is a mixture of

Gaussian processes, i.e., a Gaussian Process with a random covariance. The random covariance has a renewal structure based on a birth and death process.

We start with a birth and death process with birth rate α and death rate 1. The size of the population is a positive recurrent Markov chain. We start from 0, σ is the time of first birth and τ is the return time to 0. There are n individual life spans $\{[\sigma_i, \tau_i]\}$ such $\cup_i[\sigma_i, \tau_i] = [\sigma, \tau]$ where σ is the time of the first birth and τ is he time of the last death.

We denote this state space by X_{n} and the state space of a single excursion is $X = U_{n\geq 1} X_n$.

We have the probability measure Ω_α (dξ) on X with $\xi \in X$ consists of $n = n(\xi)$ life spans. We augment X_n by R^n and $y_n = X_n \times R^n$ and $y = \cup_{n\geq 1} y_n$ We define a measure $\widehat{\Omega}_\alpha$ on y as

$$e^{-\lambda(\alpha)\tau}[2\pi]^{-\frac{n(\xi)}{2}}\left[\Delta\left(\xi, u_1, \ldots, u_{n(\xi)}\right)\right]^{\frac{3}{2}}\Omega_\alpha\left(\mathrm{d}\xi, \mathrm{d}u_1, \ldots, \mathrm{d}u_{n(\xi)}\right)$$

where $\Delta(\xi, u_1, \ldots, u_n)$ is the determinant of the $n \times n$ matrix $\{w_{i,j}\}$, $w_{ij} = \left|U_i \cap U_j\right| u_i u_j$ where $U_i = [\sigma_i, \tau_i]$ and $|A|$ is the lebesgue measure of A. $\lambda(\alpha)$ is chosen so that

$$\int_y e^{-\lambda(\alpha)\tau}(2\pi)^{-\frac{n(\xi)}{2}}\Delta\left(\xi, u_1, \ldots, u_{n(\xi)}\right)\Omega_\alpha(\mathrm{d}\xi)\mathrm{d}u_1, \ldots, \mathrm{d}u_{n(\xi)} = 1$$

We have an excursion with distribution $\widehat{\Omega}$ and the initial dormant period with exponential distribution has mean α^{-1}. Followed by an excursion starting with the first birth and ending with the death of the last individual. This renewal process has a stationary version Γ_α. There is a Gaussian process with the following covariance structure. The increments are independent over disjoint excursions. It is white noise in a dormant interval. On an active interval it is a Gaussian $\beta_\xi, u_1, \ldots, u_{n(\xi)}$ with the following RN derivative with respect to white noise P.

$$\left[\Delta\left(\xi, u_1, \ldots, u_{n(\xi)}\right)\right]^{\frac{3}{2}}\exp\left[-\frac{1}{2}\sum u_i^2|x(\tau_i) - x(\sigma_i)|^2\right]$$

The central limit theorem follows from the ergodic theorem for the renewal process. The variance is

$$\sigma^2(\alpha) = \frac{\alpha^{-1} + \frac{1}{3}E^{\widehat{\Omega}}E^{\beta_{\xi,u_1,\ldots,u_{n(\xi)}}}\left[|x(\tau) - x(\sigma)|^2\right]}{\alpha^{-1} + E^{\widehat{\Omega}_\alpha}[\tau - \sigma]}$$

If we define $\tilde{Q}_{T,\epsilon}$ as the distribution

$$d\tilde{Q}_{T,\epsilon} = \frac{1}{Z(T,\epsilon)} \exp\left[\epsilon \iint\limits_{-T \le s \le t \le T} \frac{e^{-\epsilon|t-s|}}{|x(t)-x(s)|}\right] \mathrm{d}P$$

It is a rescaled version of $Q_{\alpha,T}$ with $\in = \alpha^{-2}$.

$$\lim_{\epsilon \to 0} \lim_{T \to \infty} Q_{T,\epsilon} = Q^*$$

where Q^* is increments of a stationary diffusion with generator

$$\frac{1}{2}\Delta + \frac{\nabla\phi}{\phi}.\nabla$$

ϕ is the minimizer of the variational problem

$$\sup_{\|\phi\|_2=1} \left[\iint \frac{\phi^2(x)\phi^2(y)}{|x-y|} \mathrm{d}x\mathrm{d}y - \frac{1}{2}\int \|\nabla\phi\|^2 \mathrm{d}x\right]$$

which is unique modulo translation. What remains is the behavior of $\sigma^2(\alpha)$ for large α. The conjecture is

$$\sigma^2(\alpha) \simeq c\alpha^{-4}$$

We have some ideas but needs work.

$$\begin{aligned}
\mathrm{d}Q_{\propto T} &= \exp\left[\propto \iint\limits_{-T \le s \le t \le T} \Psi(T,\omega)\mathrm{d}s\mathrm{d}t\mathrm{d}P\right] \\
&= [Z(\propto, T)]^{-1} \sum_n \alpha^n n! \left[\iint\limits_{-T \le s \le t \le T} \Psi(T,\omega)\mathrm{d}s\mathrm{d}t\right]^n \\
&= [Z(\propto, T)]^{-1} \sum_n \frac{\alpha^n}{n!} \int \cdots \iint\limits_{-T \le s_i \le t_i \le T} (2\pi)^{-\frac{n}{2}}
\end{aligned}$$

$$\exp\left[-\sum |t_i - s_i| - \frac{1}{2}\sum u_i^2 |x(t_i) - x(s_i)|^2\right] \Pi\, \mathrm{d}t_i \mathrm{d}s_i \mathrm{d}u_i$$

We have a convex combination of Gaussians. Need to have the normalization factor which is missing.

$$E^P \exp\left[-\frac{1}{2}\sum u_i^2 |x(t_i) - x(s_i)|^2\right] = [\Delta(\xi, u_1, \ldots, u_n)]^{-\frac{3}{2}}$$

Missing normalization is part of the weight. Poisson point process ξ on $-T \le s \le t \le T$, with intensity $\alpha e^{-|t-s|} \mathrm{d}s\mathrm{d}t$ having $n(\xi)$ intervals $[s_i, t_i]$, with an additional weight of $(2\pi)^{-\frac{n}{2}} \left[\Delta\left(\xi, u_1, \ldots, u_{n(\xi)}\right)\right]^{-\frac{3}{2}}$

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C. R. Rao: A Life in Statistics

B. L. S. Prakasa Rao

When I think of modern statistics, Dr C.R. Rao features on the top of the list. He once said that statistics is the technology of finding the invisible and measuring the immeasurable.

—Abdul Kalam, Bharat Ratna.

Calyampudi Radhakrishna Rao or C. R. Rao needs no introduction to statisticians, mathematicians, scientists or engineers. In the volume *Glimpses of Indian Statistical Heritage* [1], Rao wrote an autobiographical account highlighting the circumstances and influences that led him to a career in statistics and probability. He titled his autobiographical account as *Statistics as a Last Resort*. It is appropriate to mention that he came into statistics by chance. By spending a life time, putting chance to work, he has built an inspiring legacy.

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https://doi.org/10.1007/978-981-33-6991-7_5

Early Life as a Student

Rao was born on 10 September 1920 in Huvvina Hadagalli, then in the integrated Madras Province, but now in the state of Karnataka. His father C. Doraswamy Naidu was an Inspector of Police with reputation in CID work, and his mother was A. Laxmikanthamma. Rao grew up in a family environment. He was admitted to Class 2 in 1925 when he was only five years old and was able to memorize multiplication tables up to 16 by 16 (needed for monetary transactions during the British rule with monetary denominations of Rupees, 16 annas for a rupee, 12 kanis for an anna and 4 dammidies for a kani). Naidu's job required the family to move from place to place every three years. Rao completed classes 2 and 3 in Gudur, classes 4 and 5 in Nuzvid and the first and second forms in Nandigama, all in the present state of Andhra Pradesh. After retirement, Rao's father decided to settle down in Visakhapatnam, a coastal city in Andhra Pradesh. Rao finished his high school and obtained his first college degree B A (Hons), with a first class and first rank, in Visakhapatnam. His early childhood involved frequent moves, but that did not affect him. His parents provided him guidance and environment conducive to excellence in studies and installed work ethics that enabled him to achieve higher goals in life. As a student, his ambition was to keep on learning. He said that he has inherited his father's analytical ability and his mother's zeal and industry. His mother was instrumental in installing a sense of discipline in him. In *Statistics and Truth: Putting Chance to Work* [2], Rao acknowledges her contributions to his life with the dedication: 'For instilling in me the quest for knowledge, I owe to my mother, A. Laxmikanthamma, who, in my younger days, woke me up every day at four in the morning and lit the oil lamp for me to study in the quiet hours of the morning when the mind is fresh'.

Rao developed research interest in mathematics when he was a student of the B.A (Hons) degree course at the age of 17 in the Andhra University. He used to solve problems posed in the journal *Mathematics Student* and was happy to see that his name was mentioned as one of those who solved the problem. His most inspiring teacher was a Cambridge trained mathematician, Vommi Ramaswami, who was the head of the department of mathematics. Rao obtained the B.A (Hons) degree at the age of 19 and wanted to pursue a research career in mathematics. With a first class and first rank in the degree examination, Rao thought he would qualify for a scholarship for doing research in mathematics. However, he did not get the scholarship for bureaucratic reasons. He decided to search for a job and saw an advertisement for a mathematician for the army survey unit to work in North Africa during the Second World War. He went to Calcutta and appeared for an interview for the job which eluded him. During his stay in Calcutta, he met one Subramanian who was employed in Bombay, but had been sent to Calcutta for training in statistics at the Indian Statistical Institute (ISI). This chance encounter led Rao to join the training programme in statistics at ISI hoping that with some additional qualification he could get some job.

40 Years at the Indian Statistical Institute

Rao joined ISI in 1941 at the age of 20 and started doing research by himself and publishing papers. He received M.A. degree in Statistics from Calcutta University in 1943 with a first class, first rank and a high percentage of marks which remains unbroken up to now. With two Master's degrees, Rao was given the position of a research scholar at ISI in 1943 and a part-time job in Calcutta University to teach a course in statistics. He continued to do research by himself on a variety of topics in combinatorics and estimation of parameters and publish papers.

A request from the Department of Anthropology, Cambridge University, was sent to ISI in 1946 to send a person to analyse measurements made on human skeletons brought from Jebel Maya in North Africa by the University Museum of Archaeology and Anthropology to trace the origin of the people who lived there using the method of Mahalanobis D-square statistic. The analysis of multiple measurements was not well developed at that time. Rao was sent to Cambridge by Mahalanobis as he had the required expertise. Rao worked in Cambridge for two years (1946–48) as a visiting scholar at the Cambridge University Museum of Archaeology and Anthropology and developed some new methods of analysis of multiple measurements and used them to analyse the data. The results of his work on the skeletal material were published in the book *Ancient Inhabitants of Jebel Maya* [3]. During this period, while working in the museum, Rao also registered for PhD degree under the supervision of Ronald A. Fisher, a distinguished statistician. Rao received his PhD in 1948 based on the new multivariate methodology, 'Multivariate Analysis Of Variance (MANOVA)' generalizing (ANOVA), and other multivariate tests, were developed by him while analysing skeletal data. Cambridge University awarded him the ScD degree in 1974 based on a peer review of his publications. In 1974, he was made an Honorary Life Fellow of King's College, Cambridge, which is a rare honour.

He returned to ISI in 1948 after two years' stay in Cambridge and was appointed as Professor at the young age of 28 'in recognition of his creativity'. He worked in ISI in various positions as the Head of Research and Training School (RTS), Director of RTS, Director of ISI, Jawaharlal Nehru Professor and National Professor for a period of 40 years and took mandatory retirement at the age of 60, and he was still active in research. He had published 201 research papers during the 40 years of employment at ISI. He wanted a suitable job in India without any administrative responsibilities to continue his research work. As this was not possible, he accepted positions of distinguished professorships offered to him with minimal teaching responsibilities in USA. He worked for another 30 years as University Professor at the University of Pittsburgh for 8 years and as Eberly Chair Professor of Statistics at The Pennsylvania State University for 13 years continuing his research in diverse areas of statistics. He retired from Penn State at the age of 81, but continued doing research as Director of the Center for Multivariate Analysis at Penn State until 2008. At present, at the age of 100, he has the position as a Research Professor at the University at Buffalo. He published 201 research papers while working in ISI and more than 275 while working in USA.

Development of Statistical Education and Training at ISI

> For its (ISI) educational programs, the institute needs not only leaders of mathematical thought like Professor Rao, who can uphold and maintain the high place in the world opinion that Indians have already won.
>
> —Sir Ronald A. Fisher,
>
> The Father of Modern Statistics

At ISI, Rao developed a variety of courses to train statisticians to work in different applied areas. The students and trainees, who were deputed by research, government and industrial organizations to study at ISI, were given, in addition to formal lectures, on the job experience in design of experiments, biostatistics, industrial quality control and other areas. Rao established research units in ISI to work on special projects in subjects such as economics, sociology, psychology, genetics, anthropology, geology and related areas. The idea of establishing these applied research units is to provide interaction between statisticians and scientists to promote the application of statistical methods in research in other areas and to develop new statistical methods motivated by real problems. The ISI served as a meeting place for scientists from all over the world to do collaborative and interdisciplinary research with statistics as the common bond. Some of the famous visitors who spent a few weeks at the ISI are Norbert Wiener (USA), Academician Kolmogorov (Russia), R. A. Fisher (UK), Ragnar Frisch (Norway) and Y. Linnik (Russia). They gave lectures and interacted with research scholars.

Degree Courses at ISI

Rao developed numerous courses in statistics at ISI over the years which were later converted into bachelor's and master's degree, when ISI was declared as an Institute of National Importance by an act of Parliament in 1959 and empowered to offer courses of study leading to degrees in statistics. Rao worked out programmes for undergraduate and postgraduate degree courses leading to B.Stat. and M.Stat. degrees. He also initiated the PhD programme in theoretical statistics and probability. The late Prof. D. Basu, who is well known for his seminal contributions to statistics, was the first PhD in theoretical statistics guided by Rao. Over the years of his professorship at universities, Rao guided the research work of over fifty students for PhD, who in turn have produced about 450 PhDs up to now. The training and research activities developed by Rao earned for ISI a place 'not far from the centre of the statistical map of the world'.

Development of National Statistical System

Under the direction of P. C. Mahalanobis, Rao helped in establishing the State Statistical Bureaus. The Indian National Statistical System, with the Central Statistical Organization and State Statistical Bureaus, is considered to be one of the best in the world, thanks to the efforts of Mahalanobis and Rao. During the early years, when Rao was at ISI, there was no ministry for statistics. Problems related to the development of statistics were under the administrative control of the Prime Minister. Mahalanobis was appointed as honorary statistical advisor to the cabinet in 1949. Pandit Jawaharlal Nehru, who was the Prime Minister at that time, was greatly interested in development of statistics for economic planning. He visited ISI a number of times at the invitation of Mahalanobis, and Rao had the opportunity of discussing with him the national statistical system and training of statisticians to work in state statistical bureaus.

Rao was a member of several government committees for development of national statistical systems, statistical education and research in India. Some of them are, Chairman of the Committee on Statistics (1962–1969), Chairman of the Demographic and Communication for Population Control (1968–1969), Chairman of the Committee on Mathematics, Atomic Energy Commission (AEC) (1969–1978), Member of COST (Committee on Science and Technology, 1969–1971), Member of Justice Sarkar Committee to enquire into the overall functioning of CSIR.

Initiation of Research in Econometrics in India

Rao published a paper in *Econometrica* in 1947 answering a problem raised by Ragnar Frisch, an econometrician [4]. He continued his research in statistics with applications to problems in econometrics. He developed the Delhi branch of the ISI as a centre for research in econometrics with emphasis on economic planning. He founded the Indian Econometric Society in 1971 and developed it over a period of 5 years.

International Statistical Education Centre

Rao played a significant role in establishing an International Statistical Educational Center (ISEC) at ISI for training students and statisticians deputed from developing countries. For several years, he was the Chairman of the Board of Directors of ISEC.

Research at ISI and USA

C. R. Rao is a great name from the golden age of statistics. His work was done in India and his intellect shaped statistics worldwide.

—Julian Champkin.

Editor of the journal Significance.

Rao has authored 14 books and more than 475 research papers. Two of his books have been translated into several European, Japanese, Taiwan, Mainland Chinese and Turkish languages.

Research During the Period 1945–1950 at ISI

Rao's career in statistics is dotted with remarkable achievements. Two of his papers, written during the forties, were reproduced [5], in the book *Breakthroughs in Statistics, 1890–1990*. One was published [6] at the age of 25, and another was published [7] at the age of 27. The first paper opened up several new areas of research and generated several technical terms bearing his name such as Cramer–Rao inequality and Rao–Blackwellization, which are basic to estimation theory and appear in text books on statistics, engineering and econometrics. Cramer–Rao inequality is listed as a technical term in the McGraw-Hill *Dictionary of Scientific and Technical Terms*, Fifth edition, 1994. In a recent book [8], the author B. Roy says, 'The Heisenberg Un certainty Principle is an expression of Cramer–Rao Inequality of classical measurement theory, as applied to position determination'. The quantum physicists derived what is termed as quantum Cramer–Rao Bound (1998) which provides a sharper version of Heisenberg Principle of Uncertainty. Rao–Blackwellization provides a method by which an unbiased estimator can be improved in efficiency, when a sufficient statistics exists. Results obtained by other authors based on Rao's paper and named after Rao are global (Bayesian) Cramer–Rao Bound (1968), Complexified and Intrinsic Cramer–Rao Bound (2005), Rao–Blackwellized Particle Filters (1996), Stereological Rao–Blackwell Theorem (1995), Rao–Blackwell versions of cross validation and nonparametric bootstrap and Cramer–Rao Function.

In the same 1945 paper, Rao proposed a differential geometric foundation for statistics by introducing a quadratic differential metric in the space of probability measures. The idea of connecting statistics and differential geometry was too early at that time. However, half a century later, his idea has been developed to become one of the most active and important work topics in information science connecting statistics, information theory, control theory and statistical physics. The concept of distance between two probability measures introduced by Rao, using differential geometric concepts is known as Rao distance. The metric is known as Fisher–Rao metric as Rao used Fisher information matrix in defining the quadratic differential metric. Some technical terms arising out of papers by others and named after Rao are Rao measure and, Cramer–Rao functional. The article received accolades from various sources.

> The article focuses on an important of the world renowned Indian statistician, Calyampudi Radhakrishna Rao. In 1945, C.R. Rao (25 years old then) published a path breaking paper, which had profound impact on subsequent statistical research. It opens up a novel paradigm by introducing differential geometric modeling ideas to the field of statistics. In recent years, this contribution has lead to the birth of a flourishing field of information geometry.
>
> —Frank Nielsen.
>
> Ecole Polytechnique, France.

Some technical terms bearing Rao's name in research papers by others are 'Rao measure' and 'Cramer–Rao functional'. The second breakthrough paper published in the *Proc. Cambridge Philos. Soc.* introduced a new asymptotic test, termed as Rao's score test, as an alternative to the likelihood ratio and Wald tests, the three known as holy trinity. The test appears in books on econometrics and its merits are discussed in various conferences. Some features of this test are discussed in a paper by Chandra and Joshi [9]. Several papers appeared from 1983, describing some good features of this test.

The combinatorial arrangements called orthogonal arrays were developed in a series of papers in the 1940s by Rao. A general formulation of orthogonal arrays and their use as experimental designs was given by Rao, which was accepted by the editor 'as a fresh and original piece of work' [10]. Orthogonal arrays have received wide applications in industrial experimentation to determine the optimum mix of factors using observations on small number of factor combinations. Taguchi, who learnt about it during his visit to ISI, made extensive use of orthogonal arrays in what is now known as the Taguchi methods in industry for determining an optimum combination of factors, which gives a high output and is robust to environmental changes. An article in *Forbes Magazine* [11] refers to orthogonal arrays as a new mantra in a variety of industrial establishments in USA.

In three papers, i.e. 'Tests with discriminant functions in multivariate analyses' [12] on the choice of minimum set of measurements for analysis; 'Utilization of multiple measurements in problems of biological classification' [13], which was the first attempt to represent high-dimensional data in a two- or three-dimensional graph and 'Tests of significance in multivariate analysis' [14], Rao laid the foundation of modern theory of multivariate methodology.

> The 1940s were ungrudgingly C.R. Rao's. His 1945 paper, which contains the Cramer–Rao Inequality, Rao–Blackwell Theorem, and the beginning of differential geometry of parameter space will guarantee that, even had he done nothing else – but there was much else.
>
> —Terry Speed.
>
> The Walter and Eliza Hall Institute of Medical Research, Melbourne.

All these papers contributed to the development of statistics as an independent discipline.

> The first half of the twentieth century was the golden age of statistical theory, during which our field grew from ad hoc origins similar to the current state of computer science to a firmly

grounded mathematical science. Men of the intellectual caliber of Fisher, Neyman, Pearson, Hotelling, Wald, Cramer, and Rao were needed to bring statistical theory to maturity.

—Brad Efron.

Stanford University, USA.

Research During the Period 1950–1980 at ISI

An estimator is said to be first-order efficient if its asymptotic variance attains the Cramer–Rao lower bound. Under some conditions, the first-order efficiency holds for a large class of estimators. In order to choose a subclass of first-order efficient estimators which are better than others, Rao introduced a criterion called second-order efficiency [15]. This is the first paper, which initiated studies on higher order asymptotics.

Rao used the idea of canonical correlations in estimating the dominant factors which explain the correlation between measurements [16]. This method is known as Rao's canonical factor analysis.

Rao made significant contributions to results on characterization of probability distributions. These results are described in the book, *Characterization Problems of Mathematical Statistics* [17]. Some of the technical terms arising out of characterization of probability distributions are Rao's Damage Model (1963), Rao–Rubin theorem (1964), Kagan, Linnik and Rao Theorems (1963). Research in this area was continued during his stay in USA.

Research During 1980–2000 at USA

Rao in collaboration with Jacob Burbea introduced a series of new measures of information and diversity measures and studied their properties [18]. Rao developed analysis of diversity (ANODIV), generalizing analysis of variance (ANOVA). He introduced what is termed as Rao's quadratic entropy, as a general measure of variance, which is used by ecologists [19].

Rao developed the concept of cross-entropy [20].

Rao's quadratic entropy fulfills all a priori criteria and it surpasses other proposed indices, because it includes species abundance and more than one trait. Therefore, it seems to be an improvement compared to other measures of functional diversity that are currently available.RRao's quadratic entropy fulfills all a priori criteria and it surpasses other proposed indices, because it includes species abundance and more than one trait. Therefore, it seems to be an improvement compared to other measures of functional diversity that are currently available.

—Z. Botta Duket, Hungarian Academy of Sciences.

Rao continued his research in USA on characterization of probability distributions in collaboration with Khatri and Shanbhag. The results are summarized in the book, *Choquet–Deny Functional Equations with Applications to Stochastic Models* [21]. In the area of functional equations in mathematics, he introduced a new equation called the integrated Cauchy functional equation. This equation provided a general technique for characterizing probability measures and solving problems of stochastic modelling of data for statistical analysis [22]. Matrix theory is another branch of mathematics he used in the discussion of statistical problems, which in turn gave impetus to research on matrices. His most important contribution to the theory of matrices is the concept of generalized inverse of a matrix (singular or not). This has become a valuable tool in developing unified theory for linear stochastic models used in prediction problems. A very general definition of inverse of a matrix, singular or not, was discussed in Ref. [23].

Using generalized inverse of a matrix, Rao provided a general technique for characterizing probability measures and solving problems of stochastic modelling of data for statistical analysis. He provided a unified theory of least squares for the linear model

$$Y = X\beta + \varepsilon \text{ with } \mathrm{E}(\varepsilon) = 0$$

and covariance matrix V of ε, where X and V may be rank-deficient. The regression coefficients β are estimated by minimizing the quadratic form

$$(Y - X\beta)' M (Y - X\beta),$$

where M is any generalized inverse of the matrix, as described in Ref. [24]. Rao also generalized what are known as Kantorovich inequalities on matrices for use in statistics which opened a new area of research in matrix algebra [25, 26].

Rao's area of research covered wide fields in statistical theory and practice and some aspects of matrix theory needed to express statistical results in their generality.

A few other technical terms which promoted research by others are, Rao's paradox in sample surveys, Rao's paradox in multivariate analysis, Rao's covariance structure, Geary–Rao theorem, Khatri–Rao product of matrices and Khatri–Rao subspace.

A Place in the History of Statistics

When Rao joined the ISI in early forties, statistics was not considered as an independent subject and no university offered courses at the masters level. Rao's contributions in the forties earned for him a place in history of statistics as one who contributed to the development of statistics as an independent discipline.

> C. R. Rao is among the worldwide leaders in statistical science over the last five decades. His research, scholarship and professional service had a profound influence in the theory and applications of statistics and are incorporated into standard references for statistical study

and practice. Rao's contributions to statistical theory earned him a place in the history of statistics.

—Samuel Karlin.

US National Academy of Sciences.

Rao is the only Asian listed in all web-sites on history of statistics, giving lists of persons with photographs and a summary of their contributions: figures from the *History of Probability and Statistics* contributed by Aldrich (UK) giving a list of 35 major contributors from sixteenth century, *Chronology of Probabilists and Statisticians*, University of Texas, USA, giving a list of 57 major contributors from sixteenth to twentieth centuries and *Statisticians in History* by American Statistical Association giving a list of 52 contributors.

Highest Awards Given to a Statistician

Samuel Wilks Medal of American Statistical Association 1989, the highest award given to a statistician in USA, 'for major contributions to the theory of multivariate statistics and applications of that theory to problems of biometry; for worldwide activities as advisor to national and international organizations; for long time conscientious as a teacher, editor, author and founder of academic institutions; and for the great influence he has had on the application of statistical thinking in different scientific disciplines, embodying over a career of more than 40 years the spirit and ideals of Samuel S. Wilks'.

National Medal of Science 2002, the highest award given to a scientist in USA: Awarded by the president of USA with the citation 'as a prophet of new age for his contributions to the foundations of statistical theory and multivariate statistical methodology and their applications, enriching the physical, biological, mathematical, economic and engineering sciences'.

India Science Award 2009, the highest recognition given to a scientist in India, 'for major contribution(s) of a path-breaking nature'. The award, given by the Prime Minister of India, carries a gold medal and cash prize of Rs 25 lakhs.

Guy Medal in Gold of the Royal Statistical Society 2011, the highest award given to a statistician in UK: This award is given once in 3 years to 'those who are judged to have merited a significant mark of distinction by reason of their innovative contribution to theory or application of statistics'. This is the first time the medal was given to an Asian and second time given to a non-British citizen during the last 118 years since the inception of the medal.

Mahalanobis Birth Centenary Gold Medal 1996, awarded by the Indian Science Congress Association.

Bhatnagar Award 1963, of the Indian Council of Scientific and Industrial Research for contributions to science.

International Mahalanobis Prize 2003, 'for lifetime achievement in statistics and the promotion of best statistical practice' awarded by the International Statistical Institute.

The Ministry of Statistics and Programme Implementation, The Government of India has instituted a National award in honour of C. R. Rao, the renowned statistician of the country.

Professional Awards

Membership in Academies.

Rao received recognition from all statistics societies for his pioneering contributions to statistical theory and applications. He was elected to Royal Society (FRS, UK Academy of Sciences); the National Academy of Sciences (USA); American Academy of Arts and Science; Indian National Science Academy; Indian Academy of Sciences; National Academy of Sciences, India; Lithuanian Academy of Sciences, and Third World Academy of Sciences. He was made an Honorary Member of the European Academy of Sciences, the International Statistical Institute, International Biometric Society, Royal Statistical Society (UK), Finnish Statistical Society, Portuguese Statistical Society, Institute of Combinatorics and Applications, American Statistical Association,\ and World Innovation Foundation. He is a Life Fellow of Kings College, Cambridge. He has been the president of the International Statistical Institute, the first one from Asia; Institute of Mathematical Statistics, USA, the first one from outside USA and the International Biometric Society, the first one from Asia.

Honorary doctorate degrees.

Rao's early research in the forties of the last century on statistical theory and practice brought him international recognition. He had the opportunity of visiting several countries to attend conferences, give lectures and collaborate with noted statisticians for joint research.

Rao was awarded 38 Honorary Doctorate degrees from universities in 19 countries, spanning six continents: Europe (10 countries, 11 degrees): Germany, Russia, Switzerland, Poland, Serbia, Spain, Finland, Portugal, Greece and Cyprus; North America (2 countries, 7 degrees): USA and Canada; South America (2 countries, 2 degrees): Brazil and Peru; Asia (3 countries, 15 degrees): India, Sri Lanka and Philippines; Australia (1 degree) and Africa (1 degree).

Rao is purely an Indian product who had all his education in India and who did all his research by himself without any guidance by others. The booklet on some famous scientists of modern India by TIFR lists Rao as 'one of those who have been instrumental in building up the vast and rich scientific culture of modern India with no infrastructure and with little support from the government'.

He is not just a statistician. He has a lot of other interests. He has interest in music and dance and pursues his hobbies of photography and gardening. When Rao moved from Calcutta to Delhi in 1970 to be at the Indian Statistical Institute, Delhi, he was surprised to find there was no dance school to teach Kuchipudi dance style and Kuchipudi did not receive the same status as Bharatanatyam, Kathak, Kathakali and Odissi. He started the Kuchipudi Dance Academy at Delhi and was its first president. The academy organized regular performances in Kuchipudi dance style. He was the president of the academy until he left for USA in 1979.

Acknowledgements I have received help from several sources in preparing this article. I would like to especially mention the material provided by the Indian Academy of Sciences and the book *Glimpses of Indian Statistical Heritage*[1]. I would also like to thank Prof. C. R. Rao for his help.

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A Secure End-to-End Verifiable E-Voting System Using Zero-Knowledge Proof and Blockchain

Somnath Panja and Bimal Roy

Conducting an election is an important and challenging task of a government to establish the democracy in the country. The constraints of conducting a secure election are remarkably strict. In the last two decades, direct-recording electronic (DRE) machines are extensively used for voting at polling stations around the world. In his seminal work [1] in 2004, Chaum proposed an end-to-end (E2E) verifiable e-voting system. Since then, the academic research on E2E verifiable e-voting system has soared. Informally, the notion of E2E verifiability refers to three properties: First, each voter can verify that her vote has been cast as intended. Secondly, she can verify that her vote has been recorded as cast. Thirdly, anyone can verify that all votes are tallied as recorded. However, in traditional paper-based voting system, a voter cannot verify how her vote has been recorded and tallied in the election. This paper presents a secure DRE-based E2E verifiable e-voting system without tallying authorities using blockchain.

There are some e-voting systems in the literature, which achieves E2E verifiability. Hao et al. proposed an e-voting system DRE-i [2] that satisfies E2E verifiability property without tallying authority. However, in DRE-i, as part of their pre-computation strategy, the pre-computed data needs to be securely stored and accessed during the voting phase. This introduces some security issues in case of an intrusive attack. If an attacker gets access to the secure storage module, the privacy of all the ballots will be lost. To circumvent this issue, Hao et al. proposed another E2E verifiable e-voting system, called DRE-ip [5]. DRE-ip also achieves stronger privacy guarantee than DRE-i. However, both the DRE-i and DRE-ip systems require a secure public bulletin board (BB) to display the recorded ballot receipts. If the public bulletin board is not secure, the integrity of the system may be compromised. In this paper, we

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A. Chaudhuri et al. (eds.), *A Tribute to the Legend of Professor C. R. Rao*,
Indian Statistical Institute Series,
https://doi.org/10.1007/978-981-33-6991-7_6

use a public blockchain as a public bulletin board. During the voting phase, all the receipts are posted on the public blockchain so that a voter can verify the her ballot is recorded on the blockchain, and then, anyone can verify that all votes are tallied as recorded on the public blockchian. The system prevents voter coercion, i.e., a voter cannot prove to an adversary how she has voted. Our system uses an ElGamal encryption algorithm whose secret key is either unknown or deleted securely during the setup phase. An ElGamal encryption of the voter's choice of candidate is given to the voter as a printed receipt. She cannot prove anyone her choice of candidate using her receipt. Consequently, this kind of coercion is avoided in our system.

Generally, in an election, a DRE machine is allocated for all voters residing in a small region to cast their votes at the polling station. Thus, revealing the tally from a DRE machine discloses the voter distribution of a small region. This is also a breach of voter's privacy to some extent. We propose a secure multi-party computation with zero-knowledge proof to compute the sum of all tallies from all DRE machines in a verifiable manner so that the tally from each DRE machine remains secret.

Since S. Nakamoto's works on Bitcoin [4], research on blockchain technology has become a thriving field. There are some decentralized e-voting systems proposed in the literature that uses Bitcoin and Ethereum [3] blockchian. In this paper, we use a blockchain to store the recorded ballots. This enables each voter to verify that their ballots are recorded as cast and tallied as recorded.

In the voter registration phase, voters provide their personal information including voter identification number and her biometric (e.g., fingerprint) to an authenticated officer. The information provided by the voters will be matched during the voter authentication phase of the election. The voter registration, voter authentication and storage of these information can be implemented using cloud technology or blockchain or combination of both.

In this system, each voter's vote is encrypted using ElGamal encryption. The secret key of the encryption is either known or securely deleted during the setup phase. The DRE also uses a digital signature scheme to authenticate each ballot. The public keys of the ElGamal encryption and digital signature algorithm are posted on the public bulletin board or blockchain.

The eligibility of each voter must be properly verified during the voter authentication phase. After proving her eligibility to vote, she obtains a random token and enters a polling booth. She uses her token to authenticate herself at the DRE machine and starts voting. The voter can cast her vote on a touch screen DRE machine. The voter performs following two steps to cast her vote.

1. In step 1, the voter goes to the DRE machine at the polling station. She selects her choice of candidate on the screen. The voting machine prints an encryption of the vote on her receipt along with the ballot serial number i and a zero-knowledge proof of well-formedness of her ballot. The receipt is digitally signed by the DRE to prove its authenticity. The signed receipt is provided to the voter. The authenticity of the receipt can be verified by anyone using its signature and the public keys. The printed receipt serves as the commitment that cannot be altered.

2. In step 2, the DRE asks the voter whether she wishes to audit or confirm her vote. If she chooses to confirm her selection, the DRE prints a 'confirm' message on a receipt and issues her the signed receipt. If she chooses to audit her selection, the DRE then prints her vote, the randomness used to generate the encryption of her vote and mark this receipt as 'audited'. The receipt is signed by the DRE. The system will then return to step 1 to enable the voter to select her choice again. The voter is allowed to challenge the system by auditing her vote as many times as she wishes, but is allowed to cast only one 'confirmed' vote. In practice, there may be an upper limit on the number of times that a voter can audit her vote.

The audit and confirm option in step 2 is used to gain vote casting assurance while using the DRE machine so that a voter can be ensured that her vote has been cast as intended. Note that, in step 1, the DRE machine commits to a value printed on her receipt and then gives her an option to challenge (i.e., either audit or confirm) the voting machine. If she wishes to challenge the DRE by selecting the audit option, the DRE prints her vote and the randomness used to encrypt her vote on another receipt and provide it to the voter in step 2 of the voting process. By the randomness and her vote, she can reconstruct the encryption and verify that the commitment (in a receipt) given in step 1 is correct or not. If the DRE records a different vote other than the voter's choice, it will be caught if the voter chooses to audit her vote. Therefore, it is highly unlikely that the DRE can cheat on a large scale without being detected. Thus, the system ensures the cast as property of the E2E verifiability, i.e., voters can verify that her vote is cast as intended. This idea of auditing is originated from the Benaloh's idea of voter-initiated auditing.

The DRE sends all these ballots (both audited and confirmed) to a public bulletin board (i.e., blockchain) via an authenticated channel. The blockchain stores all these ballots. Voters can check that her receipts are posted on the blockchain. This ensures that her ballot has been recorded as cast.

In the tallying phase, the DRE publishes the outcome of a one-way function of the sum of all the randomness used to generate the encryption of confirmed votes and the tally on the blockchain (i.e., the public BB). The public must verify the well-formedness and the authenticity of each ballot by verifying the zero-knowledge proofs and the signature attached to each ballot. Since all the encrypted vote is posted on the blockchain and tallied result is also posted on the blockchain, anyone can verify that the votes have been tallied as recorded on the blockchain.

Secure multi-party computations to compute the sum of all tallies from all DRE machines. We now briefly describe the secure multi-party computation method to compute the sum of all tallies from all DRE machines. Without loss of generality, let us assume that there are only three DRE machines having their individual private tallies p_1, p_2 and p_3, respectively. The first DRE chooses p_{11} and p_{12} uniformly and independently from $\mathbb{Z}_q$ and calculates $p_{13} = (p_1 - p_{11} - p_{12})$. Then, the first DRE sends p_{12} and p_{13} to the second and the third DRE, respectively, in encrypted format over an authenticated channel. Similarly, the second DRE chooses p_{21} and p_{22} uniformly and independently from $\mathbb{Z}_q$ and calculates $p_{23} = (p_2 - p_{21} - p_{22})$. Then, the second sends p_{21} and p_{23} to the first and third DRE, respectively, in encrypted format over an authenticated channel. Similarly, the third DRE chooses p_{31} and p_{32} uniformly and independently from $\mathbb{Z}_q$ and calculates $p_{33} = (p_3 - p_{31} - p_{32})$. The third DRE sends p_{31} and p_{32} to the first and second DRE, respectively, in encrypted format over an authenticated channel. Then after receiving p_{21} and p_{31} from the second and the third DRE, respectively, the first DRE calculates $P_1 = p_{11} + p_{21} + p_{31}$. Similarly, after receiving p_{12} and p_{32} from the first and the third DRE, respectively, the second DRE calculates $P_2 = p_{12} + p_{22} + p_{32}$. Similarly, after receiving p_{13} and p_{23} from the first and the second DRE, respectively, the third DRE calculates $P_3 = p_{13} + p_{23} + p_{33}$. All the three DRE machines send their P_i's to a fourth party who computes $P = P_1 + P_2 + P_3$. Note that P is the sum of three DRE machines private tallies, i.e., $P = p_1 + p_2 + p_3$. We also use zero-knowledge proofs to prove that all the DRE machines follow the protocol honestly and compute the final tally correctly. Note that the private tallies p_i of each DRE machine remain secret while computing their sum, for all $i \in \{1, 2, 3\}$. Thus, untrusted parties can compute the sum of their private inputs.

The system provides an efficient and practical DRE-based voting solution preserving secrecy of ballots and integrity of the system. The public bulletin board can be a blockchain [6].

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Glimpses from the Life and Work of Dr. C.R. Rao: A Living Legend in Statistics

Anil Bera and Priyasmita Ghosh

Prologue and Family

As per the Indian mythology, Krishna, the eighth avatar of Lord Vishnu, was the eighth child of Devaki and Vasudeva. Coincidentally, our Calyampudi Radhakrishna Rao was the eighth offspring of his parents, mother A. Laxmikantamma, and father C.D Naidu (1879–1940). Although Rao was universally addressed as Dr Rao by all his students and colleagues at the Indian Statistical Institute (ISI), where the "Professor" title was exclusively reserved for Prasanta Chandra Mahalanobis (PCM), the history of his birthname is quite interesting. Not only that he was named after Lord Krishna, but also his middle name symbolizes the purest form of romance. Rao always claims that though he is a romantic by temperament, yet all his romance is somehow in the wrong place, that is in his head instead of his heart!

Continuing with the spirit of Radha and Krishna, let us consider the following excerpt from the lyrical Sanskrit poem *Gita Govinda*, by the twelfth century poet Jayadeva[1]:

"And, led by Radha's spirit,
The feet of Krishna found the road alright;
Wherefore, in bliss which all hearts inherit,
Together taste they Love's divine delight."

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A. Chaudhuri et al. (eds.), *A Tribute to the Legend of Professor C. R. Rao*,
Indian Statistical Institute Series,
https://doi.org/10.1007/978-981-33-6991-7_7

The significance of the above lines will indeed constitute the essence of this essay, i.e., one cannot really talk about the life and work of Dr. Rao without referring to PCM and ISI. In fact, in a way, PCM, Rao and ISI form a trinity where PCM can be viewed as the father, Rao the son and statistics (or more specifically ISI), the holy spirit. Let us take some liberty in paraphrasing the above excerpt from Jayadeva in this context as:

So PCM guided, and led by Statistics' spirit,
The feet of CR Rao found the road alright;
Wherefore, in "numbers" which all hearts inherit,
Together they uplift ISI's sight.

In 1989, Rao wrote a book *"Statistics and Truth: Putting Chance to Work"* [CSIR Ramanujan Memorial Lectures], and nothing is more fitting than his own life as the best empirical evidence of *"Putting Chance to Work."* The story of his joining the ISI in 1941 as a "statistics trainee" by a chance encounter is well known. However, such an inspirational story cannot be recounted enough. It was a hot summer day of June 1940. The World War II (WWII) was raging in full swing with its devastation and at the same time promising jobs to the vast unemployed youths of India amid all that chaos. Rao not yet 20 then set out on a 500-mile train journey from the coastal city of Visakhapatnam, India to Calcutta, the second largest city of the British Empire, after obtaining a first class first degree in mathematics and with a glimmer of hope of finding a job in the military. Rao was not so "lucky"; he was deemed too young for the job. However, while in Calcutta, through a chance encounter, he visited the ISI founded in 1931 by PCM, a Cambridge-trained physicist. As a last resort, he applied for the one-year training program in statistics at ISI, with a letter of recommendation from the vice chancellor of Andhra University, Professor V. S. Krishna, who was known to PCM. Very promptly, he received a positive reply from PCM admitting him to the one-year program of the ISI from January 1, 1941. And as we all know the rest is history. In fact, a very long history. Rao did not get the job he came for but found something that would keep him engaged for the next 80 years of his life and the engagement still continues.

Below we present photographs providing details about Rao's lineage and his big family in 1931, respectively.

Order (Statistic)	Year	Name
1	1909	Laxmi Narasmma (Girl)
2	1911	Sanjeevamma (Girl)
3	1912	A male child
4	1913	Sakuntalamma (Girl)
5	1915	A male child
6	1916	Thippanna – “Garbage Heap” (Boy!!)
7	1917	Venkateswara (Boy)
8	**1920**	**Radhakrishna**
9	1922	Ramachandra (Boy)
10	1925	Neelavati (Girl)

With family 1931, sitting on floor (*from l to r*): Thiappanna, Venkateswara, Radhakrishna and Ramachandra. Sitting on chairs (*from l to r*): Sanjeevamma, Chellamma (*father's sister*), father C.D. Naidu (*with grandson on his lap*), mother Laxmikanthamma (*with Neelavati on her lap*) and mother's grandmother. Standing (*from l to r*): Policeman in attendance to C.D. Naidu, Narsamma, orderly in attendance to C.D. Naidu, Sakuntalammma and a poor boy staying with the family.

Publications: Two Masterpieces by 30

After Rao passed M.A. in statistics with record marks in 1943, PCM offered him a part-time lectureship at the Calcutta University (CU). In 1944, Rao was giving a course on estimation to the senior students of the master's class at CU, where he mentioned without proof Fisher's information inequality for the *asymptotic* variance of a consistent estimate. There a bright young student Vinayak Mahadev Dandekar (VMD) (1920–1995) who was of the same age as Rao raised the question whether such an inequality exists in *finite sample.* Rao did not know the answer in the class; however, at night in his tiny apartment, with a simple application of the Cauchy-Schwartz inequality, Rao solved the problem and discussed the result in the next class!

While writing a note on this result for publication, Rao discovered many related results. Since the publication of *Sankhyā,* a journal started by PCM in 1933, was suspended during the WWII, the paper was published in the *Bulletin of the Calcutta Mathematical Society* (1945) [1]. Quite interestingly, the same WWII brought Rao from Vishakhapatnam to Calcutta and eventually to ISI! Instead of asking what is in this paper, one should ask what is not? It has (i) Cramér–Rao inequality, (ii) Rao–Blackwell theorem, (iii) Differential geometry for the first time in statistical literature, and finally, (iv) Fisher–Rao metric. Any first graduate course in statistical inference will be incomplete without the above two results, (i) and (ii). Here, we think, one should not overlook the origin of Rao's 1945 paper, i.e., the question raised by the student VMD. He later became the Director of Gokhale Institute of Politics and Economics, Pune, and did pioneering work on poverty, unemployment and income inequality in India. Just imagine what would happen, if in classrooms we have students like VMD and also teachers like Rao!

In his second masterpiece that appeared in the *Proceedings of The Cambridge Philosophical Society* (1948) [2], Rao introduced altogether a new test principle, famously known as Rao's score (RS) test, as an alternative to Neyman–Pearson likelihood ratio (LR) and Wald (W) tests. Thus, we now have another "holy trinity" in statistics, namely LR, W and RS. The seed of the paper was planted in Calcutta in 1944. That too is an interesting story.

S. J. Poti who joined the ISI as a teaching assistant found himself without a place to stay in Calcutta. Observing Poti's plight, Rao offered to share his small single room in the fourth floor of a building. The room had no kitchen, and they shared a bathroom and toilet with all the tenants of the fourth floor. They had no cots and would spread their beds on the floor to sleep at night and then roll the beds up in the morning. The two used to go for long walks in the evening, often discussing problems in statistics. One day Poti asked Rao whether the Neyman–Pearson (NP) theory could be used to test a hypothesis about a parameter when the alternative is one-sided. Rao gave an immediate solution that was published as a note in *Sankhyā* (1946) [3], which using the NP lemma showed that a locally most powerful (LMP) test must be based on the score function, i.e., the derivative of the loglikelihood with respect to the parameter, evaluated *under the null hypothesis*. This little obscure note can be viewed as a precursor to the Rao's pathbreaking 1948 paper, though the general idea for RS test evolved in natural way while Rao was analyzing some genetic data. The problem was estimation of a linkage parameter using data sets from different experiments designed in such a way that each data set had information on the same linkage parameter. Thus, like most of Rao's work, the RS test is an example of a statistical method motivated by a practical problem.

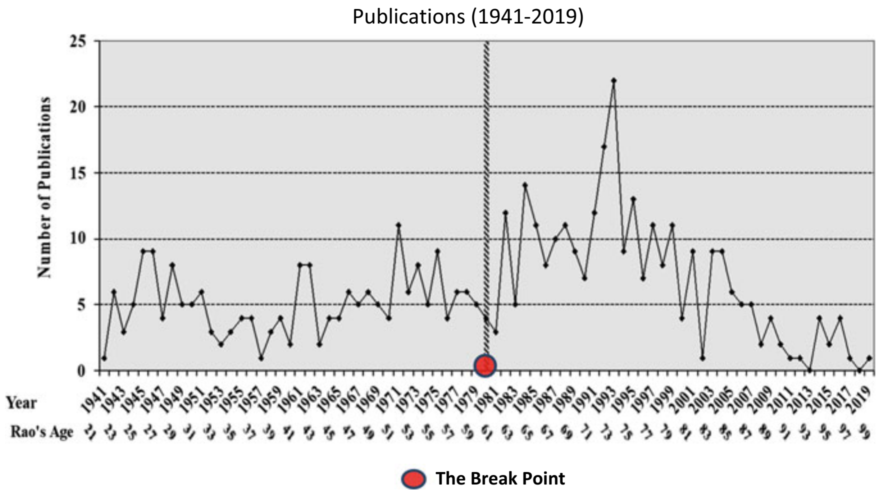
The importance of RS test in statistics and econometrics cannot be overstated; it is one of the most useful tools in evaluating and testing statistical and econometric models. Also, there are many well-known tests in the literature that were suggested long before 1948, whose theoretical foundation can now be buttressed by RS test principle. As mentioned above, RS test requires estimation of the score function only *under the null*, thus facilitating a huge simplification of the final form of the test statistic. In most applications, RS statistic has a simple close-form expression which is quite unthinkable, in many cases, for the LR and W tests.

One of us (Bera) has a first-hand experience in using the RS test in devising the following widely used test for normality, popularly known as the Jarque–Bera (JB) statistic first published in 1980 [4]:

$$\mathrm{JB} = \sqrt{n}\left[\frac{\left(\sqrt{b_1}\right)^2}{6} + \frac{(b_2 - 3)^2}{24}\right],$$

where n is the sample size and $\sqrt{b_1}$ and b_2 are, respectively, the sample skewness and kurtosis. JB was derived using the Pearson system of curves as alternative (to normality) hypothesis, for which LR and W are almost impossible to apply. Not only the RS principle leads to a neat and beautiful expression, but also it uncovers the optimality implications, namely LMP, of using sample skewness and kurtosis. JB and its various extensions have around 8,000 citations. It will not be an exaggeration to say that Bera's whole livelihood had been based on RS.

We can pick up each paper of Rao and go through its origin and underlying history; however, that would be a very long exercise, not suitable for this essay. Thus, to cut the long story short, below we present a graph of Rao's number of publications over the last eight decades (1941–2019).



Clearly, there is a structural break around the year 1980 when he took the mandatory retirement from ISI at the age of 60 and move to the University of Pittsburgh, USA. Simple counting says, he published 203 papers by that time, and another 271 papers over the last 40 years. Dr. Rao's research had significance beyond statistics and econometrics, like the Quantum Cramér-Rao bound providing sharper versions of Heisenberg's Principle in Quantum Physics. He has made a major contribution to the combinatorial theory of design by extending the notion of orthogonal Latin squares through the notion of orthogonal arrays [5]. In the last two decades, he has also touched upon nonlinear methods, resampling methods, neural networks, and data mining. It is inspirational to see how a man who is about to cross a century is determined to stay up-to-date in the main stream of modern statistical learning and data mining.

Rao, The Master Guide

After Raj Chandra Bose and Samarendra Nath Roy left ISI in 1949 and 1950, respectively, and PCM got busy with national planning, Rao became the natural successor to the leadership of research and training of the institute and was the doctoral thesis adviser of many bright students. Over his full academic life, Rao has directed the research work of more than 50 Ph.D. students who in turn produced more than 350 Ph.D.'s.

About our very own Professor Ranga Rao at UIUC [6], we note, "Ramaswamy Ranga Rao is a prominent Indian mathematician. He finished his Ph.D. under the supervision of C.R. Rao at ISI, Calcutta. He was one of the 'famous four' students of Rao in ISI during 1956–1963. Ranga Rao is now professor emeritus of mathematics at University of Illinois. He made fundamental contributions to statistics, Lie groups, and Lie algebras." The four, K. R. Parthasarathy (KRP), Veeravalli S. Varadarajan (VSV), S. R. Srinivasa Varadhan (SRSV) and Ramaswamy Ranga Rao (RRR), are known as fabulous (fav) four in ISI legend. VSV joined the institute in 1956 and in a marked departure, instead of statistics, he decided to work on probability theory, and started his research career by learning the necessary mathematics by himself within a year or so. RRR and KRP became partners of VSV, and quite coincidentally like VSV, they also had B. Sc. Honors degree in Mathematics from the University of Madras. VSV completed his Ph.D. in 1959, and in the same year, SRSV came to ISI as a Ph.D. student [7]. SRSV later went on to receive the Abel prize of the Norwegian Academy of Sciences (considered as an equivalent to the Nobel prize, there being no Nobel prize for Mathematics) in 2007. Like each of the previous three, he also did a B.Sc. Honors from Madras before coming to Calcutta. Although, at times helped by the brilliant statistician Raj Bahadur, it was Rao who provided the encouragement and guidance to this "fav four." Rao created a wonderful academic atmosphere for young scholars to carry out their perennial discussion on mathematics, most of the time conducted in Tamil. There is a funny anecdote about a non-mathematician and non-Tamilian[6], who claimed to have learnt a Tamil word "ergodicity" from overhearing this intense discourse taking place among the "fav four."

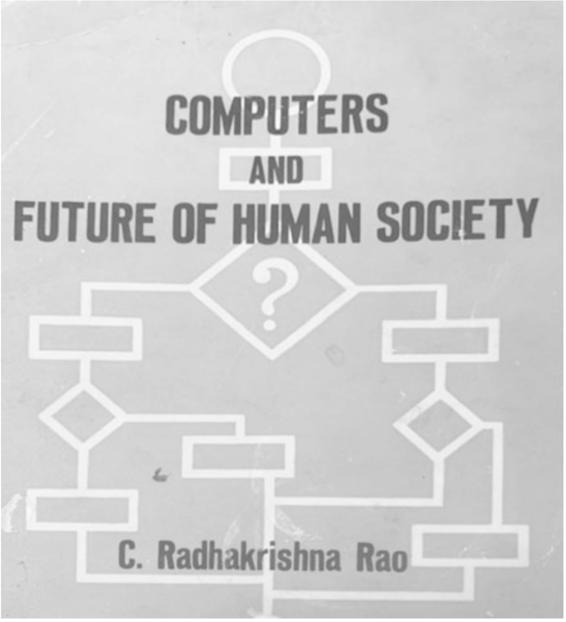
Dr. Rao had an unorthodox modus operandi while guiding Ph.D. students. His first Ph.D. student was Debabrata Basu whom he recruited in 1950. The celebrated Basu's theorem arose out of a question Rao put to Basu, about the existence of maximal ancillary statistic. Basu proved the nonexistence of such a statistic and in addition established the independence of an ancillary statistic and the minimal sufficient statistic. In a recent interview in "*Bhāvanā"* [8], KRP mentioned that on advice of C.R. Rao he read the "*Mathematical Foundation of Information Theory*" by A.Y. Khinchin [9], but was not satisfied with one particular chapter. KRP rewrote that chapter which involved learning some ergodic theory and about the probability measures invariant under transformation. KRP sent it to Joseph L. Doob of the University of Illinois at Urbana Champaign (UIUC) and who in turn gave it to measure theorist, John C. Oxtoby. Oxtoby wrote and published a paper giving full credit to KRP [10]. In those days, there were no regular fellowships at ISI and whatever Rao decided was the scholarship amount. When KRP showed Oxtoby's paper to C.R. Rao, his scholarship was doubled. That was the kind of incentive Rao provided to his Ph.D. students.

During the 1962 ISI convocation time, Dr. Rao asked KRP and SRSV to suggest a prominent scientist from USSR who could be invited. The duo suggested the most famous of all, the legendary probability theorist A.N. Kolmogorov (ANK). Rao's response was simply, "No problem. We will get him." Rao put a word to PCM who in turn put the right word at the right time to the USSR Academy of Sciences. ANK, possibly the most important visitor that ISI has ever received, came on April 14, 1962, and stayed till May 12. ANK's visit testifies the importance Rao paid to students' needs and suggestions. The visit was a huge moral booster for not only the fledgling young ISI probabilists but also for the whole of India, all the more as ANK rarely traveled outside USSR.

Rao's guidance went far beyond Basu and the fab-four. During 1960s, Rao had five or six students working for Ph.D. at any given time, including I.M. Chakraborti in design of experiments; R.G. Laha in characterization problems; J. Roy in multivariate analysis; A.C. Das and Des Raj in survey sampling and A. Matthai on quality control. Very substantially if not wholly, the cluster of these brilliant and creative Ph.D. scholars of C.R. Rao enthroned the ISI to its golden period during 1960s.

Rao's Foresight on Computer Technology

Long before the era of digital revolution, Rao wrote the book "*Computers and Future of Human Society*" published in 1969 [11]. Being a visionary, he anticipated that the path to economic development and prosperity is tied to the use of the computers, at a time when less than 1% of the world knew the use of computers. In his words, "We need computers to guard our land frontiers and the long coasts, save us from floods and fury of storms, defend ourselves against external aggression, improve our agriculture to feed the teeming millions, give the best education to our children, help in better medical diagnosis and save the lives of patients, and provide the people with necessary comforts, facilities and opportunities in life."



Published in 1969

He traced the history of computers, both hardware and software and analyzed questions like: "Is the human brain superior to the computer? Are we likely to succeed in producing a robot identical with human being in all respects except in origin?" that were much ahead of their time. He laid out that computers can prove to be a great boon for a developing country like India, to close the productivity gap with some of the advanced countries of the world. But he also admitted that there is a dangerous possibility that robots controlled by computers would run factories, and today it is not a secret that many jobs have been lost to automation.

He pretty much forecasted the future of the world lies in computer programming, and we are now witnessing that every day in our lives. The field of computer programming, especially machine learning, is experiencing exponential growth today, while in Rao's time, computers were hunks of machinery, the size of a room. Rao wrote about Anne (shown in the photo above), "She is looking forward to the day when such a console typewriter in installed in her home as telephones are now. She can utilize the computer which may be far away sitting at her home and any time." Rao anticipated that the day is not far when a computer console will be installed in every home and almost envisioned that one will be able to utilize it at any time to get access to any type of information, like checking email, surfing Internet and having all kind of data at fingertips. Rao perceived the world with computers in ways that we never thought was possible.

Rao visited the UIUC, in 1953–1954 as a visiting research Professor of Mathematical Statistics. During his stay at UIUC, he had access to Illiac, the first digital computer in the United States. He took a course on programming using machine language and started using the computer to do his own computations. The UIUC gave him two students to work on developing computer programs for statistical methods. But unfortunately, on returning back to India, Rao did not have an opportunity to use computers, and his research involving heavy computations was not possible with hand-driven desk computers. The ISI had acquired a digital computer, but the workers' union at the ISI protested the use of computers. This obstructed the possibility of creation of computer-intensive statistical techniques in India well before others did. This was such a big disappointment for Rao that he was even ready to resign; however, PCM and other administrators of the ISI persuaded him to withdraw the resignation.

The Women Behind the Man

Rao dedicated his book "*Statistics and Truth: Putting Chance to Work,*" to his mother A. Laxmikantamma, and credited her "For instilling the quest of knowledge" in him and "who woke him up "every day at four in the morning and lit the oil lamp to study in the quiet hours of morning when the mind is fresh." Rao remembers his mother as a great disciplinarian, and she controlled the daily activities of her children, prescribing the time for playing, studying, and sleeping. This regiment later helped him in leading a disciplined and successful life. He had very little contact with his father until he retired as he was a police officer that required him to work at different locations. Rao was fortunate to have parents who fostered his innate abilities with proper guidance, provided an environment conducive to study, and gave a framework for ethics of life. He himself admits that genetics played an important role in his achievements and goes on to say "I inherited my father's analytical ability and my mother's tenacity and industry."

Top: Rao with his wife Bhargavi, 1954

Right: Rao's daughter Tejaswini, striking a pose.

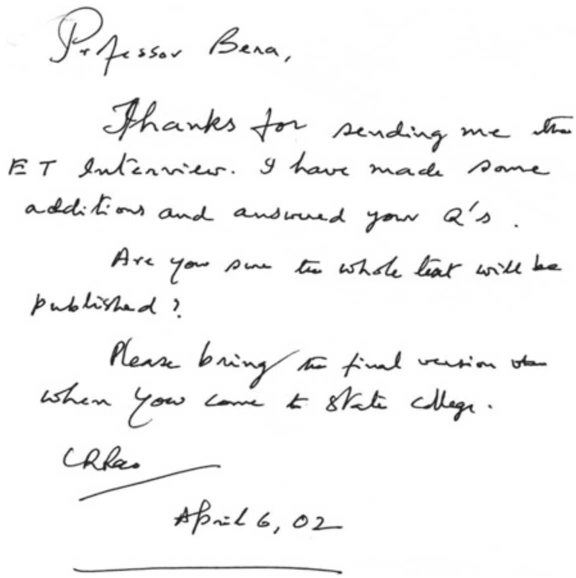
On returning from Cambridge to India in August 1948, Rao got married on September 9, a day before his 28th birthday. In the initial days of their marriage, Bhargavi struggled to keep up with the activities of Rao; however, soon she realized the importance of the work that he was doing and adjusted herself to the life of an academician. She supported Rao and provided him with an environment at home to pursue his research peacefully. She had two master's degrees, one in history from the Benaras Hindu University in India and another in psychology from the UIUC. In Calcutta, she worked for some years as a high school teacher and for a number of years as a lecturer. Rao and his wife were happily married for over 65 years, before she passed away in 2016.

Mrs. Bhargavi Rao narrated a funny anecdote about her husband, in the book, "*Putting Chance to Work; A Life in Statistics: A Biography of C.R. Rao*" by Nalini Krishnankutty [12], published in 1996. Once a visiting scientist came from the Soviet Academy of Sciences, to meet with Dr. Rao personally. He knocked on the door to be opened by Mrs. Rao and enquired about Dr. Rao. He was amazed that like the Soviet scientists, Dr Rao did not live in a big guarded house. She told him that her husband was downstairs near the car. He replied that he had only seen a mechanic working on the car, but not C.R Rao. Then, Mrs. Rao informed him that it was not just any mechanic, he had actually seen The C.R. Rao. The visitor returned in the evening and saw Rao playing badminton with his colleagues, and the next day he went to the ISI office to meet the Director of ISI. To his surprise, Dr. Rao was the Director, sitting in a room resembling a cubicle with only the flap door closed and easily accessible to everyone. Dr. Rao invited him home for dinner and welcome him like a perfect host, with vodka and Russian caviar. The visitor while leaving the house exclaimed, "I have seen the mechanic, the athlete, the scholar and the perfect host, all in one day." Like Lord Krishna had (108) names and roles—Dr. Rao wore many hats and that too so successfully.

Rao's daughter Tejaswini is an accomplished Indian classical (Kuchipudi) dancer and runs a dance school. She has a Ph.D. in nutrition and is a Professor at SUNY, Buffalo. In 1970, Rao moved to ISI, Delhi, and was surprised to find that there was no dance school to teach Kuchipudi. Because of his interest in Indian classical dance, Rao started a Kuchipudi Dance Academy in Delhi and was its President until he left for the USA in 1979. Rao and his wife admitted Tejaswini in a dance school at the age of eight. When she performed in public, Rao often supervised the performances, the music, the introductions, and the lighting. No matter how busy he was he would always take the time to ensure the performance was well planned. He made her do demonstrations of abhinaya in English translations of her dance music. This is very common now during the Indian dance performances, but it was unheard of in the early 60s. So in this matter too, Rao was well ahead of his time. Rao had an artistic vision, and dancers today follow techniques that were made to Tejaswini more than half a century earlier. Tejaswini now regrets that she did not take on some of his suggestions and act on them more seriously. Currently, Rao is taken care of by Tejaswini at her Buffalo home.

Tejaswini admits that she and her brother Veerendra did not know much about their father during their stay in India, as Rao never talked much about himself. It is only after he moved to the USA that they learned about him as a scientist and as a person who is a very sensitive and caring person, with a great but subtle sense of humor. His family wished that he would take some time off to enjoy with them; however, Rao's greatest pleasure in life comes from being immersed in work and research.

Rao Modesty and Magic


Professor Bera,

Thanks for sending me the ET Interview. I have made some additions and answered your Q's.

Are you sure the whole text will be published?

Please bring the final version when you come to State College.

CRRao

April 6, 02

One of us (Bera) was involved in preparing the *Econometric Theory (ET)* Interview with Dr. Rao [13], at the invitation of the Editor: Professor Peter C.B. Phillips (PCBP). As the interview went along, its size grew longer and larger, and at one stage, it was almost 100 typed pages. Rao became worried, and his modesty was expressed as "Are you sure the whole text will be published?," as in the handwritten note below.

With a lot of trepidation, Bera submitted the interview to *ET* on May 31, 2002, expecting a response in a few weeks, from the Editor, most probably with a lot of suggestions to cut down the size. Quite surprisingly, within a week, apt came the Editor's letter of acceptance without a single correction (reproduced below)! This is simply Rao magic.

RE: Rao Interview - Message Page 1 of 2

WebMail - RE: Rao Interview

Delete | File | Create | Reply | Reply All | Forward | Previous | Next | Options | Index | Help

Date Sent: Wednesday, June 05, 2002 11:52 AM

From: "Peter C. B. Phillips" <peter.phillips@yale.edu> Add to Address Book

To: a-bera <a-bera@uiuc.edu>

Cc: Mary Moulder <mary.moulder@yale.edu>

Subject: RE: Rao Interview

Status: ☐ Urgent ☐ New

Dear Anil:

I have now had time to read the Interview with C. R. Rao carefully.

It is truly outstanding, extremely thorough and has some wonderful new material and historical insights in it. Both you and Dr Rao have done a superb job. In fact, I can tell you that this is the first interview I have accepted without a single correction. I couldn't find anything that I would change.

So, congratulations to you and Dr Rao for this marvellous interview. It will be a superb historical record and document for scholars in the future, as well as a tremendously good read for us all in econometrics and statistics.

The interview is accepted herewith. It will probably appear in the second or third issue next year and you'll get proofs some months before. You might like to send in a final copy with your preferred address for proofs and contact numbers detailed in the file and on the front page.

Please send along Dr Rao's email address and I'll write him a separate note of thank you. Please, in addition, convey my personal thanks to him for this marvellous interview and all the time and effort he has expended on it.

Every best wish.

Peter

At 05:16 PM 10/4/01 -0500, you wrote:
>Dear Peter:
>
>I hope this finds you in best of health.

https://webmail.uiuc.edu/MBX/a-bera@staff.uiuc.edu/ID=3D4B13D9/MSG:1 6/5/2002

Not only that, the Editor also sent the following email to Rao:

Peter C B Phillips, 11:54 AM 09/13/2002 -0400, Thank you

X-Sender: pcb@pcb.mail.yale.edu
X-Mailer: Windows Eudora Pro Version 3.0.1 (32)
Date: Fri, 13 Sep 2002 11:54:57 -0400
To: crr1@email.psu.edu
From: Peter C B Phillips <peter.phillips@yale.edu>
Subject: Thank you
Cc: a-bera@uiuc.edu
X-YaleITSMailFilter: Version 1.0c (attachment(s) not renamed)

Dear Professor Rao:

I am writing to thank you for the time and effort you put into doing the Interview with Anil Bera for the journal Econometric Theory.

The interview is a delight to read, is thoughtful, wideranging, and contains some wonderful historical insights. In additional to being a marvellous interview, I believe it will be a superb historical record and document for scholars in the future. I am confident it will be read and enjoyed by all econometricians and, I'm sure, the much larger community of statisticians.

Thank you again. And congratulations to you and Anil Bera for such a superb job. Your interview will appear in the second issue of 2003.

With respect and best regards,

Peter C. B. Phillips

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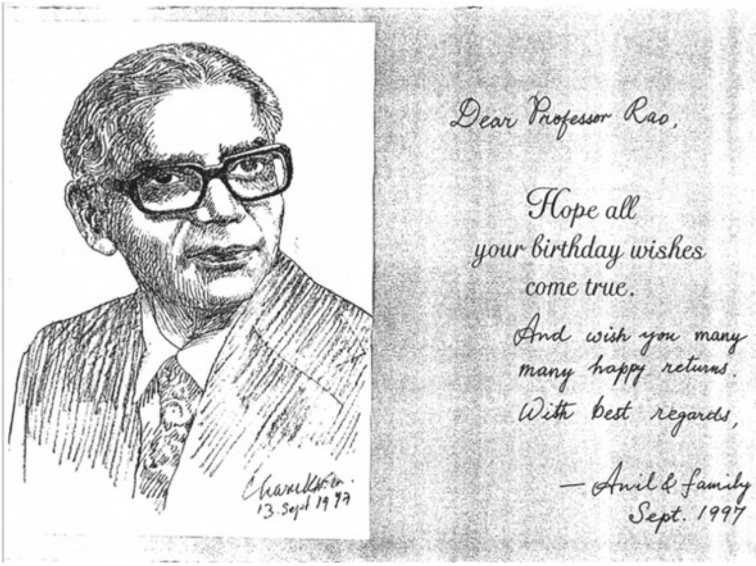
Here, quite fittingly, the Editor marvels at the historical insights provided by Rao that will be read and enjoyed by not only econometricians but the wider community of statisticians.

Epilogue and Celebrations: Given that my academic career started with RS Test and still continues, I (Bera) have always been in touch with Dr. Rao for research and on special occasions. None of my emails/letters has gone without a prompt reply from Dr. Rao. I wish I were equally prompt with his letters and emails. On the occasion of his 77th birthday in September 1997, I sent him Birthday wishes with a sketch by eminent Indian artist Charu Khan. Rao appreciated it very much and sent a warm reply back.



Rao's sketch by painter Charu Khan.

Rao is recognized internationally as a pioneer who laid the foundation of modern statistics, with multifaceted distinctions as a mathematician, researcher, scientist, and teacher. His contributions to mathematics and to the theory and application of statistics have become part of graduate and postgraduate courses in statistics, econometrics, electrical engineering, and many other disciplines at most universities throughout the world. Rao is loved and respected all over the world. His birthdays are celebrated by the entire econometrics and statistics community with great enthusiasm, internationally.

For his outstanding research contributions, C.R. Rao has been honored with the establishment of an institute named after him: C.R. Rao Advanced Institute for Mathematics, Statistics and Computer Science (C.R. Rao AIMSCS), in Hyderabad, India. It is a fitting tribute to Rao, since he hails from this part of India. The institute is engaged in cutting edge research in the areas of Statistics, Mathematics, Computer Science, Wireless Communication, and interdisciplinary fields. Fittingly, the C.R. Rao Birth Centenary Celebrations started at the AIMSCS on his 99th birthday, September 10, 2019, with a video message from Rao himself (a screenshot of that is given below).

In his message, Rao said, "I was fortunate to have made some fundamental contributions to the field of statistics and to see the impact of my work in furthering research. In my lifetime, I have seen statistics grow into a strong independent field of study based on mathematical, and more recently computational, tools. Its importance has spread across numerous areas such as business, economics, health and medicine, banking, management, physical, natural, and social sciences. Statistics is the science of learning from data. Today is the age of data revolution. There is therefore, a heightened need for statistics- both in terms of training in statistics to help analyze and interpret the data, and in terms of research to answer new questions arising from the data. [14]" This succinctly sums up the long statistical journey of C.R. Rao and also indicates that it is the Data Science what is stored for the future of Statistics.

The 2019 International Indian Statistical Association (IISA) Conference at Mumbai, India, on Dec 26th–30th, 2019, brought together statisticians worldwide from academia, industry, government, and research institutions to explore the latest developments and challenges in the era of Data Science and Statistical Learning. For that occasion Tapan Nayak, a former Ph.D. student of Dr. Rao organized the C. R. Rao Honorary Session with following four presentations:

A new all-purpose generic multivariate transformation with applications in multivariate modeling and missing value imputation, Ravindra Khattree, Oakland University.

Analyzing periodic and nearly periodic data: Statistical perspectives, Debasis Kundu, Indian Institute of Technology, Kanpur.

Shailaja Suryawanshi, Merck & Co.

The Trinity: Professor Mahalanobis, Dr. Rao, and the ISI, Anil Bera, University of Illinois at Urbana-Champaign (UIUC).

Incidentally, Shailaja was the last Ph.D. student of Dr. Rao to graduate from the Pennsylvania State University in 1996 with a dissertation titled, *Analysis of High-Dimensional Data in Problems of Regression and Discrimination, with Applications to Size and Shape Analysis*. The Session was very well attended followed by a very lively discussion.

Of course, there will be many more tributes coming to Rao in the coming months and years analyzing his research contributions. However, we should not forget the man, or the "life" behind all the work—the humble and the unassuming person. Here, we can only borrow a couple of lines from Rabindranath Tagore to express our feelings for Rao, "You are greater than your achievements. The chariot of your life leaves your achievements behind, time after time."

তোমার কীর্তির চেয়ে তুমি যে মহৎ, তাই তব জীবনের রথ
পশ্চাতে ফেলিয়া যায় কীর্তিরে তোমার, বারম্বার ।

Acknowledgements We would like to thank Professor Arijit Choudhuri for his kind invitation to contribute an essay on the occasion of C.R. Rao centenary. We are also thankful to Yufan Leiluo for his comments and suggestions on an earlier draft of this essay. All omissions and misreadings are ours.

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My Memorable Interactions with Professor C. R. Rao

J. N. K. Rao

I was doing a Master degree in Statistics from Bombay University during the period 1954–1956. In the first year itself, I was exposed to the famous Cramer–Rao lower bound on the variance of any unbiased estimator of a parameter of interest and the Rao–Blackwell theorem relating sufficient statistics to unbiased estimators by showing that conditioning any unbiased estimator on a sufficient statistic leads to uniformly better unbiased estimator. Professor Rao was 25 years old when he did the above seminal work. I was also inspired by C. R. Rao's pioneering 1952 Wiley book *Advanced Statistical Methods in Biometric Research* which was regarded as a bible for the theory of mathematical statistics. In the summer of 1955, I went to my home town Eluru in Andhra Pradesh for holidays. I had great passion for cricket although I never excelled in cricket. I had a close cricket friend in Eluru and I was playing cricket with him one fine day. I talked to him about my graduate studies and in particular mentioned that C. R. Rao was my hero in statistics. I was frozen when he told me that Professor Rao was his maternal uncle and that he was passing through Eluru railway station the next day on his way back to Calcutta after visiting Madras to conduct some examinations there. He also asked me if I would be interested in meeting him. Of course I said yes but I could not sleep the whole night thinking about meeting the great statistician. The next day morning I met Dr. Rao at the railway station. Even though the train stopped for only ten minutes at the station and several family members were present including his sister, Dr. Rao was kind enough to speak to me after his nephew told him that I was doing a Master degree in statistics. I will never forget my first interaction with him. Dr. Rao's nephew told me later that two famous test cricketers, the legendary C. K. Naidu and his younger brother C. S. Naidu, were Dr. Rao's uncles. This was an exciting news to me given my passion for cricket.

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A. Chaudhuri et al. (eds.), *A Tribute to the Legend of Professor C. R. Rao*,
Indian Statistical Institute Series,
https://doi.org/10.1007/978-981-33-6991-7_8

I joined the Indian Statistical Institute (ISI) after my Master's degree in 1956 to do a Diploma degree and I had the privilege of attending a class taught by Dr. Rao. However, I left ISI after a month to pursue a research career elsewhere. In 1967, I was a full professor at Texas A&M University and well settled. But I needed to visit India for an extended period due to pressing family problems. I wrote to Dr. Rao that I would be interested in visiting ISI and he promptly offered a visiting professor position for one year and I joined ISI in July 1968. I was fortunate to associate myself closely with Dr. Rao and we had many stimulating conversations in the Research and Training coffee room as well as when we were walking around the pond in Amrapali compound. Mrs. Bhargavi Rao and my wife Neela became very good friends and we had several family gatherings during my stay at ISI.

During my stay at ISI, Professor Rao invited me to give a seminar on my joint work with Professor H. O. Hartley on variance estimation under stratified random sampling with one unit per stratum. In this case, standard design-based methods for variance estimation of a stratified mean are not applicable, and it is necessary to resort to models. We used a linear regression model with unequal error variances and expressed the variance of the stratified mean as a linear combination of the error variances. We then developed a new method of estimating the error variances which in turn led to a new variance estimator of the stratified mean. We submitted the paper for publication before I left for India. After my talk, Professor Rao was excited and told me that he could establish optimality properties for our method. This led to his well-known MINQUE method (C. R. Rao, 1970) and my paper was published in 1969 (Hartley et al. 1969). He was kind enough to mention in his paper that "The motivation for writing this article is a recent contribution by Hartley, Rao and Kiefer (1969) who obtained unbiased estimators when all the variances are unequal." Subsequently, Professor Rao studied MINQUE theory extensively in a series of papers including extensions to linear mixed models.

In the 1960s, V. P. Godambe was making important contributions to the foundations of survey sampling inference using the distinct labels of the sample units as part of the observed data. This setup led to a flat likelihood function providing no information on the non-sampled units. To overcome this, difficulty Hartley and I proposed that some aspects of the sample data, depending on the situation at hand, need to be ignored to arrive at an informative nonparametric likelihood (Hartley and Rao, 1968), now called empirical likelihood (Owen, 1988). Dr. Rao invited me to give several lectures on the foundations of inference in survey sampling including the Hartley–Rao approach. He attended every lecture and participated actively. Professor Rao wrote a nice article later (C. R. Rao, 1971) addressing the flat likelihood problem and his views seem to agree with the Hartley–Rao approach: "In situations like the one we are considering where the full likelihood does not satisfy our purpose, we may have to depend on a statistic which for every observed value supplies information (however poor it may be) on parameters of interest."

After leaving India in 1969, I kept in touch with Professor Rao and also met him at the 1971 International Statistical Institute meetings in Washington DC. I was also fortunate to meet him at several statistical meetings in North America and Europe after he left India to take up a position in USA. He was kind enough to invite me to give a plenary talk at a conference in San Antonio in 1990 on the occasion of his seventieth birthday. He also invited my son Sunil, who works on modern statistical methods, to give several talks at a conference where the famous Leo Breiman was the principal speaker.

My wife and I visited Dr. Rao and Mrs. Rao in Buffalo on our way to attending a 2014 conference celebrating Malay Ghosh's 70th birthday in Washington DC. His daughter, Tejaswini, insisted that we stay at their house and it was a memorable visit. As soon as we entered the house, Dr. Rao took me straight to his study room and showed the draft of a paper he was writing on the computer. It is truly amazing that he continued to be active in research at 94.

It is my great privilege and pleasure interacting with Professor C. R. Rao over the years. My wife Neela and my son Sunil join me in sending our greetings and very best wishes on the occasion of his 100th birthday.

Further Non-subjective Priors for Wrapped Cauchy Distributions

Malay Ghosh and Ruoyang Zhang

Introduction

It is our great honor and pleasure to contribute to this volume celebrating the centenary of Professor C.R. Rao. Professor Rao is indeed a living legend in statistics. His contribution encompasses virtually every single area of our field. His pioneering contribution has inspired several generations of statisticians and will continue doing so for years ahead.

Back to the subject matter of this paper, wrapped Cauchy distributions occupy a very prominent place for the analysis of directional data. These distributions have found multiple applications in image processing, proteomics, geological data, just to name a few. Ghosh et al. (3) performed a Bayesian analysis for such distributions using some non-subjective priors, in particular, reference priors and quantile matching priors. One of the virtues of such priors is that they achieve Bayes–frequentist synthesis by matching asymptotically coverage probabilities of Bayesian credible intervals and frequentist confidence intervals. We want to explore in this article other matching priors, namely highest posterior density (HPD) matching priors and matching priors, associated with likelihood ratio (LR) statistics (Ghosh and Mukerjee 2, Rao and Mukerjee 4).

The priors are derived in Sect. The Priors. They are derived by solving certain differential equations. While these differential equations are different from those encountered in relation to quantile matching priors, they often provide the same solution as the latter. Thus, some of these priors meet several matching criteria and can be recommended for use in practice. Section Simulation Study of this paper involves some simulations and demonstrates that the matching is accomplished even for fairly small sample size.

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A. Chaudhuri et al. (eds.), *A Tribute to the Legend of Professor C. R. Rao*,
Indian Statistical Institute Series,
https://doi.org/10.1007/978-981-33-6991-7_9

The Priors

The pdf of a wrapped Cauchy distribution is given by

$$f(\theta) = \frac{1-\rho^2}{2\pi[1-2\rho\cos(\theta-\mu)+\rho^2]}. \tag{1}$$

It may be noted that a wrapped Cauchy distribution is a member of a symmetric wrapped stable family of circular densities with pdf given by

$$f(\theta) = (2\pi)^{-1}\sum_{k=1}^{\infty}(\rho^k)^\alpha \cos[k(\theta-\mu)],\ \ \rho \in [0,1), 0 \le \mu, \theta \le 2\pi.$$

The case $\alpha = 1$, on simplification leads to the wrapped Cauchy pdf given in (1). The case $\alpha = 2$ leads to the wrapped normal distribution.

The derivation of priors for both μ and ρ is based on the Fisher information matrix. From Ghosh et al. (3), it follows that this matrix is given by

$$I(\mu,\rho) = \text{diag}(\frac{2\rho^2}{(1-\rho^2)^2}, \frac{2}{(1-\rho^2)^2}) = (I_{11}(\rho), I_{22}(\rho)), \text{ say.} \tag{2}$$

For deriving the HPD or LR test-based matching priors, we need the following Lemma derived in Ghosh et al. (2019).

Lemma 1 $E(\frac{\partial logf}{\partial\mu})^3 = E(\frac{\partial logf}{\partial\rho})^3 = 0$, $E(\frac{\partial^3 logf}{\partial\mu\partial\rho^2}) = 0$, $E(\frac{\partial^3 logf}{\partial\mu^2\partial\rho}) = -\frac{2\rho(1+\rho^2)}{(1-\rho^2)^3}$.

We need also an additional result for the derivation of the necessary priors.

Lemma 2 $E(\frac{\partial^3 logf}{\partial\rho^3}) = -\frac{12\rho}{(1-\rho^2)^3}$.

Proof Differentiating both sides of the identity $\frac{2}{(1-\rho^2)^2} = I_{22}(\rho) = \int -\frac{\partial^2 \log f}{\partial\rho^2} f d\theta$ with respect to ρ, one gets

$$\frac{8\rho}{(1-\rho^2)^3} = E(-\frac{\partial^3 \log f}{\partial\rho^3}) + E(-\frac{\partial^2 \log f}{\partial\rho^2}\cdot\frac{\partial \log f}{\partial\rho}). \tag{3}$$

We also have the third Bartlett identity

$$E(\frac{\partial \log f}{\partial\rho})^3 + 3E(\frac{\partial^2 \log f}{\partial\rho^2}\cdot\frac{\partial \log f}{\partial\rho}) + E(\frac{\partial^3 \log f}{\partial\rho^3}) = 0. \tag{4}$$

Now multiplying (3) by 3 and adding to (4), in view of Lemma 1, one gets the result.

We are now in a position to derive HPD matching priors individually for both μ and ρ, while treating the other as the nuisance parameter. As in Ghosh et al. (3), we consider priors of the form $\pi(\mu, \rho) = g(\rho)$, since the location parameter μ requires only a uniform prior over $[0, 2\pi]$ and the Fisher information matrix also does not involve μ.

First, we consider HPD matching prior for μ treating ρ as the nuisance parameter. Due to orthogonality of of μ and ρ, one can apply Theorem 4.4.1 in page 75 of Datta and Mukerjee (1) and solve the differential equation

$$\frac{d}{d\rho}[g(\rho)I_{11}^{-1}I_{22}^{-1}\frac{\partial^3 \log f}{\partial\rho^3}] = 0. \tag{5}$$

This, on simplification, leads to the prior $g(\rho) \propto \rho(1-\rho^4)^{-1}$. It is interesting to note that this prior is the same as the second-order quantile matching prior for μ as obtained in Ghosh et al. (3).

Next consider the case when ρ is the parameter of interest and μ is the nuisance parameter. Again appealing to Theorem 4.4.1 of Datta and Mukerjee and Lemma 1, one needs to solve the differential equation

$$\frac{d}{d\rho}[g(\rho)\frac{(1-\rho^2)^4}{4}(-\frac{12\rho}{(1-\rho^2)^3}) - \frac{d^2}{d\rho^2}[g(\rho)(1-\rho^2)^2/2] = 0. \tag{6}$$

Writing $h(\rho) = (1-\rho^2)g(\rho)$, (6) simplifies to

$$\frac{d}{d\rho}[-2\rho h(\rho) - (1/2)(1-\rho^2)h'(\rho)] = 0. \tag{7}$$

Solving $h'(\rho)/h(\rho) = -\frac{4\rho}{1-\rho^2}$, one gets $h(\rho) \propto (1-\rho^2)^2$, or equivalently $g(\rho) \propto 1-\rho^2$.

For LR statistic matching prior, when μ is the parameter of interest and ρ is the nuisance parameter, an appeal to Theorem 5.5.2 of Datta and Mukerjee and Lemma 1 leads to the same differential equation as given in (5), leading thereby to the same prior $g(\rho) \propto \rho(1-\rho^4)^{-1}$.

Next for the LR statistic matching prior, when ρ is the parameter of interest, and μ is the nuisance parameter, once again from Theorem 5.5.2 of Datta and Mukerjee, one gets the differential equation

$$\frac{d}{d\rho}[\frac{(1-\rho^2)^2}{2}\{g'(\rho) - g(\rho)u(\rho)\}] = 0, \tag{8}$$

where

$$u(\rho) = E[\frac{(1-\rho^2)^2}{2}(\frac{\partial^2 \log f}{\partial\rho^2})(\frac{\partial \log f}{\partial\rho})] - \frac{(1-\rho^2)^2}{2\rho^2}(-\frac{2\rho(1+\rho^2)}{(1-\rho^2)^3}.$$

Adding (3) and (4), $E[(\frac{\partial^2 \log f}{\partial\rho^2})(\frac{\partial \log f}{\partial\rho})] = \frac{4\rho}{(1-\rho^2)^3}$. Then, on simplification, $u(\rho) = \frac{1+3\rho^2}{\rho(1-\rho^2)}$. Thus, one needs to solve the differential equation

$$g'(\rho)/g(\rho) = \frac{1+3\rho^2}{\rho(1-\rho^2)} = \frac{1}{\rho} + \frac{4\rho}{1-\rho^2}.$$

This leads to the solution $\log g(\rho) = \text{constant} + \log(\rho) - 2\log(1-\rho^2)$ which gives the prior $g(\rho) \propto \rho(1-\rho^2)^{-2}$. Incidentally, this is also Jeffreys' prior as found in Ghosh et al. (3).

Simulation Study

In view of the calculations of Sect. The Priors, we obtain the following three priors listed in Table 1.

In this part, we simulate i.i.d samples from a wrapped Cauchy distribution with parameters $\rho_0 = 0.7$ and $\mu_0 = \pi/2$. We investigate frequentist coverage probabilities of one-sided Bayesian credible intervals under the four priors listed in Table 1.

The posterior distribution is simulated via Gibbs sampler with each conditional distribution sampled by the accept–reject method. We use the uniform distribution as the proposal distribution in the accept–reject algorithm. Here, we report the results for sample size $n = 5$ and $n = 50$.

Tables 2 and 3 show the coverage probabilities for μ and ρ, respectively, when the sample sizes are 5 and 50, and the target coverage probability is $1-\alpha$. We can see that π_3 is conservative, especially when $n = 5$. On the other hand, π_2, While performing poorly for $n = 5$, does much better when $n = 50$.

Table 1 List of priors derived in Sect. The Priors

Prior	Category	Parameter of interest	Nuisance parameter
$\pi_1(\mu,\rho) \propto \rho(1-\rho^4)^{-1}$	HPD matching prior, LR statistic matching prior	μ	ρ
$\pi_2(\mu,\rho) \propto 1-\rho^2$	HPD matching prior	ρ	μ
$\pi_3(\mu,\rho) \propto \rho(1-\rho^2)^{-2}$	LR statistic matching prior	ρ	μ

Table 2 Coverage Probabilities of μ when $n = 5$ and $n = 50$

	$n = 5$			$n = 50$		
$1 - \alpha$	π_1	π_2	π_3	π_1	π_2	π_3
0.99	0.99	1.00	1.00	0.98	0.98	0.99
0.98	0.98	1.00	1.00	0.97	0.98	0.98
0.97	0.96	0.99	0.99	0.95	0.97	0.97
0.96	0.95	0.99	0.99	0.94	0.96	0.96
0.95	0.94	0.98	0.98	0.94	0.95	0.95

Table 3 Coverage Probabilities of ρ when $n = 5$ and $n = 50$

	$n = 5$			$n = 50$		
$1 - \alpha$	π_1	π_2	π_3	π_1	π_2	π_3
0.99	1.00	0.97	1.00	0.99	0.98	0.99
0.98	1.00	0.92	1.00	0.98	0.97	0.99
0.97	1.00	0.89	1.00	0.97	0.95	0.98
0.96	0.99	0.85	1.00	0.96	0.94	0.98
0.95	0.99	0.82	1.00	0.95	0.93	0.97

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Aspects of Inductive Inference in Statistics and Machine Learning

Palash Sarkar

Introduction

Acquisition of knowledge constitutes a fundamental human activity. To address this question, it may appear that at the outset one should put down a definition of knowledge. We will, however, side-step the issue of defining knowledge. Instead, we will assume that people have at least some idea of what constitutes knowledge. The issue that we consider is methods for acquiring knowledge. Even without having a definition of knowledge, one would agree that for the acquired knowledge to be reliable, the methods for acquiring knowledge should also be reliable. In other words, if a method for acquiring knowledge is not reliable, then the acquired knowledge cannot also be considered reliable.[1]

The question of what constitutes reliable methods for acquiring knowledge has been considered by philosophers since ancient times. Typically, discussions in philosophy books talk about valid methods of acquiring knowledge. For the purpose of this article, we will conflate 'reliable' with 'valid'.[2] The methods of acquiring knowledge that have been identified by philosophers are the following: perception; inductive inference; deductive inference; analogy and comparison; and, testimony (of authority).

[1]There are several issues here. For example, is it proper to talk about 'unreliable knowledge'; i.e., if something is unreliable, can it be considered to be knowledge? A related issue is that of quantifying unreliable knowledge by assigning a score of unreliability. Of course, a statement such as 'event X holds with 80% probability' is a definite statement about uncertainty. While these are interesting questions, discussion of these issues are outside the scope of this article.

[2]The fine distinction between a 'reliable method' and a 'valid method' and by implication the fine distinction between 'reliable knowledge' and 'valid knowledge' is again outside the scope of this article.

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A. Chaudhuri et al. (eds.), *A Tribute to the Legend of Professor C. R. Rao*,
Indian Statistical Institute Series,
https://doi.org/10.1007/978-981-33-6991-7_10

Our target of discussion is inductive inference and its relation to statistics and machine learning. Before moving on to the details of inductive inference, we mention a few words about the other methods. Perception primarily refers to the use of sense organs. A deductive inference essentially refers to a statement that can be derived from a set of statements using the rules of logic. Knowledge acquisition by analogy refers to the translation of ideas from one context to another by an appeal to the similarity of the two contexts. Knowledge from testimony is acquired by accepting a statement from some person who can be trusted.

Statistical inference refers to a wide collection of techniques which aim to derive information from observations. The methods of statistical inference involve mathematical and computational methods. The role of mathematics is to formally prove that statistical inference methods achieve their stated goals. This is essentially an application of the deductive inference method mentioned above. Computational methods are useful in actually implementing a statistical inference technique, or, more theoretically, for understanding the difficulties and limitations for the implementation of various inference techniques. The justification or suitability of a particular statistical method, however, is neither mathematical nor computational. Such justification is usually provided as an appeal to intuition.

Machine learning techniques have a similar goal as statistical inference techniques. In many cases, well-studied statistical techniques are classified as machine learning methods. One way in which machine learning perhaps differs from statistics is in the greater emphasis laid on computational issues in machine learning methods. Similar to statistical inference, justifications for various machine learning methods are also usually stated as being intuitive.

In this paper, we aim to provide perspectives on some statistical and machine learning techniques from the viewpoint of inductive inference. To this end, in the first part, we provide an overview of inductive inference and its various characteristics. This will essentially summarise prior thoughts on inductive inference which are relevant to statistics. In the second part, we will look at a number of techniques from statistics and machine learning. The goal will be to point out why such techniques are essentially inductive inferences and in particular which characteristics of the methodology of inductive inference can be found in these techniques.

One may question the usefulness of the present work, especially since it does not provide any new result in statistics and machine learning. A response would be that explicitly seeing inductive inference at work within statistics and machine learning satisfies a basic intellectual curiosity. Perhaps more importantly, fundamental philosophical issues regarding the validity of inductive inferences can also be seen to apply to statistical inferences and machine learning techniques. This would lead to replacing an aura of definiteness by an umbra of doubt or uncertainty. In more concrete terms, explicitly identifying the connections between the two areas will perhaps lead to a more productive two-way flow of ideas.

The relation between inductive inference and statistics is known (but perhaps not as well known as it should possibly be). In 1956, Mahalanobis had commented that 'statistics is the universal tool of inductive inference'. A short paper by Fisher [6] had commented on the relation between statistical methods and induction. The preface of the fascinating book on statistical thought by Chatterjee [3] mentions that the book 'views the problem of statistical induction in a wider philosophical context'. More general connections between statistics and philosophy have been explored by a galaxy of authors in a complied volume edited by Bandyopadhyay and Forster [1]. The connection between machine learning and induction has been explored by Harman and Kulkarni [7]. Finally, we would like to mention the book [11] entitled 'Statistics and Truth: Putting Chance to Work', by C. R. Rao which provides an excellent overview of the nature and role of statistics in various fields of human activity. In particular, we note that the preface of the book suggests that statistics provides a method for codifying inductive reasoning.

In view of the above, our work may be seen as a continuation of the line of thought connecting statistics and machine learning to induction and more generally philosophical issues. We would like to mention that, to the best of our knowledge, our approach of considering specific statistical inference and machine learning techniques to bring out in details the features of inductive inference therein is not present in the above-mentioned works. So, our work does offer something new to a reader interested in the connection between inductive inference and statistics/ machine learning.

Inductive Inference

Inductive inference as a method of acquiring knowledge has been studied in both Western and Indian philosophies. Putting down a precise definition of 'inductive inference' is rather difficult. The term refers to a broad set of inference mechanisms which loosely speaking may be construed as inferring something about unperceived situations from perceived information. We illustrate a few inductive inference methods through examples.

Statements such as 'the Sun rises in the East', or, 'all human beings are mortal' are derived based on observations. These are examples of enumerative induction or universal inference, i.e. inference from particular observations to a universal statement. More generally, these are of the following type. Instances $a_1, a_2, \ldots, a_n$ which are all F's are also observed to be G's; from this a general principle 'all *F's* are *G's* is inferred.

Inductive inference need not only be from particular to the universal. For example, from 'all observed rubies have been red' inferring 'the next yet to be found ruby will also be red' is an example of inductive inference where the premise is general and the conclusion is particular.

It is not necessary that an inductive inference will have a universal statement. For example, from 'Mercury is spherical, Venus is spherical, Earth is spherical, …' inferring 'the next yet to be discovered planet will also be spherical' does not involve any universal step. This is called a singular predictive inference which moves from particular premises to a particular conclusion.

The Problem of Induction

In Western philosophy, Hume made the most influential contribution to the study of inductive inference. The most disturbing question about induction is whether it is justified. For example, what are the justifications for the above examples of inductive inference? Clearly, these inferences cannot be justified using deductive methods. For example, the argument.

((all observed F's have also been G's) and (a is an F)) imply (a is a G).
is invalid; i.e., there exists a model in which both the premises are true, but the conclusion is false.

Hume had identified that any justification of inductive inference must necessarily be inductive leading to a circularity of argument or *petitio principii*. This was summed up in the following famous statement by Hume (1738) in 'A Treatise of Human Nature': 'instances, of which we have had no experience, must resemble those, of which we have had experience, and that the course of nature continues always uniformly the same'. This underlines that inductive inferences assume that nature continues uniformly. The statement that nature continues uniformly itself requires justification, and this justification can only be obtained through induction leading to a circularity of argument.

The problem of induction is the question of how to distinguish reliable from unreliable inductive inferences? This is a conundrum which is yet to be satisfactorily resolved despite efforts by philosophers such as Karl Popper and others. Perhaps the question does not even have a resolution. The inability to resolve the problem of induction has at least two implications. The first is methodological; i.e. there is no method or procedure which may be applied to distinguish good from bad inductive inferences. The second question is more fundamental in that there is possibly no objective difference between reliable and unreliable inductive inferences. While the unresolved problem of induction is a philosophical irritant, in practice, inductive inferences are regularly made. We refer to [9] for description of the problem of induction.

The method of induction has been studied in Indian philosophy. The main criticism against inductive inference is that of circularity. This was identified by the Cārvāka school of thought. Chapter 1 of [2] provides an excellent account of the Cārvāka criticism of induction. We also refer to [12] for a description of how the Cārvāka school of thought anticipated some modern notions.

The Principle of Simplicity

Often called Occam's razor, the principle of simplicity is the idea that the simplest among several available options should be chosen. For example, the simplest among several competing hypotheses suggested by observations should be chosen. Another example would be to choose the simplest among several models. The simplicity principle is ubiquitous in human reasoning.

One may look for an objective justification of the principle of simplicity. This, however, is hard to find. One possible justification could be that this principle has proved to be correct in the past, and so, it can be used in future. Such a justification is essentially an inductive inference. An eloquent criticism of the justification of the principle of simplicity by appealing to the past has been made by Bertrand Russell in the book 'On Scientific Method in Philosophy' (1914). He remarks: 'But it is just this characteristic of simplicity in the laws of nature hitherto discovered which it would be fallacious to generalise, for it is obvious that simplicity has been a part cause of their discovery, and can, therefore, give no ground for the supposition that other undiscovered laws are equally simple'.

There are several other troublesome issues. The principle of simplicity tacitly assumes that the options are known, that it is possible to compare any two options with regard to simplicity, and that the set of options has a unique simplest member. All of these issues can be stated in a more formal framework though we are not aware of any place where such formalisation has been done. Nevertheless, it is not our purpose here to get into a detailed formal investigation of the principle of simplicity. We note two points. In practice, the principle of simplicity is universally applied and that its only possible justification arises as an inductive inference.

Inference to the Best Explanation

Philosophers distinguish between three types of inferences, namely deductive, inductive and abductive [4]. Abductive inference is also called inference to the best explanation (IBE). A standard example of IBE is the following. Suppose on waking up in the morning, one finds the outside to be wet. From this, one infers that it had rained in the night. This inference is the best one which explains the observation. In theory, it is possible to make other inferences such as water was sprayed from a low flying aircraft, but, would not be considered the best inference. Of course, the inference that it rained in the night is also the simplest of explanations, so this particular example is also an example of the application of the principle of simplicity.

The idea of abductive inference or abductive logic was introduced by Charles Sanders Peirce. He considered abductive inference to be a form of non-deductive inference which is different from inductive inference. The notion of abductive inference has been closely studied. We refer to [4] for an introduction to the various issues.

The validity of IBE can be questioned. As in the case of the principle of simplicity, one may ask for an objective justification of IBE. Such a justification may be provided by considering past applications; i.e., IBE has proved to be true in the past and so it will be true in the future. This is again an appeal to inductive inference. So, while IBE is considered to be different from inductive inference, its justification seems to rely on inductive inference. So, IBE (and also the principle of simplicity) may be considered to be second-order induction.

Pragmatism

Roughly speaking, pragmatism refers to the idea that among various options, choose the one which is most useful. It may turn out that the most useful is also the simplest or the best. For example, among various techniques that in theory can be employed to analyse a situation, use the one which is the easiest to apply.

As in the case of simplicity and IBE, justification for pragmatism arises from an appeal to induction. Further, the issue of determining the most useful option has difficulties similar to that of determining the simplest or the best option.

Features of Inductive Inference

It has been mentioned that no objective justification of inductive inference has been found till date. Nevertheless, investigations have identified several features that can be seen in various inductive inferences. We briefly discuss these below.

Ampliative: This is intended to mean that the conclusion of the inference has more content that its premise. For example, in universal inference, the premise consists of some observations while the conclusion is a universal statement. This is a distinctive feature of inductive inference as opposed to deductive inference in which there is no amplification of the logical content of the conclusion beyond what is contained in the premises.

Contingent: The conclusion of an inductive inference does not follow as a necessary condition of its premise. In other words, it cannot be logically said that if the premise holds then the conclusion must also hold.

Non-monotonic: Inductive inference is based upon perceived information. An inference which is made from some amount of perceived information may become invalid if additional information becomes available.

Non-preservation of truth: It is possible that the premises of an inductive inference are true, yet the conclusion is false. For example, in an enumerative induction, the individual premises are observations and are true. The universal conclusion, however, could be false since it may not hold for some hitherto unobserved instance.

Statistical Methods and Inductive Inference

In this section, we consider some basic statistical notions and point out aspects of induction that are implicit in such notions. This provides a better understanding of the link between the more philosophical notion of inductive inference and statistical methods.

Sufficient statistic: The notion of sufficient statistic is a basic notion in statistical inference. Given observations $X_1, \ldots, X_n$ following some known distribution with an unknown parameter θ, a statistic T is a function $T\ (X_1, \ldots, X_n)$ of the observations. In the words of Fisher [5], a statistic is sufficient for an unknown parameter if 'no other statistic that can be calculated from the same sample provides any additional information as to the value of the parameter'. The notion of sufficient statistic provides a simple example of non-monotonicity. It is easy to construct examples where a statistic T is sufficient for a parameter using observations $X_1, \ldots, X_n$, but is no longer sufficient if an additional observation X_{n+1} becomes available. So, it is important to use all available data and it is usually assumed that the number of observations is known and fixed.

Maximum likelihood estimate (MLE): Given data $x_1, \ldots, x_n$, the likelihood function $L(\theta;\ x_1,\ldots,x_n)$ is a function of an unknown parameter θ which gives the probability of obtaining the sample $x_1,\ldots,x_n$ given the value of the parameter. Once the data is available, an estimate of the value of the parameter θ is desired. The MLE $\hat{\theta}$ of θ is the value which maximises the likelihood function. In other words, for $\theta = \hat{\theta}$, the probability of observing the data $x_1, \ldots, x_n$ is maximised. The justification for using MLE is implicitly based on abduction, or inference to the best explanation (IBE). The rationale is that since $\hat{\theta}$ maximises the probability of observing the data, it is the best explanation for observing the data.

Null hypothesis testing: A null hypothesis H_0 to be tested is formulated. This formulation involves defining a test statistic. A number of observations are made which provide the data using which H_0 is to be tested. The p-value of the test is the probability that under H_0 the test statistic equals the observation or more extreme. The null hypothesis H_0 is rejected at α level of significance if the p-value is less than α.

Several features of inductive inference can be identified in the procedure. First, the procedure is ampliative. The premise of the inference mechanism is the data while the conclusion is about the hypothesis. So, the conclusion has more content than the premise. Second, null hypothesis testing is non-monotonic; i.e., a hypothesis which was not priorly rejected can become rejected with the availability of additional data. So, the non-rejection of H_0 does not imply it is established. In the words of Fisher [6], '[i]t is a fallacy, …, to conclude from a test of significance that the null hypothesis is thereby established; at most it may be said to be confirmed or strengthened'. The third aspect of induction arises in the choice of α. Justification for choosing a value for α is based on this value being used in various other situations. This justification is essentially an inductive inference that the value of α which has been appropriate in other situations will also be appropriate for the situation at hand.

Prediction Error: Let X be a real-valued input random variable, i.e. a predictor or a feature, and let Y be a real-valued output random variable, i.e. the response or the dependent variable. Let $\Pr[X, Y]$ be the joint distribution of X and Y. A basic statistical technique is regression, i.e. to obtain a function $f(X)$ of X which can be used to predict Y given X. A loss function is used to measure the efficacy of the prediction.

The most common and convenient loss function is the squared error loss, i.e. $L(Y, f(X)) = (Y - f(X))^2$. Using this loss function, the expected (squared) prediction error is defined to be EPE(f) $= E(Y - f(X))^2$. The goal is to choose f such that EPE(f) is minimised. The EPE can be simplified as $EPE(f) = E_X E_{Y|X}\left[\left(Y - f(X)^2\right)|X\right]$, and so to minimise EPE(f) it is sufficient to perform point-wise minimisation, i.e. $f(x) = argmin_c \ E_{Y|X}\left((Y - c)^2|X = x\right)$. The solution is $f(x) = E(Y|X = x)$. So, we have that the conditional expectation, which is also called the regression function, provides the best prediction of Y at a point x.

The notion of 'best' in the above is with respect to the squared error loss. One may ask for a justification of using the squared error loss. For example, the loss function could have been defined as $|Y - f(X)|$, in which case the solution would turn out to be the conditional median, i.e. median($Y|X = x$). Is there any a priori reason to prefer squared error over absolute error? A sort of justification forwarded in [8] (the descriptions of squared error and median error are also from [8]) is that 'squared error is analytically convenient and the most popular'. The reason for its popularity is perhaps based on analytical convenience, so the main justification for using the squared error is that it is analytically convenient. This is a pragmatic consideration.

We may take a moment to reflect on qualitative aspects of this issue. Regression forms an important technique of statistical decision theory and machine learning. Outputs of a prediction function will conceivably be used to arrive at decisions which can have major social effects. The decisions and their social consequences then depend upon the actual choice of the prediction function. So, for example, using a prediction function based on squared error can lead to a decision which is different from a decision which is arrived at by using a prediction function based on absolute error. The justification for such a difference in decisions would really be the analytical convenience of the squared error. In other words, the comparative simplicity of being able to mathematically handle one expression over another can lead to wholly different social consequences.

Model Selection: Suppose $x_1, \ldots, x_n$ constitute the data. Further, suppose that there is a set of models $M_1, \ldots, M_m$ and the goal is to choose one of the models based on the data. This is a typical setting of inductive inference, where from particular observations, one infers a general statement. In this particular setting, the inference is somewhat restricted in the sense that the requirement is to choose one among a finite set of models. One may ask as to how the set of models have been determined? The answer would typically be a combination of the following justifications: from previous experience, usefulness, simplicity. All of these justifications themselves are inductive inferences.

A model is determined by its parameters. Suppose the parameter vector for the ith model is $\boldsymbol{\theta}_i$, $i = 1, \ldots, m$. Further, suppose that the dimension of $\boldsymbol{\theta}_i$ is k_i, i.e. the ith model is determined by k_i parameters. Let $L_i(\boldsymbol{\theta}_i; x_1, \ldots, x_n)$ be the likelihood function for the ith model.

Two standard ways of assigning scores to models are the Akaike information criterion (AIC) and Bayesian information criterion (BIC). For the ith model, these are defined as follows. Let $\hat{\theta}_i$ be the MLE for $\boldsymbol{\theta}_i$.

$$\text{AIC}(M_i) = 2k_i - 2\ln L_i(\hat{\boldsymbol{\theta}}_i; x_1, \ldots, x_n);$$
$$\text{BIC}(M_i) = (\ln n)k_i - 2\ln L_i(\hat{\boldsymbol{\theta}}_i; x_1, \ldots, x_n).$$

The procedure for selecting a model using AIC (resp. BIC) is to compute the AIC (resp. BIC) scores for all the models and choose the one with the minimum score. Both the AIC and the BIC scores use the MLE $\hat{\boldsymbol{\theta}}_i$, and also both penalise models having a large number of parameters. The extent of the penalty is smaller for the AIC score than the BIC score.

Implicit in the above definitions of the AIC and BIC scores are two aspects of inductive inference. The first is the use of MLE. As discussed earlier, the justification for using MLE is inference to the best explanation. The second aspect is that of penalising more complex models,, or in other words preferring simpler models. Again, as discussed earlier, simplicity is justified by induction.

The AIC score for the *i*th model is derived through Taylor series approximations of the Kullback–Leibler (KL) divergence of the density of the *i*th model from that of the correct density. The goal of model selection based on the AIC score is essentially to choose the model for which the KL divergence is the minimum. The choice of KL divergence for use in model selection is itself based on induction; i.e., the KL divergence has proved to be useful in various other settings, and so it should also be useful for model selection.

The derivation of the BIC score is based on Bayes theorem. Suppose a prior probability p_i is assigned to model M_i. Also, consider the observations to be random variables $X_1,\ldots,X_n$ drawn from an unknown distribution. From Bayes theorem, $\Pr(M_i|X_1, \ldots, X_n)$ is proportional to $\Pr(X_1, \ldots,X_n|M_i)p_i$. The BIC score is arrived at through approximations of the last expression. The goal of the BIC score is to maximise the probability $\Pr(M_i|X_1, \ldots,X_n)$, i.e. the probability of M_i given the observations $X_1, \ldots,X_n$. This is an example of inference to the best explanation.

Machine Learning and Inductive Inference

Machine learning is a broad term used to denote a variety of techniques whose goal is to gather information from data. The data consists of pairs $(\mathbf{x}_1, y_1), \ldots, (\mathbf{x}_n, y_n)$ where $\mathbf{x}_1, \ldots, \mathbf{x}_n$ are feature vectors and $y_1, \ldots,y_n$ are the labels associated to $\mathbf{x}_1, \ldots, \mathbf{x}_n$, respectively. The feature vectors are drawn from some distribution which is typically unknown, so that the analysis is done in a non-parametric setting. Depending on the nature of the labels, two kinds of problems are identified. If the y_i*'s* are elements of $\{0, 1\}$(or some finite non-empty set), then we have a classification problem, while if the y_i*'s* are elements of R, then we have an estimation problem. Given the pairs (X_i, y_i), $i = 1, \ldots, n$, the goal is to 'learn' some rules so that given a new feature vector x, it is possible to provide the corresponding y. This problem is called supervised learning, since there is a learning phase where the given pairs are used to learn a rule. There is another problem called unsupervised learning which does not have a learning phase. We do not consider unsupervised learning in this brief discussion. Statistics forms the basic theory for machine learning techniques. We refer to [8, 10, 13] for good introductions to the subject.

From the overview of machine learning stated above, the entire field essentially amounts to inductive inference. A particular machine learning method is a particular kind of inductive inference. Machine learning procedures measure performance by bounds on the error in estimation or classification. This often involves sophisticated mathematical machinery. Such mathematical rigour may lead one to believe that the problem of induction (i.e. providing justification for employing a particular induction procedure) discussed earlier has been addressed, if not fully at least partially. Such an assumption, however, would be incorrect. In the various machine learning procedures, aspects of inductive inference are implicitly used in such a manner that any justification of the learning procedure would amount to circular reasoning. Below we consider examples of implicit inductive inferences in two well-known machine learning procedures.

Neural Networks

Neural networks are complex models for learning from data. They come in various forms. Our brief consideration of neural networks is based on the treatment in [10].

Consider the problem of classification. A multilayer feed forward neural network for this problem can be visualised in the following manner. The network consists of computation units. The units are organised into layers. There is an input layer and an output layer. The output layer has a single unit. Each unit receives some inputs and provides a single output. The single unit in the output layer provides the output of the network which is a binary value. The units in the input layer receive the input to the network which is a feature vector. Connections between the units are as follows. Suppose there are $k + 1$ layers $L_0,\ldots,L_k$, where L_0 is the input layer and L_k is the output layer. The output of any unit in layer L_i is provided as input to one or more units in layer L_{i+1}, for $i = 0, \ldots, k - 1$. These connections are considered to be directed arcs from one unit to another. So, information flows from the input of the network to the output of the network, i.e. the network maps a feature vector to a binary value.

Weights are associated to each of the arcs in the network. Suppose u is a unit in a layer L_i other than the input layer, i.e. $i \geq 1$. Further, suppose that outputs of the units $u_1, \ldots, u_k$ of the previous layer are provided as input to u. So, there are k arcs connecting $u_1, \ldots, u_k$ to u. Let the weights associated to these arcs be $w_1, \ldots, w_k$. On a particular input x to the entire network, suppose that the outputs of $u_1,., u_k$ are $b_1,\ldots,$ b_k, respectively. Then, the consolidated input to u is the weighted sum $a = w_1b_1 + \ldots + w_kb_k$. The computation done by u is on the value a to produce an output b; if L_i is not the output layer, i.e., $i < k$, then the value b propagates to units in layer L_{i+1} to which u is connected.

The computation done by all the units in the network are the same. Each unit computes a sigmoidal function, i.e., a function from R to R which is bounded and has non-negative derivative at all real values. Various sigmoidal functions are known and used in neural network computations. The output unit additionally applies a thresholding at the mid-point of the two bounds of the sigmoidal function to convert the real value to a binary value.

The network 'learns' by modifying the weights associated to the arcs. Suppose the set of arcs is A. Initially, the learning process starts out with a weight assignment $\{w_a^{(0)}\}_{a\in A}$. After the first pair (X_1, y_1) is processed, the weights are updated to $\{w_a^{(1)}\}_{a\in A}$; in general, after the ith pair (X_i, y_i) is processed, the weights are updated to $\{w_a^{(i)}\}_{a\in A}$, $i = 1, \ldots, n$. So, at the end of the learning procedure, the final weights are $\{w_a^{(n)}\}_{a\in A}$. At this point, the network is ready to perform classification of new inputs. On any new feature vector X, the corresponding classification value y is the output of the network when fed with X.

The goal of learning is to minimise misclassification error. This implies that learning is not perfect; i.e., from a finite number of samples, it is not possible to predict all future outputs in an error-free manner. Suppose the training data (X_1, y_1),…, (X_{i-1}, y_{i-1}) have already been processed and the arc weights are $\{w_a^{(i-1)}\}_{a\in A}$. The next training data is (X_i, y_i). The vector X_i is provided as input to the network with arc weights $\{w_a^{(i-1)}\}_{a\in A}$, and the output $y_i^{'}$ of the network is computed. If $y_i = y_i^{'}$, then there is no error and the arc weights $\{w_a^{(i)}\}_{a\in A}$ are taken to be the arc weights $\{w_a^{(i-1)}\}_{a\in A}$. On the other hand, if $y_i \neq y_i^{'}$,, then the network has made an error. This necessitates updating the arc weights $\{w_a^{(i-1)}\}_{a\in A}$ to obtain the arc weights $\{w_a^{(i)}\}_{a\in A}$.

The goal of the updation procedure is to minimise the training error. Doing this in an absolute sense would require knowledge of the entire error surface. Since the error surface can be complex, it is not feasible to minimise over the entire surface. Instead, the updation procedure attempts to minimise the error using a procedure called gradient descent. This results in local minima which may be different from the global minima. The algorithm resulting from the application of gradient descent to update the arc weights is called the back propagation algorithm.

Let us now consider whether a neural network provides a formal justification of inductive inference. It indeed provides a formal description of a method for obtaining a general rule from available information. One may argue that since the prediction of a neural network is not guaranteed to be correct, this itself shows that it is not a reliable induction. A response to this argument would be that being able to predict an outcome with a guaranteed bound on error itself counts as knowledge; i.e., knowledge may be probabilistic in nature.

We do, however, think that the neural network methodology does not provide a justification for the method of induction. For the sake of concreteness, we focus on the multi-layer feed forward network with weights computed using the back propagation algorithm as outlined above. The back propagation algorithm is one particular embodiedment of the gradient descent methodology. So, one may question the justification for using gradient descent to minimise error. As mentioned above, this does not guarantee that the error obtained is the global minimum. It only guarantees that the error is locally minimum. So, even assuming knowledge to be probabilistic in nature, how does one become sure that the correct probabilistic knowledge has been attained? The justification provided in [10] is that 'the algorithm has been found to work well in practice and is the most widely used training algorithm for multilayer networks'. So, the argument being made here is that the algorithm has worked well on previous occasions and hence is expected to work well in the future. This, of course, is an inductive inference. So, if we consider multilayer neural network as a method for induction, then to justify it we need to take recourse to induction, leading to circularity in argument.

Support Vector Machines

We consider the support vector machine (SVM) method for the classification problem. Our discussion is based on the description given in [10]. There are two key ideas in SVM. The first is that of using a linear separator with the maximum margin, and the second is that of mapping the feature space to a high-dimensional space and using a linear separator in the high-dimensional space.

Suppose the training data $(X_1, y_1), \ldots, (X_n, y_n)$ is linearly separable. If the data is not linearly separable, then the idea mentioned below is augmented using the notion of slack variables. Linear separability of the training data means that there is a hyperplane such that all training data having label 0 falls on one side of the hyperplane while all training data having label 1 falls on the other side. There is no unique hyperplane which separates the data. In fact, there will be inifinitely many hyperplanes any of which can act as a separator for the data. The first key idea of SVM is to choose a hyperplane having the maximum margin. Given a hyperplane h, let d_i be the minimum distance of h from all points labelled i, for $i = 0, 1$. Then, the margin of h is $d_0 + d_1$. The goal of SVM is to choose h such that $d_0 + d_1$ is maximised. The maximisation problem is formulated in terms of the training data and the solution to the maximisation problem constitutes the learning phase of the method.

The second key idea is to map to a high-dimensional space and apply the maximum margin separation in that space. This is done by applying a nonlinear function to the feature vector. The actual mappings that are used are in terms of the so-called kernel functions. The optimisation problem in the high-dimensional space can be expressed in terms of the kernel functions. There are several possibilities for choosing the kernel function. Each choice gives rise to one particular SVM method.

From the point of view of induction, one would look for justifications of two issues. First would be the rationale for obtaining a maximum margin linear separator. While the idea is intuitively appealing, it is not clear that such a separator is necessary for achieving minimum error. Like other inductive methods, justification for using a maximum margin separator is based on the idea being useful in previous cases and from that inferring that it is likely to be useful for future applications. This is again an inductive inference in itself. The other issue would be the choice of kernel function. Since there is no clear cut choice of a particular kernel, any justification for choosing a kernel would be based on appeal to other successful applications sharing similar characteristics and inferring that it is likely to be useful for the application at hand. Once again, this is an inductive inference.

Concluding Remarks

In the preceding sections, we have tried to make explicit certain inductive inferences which are implicit in various statistical and machine learning methods. The problem of induction or, more specifically, the problem of justifying inductive inferences is a question of major philosophical interest. As mentioned earlier, it may appear that the formalism introduced by statistics and machine learning provides justification for inductive inference.

This idea has been explicitly mentioned on Page 7 by Kulkarni and Harman [10] where they comment: '… statistical learning theory provides partial deductive mathematical justifications for certain inductive methods, given certain assumptions'. This is a carefully crafted sentence which suggests that specific inductive inferences can be justified by statistical learning theory, but also adds the safeguard of 'given certain assumptions'. In the previous section, we have argued that neural networks and support vector machines are inductive inference mechanisms which cannot be justified without getting into a circular argument. If one were to question the 'given certain assumptions' clause in the assertion by Kulkarni and Harman, then we would be led to ask for justifications of such assumptions. It is very likely that such justifications will involve employing an inductive inference, so that we get back to a circular reasoning. Since, Kulkarni and Harman do not specify assumptions for particular learning methods, we are unable to pinpoint the circularity that would arise from such reasoning.

So, it is our case that looking at some specific examples of the use of inductive inference in statistics and machine learning, we are emboldened to state that almost all of statistics and machine learning essentially consist of inductive inference. A discerning reader may immediately note that in making such an assertion, we have ourselves made an inductive inference. So, whether our inference is justified is a specific case of the problem of induction. Consequently, this leaves open the possibility that there is indeed some statistical and/or machine learning method which justifies induction. At present, all we can say is that we are unaware of any such method.

We have repeatedly mentioned that trying to justify induction leads to a circular argument. In other words, if one tries to justify a particular inductive inference, then one is led to assuming that another inductive inference is valid, leading to a circularity of reasoning. Let us look a little more closely at this. Suppose, we are looking for a justification of a particular inductive inference, say I_1. In the process, suppose we are led to assuming the validity of a certain other inductive inference, say I_2. So, if

I_2 is valid, then so is $\boldsymbol{I_1}$. In other words, we have reduced the problem of justifying $\boldsymbol{I_1}$ to that of justifying I_2. If we denote the problem of **'justifying $\boldsymbol{I_1}$'** by $\boldsymbol{P_1}$ and that of 'justifying I_2' by P_2, then we have been able to reduce problem $\boldsymbol{P_1}$ to problem $\boldsymbol{P_2}$.

The notion of reducing one problem to another is present in mathematics and many areas of computer science. Since computer science deals with problems of varying complexity, it is of interest to identify classes of problems of similar complexity and study separation between such complexity classes. The fundamental technique in such study is that of reduction. In fact, for each class, one tries to identify one problem (or a set of problems) which capture the complexity of the class in the sense that being able to solve this problem will lead to a solution to all problems in the class. Such a problem is called complete for the class.

With this background in mind, it may be of interest to form classes of inductive inferences, where all inductive inferences in a particular class have similar characteristics. The problem of justifying an inductive inference can then be reduced to one inference (or a set of inferences) in the class. The ultimate aim of such an exercise would be to try and identify one or more inductive inferences such that if it is possible to find justifications for such inductive inferences, then the entire problem of justifying inductive inference would be solved. This would not solve the problem of inductive per se, but it would focus attention on a few kinds of inferences which are of some fundamental nature. Hopefully, this would lead to a better understanding of the problem of induction.

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"To the seers, our ancestors"

Amartya Kumar Dutta

C.R. Rao has been described by the American Statistical Association as

> a living legend whose work has influenced not just statistics, but has had far reaching implications for fields as varied as economics, genetics, anthropology, geology, national planning, demography, biometry, and medicine.

In 2002, the then President of the USA conferred on him the "National Medal of Science", the highest award given to a scientist in the country. The citation described him

> as a prophet of new age for his contributions to the foundations of statistical theory and multivariate statistical methodology and their applications, enriching the physical, biological, mathematical, economic, and engineering sciences.

The "prophet of new age", who has received the highest scientific honour from a nation which has been at the vanguard of scientific research during his time, is, in the words of B.L.S. Prakasa Rao [2]:

> purely an Indian product who had all his education in India and who did all his research by himself without any guidance by others.

This legacy reminds one of a legendary scientist from an earlier generation: C.V. Raman, who too had all his education in India, was entirely self-taught and had made a huge impact in world science from the scientific research he had accomplished in Kolkata.

Of course, C.R. Rao had worked in Cambridge for two years (1946–48) and had received his Ph.D. under the supervision of Ronald A. Fisher. But, it was not a case of a typical M.Sc. student applying for Ph.D., getting selected, trained and mentored. Already in 1945, at the age of 25, C.R. Rao had published one of his seminal

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A. Chaudhuri et al. (eds.), *A Tribute to the Legend of Professor C. R. Rao*,
Indian Statistical Institute Series,
https://doi.org/10.1007/978-981-33-6991-7_11

papers which had generated terms like "Cramer-Rao inequality" and the "Rao-Blackwellization" and which has been reproduced in the volume "*Breakthroughs in Statistics, 1890–1990*". We may recall the background for his two-year stint at Cambridge.

The Cambridge University Museum of Archaeology and Anthropology had acquired human skeletons, brought from Jebel Moya (in Sudan in northeastern Africa). In 1946, the Department of Anthropology, Cambridge University, sent a request to the Indian Statistical Institute (ISI) to send an expert who could use the D^2-statistic techniques of P.C. Mahalanobis to analyze the Jebel Moya skeletons and trace the origins of those people. C.R. Rao had the required expertise and Mahalanobis sent him to Cambridge in response to the above request. At the University Museum, C.R. Rao developed new methods of analysis of multiple measurements and used them to analyze the skeletal data. His Ph.D. thesis was based on his new multivariate methodology MANOVA and other multivariate tests he invented while analyzing the data. The results of his work are recorded in his book *Ancient Inhabitants of Jebel Moya*, published by Cambridge University Press in 1955.

C.R. Rao had remained in India, at ISI, till around a year before his scheduled retirement in 1980. With his gentle humour, Prof. Rao once remarked to me at the ISI Guest House as to why he had to shift to the USA (during the late 1970s):

> In 1980, I had turned sixty and the Government of India decided that I was no more of any use.

I mention here that, after the Shanti Swarup Bhatnagar Prize, the highest scientific recognition in India, was instituted in 1958, Profs. K. Chandrasekharan (TIFR) and C.R. Rao (ISI) became its first recipients in the Mathematical Sciences (1959). It is perhaps symbolic of the role the two leading institutes would be playing in the mathematical sciences in India after Independence. On receiving the Bhatnagar Prize from Prime Minister Jawaharlal Nehru in 1963, C.R. Rao donated the entire prize money to the National Defence Fund saying that

> The country's need is greater than that of an individual scientist.

I may now share with the readers a personal anecdote about the "living legend". In December 2003, there was a Mathematics Conference in the Indian Institute of Science (IISc), Bangalore, organised jointly by India and the American Mathematical Society (AMS). I was a speaker in both the Commutative Algebra as well as the History of Mathematics sessions. As that was my first participation in a conference on History of Mathematics, I was particularly excited.

C.R. Rao too had been at the Conference during the initial days. The night before he was leaving, I had a brief conversation with him. I described, with enthusiasm, certain aspects of ancient Indian mathematics which I was trying to bring out. He listened and then, with a piercing look at my eyes, he uttered a few simple words, slowly and softly:

> You are doing a very important work.

There was an unsaid but forceful gesture in his eyes and in his very sage-like presence, conveying an unmistakable message, "Carry on."

The episode may sound banal to a reader, but it had a decisively inspirational impact on me. Those were the days when my forays into the history of ancient Indian mathematics used to generate adverse reactions ranging from ridicule to hostility. The sincere encouragement from the great man, therefore, could not have been better-placed and better-timed. Even now, the memory of that episode acts as a sustaining influence to carry on *niṣkāma karma*, in spite of the cynical realisation that a large section of our educated elite prefers to cling to their ignorance about Indian mathematical or scientific heritage.

That C.R. Rao's statement could not have been a mere polite patronising gesture to an immature youngster can be seen from the fact that C.R. Rao himself had written a short article on Statistics in Ancient India in the magazine Science Today [3]. In the article, he highlighted the glimmerings of the concept of "sampling" that comes out in an episode from the epic Mahābhārata. This stands out in sharp contrast to the pathetic comments made about mathematics in the pre-Common Era by some of our famous Indian intellectuals, in spite of the great mathematical ideas shining clearly from the extant treatises of the era. (See [1], Section 4].)

The reverence that C.R. Rao has for ancient Indian savants can also be discerned from the following sentence in his homage to P.C. Mahalanobis in [4]:

> He worked with great devotion to the last breath of his life inspired by higher ideals and with the same missionary spirit that sustained the great sages of India.

C.R. Rao continues:

> In India — as elsewhere — we do have some learned people but most of them do not care to put their knowledge to action. On the other hand, we have a very large number of persons who act with haste unaware of their insufficient knowledge for their own gain. Both categories cause immense harm to themselves and to the society they live in. Very few persons act with adequate knowledge, not seeking the fruits of their labours, which only can bring good to the world; this distinguishes the wise from the rest of humanity. Professor Mahalanobis belonged to this tiny minority of chosen people, who are born with a mission in life, who are guided by the divine light, who understand what is to be done and act bravely with convictions.

Much has been said about the greatness of P.C. Mahalanobis and his contributions, but it will be difficult to come across a passage which expresses the quintessence of Mahalanobis with as much power and illumination.

The homage by C.R. Rao reminds us of a verse in *Ṛgveda*. Indian traditions hold the *Ṛgveda*, the oldest Veda, with deep reverence as the source of all that was great and valuable in later periods. But, the *Ṛgveda* itself pays the homage (Book 10.14.15):

> idaṁ nama ṛṣibhyaḥ pūrvajebhyaḥ pūrvebhyaḥ pathikṛdbhyaḥ
>
> Prostrations to the seers of ancient times, our ancestors who carved the path.

C.R. Rao embodies a living demonstration of the ancient Indian tradition of Gratitude and *Śraddhā*, qualities which appear to be getting atrophied in a large section of the Indian intellectual community.

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A Tiny Tribute to a Towering Statistician

S. P. Mukherjee

Fully aware of the fact that I know only a little about the scholastic contributions and professional achievements of Professor Calyampudi Radhakrishna Rao (popularly mentioned as C. R. Rao)—the living legend of Statistics—and that I could have the opportunity to interact with him only on a few occasions, I simply try to recreate some of the events that left indelible impressions on me about this doyen among Indian mathematicians and statisticians. As students of Statistics in the Presidency College, Calcutta, we had heard a lot about Rao from our teachers and had read the famous Rao-Cramer bound for the variance of an estimator. We also had the good fortune to receive lessons from late Professor Anil Kumar Bhattacharyya who had been a teacher of Rao during the latter's postgraduate studies in the University of Calcutta. In fact, we also learnt about the Bhattacharya bound as a generalization of the Rao-Cramer bound.

The first time I met Professor Rao was during a Conference of the Indian Mathematical Society held in 1959 in the then Senate Hall of the Calcutta University. We were final year M.Sc. students and were asked to work as volunteers. We were eagerly waiting for the arrival of Rao as one of the few statisticians to address the conference. After Rao entered the hall, we found our revered teacher Hari Kinkar Nandi proceeding to warmly embrace Rao. (Incidentally, we knew that Rao and Nandi were classmates in the first batch of M.Sc. statistics course in the University of Calcutta that Rao had occupied the first position in the examination, Nandi ranked second. We had also learnt that the two had not met for quite some time before this event.) Prof. Rao spoke on a problem in combinatorial mathematics which—to the extent I can recollect—was entitled as the School Girl Problem. His delivery kept the audience spellbound.

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A. Chaudhuri et al. (eds.), *A Tribute to the Legend of Professor C. R. Rao*,
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https://doi.org/10.1007/978-981-33-6991-7_12

We came to know more about the contributions of Rao in various directions including probability distributions and their characterizations, linear estimation and regression analysis, Rao-Blackwellisation of estimators, orthogonal arrays and their uses, multivariate data analysis and a host of other topics. He was the man behind research and teaching activities in the Indian Statistical Institute. Notwithstanding his continuous and high-level engagement with statistical theory and methods, he was equally concerned about meaningful applications of statistics in different scientific enquiries. His book on Linear Statistical Inference became a "must", and his contributions to multivariate statistical analysis and its applications with genetic and anthropometric data became widely known. "Advanced Statistical Methods in Biometric Research" was a household treasure in the community of those who studied and applied statistics those days.

My second and a brief meeting with Rao was sometime in late sixties. I was asked by the Department of Statistics in the Calcutta University where I had started teaching in 1964 to meet Professor Linnik, then visiting the Indian Statistical Institute and to accompany him to our department where he had been invited to deliver a lecture. I went to Amrapali where Professor Linnik was staying as a guest of late Professor P. C. Mahalanobis. Professor Rao was there, and I introduced myself. I could find in him a composed and soft-spoken gentleman. Professor Linnik spoke on the three-point characterization of the normal distribution (given out in Linnik, Zinger and Rao).

I could see Professor Rao as a distinguished speaker in some conference or the other within the country, but never ventured to communicate with him. I remember that during the 1970 session of the Indian Science Congress held at the Indian Institute of Technology, Kharagpur Professor Rao was staying with his on Veera as guests of late Professor A. K. Gayen (who was one among the few Indian mathematicians who had earned a doctoral degree in Statistics from the University of Cambridge and joined the IIT Department of Mathematics on his return from England). Late professor Anadi Ranjan Roy—one-year junior to Professor Rao in the Calcutta University Department of Statistics—was presiding over the Section of Statistics and had invited Professor Rao to deliver a Special Lecture in his section. I along with my colleague Arijit Chaudhuri was also staying in the same bungalow. But we did not interact much with Professor Rao. We could only gauge how amiable and friendly a person he was. Arijit found some time to play cricket with Veera.

Years passed and I got married in 1973. I came to know from my wife that she had been a junior colleague of Mrs. Bhargavi Rao (wife of Professor Rao) in the J.D. Birla Institute and that Tejaswini (daughter of Rao's) had been a student of my wife. Mrs. Rao would often drop my wife (of course, before her marriage) at Maniktala on her way back from the college to the ISI. A couple of years later, Professor Rao and his wife had come to attend a session of the Indian Science Congress, and I met them to tell that my wife still remembers their affection. I also conveyed to Mrs. Rao information about some of her former colleagues in the Calcutta College. Thereafter, whenever we met in India Professor Rao and his wife would enquire about my wife. I could find a soft corner for near ones in their hearts. As General Secretary of the Indian Science Congress Association, I had the responsibility to look after their accommodation, health and related arrangements during the Science

Congress Session in 1996 when Professor Rao was to receive the P. C. Mahalanobis Birth Centenary Award (Gold Medal), and I enjoyed talking with Professor Rao on a wide array of topics around development of Science and Technology in India. Much later, during our trip to USA and Canada, we were to stay in Buffalo as guests of Teja, but had to cancel the visit to her place (coming from Hamilton) because of severe transport dislocation caused by the 9/11 tragedy.

I was greatly impressed by the book Statistics and Truth written by Professor Rao and published—quite rightly—by the Council of Scientific and Industrial Research. Rao's inquest into ethics of scientific research is a wonderful venture.

Nearly a decade back, I was visiting Europe on a lecture tour and stayed for a couple of days with Dr. Shalabh of IIT, Kanpur, the visiting the Ludwig-Maximilians University in Munich. Shalabh had become that time the third author for the Third Edition (2007) of the book entitled "Linear Models and Generalizations" by Rao et al. He told me that as the junior author required to add some materials over the content of the earlier edition, he was to prepare and forward his part to Professor Rao within the stipulated time and that Professor Rao would meticulously go through those materials and come back without much of a delay to put forth his views and suggestions for improvement. Sometime later, Shalabh presented a hard copy of the book published by Springer. I read through the volume carefully, sometimes on more than one occasions. I was fascinated by the content of Chap. 5 on "exact and Stochastic Linear restrictions" since I could get a lead to answer some of the questions which had vexed my mind all the time as I looked at some problem in industrial research where regression coefficients are required to satisfy some constraints, usually by way of chance constraints. Not much recourse has been taken in this volume to mathematical programming for the purpose of obtaining estimates of these restricted regression parameters. I could locate quite a few mistakes (not just printer's devil) including some misleading statements and use of inconsistent notations. Initially, I thought of forwarding the list to Professor Rao for necessary action at his end. I later felt that this would be rather an embarrassment to that great academician at an advanced age where co-authors should better accept the responsibility for any shortcomings. And I just forwarded the list to Shalabh.

Some years back, I was requested by the Director of the ISI to become the Chairman of the Board of Directors for the International Centre for Statistical Education (within the ISI). I came to know that Professor Rao was stepping down from this office after having been the Chairman since the demise of the founder Professor P. C. Mahalanobis. I told the Director that there would be more competent persons to become the Chairman of the Board. I was told that Professor Rao had in the last meeting in which he acted as the Chairman in absentia had suggested that I should be approached first with a request to take up the responsibility, and I could not say "No". A few days after, I accepted the request, I felt highly elated to receive a letter from Professor Rao—the only one I ever received from him—suggesting that I should try to re-establish the linkage between the International Statistical Institute and the ISEC and to secure recognition and support from the Third World Academy of Science. I tried to initiate some steps in this direction, but have not been able to secure any concrete result.

What has made Professor C. R. Rao somewhat different and more respectable to me—compared to some other Indian statisticians who also attained eminence in terms of their academic contributions—is that he spent most of his active academic life in India trying to raise the standard of teaching, research and promotional work related to Statistics in India, had an abiding interest in the development of the Indian Statistical Institute

As we all know, C. R. Rao has a large number of publications to his credit, including 14 books, quite a few edited volumes on a wide range of (published as handbook of Statistics by North Holland) and a large number of research articles some of which contained path-breaking concepts, methods and techniques. Most of his publications deal with the subject at an advanced level and are really addressed to serious readers. "Statistics and Truth: Putting Chance to Work" is again a volume meant for seekers of truth behind contributions in scientific research, looked back critically in terms of credibility and validity of evidences and their treatment by the scientists.

C. R. Rao inspired many a scholar in Mathematics and Statistics through his significant and seminal contributions to these subjects and through direct teaching, counselling and efforts to promote the development and use of Statistics in the country as also elsewhere. He guided a large number of scholars in their doctoral work, including some who earned a lot of recognition. His foray into a wide array of topics encouraged many to take up studies on several less-trodden paths, without possibly interacting with him directly. And the author of this note is one among many who can be put in this category.

Midway in my academic life, I found interest in Optimization Theory and its Applications in Statistics. And I gave the title "Is the Statistician just an optimizer?" to the lecture I delivered as the President of the Section of Statistics in the Indian Science Congress Session in Jaipur in 1994. The content of my presentation was, in some sense, motivated by the following sentences in a Foreword written by Rao for the Book Mathematical Programming in Statistics by T. S. Arthanari and Y. Dodge. "The classical optimization methods based on differential calculus are too restrictive, and are either inapplicable or difficult to apply in many situations that arise in statistical work. This, together with the lack of suitable numerical algorithms for solving optimizing equations, has placed some limitations on the choice of objective functions and constraints and led to the development and use of some inefficient statistical procedures." In fact, Rao himself used optimization methods in his works related to regression analysis in its comprehensive form.

I also derived inspiration from the contributions of C. R. Rao in the area of characterizations of probability distributions, and I read with great interest "Characterization Results in Mathematical Statistics" by Rao, Linnik, and Zinger and tried to make use of certain results contained there in proving characterization results for some distributions in reliability analysis.

I end with my sincere regards for a gentleman of great erudition with a commitment to the subject he espoused.

Our Dr. Rao

Shibdas Bandyopadhyay

It was in July 1964. All the new students and scholars were to have their first class in the auditorium of the institute in the geology building. We were told that professor would take the class. The dais was decorated with scientific items.

With overflowing audience, including our new faces, it was like a festive event. We heard that Dr. Rao will also attend.

I saw Dr. Rao there but did not meet with him then.

We came to know that professor C. R. Rao is Dr. Rao [and professor P. C. Mahalanobis was professor] in the institute.

We used to have our classes, except science subjects, in room 3.1 on the third floor of rts building. Dr. Rao, who was then the head of the research and training school [rts], also had his office on the same floor. We used to see him walk along the third floor corridor often. Those days, bells would ring after each period [of 50 min]. Students would, like all students, would talk while waiting for the next teacher. Occasionally, when the next teacher was yet to arrive, Dr. Rao would come into the classroom. He would ask about the teacher, subject being taught and his teaching related issues. Once in a while, we would see the teacher waiting outside while Dr. Rao was in the class.

Most of us used to stay in 205 hostels. Dr. Rao used to live on 205 campus. We got familier with Dr. Rao on our way to classes and also on campus. Most evenings, Dr. Rao would walk around 205 campus pond with few others joining him. During hostels day and holi every year there were more campus community interactions. Familiarities increased from good wishes from us to his recognition of us by names during the years.

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A. Chaudhuri et al. (eds.), *A Tribute to the Legend of Professor C. R. Rao*,
Indian Statistical Institute Series,
https://doi.org/10.1007/978-981-33-6991-7_13

The first time I really met with Dr. Rao was in 1967. Every B. Stat. student used to opt, in writing, for M. Stat. specialisation. Allotment would be on the basis of an interview. Students would be called one by one on the basis of final year B. Stat. roll numbers. My roll number was 2 but I was not called for interviews. At the end, I was advised to meet with Dr. Rao. I met with him.

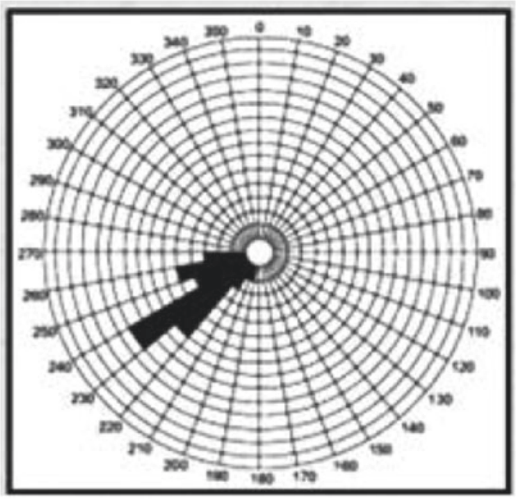
I was the only B. Stat. allotted a new specialisation starting that year; Dr. Rao said that my training was appropriate for it; the area of specialisation had a lot of demand in the US. The end with the first meet.

Dr. Rao would occasionally take our classes. One of those days he brought a book in the class and asked us to read from a random page. None of us could. It was a translated version of his linear statistical inference book in Russian language. That was a friendly way to talk to him.

After my M. Stat. in 1968, I became a project assistant officially. A project assistant used to get monthly stipend of Rs. 300/- whereas a research scholar's stipend used to be Rs. 250/-. Officially, I was under the dean of studies but I was to report to Dr. Rao for research work and Dr. J. Roy of the computer science unit for project related work.

On his advice, I started reading research reports. Dr. Rao wanted me to report my progress periodically. He would give time for discussions. He was sure to be available whenever I sought time. I started to work on circular distributions. He, however, gave opportunity and encouraged work in different areas.

One of those days, I was asked by Dr. J. Roy to meet with the professor. Before meeting the professor, I had a discussion with Dr. Rao. I came to know that I was to meet Mrs. Haldane in Hyderabad to look into her data on turtles that were collected from the coastal areas of Orissa. Further, I was to combine my visit to have a firsthand exposure to circular distributions by making field visits with a research team from the geological research unit of the institute doing field work in Andhra Pradesh. I did both visits. I do not know if my visit to Mrs. Haldane was any help to her, I collected some circular data from the river bed that I analysed [see the rose diagram] to publish my first research paper in a journal. I continued my research in the area of circular distributions and produced another technical report on the measures of circular association.



ROSE DIAGRAM TO IDENTIFY PALIOCURRENT DIRECTION

Political atmosphere was getting unstable by the end of 1969. A number of ISI faculty moved out of Calcutta. Dr. Rao was also to move to Delhi. I was confused with several suggestions to move forward with my research. I chose to do Ph.D. from the US. I left India in 1970.

Upon my return to India in July 1975, I joined ISI. Other than sample survey projects, I started teaching degree courses and ISEC courses. At that time, Dr. Rao was the chairman of the ISEC board of directors. Events would be organised for ISEC trainees and teachers to interact with Dr. Rao. I enjoyed interacting with Dr. Rao during such events. There were several such periodic interactions.

In 1980, we organised a symposium on advances in methods of data collection. The symposium was inaugurated by Dr. Rao. In the following picture, Dr. Rao is seated in the first row with five participants each on either side; the author is in the third row with four participants on either side.

That year Dr. Rao retired from the institute. He, however, remained the ISEC chairman and my interaction with Dr. Rao continued. During my long association with ISEC trainees, I was told by the trainees in early 80s that a certificate was inappropriate for the 10-month-long ISEC regular course. Many felt that they learnt significantly in 10 months in the regular course, but, with a certificate they were unable to apply for advanced studies. When I was made the member-secretary of the ISEC Board of Directors in July 1983 with Dr. Rao, its Chairman, I started negotiating with the Govt. of India to upgrade the certificate to a diploma. The 10-month regular course had content and potential for higher studies in different areas of statistics. I proposed to change the certificate to a junior diploma. During my period as member-secretary, I used to send frequent reports on ISEC to all members of the board of directors. I discussed the issue of upgrading certificate to diploma. I got Dr. J. Durbin, Dr. J. Gani and Dr. E. Lunenberg of the International Statistical Institute to check the status of ISEC certificate. After a lot of discussions on the content and duration, it was agreed by ISI Director, Government of India and the International Statistical Institute to call the diploma a 'Statistical Training Diploma' from the 1986–87 term.

The computing facility available to ISEC trainees was limited. It was the time when micro-computers were coming to Indian market. Dr. Rao wanted a micro-computer with a printer exclusively for the trainees. I was unable to raise the cost of micro-computer within the institute. I wanted to wait for cost reduction but Dr. Rao did not. With Dr. Rao's help and additional fund from Dr. Gani's applied probability trust and from the International Statistical Institute, one DCM TANDY micro-computer was purchased.

After I left the position of the ISEC member-secretary in 1987, my interaction with Dr. Rao was mostly limited through ISEC only as an ISEC teacher. I continued to meet with him at several other Kolkata ISI occasions. A function was held to honour Dr. Rao on 28 January 2003 by Stat-Math unit. I was very proud to present a memento to Dr. Rao at an informal gathering in front of room 3.0. Mrs. Rao is seen on Dr. Rao's left in [picture-2]. A moment of joy was when Dr. Rao lifted the memento with two hands above his head, in [picture-3], like a tennis grand slam winner.

The institute was celebrating the golden jubilee of ISI degree courses. I was responsible to organise several events. I took the opportunity to invite the current and all former deans of studies for a function. I also invited Dr. Rao. Dr. Rao was present on the dais along with deans of studies; the author is on Dr. Rao's left in [picture 4].

My association with Dr. Rao continued. I would go and wish him whenever he visited Kolkata. Sampling and Official Statistics [SOSU] was a new unit created in the institute in Kolkata in 2012. I was made its head. Dr. Rao was a former president of the International Statistical Institute. We, in SOSU, wanted his opinion on future activities of the unit and also blessings. On one of his visits to Kolkata, I invited Dr. Rao to visit the new unit SOSU. We felicitated him with a bouquet of flowers. We felt that he was indeed one of us when he sat on a shelf [picture 5] by the wall. * Colour Pictures are in page 205.

I am fortunate to be associated with Dr. Rao.

A Tribute to Professor C. R. Rao

Arijit Chaudhuri

While paying regards to a hugely phenomenal savant of eminence, a disclosure of a few features of the person submitting them is axiomatically inevitable. The article 'C. R. Rao: A life in statistics' by Prof. B L S Prakash Rao (2014) contained in this volume has admirably exposed adequately diverse aspects of the life and work of Prof. C. R. Rao. So, I must refrain from trying to add anything to such an exercise. But I like to reveal here a few personal matters only by way of paying regards to Prof. C. R. Rao.

My Ph.D. research supervisor was Prof. H. K. Nandi in Calcutta University (C.U) who was a classmate of Prof. C. R. Rao in the first batch of post-graduate students of statistics in C.U. I was never a student of Prof. C. R. Rao. I passed M.A in statistics in February, 1962 from C.U. But Prof. Nandi taught us his first published book 'Advanced Statistical methods in Biometric Research' (1952, John Wiley & Sons). We learnt Rao—Cramer inequality and Rao—Blackwell theorem from this book.

Treatment of vectors and matrices, linear models, limit theorems on probability, and distribution theories in this early book appeared to us formidably tough. Just before starting this write-up, I once again glanced through its pages and the contents looked no easier even now.

Once in 1977, I received Prof. Rao's first letter asking me to contribute a research article with a 4-page limit for Bulletin of International Statistical Institute (BISI) as he was the then President of ISI. To my utter delight in 1983, I received his second letter asking me to contribute one full-fledged article to the Handbook of Statistics Volume 6 he was to jointly edit with Prof. P. R. Krishnaiah for Elsevier Publishers of North Holland. This was eventually published in 1988 with my paper turning out to be the biggest in length. In 2015, when S. L. Warner, the father of randomized response (RR) techniques (RRT), was to attain 50 years of his invention, Prof. C. R. Rao in his email asked me to be his joint editor for the Handbook of Statistics

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A. Chaudhuri et al. (eds.), *A Tribute to the Legend of Professor C. R. Rao*,
Indian Statistical Institute Series,
https://doi.org/10.1007/978-981-33-6991-7_14

Volume 34 exclusively on RRTs. He immediately agreed to my suggestion to add Prof. T. C. Christofides of Cyprus University as the third joint editor, and the volume was published in April 2016. By that time, however, I developed a closeness to him in course of my repeated correspondences procuring his unmixed collaborative participation in his aiding our publication with authorship of three of the four books with his contributed articles therein brought out by ISIREA during 2009–2017. He not only wrote for us, but emailed me in appreciation of our publications which he claimed to have read cover to cover including the articles in Bengali and not to speak of those in English.

When we sought his blessings announcing our scheme to observe his centenary with a financial and moral backing from the ISI Director, he enthusiastically emailed me his elaborate CV plus a few photographs of himself.

Let me now recount a few of mine own academic contributions through which we may mean to pay our due respects to this man of outstanding stature with lifelong academic pursuits.

I shall narrate only a few simple findings of mine or jointly with those of my students:

(i) To guard against possibilities of committing silly mistakes, not uncommon in practice, it is good to know that for the existence of an unbiased estimator for a finite population total 'the inclusion–probability of every population unit' should be necessarily positive and it is 'sufficient' for it to be so; proof is non-trivial;

(ii) for the existence of an unbiased estimator of the variance of an unbiased estimator for a finite population total its 'positivity of inclusion-probability of every pair of distinct population units,' is 'necessary' as well as 'sufficient.' Why?
$V(t) = E(t^2) -$ (Sum of squared y's + Sum of cross-products of all y_i, y_j's for distinct pairs of units).
So, unbiased estimation of Sum of y_i, y_j's, in the cross-product herein positivity of π_{ij}'s is an 'NSC.' Hence the above claim.

(iii) With survey data in hand through 'Probability Proportional to Size Sampling Without Replacement' to unbiasedly estimate the gain in efficiency over a rival 'Simple Random Sampling Without Replacement' (SRSWOR) a simple trick is to use Desraj's (1956) unbiased estimator for the 'Population total of the squared variate-values.'

(iv) To rationally set the size of an SRSWOR to ensure

$$\text{Prob}[|t - Y| \leq fY] \geq 1 - a$$

with pre-assigned proper fractional positive values of f and a one may use Chebyshev's inequality

$$\text{Prob}[|t - Y| \leq K.\text{SE}(t) \text{ of } t] \geq 1 - \frac{1}{K^2}.$$

This equivalently needs tabulating N, f, α, CV$= 100\frac{\mathrm{SE}(t)}{Y}$ to lay down the values of n, the sample-size.

(v) It is a tough task to extend (iv) to cover Warner's RR alternative estimator e replacing t. To see this, note

$$V(e) = V(t) + \frac{N^2}{n}\frac{P(1-P)}{(2P-1)^2} = \mathrm{I} + \mathrm{II}, \text{ say}$$

This is so when we employ SRSWOR of size n,
$t = N$ Sample-mean of y-values,
$e = t$ (with y's replaced by RR-based unbiased estimates of y's)
$P =$ Proportion of cards marked 'Yes' $\left(0 < P \neq \frac{1}{2} < 1\right)$ about bearing a stigmatizing characteristic, say, 'Testing HIV Positive.'
It is verifiable that the Chebyshev's approach in (iv) above gives reasonable values of n versus N, f, α, CV to contact but the solution of n correspondingly for $\mathrm{I} + \mathrm{II}$ turns out beyond limits of logic.

(vi) To cover the cases of unequal probability sampling is difficult but a few solutions we have reached.

(vii) To apply sophisticated method of sampling to audit Government financial transactions of huge dimensions, we found it convenient to apply 'Multi-stage Sampling,' choosing, e.g., district-wise PWD offices by Rao—Hartley—Cochran (RHC 1962) scheme in the first stage, with sanctioned budget amounts as size-measures and by SRSWOR each of the next four-stage units, namely the different office departments, department-wise ledger books, pages and the page-wise rows, and reading of the row-wise column entries. Unbiased estimation of totals and their variances is easy to come by [vide Chaudhuri, Arijit (2010)].

(viii) An Indian version of small area estimation (SAE) to apply to NSSO.
What is called small area estimation (SAE) or problem of developing small domain statistics arises when a suitable sample is taken from a population to estimate its total but in addition the same sample is used to estimate totals of the domains of the population of various sizes and unacceptably low efficiencies are encountered about estimates of several domain totals owing possibly to small numbers of sample units pertaining to several domains of interest.
In NSSO, rural surveys in India samples are taken in two stages from every district in each state or union territory (UT); the villages are the first stage and the households (HH) therefrom, if selected, constituting the second stage units. The survey data are used to derive 'District-level' estimates, and they are just added to provide state or UT-level estimates.
The sample-size for every district invariably turns out palpably small raising doubts about the efficiency level of state/UT-level estimates which may hopefully be improved by applying model-based SAE methods.
The Government of India in 2005 asked me to try and provide other alternative estimates for all the states/UT's, Chaudhuri (2005) provided his response

with estimates having reduced estimated coefficients of variation on applying generalized regression (Greg) method of estimation taking village 'Census-population values as exogenous regressors.' We also tried Empirical Bayes (EB) estimates which offered no tangible further improvements on the Greg estimates. The government appeared satisfied with a letter of thanks on receiving my reported alternatives.

(ix) Randomized response techniques since inception happened to be based mostly on exclusively simple random sampling with replacement (SRSWR) are seen in the early publications.

In order to convince NSSO of their applicability in Chaudhuri (2001), it showed extension of most of the known RRTs to be applicable in usual varying probability sampling situations.

(x) Observing wide discrepancies in reported amounts of rural loans 'incurred' from NSSO surveys and 'advanced' by banks from RBI accounts, I could document [vide Chaudhuri (2017)] an effort to reconcile them by dint of 'Network and Adaptive sampling'—a book-length publication being Chaudhuri, Arijit's (2015) text with this title.

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His Experiments with Truth

Dipti Prasad Mukherjee

Prologue

Tribute to the living legend: It is an honor for me to pay tribute to Prof. C R Rao (CRR), a living legend in the true sense of the term. It is indeed a rare honor to write a eulogy of a person, who at this incredible age takes less than 24 hours to reply to his emails from another continent. Not just that, here we have a man on the threshold of a century of life and activity. I salute Prof. C R Rao as the instance of a person who celebrates his life meaningfully. I must congratulate the ISI Retired Employees Association for organizing this memoir as tribute to a genius and for giving me the opportunity to express my appreciation for a true and exemplary academic.

CRR and ISI: Stories about Prof. Rao abound in the Indian Statistical Institute (ISI). Anybody spending some time at ISI comes to hear these funny anecdotes about the legend. I can share some of them. But I cannot share all as what I am narrating is meant to be a tribute to a living person. One such story is about God tossing a coin! Prof. Rao used to send his students to a hospital near Bonhooghly to record the birth of boys and girls. The idea was to derive a binary sequence (say, a sequence of 1 and 0 representing birth of a girl child and a boy child, respectively). Parallely, he asked his students to generate a random sequence of heads and tails through coin tossing. He would then demonstrate that the birth sequence of boys and girls closely resembles a random sequence. As a person interested in application driven research, I found this simple demonstration amazing.

ISI Poster Boy: I am privy to a number of presentations and reports on ISI. These presentations and reports are usually placed before government or non-government agencies or foreign institutions. Apart from contributions of the founding father of the institute, Prof. P C Mahalanobis, the most significant contributions of ISI invariably

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A. Chaudhuri et al. (eds.), *A Tribute to the Legend of Professor C. R. Rao*,
Indian Statistical Institute Series,
https://doi.org/10.1007/978-981-33-6991-7_15

start with contributions made by the institute's poster boy Prof. C R Rao. By design, the DNA of ISI is interdisciplinary. The varied research interest of Prof. Rao is a true reflection of the interdisciplinary culture of the ISI. Prof. Rao's Ph.D. thesis in 1948 investigated statistical problems of biological classifications. He was one of only six students (and probably the last) advised by Prof. Ronald Fisher (Marianna Bolla's blog on Prof. C R Rao mentions that Prof. Rao is the only Ph.D. student of Sir R A Fisher [1]). In 1968, Prof. Rao delivered the Alladi Krishnaswamy lecture in Andhra University on Computers and Future of Human Society [2]. In the lecture, he discussed a set of computer applications in experimental stages. It is remarkable that he was stressing on applications like machine-aided translation, identification of literary style of any author using computer algorithm, problems of information deluge, computer-assisted instruction, and so on, most of which are relevant even as of today [2]. Prof. Rao's achievements eco achievements in the ISI, from mathematics to statistics, and biology to computers.

Amazing wit: I remember attending Prof. Rao's lecture in the erstwhile Geology Auditorium in Baranagar campus, sometime in the nineties. I do not remember now the particular topic on which he was lecturing. What amazed me was the organized presentation he was making even at his age, which was then close to 80. The most striking thing was the sharp wit and the number of anecdotes that was packed into the presentation. I still clearly remember that his presentation was like that of a practitioner simulating not just blue-sky results but also trying to intuitively explain the essence of the theory.

Acknowledgement: This modest tribute would not have materialized without the persuasion of Prof. Arijit Chaudhuri. It is a strange coincidence that almost every day I cross paths with Prof. Chaudhuri, particularly at lunch time. As a dedicated editor of this memoir on Prof. Rao, Prof. Chaudhuri is always chasing to find a potential contributor for his volume. Prof. Chaudhuri would sometimes inspire me, sometimes cajole me, and even sometimes ignite my interest with provocative anecdotes, to write at least a short note celebrating the works of Prof. Rao. One day in late November of last year, pointing to beautiful flowering plants in front of the S N Bose Bhavan, he told me: Did you know that Prof. Rao was interested in identifying if a tree was leftist or rightist? This ignited the final spark to investigate the genius of CRR. (Later, while reading the lecture notes of Prof. Rao, I discovered that it was Prof. T A Davis who first thought about such an experiment with a coconut tree at ISI. The interested reader can refer to [3] for details.)

Statistics and Truth

Preface: I am not formally trained in statistics. However, this lack of orientation gives me an excuse to ask questions which appear either trivial or obvious to trained statisticians. I always remember former bureaucrat Mr. N Vittal's speech at the Indian Statistical Institute. He graduated in chemistry, and then entered the Indian Administrative

Service. He eventually went on to become the chairman of the telecom commission of India. During his initial engagement with the department of telecommunications, he started asking seemingly trivial questions without fully understanding their impact or anticipating the extent of their importance. He mentioned that seasoned technocrats initially thought that answers to these questions were likely to be impractical. However, technocrats, unable to avoid questions from a bureaucrat, started thinking out of the box. And that, Vittal claimed, had brought about much desired changes for the telecom sector in India. Prof. Rao's deceptively simple approach to truth, through statistics, inaugurates, in my opinion, a new way of understanding his extremely iconic vision of statistical phenomena in the world around us. I hope that my way of understanding the relationship between statistics and truth opens up a window into the genius of Prof. Rao.

Context: I reiterate, therefore, that for me anecdotes contributed to painting a vibrant picture of this legend. I was parsing Prof. Rao's publications and tried to create a snapshot of this man as it revealed itself through his writings. I discovered his series of lectures 'Statistics and Truth, putting chance to work' [3]. This was a part of the CSIR distinguished lecture series on the occasion of the Ramanujan Centenary Celebrations of 1987. Prof. Rao delivered three lectures at NPL (Delhi), ISI (Calcutta), and CLRI (Madras) in that same year.

However, this was not the first attempt to link statistics to truth. A casual Google search points to a book by the legendary Prof. Richard von Mises, 'Probability Statistics and Truth' [4]. Prof. Rao also acknowledged this book by von Mises in his lecture. Is there any paradigm shift from von Mises' understanding of truth based on probability in Rao's calibration of truth with uncertainty?

I find two striking observations on the understanding of the relationship of truth to uncertainty in the work of von Mises. I understand that random quotes from writings may be misleading. Yet, I cannot resist in reproducing two observations *verbatim* from von Mises. On one hand, von Mises is rediscovering facts or truth which is already there: *In dealing with the theory of probability, i.e., with probability calculus, I do not hope to achieve more than the results already attained by geometry, mechanics, and certain other branches of physics. That is to say, I aim at the construction of a rational theory, based on the simplest possible exact concepts, one which, although admittedly inadequate to represent the complexity of the real processes, is able to reproduce satisfactorily some of their essential properties* [4].

On the other hand, he claims to deduce uncertainty from statistical considerations: *…conclusions drawn from statistical considerations are at best uncertain and at worst misleading. I do not deny that a great deal of meaningless and unfounded talk is presented to the public in the name of statistics. But my purpose is to show that, starting from statistical observations and applying to them a clear and precise concept of probability it is possible to arrive at conclusions which are just as reliable and truthful and quite as practically useful as those obtained in any other exact science. In order to achieve this purpose, I must ask you to follow me along a road which is often laborious and by paths which at first sight may appear unnecessarily winding* [4].

Reflections: In contrast, Prof. Rao's acknowledgement of the epistemological rule of statistics is much wider and all-encompassing. Prof. Rao's conviction with statistics can be summed up in one sentence from his own words: *All methods of acquiring knowledge are statistics* [3]. He believed in the quantification of uncertainty and in doing so he claims that *we are able to raise new questions which cannot be answered by the classical or Aristotelian logic based on two alternatives "yes" and "no"* [3]. The objective management and measurement of uncertainty, according to Prof. Rao, help manage *individual and institutional activities in an optimum way by controlling, reducing, and, what is more important, making allowance for uncertainty* [3]. Through tons of observations and examples, starting from the design of experiments, to encryption of messages, to building models to solve numerous and wide-ranging practical and focused applications (solutions to disputed authorship, dating of publications, geological time scale, circadian rhythm, disputed paternity, salt in statistics, economy in blood testing, and so on [3]), he celebrates uncertainty, and thereby comes close to the idea of truth.

Is Statistics Enough for Truth?

Prof. Richard von Mises and Prof. Rao could convince me that one of the most plausible ways to find truth is statistics. But is that enough? Is statistics sufficient in defining truth? I turn around and consider statesmen and thinkers who influenced society over the years. On most crucial occasions, these stalwarts came up with brilliant and path breaking ideas based on conviction and belief in humanity in order to change a nation's course or influence a generation. Did they have enough data points to support their postulates? Did they have any statistics favoring their hypothetical adventurism? In most cases, they did not have a single observation to back up their ideas but they went ahead unheeding and took a risk!

We do not have to go too far to understand this. We are celebrating the sesquicentenary of Mahatma Gandhi this year. The Mahatma's experiments with truth are chronicled in his autobiography [5]. Almost in every page of his celebrated autobiography, the Mahatma is experimenting with an idea of a truth but all on his own terms. To quote: *My uniform experience has convinced me that there is no other God than truth* [5]. What is all-pervasive is the Mahatma's self-realization and self-confidence on each and every inference he made. In most cases, statistics were not in his favor, numbers may not have been supporting his decision, and experiments may not have been air-tight, but he was set to define his own truth and able to convince the masses about the universal applicability of that truth. To quote again from his autobiography: *To see the universal and all- pervading Spirit of Truth face to face one must be able to love the meanest of creation as oneself. And a man who aspires after that cannot afford to keep out of any field of life* [5]. And this, per Mahatma, is the defining answer to the question: Why had he joined politics? *That is why my devotion of truth has drawn me into the field of politics, and I can say without the slightest hesitation* [5].

Postscript

Greek philosophers including Plato argued in favor of the objectivity of truth. Statistics, to my understanding, is an effective tool that supports rather than negates this objectivity. Prof. Rao's lecture lucidly relates uncertainty to truth. His numerous ideas of experiments substantiated the universality of uncertainty. It would be interesting to see a discussion on Prof. Rao's bold observation that statistics defines all methods of acquiring knowledge. I invite views of philosophers and their attempt to explain the structure of knowledge acquired by genius without recourse to statistical interpretation. The fact that leaders of the world were and are still experimenting with their own definitions of truth, and in most cases ignoring cogitated statistics, has definitely influenced the course of society and well-being of individuals in civilization. Sometimes these leaders are taking the civilization to a new height and sometimes pushing humanity toward extinction. The question is as follows: Can statistics help or impede them? Does not genius need statistics? Or is statistics only for the ordinary intellect? Is reliance on statistics likely to discount the genius of crude and far-reaching vision? I rest my case with a subtle realization of the Mahatma as he narrates it in his autobiography. In his initial days, he was not convinced of life insurance as he thought *life assurance implied fear and want of faith in God* [5]. But then, the statistics of life and family won over his faith, and he finally decided to buy a life insurance to mitigate uncertainty in life, of his own and of others close to him.

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C. R. Rao

Tasos C. Christofides

I am honored that Professor Chaudhuri has asked me to prepare a short write-up for Professor C. R. Rao, a legend of modern statistical theory. I first met Professor C. R. Rao in 1998 in Athens, Greece, when he attended a conference organized by the University of Athens. What impressed me the most was that a person of his caliber, a giant of statistics, was behaving like everybody else, as an ordinary conference participant. The following year, he visited the University of Cyprus where he delivered an inspiring and well-prepared speech to students and talked about the future of statistics. I must say that based on my recollection, what he talked about and conjectured (about the state of statistics in the future) is happening right now. Usually, one expects first class researchers to care little about teaching and statistics education. However, this is not the case for Professor Rao. This conclusion does not come out from his presentation alone at that time but from the fact that during his career and especially during his tenure at the Indian Statistical Institute, he put a lot of emphasis on statistics education. His second visit to Cyprus took place in 2001, when he was awarded an honorary doctorate from the University of Cyprus. I had the privilege to present his scientific achievements during the ceremony. I must admit that it was not easy for me to prepare my speech. His numerous important results in many areas of statistics made it difficult for me to decide what is important to present and what it is not. I knew that his contributions on many areas of statistics, and among them, estimation, testing, linear models, multivariate analysis and design of experiments were very influential. But still, I was not at all sure whether in my presentation, I had included his most significant achievements. It was an exciting day and an important ceremony attended by rectors of many universities from various commonwealth countries, including of course universities from his native India. But for me, the most memorable event of the day is what happened at the dinner which

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A. Chaudhuri et al. (eds.), *A Tribute to the Legend of Professor C. R. Rao*,
Indian Statistical Institute Series,
https://doi.org/10.1007/978-981-33-6991-7_16

took place at a local fish tavern in the evening. He, an eighty-years-old person, had a detailed conversation with my nine-years-old son, on a topic I had no idea about. About Pokemons! C. R. Rao knew so many details about Pokemon figures that only a child playing with them would be able to remember. When I asked him how it was possible to know all these things about Pokemon figures, he responded with the sentence “well, I am grandfather you know”.

In the years that followed his last visit to Cyprus, I have had the opportunity to meet Professor C. R. Rao on various occasions, mainly in conferences. But I believe that the highlight of my association with him took place in 2016 via a joint editing (with Professor Arijit Chaudhuri as well) of Volume 34 of the Handbook of Statistics. I am extremely privileged that I had the chance to meet him on several important occasions and I am deeply honored that my name is associated with a person whose contributions and impact on statistics are of enormous importance and whose name appears in a significant number of results in modern statistical theory.

Professor C R Rao: Homage to the Towering Personality of Indian Statistical Institute

Nibedita Ganguly

Professor Calyampudi Radhakrishna Rao, popularly known as Professor C R Rao, is the name of veneration of a world-famous statistician, an illustrious teacher and the friend–philosopher and guide of the students of Indian Statistical Institute. He is a soft-spoken personality with deep sense of humour. Everyone used to look at his appearance with a sense of deep reverence and admiration. I was neither a student of him nor an employee of ISI during his tenure. Yet this towering personality caught the fancy of queries of me like that of other employees of ISI. During recent past, Professor C R Rao used to visit ISI every year or every alternate year in winter months. Whenever he came to ISI, he used to visit, at least once, the Reprography and Photography Unit of the Library to go through the photographs of ISI kept in this Unit and memorize previous eventful days. He had an intense memory and deep passion and interest in photography. This world-famous statistician himself checked many photograph negative reels, wrote the related events and year, wrapped those in white papers and put them back in the drawers after describing those events to me. It was a mesmerizing and unforgettable experience to an ordinary person like me.

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A. Chaudhuri et al. (eds.), *A Tribute to the Legend of Professor C. R. Rao*,
Indian Statistical Institute Series,
https://doi.org/10.1007/978-981-33-6991-7_17

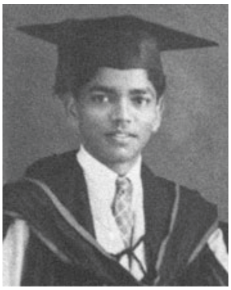
C R Rao with a photo negative roll

With Photo album

With printed photographs of ISI

In the winter of 2010, teachers and fellows of ISI had planned to felicitate Professor C R Rao, in the NAB-I hall of the Kolmogorove Bhavan, on his 90th birthday. It was also an occasion of celebration of Golden Jubilee of the graduation programme at ISI. This time, Professor Rao came to India along with his wife, Mrs. Bhargavi Rao, and his daughter, Tejaswini. We, the ‘workers’ in the Reprography and Photography Unit, had been planning to prepare a photograph album on this towering personality of ISI. The work had started on quietly.

C R Rao described in his autobiography that he was the eighth child to be born in a family of ten siblings on 10 September 1920 in Huvina Hadagali, now in Karnataka. Being the eighth child, he was given the name ‘Radhakrishna’ after God Sri Krishna. From his childhood, he was inspired to solve mathematical problems under the guidance of his father. After he did his graduation in M.A. with first-class honours in Mathematics from Andhra University in 1940, he applied for a research scholarship from Andhra University, but his application was rejected on the grounds that it had been received after the deadline. He also appeared for the competitive Indian Civil Service examinations, but it did not materialize due to underage. He applied for job as a mathematician in an army survey unit and was called to Calcutta for an interview but failed to get the job. However, this was a turning point for C R Rao. He stayed in a South Indian Hotel before his interview, and there he met a young man who was being trained in statistics at the Indian Statistical Institute. C R Rao had taken a course on probability while studying for his master’s degree at Andhra University, but he had never heard of the Indian Statistical Institute. The young man took C R Rao along with him to visit the Institute, which was located in the Physics Department of Presidency College during that time. There at ISI, he got a chance in ‘training in statistics’ which also met the need for a job and to test whether he would like research. So, C R Rao applied for the one-year training course in statistics. That was the beginning of his career in statistics ‘chosen as a last resort’.



C R Rao after graduation

After C R Rao joined the Indian Statistical Institute as ‘Institute Technician’, in a few months, the Calcutta University started the Statistics Department with Professor P C Mahalanobis as its head. C R Rao had applied for its post-graduation course. In his words,

> In July 1941, six months after I joined the institute as a trainee, the Calcutta University started a Master's degree program in Statistics, which was the first of its kind in India. The Professor was made the honorary head of the new department of statistics and the 'Institute Technicians' were made part time lecturers at the University to teach the courses. R C Bose and S N Roy were the main professors and they gave courses on the design of experiments and statistical inference; there were others, Sadashiv Sengupta and P K Bose who covered other topics. (Rao)

Among six students of the first batch of Statistics Department of Calcutta University who completed their post-graduation in statistics were Hari Kinkar Nandi and C Radhakrishna Rao.

C R Rao was awarded the degree in 1943 with first class and a gold medal. In his autobiography, C R Rao amusingly wrote that his wife later had used that gold medal as a pendant.

In those formative years, ISI required researchers and fellows in statistics. In the ISI Council Proceedings on 4 January 1943, a draft advertisement was prepared. It reads as the following,

> Applications are invited from candidates with qualifications for undertaking statistical researches for scholarships and fellowships of the Indian Statistical Institute of value ranging from Rs. 50 and Rs. 100 per month depending on qualifications tenable for one year in the first instance…

In the next proceedings of ISI Council meeting held on 12 February 1943, it was resolved that '*scholarships of Rs. 50/- not exceeding six in number be awarded in the first instance for the present, but in case any candidate is found to be of sufficient ability the amount of the scholarship may be increased to Rs. 75/- or a fellowship of Rs.100/- may be awarded as a special case*'.

All the six students of the first batch of post-graduate class of Calcutta University Statistics Department joined at ISI. C R Rao has described this in his autobiography. He wrote, 'Professor offered jobs to all of us'. He also narrated about the poor salary at ISI.

> The Professor offered jobs to all of us in the ISI as technical apprentices on a salary of Rs.75 (5 dollars) a month. With first class Master's degrees in Mathematics at the Andhra University and Statistics at the Calcutta University, I was expecting a higher salary. However, I accepted the job without 'asking for more' and joined the ISI in December 1943. To my surprise, I found that my salary was increased to Rs.150 within a month. Perhaps, the Professor meant it as a gift for accepting the earlier offer without 'asking for more'. Rao (1992)

C R Rao was appointed as a Technical Apprentice at the Indian Statistical Institute in November 1943, and, a few months later, in June 1944, he also worked as a Part-Time Lecturer at Calcutta University.

C R Rao at ISI

C R Rao did not mention the names of the members of the Selection Committee. In ISI Council Proceedings of 12 Feb 1943, it is mentioned that Prof. Levi, Prof. J C Sinha, Mr. R C Bose and Prof. P C Mahalanobis (Secretary) were appointed to make a preliminary selection of suitable candidates.

There is an error in his name in C R Rao's first appointment resolution in ISI Council Proceedings of 9 February 1944 where it is mentioned, '*that Radhakrishna Rao, Harikinkar Nandi and K C Cheryan, be awarded research scholarships of Rs. 75/- per month with effect from the dates on which they joined the Institute up to November 1943 and research fellowships of Rs. 150/- per month for one year from December 1943*'. Instead of his actual name Calyampudi Radhakrishna Rao, his name was written as 'Radhakrishna Rao' in the proceedings.

P C Mahalanobis, Padma Bibhusan and C R Rao, Padma Bhusan at ISI after being awarded by Govt. of India in 1968

Professor P C Mahalanobis was always impressed by persons with imagination and originality, but he emphasized the importance of the capacity to do routine work according to well-thought-out rules and well-ordered schemes. P C Mahalanobis always believed that routine work gave a much better grasp on details which strengthened the foundation of their ability to contribute to real research as well as thoroughness of the work. In ISI, he encouraged in group work. This had flourished mostly in 1940s.

C R Rao wrote, '*Working in the Institute was an exciting experience. One had the freedom to pursue one's ideas to get involved in the practical projects undertaken by the Institute, to accept administrative responsibilities or to devote some time to each of these activities. Mahalanobis insisted on our signing attendance register, recording the arrival and departure times, and also on maintaining a systematic diary of our daily activities. He was fond of calling the 'worker' of the Institute (the common designation which all categories of staff had for a long time), to his office and 'discussing' with him, which resulted in the frequent entry in our diaries, 'discussion with the Professor', and which covered most of our working time in the early days*'. Rudra (1991).

P C Mahalanobis had great confidence on C R Rao and his meticulousness. The decision of the selection of C R Rao to offer him the post of the Director of Research & Training School was taken in the Governing Body of the Council of Indian Statistical Institute on 25 March 1964. Interestingly, P C Mahalanobis was so keen to offer the post to C R Rao that he personally discussed the procedural matter with Shri Kumar Mitra, Additional Secretary, and Shri S K Chowdhury, Joint Secretary. According to Rudra (1991), 'Considering that PCM's success in recruiting young scientists, who almost all proved their worth in the course of their careers, is without parallel in the country'. In the Governing Body Meeting on 25 March 1964, it was agreed that '*his designation would be the Director, Research and Training School from the date of his acceptance of the post but he will be considered to be on leave without pay during the period he is gainfully employed elsewhere*' (PCM Memorial Archive). C R Rao at that time was 'gainfully employed' with the John Hopkins University, Baltimore. In ISI, the graduation course had just been started in 1962.

Inauguration of Graduate Course at ISI.

C R Rao expressed his willingness to accept the appointment. He in his acceptance letter on 19 May 1964 wrote:

> *I am extremely sorry for the delay in sending the acceptance letter. I would have preferred to be in a less responsible position although in an Indian set up some authority seems to be necessary to do anything useful. As desired by you I am accepting the post*. (PCM Memorial Archive). This letter shows CRR's humility where he wrote, '*I would have preferred to be in a less responsible position.*', and which he actually showed throughout his life. He was designated as Secretary-Director till January 1976.

Professor C R Rao was awarded with the Guy Medal in Silver in 1965. The letter dated 29 March 1965 from U N Croker, Secretary of the Royal Statistical Society, London, reads as following:

> The Council at their meeting last week were unanimous in awarding you the Society's Guy Medal in Silver for your contributions to the Society's proceedings. They very much hope that you will be willing to accept the award. Normally the presentation is made at the Annual General Meeting of the Society in June (this year June 16th). It is not of course expected that you will be in this country at that time. (PCM Memorial Archive)

In this respect, Professor Rao sent a copy of this letter to PCM, who in response wrote a note, '*I am very happy to see this. Please return it for my file*'. P C Mahalanobis was very glad at this achievement of C R Rao. Professor Rao was awarded with the Guy Medal in Gold on 29 June 2011.

C R Rao was the confidante of P C Mahalanobis. The close association between these two families of P C Mahalanobis and C R Rao is evident in few letters. Whatever be the issue, PCM used to discuss that with C R Rao and even with his wife Bhargavi Rao. In a typed long letter of four pages on 20 July 1964, PCM wrote to Rao in detail about the contemporary political situation of India, especially after the demise of Jawaharlal Nehru, about whom PCM had great respect. He wrote, '*It is now the most crucial period since independence*'. The long letter contained issues like cabinet formation under the leadership of Lal Bahadur Shastri, the new Prime Minister, political situation, activities of the Planning Commission, Five-Year Plan, food policies, etc. It seems both were of same opinion on the socio-economic issues.

In another long letter to Bhargavi Rao on 24 February 1964, P C Mahalanobis wrote about the political crises in India as well as mundane daily matters. At the end, PCM wrote to her, '*I have written at considerable length as you said in your letter that you get very little news from India. I hope you give good marks for my performance*'.

Professor Rao had many hobbies such as gardening, photography, cooking and Indian classical dance. In Calcutta, he played soccer and badminton with staff and students in the evenings. In the USA, his relaxation was walking. He wrote, '*I took to photography as a hobby after I married and found an attractive subject in my wife for experiment in photography. Some of the pictures I took have been published in newspapers and photographic magazines. My friends tell me that I take good pictures; parents often ask me to take photos of their unmarried daughters for sending them to prospective bridegrooms*'. Rao (1992)

C R Rao sorting out the photo negative reel with me

C R Rao identifying photographs with me

C R Rao at a photo exhibition

That Professor Rao was attracted towards photography was also evident from his style of handling of the negatives kept in large number in the Reprography and Photography Unit. In many instances, he himself corrected the names, events and years of the photographs and even narrated the incidents of certain photographs. C R Rao was also fascinated towards sports. He wrote, ‘w*hile in school and intermediate, I took part in debating and participated in outdoor games*’ (Rao 1992). While he was sorting out photographs on ISI sports, I got to know about an unfortunate incident of flight crash in USA and death of Professor T V Hanurav’s (then Dean of Studies), daughter at the age of 27 years.

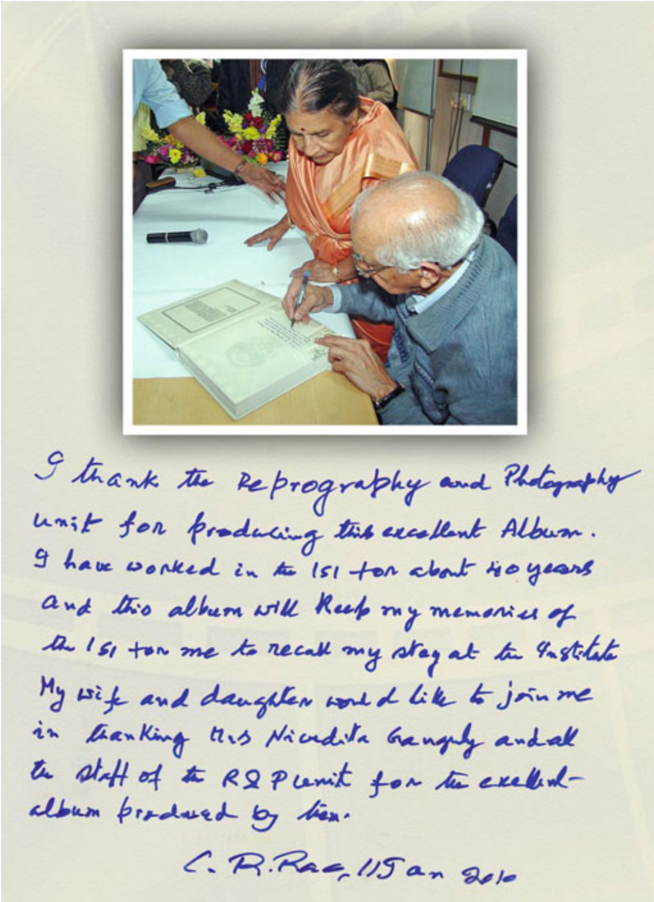
C R Rao at ISI Annual Sports

Daughter of Prof. T V Hanurav at ISI Annual Sports. C R Rao throwing stone into her pitcher

The album, ‘Silhouettes of Memories: A Pictorial Album on C. R. Rao at Indian statistical Institute’, depicting the long association of C R Rao with ISI was prepared to present him on the day of celebration of his 90th birthday. He came along with his wife and daughter. Professors present there had planned to felicitate him on his 90th birthday on that day. We in the Reprography and Photography Unit got a chance to hand over the beautiful photograph album to him. Mr. Tapas Basu and other members of the Reprography and Photography Unit worked hard to prepare this. Professor Rao’s reaction after getting the album cannot be expressed in words. The photographs can describe this.

Professor Rao with the Album in his hand with his wife Dr. Bhargavi Rao

A nostalgic feeling engrossed him when I sent this family photograph to him. It was published in *Samvadadhvam*, a house journal of ISI.



Professor C R Rao at the age of 82 in 2002, after he received the US Presidential Award for Science, told to the Times of India in an interview, '*In India no one respects you after you retire. Even the darwan will salute you only when you are in service. Even colleagues respect authority, not scholarship*'. The same tone we heard him when he wrote to P C Mahalanobis, '*I would have preferred to be in a less responsible position although in an Indian set up some authority seems to be necessary to do anything useful.* (PCM Memorial Archive). Professor Rao was Professor and Head of the Division of Theoretical Research and Training in ISI from 1949 to 1963. His designation was changed to Director of the Research and Training School in 1963. He became Director and Secretary in 1972 after the expiry of Professor P C Mahalanobis. After retirement, he became the Jawaharlal Nehru Professor and later National Professor. He led a long thriving life. In his autobiography, he wrote, '*Although I had some success in this endeavour, I have a feeling that much more could have been accomplished but for the numerous obstacles that could not be* *overcome and sensitive issues which could not be ignored in the context of the complex socio-economic-political-linguistic milieu in India*'. (Rao 1992)

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P C M Memorial Archives.

Indian Statistical Institute Council Proceedings archive.

Photo Archive, Reprography & Photography Unit, Library, ISI.

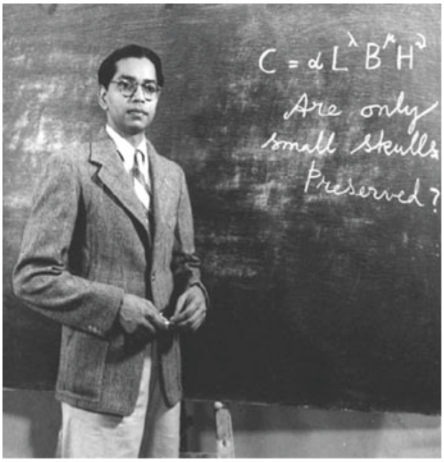
Silhouettes of Memories. (2010). A Pictorial Album on C R Rao at Indian Statistical Institute. Reprography & Photography Unit, ISI.

Rudra, A. (1991). *Prasanta Chandra Mahalanobis, a biography*. Calcutta: Oxford.

Annexure : Some of the Old Memory Collected from Different Source (Web)

School photograph taken at Nuzvid, Andhra Pradesh, in 1929. C.R. Rao, wearing a dark jacket, is first on the left in the bottom row (@2011 The Royal Statistical Society)

A. Chaudhuri et al. (eds.), *A Tribute to the Legend of Professor C. R. Rao*, Indian Statistical Institute Series,
https://doi.org/10.1007/978-981-33-6991-7



Prof. Rao giving a talk based on research work done at the Museum of Archaeology and Anthropology in Cambridge (CURRENT SCIENCE, VOL. 107, NO. 5, 896 10 SEPTEMBER 2014)

More Colour Pictures

By B. L. S. Prakasa Rao

D Basu, the first Ph D student of C. R. Rao

From the Album of SOSU, ISI

Picture-5 by S. Bandyopadhyay

By Shibdas Bandyopadhyay : From reprography unit, ISI

Picture-2

Picture-3

Picture-4

By Dr. Nibedita Ganguly:

C R Rao sorting out the photo negative reel with me

With photo album and printed photographs of ISI

~:UNITY IN DIVERSITY:~