Low c-Differential Uniformity for the Gold Function Modified on a Subfield

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1 Introduction and Basic Definitions

In [10], we defined a multiplier differential and difference distribution table (in any characteristic). There seems to be quite a bit of interest in this new notion, as it opens the possibility for a modification of the differential attack. Using this concept, we extended the notion of the Boomerang Connectivity Table in [22]. In this paper, we investigate the c-differential uniformity for the Gold function, modified on a subfield.

As customary, n is a positive integer, p is a prime number, $\mathbb{F}_{p^n}$ is the finite field with p^n elements, and $\mathbb{F}_{p^n}^* = \mathbb{F}_{p^n} \setminus \{0\}$ is the multiplicative group (for $a \neq 0$, $\frac{1}{a}$ means the inverse of a in the multiplicative group of the corresponding finite field). We let $\mathbb{F}_p^n$ be the n-dimensional vector space over $\mathbb{F}_p$. We use $\#S$ to denote the cardinality of a set S and $\bar{z}$, for the complex conjugate. We call a function from $\mathbb{F}_{p^n}$ (or $\mathbb{F}_p^n$) to $\mathbb{F}_p$ a *p-ary function* on n variables. For positive integers n and m, any map $F : \mathbb{F}_{p^n} \to \mathbb{F}_{p^m}$ (or $\mathbb{F}_p^n \to \mathbb{F}_p^m$) is called a *vectorial p-ary function*, or (n, m)-*function*. When $m = n$, F can be uniquely represented as a univariate polynomial over $\mathbb{F}_{p^n}$ (using some identification, via a basis, of the finite field with the vector space) of the form $F(x) = \sum_{i=0}^{p^n-1} a_i x^i,\ a_i \in \mathbb{F}_{p^n}$, whose *algebraic degree* is then the largest Hamming weight of the exponents i with $a_i \neq 0$.

Given a p-ary function f, the derivative of f with respect to $a \in \mathbb{F}_{p^n}$ is the p-ary function $D_a f(x) = f(x + a) - f(x),$ for all $x \in \mathbb{F}_{p^n}$, which can be naturally extended to vectorial p-ary functions.

The next concept can be defined for general (n, m)-functions, though in this paper we only consider $m = n$. For an (n, n)-function F, and $a, b \in \mathbb{F}_{p^n}$, we let $\Delta_F(a, b) = \#\{x \in \mathbb{F}_{p^n} : F(x + a) - F(x) = b\}$. We call the quantity $\delta_F = \max\{\Delta_F(a, b) : a, b \in \mathbb{F}_{p^n}, a \neq 0\}$ the *differential uniformity* of F. If $\delta_F = \delta$, then

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we say that F is differentially δ-uniform. If $\delta = 1$, then F is called a *perfect nonlinear* (*PN*) function, or *planar* function. If $\delta = 2$, then F is called an *almost perfect nonlinear* (*APN*) function. It is well known that PN functions do not exist if $p = 2$.

For a p-ary (n, m)-function $F : \mathbb{F}_{p^n} \to \mathbb{F}_{p^m}$ and $c \in \mathbb{F}_{p^m}$; the (*multiplicative*) *c-derivative* of F with respect to $a \in \mathbb{F}_{p^n}$ is the function

$$ {}_cD_aF(x) = F(x + a) - cF(x), \text{ for all } x \in \mathbb{F}_{p^n}. $$

For an (n, n)-function F, and $a, b \in \mathbb{F}_{p^n}$, we let the entries of the c-Difference Distribution Table (c-DDT) be defined by ${}_c\Delta_F(a, b) = \#\{x \in \mathbb{F}_{p^n} : F(x + a) - cF(x) = b\}$. We call the quantity

$$ \delta_{F,c} = \max \left\{ {}_c\Delta_F(a, b) \,|\, a, b \in \mathbb{F}_{p^n}, \text{ and } a \neq 0 \text{ if } c = 1 \right\} $$

the *c-differential uniformity* of F (see [2] for a particular case). If $\delta_{F,c} = \delta$, then we say that F is differentially (c, δ)-uniform (or that F has c-uniformity δ, or for short, *F is δ-uniform c-DDT*). If $\delta = 1$, then F is called a *perfect c-nonlinear* (*PcN*) function (certainly, for $c = 1$, they only exist for odd characteristic p; however, as proven in [10], there exist PcN functions for $p = 2$, for all $c \neq 1$). If $\delta = 2$, then F is called an *almost perfect c-nonlinear* (*APcN*) function. It is easy to see that if F is an (n, n)-function, that is, $F : \mathbb{F}_{p^n} \to \mathbb{F}_{p^n}$, then F is PcN if and only if ${}_cD_aF$ is a permutation polynomial.

This concept has been picked up quickly by the community and a flurry of papers started appearing [1, 17, 21–24, 27–29]. It is the purpose of this paper to investigate the c-differential uniformity for a subfield-modified (concept defined below) Gold function in the binary case. These affine modifications are occurring in many papers (see [12–14, 18–20, 26, 30], to cite just a few works).

The reader can consult [4–6, 8, 16, 25] for more on Boolean and p-ary (p is an odd prime) functions beyond what we have introduce here.

We will only consider the $p = 2$ case in this note. Given $F : \mathbb{F}_{2^n} \to \mathbb{F}_{2^n}$, and a divisor of n, say $s \,|\, n$, a fixed $t \in \mathbb{F}_{2^s}$, we let G be the $\mathbb{F}_{2^s}$-modification of F defined by

$$ G(x) = F(x) + t\left(x^{2^s} + x\right)^{2^n-1} + t = \begin{cases} F(x) + t & \text{if } x \in \mathbb{F}_{2^s} \\ F(x) & \text{if } x \notin \mathbb{F}_{2^s}. \end{cases} $$

In this paper, we consider the $\mathbb{F}_{2^s}$-modification of the Gold function only, so, $G(x) = x^{2^k+1} + t\left(x^{2^s} + x\right)^{2^n-1} + t$, $1 \leq k < n$, $\gcd(k, n) = 1$, $s \,|\, n$, $t \in \mathbb{F}_{2^s}$.

2 The c-Differential Uniformity of the Subfield Modified Gold Function

We will now state and prove our result for the c-differential uniformity of the binary $\mathbb{F}_{2^s}$-modification of the Gold function $F(x) = x^{2^k+1}$, $\gcd(n, k) = 1$, which is known to be APN under $\gcd(n, k) = 1$ (it is differentially 4-uniformity when $n \equiv 2 \pmod 4$ and $\gcd(n, k) = 2$).

Theorem *Let $G(x) = x^{2^k+1} + t\left(x^{2^s} + x\right)^{2^n-1} + t$ be the $\mathbb{F}_{2^s}$-modification of the Gold function, $1 \le k < n$, $\gcd(k, n) = 1$, $s \mid n, t \in \mathbb{F}_{2^s}$. Then, for $c \neq 1$, the c-differential uniformity of G is $\delta_{G,c} \le 3$.* □

Proof There is no need to consider $c = 0$ for the c-differential uniformity, since we can easily show that G is a permutation, and we argue that below. We assume that $G(x_1) = G(x_2)$, for some $x_1, x_2 \in \mathbb{F}_{2^n}$. If both $x_1, x_2 \notin \mathbb{F}_{2^s}$, then we get $x_1^{2^k+1} + t = x_2^{2^k+1} + t$, implying that $x_1 = x_2$ from the invertibility of F. If $x_1 \in \mathbb{F}_{2^s}, x_2 \in \mathbb{F}_{2^n}$, then $x_2^{2^k+1} = x_1^{2^k+1} + t \in \mathbb{F}_{2^s}$, implying that $x_2 \in \mathbb{F}_{2^s}$, as well, which is a contradiction. If none of x_1, x_2 are in $\mathbb{F}_{2^s}$, then again $x_1^{2^k+1} = x_2^{2^k+1}$, implying that $x_1 = x_2$.

From here on, we assume that $c \neq 0, 1$. The c-differential equation $G(x + a) - cG(x) = b$ of G at $a, b \in \mathbb{F}_{2^n}$ is

$$(x+a)^{2^k+1} + t\left((x+a)^{2^s} + (x+a)\right)^{2^n-1} + t + cx^{2^k+1} + ct\left(x^{2^s} + x\right)^{2^n-1} + ct = b,$$

which is equivalent to

$$(1+c)x^{2^k+1} + ax^{2^k} + a^{2^k}x + t\left(x^{2^s} + x + a^{2^s} + a\right)^{2^n-1} + ct\left(x^{2^s} + x\right)^{2^n-1} + a^{2^k+1} + t(1+c) + b = 0.$$

Case 1. Let $a \in \mathbb{F}_{2^s}, x \in \mathbb{F}_{2^s}$. By expanding the first term, the above equation transforms into

$$(1+c)x^{2^k+1} + x^{2^k}a + a^{2^k}x + a^{2^k+1} + t(1+c)\left(x^{2^s} + x\right)^{2^n-1} + t(1+c) + b = 0.$$

Since $x \in \mathbb{F}_{2^s}$, the equation becomes (when divided by a^{2^k+1} and by relabeling $\frac{x}{a} \mapsto x$)

$$x^{2^k+1} + \frac{1}{1+c}x^{2^k} + \frac{1}{1+c}x + d = 0, \tag{1}$$

where $d = \frac{a^{2^k+1}+t(1+c)+b}{(1+c)a^{2^k+1}}$. If $d = 0$, then $x = 0$ is a solution. The cofactor, with the relabeling $\frac{1}{x} \mapsto x$, becomes

$$x^{2^k} + x + (c+1) = 0.$$

We now need to investigate the number of solutions of this linearized polynomial. We rewrite (simplifying some parameters) a result from [7, 15]. Let $f(z) = z^{p^k} - Az - B$ in $\mathbb{F}_{p^n}$, $g = \gcd(n, k)$, $m = n/\gcd(n, k)$, and $\mathrm{Tr}_{\mathbb{F}_{p^n}/\mathbb{F}_{p^g}}$ be the relative trace from $\mathbb{F}_{p^n}$ to $\mathbb{F}_{p^g}$. For $0 \le i \le m-1$, we define $t_i = \frac{p^{nm} - p^{n(i+1)}}{p^n - 1}$, $\alpha_0 = A$, $\beta_0 = B$. If $m > 1$, then, for $1 \le r \le m-1$, we let $\alpha_r = A^{\frac{p^{k(r+1)}-1}{p^k-1}}$ and $\beta_r = \sum_{i=0}^{r} A^{s_i} B^{p^{ki}}$, where $s_i = \frac{p^{k(r+1)} - p^{k(i+1)}}{p^k - 1}$, for $0 \le i \le r-1$ and $s_r = 0$. The trinomial f has no roots in $\mathbb{F}_{p^n}$ if and only if $\alpha_{m-1} = 1$ and $\beta_{m-1} \neq 0$. If $\alpha_{m-1} \neq 1$, then it has a unique root, namely $x = \beta_{m-1}/(1 - \alpha_{m-1})$, and, if $\alpha_{m-1} = 1, \beta_{m-1} = 0$, it has p^g roots in $\mathbb{F}_{p^n}$ given by $x + \delta\tau$, where $\delta \in \mathbb{F}_{p^g}$, τ is fixed in $\mathbb{F}_{p^n}$ with $\tau^{p^k - 1} = a$ (that is, a $(p^k - 1)$-root of a), and, for any $e \in \mathbb{F}_{p^n}^*$ with $\mathrm{Tr}_g(e) \neq 0$, then $x = \frac{1}{\mathrm{Tr}_{\mathbb{F}_{p^n}/\mathbb{F}_{p^g}}(e)} \sum_{i=0}^{m-1} \left(\sum_{j=0}^{i} e^{p^{kj}} \right) A^{t_i} B^{p^{ki}}$.

For our prior case, $p = 2$, $A = 1$, $B = c + 1$, $m = n$, $g = 1$, and so, $\alpha_{n-1} = 1$, $\beta_{n-1} = \sum_{i=0}^{n-1} (c+1)^{2^{ki}}$. Thus, if $\beta_{n-1} \neq 0$, we have no roots, and if $\beta_{n-1} = 0$, we have 3 roots (we added the previous 0 root to the count). We make the observation that these roots can belong to $\mathbb{F}_{2^s}$ if we force $c \in \mathbb{F}_{2^s}$, and we take $e \in \mathbb{F}_{2^s}$ in the formula above.

If $d \neq 0$, taking $y = x + (c+1)^{-1}$ we obtain

$$y^{2^k+1} + \frac{1}{1+c}\left(1 + \frac{1}{(1+c)^{2^k-1}}\right) y + d + \frac{1}{(c+1)^2} = 0.$$

Now, let $y = \alpha z$, where $\alpha = \left(\frac{1}{1+c} + \frac{1}{(1+c)^{2^k}}\right)^{2^{-k}} = 1 + \frac{1}{(1+c)^{2^{-k}}}$ (the 2^k-root exists since $\gcd(2^k, 2^n - 1) = 1$). The previous equation becomes

$$z^{2^k+1} + z + \beta = 0, \tag{2}$$

where $\beta = \frac{d(1+c)^2 + 1}{\alpha^{2^k+1}(1+c)^2} = \frac{ca^{2^k+1} + t(1+c)^2 + b(1+c)}{a^{2^k+1}\alpha^{2^k+1}(1+c)^2}$. Assuming $\beta \neq 0$, we will be using some results of [11] (see also [3, 9]), under $\gcd(n, k) = 1$. By [11] [Theorem 1], we know that Eq. (2) has either none, one or three solutions in $\mathbb{F}_{2^n}$. In fact, the distribution of these cases for n odd (respectively, n even) is (denoting by M_ℓ the amount of equations of type (2) with ℓ solutions)

$$M_0 = \frac{2^n + 1}{3},\ M_1 = 2^{n-1} - 1,\ M_3 = \frac{2^{n-1} - 1}{3},\ \text{for } n \text{ odd},$$
$$M_0 = \frac{2^n - 1}{3},\ M_1 = 2^{n-1},\ M_3 = \frac{2^{n-1} - 2}{3},\ \text{for } n \text{ even}.$$

Then, for $n \geq 3$, $c \neq 0, 1$, and $\gcd(n, k) = 1$, and since β is linear on b, this implies that, for any β and any a, c, we can find b such that $\beta = \dfrac{ca^{2^k+1} + b(1-c)}{\alpha^{2^k+1}(1-c)^2}$, so the maximum (attainable, if they happen to be in $\mathbb{F}_{2^s}$) number of solutions for (1) in this case is 3.

Case 2. Let $a \in \mathbb{F}_{2^s}$, $x \notin \mathbb{F}_{2^s}$. Then our equation becomes (when divided by a^{2^k+1} and relabeling $\frac{x}{a} \mapsto x$)

$$x^{2^k+1} + \frac{1}{1+c}x^{2^k} + \frac{1}{1+c}x + \frac{a^{2^k+1}+b}{(1+c)a^{2^k+1}} = 0. \tag{3}$$

Arguing as above we get that the maximum number of solutions is, yet again, 3, if $b \neq a^{2^k+1}$.

Case 3. Let $a \notin \mathbb{F}_{2^s}$, $x \in \mathbb{F}_{2^s}$. As in the first case, by expanding the first term, the above equation transforms into

$$(1+c)x^{2^k+1} + ax^{2^k} + a^{2^k}x + a^{2^k+1} + tc + b = 0,$$

which, as before, is equivalent to

$$x^{2^k+1} + \frac{1}{1+c}x^{2^k} + \frac{1}{1+c}x + d = 0,$$

with $d = \frac{a^{2^k+1}+tc+b}{(c+1)a^{2^k+1}}$. A similar analysis as in the prior case renders a maximum of 3 solutions.

Case 4. Let $a \notin \mathbb{F}_{2^s}$, $x \notin \mathbb{F}_{2^s}$, and $x + a \in \mathbb{F}_{2^s}$. The c-differential equation of G becomes

$$(1+c)x^{2^k+1} + ax^{2^k} + a^{2^k}x + a^{2^k+1} + t + b = 0,$$

which resembles the prior equations, and so, by appropriate substitutions and arguing similarly, we infer that it has a maximum of 3 solutions.

Case 5. Let $a \notin \mathbb{F}_{2^s}$, $x \notin \mathbb{F}_{2^s}$, and $x + a \notin \mathbb{F}_{2^s}$. The relevant equation is then

$$(1+c)x^{2^k+1} + ax^{2^k} + a^{2^k}x + a^{2^k+1} + b = 0,$$

which, as we got used by now, renders a maximum of 3 solutions. The theorem is shown. □

3 Concluding Remarks

In this paper, we find the c-differential uniformity of the $\mathbb{F}_{2^s}$-modification of the Gold function on $\mathbb{F}_{2^n}$, $s \mid n$, and show that its c-differential uniformity is less than or equal to 3. As we saw already, investigating questions on c-differential uniformity by this method is not a simple matter, mostly because the obtained equations need to be solved over finite fields and not many techniques have been developed for that purpose. In spite of that, it would be interesting to find other classical PN/APN functions and study their properties through the new differential. It will also be worthwhile to check into the general p-ary versions of the results from this paper, as well as other modifications of the Gold, the inverse, or other PN/APN functions.

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