# Stochastic Comparisons of Systems with Heterogeneous Log-Logistic Components

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## 1 Introduction

The log-logistic distribution, henceforth referred to as LLD, is a well-known life distribution that finds widespread application in different fields such as survival analysis, hydrology, economics, and networking.

It has been used in regression models for survival data (see Bennet 1983) and also as a parametric model for events whose failure rate increases initially and decreases later, for example, the mortality rate from cancer following diagnosis or treatment. Its application can also be seen in the field of hydrology for modeling precipitation and stream flow rates. For example, to analyze Canadian precipitation data Shoukri et al. (1988) showed that LLD is a suitable choice whereas Fahim and Smail (2006) used LLD for modeling stream flow rates. The LLD is also known as Fisk distribution in the field of economics where it has been utilized to describe the distribution of wealth or income (see Fisk 1961). In the field of computer science and networking, LLD has been used as a more accurate probabilistic model (see Gago-Benítez et al. 2013 for details).

The LLD is very similar in shape to the log-normal distribution but has the added advantage of being mathematically more tractable because of its closed form dis-

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A. K. Laha (ed.), *Applied Advanced Analytics*, Springer Proceedings in Business and Economics, https://doi.org/10.1007/978-981-33-6656-5_14

tribution function and its quite flexible hazard rate function. It is a good alternative to the Weibull, whose hazard rate function is either increasing or decreasing, i.e., monotonic, depending on the value of its shape parameter. As such, the use of the Weibull distribution may be inappropriate where the course of the disease is such that mortality reaches a peak after some finite period and then slowly declines. Additionally, the LLD is also connected to extreme value distributions. As showed by Lawless (1986), the Weibull distribution has paramount importance in reliability theory as it is the only distribution that belongs to two families of extreme value distributions, each of which has essential qualities for the study of proportional hazard and accelerated failure times. Thus, the LLD possesses the nice characteristic of being a representative of both these families.

A random variable (r.v.) $X$ is said to have the LLD with shape parameter $\alpha$ and scale parameter $\gamma$, written as $LLD(\alpha, \gamma)$, if its *probability density function* (pdf) is given by

$$f(x; \alpha, \gamma) = \frac{\alpha\gamma(\gamma x)^{\alpha-1}}{(1 + (\gamma x)^{\alpha})^2}\ ,\ x \geq 0,\ (\alpha > 0, \gamma > 0). \tag{1.1}$$

Just as one gets the log-normal and log-Pearson distributions from normal and Pearson distribution, LLD is obtained by taking the logarithmic transformation of the logistic distribution. The LLD is also a special case of the 'kappa distributions' introduced by Mielke and Johnson (1973). Another interesting fact is that LLD can also be obtained from the ratio of two independent Stacy's generalized gamma variables (see Malik 1967; Block and Rao 1973). Even though different properties of this distribution have been explored intensely by many researchers, the stochastic comparisons of their extreme order statistics have not been studied so far. This is the primary motivation behind the present work.

But first, a few words about order statistics which occupy a place of remarkable importance in both theory and practice. It play a vital role in many areas including reliability theory, economics, management science, operations research, insurance, hydrology, etc., and have received a lot of attention in the literature during the last several decades [(see, e.g., the two encyclopedic volumes by Balakrishnan and Rao (1998a, b)]. Let $X_{1:n} \leq \cdots \leq X_{n:n}$ represent the order statistics corresponding to the $n$ independent random variables (r.v.'s) $X_1, \ldots, X_n$.

It is a well-known fact that the $k$th order statistic $X_{k:n}$ represents the lifetime of a $(n - k + 1)$-out-of-$n$ system which happens to be a suitable structure for redundancy that has been studied by many researchers. Series and parallel systems, which are the building blocks of many complex coherent systems, are particular cases of a $k$-out-of-$n$ system. A series system can be regarded as a $n$-out-of-$n$ system, while a parallel system is a 1-out-of-$n$ system. In the past two decades, a large volume of work has been carried out to compare the lifetimes of the series and parallel systems formed with components from various parametric models; see Fang and Zhang (2015), Zhao and Balakrishnan (2011), Fang and Balakrishnan (2016), Li and Li (2015), Torrado (2015), Torrado and Kochar (2015), Kundu and Chowdhury (2016), Nadarajah et al. (2017), Majumder et al. (2020) and the references therein.

Here, we investigate comparison results between the lifetimes of series and parallel systems formed with LLD samples in terms of different ordering notions such as stochastic order, hazard rate order, reversed hazard rate order, and likelihood ratio order. These orders are widely used in the literature for fair and reasonable comparison (see Shaked and Shanthikumar 2007). The rest of the paper is presented as follows. Preliminary definitions and useful lemmas can be found in Sect. 2. In Sect. 3, we discuss the comparison of lifetimes of parallel systems with heterogeneous LLD components. We also study the comparison in the case of the multiple-outlier LLD model. In Sect. 4, ordering properties are discussed for the lifetimes of series systems with heterogeneous LLD components.

Throughout this article, 'increasing' and 'decreasing' mean 'nondecreasing' and 'nonincreasing,' respectively, and the notation $f(x) \overset{\text{sign}}{=} g(x)$ implies that $f(x)$ and $g(x)$ are equal in sign.

## 2 Notations, Definitions, and Preliminaries

Here, we review some definitions and various notions of stochastic orders and majorization concepts.

**Definition 1** (Shaked and Shanthikumar 2007) Let $X$ and $Y$ be two absolutely continuous r.v.'s with cumulative distribution functions (cdfs) $F(\cdot)$ and $G(\cdot)$, survival functions $\bar{F}(\cdot)$ and $\bar{G}(\cdot)$, pdfs $f(\cdot)$ and $g(\cdot)$, hazard rates $h_F(\cdot)$ and $h_G(\cdot)$, and reverse hazard rate functions $r_F(\cdot)$ and $r_G(\cdot)$, respectively.

(i) If $\bar{F}(x) \le \bar{G}(x)$ for all $x \ge 0$, then $X$ is smaller than $Y$ in the usual stochastic order, denoted by $X \le_{st} Y$.
(ii) If $\bar{G}(x)/\bar{F}(x)$ is increasing in $x \ge 0$, then $X$ is smaller than $Y$ in the hazard rate order, denoted by $X \le_{hr} Y$.
(iii) If $G(x)/F(x)$ is increasing in $x \ge 0$, then $X$ is smaller than $Y$ in the reversed hazard rate order, denoted by $X \le_{rh} Y$.
(iv) If $g(x)/f(x)$ is increasing in $x \ge 0$, then $X$ is smaller than $Y$ in the likelihood ratio order, denoted by $X \le_{lr} Y$.

From Shaked and Shanthikumar (2007), it is well established that

$$X \le_{lr} Y \implies X \le_{hr} Y \implies X \le_{st} Y$$

and

$$X \le_{lr} Y \implies X \le_{rh} Y \implies X \le_{st} Y$$

but the opposite implications do not hold in general. Also, $X \le_{hr} Y \not\Leftrightarrow X \le_{rh} Y$.

The notion of majorization is a key concept in the theory of stochastic inequalities. Let $(x_{(1)}, x_{(2)}, \ldots, x_{(n)})$ denote the components of the vector $\mathbf{x} = (x_1, x_2, \ldots, x_n)$

arranged in ascending order. Let $I^n$ be a subset of the $n$-dimensional Euclidean space $\mathbb{R}^n$, where $I \subseteq \mathbb{R}$ and $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ be a vector in $I^n$.

**Definition 2** The vector $\mathbf{x}$ is said to be majorized by the vector $\mathbf{y}$, denoted by $\mathbf{x} \overset{m}{\preceq} \mathbf{y}$, if

$$\sum_{i=1}^{j} x_{(i)} \geq \sum_{i=1}^{j} y_{(i)} \quad \text{for } j = 1, \ldots, n-1$$

and

$$\sum_{i=1}^{n} x_{(i)} = \sum_{i=1}^{n} y_{(i)}.$$

In addition, the vector $\mathbf{x}$ is said to be weakly supermajorized by the vector $\mathbf{y}$, denoted by $\mathbf{x} \preceq^w \mathbf{y}$, if

$$\sum_{i=1}^{j} x_{(i)} \geq \sum_{i=1}^{j} y_{(i)} \quad \text{for } j = 1, \ldots, n.$$

Clearly,

$$\mathbf{x} \overset{m}{\preceq} \mathbf{y} \Longrightarrow \mathbf{x} \preceq^w \mathbf{y}. \tag{2.1}$$

**Definition 3** A real-valued function $\phi : \mathbb{R}^n \to \mathbb{R}$ is said to be Schur-convex (Schur-concave) on $\mathbb{R}^n$ if $\mathbf{x} \overset{m}{\preceq} \mathbf{y}$ implies $\phi(\mathbf{x}) \ \leq (\geq)\ \phi(\mathbf{y})$ for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$.

An extensive and comprehensive review on majorization can be found in Marshall et al. (2011).

We now introduced some well-known results which will be used in the subsequent sections to prove our main theorems.

**Lemma 1** *A real-valued function $\psi$ on $I^n$ has the property*

$$\psi(\mathbf{x}) \leq \psi(\mathbf{y}) \quad \text{whenever} \quad \mathbf{x} \preceq^w \mathbf{y}$$

*if and only if $\psi$ is decreasing and Schur-convex on $I^n$.*

**Lemma 2** (Schur–Ostrowski criterion). *A continuously differentiable function $\phi : I^n \to \mathbb{R}$ is Schur-convex (Schur-concave) if and only if $\phi$ is symmetric and*

$$(x_i - x_j)\left(\frac{\partial \phi(\mathbf{x})}{\partial x_i} - \frac{\partial \phi(\mathbf{x})}{\partial x_j}\right) \geq (\leq) 0$$

*for all $i \neq j$ and $\mathbf{x} \in I^n$.*

The following two lemmas are easy to establish.

**Lemma 3** *For $x \geq 0$, the function $\kappa(x) := (1 + x^{\alpha})^{-1}$ is decreasing in $x$ for any $\alpha > 0$ and convex in $x$ for $0 < \alpha \leq 1$. Also, the function $\tau(x) := 1 - \kappa(x)$ is concave in $x$ for $0 < \alpha \leq 1$.*

**Lemma 4** *For $x \geq 0$, the function $\varphi(x) := -\alpha x^{\alpha-1}(1 + x^{\alpha})^{-3}$ is increasing in $x$ for $0 < \alpha \leq 1$.*

## 3 Order Relations for Parallel Systems

This section considers stochastic comparisons between the lifetimes of parallel systems whose components arise from two sets of heterogeneous LLD samples with a common shape parameter but different scale parameters and vice versa.

Let $X_{\gamma_i}$ for $i = 1, \ldots, n$ be $n$ independent nonnegative r.v.'s following $LLD(\alpha, \gamma_i)$ with density function given by (1.1). Let the lifetime of the parallel system formed from $X_{\gamma_1}, X_{\gamma_2}, \ldots, X_{\gamma_n}$ be $X^{\gamma}_{n:n}$. Then, its distribution and density functions are given by

$$F^{\gamma}_{n:n}(x) = \prod_{i=1}^{n} F_{\gamma_i}(x), \quad f^{\gamma}_{n:n}(x) = \prod_{i=1}^{n} F_{\gamma_i}(x) \sum_{i=1}^{n} r_{F_{\gamma_i}}(x),$$

and the corresponding reversed hazard rate function is

$$r^{\gamma}_{n:n}(x) = \frac{f^{\gamma}_{n:n}(x)}{F^{\gamma}_{n:n}(x)} = \sum_{i=1}^{n} r_{F_{\gamma_i}}(x).$$

At first, we compare two different parallel systems with common shape parameter under reversed hazard rate ordering.

**Theorem 1** *For $i = 1, 2, \ldots, n$, let $X_{\gamma_i}$ and $X_{\beta_i}$ be two sets of independent r.v.'s such that $X_{\gamma_i} \sim LLD(\alpha, \gamma_i)$ and $X_{\beta_i} \sim LLD(\alpha, \beta_i)$ where $\gamma_i, \beta_i > 0$. Then for $0 < \alpha \leq 1$,*

$$(\gamma_1, \ldots, \gamma_n) \preceq^{w} (\beta_1, \ldots, \beta_n) \Longrightarrow X^{\gamma}_{n:n} \leq_{rh} X^{\beta}_{n:n}.$$

***Proof*** Fix $x \geq 0$. The reversed hazard rate function of $X^{\gamma}_{n:n}$ is

$$r^{\gamma}_{n:n}(x) = \sum_{i=1}^{n} \alpha x^{-1} \left(1 + (\gamma_i x)^{\alpha}\right)^{-1} = \alpha x^{-1} \sum_{i=1}^{n} \kappa(\gamma_i x)$$

where $\kappa(x)$ is defined as in Lemma 3. From Lemma 1, it is sufficient to prove that, for every $x \geq 0$, $r^{\gamma}_{n:n}(x)$ is decreasing in each $\gamma_i$ and a Schur-convex function of $(\gamma_1, \ldots, \gamma_n)$. Now from the Proposition C.1 of Marshall et al. (2011), to demonstrate the Schur-convexity of $r^{\gamma}_{n:n}(x)$, it is sufficient to prove the convexity of $\kappa(x)$. Thus, using Lemma 3 the proof follows from Definition 1.

One can have the following corollary which is an easy consequence of the relation (2.1).

**Corollary 1** *For* $i = 1, 2, \ldots, n$, *let* $X_{\gamma_i}$ *and* $X_{\beta_i}$ *be two sets of independent r.v.'s such that* $X_{\gamma_i} \sim LLD(\alpha, \gamma_i)$ *and* $X_{\beta_i} \sim LLD(\alpha, \beta_i)$ *where* $\gamma_i, \beta_i > 0$. *Then for* $0 < \alpha \leq 1$,

$$(\gamma_1, \ldots, \gamma_n) \overset{m}{\preceq} (\beta_1, \ldots, \beta_n) \Longrightarrow X^{\gamma}_{n:n} \leq_{rh} X^{\beta}_{n:n}.$$

The above theorem ensures that for two parallel systems having independent LLD components with common shape parameter, the majorized scale parameter vector leads to corresponding system lifetime smaller in the sense of the reversed hazard rate ordering. In the following theorem, we investigate whether the systems are ordered under likelihood ratio ordering for the case $n = 2$.

**Theorem 2** *For* $i = 1, 2$, *let* $X_{\gamma_i}$ *and* $X_{\beta_i}$ *be two sets of independent r.v.'s such that* $X_{\gamma_i} \sim LLD(\alpha, \gamma_i)$ *and* $X_{\beta_i} \sim LLD(\alpha, \beta_i)$ *where* $\gamma_i, \beta_i > 0$. *Then for* $0 < \alpha \leq 1$,

$$(\gamma_1, \gamma_2) \overset{m}{\preceq} (\beta_1, \beta_2) \Longrightarrow X^{\gamma}_{2:2} \leq_{lr} X^{\beta}_{2:2}.$$

***Proof*** In view of Definition 1, it is enough to show that

$$\frac{f^{\beta}_{2:2}(x)}{f^{\gamma}_{2:2}(x)} = \frac{F^{\beta}_{2:2}(x)}{F^{\gamma}_{2:2}(x)} \cdot \frac{r^{\beta}_{2:2}(x)}{r^{\gamma}_{2:2}(x)} \text{ is increasing in } x. \tag{3.1}$$

From Corollary 1, we already have $F^{\beta}_{2:2}(x)/F^{\gamma}_{2:2}(x)$ is increasing in $x$ for $0 < \alpha \leq 1$. So, (3.1) implies that it only remains to show that $\psi(x) = r^{\beta}_{2:2}(x)/r^{\gamma}_{2:2}(x)$ is increasing in $x$. Now the reversed hazard rate function of $X^{\beta}_{2:2}$ is given by

$$r^{\beta}_{2:2}(x) = \alpha x^{-1}\left[(1 + (\beta_1 x)^{\alpha})^{-1} + (1 + (\beta_2 x)^{\alpha})^{-1}\right].$$

Then, $\psi(x) = \dfrac{\kappa(\beta_1 x) + \kappa(\beta_2 x)}{\kappa(\gamma_1 x) + \kappa(\gamma_2 x)}$, where $\kappa(x)$ is defined as in Lemma 3. Observe that

$$\kappa'(x) = -\alpha x^{\alpha-1}(1 + x^{\alpha})^{-2} = \alpha x^{-1}\kappa(x)\eta(x)$$

where $\eta(x) = \kappa(x) - 1$. Differentiating $\psi(x)$ with respect to $x$, we get

$$\begin{aligned}
\psi'(x) &\overset{\text{sign}}{=} \left[\kappa'(\beta_1 x) + \kappa'(\beta_2 x)\right][\kappa(\gamma_1 x) + \kappa(\gamma_2 x)] - [\kappa(\beta_1 x) + \kappa(\beta_2 x)]\left[\kappa'(\gamma_1 x) + \kappa'(\gamma_2 x)\right] \\
&\overset{\text{sign}}{=} [\kappa(\beta_1 x)\eta(\beta_1 x) + \kappa(\beta_2 x)\eta(\beta_2 x)][\kappa(\gamma_1 x) + \kappa(\gamma_2 x)] \\
&- \ [\kappa(\beta_1 x) + \kappa(\beta_2 x)][\kappa(\gamma_1 x)\eta(\gamma_1 x) + \kappa(\gamma_2 x)\eta(\gamma_2 x)]
\end{aligned}$$

Thus showing that $\psi(x)$ is increasing in $x$, i.e., $\psi^{'}(x) \geq 0 \; \forall \; x \geq 0$, is equivalent to proving

$$\phi(\beta_1, \beta_2) = \frac{\kappa(\beta_1 x)\eta(\beta_1 x) + \kappa(\beta_2 x)\eta(\beta_2 x)}{\kappa(\beta_1 x) + \kappa(\beta_2 x)}$$

is Schur-convex in $(\beta_1, \beta_2)$. Now, the function $\varphi(x)$ defined in Lemma 4 turns out to be $\kappa(x)\eta'(x)$, where $\kappa(x)$ and $\eta'(x)$ are defined as before. We thus have

$$\begin{aligned}\frac{\partial \phi}{\partial \beta_1} &\overset{\text{sign}}{=} \left[\kappa'(\beta_1 x)\eta(\beta_1 x) + \kappa(\beta_1 x)\eta'(\beta_1 x)\right]\left[\kappa(\beta_1 x) + \kappa(\beta_2 x)\right] \\ &\quad - \left[\kappa(\beta_1 x)\eta(\beta_1 x) + \kappa(\beta_2 x)\eta(\beta_2 x)\right]\kappa'(\beta_1 x) \\ &= \kappa'(\beta_1 x)\kappa(\beta_2 x)\left[\eta(\beta_1 x) - \eta(\beta_2 x)\right] + \varphi(\beta_1 x)\left[\kappa(\beta_1 x) + \kappa(\beta_2 x)\right].\end{aligned}$$

and

$$\frac{\partial \phi}{\partial \beta_2} \overset{\text{sign}}{=} \kappa(\beta_1 x)\kappa'(\beta_2 x)\left[\eta(\beta_2 x) - \eta(\beta_1 x)\right] + \varphi(\beta_2 x)\left[\kappa(\beta_1 x) + \kappa(\beta_2 x)\right].$$

Thus,

$$\begin{aligned}\frac{\partial \phi}{\partial \beta_1} - \frac{\partial \phi}{\partial \beta_2} &\overset{\text{sign}}{=} \left[\eta(\beta_1 x) - \eta(\beta_2 x)\right]\left[\kappa'(\beta_1 x)\kappa(\beta_2 x) + \kappa'(\beta_2 x)\kappa(\beta_1 x)\right] \\ &\quad + \left[\kappa(\beta_1 x) + \kappa(\beta_2 x)\right]\left[\varphi(\beta_1 x) - \varphi(\beta_2 x)\right].\end{aligned}$$

From Lemma 4, $\varphi(x)$ is increasing in $x$ for $0 < \alpha \leq 1$. This together with the observation $\beta_1 \leq \beta_2$ and the facts that $\kappa(x)$ and $\eta(x)$ are decreasing functions of $x$ yields

$$(\beta_1 - \beta_2)\left(\frac{\partial \phi}{\partial \beta_1} - \frac{\partial \phi}{\partial \beta_2}\right) \geq 0.$$

Hence, from Lemma 2 the theorem follows.

It is worth mentioning here that for $\alpha > 1$ the above result may not hold, as the next example shows.

**Example 1** Let $(X_{\gamma_1}, X_{\gamma_2})$ and $(X_{\beta_1}, X_{\beta_2})$ be two sets of vectors of heterogeneous LLD r.v.'s with shape parameter $\alpha = 1.5$ and scale parameters $(\gamma_1, \gamma_2) = (0.5, 1.5)$ and $(\beta_1, \beta_2) = (0.3, 1.7)$. Then obviously $(\gamma_1, \gamma_2) \overset{m}{\preceq} (\beta_1, \beta_2)$ but $f_{2:2}^{\beta}(x)/f_{2:2}^{\gamma}(x)$ is not monotonic as is evident from Fig. 1. Hence in Theorem 2, the restriction over $\alpha$ is necessary to get the $\leq_{lr}$ order comparison.

Next theorem shows that the likelihood ratio order holds among two parallel systems formed with heterogeneous LLD components where heterogeneity occurs in terms of scale parameters.

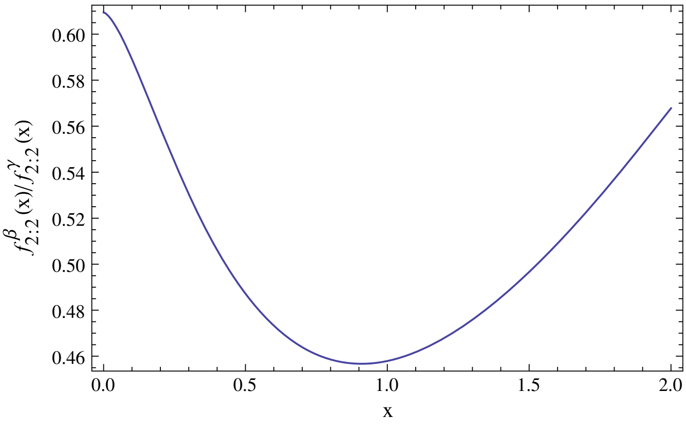


**Fig. 1** Plot of $f^{\beta}_{2:2}(x)/f^{\gamma}_{2:2}(x)$ when $\alpha = 1.0$, $\gamma = (0.5, 1.5)$, and $\beta = (0.3, 1.7)$

**Theorem 3** *Let $X_{\gamma_1}$, $X_\gamma$ be independent r.v.'s with $X_{\gamma_1} \sim LLD(\alpha, \gamma_1)$ and $X_\gamma \sim LLD(\alpha, \gamma)$ where $\gamma_1, \gamma > 0$. Let $Y_{\gamma^*}$, $Y_\gamma$ be independent r.v.'s with $Y_{\gamma^*} \sim LLD(\alpha, \gamma^*)$ and $Y_\gamma \sim LLD(\alpha, \gamma)$ where $\gamma^*, \gamma > 0$. Suppose that $\gamma^* = min(\gamma, \gamma_1, \gamma^*)$ then for any $\alpha > 0$,*

$$(\gamma_1, \gamma) \preceq^{w} (\gamma^*, \gamma) \Longrightarrow X_{2:2} \leq_{lr} Y^*_{2:2}.$$

***Proof*** The reversed hazard rate function of $X_{2:2}(x)$ has the form

$$r_{2:2}(x) = \alpha x^{-1}\left[(1 + (\gamma_1 x)^\alpha)^{-1} + (1 + (\gamma x)^\alpha)^{-1}\right].$$

Let $\psi(x) = \dfrac{r^*_{2:2}(x)}{r_{2:2}(x)} = \dfrac{(1 + (\gamma x)^\alpha)^{-1} + (1 + (\gamma^* x)^\alpha)^{-1}}{(1 + (\gamma x)^\alpha)^{-1} + (1 + (\gamma_1 x)^\alpha)^{-1}}$. Now utilizing Eq. (3.1) and Theorem 1, it only remains to show that $\psi(x)$ is increasing in $x$, i.e., $\psi'(x) \geq 0, \ \forall x \geq 0$. Now differentiating $\psi(x)$ with respect to $x$ and using the functions $\kappa(x)$ and $\eta(x)$ defined earlier, we get

$$\begin{aligned}
\psi'(x) &\overset{\text{sign}}{=} \left[(1 + (\gamma_1 x)^\alpha)^{-1} + (1 + (\gamma x)^\alpha)^{-1}\right]\left[-(\gamma x)^\alpha (1 + (\gamma x)^\alpha)^{-2} - (\gamma^* x)^\alpha (1 + (\gamma^* x)^\alpha)^{-2}\right] \\
&- \left[(1 + (\gamma x)^\alpha)^{-1} + (1 + (\gamma^* x)^\alpha)^{-1}\right]\left[-(\gamma x)^\alpha (1 + (\gamma x)^\alpha)^{-2} - (\gamma_1 x)^\alpha (1 + (\gamma_1 x)^\alpha)^{-2}\right] \\
&= \left[\kappa(\gamma x)\eta(\gamma x) + \kappa(\gamma^* x)\eta(\gamma^* x)\right]\left[\kappa(\gamma_1 x) + \kappa(\gamma x)\right] \\
&- \left[\kappa(\gamma x) + \kappa(\gamma^* x)\right]\left[\kappa(\gamma_1 x)\eta(\gamma_1 x) + \kappa(\gamma x)\eta(\gamma x)\right] \\
&= \kappa(\gamma_1 x)\kappa(\gamma^* x)\left[\eta(\gamma^* x) - \eta(\gamma_1 x)\right] + \kappa(\gamma_1 x)\kappa(\gamma x)\left[\eta(\gamma x) - \eta(\gamma_1 x)\right] \\
&+ \kappa(\gamma x)\kappa(\gamma^* x)\left[\eta(\gamma^* x) - \eta(\gamma x)\right].
\end{aligned}$$

Since $(\gamma_1, \gamma) \preceq^{w} (\gamma^*, \gamma)$ and $\gamma^* = \min(\gamma, \gamma_1, \gamma^*)$, two cases may arise:
**Case I:** $\gamma^* \leq \gamma \leq \gamma_1$. It can be easily seen that $\psi'(x) \geq 0$, using the facts $\kappa(x) \geq$

$0\ \forall x \geq 0$ and $\eta(x)$ is decreasing in $x$.
**Case II:** $\gamma^* \leq \gamma_1 \leq \gamma$. Again utilizing the above facts, we have

$$\begin{aligned}\psi'(x) &\geq \kappa(\gamma_1 x)\kappa(\gamma x)\left[\eta(\gamma^* x) - \eta(\gamma_1 x)\right] + \kappa(\gamma_1 x)\kappa(\gamma x)\left[\eta(\gamma x) - \eta(\gamma_1 x)\right] \\ &\quad + \kappa(\gamma x)\kappa(\gamma_1 x)\left[\eta(\gamma^* x) - \eta(\gamma x)\right] \\ &= 2\kappa(\gamma x)\kappa(\gamma_1 x)\left[\eta(\gamma^* x) - \eta(\gamma_1 x)\right] \geq 0.\end{aligned}$$

Thus in both the cases, one has $\psi(x)$ is increasing in $x$. Hence, the theorem follows.

Now we establish a comparison between parallel systems based on two sets of heterogeneous LLD r.v.'s with common scale parameter and majorized shape parameters according to stochastic ordering.

**Theorem 4** *For $i = 1, 2, \ldots, n$, let $X_{\alpha_i}$ and $X_{\beta_i}$ be two sets of independent r.v.'s with $X_{\alpha_i} \sim LLD(\alpha_i, \gamma)$ and $X_{\beta_i} \sim LLD(\beta_i, \gamma)$ where $\alpha_i, \beta_i > 0$. Then for any $\gamma > 0$,*

$$(\alpha_1, \ldots, \alpha_n) \overset{m}{\preceq} (\beta_1, \ldots, \beta_n) \Longrightarrow X^{\alpha}_{n:n} \leq_{st} X^{\beta}_{n:n}.$$

***Proof*** The distribution function of $X^{\alpha}_{n:n}$ is

$$F^{\alpha}_{n:n}(x) = \prod_{i=1}^{n} F_{\alpha_i}(x) = \prod_{i=1}^{n} (\gamma x)^{\alpha_i}(1 + (\gamma x)^{\alpha_i})^{-1} = \prod_{i=1}^{n} \zeta_{\gamma x}(\alpha_i)$$

where $\zeta_x(\alpha) = x^{\alpha}/(1 + x^{\alpha})$, $x, \alpha > 0$. From Definition 1, we have to show that $F^{\alpha}_{n:n}(x)$ is Schur-concave in $(\alpha_1, \ldots, \alpha_n)$. Proposition E.1. of Marshall et al. (2011) implies that it is sufficient to check the concavity of $\log_e \zeta_x(\alpha)$, in order to establish the Schur-concavity of $F^{\alpha}_{n:n}(x)$. Observe that the function $\log_e \zeta_x(\alpha)$ is concave in $\alpha$ for all $\gamma > 0$. Hence, $F^{\alpha}_{n:n}(x)$ is Schur-concave in $(\alpha_1, \ldots, \alpha_n)$.

Next, we investigate whether the above result can be generalized to the case of reversed hazard rate ordering. Consider the following example:

**Example 2** Let $X_{\alpha_i} \sim LLD(\alpha_i, \gamma)$ and $X_{\beta_i} \sim LLD(\beta_i, \gamma)$ for $i = 1, 2$, where $(\alpha_1, \alpha_2) = (2.5, 1.5)$ and $(\beta_1, \beta_2) = (1, 3)$ with common scale parameter $\gamma = 2$. Obviously $(\alpha_1, \alpha_2) \overset{m}{\preceq} (\beta_1, \beta_2)$ but $X^{\alpha}_{2:2} \nleq_{rh} X^{\beta}_{2:2}$ which can be easily verified by the plot of corresponding reversed hazard rate functions in Fig. 2.

Next, we investigate the likelihood ratio ordering on maximum-order statistics arising from multiple-outlier LLD samples. Here, it is pertinent to mention that a multiple-outlier model is a set of independent r.v.'s $X_1, X_2, \ldots, X_n$ such that $X_i \overset{st}{=} X$, $i = 1, 2, \ldots, p$ and $X_i \overset{st}{=} Y$, $i = p+1, p+2, \ldots, p+q = n$ where $1 \leq p < n$ and $X_i \overset{st}{=} X$ means that $X_i$ and $X$ are identically distributed. In summary, the set of r.v.'s $X_1, X_2, \ldots, X_n$ is said to constitute a multiple-outlier model if two sets of r.v.'s

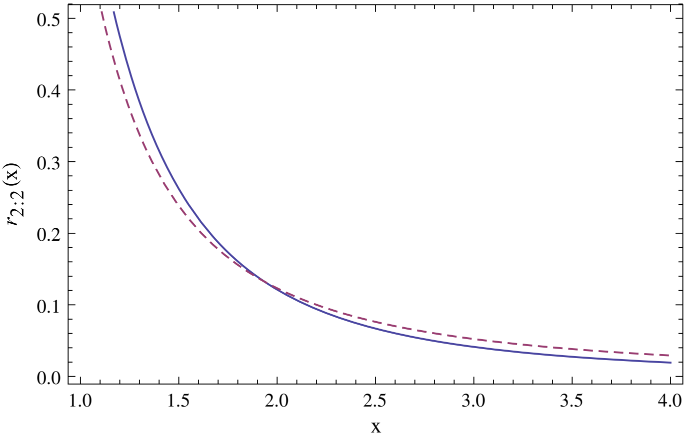


**Fig. 2** Plot of the reversed hazard rate function of $X^{\alpha}_{2:2}$ (continuous line) and $X^{\beta}_{2:2}$ (dashed line) when $\gamma = 2$, $(\alpha_1, \alpha_2) = (2.5, 1.5)$ and $(\beta_1, \beta_2) = (1, 3)$

$X_1, X_2, \ldots, X_p$ and $X_{p+1}, X_{p+2}, \ldots, X_{p+q}$ are homogeneous among themselves and heterogeneous between themselves. For more details on multiple-outlier models, see Balakrishnan (2007).

The following two theorems present versions of Theorems 2 and 3 in the context of multiple-outlier models.

**Theorem 5** *Let $X_1, X_2, \ldots, X_n$ be independent r.v.'s following the multiple-outlier LLD model such that $X_i \sim LLD(\alpha, \gamma_1)$ for $i = 1, 2, \ldots, p$ and $X_j \sim LLD(\alpha, \gamma_2)$ for $j = p+1,\ p+2, \ldots, n$ with $\gamma_1, \gamma_2 > 0$. Let $Y_1, Y_2, \ldots, Y_n$ be another set of independent r.v.'s following the multiple-outlier LLD model such that $Y_i \sim LLD(\alpha, \beta_1)$ for $i = 1, 2, \ldots, p$ and $Y_j \sim LLD(\alpha, \beta_2)$ for $j = p+1, p+2, \ldots, n$ with $\beta_1, \beta_2 >$ 0. Then for* $0 < \alpha \leq 1$,

$$(\underbrace{\gamma_1, \ldots, \gamma_1}_{p}, \underbrace{\gamma_2, \ldots, \gamma_2}_{q}) \overset{m}{\preceq} (\underbrace{\beta_1, \ldots, \beta_1}_{p}, \underbrace{\beta_2, \cdots, \beta_2}_{q}) \Longrightarrow X^{\gamma}_{n:n} \leq_{lr} Y^{\beta}_{n:n} \text{ where } p+q=n.$$

***Proof*** In view of Theorem 2, an equivalent form of (3.1) for this model enables us to complete the proof by simply showing that $r^{\beta}_{n:n}(x)/r^{\gamma}_{n:n}(x)$ is increasing in $x$. Here, the reversed hazard rate of $Y^{\beta}_{n:n}$ is

$$r^{\beta}_{n:n}(x) = \alpha x^{-1}\left[p(1+(\beta_1 x)^{\alpha})^{-1} + q(1+(\beta_2 x)^{\alpha})^{-1}\right]$$

where $p + q = n$. Then,

$$\psi(x) = \frac{r^{\beta}_{n:n}(x)}{r^{\gamma}_{n:n}(x)} = \frac{p\kappa(\beta_1 x) + q\kappa(\beta_2 x)}{p\kappa(\gamma_1 x) + q\kappa(\gamma_2 x)}$$

where $\kappa(x)$ is defined as in Lemma 3. Note that, for $x \geq 0$

$$\begin{aligned}\psi'(x) &\overset{\text{sign}}{=} \left[p\kappa(\gamma_1 x) + q\kappa(\gamma_2 x)\right]\left[p\kappa'(\beta_1 x) + q\kappa'(\beta_2 x)\right] \\ &- \left[p\kappa(\beta_1 x) + q\kappa(\beta_2 x)\right]\left[p\kappa'(\gamma_1 x) + q\kappa'(\gamma_2 x)\right] \\ &\overset{\text{sign}}{=} \left[p\kappa(\beta_1 x)\eta(\beta_1 x) + q\kappa(\beta_2 x)\eta(\beta_2 x)\right]\left[p\kappa(\gamma_1 x) + q\kappa(\gamma_2 x)\right] \\ &- \left[p\kappa(\beta_1 x) + q\kappa(\beta_2 x)\right]\left[p\kappa(\gamma_1 x)\eta(\gamma_1 x) + q\kappa(\gamma_2 x)\eta(\gamma_2 x)\right]\end{aligned}$$

where $\eta(x)$ is defined as in the proof of Theorem 2. To show $\psi(x)$ is increasing in $x$, i.e.,

$$\frac{p\kappa(\beta_1 x)\eta(\beta_1 x) + q\kappa(\beta_2 x)\eta(\beta_2 x)}{p\kappa(\beta_1 x) + q\kappa(\beta_2 x)} \geq \frac{p\kappa(\gamma_1 x)\eta(\gamma_1 x) + q\kappa(\gamma_2 x)\eta(\gamma_2 x)}{p\kappa(\gamma_1 x) + q\kappa(\gamma_2 x)},$$

it is sufficient to show that the function $\phi(\beta_1, \beta_2) = \dfrac{p\kappa(\beta_1 x)\eta(\beta_1 x) + q\kappa(\beta_2 x)\eta(\beta_2 x)}{p\kappa(\beta_1 x) + q\kappa(\beta_2 x)}$ is Schur-convex in $(\beta_1, \beta_2)$.
Now, differentiating $\phi(\beta_1, \beta_2)$ with respect to $\beta_1$, we obtain

$$\begin{aligned}\frac{\partial\phi}{\partial\beta_1} &\overset{\text{sign}}{=} \left[\kappa'(\beta_1 x)\eta(\beta_1 x) + \kappa(\beta_1 x)\eta'(\beta_1 x)\right]\left[p\kappa(\beta_1 x) + q\kappa(\beta_2 x)\right] \\ &- \left[p\kappa(\beta_1 x)\eta(\beta_1 x) + q\kappa(\beta_2 x)\eta(\beta_2 x)\right]\kappa'(\beta_1 x) \\ &= q\kappa'(\beta_1 x)\kappa(\beta_2 x)\left[\eta(\beta_1 x) - \eta(\beta_2 x)\right] + \varphi(\beta_1 x)\left[p\kappa(\beta_1 x) + q\kappa(\beta_2 x)\right]\end{aligned}$$

where $\varphi(x)$ is as defined in Theorem 2. By interchanging $\beta_1$ and $\beta_2$, we obtain

$$\frac{\partial\phi}{\partial\beta_2} \overset{\text{sign}}{=} p\kappa(\beta_1 x)\kappa'(\beta_2 x)\left[\eta(\beta_2 x) - \eta(\beta_1 x)\right] + \varphi(\beta_2 x)\left[p\kappa(\beta_1 x) + q\kappa(\beta_2 x)\right].$$

Now,

$$\begin{aligned}\frac{\partial\phi}{\partial\beta_1} - \frac{\partial\phi}{\partial\beta_2} &\overset{\text{sign}}{=} \left[\eta(\beta_1 x) - \eta(\beta_2 x)\right]\left[q\kappa'(\beta_1 x)\kappa(\beta_2 x) + p\kappa'(\beta_2 x)\kappa(\beta_1 x)\right] \\ &+ \left[p\kappa(\beta_1 x) + q\kappa(\beta_2 x)\right]\left[\varphi(\beta_1 x) - \varphi(\beta_2 x)\right]\end{aligned}$$

Since $\beta_1 \leq \beta_2$ and $\kappa(x)$ and $\eta(x)$ are decreasing in $x$ and $\varphi(x)$ is increasing in $x$, we have, for $0 < \alpha \leq 1$

$$(\beta_1 - \beta_2)\left(\frac{\partial\phi}{\partial\beta_1} - \frac{\partial\phi}{\partial\beta_2}\right) \geq 0.$$

Hence, $\phi(\beta_1, \beta_2)$ is Schur-convex in $(\beta_1, \beta_2)$ and consequently the theorem follows.

**Theorem 6** *Let $X_1, X_2, \ldots, X_n$ be independent r.v.'s following the multiple-outlier LLD model such that $X_i \sim LLD(\alpha, \gamma_1)$ for $i = 1, 2, \ldots, p$ and $X_j \sim LLD(\alpha, \gamma)$ for $j = p+1, p+2, \ldots, n$ with $\gamma_1, \gamma > 0$. Let $Y_1, Y_2, \cdots, Y_n$ be another set of inde-*

*pendent r.v.'s following the multiple-outlier LLD model such that $Y_i \sim LLD(\alpha, \gamma^*)$ for $i = 1, 2, \ldots, p$ and $Y_j \sim LLD(\alpha, \gamma)$ for $j = p+1, p+2, \ldots, n$ with $\gamma^*, \gamma > 0$. Suppose that $\gamma^* = min(\gamma, \gamma_1, \gamma^*)$ then for any $\alpha > 0$,*

$$(\underbrace{\gamma_1, \ldots, \gamma_1}_{p}, \underbrace{\gamma, \ldots, \gamma}_{q}) \overset{w}{\preceq} (\underbrace{\gamma^*, \ldots, \gamma^*}_{p}, \underbrace{\gamma, \ldots, \gamma}_{q}) \Longrightarrow \frac{r^*_{n:n}(x)}{r_{n:n}(x)} \text{ is increasing in } x,$$

*where $p + q = n$.*

***Proof*** The reversed hazard function of $X_{n:n}$ is

$$r_{n:n}(x) = \frac{\alpha}{x}\left[\frac{p}{1+(\gamma_1 x)^\alpha} + \frac{q}{1+(\gamma x)^\alpha}\right] \text{ where } p + q = n.$$

Let $\psi(x) = \dfrac{r^*_{n:n}(x)}{r_{n:n}(x)} = \dfrac{q(1+(\gamma x)^\alpha)^{-1} + p(1+(\gamma^* x)^\alpha)^{-1}}{q(1+(\gamma x)^\alpha)^{-1} + p(1+(\gamma_1 x)^\alpha)^{-1}}$. To show $\psi(x)$ is increasing in $x$, we consider

$$\begin{aligned}
\psi'(x) &\overset{\text{sign}}{=} \left[\frac{p}{1+(\gamma_1 x)^\alpha} + \frac{q}{1+(\gamma x)^\alpha}\right]\left[\frac{-q(\gamma x)^\alpha}{(1+(\gamma x)^\alpha)^2} + \frac{-p(\gamma^* x)^\alpha}{(1+(\gamma^* x)^\alpha)^2}\right] \\
&- \left[\frac{q}{1+(\gamma x)^\alpha} + \frac{p}{1+(\gamma^* x)^\alpha}\right]\left[\frac{-q(\gamma x)^\alpha}{(1+(\gamma x)^\alpha)^2} + \frac{-p(\gamma_1 x)^\alpha}{(1+(\gamma_1 x)^\alpha)^2}\right] \\
&= \left[q\kappa(\gamma x)\eta(\gamma x) + p\kappa(\gamma^* x)\eta(\gamma^* x)\right]\left[p\kappa(\gamma_1 x) + q\kappa(\gamma x)\right] \\
&- \left[q\kappa(\gamma x) + p\kappa(\gamma^* x)\right]\left[p\kappa(\gamma_1 x)\eta(\gamma_1 x) + q\kappa(\gamma x)\eta(\gamma x)\right] \\
&= p^2\kappa(\gamma_1 x)\kappa(\gamma^* x)\left[\eta(\gamma^* x) - \eta(\gamma_1 x)\right] + pq\kappa(\gamma_1 x)\kappa(\gamma x)\left[\eta(\gamma x) - \eta(\gamma_1 x)\right] \\
&+ pq\kappa(\gamma x)\kappa(\gamma^* x)\left[\eta(\gamma^* x) - \eta(\gamma x)\right].
\end{aligned}$$

Now using the facts that $\eta(x)$ is decreasing in $x$, $\kappa(x) \geq 0 \;\forall x > 0$ and $\gamma^* = min(\gamma, \gamma_1, \gamma^*)$, it is easy to show the following: If $\gamma^* \leq \gamma \leq \gamma_1$, then $\psi'(x) \geq 0 \;\forall x > 0$. Also, if $\gamma^* \leq \gamma_1 \leq \gamma$, we have

$$\begin{aligned}
\psi'(x) &\geq p^2\kappa(\gamma_1 x)\kappa(\gamma x)\left[\eta(\gamma^* x) - \eta(\gamma_1 x)\right] + pq\kappa(\gamma_1 x)\kappa(\gamma x)\left[\eta(\gamma x) - \eta(\gamma_1 x)\right] \\
&+ pq\kappa(\gamma_1 x)\kappa(\gamma x)\left[\eta(\gamma^* x) - \eta(\gamma x)\right] \\
&= np\kappa(\gamma x)\kappa(\gamma_1 x)\left[\eta(\gamma^* x) - \eta(\gamma_1 x)\right] \geq 0.
\end{aligned}$$

Thus in both the cases, $\psi'(x) \geq 0 \;\forall x > 0$ and the theorem follows.

Observe that if $(\gamma_1, \gamma) \preceq^w (\gamma^*, \gamma)$ where $\gamma^* = \min(\gamma, \gamma_1, \gamma^*)$ then the parallel system formed by $LLD(\alpha, \gamma_1)$ and $LLD(\alpha, \gamma)$ has the smaller lifetime than the system formed with $LLD(\alpha, \gamma^*)$ and $LLD(\alpha, \gamma)$ in the reverse hazard rate sense for any shape parameter $\alpha > 0$. Using this fact together with the result in Theorem 1.C.4. of Shaked and Shanthikumar (2007), one can get the following result.

**Theorem 7** *Let $X_1, X_2, \ldots, X_n$ be independent r.v.'s following the multiple-outlier LLD model such that $X_i \sim LLD(\alpha, \gamma_1)$ for $i = 1, 2, \ldots, p$ and $X_j \sim LLD(\alpha, \gamma)$ for $j = p+1, p+2, \ldots, n$ with $\gamma_1, \gamma > 0$. Let $Y_1, Y_2, \ldots, Y_n$ be another set of independent r.v.'s following the multiple-outlier LLD model such that $Y_i \sim LLD(\alpha, \gamma^*)$ for $i = 1, 2, \ldots, p$ and $Y_j \sim LLD(\alpha, \gamma)$ for $j = p+1, p+2, \ldots, n$ with $\gamma^*, \gamma > 0$. Suppose that $\gamma^* = min(\gamma, \gamma_1, \gamma^*)$ then for any $\alpha > 0$,*

$$(\underbrace{\gamma_1, \ldots, \gamma_1}_{p}, \underbrace{\gamma, \ldots, \gamma}_{q}) \overset{w}{\preceq} (\underbrace{\gamma^*, \ldots, \gamma^*}_{p}, \underbrace{\gamma, \ldots, \gamma}_{q}) \Longrightarrow X_{n:n} \leq_{lr} Y^*_{n:n} \text{ where } p + q = n.$$

## 4 Order Relations for Series System

In this section, our main aim is to compare two series systems formed with independent heterogeneous LLD samples either having common shape parameter but different scale parameters or conversely.

Let $X^{\gamma}_{1:n}$ denote the lifetime of the series system formed with $n$ independent non-negative r.v.'s $X_{\gamma_1}, X_{\gamma_2}, \ldots, X_{\gamma_n}$, where each $X_{\gamma_i} \sim LLD(\alpha, \gamma_i)$. Then, its survival and density functions are given by

$$\bar{F}^{\gamma}_{1:n}(x) = \prod_{i=1}^{n} \bar{F}_{\gamma_i}(x), \quad f^{\gamma}_{1:n}(x) = \prod_{i=1}^{n} \bar{F}_{\gamma_i}(x) \sum_{i=1}^{n} h_{F_{\gamma_i}}(x),$$

and the corresponding hazard rate function is

$$h^{\gamma}_{1:n}(x) = \frac{f^{\gamma}_{1:n}(x)}{\bar{F}^{\gamma}_{1:n}(x)} = \sum_{i=1}^{n} h_{F_{\gamma_i}}(x).$$

The following theorem shows that under a certain condition on the shape parameter, one can compare the lifetimes of two series systems with independent LLD components according to hazard rate ordering.

**Theorem 8** *For $i = 1, 2, \ldots, n$, let $X_{\lambda_i}$ and $X_{\beta_i}$ be two sets of independent r.v.'s with $X_{\gamma_i} \sim LLD(\alpha, \gamma_i)$ and $X_{\beta_i} \sim LLD(\alpha, \beta_i)$ where $\gamma_i, \beta_i > 0$. Then for $0 < \alpha \leq 1$,*

$$(\gamma_1, \ldots, \gamma_n) \overset{m}{\preceq} (\beta_1, \ldots, \beta_n) \Longrightarrow X^{\gamma}_{1:n} \leq_{hr} X^{\beta}_{1:n}.$$

***Proof*** Fix $x \geq 0$. The hazard rate function of $X^{\gamma}_{1:n}$ is

$$h^{\gamma}_{1:n}(x) = \sum_{i=1}^{n} \alpha x^{-1} (\gamma_i x)^{\alpha} (1 + (\gamma_i x)^{\alpha})^{-1} = \sum_{i=1}^{n} \alpha x^{-1} \tau(\gamma_i x)$$

where $\tau(x)$ is as defined in Lemma 3 and is concave in $x$ for $0 < \alpha \leq 1$. It follows from Proposition C.1. of Marshall et al. (2011) that $\sum_{i=1}^{n} \tau(\gamma_i x)$ is Schur-concave. This completes the proof.

From the theory of stochastic ordering, we have $\leq_{rh} \Longrightarrow \leq_{st}$. Thus from the above theorem, it is clear that the result is also valid in the sense of stochastic ordering. The next question that arises naturally is whether the comparison can be extended to likelihood ratio ordering, i.e., if a version of Theorem 3 for comparison in the sense of likelihood ratio ordering is valid in the context of series systems. The following example gives the answer in the negative.

**Example 3** Let $X_{\gamma_i} \sim LLD(\alpha, \gamma_i)$ and $X_{\beta_i} \sim LLD(\alpha, \beta_i)$ for $i = 1, 2$ where the scale parameters are $(\gamma_1, \gamma_2) = (0.5, 1.5)$ and $(\beta_1, \beta_2) = (0.3, 1.7)$, respectively. Now the plot of $f_{1:n}^{\beta}(x)/f_{1:n}^{\gamma}(x)$ for the common shape parameters $\alpha = 0.5$ and $\alpha = 1.5$ is given in Figs. 3a, b, respectively. Obviously in both the cases, $(\gamma_1, \gamma_2) \overset{m}{\preceq} (\beta_1, \beta_2)$ holds but $X_{2:2}^{\gamma} \nleq_{lr} X_{2:2}^{\beta}$ since in both the cases $f_{1:2}^{\beta}(x)/f_{1:2}^{\gamma}(x)$ is not a monotonic function.

Now we consider series systems having heterogeneous LLD components with common scale parameter and different shape parameters (which are also majorized) and investigate similar results.

**Theorem 9** *For $i = 1, 2, \ldots, n$, let $X_{\alpha_i}$ and $X_{\beta_i}$ be two sets of independent r.v.'s with $X_{\alpha_i} \sim LLD(\alpha_i, \gamma)$ and $X_{\beta_i} \sim LLD(\beta_i, \gamma)$ where $\alpha_i, \beta_i > 0$. Then for any $\gamma > 0$,*

$$(\alpha_1, \ldots, \alpha_n) \overset{m}{\preceq} (\beta_1, \ldots, \beta_n) \Longrightarrow X_{1:n}^{\alpha} \geq_{st} X_{1:n}^{\beta}.$$

***Proof*** The survival function of $X_{1:n}^{\alpha}$ is

$$\bar{F}_{1:n}^{\alpha}(x) = \prod_{i=1}^{n} \bar{F}_{\alpha_i}(x) = \prod_{i=1}^{n} (1 + (\gamma x)^{\alpha_i})^{-1} = \prod_{i=1}^{n} \upsilon_{\gamma x}(\alpha_i)$$

where $\upsilon_{\gamma x}(\alpha_i) = (1 + (\gamma x)^{\alpha_i})^{-1}$. To establish the result, it is enough to show that $\bar{F}_{1:n}^{\alpha}(x)$ is Schur-concave in $(\alpha_1, \ldots, \alpha_n)$. Observe that the function $\log_e \upsilon_{\lambda x}(\alpha)$ is concave in $\alpha$ for all $\gamma > 0$. Then, an argument similar to that of Theorem 4 yields the result.

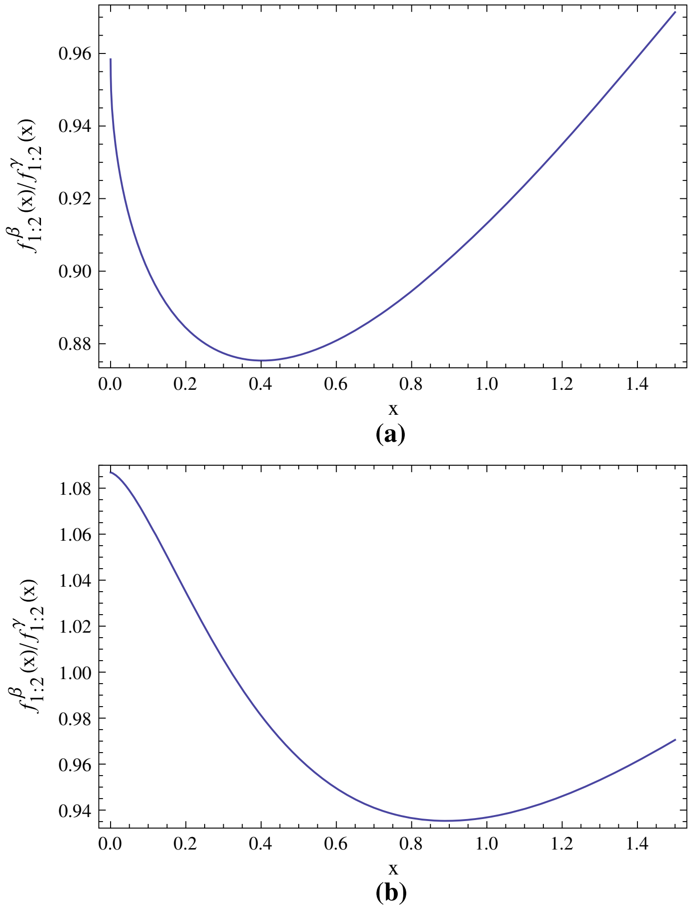


**Fig. 3** Plot of $f^{\beta}_{1:2}(x)/f^{\gamma}_{1:2}(x)$ when $\alpha = 0.5$ (Sub-Fig. **a**) and $\alpha = 1.5$ (Sub-Fig. **b**) for $= (0.5, 1.5)$, and **fi** $= (0.3, 1.7)$

In this type of series system model when we compare further, the following example illustrates that no such comparison can be made in the sense of hazard rate ordering.

**Example 4** Figure 4 illustrates that stochastic comparison between lifetimes of two series systems $X^{\alpha}_{1:2}$ and $X^{\beta}_{1:2}$ with LLD components having common scale parameter $\gamma = 1$ and majorized shape parameters $(\alpha_1, \alpha_2) = (2.5, 1.5)$ and $(\beta_1, \beta_2) = (1, 3)$ is not ordered in the sense of hazard rate ordering.

**Acknowledgements** We thank the anonymous reviewer for his/her helpful comments which have substantially improved the presentation of the paper. The authors are also grateful to Prof. Arnab K. Laha, IIMA, for his constant encouragement and words of advice.

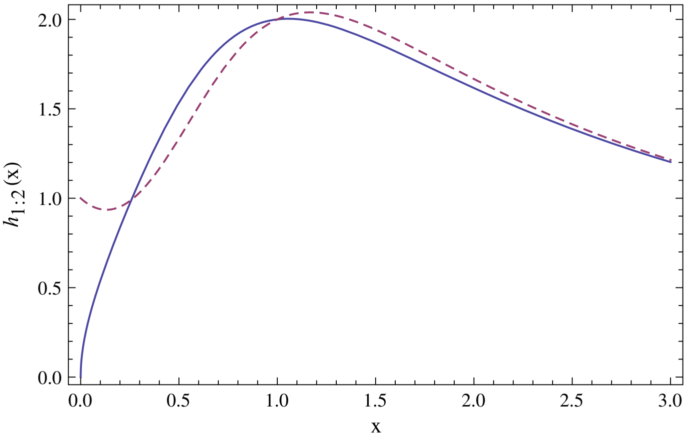


**Fig. 4** Plot of the hazard rate function of $X^{\alpha}_{1:2}$ (continuous line) $X^{\beta}_{1:2}$ (dashed line) when $\gamma = 1$, $(\alpha_1, \alpha_2) = (2.5, 1.5)$ and $(\beta_1, \beta_2) = (1, 3)$

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