Parametric Oscillations of Viscoelastic Orthotropic Rectangular Plates of Variable Thickness

Rustamkhan Abdikarimov, Bakhodir Normuminov, Dadakhan Khodzhaev, and Davron Yulchiyev

Abstract A mathematical model of the problem of parametric vibrations of viscoelastic rectangular orthotropic plates of variable thickness under periodic load is given in the paper on the basis of the Kirchhoff–Love hypothesis in a geometrically nonlinear statement. The mathematical model of this problem is constructed taking into account the propagation of elastic waves. Using the Bubnov–Galerkin method, based on a polynomial approximation of deflection and displacements, the problem is reduced to solving systems of nonlinear integro-differential equations with variable coefficients. The effects of viscoelastic properties of the material and changes in thickness on the oscillation process are studied.

Keywords Rectangular plate · Variable thickness · Viscoelasticity · Orthotropy · Parametric vibrations · Mathematical model · Relaxation kernel · Integro-differential equation · Numerical method

1 Introduction

Plates and shells of variable thickness are widely introduced in various fields of technology. This is primarily due to the requirements for strength, durability, and design of thin-walled elements of modern structures. Along with thin-walled structural elements from traditional metal materials, structures made of composite materials are widely used; this leads to the need to consider structures with homogeneous and inhomogeneous material properties. The study of problems for plates and shells of variable thickness is a very difficult task and sometimes faces insurmountable difficulties. On the one hand, this is connected with the solution of rather cumbersome equations, which are obtained in mathematical modeling, to reflect the real

R. Abdikarimov
60A A.Temur Str., Tashkent Institute of Finance, Tashkent 100000, Uzbekistan

B. Normuminov · D. Khodzhaev (✉) · D. Yulchiyev
Tashkent Institute of Irrigation and Agricultural Mechanization Engineers, 39 Kari Niyazov str., Tashkent 100000, Uzbekistan
e-mail: dhodjaev@mail.ru

N. Voinov et al. (eds.), *Proceedings of International Scientific Conference on Telecommunications, Computing and Control*, Smart Innovation, Systems and Technologies 220, https://doi.org/10.1007/978-981-33-6632-9_9

mechanical essence of the process of this problem. And on the other hand, it is connected with certain computational difficulties, i.e., the lack of suitable universal numerical methods for solving the obtained equations, and as a result, unified computational algorithms. The widespread use of personal computers and software products for solving similar problems of the theory of plates and shells of variable rigidity contributes to the increasing use of numerical analysis methods.

A number of papers [1, 2] are devoted to studying the behavior of plates and shells of constant thickness under dynamic loads in an elastic statement, and there a detailed review of the results of these studies can be found.

In [3] derives accurately, for the first time, the nonlinear damping from a fractional viscoelastic standard solid model by introducing geometric nonlinearity in it.

Theoretical and experimental nonlinear vibrations of thin rectangular plates and curved panels subjected to out-of-plane harmonic excitation are investigated in [4]. Experiments have been performed on isotropic and laminated sandwich plates and panels with supported and free boundary conditions.

Nonlinear vibrations of viscoelastic thin rectangular plates subjected to normal harmonic excitation in the spectral neighborhood of the lowest resonances are investigated in [5].

A review of publications devoted to the study of the behavior of plates and shells of smoothly variable thickness shows that at present, the behavior of such structural elements is insufficiently studied taking into account all the noted significant factors [6–11].

Studies of parametric vibrations of thin-walled structures have become a separate area of research in the mechanics of a deformable rigid body. They are widely applied to various mechanical systems, in particular to plates and shells.

In [12], a numerical–analytical method was proposed for studying parametric oscillations of plates under the action of static and periodic loads.

In [13–16], the results of a study of dynamic stability of various types of plates subjected to harmonic loading with and without nonlinearity are presented.

An analysis of the available literature showed [17–19] that there are almost no publications devoted to the study of nonlinear vibrations and dynamic stability of thin-walled structures such as viscoelastic plates and shells of variable thickness. In this paper, nonlinear parametric oscillations of viscoelastic orthotropic rectangular plates of variable thickness are numerically investigated. Based on the algorithm for the problem solution, a program was compiled in the Delphi programming environment.

2 Materials and Methods

Consider a viscoelastic orthotropic rectangular plate of variable thickness $h = h(x, y)$ with sides a and b under the action of axial dynamic loads. Let the plate undergo dynamic loading along side a with a periodic load $P(t) = P_0 + P_1 \cos(\Theta t)$ ($P_0, P_1 = \text{const}$, Θ is the frequency of external periodic load). A mathematical model of the problem is constructed in a geometrically nonlinear statement according to the classical Kirchhoff–Love theory. We assume that the plate has initial deflections $w_0 = w_0(x, y)$.

In this case, physical dependence between stresses $\sigma_x, \sigma_y, \tau_{xy}$ and strains $\varepsilon_x, \varepsilon_y, \gamma_{xy}$ is taken in the form [2, 20]:

$$\sigma_x = B_{11}(1 - \Gamma_{11}^*)\varepsilon_x + B_{12}(1 - \Gamma_{12}^*)\varepsilon_y, \; (x \leftrightarrow y, 1 \leftrightarrow 2),$$
$$\tau_{xy} = 2B(1 - \Gamma^*)\gamma_{xy}, \tag{1}$$

where Γ^*, Γ_{ij}^* are the integral operators with the relaxation kernels $\Gamma(t)$ and $\Gamma_{ij}(t)$, respectively:

$$\Gamma^*\phi = \int_0^t \Gamma(t-\tau)\phi(\tau)\mathrm{d}\tau, \; \Gamma_{ij}^*\phi = \int_0^t \Gamma_{ij}(t-\tau)\phi(\tau)\mathrm{d}\tau, \; i, j = 1, 2,$$

$$B_{11} = \frac{E_1}{1-\mu_1\mu_2}, \;\; B_{22} = \frac{E_2}{1-\mu_1\mu_2}, \;\; B_{12} = B_{21} = \mu_1 B_{22} = \mu_2 B_{11}, \;\; B = \frac{G}{2},$$

E_1, E_2 are the elastic moduli in the direction of the axes x and y; G is the shear modulus; μ_1, μ_2 are Poisson's ratios; here and hereafter, the symbol $(x \leftrightarrow y, \; 1 \leftrightarrow 2)$ indicates that the remaining relations are obtained by circular substitution of indices.

The relationship between strains in the middle surface $\varepsilon_x, \varepsilon_y, \gamma_{xy}$ and displacements u, v, w in x, y, z directions, taking into account initial irregularities, is taken in the form [2]:

$$\varepsilon_x = \frac{\partial u}{\partial x} + \frac{1}{2}\left[\left(\frac{\partial w}{\partial x}\right)^2 - \left(\frac{\partial w_0}{\partial x}\right)^2\right],$$
$$\varepsilon_y = \frac{\partial v}{\partial y} + \frac{1}{2}\left[\left(\frac{\partial w}{\partial y}\right)^2 - \left(\frac{\partial w_0}{\partial y}\right)^2\right],$$
$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} + \frac{\partial w}{\partial x}\frac{\partial w}{\partial y} - \frac{\partial w_0}{\partial x}\frac{\partial w_0}{\partial y} \tag{2}$$

Bending M_x, M_y and torques H with (2) have the form [2, 20]:

$$M_x = -\frac{h^3}{12}\left[B_{11}(1-\Gamma_{11}^*)\frac{\partial^2(w-w_0)}{\partial x^2} + B_{12}(1-\Gamma_{12}^*)\frac{\partial^2(w-w_0)}{\partial y^2}\right],$$
$$(x \leftrightarrow y, 1 \leftrightarrow 2),$$
$$H = -\frac{Bh^3}{3}(1-\Gamma^*)\frac{\partial^2(w-w_0)}{\partial x \partial y}. \tag{3}$$

Substituting (1) and (3) into equation of motion [2]

$$\frac{\partial N_x}{\partial x} + \frac{\partial N_{xy}}{\partial y} + p_x - \rho h \frac{\partial^2 u}{\partial t^2} = 0,\ \frac{\partial N_{xy}}{\partial x} + \frac{\partial N_y}{\partial y} + p_y - \rho h \frac{\partial^2 v}{\partial t^2} = 0$$
$$\frac{\partial M_x}{\partial x^2} + \frac{\partial^2 M_y}{\partial y^2} + 2\frac{\partial^2 H}{\partial x \partial y} + \frac{\partial}{\partial x}\left(N_x \frac{\partial w}{\partial x} + N_{xy}\frac{\partial w}{\partial y}\right)$$
$$+ \frac{\partial}{\partial y}\left(N_{xy}\frac{\partial w}{\partial x} + N_y \frac{\partial w}{\partial y}\right) + P_x(t)\frac{\partial^2 w}{\partial x^2} + q - \rho h \frac{\partial^2 w}{\partial t^2} = 0 \quad (4)$$

we get a system of integro-differential equations in partial derivatives of the form:

$$h\left[B_{11}\left(1-\Gamma_{11}^*\right)\frac{\partial \varepsilon_x}{\partial x} + B_{12}\left(1-\Gamma_{12}^*\right)\frac{\partial \varepsilon_y}{\partial x} + 2B\left(1-\Gamma^*\right)\frac{\partial \varepsilon_{xy}}{\partial y}\right]$$
$$+ \frac{\partial h}{\partial x}\left[B_{11}\left(1-\Gamma_{11}^*\right)\varepsilon_x + B_{12}\left(1-\Gamma_{12}^*\right)\varepsilon_y\right] + 2B\frac{\partial h}{\partial y}\left(1-\Gamma^*\right)\varepsilon_{xy} - \rho h \frac{\partial^2 u}{\partial t^2} = 0,$$
$$h\left[B_{22}\left(1-\Gamma_{22}^*\right)\frac{\partial \varepsilon_y}{\partial y} + B_{21}\left(1-\Gamma_{21}^*\right)\frac{\partial \varepsilon_x}{\partial y} + 2B\left(1-\Gamma^*\right)\frac{\partial \varepsilon_{xy}}{\partial x}\right]$$
$$+ 2B\frac{\partial h}{\partial x}\left(1-\Gamma^*\right)\varepsilon_{xy} + \frac{\partial h}{\partial y}\left[B_{21}\left(1-\Gamma_{21}^*\right)\varepsilon_x + B_{22}\left(1-\Gamma_{22}^*\right)\varepsilon_y\right] - \rho h \frac{\partial^2 v}{\partial t^2} = 0,$$
$$D\left[B_{11}\left(1-\Gamma_{11}^*\right)\frac{\partial^4 (w-w_0)}{\partial x^4} + \left(8B\left(1-\Gamma^*\right) + B_{12}\left(1-\Gamma_{12}^*\right) + B_{21}\left(1-\Gamma_{21}^*\right)\right)\right.$$
$$\left.\frac{\partial^4 (w-w_0)}{\partial x^2 \partial y^2} + B_{22}\left(1-\Gamma_{22}^*\right)\frac{\partial^4 (w-w_0)}{\partial y^4}\right]$$
$$+ \frac{\partial^2 D}{\partial x^2}\left(B_{11}\left(1-\Gamma_{11}^*\right)\frac{\partial^2 (w-w_0)}{\partial x^2} + B_{12}\left(1-\Gamma_{12}^*\right)\frac{\partial^2 (w-w_0)}{\partial y^2}\right)$$
$$+ 2\frac{\partial D}{\partial x}\left[B_{11}\left(1-\Gamma_{11}^*\right)\frac{\partial^3 (w-w_0)}{\partial x^3} + \left(B_{12}\left(1-\Gamma_{12}^*\right) + 4B\left(1-\Gamma^*\right)\right)\frac{\partial^3 (w-w_0)}{\partial x \partial y^2}\right]$$
$$+ 2\frac{\partial D}{\partial y}\left[B_{22}\left(1-\Gamma_{22}^*\right)\frac{\partial^3 (w-w_0)}{\partial y^3} + \left(B_{21}\left(1-\Gamma_{21}^*\right) + 4B\left(1-\Gamma^*\right)\right)\frac{\partial^3 (w-w_0)}{\partial x^2 \partial y}\right]$$
$$+ \frac{\partial^2 D}{\partial y^2}\left(B_{22}\left(1-\Gamma_{22}^*\right)\frac{\partial^2 (w-w_0)}{\partial y^2} + B_{21}\left(1-\Gamma_{21}^*\right)\frac{\partial^2 (w-w_0)}{\partial x^2}\right)$$
$$+ 8\frac{\partial^2 D}{\partial x \partial y}B\left(1-\Gamma^*\right)\frac{\partial^2 (w-w_0)}{\partial x \partial y} - \frac{\partial w}{\partial x}\left\{h\left[B_{11}\left(1-\Gamma_{11}^*\right)\frac{\partial \varepsilon_x}{\partial x} + B_{12}\left(1-\Gamma_{12}^*\right)\frac{\partial \varepsilon_y}{\partial x}\right.\right.$$
$$\left.\left.+2B\left(1-\Gamma^*\right)\frac{\partial \varepsilon_{xy}}{\partial y}\right] + \frac{\partial h}{\partial x}\left[B_{11}\left(1-\Gamma_{11}^*\right)\varepsilon_x + B_{12}\left(1-\Gamma_{12}^*\right)\varepsilon_y\right] + 2B\frac{\partial h}{\partial y}\left(1-\Gamma^*\right)\varepsilon_{xy}\right\}$$
$$- h\frac{\partial^2 w}{\partial x^2}\left[B_{11}\left(1-\Gamma_{11}^*\right)\varepsilon_x + B_{12}\left(1-\Gamma_{12}^*\right)\varepsilon_y\right] - \frac{\partial w}{\partial y}\left\{h\left[B_{22}\left(1-\Gamma_{22}^*\right)\frac{\partial \varepsilon_y}{\partial y}\right.\right.$$
$$\left.+ B_{21}\left(1-\Gamma_{21}^*\right)\frac{\partial \varepsilon_x}{\partial y} + 2B\left(1-\Gamma^*\right)\frac{\partial \varepsilon_{xy}}{\partial x}\right] + 2B\frac{\partial h}{\partial x}\left(1-\Gamma^*\right)\varepsilon_{xy}$$
$$\left.+\frac{\partial h}{\partial y}\left[B_{21}\left(1-\Gamma_{21}^*\right)\varepsilon_x + B_{22}\left(1-\Gamma_{22}^*\right)\varepsilon_y\right]\right\} - h\frac{\partial^2 w}{\partial y^2}\left[B_{21}\left(1-\Gamma_{21}^*\right)\varepsilon_x\right.$$

$$+B_{22}\left(1-\Gamma_{22}^{*}\right)\varepsilon_y\Big]-4h\frac{\partial^2 w}{\partial x\partial y}B\left(1-\Gamma^{*}\right)\varepsilon_{xy}+P_x(t)\frac{\partial^2 w}{\partial x^2}+\rho h\frac{\partial^2 w}{\partial t^2}=q \tag{5}$$

The system of Eq. (5) with the corresponding boundary and initial conditions describes the motion of a viscoelastic orthotropic rectangular plate of variable thickness under the action of a periodic load $P(t) = P_0 + P_1\cos(\Theta t)$ taking into account initial imperfections.

In calculations, the singular kernels of the Koltunov–Rzhanitsyn type [21] are used as relaxation kernels $\Gamma(t), \Gamma_{ij}(t), i, j = 1, 2$:

$$\Gamma(t) = Ae^{-\beta t}t^{\alpha-1}, (0 < \alpha < 1), \Gamma_{ij}(\mathrm{t}) = A_{ij}e^{-\beta_{ij}t}t^{\alpha_{ij}-1}, \left(0 < \alpha_{ij} < 1\right) \tag{6}$$

Let the plate thickness change according to the following law $h(x) = \frac{1}{2}h_0(1+\alpha * x)$; i.e., a linear increase in the plate thickness is observed (Fig. 1). Here, α^* is a parameter characterizing the variability of the thickness; h_0 is the plate thickness corresponding to $\alpha^* = 0$.

A solution to system (5) satisfying the boundary conditions of the problem is sought with respect to the displacements u and v, and deflection w in the form

$$u(x, y, t) = \sum_{n=1}^{N}\sum_{m=1}^{M} u_{nm}(t)\phi_{nm}(x, y), v(x, y, t) = \sum_{n=1}^{N}\sum_{m=1}^{M} v_{nm}(t)\phi_{nm}(x, y),$$
$$w(x, y, t) = \sum_{n=1}^{N}\sum_{m=1}^{M} w_{nm}(t)\psi_{nm}(x, y) \tag{7}$$

Substituting (7) into the system of Eq. (5) and performing the Bubnov–Galerkin procedure, taking into account dimensionless quantities

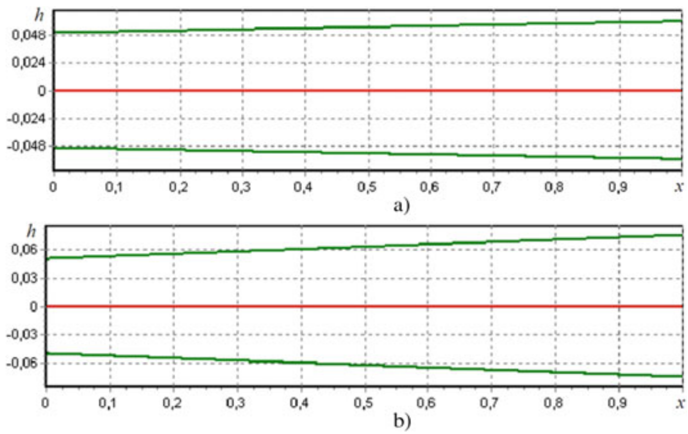

Fig. 1 Change in plate thickness depending on parameter α^*: **a** $\alpha^* = 0.2$; **b** $\alpha^* = 0.5$

$$\frac{u}{h_0}, \frac{v}{h_0}, \frac{w}{h_0}, \frac{w_0}{h_0}, \frac{x}{a}, \frac{y}{b}, \frac{h}{h_0}, \lambda = \frac{a}{b}, \delta = \frac{b}{h_0}, q^* = \frac{q}{E}\left(\frac{b}{h_0}\right)^4, \frac{\Theta}{\omega}, \omega t$$

and maintaining the previous notations, the following system of basic resolving nonlinear integro-differential equations is obtained

$$\begin{aligned}
&\sum_{n=1}^{N}\sum_{m=1}^{M} a_{k\ln m}\ddot{u}_{nm} - \eta_1\left\{\sum_{n=1}^{N}\sum_{m=1}^{M}\{[(1-\Gamma_{11}^*)d_{1k\ln m} + (1-\Gamma^*)d_{2k\ln m}]u_{nm}\right.\\
&+[(1-\Gamma_{12}^*)d_{3k\ln m} + (1-\Gamma^*)d_{4k\ln m}]v_{nm}\}\\
&\left.+\sum_{n,i=1}^{N}\sum_{m,j=1}^{M}\left[(1-\Gamma_{11}^*)d_{7k\ln mij} + (1-\Gamma_{12}^*)d_{8k\ln mij} + (1-\Gamma^*)d_{9k\ln mij}\right](w_{nm}w_{ij} - w_{0nm}w_{0ij})\right\} = 0,\\
&\sum_{n=1}^{N}\sum_{m=1}^{M} b_{k\ln m}\ddot{v}_{nm} - \eta_2\left\{\sum_{n=1}^{N}\sum_{m=1}^{M}\{[(1-\Gamma_{21}^*)e_{1k\ln m} + (1-\Gamma^*)e_{2k\ln m}]u_{nm}\right.\\
&+[(1-\Gamma_{22}^*)e_{3k\ln m} + (1-\Gamma^*)e_{4k\ln m}]v_{nm}\}\\
&\left.+\sum_{n,i=1}^{N}\sum_{m,j=1}^{M}\left[(1-\Gamma_{22}^*)e_{7k\ln mij} + (1-\Gamma_{21}^*)e_{8k\ln mij} + (1-\Gamma^*)e_{9k\ln mij}\right](w_{nm}w_{ij} - w_{0nm}w_{0ij})\right\} = 0\\
&\sum_{n=1}^{N}\sum_{m=1}^{M} c_{k\ln m}\ddot{w}_{nm} + \eta_3\sum_{n=1}^{N}\sum_{m=1}^{M} p_{k\ln m}^2(1-2\mu_{k\ln m}\cos\Theta t)w_{nm}\\
&-\eta_3\left\{\sum_{n=1}^{N}\sum_{m=1}^{M}\{[\Gamma_{11}^* f_{5k\ln m} + \Gamma_{12}^* f_{6k\ln m} + \Gamma_{22}^* f_{7k\ln m} + \Gamma_{21}^* f_{8k\ln m} + \Gamma^* f_{9k\ln m}]w_{0nm}\}\right.\\
&-\eta_3\left\{\sum_{n,i=1}^{N}\sum_{m,j=1}^{M} w_{nm}\left\{\left[(1-\Gamma_{11}^*)\xi_{1k\ln mij} + (1-\Gamma_{21}^*)\xi_{2k\ln mij}\right.\right.\right.\\
&\left.+(1-\Gamma^*)\xi_{3k\ln mij}\right]u_{ij} + \left[(1-\Gamma_{22}^*)\xi_{4k\ln mij} + (1-\Gamma_{12}^*)\xi_{5k\ln mij} + (1-\Gamma^*)\xi_{6k\ln mij}\right]v_{ij}\}\\
&+\sum_{n,i,r=1}^{N}\sum_{m,j,s=1}^{M} w_{nm}\left\{(1-\Gamma_{11}^*)g_{5k\ln mijrs} + (1-\Gamma_{12}^*)g_{6k\ln mijrs} + (1-\Gamma_{22}^*)g_{7k\ln mijrs}\right.\\
&\left.+(1-\Gamma_{21}^*)g_{8k\ln mijrs} + (1-\Gamma^*)g_{9k\ln mijrs}\right\}(w_{ij}w_{rs} - w_{0ij}w_{0rs})\Big\} = 0\\
&u_{nm}(0) = u_{0nm}, \dot{u}_{nm}(0) = \dot{u}_{0nm}, v_{nm}(0) = v_{0nm}, \dot{v}_{nm}(0) = \dot{v}_{0nm},\\
&w_{nm}(0) = w_{0nm}, \dot{w}_{nm}(0) = \dot{w}_{0nm}
\end{aligned} \tag{8}$$

where the constant coefficients entering this system are related to coordinate functions and their derivatives:

$$p_{klnm}^2 = f_{5klnm} + f_{6klnm} + f_{7klnm} + f_{8klnm} + f_{9klnm} - 4\pi^2\lambda^2 p_{klnm}^*\delta_0;$$
$$\mu_{klnm} = \frac{2\pi^2\lambda^2 p_{k\ln m}^*}{p_{klnm}^2}\delta_1.$$

Based on the developed algorithm, a program in the Delphi algorithmic language was compiled.

3 Results and Discussion

Integration of system (8) was carried out using a numerical method based on the use of quadrature formulas [17]. The calculation results for various physical and geometric parameters are shown in graphs, Figs. 2 and 3. The dependence of the change in thickness has the following form: $h = 1 + \alpha^* x$, $h_0 = h(0) = const$, where α^* is the parameter of thickness change.

The effect of inhomogeneous material properties on the plate behavior was studied (Fig. 2).

As seen from the figure, an increase in parameter $\Delta = \sqrt{E_1/E_2}$ determining the degree of anisotropy (curve 1—$\Delta = 1$; curve 2—$\Delta = 1.5$, and curve 3—$\Delta = 2$) leads to a rapid increase in the amplitude of oscillations.

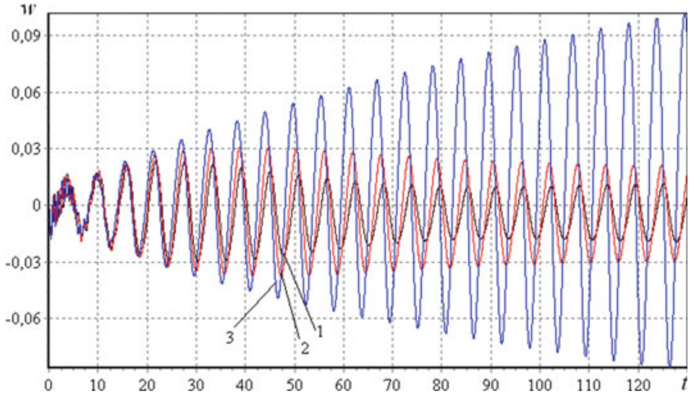

Fig. 2 Dependence of deflections on time at = 1 (1); 1.5 (2); 2 (3)

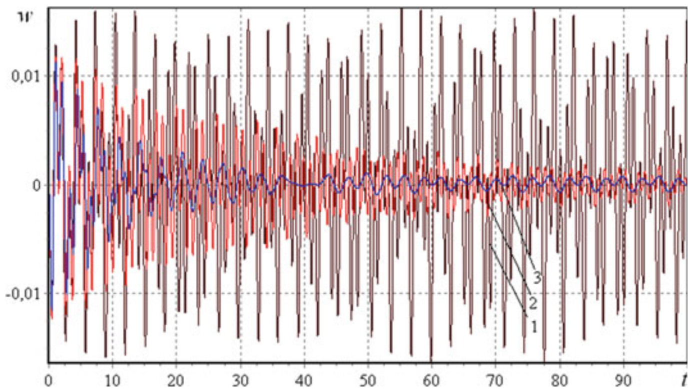

Fig. 3 Dependence of deflections on time

In Fig. 3, various curves correspond to the results obtained by various theories. Curve 1 corresponds to the elastic case, curve 2 to the results obtained taking viscosity into account in shear direction only $\left(A = 0.05,\ A_{ij} = 0,\ i, j = 1, 2\right)$, and curve 3 to the case when viscosity is taken into account in all directions $\left(A = A_{ij} = 0.05,\ i, j = 1, 2\right)$.

The results obtained confirm the need to take into account the viscoelastic properties of the material not only in shear direction, but in other directions as well.

4 Conclusion

A mathematical model, method, and computer program have been developed for evaluating the parametric vibrations of a viscoelastic orthotropic rectangular plate of variable thickness, taking into account geometric nonlinearity under the action of periodic loads.

The effect of the change in physicomechanical and geometric parameters of the plate material on the amplitude–time characteristics and stress–strain state is estimated.

The method proposed in this work can be used for various types of thin-walled structures, such as plates, panels, and shells of variable thickness.

References

1. Bolotin, V.V.: The Dynamic Stability of Elastic Systems. Holden-Day, San Francisco (1964)
2. Volmir, A.S.: The nonlinear Dynamics of Plates and Shells. Foreign Technology Division Wright-Patterson Air Force, USA, Ohio (1974)
3. Amabili, M.: Nonlinear damping in large-amplitude vibrations: modelling and experiments. Nonlinear Dyn. 93 (2018). https://doi.org/10.1007/s11071-017-3889-z
4. Amabili, M., Alijani, F., Delannoy, J.: Damping for large-amplitude vibrations of plates and curved panels, part 2: Identification and comparisons. Int. J. Non. Linear. Mech. 85 (2016). https://doi.org/10.1016/j.ijnonlinmec.2016.05.004
5. Amabili, M.: Nonlinear vibrations of viscoelastic rectangular plates. J. Sound Vib. **362**, 142–156 (2016). https://doi.org/10.1016/j.jsv.2015.09.035
6. Karpov, V.V.: Geometrically Nonlinear Problems for Plates and Shells and Methods for Solving Them. Publishing house ASV; SPbSACU, SPb (1999)
7. Karpov, V.V., Semenov, A.A.: Mathematical models and algorithms for studying strength and stability of shell structures. J. Appl. Ind. Math. **11**, 70–81 (2017). https://doi.org/10.1134/S1990478917010082
8. Tyukalov, Y.Y.: Finite element models in stresses for plane elasticity problems. Mag. Civ. Eng. **77**, 23–37 (2018). https://doi.org/10.18720/MCE.77.3
9. Tyukalov, Y.Y.: Calculation method of bending plates with assuming shear deformations. Mag. Civ. Eng. (2019). https://doi.org/10.18720/MCE.85.9
10. Mochalin, A.A.: Modeling free oscillations of an isotropic cylindrical shell with a variable thickness and density. J. Mach. Manuf. Reliab. **44**, 434–438 (2015). https://doi.org/10.3103/S1052618815030127

11. Kochurov, R., Avramov, K.: V: On effect of initial imperfections on parametric vibrations of cylindrical shells with geometrical non-linearity. Int. J. Solids Struct. **49**, 537–545 (2012). https://doi.org/10.1016/j.ijsolstr.2011.10.023
12. Kurpa, L.V., Mazur, O.S., Tkachenko, V.V.: Parametric vibration of multilayer plates of complex shape. J. Math. Sci. **203**, 165–184 (2014). https://doi.org/10.1007/s10958-014-2098-2
13. Darabi, M., Ganesan, R.: Nonlinear dynamic instability analysis of laminated composite thin plates subjected to periodic in-plane loads. Nonlinear Dyn. **91**, 187–215 (2018). https://doi.org/10.1007/s11071-017-3863-9
14. Huynh, H.Q., Nguyen, H., Nguyen, H.L.T.: Non-linear parametric vibration and dynamic instability of laminated composite plates using extended dynamic stiffness method. J. Eng. Technol. **6**, 170–185 (2017)
15. Kumar, R., Dutta, S.C., Panda, S.K.: Linear and non-linear dynamic instability of functionally graded plate subjected to non-uniform loading. Compos. Struct. **154**, 219–230 (2016). https://doi.org/10.1016/j.compstruct.2016.07.050
16. Kumar, R., Mondal, S., Guchhait, S., Jamatia, R.: Analytical approach for dynamic instability analysis of functionally graded skew plate under periodic axial compression. Int. J. Mech. Sci. **130**, 41–51 (2017). https://doi.org/10.1016/j.ijmecsci.2017.05.050
17. Abdikarimov, R.A., Khodzhaev, D.A.: Computer modeling of tasks in dynamics of viscoelastic thin-walled elements in structures of variable thickness. Mag. Civ. Eng. 83–94 (2014). https://doi.org/10.5862/MCE.49.9
18. Khodzhaev, D., Abdikarimov, R., Vatin, N.: Nonlinear oscillations of a viscoelastic cylindrical panel with concentrated masses. In: MATEC Web of Conferences (2018). https://doi.org/10.1051/matecconf/201824501001
19. Abdikarimov, R., Khodzhaev, D., Vatin, N.: To calculation of rectangular plates on periodic oscillations. In: MATEC Web of Conferences (2018). https://doi.org/10.1051/matecconf/201824501003
20. Ilyushin, A.A.: Plasticity. Foundations of General Mathematical Theory. Lenand, Moscow (2016)
21. Mal'tsev, L.E.: The analytical determination of the Rzhanitsyn-Koltunov nucleus. Mech. Compos. Mater. **15**, 131–133 (1979)