Regularity of Solutions of Obstacle Problems –Old & New–

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Abstract Two kinds of machinery to show regularity of solutions of bilateral/unilateral obstacle problems are presented. Some generalizations of known results in the literature are included. Several important open problems in the topics are given.

Keywords Bilateral/Unilateral obstacle problem · Regularity of solutions · Bernstein method · Bellman-Isaacs equation · Penalization · Fully nonlinear elliptic equation · Weak Harnack inequality · L^p viscosity solution

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1 Introduction

In this survey, we overview regularity of solutions of obstacle problems associated with second-order uniformly elliptic partial differential equations (PDE for short). Particularly, we show two different arguments to obtain estimates on solutions of obstacle problems due to maximum principles. On the other hand, there have appeared a huge amount of results concerning on regularity of solutions of variational inequalities, whose typical example is the obstacle problem. However, our methods here do not rely on integration by parts.

One of techniques here is the so-called Bernstein method, which is relatively old, while the other is quite a new one. Inspired by an idea in [20], we have found an interesting argument in [42], which can be applied to fully nonlinear PDE with unbounded coefficients and inhomogeneous terms.

According to [52], it seems that Fichera [24, 25] first studied the Signorini problem as a variational inequality, where a free boundary arises on the boundary of domains.

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Stampacchia in [54] announced variational inequalities in Hilbert spaces as a modification of Lax-Milgram theorem. Later, Lions-Stampacchia in [46] introduced unilateral obstacle problems in the whole domain as an example of minimization problems associated with energy functionals over closed convex sets.

Afterwards, several regularity results on solutions of variational inequalities appeared in [6, 7, 27, 44].

We shall first consider a minimizing problem of given energies under restrictions. Fix a bounded domain $\Omega \subset \mathbb{R}^n$ with smooth boundary $\partial\Omega$. For a given $\psi \in C(\overline{\Omega})$, which is called an upper obstacle, we set a closed convex set

$$K^{\psi} := \{u \in H_0^1(\Omega) \mid u \leq \psi \text{ a.e. in } \Omega\},$$

where $H_0^1(\Omega)$ is the closure of $C_0^{\infty}(\Omega)$ with respect to $H^1(\Omega)$ norm.

For any fixed $f \in L^2(\Omega)$, by setting our energy

$$E[u] := \int_{\Omega} \left(\frac{1}{2}|Du|^2 - fu \right) dx$$

for $u \in K^{\psi}$, it is known that there is a unique $u \in K^{\psi}$ such that

$$E[u] = \min_{v \in K^{\psi}} E[v].$$

Formally, we observe that

$$\begin{cases} -\Delta u \leq f \text{ in } \Omega, \\ \quad u \leq \psi \text{ in } \Omega, \\ -\Delta u = f \text{ in } \{x \in \Omega \mid u(x) < \psi(x)\}. \end{cases}$$

Hence, we may write down this problem as a Bellman equation

$$\max\{-\Delta u - f, u - \psi\} = 0 \quad \text{in } \Omega \tag{1.1}$$

under the Dirichlet condition $u = 0$ on $\partial\Omega$.

Obstacle problems arise in various settings both from purely mathematical interests and from their rich applications. For later topics, we only refer to some text books [3, 26, 34, 45, 53, 56] because it is too wide for this article to mention these issues. We will concentrate on regularity of solutions of obstacle problems but not on regularity of the free boundary, which may be more interesting subject. See [11, 14, 28] and references therein for this topics.

It is worth mentioning that for (1.1), we can only expect solutions to belong to $W^{2,\infty}(\Omega)$ in general even if ψ and f are smooth enough. The first example is a simple one.

Example 1.1 Let $\Omega := (-\frac{5}{4}, \frac{5}{4})$ for $n = 1$, and $\psi(x) = x^2 - 1$. We easily see that

$$u(x) := \begin{cases} |x| - \frac{5}{4} \ (\frac{1}{2} < |x| \le \frac{5}{4}), \\ x^2 - 1 \ \ (|x| \le \frac{1}{2}), \end{cases}$$

satisfies

$$\max\left\{-\frac{d^2u}{dx^2}, u - \psi\right\} = 0 \quad \text{a.e. in } \Omega$$

under the Dirichlet condition $u(\pm\frac{5}{4}) = 0$. We notice that this u is not twice differentiable at $x = \pm\frac{1}{2}$.

We next show the other example when there is a 0th order term of unknown functions.

Example 1.2 Let Ω and ψ be the same ones as in Example 1.1. For the inhomogeneous term $f \in C^2(\overline{\Omega})$, we choose

$$f(x) = \begin{cases} |x| - \frac{5}{4} & (\frac{1}{4} < |x| \le \frac{5}{4}), \\ -8x^4 + 3x^2 - \frac{37}{32} & (|x| \le \frac{1}{4}). \end{cases}$$

It is easy to verify that the same function u in Example 1.1 satisfies

$$\max\left\{-\frac{d^2u}{dx^2} + u - f, u - \psi\right\} = 0 \quad \text{a.e. in } \Omega.$$

We next consider a minimizing problem under the other kind of restriction. Given two obstacles $\varphi, \psi \in C(\overline{\Omega})$ satisfying the compatibility condition

$$\varphi \le \psi \quad \text{in } \Omega, \quad \text{and} \quad \varphi \le 0 \le \psi \quad \text{on } \partial\Omega, \tag{1.2}$$

we introduce the closed convex set

$$K_\varphi^\psi := \{u \in H_0^1(\Omega) \mid \varphi \le u \le \psi \text{ a.e. in } \Omega\}.$$

Again, it is known that there is a unique $u \in K_\varphi^\psi$ such that

$$E[u] = \min_{v \in K_\varphi^\psi} E[v].$$

We observe that u satisfies at least formally

$$\min\{\max\{-\Delta u - f, u - \psi\}, u - \varphi\} = 0 \quad \text{in } \Omega. \tag{1.3}$$

This is a bilateral obstacle problem, which is an Isaacs equation while (1.1) is called a Bellman equation for unilateral obstacle problems.

Because of (1.2), it is easy to see formally that (1.3) is equivalent to the following PDE:

$$\max\{\min\{-\Delta u - f, u - \varphi\}, u - \psi\} = 0 \quad \text{in } \Omega.$$

Using the standard Euclidean inner product $\langle \cdot, \cdot \rangle$, we consider the energy

$$E[u] := \int_\Omega \left(\frac{1}{2} \langle ADu, Du \rangle + \frac{1}{2} cu^2 - fu \right) dx,$$

where $A := (a_{ij}) : \Omega \to S^n$ is positively definite; $\exists \theta > 0$ such that

$$\langle A(x)\xi, \xi \rangle \geq \theta |\xi|^2 \quad \text{for any } \xi \in \mathbb{R}^n \text{ and } x \in \Omega. \tag{1.4}$$

Here and later S^n denotes the set of real-valued symmetric matrices of order n.

When $a_{ij} \in C^1(\overline{\Omega})$ for simplicity, the minimizer of $E[\cdot]$ over $H_0^1(\Omega)$ formally satisfies

$$Lu = f \quad \text{in } \Omega,$$

where

$$Lu := -\text{Tr}(AD^2 u) + \langle b, Du \rangle + cu.$$

Here, we set

$$b := (b_1, \ldots, b_n) = -\left(\sum_{j=1}^n \frac{\partial a_{1j}}{\partial x_j}, \ldots, \sum_{j=1}^n \frac{\partial a_{nj}}{\partial x_j} \right).$$

Hence, as before, we derive the Bellman equation associated with the minimization of $E[\cdot]$ over K^ψ:

$$\max\{Lu - f, u - \psi\} = 0 \quad \text{in } \Omega.$$

Throughout this paper, we shall suppose that there is $M_c > 0$ such that

$$0 \leq c(x) \leq M_c \quad \text{for } x \in \overline{\Omega}. \tag{1.5}$$

If we suppose that c is positive in $\overline{\Omega}$, then particularly, L^∞ estimates become easier to prove. In fact, under (1.5), we need a perturbation function such as w in Proposition 2.1. We choose $R_0 > 0$ such that

$$\Omega \subset B_{R_0}. \tag{1.6}$$

Here and later, we set $B_r := \{y \in \mathbb{R}^n \mid |x| < r\}$, and $B_r(x) := x + B_r$ for $x \in \mathbb{R}^n$.

In this survey, we are concerned with regularity of solutions for obstacle problems, where the PDE part is given by the above linear second-order uniformly elliptic operator L or Bellman-Isaacs ones. We will always assume that the existence of (approximate) solutions of each obstacle problem. In Sects. 2 and 3, using Bernstein method, we obtain (local) $W^{2,\infty}(\Omega)$ estimates on solutions of approximate equations via penalization. We consider the case when the PDE part is linear with bilateral obstacles in Sect. 2 while we deal with Bellman equations with bi- and unilateral obstacles in Sect. 3. In Sect. 4, to show the Hölder continuity of the first derivative, we apply the weak Harnack inequality to solutions of bilateral obstacle problems, where the main PDE part can be of Isaacs type, and moreover, coefficients and inhomogeneous terms can be unbounded. Since fully nonlinear PDE contain 0th order terms in Sect. 4, we need to modify basic tools such as the Aleksandrov-Bakelman-Pucci (ABP for short) maximum principle, weak Harnack inequality and local maximum principle to PDE with 0th order terms. In Appendix, we present those for the reader's convenience.

2 A Linear Operator Case

Although some results in this section will be generalized in Sect. 3, we will present those to clarify our basic argument.

In this section, for coefficients in the linear operator L, and obstacles, we impose that

$$a_{ij}, b_i, f, c, \varphi, \psi \in C^2(\overline{\Omega}). \tag{2.1}$$

To introduce penalty equations, we need $\beta \in C^2(\mathbb{R})$ such that

$$\begin{cases} (i) & \beta(t) = 0 \text{ for } t \leq 0, \\ (ii) & \beta(t) \text{ grows linearly } t \gg 1, \\ (iii) & \beta' \geq 0 \text{ and } \beta'' \geq 0 \text{ in } \mathbb{R}. \end{cases} \tag{2.2}$$

For instance, it is easy to verify that $\beta \in C^2(\mathbb{R})$ defined by

$$\beta(t) := \begin{cases} 0 & \text{for } t \leq 0, \\ -t^4 + 4t^3 & \text{for } t \in (0, 2), \\ 16(t-1) & \text{for } t \geq 2 \end{cases}$$

satisfies all the properties in (2.2).

For $\varepsilon \in (0, 1)$, we will use $\beta_\varepsilon(t) := \beta(t/\varepsilon)$ for $t \in \mathbb{R}$. Furthermore, we easily observe that

$$\text{there is } \hat{C} > 0 \text{ such that } -\hat{C} \leq \beta_\varepsilon(t) - t\beta'_\varepsilon(t) \leq 0. \tag{2.3}$$

We shall consider approximate equations with penalized terms:

$$Lu + \beta_\varepsilon(u - \psi) - \beta_\varepsilon(\varphi - u) = f \quad \text{in } \Omega \tag{2.4}$$

under the Dirichlet condition

$$u = 0 \quad \text{on } \partial\Omega. \tag{2.5}$$

Hereafter, we will use the notations: for $t, s \in \mathbb{R}$,

$$t \vee s := \max\{t, s\} \quad \text{and} \quad t \wedge s := \min\{t, s\}.$$

For simplicity, we will write

$$u_{x_i},\ u_{x_i x_j},\ etc. \quad \text{for} \quad \frac{\partial u}{\partial x_i},\ \frac{\partial^2 u}{\partial x_i \partial x_j},\ etc.,\ respectively.$$

We also use the summation convention for repeated indices, e.g..

$$a_{ij} u_{x_i x_j} = \sum_{i,j=1}^{n} a_{ij} u_{x_i x_j}.$$

Proposition 2.1 (L^∞ estimates) *Assume* (1.2)*,* (1.4)*,* (1.5) *and* (2.1)*. Let* $u^\varepsilon \in C(\overline{\Omega}) \cap C^2(\Omega)$ *be solutions of* (2.4) *satisfying* (2.5)*. Then, there is* $\hat{C} > 0$ *such that*

$$-\hat{C} \max_{\overline{\Omega}} f^- - \max_{\overline{\Omega}} \psi^- \le u^\varepsilon \le \max_{\overline{\Omega}} \varphi^+ + \hat{C} \max_{\overline{\Omega}} f^+ \quad in\ \overline{\Omega} \quad for\ \varepsilon \in (0, 1).$$

Proof We shall only prove the second inequality since the first one can be shown similarly. We shall write u for u^ε for simplicity.

Setting $C_0 := \max_{\overline{\Omega}} \varphi^+ \ge 0$ and $C_1 := \max_{\overline{\Omega}} f^+$, we shall suppose

$$\Theta := \max_{\overline{\Omega}} \{u - C_0 - \mu(C_1 + \delta) w\} > 0.$$

Here $\mu > 0, \delta \in (0, 1)$ and $w(x) := e^{2\gamma R_0} - e^{\gamma(x_1 + R_0)} > 0$ for $x = (x_1, \dots, x_n) \in \Omega$, where $\gamma \ge 1$, and $R_0 > 0$ is from (1.6).

By letting $\hat{x} \in \overline{\Omega}$ satisfy $\Theta = u(\hat{x}) - C_0 - \mu(C_1 + \delta) w(\hat{x})$, (2.5) yields $\hat{x} \in \Omega$. Hence, at $\hat{x} = (\hat{x}_1, \dots, \hat{x}_n) \in \Omega$, the weak maximum principle implies

$$\begin{aligned} 0 &\le -a_{ij} u_{x_i x_j} + b_i u_{x_i} + \mu(C_1 + \delta)\gamma e^{\gamma(\hat{x}_1 + R_0)}(-a_{11}\gamma + b_1) \\ &\le f - cu - \overline{\beta_\varepsilon} + \underline{\beta_\varepsilon} + \mu(C_1 + \delta)\gamma e^{\gamma(\hat{x}_1 + R_0)}(-\theta\gamma + |b_1|). \end{aligned} \tag{2.6}$$

Here and later, to distinguish composite functions $\beta_\varepsilon(u - \psi)$ and $\beta_\varepsilon(\varphi - u)$, we use the following notation:

$$\overline{\beta_\varepsilon}(\cdot) := \beta_\varepsilon(u(\cdot) - \psi(\cdot)) \quad \text{and} \quad \underline{\beta_\varepsilon}(\cdot) := \beta_\varepsilon(\varphi(\cdot) - u(\cdot)).$$

Thus, for a fixed $\gamma := (\max_{\overline{\Omega}} |b_1| + \theta)/\theta$, (2.6) together with (1.5) implies

$$\theta\mu(C_1+\delta)\gamma \le f - c\{C_0+\mu(C_1+\delta)w\} + \underline{\beta_\varepsilon} \le f + \underline{\beta_\varepsilon} \quad \text{at } \hat{x}.$$

Since $\varphi - u \le \varphi - C_0 - \mu(C_1+\delta)w \le \varphi - C_0 \le 0$ at $\hat{x}$, this inequality yields

$$\theta\mu(C_1+\delta)\gamma \le f(\hat{x}),$$

which is a contradiction for $\mu > 1/(\theta\gamma)$. Therefore, for fixed $\mu, \gamma > 0$ in the above, we have $\Theta \le 0$, which concludes the proof. □

We notice that in the above proof, we do not need the whole of (2.1) but we do not present "minimal" hypotheses on regularity of given functions for the sake of presentations.

Proposition 2.2 ($W^{2,p}$ estimates) *Assume* (1.2)*,* (1.4)*,* (1.5) *and* (2.1)*. Let* $u^\varepsilon \in C^2(\overline{\Omega})$ *be solutions of* (2.4) *satisfying* (2.5)*. Then, there is* $\tilde{C} > 0$ *such that for* $\varepsilon \in (0,1)$*,*

$$\begin{cases} \|\overline{\beta_\varepsilon}(u^\varepsilon - \psi)\|_{L^\infty(\Omega)} \le \max_{\overline{\Omega}} f^+ + M_c \max_{\overline{\Omega}} \psi^- + \tilde{C}\|D\psi\|_{W^{1,\infty}(\Omega)}, \\ \|\underline{\beta_\varepsilon}(\varphi - u^\varepsilon)\|_{L^\infty(\Omega)} \le \max_{\overline{\Omega}} f^- + M_c \max_{\overline{\Omega}} \varphi^+ + \tilde{C}\|D\varphi\|_{W^{1,\infty}(\Omega)}. \end{cases} \tag{2.7}$$

In particular, for each $p \in (1,\infty)$*, there is* $\tilde{C}_p > 0$ *such that*

$$\|u^\varepsilon\|_{W^{2,p}(\Omega)} \le \tilde{C}_p \quad \textit{for } \varepsilon \in (0,1). \tag{2.8}$$

Proof We shall only show the bound for $\overline{\beta_\varepsilon}$ since we can prove the other one similarly. We shall simply write u for u^ε again.

Suppose that $\Theta := \max_{\overline{\Omega}} \overline{\beta_\varepsilon} > 0$. In view of the second inequality of (1.2), we can choose $\hat{x} \in \Omega$ such that $\Theta = \overline{\beta_\varepsilon}(u(\hat{x}) - \psi(\hat{x}))$. Since $\overline{\beta_\varepsilon}$ is nondecreasing, we see that $u - \psi$ attains its maximum at $\hat{x} \in \Omega$. Hence, we have at $\hat{x}$,

$$\begin{aligned} 0 &\le -a_{ij}(u-\psi)_{x_i x_j} + b_i(u-\psi)_{x_i} \\ &= f - cu - \overline{\beta_\varepsilon} + \underline{\beta_\varepsilon} + a_{ij}\psi_{x_i x_j} - b_i\psi_{x_i} \\ &\le f - c\psi - \overline{\beta_\varepsilon} + \underline{\beta_\varepsilon} + C\|D\psi\|_{W^{1,\infty}(\Omega)}. \end{aligned}$$

Here and later, C denotes the various positive constant depending only on known quantities.

Note that the first inequality of (1.2) yields

$$(\varphi - u)(\hat{x}) \le (\psi - u)(\hat{x}) < 0.$$

Therefore, we have $0 \le \overline{\beta_\varepsilon} \le \overline{\beta_\varepsilon}(\hat{x}) \le \max_{\overline{\Omega}} f^+ + M_c \max_{\overline{\Omega}} \psi^- + C\|D\psi\|_{W^{1,\infty}(\Omega)}$ in $\overline{\Omega}$, where $M_c > 0$ is the constant in (1.5) □

Remark 2.3 When we consider Bellman operators in Sect. 3, the L^∞ estimate on the penalty terms for obstacles does not imply (2.8) because we will have one more penalty term, which cannot be evaluated by the above argument.

Now, we show local $W^{2,\infty}$ estimates on solutions of (2.4). Our argument is more or less standard though we do not know if the next proposition has appeared somewhere to our knowledge.

Proposition 2.4 (Local $W^{2,\infty}$ estimates) *Assume* (1.2), (1.4), (1.5) *and* (2.1). *Let* $u^\varepsilon \in C^4(\Omega)\cap C^1(\overline{\Omega})$ *be solutions of* (2.4). *Then, for each compact set* $K \Subset \Omega$, *there is* $\tilde{C}_K > 0$ *independent of* $\varepsilon \in (0,1)$ *such that*

$$\max_K |D^2u^\varepsilon| \le \tilde{C}_K.$$

Proof Choose $\zeta \in C_0^\infty(\Omega)$ such that

$$0 \le \zeta \le 1 \quad \text{in } \Omega, \quad \text{and} \quad \zeta = 1 \quad \text{on } K.$$

Putting $M := \max_\Omega \zeta|D^2u^\varepsilon|$, we may suppose $M \ge 1$.

Writing u and β for u^ε and β_ε, respectively, we set

$$V := \zeta^2|D^2u|^2 + \gamma M\{\beta(u-\psi)+\beta(\varphi-u)\} + \gamma M|Du|^2.$$

We shall write $\overline{\beta} := \beta(u-\psi)$ and $\underline{\beta} := \beta(\varphi-u)$ again for simplicity. In the proceeding calculations, we shall more simply write u_{ij}, u_{ijk}, $a_{ij,k}$ etc. for $u_{x_ix_j}$, $u_{x_ix_jx_k}$, $(a_{ij})_{x_k}$ etc., respectively.

We may suppose that $\max_{\overline{\Omega}} V = V(\hat{x}) > 0$ for some $\hat{x} \in \Omega$. By setting $L_0\xi := -a_{ij}\xi_{ij} + b_i\xi_i$, since $L_0V(\hat{x}) \ge 0$ by the weak maximum principle, at $\hat{x}$, we have

$$\begin{aligned}
0 \le -a_{ij} & \left\{ \begin{array}{l} 2\zeta\zeta_{ij}|D^2u|^2 + 2\zeta_i\zeta_j|D^2u|^2 + 8\zeta\zeta_i u_{k\ell}u_{k\ell j} + 2\zeta^2 u_{k\ell}u_{k\ell ij} \\ +2\zeta^2 u_{k\ell i}u_{k\ell j} + \gamma M\overline{\beta}''(u-\psi)_i(u-\psi)_j + \gamma M\overline{\beta}'(u-\psi)_{ij} \\ +\gamma M\underline{\beta}''(\varphi-u)_i(\varphi-u)_j + \gamma M\underline{\beta}'(\varphi-u)_{ij} + 2\gamma M u_k u_{kij} \\ +2\gamma M u_{ki}u_{kj} \end{array} \right\} \\
& +b_i \left\{ \begin{array}{l} 2\zeta\zeta_i|D^2u|^2 + 2\zeta^2 u_{k\ell}u_{k\ell i} + \gamma M\overline{\beta}'(u-\psi)_i + \gamma M\underline{\beta}'(\varphi-u)_i \\ +2\gamma M u_k u_{ki} \end{array} \right\} \\
& \le -2\theta(\zeta^2|D^3u|^2 + \gamma M|D^2u|^2) - \gamma M\theta(\overline{\beta}''|D(u-\psi)|^2 + \underline{\beta}''|D(\varphi-u)|^2) \\
& \quad +C(|D^2u|^2 + \zeta|D^2u||D^3u|) + \gamma M\overline{\beta}'L_0(u-\psi) + \gamma M\underline{\beta}'L_0(\varphi-u) \\
& \quad +2\zeta^2 u_{k\ell}L_0u_{k\ell} + 2\gamma M u_k L_0 u_k.
\end{aligned}$$

By Young's inequality, at $\hat{x}$, we have

$$\begin{aligned}
I_0 &:= \theta\zeta^2|D^3u|^2 + \theta\gamma M\{|D^2u|^2 + \overline{\beta}''|D(u-\psi)|^2 + \underline{\beta}''|D(\varphi-u)|^2)\} \\
&\le \gamma M\{\overline{\beta}'L_0(u-\psi) + \underline{\beta}'L_0(\varphi-u)\} + 2\zeta^2 u_{k\ell}L_0u_{k\ell} + 2\gamma M u_k L_0 u_k \\
&=: I_1 + I_2 + I_3
\end{aligned}$$

for large $\gamma > 1$.

Since (2.8) for $p > n$ implies the $W^{1,\infty}$ estimates on u, we will not mention the dependence on $\|u\|_{W^{1,\infty}(\Omega)}$ in the calculations below. In order to estimate I_3, we differentiate (2.4) with respect to x_k to obtain

$$L_0 u_k = f_k + a_{ij,k} u_{ij} - b_{i,k} u_i - c u_k - c_k u - \overline{\beta}'(u-\psi)_k + \underline{\beta}'(\varphi - u)_k.$$

Thus, we have

$$\begin{aligned} I_0 \le\ & C\gamma M(1+|D^2u|) + I_1 + I_2 \\ & + \gamma M\{\overline{\beta}'(-|Du|^2 + |D\psi|^2) + \underline{\beta}'(-|Du|^2 + |D\varphi|^2)\}. \end{aligned} \tag{2.9}$$

To estimate I_2, we differentiate (2.4) with respect to x_k and x_ℓ to obtain

$$\begin{aligned} L_0 u_{k\ell} = & f_{k\ell} + a_{ij,k\ell} u_{ij} + a_{ij,k} u_{ij\ell} + a_{ij,\ell} u_{ijk} - b_{i,k\ell} u_i - b_{i,k} u_{i\ell} - b_{i,\ell} u_{ik} \\ & - \overline{\beta}'(u-\psi)_{k\ell} - \overline{\beta}''(u-\psi)_k (u-\psi)_\ell + \underline{\beta}'(\varphi - u)_{k\ell} \\ & + \underline{\beta}''(\varphi - u)_k (\varphi - u)_\ell. \end{aligned}$$

Hence, we have

$$\begin{aligned} I_2 \le\ & \theta\zeta^2|D^3u|^2 + C(1+|D^2u|^2) + 2M\{\overline{\beta}''|D(u-\psi)|^2 + \underline{\beta}''|D(\varphi-u)|^2\} \\ & + \zeta^2\{\overline{\beta}'(-|D^2u|^2 + |D^2\psi|^2) + \underline{\beta}'(-|D^2u|^2 + |D\varphi|^2)\}. \end{aligned}$$

Thus, inserting this in (2.9) with $\gamma \ge 2/\theta$, we have

$$\begin{aligned} \theta\gamma M|D^2u|^2 \ \le\ & C\gamma M(1+|D^2u|) + C(1+|D^2u|^2) \\ & + \overline{\beta}' \left\{ \begin{array}{c} -\zeta^2(|D^2u|^2 - |D^2\psi|^2) - M(|Du|^2 - |D\psi|^2) \\ + \gamma M(f - u - \overline{\beta} + \underline{\beta} - L_0\psi) \end{array} \right\} \\ & + \underline{\beta}' \left\{ \begin{array}{c} -\zeta^2(|D^2u|^2 - |D^2\varphi|^2) - M(|Du|^2 - |D\varphi|^2) \\ + \gamma M(-f + u + \overline{\beta} - \underline{\beta} + L_0\varphi) \end{array} \right\} \\ =: \ & J_1 + J_2 + J_3. \end{aligned}$$

<u>Case 1 : $J_2 \le 0$ and $J_3 \le 0$</u>: In this case, for a largely fixed $\gamma \gg 2/\theta$, we immediately have

$$|D^2u|^2(\hat{x}) \le C,$$

which together with Propositions 2.1 and 2.2 implies

$$M^2 \le V(\hat{x}) \le C(1+M).$$

Case 2 : $J_2 > 0$ or $J_3 > 0$: We shall only consider the case of $J_2 > 0$ since the other one can be shown similarly. In view of (2.7), we see that

$$\zeta^2|D^2u|^2(\hat{x}) \le C(1+M),$$

which yields

$$M^2 \le V(\hat{x}) \le C(1+M).$$

Therefore, M is bounded independently from $\varepsilon \in (0,1)$. □

Remark 2.5 We note that our choice of auxiliary functions V does not work for Bellman operators in Sect. 3. Instead, we will barrow a different one from [23], which can be applied only to unilateral obstacle problems.

As mentioned in Sect. 1, Jensen in [32] showed $W^{2,\infty}(\Omega)$ estimates under additional assumptions on the coefficients. Here, in order to simplify the argument, we shall obtain the $W^{2,\infty}$ bound near the flat boundary under additional assumptions. Setting $x' = (x_1, \dots, x_{n-1}) \in \mathbb{R}^{n-1}$, we suppose that Ω satisfies

$$\begin{cases} \Omega \cap B_1 = \{x = (x', x_n) \mid |x| < 1, x_n > 0\}, \\ \partial\Omega \cap B_1 = \{(x', 0) \mid |x'| < 1\}. \end{cases} \tag{2.10}$$

To show $W^{2,\infty}$ estimates near $\partial\Omega$ for bilateral obstacle problems, we follow the argument in [31].

Theorem 2.6 *Assume* (1.2)*,* (1.4)*,* (1.5)*,* (2.1) *and* (2.10)*. Assume also that*

$$a_{in} = 0 \quad \text{on } \partial\Omega \cap B_1. \tag{2.11}$$

Let $u^\varepsilon \in C^4(\overline{\Omega})$ *be solutions of* (2.4)*. Then, there is* $\hat{C} > 0$ *such that*

$$|D^2u^\varepsilon| \le \hat{C} \quad \text{in } \overline{\Omega} \cap B_{\frac{1}{2}}.$$

Remark 2.7 Under hypothesis (2.11), we note that

$$-a_{nn}u^\varepsilon_{nn} + b_n u^\varepsilon_n = f \quad \text{on } \partial\Omega \cap B_1 \tag{2.12}$$

since $u^\varepsilon_i = u^\varepsilon_{ij} = 0$ for $1 \le i, j \le n-1$ on $\partial\Omega \cap B_1$ by (2.5).

Proof As before, we shall write u for u^ε, and use other simplified notations.

We choose $\eta \in C_0^\infty(B_1)$ such that

$$\begin{cases} 0 \leq \eta \leq 1 & \text{in } B_1, \\ \eta = 1 & \text{in } B_{\frac{1}{2}}, \\ \eta_{x_n} = 0 & \text{on } \partial\Omega \cap B_1. \end{cases} \tag{2.13}$$

Setting

$$v_{ij} := \begin{cases} u_{ij} & \text{for } (i,j) \neq (n,n), \\ u_{nn} - \hat{b}_n u_n + \hat{f} & \text{for } (i,j) = (n,n), \end{cases}$$

where $\hat{b}_n = b_n/a_{nn}$ and $\hat{f} = f/a_{nn}$, we define

$$|D^2 v|^2 := \sum_{i,j=1}^{n} v_{ij}^2 = \sum_{(i,j)\neq(n,n)} u_{ij}^2 + (u_{nn} - \hat{b}_n u_n + \hat{f})^2.$$

Consider W defined by

$$W := e^{Ax_n}\eta^2|D^2 v|^2 + \gamma M(\overline{\beta} + \underline{\beta}) + \gamma M|Du|^2,$$

where $M := \max_{\overline{\Omega}} \eta|D^2 u|$, and $A, \gamma > 1$ will be fixed. We may suppose $M \geq 1$.

Let $\hat{x} = (\hat{x}_1, \hat{x}_2, \ldots, \hat{x}_n) \in \overline{\Omega} \cap \overline{B}_1$ be a point such that $\max_{\overline{\Omega}\cap\overline{B}_1} W = W(\hat{x}) > 0$. Because of $W(\hat{x}) > 0$, we may also assume that $\hat{x} \in \overline{\Omega} \cap B_1$.

Since the argument in the proof of Proposition 2.4 can be applied to the case when $\hat{x} \in \Omega \cap B_1$ with some minor changes, we may suppose $\hat{x} \in \partial\Omega \cap B_1$, and we will obtain a contradiction. Since $|D^2 v|^2 = 2\sum_{i=1}^{n-1} u_{in}^2$ at $\hat{x}$, (2.5) implies

$$W_n = 2e^{A\hat{x}_n}\eta^2 \sum_{i=1}^{n-1}(Au_{in}^2 + 2u_{in}u_{inn}) + 2\gamma M u_n(\hat{b}_n u_n - \hat{f}).$$

By noting $u_{inn} = (\hat{b}_n u_n - \hat{f})_i$ at $\hat{x}$, this equality implies

$$\begin{aligned} W_n &\geq 2e^{A\hat{x}_n}\eta^2 \left\{ (A - C)\sum_{i=1}^{n-1} u_{in}^2 - C \right\} - CM \\ &\geq 2e^{A\hat{x}_n} \left\{ \eta^2 (A - C)\sum_{i=1}^{n-1} u_{in}^2 - CM \right\} \\ &\geq 2e^{A\hat{x}_n}(\eta^2|D^2 v|^2 - CM) \end{aligned}$$

for a fixed $A > 1$. If the right hand side of the above is non-positive, then we have

$$\eta^2|D^2 v|^2(\hat{x}) \leq CM,$$

which implies the uniform bound of M independent of $\varepsilon \in (0,1)$. Therefore, we have $W_n(\hat{x}) > 0$ but this implies that $\hat{x}$ is not the maximum of W, which is a contradiction. □

Following [31], we give a sufficient condition to derive (2.11). We use the following notation:

$$B_r^+ := \{x = (x_1, \dots, x_n) \in B_r \mid x_n > 0\}.$$

Although B_1^+ is not a smooth domain, considering an appropriate smooth domain $\Omega \supset B_1^+$, we may assume ∂B_1^+ is smooth. The next proposition yields (2.11).

Proposition 2.8 *Suppose that there is* $\alpha \in (0,1)$ *such that*

$$a_{ij} \in C^{3,\alpha}(\overline{B}_1^+) \quad \text{for } 1 \le i, j \le n.$$

There is a C^4*-diffeomorphism* $T = (T_1, \dots, T_n) : \overline{B}_1^+ \to T(\overline{B}_1^+)$ *such that* $T_k \in C^{4,\alpha}(\overline{B}_1^+)$ *such that*

$$\hat{a}_{kl}(y) = \sum_{i,j=1}^{n} a_{ij}(T^{-1}(y)) \frac{\partial T_k}{\partial x_i}(x) \frac{\partial T_\ell}{\partial x_j}(x)$$

and

$$\hat{a}_{kn}(y', 0) = 0 \quad (1 \le k \le n-1), \quad \text{for } T^{-1}(y', 0) \in \overline{B}_1^+.$$

Proof. We begin with considering the following PDE

$$-a_{ij}(x) u_{x_i x_j} + b_i(x) u_{x_i} + c(x) u + \beta_\varepsilon(u - \psi) - \beta_\varepsilon(\varphi - u) = f(x) \quad \text{in } B_1^+$$

such that $u(x) = 0$ for $x = (x_1, \dots, x_{n-1}, 0) \in \overline{B}_1^+$. Consider the change of variable $\hat{T} = (\hat{T}^1, \dots, \hat{T}^n) : \overline{B}_+^1 \to \mathbb{R}^n$ defined by

$$y_k = \hat{T}^k(x) = \begin{cases} x_k + T^k(x) - T^k(x', 0) & \text{for } x = (x', x_n) \in \overline{B}_1^+,\ 1 \le k \le n-1, \\ x_n & \text{for } x = (x', x_n) \in \overline{B}_1^+,\ k = n. \end{cases}$$

Here, $T = (T^1, \dots, T^n) \in C^{4,\alpha}(\overline{B}_1^+; \mathbb{R}^n)$ is the solution of

$$\begin{cases} -\Delta T^k + T^k = 0 & \text{in } B_1^+, \\ \langle DT^k, \nu \rangle = \dfrac{a_{kn}}{a_{nn}} & \text{on } \partial B_1^+, \end{cases} \tag{2.14}$$

where ν is the outward unit normal of ∂B_1^+.

It is easy to rewrite the equation for $v(y) := u(x)$ with this new variable $y = \hat{T}(x)$:

$$-\hat{a}_{ij}(y)v_{y_iy_j} + \hat{b}_i(y)u_{y_i} + \hat{c}(y)v + \beta_\varepsilon(v - \hat{\psi}) - \beta_\varepsilon(\hat{\varphi} - v) = \hat{f}(y),$$

where $\hat{c}(y) = c(x)$, $\hat{f}(y) = f(x)$, $\hat{\psi}(y) = \psi(x)$, $\hat{\varphi}(y) = \varphi(x)$,

$$\hat{a}_{ij}(y) = \sum_{k,\ell=1}^{n} a_{k\ell}(x)\hat{T}^i_{x_k}(x)\hat{T}^j_{x_\ell}(x),$$

and

$$\hat{b}_i(y) = \sum_{k=1}^{n} b_i(x)\hat{T}^i_{x_k}(x) - \sum_{k,\ell=1}^{n} a_{k\ell}(x)\hat{T}^i_{x_kx_\ell}(x).$$

In view of the boundary condition of (2.5), it is immediate to verify that for $1 \le i \le n-1$,

$$\begin{aligned}\hat{a}_{in}(y', 0) &= \sum_{k,\ell=1}^{n} a_{k\ell}(x', 0)\hat{T}^i_{x_k}(x', 0)\hat{T}^n_{x_\ell}(x', 0)\\ &= \sum_{k=1}^{n} a_{kn}(x', 0)\hat{T}^i_{x_k}(x', 0)\\ &= a_{in}(x', 0) + a_{nn}(x', 0)\hat{T}^i_{x_n}(x', 0) = 0. \qquad \square\end{aligned}$$

Open question 1: Is it possible to obtain $W^{2,\infty}(\Omega)$ estimates with no extra assumption (2.11) on a_{ij}?

3 A Bellman Type Operator Case

In this section, we obtain $W^{1,\infty}$ bounds for solutions of bilateral obstacle problems when the PDE part is of Bellman type. However, we do not know if we can show further estimates on the second derivative of solutions of penalized systems below for bilateral obstacle problems. Thus, following [43], we will discuss local $W^{2,\infty}$ estimates on solutions of unilateral obstacle problems for Bellman equations.

3.1 Bilateral Obstacles

We first consider the following bilateral obstacle problems

$$\min\{\max\{F(x, u, Du, D^2u), u - \psi\}, u - \varphi\} = 0 \quad \text{in } \Omega, \tag{3.1}$$

where $F : \overline{\Omega} \times \mathbb{R} \times \mathbb{R}^n \times S^n \to \mathbb{R}$ is defined by

$$F(x, r, \xi, X) := \max_{k\in\mathcal{N}}\{-\mathrm{Tr}(A^k(x)X) + \langle b^k(x), \xi\rangle + c^k(x)r - f^k(x)\}. \tag{3.2}$$

Here, by letting $N \geq 2$ be a fixed integer, for $k \in \mathcal{N} := \{1, 2, \ldots, N\}$, functions $A^k = (a^k_{ij}) : \overline{\Omega} \to S^n, b^k = (b^k_i) : \overline{\Omega} \to \mathbb{R}^n, c^k : \overline{\Omega} \to \mathbb{R}$ and $f^k : \Omega \to \mathbb{R}$ are given. We will use linear operators

$$L^k u := -\mathrm{Tr}(A^k(x)D^2u) + \langle b^k(x), Du\rangle + c^k(x)u.$$

As in Sect. 2, we suppose that there is $\theta > 0$ such that

$$\langle A^k(x)\xi, \xi\rangle \geq \theta|\xi|^2 \quad \text{for any } \xi \in \mathbb{R}^n \text{ and } (x, k) \in \Omega \times \mathcal{N}, \tag{3.3}$$

and there is $M_c > 0$ such that

$$0 \leq c^k \leq M_c \quad \text{in } \overline{\Omega} \quad \text{for } k \in \mathcal{N}. \tag{3.4}$$

Following [22], we introduce a system of PDE via penalization: for $k \in \mathcal{N}$,

$$\begin{cases} L^k u^k + \beta_\varepsilon(u^k - u^{k+1}) + \beta_\varepsilon(u^k - \psi) - \beta_\varepsilon(\varphi - u^k) = f^k \text{ in } \Omega, \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad u^k = 0 \quad \text{on } \partial\Omega, \end{cases} \tag{3.5}$$

where $u^{N+1} := u^1$ and β_ε is given in Sect. 2. In order to distinguish three β_ε in (3.5), we will simply write

$$\begin{cases} \beta^k(x) := \beta^k_\varepsilon(x) = \beta_\varepsilon(u^k(x) - u^{k+1}(x)), \\ \overline{\beta}^k(x) := \overline{\beta}^k_\varepsilon(x) = \beta_\varepsilon(u^k(x) - \psi(x)), \\ \underline{\beta}^k(x) := \underline{\beta}^k_\varepsilon(x) = \beta_\varepsilon(\varphi(x) - u^k(x)). \end{cases}$$

For given functions, we suppose that

$$a^k_{ij}, b^k_i, f^k, c^k, \psi, \varphi \in C^2(\overline{\Omega}) \quad \text{for } 1 \leq i, j \leq n, \text{ and } k \in \mathcal{N}. \tag{3.6}$$

Setting

$$\overline{f} := \max_{k\in\mathcal{N}} f^{k,+}, \quad \text{and} \quad \underline{f} := \max_{k\in\mathcal{N}} f^{k,-},$$

we have the L^∞ estimates on $u^{k,\varepsilon}$ independent of $(\varepsilon, k) \in (0, 1) \times \mathcal{N}$.

Proposition 3.1 (L^∞ estimates) *Assume* (1.2), (3.3) *and* (3.6). *Let* $u^\varepsilon := (u^{k,\varepsilon}) \in C^2(\overline{\Omega}; \mathbb{R}^N)$ *be solutions of* (3.5). *Then, there is* $\hat{C} > 0$ *such that*

$$-\hat{C}\max_{\overline{\Omega}} \underline{f} - \max_{\overline{\Omega}} \psi^- \leq u^{k,\varepsilon} \leq \max_{\overline{\Omega}} \varphi^+ + \hat{C}\max_{\overline{\Omega}} \overline{f} \quad \text{in } \overline{\Omega} \quad \text{for } (\varepsilon, k) \in (0, 1) \times \mathcal{N}.$$

Proof Setting $C_0 := \max_{\overline{\Omega}} \psi^-$ and $C_1 := \max_{\overline{\Omega}} \underline{f}$, we suppose

$$\min_{k\in\mathcal{N},x\in\overline{\Omega}} u^{k,\varepsilon}(x) + C_0 + \mu(C_1+\delta)w(x) < 0.$$

Here, $\delta > 0$ will be sent to 0 in the end, and w is the function in the proof of Proposition 2.1; $w(x) := e^{2\gamma R_0} - e^{\gamma(x_1+R_0)} > 0$ in $\Omega \subset B_{R_0}$, where $\gamma > 0$ will be fixed later. Dropping $\varepsilon > 0$ from $u^{k,\varepsilon}$ and β_ε, we may assume that there is $\hat{x} \in \Omega$ such that

$$u^1(\hat{x}) + C_0 + \mu(C_1+\delta)w(\hat{x}) = \min_{k\in\mathcal{N},x\in\overline{\Omega}} \{u^k(x) + C_0 + \mu(C_1+\delta)w(x)\} < 0.$$

By setting $\gamma := \left(\max_{k\in\mathcal{N},x\in\overline{\Omega}} |b_1^k| + \theta\right)/\theta$, the weak maximum principle implies that at $\hat{x} \in \Omega$,

$$\begin{aligned}
0 &\geq -a_{ij}^1 u_{ij}^1 + b_i^1 u_i^1 + \mu(C_1+\delta)e^{\gamma(\hat{x}_1+R_0)}\gamma(\gamma a_{11}^1 - b_1^1) \\
&\geq f^1 - c^1 u - \beta(u^1-u^2) - \beta(u^1-\psi) + \theta\mu(C_1+\delta)\gamma \\
&\geq -\underline{f} + c^1\{C_0 + \mu(C_1+\delta)w\} - \beta(u^1-u^2) - \beta(u^1-\psi) + \mu(C_1+\delta)\gamma.
\end{aligned}$$

Since $u^1 \leq u^2$ and $u^1 - \psi \leq 0$ at $\hat{x}$, these observation yield

$$\underline{f}(\hat{x}) \geq \theta\mu(C_1+\delta)\gamma,$$

which gives a contradiction when $\mu > 1/(\theta\gamma)$. Therefore, we conclude the proof of the first inequality.

The second inequality can be shown more easily since we may avoid the penalty term $\beta_\varepsilon(u^k - u^{k+1})$ in the opposite inequalities. □

Next, we show L^∞ estimates on $\beta_\varepsilon(u^k - \psi)$ and $\beta_\varepsilon(\varphi - u^k)$ independent of $(\varepsilon, k) \in (0,1) \times \mathcal{N}$.

Proposition 3.2 (L^∞ estimates on penalty terms) *Assume* (1.2)*,* (3.3) *and* (3.6)*. Let* $u^\varepsilon := (u^{k,\varepsilon}) \in C^2(\overline{\Omega};\mathbb{R}^N)$ *be solutions of* (3.5)*. Then, there exists* $\tilde{C}_1 > 0$ *such that for* $\varepsilon \in (0,1)$ *and* $k \in \mathcal{N}$*,*

$$\begin{cases}
\|\beta_\varepsilon(u^{k,\varepsilon}-\psi)\|_{L^\infty(\Omega)} \leq \max\limits_{\overline{\Omega}} \overline{f} + M_c \max\limits_{\overline{\Omega}} \psi^- + \tilde{C}_1\|D\psi\|_{W^{1,\infty}(\Omega)}, \\
\|\beta_\varepsilon(\varphi-u^{k,\varepsilon})\|_{L^\infty(\Omega)} \leq \max\limits_{\overline{\Omega}} \underline{f} + M_c \max\limits_{\overline{\Omega}} \varphi^+ + \tilde{C}_1\|D\varphi\|_{W^{1,\infty}(\Omega)}.
\end{cases}$$

Proof We shall write u^k for $u^{k,\varepsilon}$ as before. By the same reason in the proof of Proposition 3.1, we shall only show the estimates on $\beta_\varepsilon(\varphi - u^k)$.

Suppose $\max_{\overline{\Omega},\mathcal{N}} \underline{\beta}^k = \underline{\beta}^1(x_0) > 0$ for some $x_0 \in \Omega$. Thus, we may assume $\max_{\overline{\Omega},\mathcal{N}}(\varphi - u^k) = (\varphi - u^1)(x_0) > 0$. Hence, at $x_0 \in \Omega$, we have

$$0 \leq -a_{ij}^1(\varphi-u^1)_{ij} + b_i^1(\varphi-u^1)_i \leq -f^1 + c^1u^1 + \beta^1 + \overline{\beta}^1 - \underline{\beta}^1 + C\|D\varphi\|_{W^{1,\infty}(\Omega)}.$$

Since $u^1 - u^2 \leq 0$, and $\varphi - u^1 > 0$ at x_0, we have

$$\underline{\beta}^1 \leq -f^1 + c^1\varphi + C\|D\varphi\|_{W^{1,\infty}(\Omega)},$$

which concludes the assertion as in the proof of Proposition 2.2. □

Remark 3.3 Notice that we cannot apply the above argument to obtain L^∞-estimates on $\beta_\varepsilon(u^{k,\varepsilon} - u^{k+1,\varepsilon})$. Therefore, unlike Proposition 2.2, we cannot obtain $W^{2,p}$ estimates on $u^{k,\varepsilon}$.

For further regularity, we first obtain the estimate of first derivatives on $\partial\Omega$ in Proposition 3.4 below. To this end, we shall use $W^{1,\infty}$ estimates on approximate solutions of the associated unilateral obstacle problems via penalization.

Proposition 3.4 (Gradient estimates on $\partial\Omega$) *Assume* (1.2), (3.3) *and* (3.6). *Let* $u^\varepsilon := (u^{k,\varepsilon}) \in C^1(\overline{\Omega}; \mathbb{R}^n) \cap C^2(\Omega; \mathbb{R}^n)$ *be solutions of* (3.5). *Then, there exists* $\tilde{C}_2 > 0$ *such that for* $\varepsilon \in (0, 1)$ *and* $k \in \mathcal{N}$,

$$\|Du^{k,\varepsilon}\|_{L^\infty(\partial\Omega)} \leq \tilde{C}_2.$$

Proof Because $u^{k,\varepsilon} = 0$ on $\partial\Omega$, we only need the estimate

$$\left|\frac{\partial u^{k,\varepsilon}}{\partial n}(z)\right| \leq C \quad \text{for any } z \in \overline{\Omega},$$

where $n = n(z) \in \partial B_1$ denotes the outward unit vector at $z \in \partial\Omega$.

Let $v^\varepsilon = (v^{k,\varepsilon}) : \overline{\Omega} \to \mathbb{R}^N$ be the unique solution of the penalized system of the following unilateral obstacle problem.

$$\begin{cases} L^k v^k + \beta_\varepsilon(v^k - v^{k+1}) + \beta_\varepsilon(v^k - \psi) = f^k \text{ in } \Omega, \\ \hfill v^k = 0 \quad \text{on } \partial\Omega. \end{cases} \tag{3.7}$$

Due to Lemmas 2.1, 2.2 and 3.1 in [43], we find $\hat{C}_1 > 0$, and for each compact $K \Subset \Omega$, $\hat{C}_1(K) > 0$ such that

$$\|v^{k,\varepsilon}\|_{W^{1,\infty}(\Omega)} \leq \hat{C}_1, \quad \text{and} \quad \|D^2 v^{k,\varepsilon}\|_{L^\infty(K)} \leq \hat{C}_1(K). \tag{3.8}$$

We claim that

$$v^{k,\varepsilon} \leq u^{k,\varepsilon} \quad \text{in } \overline{\Omega} \quad \text{for } (\varepsilon, k) \in (0, 1) \times \mathcal{N}.$$

Indeed, if we suppose $\Theta := \max_{\overline{\Omega},\mathcal{N}}(v^{k,\varepsilon} - u^{k,\varepsilon} - \delta w) > 0$, where $\delta > 0$ will be sent to 0, and w is the function in Proposition 2.1, then we may suppose $\Theta = (v^{1,\varepsilon} - u^{1,\varepsilon} - \delta w)(\hat{x})$ for some $\hat{x} \in \Omega$. Hence, at $\hat{x}$, we have

$$0 \le -a^1_{ij}(v^{1,\varepsilon} - u^{1,\varepsilon})_{ij} + b^1_i(v^{1,\varepsilon} - u^{1,\varepsilon})_i + \delta\gamma e^{\gamma\hat{x}_1}(-\theta\gamma + |b^1_1|)$$
$$\le -c^1(v^{1,\varepsilon} - u^{1,\varepsilon}) - \beta_\varepsilon(v^{1,\varepsilon} - v^{2,\varepsilon}) - \beta_\varepsilon(v^{1,\varepsilon} - \psi) + \beta_\varepsilon(u^{1,\varepsilon} - u^{2,\varepsilon})$$
$$+\beta_\varepsilon(u^{1,\varepsilon} - \psi) - \beta_\varepsilon(\varphi - u^{1,\varepsilon}) - \theta\delta\gamma$$

provided $\gamma \ge (\max_{\mathcal{N},\overline{\Omega}} |b^k_1| + \theta)/\theta$. Since $v^{1,\varepsilon} > u^{1,\varepsilon}$ and $v^{1,\varepsilon} - v^{2,\varepsilon} \ge u^{1,\varepsilon} - u^{2,\varepsilon}$ at $\hat{x}$, we immediately obtain a contradiction. Therefore, we have

$$v^{k,\varepsilon} \le u^{k,\varepsilon} + \delta w \quad \text{in } \overline{\Omega},$$

which concludes the claim by sending $\delta \to 0$. Therefore, we have

$$\frac{\partial u^{k,\varepsilon}}{\partial n}(z) \le \frac{\partial v^{k,\varepsilon}}{\partial n}(z) \le \hat{C}_1 \quad \text{for any } z \in \partial\Omega. \tag{3.9}$$

On the other hand, for each $k \in \mathcal{N}$, we next let $w^{k,\varepsilon}$ be solutions of

$$\begin{cases} L^k u - \beta_\varepsilon(\varphi - u) = f^k & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

We claim that for $(\varepsilon, k) \in (0,1) \times \mathcal{N}$,

$$u^{k,\varepsilon} \le w^{k,\varepsilon} \quad \text{in } \overline{\Omega}.$$

Indeed, assuming $\max_{\overline{\Omega},\mathcal{N}}(u^{k,\varepsilon} - w^{k,\varepsilon} - \delta w) = (u^{1,\varepsilon} - w^{1,\varepsilon} - \delta w)(\hat{x}) > 0$ for some $\hat{x} \in \Omega$, at $\hat{x}$, we have

$$0 \le -a^1_{ij}(u^{1,\varepsilon} - w^{1,\varepsilon})_{ij} + b^1_i(u^{1,\varepsilon} - w^{1,\varepsilon})_i + \delta\gamma e^{\gamma\hat{x}_1}(-\theta\gamma + |b^1_1|)$$
$$< -\beta^1_\varepsilon(u^{1,\varepsilon} - u^{2,\varepsilon}) - \beta^1_\varepsilon(u^{1,\varepsilon} - \psi) + \beta_\varepsilon(\varphi - u^{1,\varepsilon}) - \beta_\varepsilon(\varphi - w^{1,\varepsilon}) - \theta\delta\gamma$$
$$< 0$$

for large $\gamma > 1$ as before. Hence, the same argument to obtain (3.9) implies

$$\frac{\partial u^{k,\varepsilon}}{\partial n}(z) \ge \frac{\partial w^{k,\varepsilon}}{\partial n}(z) \quad \text{for any } z \in \partial\Omega. \tag{3.10}$$

By the same argument as in the proof of Proposition 2.2, we find $\tilde{C} > 0$ such that

$$0 \le \beta_\varepsilon(\varphi - w^{k,\varepsilon}) \le \tilde{C} \quad \text{in } \overline{\Omega} \text{ and for } (\varepsilon, k) \in (0,1) \times \mathcal{N},$$

which implies

$$\max_{k\in\mathcal{N}} \|Dw^{k,\varepsilon}\|_{L^\infty(\Omega)} \le C \quad \text{for any } \varepsilon \in (0,1).$$

This together with (3.9) and (3.10) concludes the assertion. □

Now, we shall use Bernstein method to derive $W^{1,\infty}(\Omega)$ estimates on $u^{k,\varepsilon}$.

Proposition 3.5 *Assume* (1.2)*,* (3.3) *and* (3.6)*. Let* $u^\varepsilon = (u^{k,\varepsilon}) \in C^1(\overline{\Omega}; \mathbb{R}^N) \cap C^3(\Omega; \mathbb{R}^N)$ *be solutions of* (3.5)*. There exists* $\tilde{C}_3 > 0$ *such that*

$$\max_{k\in\mathcal{N}} \|u^{k,\varepsilon}\|_{W^{1,\infty}(\Omega)} \le \tilde{C}_3 \quad \textit{for } \varepsilon \in (0,1).$$

Proof We shall drop ε from $u^{k,\varepsilon}$. Set

$$V^k(x) := |Du^k|^2 + \gamma(u^k)^2.$$

In view of Proposition 3.4, we may suppose that

$$\max_{\mathcal{N},\overline{\Omega}} V^k = V^1(\hat{x}) > 0$$

for some $\hat{x} \in \Omega$. We shall write u and v for u^1 and u^2, respectively. Furthermore, we shall write β, $\overline{\beta}$ and $\underline{\beta}$ for β^1, $\overline{\beta}^1$ and $\underline{\beta}^1$, respectively.

We then have at $\hat{x} \in \Omega$,

$$\begin{aligned}
0 &\le -2a^1_{ij}(u_{ki}u_{kj} + u_k u_{kij} + \gamma u_i u_j + \gamma u u_{ij}) + 2b^1_i(u_k u_{ki} + u u_i)\\
&\le -2\theta(|D^2u|^2 + \gamma|Du|^2) + 2\gamma u(f^1 - c^1 u - \beta - \overline{\beta} + \underline{\beta})\\
&\quad +2u_k \left\{ \begin{array}{l} f^1_k + a^1_{ij,k}u_{ij} - b^1_{i,k}u_i - c^1_k u - c^1 u_k \\ -\beta'(u-v)_k - \overline{\beta}'(u-\psi)_k + \underline{\beta}'(\varphi - u)_k \end{array} \right\}\\
&\le -\gamma\theta|Du|^2 + C + \beta'(-|Du|^2 + |Dv|^2 - \gamma u^2 + \gamma v^2)\\
&\quad +\overline{\beta}'(-|Du|^2 + |D\psi|^2 - \gamma u^2 + \gamma\psi^2)\\
&\quad +\underline{\beta}'(-|Du|^2 + |D\varphi|^2 - \gamma u^2 + \gamma\varphi^2)
\end{aligned}$$

for large $\gamma > 1$. We use (2.3) to obtain the last inequality in the above.

Since we may suppose the last two terms are non-positive and $V^1 \ge V^2$ at $\hat{x}$, we have $\gamma\theta|Du(\hat{x})|^2 \le C$, which concludes the assertion. □

Since we do not know L^∞ estimates on $\beta_\varepsilon(u^k - u^{k+1})$, it seems difficult to find a weak (or viscosity) solution of (3.1) only with $W^{1,\infty}$ estimates. Thus, we shall switch to unilateral obstacle problems.

3.2 Unilateral Obstacles

In order to show local $W^{2,\infty}$ estimates on solutions of obstacle problems, we shall restrict ourselves to consider unilateral obstacle ones;

$$\begin{cases} \max\{F(x,u,Du,D^2u), u - \psi\} = 0 \text{ in } \Omega, \\ \qquad\qquad\qquad\qquad\qquad u = 0 \text{ on } \partial\Omega, \end{cases} \tag{3.11}$$

where F is of Bellman type defined in (3.2).

Lenhart in [43] showed the $W^{2,\infty}_{loc}(\Omega)$ estimates on solutions of (3.11). We will recall the argument here.

We notice that proceeding arguments for $W^{2,\infty}$ estimates can not be applied to the following unilateral obstacle problem with the same F because the PDE below is of Isaacs type:

$$\min\{F(x, u, Du, D^2u), u - \varphi\} = 0 \quad \text{in } \Omega. \tag{3.12}$$

Open question 2: Is it possible to obtain (local) $W^{2,\infty}$ estimates on solutions of (3.12)?

In place of (1.2), we only need to suppose

$$\psi \geq 0 \quad \text{on } \partial\Omega. \tag{3.13}$$

The penalized system of (3.11) is as follows: for $u^\varepsilon = (u^{k,\varepsilon})$,

$$\begin{cases} L^k u^{k,\varepsilon} + \beta_\varepsilon(u^{k,\varepsilon} - u^{k+1,\varepsilon}) + \beta_\varepsilon(u^{k,\varepsilon} - \psi) = f^k \text{ in } \Omega, \\ \hfill u^{k,\varepsilon} = 0 \quad \text{on } \partial\Omega, \end{cases} \tag{3.14}$$

where $u^{N+1,\varepsilon} := u^{1,\varepsilon}$.

It is easy to establish the next lemma by following the proofs of Propositions 3.1, 3.2 and 3.5. We note that the Bernstein method with the standard barrier argument can also work for the Bellman equation with unilateral obstacles. We refer to Lemma 2.1 in [43] for the details.

Lemma 3.6 *There exists* $\hat{C} > 0$ *such that*

$$\|u^{k,\varepsilon}\|_{W^{1,\infty}(\Omega)} + \|\beta_\varepsilon(u^{k,\varepsilon} - \psi)\|_{L^\infty(\Omega)} \leq \hat{C} \quad \text{for } (\varepsilon, k) \in (0, 1) \times \mathcal{N}.$$

Following the argument in [43] with a bit simpler auxilialy function V below than that there, we establish $W^{2,\infty}_{loc}(\Omega)$ estimates.

Theorem 3.7 (Local $W^{2,\infty}$ estimates) *Assume* (3.3)*,* (3.6) *and* (3.13)*. Let* $u^\varepsilon = (u^{k,\varepsilon}) \in C^4(\Omega : \mathbb{R}^N) \cap C^1(\overline{\Omega} : \mathbb{R}^N)$ *be solutions of* (3.14)*. Then, for each compact* $K \Subset \Omega$*, there is* $C_K > 0$ *such that*

$$\max_{x\in K, k\in\mathcal{N}} |D^2u^{k,\varepsilon}(x)| \leq C_K \quad \text{for } \varepsilon \in (0, 1).$$

Proof We shall simply write u^k for $u^{k,\varepsilon}$ again.

Let $\zeta \in C^\infty_0(\Omega)$ be the same function as in the proof of Proposition 2.4. Putting $M^k = \max_{\overline{\Omega}} \zeta|D^2u^k|$, we may suppose $M = \max_{\mathcal{N}} M^k = \zeta(\hat{z})|D^2u^1(\hat{z})| \geq 1$ for some $\hat{z} \in \Omega$. By change of variables using the orthogonal matrix B such that $BA^1(\hat{z})^tB = (\alpha_k\delta_{k\ell})$, we may suppose that

$$L^1u^1(\hat{z}) = -\alpha_k u^1_{kk}(\hat{z}) + b^1_k(\hat{z})u^1_k(\hat{z}) + c^1(\hat{z})u^1(\hat{z})$$

for some $\alpha_k \geq \theta$. For each $i \in \mathcal{N}$, setting

$$V^i := \zeta^2|D^2u^i|^2 + \gamma M\zeta^2\alpha_k u^i_{kk} + \gamma M|Du^i|^2,$$

we may suppose that $\max_{\mathcal{N},\overline{\Omega}} V^i = V^{i_0}(\hat{x}) > 0$ for some $\hat{x} \in \Omega$ and $i_0 \in \mathcal{N}$.

We note that

$$\begin{aligned} M^2 = \zeta^2|D^2u^1|^2(\hat{z}) &\leq V^{i_0}(\hat{x}) - \gamma M\zeta^2\alpha_k u^1_{kk}(\hat{z}) \\ &\leq V^{i_0}(\hat{x}) + \gamma M\zeta^2(f^1 - b^1_i u^1_i - c^1u^1)(\hat{z}). \end{aligned}$$

Thus, for a fixed $\gamma > 1$, once we obtain

$$|D^2u^{i_0}|^2(\hat{x}) \leq CM, \tag{3.15}$$

then we have

$$M^2 \leq V^{i_0}(\hat{x}) + CM \leq CM(1 + \sqrt{M}),$$

which concludes the assertion.

We shall write a_{ij}, b_i, c, V, u and v for $a^{i_0}_{ij}, b^{i_0}_i, c^{i_0}, V^{i_0}, u^{i_0}$ and u^{i_0+1}, respectively, for simplicity. The weak maximum principle yields, at $\hat{x}$,

$$\begin{aligned} 0 &\leq -a_{ij}V_{ij} + b_iV_i \\ &= -a_{ij}\left\{\begin{array}{l} 2\zeta\zeta_{ij}|D^2u|^2 + 2\zeta_i\zeta_j|D^2u|^2 + 8\zeta\zeta_i u_{k\ell}u_{k\ell j} + 2\zeta^2u_{k\ell}u_{k\ell ij} \\ +2\zeta^2u_{k\ell i}u_{k\ell j} + 2\gamma M\zeta\zeta_{ij}\alpha_k u_{kk} + 2\gamma M\zeta_i\zeta_j\alpha_k u_{kk} \\ +4\gamma M\zeta\zeta_i\alpha_k u_{kkj} + \gamma M\zeta^2\alpha_k u_{kkij} + 2\gamma M u_k u_{kij} + 2\gamma M u_{ki}u_{kj} \end{array}\right\} \\ &\quad +b_i\left\{\begin{array}{l} 2\zeta\zeta_i|D^2u|^2 + 2\zeta^2u_{k\ell}u_{k\ell i} + 2\gamma M\zeta\zeta_i\alpha_k u_{kk} + \gamma M\zeta^2\alpha_k u_{kki} \\ +2\gamma M u_k u_{ki} \end{array}\right\}. \end{aligned}$$

Hence, setting $L_0v := -a_{ij}v_{ij} + b_iv_i$, at $\hat{x}$, we have

$$\begin{aligned} &2\theta(\zeta^2|D^3u|^2 + \gamma M|D^2u|^2) \\ &\leq C(|D^2u|^2 + \zeta|D^2u||D^3u| + \gamma M|D^2u| + \gamma M\zeta|D^3u|) \\ &\quad +\gamma M\zeta^2\alpha_k L_0u_{kk} + 2\zeta^2u_{k\ell}L_0u_{k\ell} + 2\gamma M u_k L_0u_k \\ &=: I_1 + I_2 + I_3 + I_4. \end{aligned}$$

By the definition of I_2 and I_3, we have

$$
\begin{aligned}
I_2 + I_3 = \gamma M \zeta^2 \alpha_k & \left\{ \begin{array}{l} f_{kk} + a_{ij,kk} u_{ij} + 2a_{ij,k} u_{ijk} - b_{i,kk} u_i - 2b_{i,k} u_{ik} \\ -c_{kk} u - 2c_k u_k - c u_{kk} - \beta''(u-v)_k^2 \\ -\beta'(u-v)_{kk} - \overline{\beta}''(u-\psi)_k^2 - \overline{\beta}'(u-\psi)_{kk} \end{array} \right\} \\
+2\zeta^2 u_{k\ell} & \left\{ \begin{array}{l} f_{k\ell} + a_{ij,k\ell} u_{ij} + 2a_{ij,k} u_{ij\ell} - b_{i,k\ell} u_i - 2b_{i,k} u_{i\ell} \\ -c_{k\ell} u - c_k u_\ell - c_\ell u_k - c u_{k\ell} - \beta''(u-v)_k (u-v)_\ell \\ -\beta'(u-v)_{k\ell} - \overline{\beta}''(u-\psi)_k (u-\psi)_\ell - \overline{\beta}'(u-\psi)_{k\ell} \end{array} \right\} \\
\le \gamma M \zeta^2 & \left\{ \begin{array}{l} C(1 + |D^2 u| + |D^3 u|) - \theta \beta'' |D(u-v)|^2 \\ -\alpha_k \beta'(u-v)_{kk} - \theta \overline{\beta}'' |D(u-\psi)|^2 - \alpha_k \overline{\beta}'(u-\psi)_{kk} \end{array} \right\} \\
+\zeta^2 & \left\{ \begin{array}{l} C|D^2 u|(1 + |D^2 u| + |D^3 u|) + 2\beta'' |D(u-v)|^2 |D^2 u| \\ +\beta'(-|D^2 u|^2 + |D^2 v|^2) + 2\overline{\beta}'' |D(u-\psi)|^2 |D^2 u| \\ +\overline{\beta}'(-|D^2 u|^2 + |D^2 v|^2) \end{array} \right\}.
\end{aligned}
$$

Moreover, I_4 is estimated by

$$
\begin{aligned}
I_4 &\le 2\gamma M u_k \{ f_k + a_{ij,k} u_{ij} - b_{i,k} u_i - c_k u - c u_k - \beta'(u-v)_k - \overline{\beta}'(u-\psi)_k \} \\
&\le \gamma M \{ C(1 + |D^2 u|) + \beta'(-|Du|^2 + |Dv|^2) + \overline{\beta}'(-|Du|^2 + |D\psi|^2) \}.
\end{aligned}
$$

Hence, these inequalities together with Young's inequality give

$$
\begin{aligned}
& \theta(\zeta^2 |D^3 u|^2 + \gamma M |D^2 u|^2) \\
\le\; & I_1 + C\gamma M(\gamma M + |D^2 u|) + M(2 - \gamma\theta)\zeta^2 \beta'' |D(u-v)|^2 \\
& + M(2 - \gamma\theta)\zeta^2 \overline{\beta}'' |D(u-\psi)|^2 + \beta'(-V^{i_0} + V^{i_0+1}) \\
& + \overline{\beta}'(-V^{i_0} + \zeta^2 |D^2 \psi|^2 + \gamma M \zeta^2 \alpha_k \psi_{kk} + \gamma M |D\psi|^2).
\end{aligned}
$$

Note $V^{i_0} \ge V^{i_0+1}$ at $\hat{x}$. Furthermore, we may suppose $0 \ge -V^{i_0} + \zeta^2 |D^2 \psi|^2 + \gamma M \zeta^2 \alpha_k \psi_{kk} + \gamma M |D\psi|^2$ at $\hat{x}$. Thus, taking $\gamma \ge 2/\theta$, we have

$$
\begin{aligned}
\theta(\zeta^2 |D^3 u|^2 + \gamma M |D^2 u|^2) &\le I_1 + C\gamma M(\gamma M + |D^2 u|) \\
&\le C(1 + |D^2 u|^2 + \gamma^2 M^2) + \theta \zeta^2 |D^3 u|^2.
\end{aligned}
$$

Remembering $M, \gamma \ge 1$, we have

$$
(\theta\gamma M - C)|D^2 u|^2(\hat{x}) \le C(1 + \gamma^2 M^2),
$$

which implies

$$
\theta\gamma M |D^2 u|^2(\hat{x}) \le C\gamma^2 M^2
$$

provided $\theta\gamma M \ge 2C$. This yields (3.15). □

<u>**Open question 3**</u>: Is it possible to obtain $W^{2,\infty}(\Omega)$ or $W^{2,p}(\Omega)$ estimates for Bellman equations with unilateral obstacles under additional conditions if necessary?
<u>**Open question 4**</u>: Is it possible to obtain local $W^{2,\infty}_{loc}(\Omega)$ estimates for Bellman equations with bilateral obstacles?

4 A Fully Nonlinear Operator Case

In Sects. 2 and 3, thanks to Bernstein method, we establish estimates on solutions of approximate PDE (or systems of PDE), which present the existence of (strong) solutions belonging to the associated function spaces (i.e. $W^{2,\infty}(\Omega)$ or $W^{2,\infty}_{loc}(\Omega)$). See [43, 47] for the details. We also refer to [2] for a modern version of Bernstein method.

We note that there is a fully nonlinear uniformly elliptic equation which does not have classical solutions. See [48]. Furthermore, in [49], it is shown that there exists a viscosity solution of a fully nonlinear uniformly elliptic PDE whose second derivative is not bounded. On the other hand, we also know there is a classical solution of a special Isaacs equation consisting of three linear operators in [9].

In this section, we study more general PDE such as Isaacs equations with bilateral obstacles, and with unbounded, possibly discontinuous coefficients and inhomogeneous terms. In fact, to our knowledge, we do not know any regularity results for obstacle problems of Isaacs equations via penalization. In order to see a difficulty in the study of Isaacs equations via penalization, let us consider approximate Isaacs equations with no obstacles via penalization:

$$L^{k,\ell}u^{k,\ell} + \beta_\varepsilon(u^{k,\ell} - u^{k+1,\ell}) - \beta_\varepsilon(u^{k,\ell+1} - u^{k,\ell}) = f^{k,\ell} \quad \text{in } \Omega, \tag{4.1}$$

where $u^{M+1,\ell} = u^{1,\ell}$ for $\ell \in \mathcal{N}$ and $u^{k,N+1} = u^{k,1}$ for $k \in \mathcal{M}$. Here, by setting $\mathcal{M} := \{1, \dots, M\}$ and $\mathcal{N} := \{1, \dots, N\}$, $u^{k,\ell} : \overline{\Omega} \to \mathbb{R}$ for $(k, \ell) \in \mathcal{M} \times \mathcal{N}$ are unknown functions, and linear operators are defined by

$$L^{k,\ell}\zeta := -\mathrm{Tr}(A^{k,\ell}(x)D^2\zeta) + \langle b^{k,\ell}(x), D\zeta\rangle + c^{k,\ell}(x)\zeta,$$

where given functions $A^{k,\ell} : \overline{\Omega} \to S^n$, $b^{k,\ell} : \overline{\Omega} \to \mathbb{R}^n$ and $c^{k,\ell} : \overline{\Omega} \to [0, \infty)$ satisfy enough regularity.

If we obtain L^∞_{loc} estimates on $\beta_\varepsilon(u^{k,\ell} - u^{k+1,\ell})$ and $\beta_\varepsilon(u^{k,\ell+1} - u^{k,\ell})$, then it is easy to verify that $u^{k,\ell}_\varepsilon$ converge to a single limit u as $\varepsilon \to 0$ (along a subsequence if necessary), which is a solution of

$$\min_{\ell\in\mathcal{N}} \max_{k\in\mathcal{M}} \{L^{k,\ell}u - f^{k,\ell}\} = 0 \quad \text{in } \Omega. \tag{4.2}$$

However, it is difficult to show L^∞ estimates on the first and second penalty terms. In fact, in a pioneering work [47], we first derive $W^{2,\infty}$ estimates on solutions of penalized problems for Bellman equations (i.e. $N = 1$), and then this gives L^∞ bounds for the penalty term. Moreover, Bernstein method does not work to obtain $W^{2,\infty}$ estimates on solutions of (4.1). Furthermore, even if we establish $W^{2,\infty}$ estimates on approximate solutions, since we have two penalty terms with opposite signs in (4.1), we still do not know if solutions of the system (4.1) converge to a single solution of (4.2).

Open question 5: Is it possible to obtain a weak/viscosity solution of (4.2) satisfying (2.5) via penalization?

If we restrict ourselves to try to establish $C^{1,\gamma}$ estimates on solutions of bilateral obstacle problems for $\gamma \in (0, 1)$, then we can accomplish such estimates even when F is of Isaacs type;

$$G(x, r, \xi, X) := \min_{\ell \in \mathcal{N}} \max_{k \in \mathcal{M}} \left\{ -\mathrm{Tr}(A^{k,\ell}(x)X) + \langle b^{k,\ell}(x), \xi \rangle + c^{k,\ell}(x)r \right\}.$$

Moreover, since we do not need systems of PDE via penalization, we may deal with compact sets $\mathcal{M}, \mathcal{N}$ in $\mathbb{R}^m$ for some $m \in \mathbb{N}$. Furthermore, since we will not differentiate PDE (because it is impossible!), it is possible to treat discontinuous coefficients and inhomogeneous terms. In this procedure, we need to show the existence of weak/viscosity solutions of Isaacs equations with obstacles by a different method. We only refer to [16] and [42] for the existence issue.

This section is based on a recent work by the author and Tateyama in [42].

4.1 Equi-Continuity

Modifying arguments by Duque in [20], we present an idea to apply the weak Harnack inequality to obtain estimates on solutions of obstacle problems when the PDE part may be fully nonlinear. Here, the terminology fully nonlinear means that the mapping $(\xi, X) \in \mathbb{R}^n \times S^n \to G(x, r, \xi, X) \in \mathbb{R}$ is neither convex nor concave for each $(x, r) \in \Omega \times \mathbb{R}$.

In what follows, we suppose that

$$\begin{cases} (i) & G(x, 0, 0, O) = 0 \text{ for } x \in \Omega, \\ (ii) & \mathcal{P}^-(X - Y) \le G(x, r, \xi, X) - G(x, r, \xi, Y) \le \mathcal{P}^+(X - Y) \\ & \text{for } x \in \Omega, r \in \mathbb{R}, \xi \in \mathbb{R}^n, X, Y \in S^n, \\ (iii) & \text{there is } \mu \in L^q(\Omega) \text{ such that } q > n, \text{ and} \\ & |G(x, r, \xi, X) - G(x, r, \eta, X)| \le \mu(x)|\xi - \eta| \\ & \text{for } x \in \Omega, r \in \mathbb{R}, \xi, \eta \in \mathbb{R}^n, X \in S^n, \\ (iv) & \text{there is } c_0 \in C(\overline{\Omega}) \text{ such that } c_0 \ge 0 \text{ in } \Omega, \text{ and} \\ & G(x, r, \xi, X) - G(x, s, \xi, X) \ge c_0(x)(r - s) \\ & \text{for } x \in \Omega, r, s \in \mathbb{R}, \xi \in \mathbb{R}^n, X \in S^n, \\ (v) & f \in L^p(\Omega) \text{ for } q \ge p > p_0. \end{cases} \tag{4.3}$$

Here, $p_0 \in [\frac{n}{2}, n)$ is the so-called Escauriaza's constant in [21], and for a fixed $\theta \in (0, 1]$, Pucci operators $\mathcal{P}^\pm : S^n \to \mathbb{R}$ are defined as follows:

$$\mathcal{P}^+(X) := \max\{-\mathrm{Tr}(AX) \mid A \in S^n_\theta\} \text{ and } \mathcal{P}^-(X) := \min\{-\mathrm{Tr}(AX) \mid A \in S^n_\theta\},$$

where $S^n_\theta := \{X \in S^n \mid \theta I \le X \le \theta^{-1} I\}$. Under hypotheses (i)–(iv) in (4.3), we easily verify that

$$\mathcal{P}^-(X) - \mu(x)|\xi| + c_0(x)r \le G(x, r, \xi, X) \le \mathcal{P}^+(X) + \mu(x)|\xi| + c_0(x)r$$

for $x \in \Omega$, $r \in \mathbb{R}$, $\xi \in \mathbb{R}^n$ and $X \in S^n$.

In a celebrated paper [10] by Caffarelli, it has turned out that to establish the regularity of viscosity solutions of fully nonlinear uniformly elliptic PDE

$$G(x, u, Du, D^2u) = f(x) \quad \text{in } \Omega,$$

instead of this equation, it is essential to study extremal inequalities:

$$G^-(x, u, Du, D^2u) \le f^+(x) \quad \text{and} \quad G^+(x, u, Du, D^2u) \ge -f^-(x),$$

where $G^\pm(x, r, \xi, X) := \mathcal{P}^\pm(X) \pm \mu(x)|\xi| \pm c_0(x)r^\pm$.

Furthermore, according to [10] again, the key for the regularity theory is the weak Harnack inequality for supersolutions.

We recall the definition of L^p-viscosity solutions of

$$H(x, u, Du, D^2u) = 0 \quad \text{in } \Omega, \tag{4.4}$$

where $H : \Omega \times \mathbb{R} \times \mathbb{R}^n \times S^n \to \mathbb{R}$ is given (not necessarily continuous).

Definition 4.1 We say that $u \in C(\Omega)$ is an L^p viscosity subsolution (resp., supersolution) of (4.4) if it follows that

$$\lim_{r\to 0} ess. \inf_{B_r(x)} H(y, u(y), D\zeta(y), D^2\zeta(y)) \le 0$$

$$\left(\text{resp., } \lim_{r\to 0} ess. \sup_{B_r(x)} H(y, u(y), D\zeta(y), D^2\zeta(y)) \ge 0\right)$$

whenever for any $\zeta \in W^{2,p}_{loc}(\Omega)$, $u - \zeta$ attains its local maximum (resp., minimum) at $x \in \Omega$. Finally, we say that $u \in C(\Omega)$ is an L^p viscosity solution of (4.4) if it is both of an L^p viscosity subsolution and an L^p viscosity supersolution of (4.4).

Throughout this section, we at least suppose that

$$\varphi, \psi \in C(\overline{\Omega}) \tag{4.5}$$

satisfy (1.2). Under hypotheses (4.3), (4.5) and (1.2), we consider

$$\min\{\max\{G(x, u, Du, D^2u) - f(x), u - \psi(x)\}, u - \varphi(x)\} = 0 \quad \text{in } \Omega \tag{4.6}$$

under the Dirichlet condition (2.5). It is immediate to see that if $u \in C(\Omega)$ is an L^p viscosity subsolution (resp., supersolution) of (4.6) with this G, then it is an L^p viscosity subsolution (resp., supersolution) of

$$\min\{\max\{G^-(x, u, Du, D^2u) - f^+(x), u - \psi(x)\}, u - \varphi(x)\} = 0$$

$$\big(\text{resp.,}\ \ \min\{\max\{G^+(x, u, Du, D^2u) + f^-(x), u - \psi(x)\}, u - \varphi(x)\} = 0\big)$$

in Ω. We will only use these information in the argument below.

We recall a reasonable result without proof.

Proposition 4.2 (Proposition 2.9 in *[42]*) *Under the same hypotheses as in Theorem 4.3, if $u \in C(\Omega)$ is an L^p viscosity subsolution (resp., supersolution) of* (4.6)*, then it follows that*

$$u \le \psi \quad (\textit{resp., } u \ge \varphi) \quad \textit{in } \Omega.$$

In what follows, we call ω a modulus of continuity of functions if

$$\omega \in C([0, \infty)) \text{ is nondecreasing, and } \omega(0) = 0.$$

We also use the notation A^i for the set of interior points of $A \subset \mathbb{R}^n$.

Theorem 4.3 (Theorem 2.10 in *[42]*) *Assume* (4.3)*,* (4.5) *and* (1.2)*. Then, there exists a modulus of continuity ω such that for any L^p viscosity solution of* (4.6) *satisfying* (2.5)*, it follows that*

$$|u(x) - u(y)| \le \omega(|x - y|) \quad \textit{for any } x, y \in \overline{\Omega}.$$

Moreover, if we suppose $\varphi, \psi \in C^\alpha(\overline{\Omega})$ for some $\alpha \in (0, 1)$, then there are $\hat{C} > 0$ and $\hat{\alpha} \in (0, \alpha]$ such that for any L^p viscosity solution of (4.6) *satisfying* (2.5)*, it follows that*

$$|u(x) - u(y)| \le \hat{C}|x - y|^{\hat{\alpha}} \quad \textit{for any } x, y \in \overline{\Omega}.$$

For $r > 0$ and $x \in \mathbb{R}^n$, we define closed cubes as follows:

$$Q_r := \left[-\frac{r}{2}, \frac{r}{2}\right]^n, \quad Q_r(x) := x + Q_r.$$

Proof We shall only give a proof for local estimates since we can modify the argument below by using the weak Harnack inequality near $\partial\Omega$. See [29] for its usage.

Fix any $K \Subset \Omega$. We shall divide K by

$$K^+ := \{x \in K \mid u(x) = \psi(x)\}, \quad K^- := \{x \in K \mid u(x) = \varphi(x)\},$$

and $K_0 := K \setminus (K^+ \cup K^-)$.

It is standard to show the assertion when $x, y \in K_0^i$. See [42] for the details.

Let ω_0 be the modulus of continuity of obstacles;

$$|\psi(x)-\psi(y)| \vee |\varphi(x)-\varphi(y)| \le \omega_0(|x-y|) \quad \text{for } x, y \in \overline{\Omega}.$$

Fix any $\hat{x} \in K$. We may suppose $\hat{x} = 0$ by translation. For $r \in (0, d_0/(2\sqrt{n}))$, where $d_0 := \text{dist}(\partial\Omega, K)$, we set

$$\overline{u} := u \vee (\varphi(0) + \omega_0(2\sqrt{n}r)) \quad \text{and} \quad \underline{u} := u \wedge (\psi(0) - \omega_0(2\sqrt{n}r)).$$

Notice that $\varphi(0) + \omega_0(2\sqrt{n}r) \ge \varphi$ and $\psi \ge \psi(0) - \omega_0(2\sqrt{n}r)$ in $Q_{4r} \subset \Omega$. It is standard to see that $\overline{u}$ and $\underline{u}$ are, respectively, an L^p viscosity subsolution and supersolution of

$$G^-(x, u, Du, D^2u) - f^+(x) = 0 \quad \text{and} \quad G^+(x, u, Du, D^2u) + f^-(x) = 0 \quad \text{in} Q_{4r}.$$

For $s \in (0, d_0)$, set

$$M_s := \sup_{Q_s} \overline{u} \quad \text{and} \quad m_s := \inf_{Q_s} \underline{u}.$$

We then define

$$\overline{U} := M_{4r} - \overline{u} \quad \text{and} \quad \underline{U} := \underline{u} - m_{4r} \quad \text{for } r \in (0, d_0/(2\sqrt{n})).$$

It is immediate to see that $\overline{U}$ and $\underline{U}$ are nonnegative L^p viscosity supersolutions of

$$\mathcal{P}^+(D^2u) + \mu|Du| + c_0u + f^{\pm} = 0 \quad \text{in } Q_{4r}.$$

Since $\|\mu\|_{L^n(Q_{4r})} \le \|\mu\|_{L^q(Q_{4r})}(2\sqrt{n}r)^{1-\frac{n}{q}}$, we can apply Proposition 5.7 in Appendix with the standard scaling to have

$$\left(\int_{Q_r} \overline{U}^{\varepsilon_0} dx\right)^{\frac{1}{\varepsilon_0}} \le Cr^{\frac{n}{\varepsilon_0}} \left(\inf_{Q_r} \overline{U} + r^{\alpha_0}\|f^+\|_{L^{p\wedge n}(Q_{4r})}\right),$$
$$\left(\int_{Q_r} \underline{U}^{\varepsilon_0} dx\right)^{\frac{1}{\varepsilon_0}} \le Cr^{\frac{n}{\varepsilon_0}} \left(\inf_{Q_r} \underline{U} + r^{\alpha_0}\|f^-\|_{L^{p\wedge n}(Q_{4r})}\right),$$

where $\alpha_0 := 2 - \frac{n}{p\wedge n} \in (0, 1]$. By noting $M_{4r} - m_{4r} = \overline{U} + (\overline{u} - u) + (u - \underline{u}) + \underline{U} \le \overline{U} + 4\omega_0(2\sqrt{n}r) + \underline{U}$, the above inequalities imply

$$M_{4r} - m_{4r} \le C\left(\inf_{Q_r} \overline{U} + \inf_{Q_r} \underline{U} + r^{\alpha_0} + \omega_0(2\sqrt{n}r)\right),$$

which gives a decay estimate of oscillations:

$$M_r - m_r \le \theta_0(M_{4r} - m_{4r}) + r^{\alpha_0} + \omega_0(2\sqrt{n}r).$$

Since $u(x) - u(y) \le \overline{u}(x) - \underline{u}(y)$, it is standard (e.g. in [29]) to obtain equi-continuity of u.

If φ and ψ are Hölder continuous, then the above estimate implies the Hölder continuity with some exponent. □

4.2 $C^{1,\gamma}$ Estimates

Now, assuming that there is $\hat{\gamma} \in (0, 1)$ such that

$$\varphi, \psi \in C^{1,\hat{\gamma}}(\Omega), \tag{4.7}$$

we will suppose (4.3) but $q \ge p > n$ in (v). Under this assumption, we will use the Hölder exponent

$$\gamma_0 := \min\left\{1 - \frac{n}{p}, \hat{\gamma}\right\} \in (0, 1).$$

For simplicity, we will also suppose

$$\varphi < \psi \quad \text{in } \Omega. \tag{4.8}$$

For G in (4.3), we use the notation:

$$\theta(x, y) := \sup_{X \in S^n} \frac{|G(x, 0, 0, X) - G(y, 0, 0, X)|}{1 + \|X\|} \quad \text{for } x, y \in \Omega.$$

Theorem 4.4 *Assume (4.3) replaced by $q \ge p > n$ in (v), (4.7) and (4.8). For any $K \Subset \Omega$, there exist $\hat{C}_K > 0$, $\gamma \in (0, \gamma_0]$, $r_0 \in (0, dist(K, \partial\Omega))$, and $\delta_0 > 0$ such that if $u \in C(\Omega)$ is an L^p viscosity solution of (4.6), and if*

$$r^{-1}\|\theta(y, \cdot)\|_{L^n(B_r(y))} \le \delta_0 \quad \textit{for } r \in (0, r_0) \textit{ and } y \in N_K, \tag{4.9}$$

where by setting $C_K[u] := \{x \in K \mid u(x) = \varphi(x) \text{ or } u(x) = \psi(x)\}$, we define the non-coincidence set by $N_K[u] := \{x \in K \mid dist(x, C_K[u]) > 0\}$, then it follows that

$$|Du(x) - Du(y)| \le \hat{C}_K|x - y|^{\gamma} \quad \textit{for } x, y \in K.$$

Proof Following the argument in the proof of Proposition 5.1 in [42], we can find $\gamma_1 \in (0, 1)$ such that

$$|Du(x) - Du(y)| \le C|x - y|^{\gamma_1} \quad B_r(x) \subset N_K[u] \text{ for some } r > 0. \tag{4.10}$$

In fact, we need some modification of the standard argument in [10] since our PDE contains unbounded ingredients. See Sect. 5.1 in [42] for the details. We only need (4.9) to prove this fact.

We shall show the assertion near the coincidence set. Thus, we shall fix $z \in K$ such that $u(z) = \varphi(z)$. Again, we may suppose $z = 0$ by translation. We will show that

$$|u(x) - u(0) - \langle D\varphi(0), x\rangle| \le Cr^{1+\gamma_0} \quad x \in Q_{\frac{r}{4}},$$

which implies that u is differentiable at 0, $Du(0) = D\varphi(0)$, and moreover,

$$|Du(x) - Du(0)| \le C|x|^{\gamma_0} \quad \text{for } x \in Q_{\frac{r}{4}}.$$

We refer to [1] for its readable proof.

Setting $v := u - \varphi(0) - \langle D\varphi(0), x\rangle + Ar^{1+\hat{\gamma}}$ for large $A > 0$, we claim that v is a nonnegative L^p viscosity supersolution of

$$\mathcal{P}^+(D^2u) + \mu|Du| + c_0 v + g^- = 0 \quad \text{in } Q_{4r},$$

where $g^-(x) := f^-(x) + |D\varphi(0)|\mu(x) + c_0\{\varphi(0) + \langle D\varphi(0), x\rangle\}$. Considering $\hat{v} := v\left(\inf_{Q_r} u + \delta_0^{-1}\|g^-\|_{L^{p\wedge n}(Q_{4r})}\right)^{-1}$, we note that we may apply Proposition 5.7 to find $\varepsilon_0 > 0$ such that

$$\begin{aligned} r^{-\frac{n}{\varepsilon_0}}\|v\|_{L^{\varepsilon_0}(Q_r)} &\le C\left(\inf_{Q_r} v + r^{2-\frac{n}{p}}\|g^-\|_{L^n(Q_{4r})}\right) \\ &\le C(v(0) + r^{2-\frac{n}{p}}) \\ &\le Cr^{1+\gamma_0}. \end{aligned} \tag{4.11}$$

For large $\nu > 1$, it is easy to verify that $w := v \vee (\nu Ar^{1+\hat{\gamma}})$ is an L^p viscosity subsolution of

$$\mathcal{P}^-(D^2u) - \mu|Du| - g^+ = 0 \quad \text{in } Q_{4r},$$

where $g^+ = f^+ + |D\varphi(0)|\mu - c_0\{\varphi(0) + \langle D\varphi(0), x\rangle - Ar^{1+\hat{\gamma}}\}$. In view of Proposition 5.8, we have

$$\sup_{Q_{\frac{r}{4}}} v \le \tilde{C}\left\{r^{-\frac{n}{\varepsilon_0}}\left(\int_{Q_r} w^{\varepsilon_0}dx\right)^{\frac{1}{\varepsilon_0}} + r^{2-\frac{n}{p}}\|f^+ + \mu\|_{L^n(Q_{4r})}\right\},$$

where $\tilde{C} = \tilde{C}(\varepsilon_0) > 0$. Hence, by (4.11), we have

$$v \le Cr^{1+\gamma_0} \quad \text{in } Q_{\frac{r}{4}}.$$

The opposite inequality is trivial because Proposition 4.2 yields

$$u(x) - \varphi(0) - \langle D\varphi(0), x\rangle \geq \varphi(x) - \varphi(0) - \langle D\varphi(0), x\rangle \geq -Cr^{1+\hat{\gamma}} \geq -Cr^{1+\gamma_0}$$

for $|x| \leq r$.

Now, we shall combine two cases to establish the estimate. For $x, y \in N_K$, we may assume $0 < \mathrm{dist}(y, C_K[u]) \leq \mathrm{dist}(x, C_K[u])$. Choose $\hat{x}, \hat{y} \in C_K[u]$ such that

$$|x - \hat{x}| = \mathrm{dist}(x, C_K[u]) \geq |y - \hat{y}| = \mathrm{dist}(y, C_K[u]).$$

Case 1 : $|x - y| < \frac{1}{2}|x - \hat{x}|$. In this case, by (4.10), we have

$$|Du(x) - Du(y)| \leq C|x - y|^{\gamma_1}.$$

Case 2 : $|x - y| \geq \frac{1}{2}|x - \hat{x}| \geq \frac{1}{2}|y - \hat{y}|$. We may suppose that $(u - \varphi)(\hat{x}) = (u - \varphi)(\hat{y})$ (or $(u - \psi)(\hat{x}) = (u - \psi)(\hat{y})$) because $\psi(x) - \varphi(y) \geq \tau_0 > 0$ for $y \in B_r(x) \cap K$ with small $r > 0$.

Thus, due to the above observation, we have

$$\begin{aligned}
&|Du(x) - Du(y)| \\
&\leq |Du(x) - Du(\hat{x})| + |Du(\hat{x}) - Du(\hat{y})| + |Du(\hat{y}) - Du(y)| \\
&\leq C|x - \hat{x}|^{\gamma_0} + |D\varphi(\hat{x}) - D\varphi(\hat{y})| + C|y - \hat{y}|^{\gamma_0} \\
&\leq C|x - y|^{\gamma_0} + C|\hat{x} - \hat{y}|^{\hat{\gamma}} \\
&\leq C|x - y|^{\gamma_0}
\end{aligned}$$

because $|\hat{x} - \hat{y}|^{\hat{\gamma}} \leq |\hat{x} - x|^{\hat{\gamma}} + |x - y|^{\hat{\gamma}} + |y - \hat{y}|^{\hat{\gamma}}$ and $\gamma_0 \leq \hat{\gamma}$. □

Open question 6: What is a sufficient condition to obtain $W^{2,\infty}_{loc}(\Omega)$ or $W^{2,p}_{loc}(\Omega)$ estimates on solutions of Isaacs equations with obstacles?

5 Appendix

In [38, 39], we established the ABP maximum principle and weak Harnack inequality for L^p viscosity solutions only when the PDE does not contain 0th order terms for the sake of simplicity. Since in Sect. 4 we obtain the results assuming (4.3), which allows the PDE to admit 0th order terms, we shall give the ABP maximum principle and weak Harnack inequality for those.

The ABP maximum principle can be proved immediately due to known results.

Proposition 5.1 *Assume $\mu \in L^q(\Omega)$, $f \in L^p(\Omega)$ for $q > n$ and $q \geq p > p_0$. Assume also that $c_0 \in C(\overline{\Omega})$ is nonnegative in $\overline{\Omega}$. Then, there exists a universal constant $C_0 > 0$ (depending on $\|\mu\|_{L^q(\Omega)}$) such that if $u \in C(\overline{\Omega})$ is an L^p viscosity subsolution (resp., supersolution) of*

$$\mathcal{P}^-(D^2u) - \mu(x)|Du| - c_0(x)u^- - f^+(x) = 0 \quad \text{in } \Omega \tag{5.1}$$

$$\left(\mathcal{P}^+(D^2u) + \mu(x)|Du| + c_0(x)u^+ + f^-(x) = 0 \quad \text{in } \Omega\right),$$

then it follows that

$$\max_{\Omega} u \leq \max_{\partial\Omega} u^+ + C_0 d_\Omega^{2-\frac{n}{p}} \|f^+\|_{L^{p\wedge n}(\Omega^+[u])} \tag{5.2}$$

$$\left(\text{resp.,} \quad \min_{\Omega} u \geq -\max_{\partial\Omega} u^- - C_0 d_\Omega^{2-\frac{n}{p}} \|f^-\|_{L^{p\wedge n}(\Omega^-[u])}\right),$$

where $\Omega^\pm[u] := \{x \in \Omega \mid \pm u(x) > \max_{\partial\Omega} u^\pm\}$ *and* $d_\Omega := \sup\{|x-y| \mid x, y \in \Omega\}$.

Proof We shall only show the first assertion. It is immediate to verify that u is an L^p viscosity subsolution of

$$\mathcal{P}^-(D^2u) - \mu(x)|Du| - f^+(x) = 0 \quad \text{in } \Omega^+[u].$$

Hence, we can apply Proposition 2.8 and Theorem 2.9 in [38] to conclude our proof. □

We next show the weak Harnack inequality. We first present a decay of distribution functions of L^p viscosity supersolutions.

Lemma 5.2 (cf. Theorem 2.3 in [41]) *Assume the same hypotheses in Proposition 5.1. There are* $r_0, \delta_0 > 0$ *and* $A \geq 1$ *such that for any nonnegative* L^p *viscosity supersolution of*

$$\mathcal{P}^+(D^2u) + \mu(x)|Du| + c_0(x)u - f(x) = 0 \quad \text{in } Q_4,$$

if $\inf_{Q_1} u \leq 1$ *and* $\|\mu\|_{L^{p\wedge n}(Q_4)} \vee \|f^-\|_{L^{p\wedge n}(Q_4)} \leq \delta_0$, *then we have*

$$|\{x \in Q_1 \mid u(x) > t\}| \leq \frac{A}{t^{r_0}} \quad \text{for } t > 1.$$

Remark 5.3 It is trivial that the conclusion holds true for any $t > 0$ since $A \geq 1$.

Remark 5.4 The assertion is known in [39] when $c_0 \equiv 0$. In fact, in our case, we do not know if the strong maximum principle holds when the coefficient to the first derivative (i.e. μ) is unbounded. Therefore, we will use an auxiliary function φ_0, which is a strong solution of PDE with no first derivative terms. We notice that if we add $\mu|D\varphi_0|$ in the left hand side of (5.3), then we cannot show (5.4) below. We will then have μ in the inhomogeneous term which is small in L^n norm.

Proof In view of Proposition 2.4 in [40] with some modifications as in the proof of Lemma 4.2 in [39], there exists $\varphi_0 \in W^{2,p'}(Q_4 \setminus Q_1) \cap C(Q_4 \setminus Q_1^i)$ for any $p' > n$ such that

$$\begin{cases} \mathcal{P}^-(D^2u) + c_0(x)u = 0 & \text{in } Q_4^i \setminus Q_1, \\ u = 0 & \text{on } \partial Q_4, \\ u = -1 & \text{on } \partial Q_1. \end{cases} \tag{5.3}$$

Since φ is also an $L^{p'}$ viscosity solution of the PDE in the above, if we suppose $\sup_{Q_4 \setminus Q_1} \varphi_0 > 0$ or $\inf_{Q_4 \setminus Q_1} \varphi_0 < -1$, then this contradicts to the definition of $L^{p'}$ viscosity solution. Thus, we have $-1 \le \varphi_0 \le 0$ in $Q_4 \setminus Q_1$.

Furthermore, we claim that there is $\theta_0 > 0$ such that

$$\varphi_0 \le -\theta_0 \quad \text{in } Q_3 \setminus Q_1. \tag{5.4}$$

Although the proof of (5.4) is known in [33] for instance, we will give a proof of this claim for the reader's convenience in the end.

Extending φ_0 appropriately in Q_1, for large $\lambda > 1$, we may suppose that $\varphi := \lambda\varphi_0 \in W^{2,p'}(Q_4)$ is an $L^{p'}$ strong solution of

$$\mathcal{P}^-(D^2u) + c_0u = \xi \quad \text{in } Q_4$$

such that $\varphi \le -2$ in Q_3, where $\xi \in L^q(Q_1)$ satisfies $\xi = 0$ in $Q_4 \setminus Q_1$.

We observe that $w := u + \varphi$ is an L^p viscosity supersolution of

$$\mathcal{P}^+(D^2w) + \mu|Dw| + c_0w^+ = -\mu|D\varphi| - f^- + \xi \quad \text{in } Q_4.$$

Hence, setting $\Omega := \{x \in Q_4^i \mid w(x) < 0\}$, by Proposition 5.1, we have

$$\begin{aligned} -1 \ge \inf_{Q_1} w \ge \inf_{Q_4} w = \inf_{\Omega} w \\ &\ge -C\|\mu|D\varphi| + f^- - \xi\|_{L^{p\wedge n}(\Omega)} \\ &\ge -C\left(\delta_0 + |\{x \in Q_1 \mid w(x) < 0\}|^{\frac{1}{p\wedge n}}\right). \end{aligned}$$

Therefore, for a fixed $\delta_0 > 0$, we can find $\theta_1 \in (0, 1)$ such that

$$\theta_1 \le |\{x \in Q_1 \mid u(x) \le M\}|,$$

where $M := \max_{Q_4}(-\varphi) > 1$. It is now standard by an induction argument to see that

$$|\{x \in Q_1 \mid u(x) > M^k\}| \le (1-\theta_1)^k \quad k \in \mathbb{N},$$

which implies the decay of distribution function of u. Therefore, we conclude the assertion by the standard argument. See [39] for the details.

Proof of claim (5.4) (cf. Theorem 1 in [33]) It is enough to show that $\varphi_0(x) < 0$ for $x \in Q_4^i \setminus Q_1$. Setting $K_0 := \{x \in Q_4^i \setminus Q_1 \mid \varphi_0(x) = 0\}$, we may suppose $K_0 \ne \emptyset$. We can choose $R > 0$, $z \in K_0$ and $\hat{z} \in \Omega_0 := (Q_4^i \setminus Q_1) \cap K_0^c$ such that

$$\overline{B}_R(\hat{z}) \setminus \{z\} \subset \Omega_0, \quad \text{and} \quad \partial B_R(\hat{z}) \cap K_0 = \{z\}.$$

Setting an open annulus $A_0 := \{x \in \mathbb{R}^n \mid R/2 < |x - \hat{z}| < R\}$, we introduce $\zeta(x) := \varepsilon(e^{-\beta R^2/2} - e^{-\beta|x-\hat{z}|^2/2}) \leq 0$, where $\beta > 1$ and $\varepsilon \in (0, 1)$ will be chosen later. Furthermore, we have

$$M_1 := \max_{x \in \overline{A}_0} (\varphi_0 - \zeta)(x) \geq (\varphi_0 - \zeta)(z) = 0.$$

We also note that $(\varphi_0 - \zeta)(x) < 0$ if $x \in \partial B_R(\hat{z}) \setminus \{z\}$. Now, setting $\theta_0 := \min_{x \in \partial B_{R/2}(\hat{z})}(-\varphi_0(x)) > 0$ and $\varepsilon := \theta_0/2$, we observe that

$$\max_{x \in \partial B_{R/2}(\hat{z})} (\varphi_0 - \zeta)(x) \leq -\theta_0 + \varepsilon e^{-\frac{\beta R^2}{8}} \leq -\frac{\theta_0}{2} < 0.$$

Next, assume that $\varphi_0 - \zeta$ attains its maximum at $\hat{x} \in A_0$. Since φ_0 is a viscosity subsolution of

$$\mathcal{P}^-(D^2 u) + c_0 u = 0 \quad \text{in } Q_4^i \setminus Q_1,$$

we have

$$0 \geq e^{-\frac{\beta|\hat{x}-\hat{z}|^2}{2}} \left\{\beta \mathcal{P}^-(I - \beta(\hat{x} - \hat{z}) \otimes (\hat{x} - \hat{z}))\right\} + c_0(\hat{x})\varphi_0(\hat{x}).$$

Following an argument in p. 20 of [12], since $\mathcal{P}^-(I - \beta(\hat{x} - \hat{z}) \otimes (\hat{x} - \hat{z})) \geq -\frac{n-1}{\theta} + \left(\frac{\beta R^2}{4} - 1\right)\theta \geq 1$ provided $\beta \geq \beta_0$ for some $\beta_0 > 1$, we have

$$0 \geq e^{-\frac{\beta|\hat{x}-\hat{z}|^2}{2}} (\beta - c_0(\hat{x})),$$

which yields a contradiction when $\beta > \beta_0 + \max_{x \in \overline{\Omega}} c_0$. Therefore, because $(\varphi_0 - \zeta)(z - he) \leq (\varphi_0 - \zeta)(z) = 0$ for small $h > 0$, where $e := (z - \hat{z})/|z - \hat{z}|$, we have

$$\frac{\varphi_0(z - he) - \varphi_0(z)}{-h} \geq \varepsilon \frac{e^{-\frac{\beta|z-he-\hat{z}|^2}{2}} - e^{-\frac{\beta|z-\hat{z}|^2}{2}}}{h}.$$

Sending $h \to 0+$, we have $\langle D\varphi_0(z), e\rangle = 0 \geq \varepsilon e^{-\frac{\beta R^2}{2}} \beta R > 0$, which is a contradiction. Hence, we have $K_0 = \emptyset$. □

Remark 5.5 It is possible to give precise functions φ_0 by considering larger ball $B_{2\sqrt{n}} \supset Q_4$. See [30] for such a function.

Remark 5.6 Concerning the strong maximum principle for PDE of divergence type with 0th order terms, we refer to [51] and references therein.

Now, we present our weak Harnack inequality.

Proposition 5.7 (cf. Theorem 3.1 in [39]) *Assume the same hypotheses in Proposition 5.1. There are $\varepsilon_0 > 0$, $\delta_0 > 0$ and $\hat{C} > 0$ such that for any nonnegative L^p viscosity supersolution of*

$$\mathcal{P}^+(D^2u) + \mu|Du| + c_0 u - f = 0 \quad \text{in } Q_4,$$

if $\|\mu\|_{L^{p\wedge n}(Q_4)} \le \delta_0$, then we have

$$\left(\int_{Q_1} u^{\varepsilon_0} dx\right)^{\frac{1}{\varepsilon_0}} \le \hat{C}\left(\inf_{Q_1} u + \|f^-\|_{L^{p\wedge n}(Q_4)}\right).$$

Proof In place of u, considering

$$V := \frac{u}{\inf_{Q_1} u + \delta_0^{-1}\|f^-\|_{L^{p\wedge n}(Q_4)} + \varepsilon},$$

where $\varepsilon > 0$ will be sent to 0 in the end, and $\delta_0 > 0$ will be fixed later, we may suppose $\|f^-\|_{L^{p\wedge n}(Q_4)} \le \delta_0$ and $\inf_{Q_1} u \le 1$.

In view of Lemma 5.2, we easily verify that for any $\varepsilon_0 \in (0, r_0)$, there is $\hat{C} = \hat{C}(\varepsilon_0) > 0$ such that

$$\left(\int_{Q_1} V^{\varepsilon_0} dx\right)^{\frac{1}{\varepsilon_0}} \le \hat{C},$$

which implies the conclusion by sending $\varepsilon \to 0$. □

In order to establish the Harnack inequality, we combine the weak Harnack inequality with the next local maximum principle.

Proposition 5.8 (Theorem 3.1 in *[41]*) *Assume the same hypotheses in Proposition 5.1. For any $\varepsilon > 0$, there is $\hat{C}_\varepsilon > 0$ such that for any L^p viscosity subsolution of*

$$\mathcal{P}^-(D^2u) - \mu|Du| - c_0 u^- - f = 0 \quad \text{in } Q_4, \tag{5.5}$$

we have

$$\sup_{Q_{\frac{1}{4}}} u \le \hat{C}_\varepsilon \left\{\left(\int_{Q_1} (u^+)^\varepsilon dx\right)^{\frac{1}{\varepsilon}} + \|f^+\|_{L^p(Q_4)}\right\}.$$

Since we have unbounded coefficient μ, we cannot use the standard argument as in [29]. We follow the idea of the proof of Lemma 4.4 in [12] with some modifications. We first prepare the following lemma:

Lemma 5.9 (*cf. Theorem 2.3 in [41]*) *For $q \ge p > p_0$ and $q > n$, let $f \in L^p(Q_4)$ and $\mu \in L^q(Q_4)$ be nonnegative. Assume that $u \in C(Q_4)$ is an L^p viscosity subsolution of* (5.5) *satisfying*

$$|\{x \in Q_1 \mid u(x) \geq t\}| \leq \frac{A}{t^{r_0}} \quad \textit{for } \forall t > 1, \tag{5.6}$$

where the constants $A \geq 1$ *and* $r_0 > 0$ *are from Lemma 5.2. Then, there are an integer* J, $\nu > 1$ *and* $\ell_j > 0$ $(j \geq J)$ *such that* $\sum_{j=J}^{\infty} \ell_j < \infty$, *and if* $u(x_0) \geq \nu^{j-1}$ *for* $j \geq J$ *and* $x_0 \in Q_{\frac{1}{2}}$, *then* $sup_{Q_{\ell_j}(x_0)} u \geq \nu^j$.

Proof We will fix $\nu > 1$, $J \in \mathbb{N}$ and $\ell_j \in (0, 1)$ for $j \geq J$. Suppose

$$\sup_{Q_{\ell_j}(x_0)} u \leq \nu^j,$$

then we will obtain a contradiction.

Setting $x = x_0 + \frac{\ell_j}{4} y$ for $y \in Q_4$, we define

$$v(y) := \alpha \left(1 - \frac{1}{\nu^j} u(x_0 + 4^{-1} \ell_j y) \right),$$

where $\alpha := \nu(\nu - 1)^{-1}$ (or $\nu = \alpha(\alpha - 1)^{-1}$). Thus, we immediately verify that $v \geq 0$ in Q_4, and $\inf_{Q_3} v \leq v(0) \leq \alpha(1 - \nu^{-1}) = 1$.

We next set

$$\alpha := 2(2A)^{\frac{1}{r_0}} > 1 \quad \left(i.e.\ \nu = 2(2A)^{\frac{1}{r_0}} \{2(2A)^{\frac{1}{r_0}} - 1\}^{-1} > 1 \right),$$

and

$$\ell_j := \left(\frac{2^{2n+2r_0+1} A}{\nu^{j r_0}} \right)^{\frac{1}{n}}.$$

Choose $J_0 \in \mathbb{N}$ such that

$$\alpha < (2^{2n+2r_0+1} A)^{\frac{1}{r_0}} < \nu^{J_0}.$$

Notice that $\ell_j < 1$ for $j \geq J_0$. We next choose $J_1 \geq J_0$ such that

$$\frac{\alpha}{\nu^j} \left(\frac{\ell_j}{4} \right)^{2 - \frac{n}{p \wedge n}} < 1 \quad \text{for } j \geq J_1.$$

We then see that v is a nonnegative L^p viscosity supersolution of

$$\mathcal{P}^+(D^2 u) + \hat{\mu}|Du| + \hat{c}_0 u + \hat{f} = 0 \quad \text{in } Q_4,$$

where

$$\hat{\mu}(y) = \frac{\ell_j}{4} \mu \left(x_0 + \frac{\ell_j}{4} y \right), \ \hat{c}_0 = \frac{\ell_j^2}{16} \nu^j c_0 \text{ and } \hat{f}(y) = \frac{\alpha \ell_j^2}{16 \nu^j} f^+ \left(x_0 + \frac{\ell_j}{4} y \right).$$

Because of our choice of $\alpha > 1$, ℓ_j and $J_1 \in \mathbb{N}$, we verify that for $j \geq J_1$,

$$\|\hat{\mu}\|_{L^{p\wedge n}(Q_4)} = \left(\frac{\ell_j}{4}\right)^{1-\frac{n}{p\wedge n}} \|\mu\|_{L^{p\wedge n}(Q_{\ell_j}(x_0))} \leq \left(\frac{\ell_j}{4}\right)^{1-\frac{p\wedge n}{q}} \|\mu\|_{L^q(Q_4)},$$

and

$$\|\hat{f}\|_{L^{p\wedge n}(Q_4)} = \frac{\alpha}{\nu^j}\left(\frac{\ell_j}{4}\right)^{2-\frac{n}{n\wedge p}} \|f^+\|_{L^{p\wedge n}(Q_{\ell_j}(x_0))} \leq \|f^+\|_{L^{p\wedge n}(Q_{\ell_j}(x_0))}.$$

Finally, we choose $J_2 \geq J_1$ such that $\|\hat{\mu}\|_{L^{p\wedge n}(Q_4)} \leq \delta_0$, where $\delta_0 > 0$ is the constant in Lemma 5.2.

In view of Lemma 5.2, we have

$$|\{x \in Q_1 \mid v(x) > \alpha/2\}| \leq A\left(\frac{2}{\alpha}\right)^{r_0},$$

which yields

$$\left|\left\{x \in Q_{\frac{\ell_j}{4}}(x_0) \,\middle|\, u(x) < \frac{\nu^j}{2}\right\}\right| \leq A\left(\frac{2}{\alpha}\right)^{r_0}\left(\frac{\ell_j}{4}\right)^n \leq \frac{1}{2}\left(\frac{\ell_j}{4}\right)^n.$$

However, (5.6) implies

$$\left|\left\{x \in Q_{\frac{\ell_j}{4}}(x_0) \,\middle|\, u(x) \geq \frac{\nu^j}{2}\right\}\right| \leq \left|\left\{x \in Q_1 \,\middle|\, u(x) \geq \frac{\nu^j}{2}\right\}\right| \leq A\left(\frac{2}{\nu^j}\right)^{r_0}.$$

Hence, we have

$$\frac{\ell_j^n}{2^{2n+1}} \leq A\left(\frac{2}{\nu^j}\right)^{r_0},$$

which implies a contradiction to the definition of ℓ_j. □

Proof of Proposition 5.8. We first consider the case of $\varepsilon = r_0$, where $r_0 > 0$ is the constant from Lemma 5.2.

Choose $z \in Q_{\frac{1}{4}}$ such that $u(z) = \max_{Q_{\frac{1}{4}}} u$. Setting $v(y) := u(z + sy)$ for $s > 0$, we observe that v is an L^p viscosity subsolution of

$$\mathcal{P}^-(D^2u) - \hat{\mu}|Du| - c_0u^- - \hat{f} = 0 \quad \text{in } Q_4,$$

where $\hat{\mu}(y) := s\mu(z + sy)$ and $\hat{f}(y) := s^2 f^+(z + sy)$.

Since we may suppose $\int_{Q_1}(v^+)^{r_0}dy > 0$, by setting

$$w(y) := v(y)\left\{A^{-\frac{1}{r_0}}\left(\int_{Q_1}(v^+)^{r_0}dy\right)^{\frac{1}{r_0}} + \delta_0^{-1}\|\hat{f}\|_{L^{p\wedge n}(Q_4)}\right\}^{-1},$$

it is immediate to see that

$$|\{y \in Q_1 \mid w(y) \geq t\}| \leq \frac{1}{t^{r_0}}\int_{Q_1} w^{r_0} \leq \frac{A}{t^{r_0}}.$$

Furthermore, we verify that w is an L^p viscosity subsolution of

$$\mathcal{P}^-(D^2u) - \mu|Du| - g = 0 \quad \text{in } Q_4,$$

where $g(y) := \delta_0\hat{f}(y)\|\hat{f}\|^{-1}_{L^{p\wedge n}(Q_4)}$.

Let $\nu > 1$, $J \in \mathbb{N}$ and $\ell_j \in (0, 1)$ be from Lemma 5.9. There is $\hat{J} \geq J$ such that

$$\sum_{j=\hat{J}}^{\infty} \ell_j \leq \frac{1}{8}.$$

We claim that $\sup_{Q_{\frac{1}{4}}} w \leq \nu^{\hat{J}-1}$. Indeed, if $w(x_0) \geq \nu^{\hat{J}-1}$ for some $x_0 \in Q_{\frac{1}{4}}$, then thanks to Lemma 5.9, we can choose $x_j \in Q_{\ell_j}(x_0)$ (for $j \geq \hat{J}$) such that

$$w(x_j) \geq \nu^j.$$

Since $x_j \in Q_{\frac{1}{2}}$ for $j \geq \hat{J}$, this contradicts to the continuity of $w \in C(Q_4)$. Hence, we have

$$\begin{aligned}\sup_{Q_{\frac{1}{4}}} u \leq \sup_{Q_{\frac{1}{4}}} v &\leq C\left\{\left(\int_{Q_1}(v^+)^{r_0}dx\right)^{\frac{1}{r_0}} + \|\hat{f}\|_{L^{p\wedge n}(Q_4)}\right\} \\ &\leq C\left\{\left(\int_{Q_1}(u^+)^{r_0}dx\right)^{\frac{1}{r_0}} + \|f\|_{L^{p\wedge n}(Q_4)}\right\}.\end{aligned}$$

In case when $\varepsilon > r_0$, instead of the above w, consider

$$\hat{w}(y) := v(y)\left\{A^{-\frac{1}{\varepsilon}}\left(\int_{Q_1}(v^+)^{\varepsilon}dy\right)^{\frac{1}{\varepsilon}} + \delta_0^{-1}\|\hat{f}\|_{L^{p\wedge n}(Q_4)}\right\}^{-1}.$$

Thus, we have

$$|\{y \in Q_1 \mid \hat{w}(y) \geq t\}| \leq \frac{A}{t^{\varepsilon}} \leq \frac{A}{t^{r_0}} \quad \text{for } t > 1.$$

Therefore, Lemma 5.9 implies the conclusion.

On the other hand, if $0 < \varepsilon < r_0$, then considering

$$\tilde{w}(y) := v(y)\left\{A^{-\frac{1}{r_0}}\left(\int_{Q_1}(v^+)^\varepsilon dy\right)^{\frac{1}{\varepsilon}} + \delta_0^{-1}\|\hat{f}\|_{L^{p\wedge n}(Q_4)}\right\}^{-1},$$

we have

$$|\{y \in Q_1 \mid \tilde{w}(y) \geq t\}| \leq \frac{A}{t^{r_0}}\int_{Q_1}(v^+)^{r_0}dy\left(\int_{Q_1}(v^+)^\varepsilon dy\right)^{-\frac{r_0}{\varepsilon}} \leq \frac{A}{t^{r_0}} \quad \text{for } t > 1.$$

Hence, Lemma 5.9 concludes the proof in this case. □

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