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Nonlinear Partial Differential Equations for Future Applications

Sendai, Japan, July 10–28 and October 2–6, 2017

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Preface

The proceedings contain several original surveys by invited speakers in a series of workshops entitled Partial Differential Equation and Future Applications, which was organized by Tohoku Forum for Creativity (TFC for short) supported in Tohoku University from July 2017 to October 2017, and also some research papers in related fields. The TFC program has started since 2013 and is evolving over various research fields on natural and human sciences. The above title *Nonlinear Partial Differential Equations for Future Applications* is one of the thematic programs in TFC. In our program, we focussed on nonlinear partial differential equations arising in fluid mechanics, reaction diffusion, optimal control, modern physics, material sciences, and geometry. Furthermore, in order to search for new applications, we invited experts from other areas.

Our program consists of the following workshops:

July 10–14, 2017	Evolution Equations and Mathematical Fluid Dynamics
July 17–21, 2017	Optimal Control and PDE
July 24–28, 2017	Hyperbolic and Dispersive PDE
October 2–6, 2017	Geometry and Inverse Problems*
	* in cooperation with A3 Foresight Program

The aim of this series of workshops was to introduce new and active fields of nonlinear partial differential equations (PDE for short) to young researchers, and moreover, to discover possibilities to connect related sciences with mathematics.

The purpose to publish these proceedings is, in addition, to enable the interested researchers to know valuable surveys with more detailed explanations. Moreover, we have decided to add several original papers which will be important contributions to future researches.

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Contents

An Introduction to Maximal Regularity for Parabolic Evolution Equations

Robert Denk

Abstract In this note, we give an introduction to the concept of maximal L^p-regularity as a method to solve nonlinear partial differential equations. We first define maximal regularity for autonomous and non-autonomous problems and describe the connection to Fourier multipliers and $\mathcal{R}$-boundedness. The abstract results are applied to a large class of parabolic systems in the whole space and to general parabolic boundary value problems. For this, both the construction of solution operators for boundary value problems and a characterization of trace spaces of Sobolev spaces are discussed. For the nonlinear equation, we obtain local in time well-posedness in appropriately chosen Sobolev spaces. This manuscript is based on known results and consists of an extended version of lecture notes on this topic.

Keywords Maximal regularity · Fourier multipliers · Parabolic boundary value problems · Quasilinear evolution equations

Mathematics Subject Classification Primary 35-02 · 35K90 · Secondary 42B35 · 35B65

1 Introduction

In this survey, we give an introduction to the method of maximal L^p-regularity which has turned out to be useful for the analysis of nonlinear (in particular, quasilinear) partial differential equations. The aim of this note is to present an overview on the main ideas and tools for this approach. Therefore, we are not trying to present the state of the art but restrict ourselves to relatively simple situations. At the same time,

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we focus on the mathematical presentation and not on the historical development of this successful branch of analysis. So we do not give detailed bibliographical remarks but refer to some nowadays standard literature, where more details on the history and on the bibliography can be found. This survey could serve as a basis for an advanced lecture course in partial differential equations, for instance for Ph.D. students. In fact, the present paper is based on a series of lectures given in July 2017 at the Tohoku University in Sendai, Japan, and on an advanced course for master students at the University of Konstanz, Germany, in the summer term 2019.

Although the concept of maximal regularity is classical, some main achievements for the abstract theory were obtained in the 1990s and in the first decade of the present century by, e.g., Amann (see [5, 6]) and Prüss (see [29]). The basic idea of maximal regularity is to solve nonlinear partial differential equations by a linearization approach. Let us consider an abstract quasilinear equation of the form

$$\begin{aligned} \partial_t u(t) - A(u(t))u(t) &= F(u(t)), \\ u(0) &= u_0. \end{aligned} \tag{1.1}$$

The linearization of (1.1) at some fixed function u is given by

$$\begin{aligned} \partial_t v(t) - A(u(t))v(t) &= F(u(t)), \\ v(0) &= u_0. \end{aligned} \tag{1.2}$$

In the maximal regularity approach, one tries to solve the linear equation in appropriate function spaces and to show that the solution has the optimal regularity one could expect. In this case, let $v =: S_u(F(u), u_0)$ denote the (u-dependent) solution operator of the linear Eq. (1.2). If S_u induces an isomorphism between appropriately chosen pairs of Banach spaces, then the solvability of the nonlinear equation (1.1) can be reduced to a fixed-point equation of the form $u = S_u(F(u), u_0)$. In many situations, the contraction mapping principle can be applied to obtain a unique solution of the fixed point equation and, consequently, of the nonlinear equation (1.1). In this way, typically short-time existence or existence for small data can be shown. For the long-time asymptotics and the stability of the solution, different methods have to be used. Here, we mention the monograph by Prüss and Simonett [30], which covers the abstract theory of maximal regularity, stability results, and many examples in fluid mechanics and geometry.

As mentioned above, one key ingredient in the maximal regularity approach is the choice of appropriate function spaces for the right-hand sides and the solution of the nonlinear equation. In the present note, we restrict ourselves to the L^p-setting, where the basic spaces are L^p-Sobolev spaces. (For maximal regularity in Hölder spaces, we mention the monograph by Lunardi [26].) Maximal L^p-regularity is closely related to the question of Fourier multipliers, as we will see in Sect. 3 below. Therefore, it was a breakthrough for the application of this concept, when an equivalent description for maximal regularity in terms of vector-valued Fourier multipliers and $\mathcal{R}$-sectoriality was found by Weis [34] in the year 2001.

The description of maximal L^p-regularity by $\mathcal{R}$-boundedness made it possible to show that a large class of parabolic boundary value problems have this property. As standard references for $\mathcal{R}$-boundedness and applications to partial differential operators, we mention [13] and [25]. For boundary value problems, also the question of appropriate function spaces on the boundary appears, which leads to the characterization of trace spaces. Here the trace can be taken with respect to time (for the initial value at time 0) or with respect to the space variable (for inhomogeneous boundary data). It turns out that the theory of trace spaces is highly nontrivial and connected with interpolation properties of intersections of Sobolev spaces. In this way, modern theory of vector-valued Sobolev spaces with non-integer order of differentiability enters. Results on trace spaces can be found, e.g., in [14], for a survey on vector-valued Sobolev spaces we refer to [7] and [23].

The plan of the present survey follows the topics just mentioned. In Sect. 2, we state the idea and the formal definition of maximal regularity, mentioning the graphical mean curvature flow as a prototype example. The connection to vector-valued Fourier multipliers and $\mathcal{R}$-boundedness is given in Sect. 3. In Sect. 4, we briefly summarize the main definitions of the different types of (non-integer) Sobolev spaces and give some key references. The application of the abstract concept to parabolic partial differential equations in the whole space is given in Sect. 5, the application to parabolic boundary value problems in Sect. 6. Finally, we return to nonlinear evolution equations in Sect. 7, where local well-posedness and higher regularity for the solution are discussed.

There are, of course, many topics in the context of maximal L^p-regularity which are not covered here. First, we want to mention the application of maximal regularity to stochastic partial differential equations, which leads to the notion of stochastic maximal regularity. Here, the class of radonifying operators plays an important role. A survey on stochastic maximal regularity can be found, e.g., in [33], for random sums and radonifying operators see also [24]. Another development that could be mentioned is the maximal L^p-regularity approach for boundary value problems which are not parabolic in a classical sense (as defined in Sects. 5 and 6 below). Some main applications are free boundary value problems from fluid mechanics or problems describing phase transitions like the Stefan problem. Here, the related symbols are not quasi-homogeneous, and the theory described below cannot be applied. One concept to show maximal L^p-regularity for such problems uses the Newton polygon, and we refer to [16] for more details.

2 Maximal Regularity and L^p-Sobolev Spaces

2.1 Linearization and Maximal Regularity

We start with an example of a quasilinear parabolic equation.

Example 2.1 (Graphical mean curvature flow) Let $T_0 \in (0, \infty]$, let M denote an n-dimensional parameter space, and let $X(t, \cdot)\colon M \to \mathbb{R}^{n+1}$, $t \in [0, T_0)$, be a family of regular maps. Here, regular means that the Jacobian $D_x X(t, x)$ with respect to $x \in M$ is injective for all $x \in M$ and $t \in [0, T_0)$. We set $M_t := X(t, M)$. Then the vectors $\partial_{x_1} X(t, x), \dots, \partial_{x_n} X(t, x)$ form a basis for the tangent space $T_x M_t$ at the point $X(t, x)$. In particular, we are interested in the graphical situation where $M = \mathbb{R}^n$ (or some domain in $\mathbb{R}^n$) and where X is given as the graph of some function $u\colon [0, T_0) \times \mathbb{R}^n \to \mathbb{R}$, so we have $X(t, x) = (x, u(t, x))$ for $x \in \mathbb{R}^n$ and $t \in [0, T_0)$.

Let $\nu\colon [0, T_0) \times M \to \mathbb{R}^{n+1}$ be one choice of the normal vector to M_t, so $\nu(t, x)$ is a unit vector which is orthogonal to the tangent space $T_x M$. For each $j = 1, \dots, n$, the vector $\partial_{x_j} \nu(t, x)$ is an element of $T_x M_t$, and therefore we can write

$$\partial_{x_j} \nu(t, x) = \sum_{i=1}^{n} S_{ij}(t, x) \partial_{x_i} X(t, x).$$

The matrix $S(t, x) := (S_{ij}(t, x))_{i,j=1,\dots,n}$ is called the shape operator at the point $X(t, x)$, its eigenvalues are called the principal curvatures, and its trace $H(t, x) := \operatorname{tr} S(t, x)$ is called the mean curvature.

The family of hypersurfaces $(M_t)_{t\in[0,T_0)}$ is said to move according to the mean curvature flow (see, e.g., [11] for a survey) if

$$\partial_t X(t, x) \cdot \nu(t, x) = -H(t, x)\nu(t, x) \quad ((t, x) \in [0, T_0) \times M^n).$$

In the graphical situation, one choice of the normal vector is given by

$$\nu(t, x) = \frac{1}{\sqrt{1 + |\nabla u|^2}} \begin{pmatrix} -\nabla u(t, x) \\ 1 \end{pmatrix}.$$

From this, we obtain for the mean curvature

$$H(t, x) = -\operatorname{div} \left(\frac{\nabla u(t, x)}{\sqrt{1 + |\nabla u|^2}} \right),$$

and the equation for the graphical mean curvature flow is given by

$$\partial_t u - \left(\Delta u - \sum_{i,j=1}^{n} \frac{\partial_i u \partial_j u}{1+|\nabla u|^2} \partial_i \partial_j u \right) = 0 \quad \text{in } (0, T_0), \qquad u(0) = u_0. \tag{2.1}$$

Here, u_0 is the initial value at time $t = 0$, so M_0 is given as $X(0, \mathbb{R}^n)$ with $X(0, x) = (x, u_0(x))$. As the coefficients of the second derivatives of u depend on u itself, this is an example of a quasilinear parabolic equation.

The above example can be written in the abstract form

$$\begin{aligned} \partial_t u + F(u)u &= G(u), \\ u(0) &= u_0, \end{aligned} \tag{2.2}$$

where $F(u)$ is a linear operator depending on u and $G(u)$ (which equals zero in the example) is, in general, some nonlinear function depending on u. For the linearization of (2.2), we fix some function u and are looking for a solution of the Cauchy problem

$$\begin{aligned} \partial_t v + F(u)v &= G(u), \\ v(0) &= u_0. \end{aligned} \tag{2.3}$$

Note that (2.3) is a linear equation with respect to v, and therefore it can be treated with methods from linear operator theory and semigroup theory. In general, (2.3) is a non-autonomous problem, as u and therefore also $F(u)$ still depend on time. Setting $A(t) := F(u(t))$ and $f(t) := G(u(t))$, we obtain

$$\begin{aligned} \partial_t v(t) - A(t)v &= f(t) \quad (t > 0), \\ v(0) &= u_0. \end{aligned} \tag{2.4}$$

The idea of maximal regularity consists in showing "optimal" regularity for the linearized equation. Roughly speaking, one should not loose any regularity when solving the linear equation, as the solution will be inserted into the equation in the next step of some iteration process. Considering (2.4) in an operator theoretic sense, we want to have good mapping properties of the solution operator who maps the right-hand side data f and u_0 to the solution v. For this, we have to fix function spaces for the right-hand side and the solution. So we have to choose the basic space $\mathbb{F}$ for the right-hand side f and a solution space $\mathbb{E}$ for v. The choice of the space $\gamma_t\mathbb{E}$ for the initial value u_0 will then be canonical, see below.

In case of maximal regularity, we expect a unique solution of (2.4) and a continuous solution operator S_u (depending on $A(t)$ and therefore on u)

$$S_u : \mathbb{F} \times \gamma_t\mathbb{E} \to \mathbb{E}, \ (f, u_0) \mapsto v$$

of the linear equation (2.4). Then the nonlinear Cauchy problem is uniquely solvable if and only if the fixed point equation

$$u = S_u(G(u), u_0)$$

has a unique solution $u \in \mathbb{E}$.

In many cases, one can show that the right-hand side of this fixed point equation defines a contraction, and therefore Banach's fixed point theorem (contraction mapping principle) gives a unique solution. To obtain the contraction property, one usually has to choose a small time interval or small initial data u_0. Typical applications for this method are

- the graphical mean curvature flow or more general geometric equations,
- Stefan problems describing phase transitions with a free boundary,
- Cahn-Hilliard equations,
- variants of the Navier-Stokes equation.

For a survey on the idea of maximal regularity and on the above applications, we mention the monographs [5, 29, 30].

The notion of maximal regularity depends on the function spaces in which the equation is considered. Typical function spaces for partial differential equations are Hölder spaces and L^p-Sobolev spaces. In the present survey, we restrict ourselves to L^p-Sobolev spaces, i.e., we are considering maximal L^p-regularity. Here, the basic function space for the right-hand side of (2.4) will be $f \in L^p((0,T); X)$, where X is some Banach space. In the L^p-setting, one will typically choose $X = L^p(G)$ for some domain $G \subset \mathbb{R}^n$. The aim is to show that the operator $A(t) := F(u(t))$ has, for every fixed u, maximal regularity in the sense specified below.

2.2 *Definition of Maximal L^p-Regularity*

We start with the notion of maximal L^p-regularity in the autonomous setting, i.e. for an operator A independent of t. Let X be a Banach space, and let $A\colon X \supset D(A) \to X$ be a closed and densely defined linear operator. Let $J = (0,T)$ with $T \in (0,\infty]$. We consider the initial value problem

$$\partial_t u(t) - Au(t) = f(t) \quad (t \in J), \tag{2.5}$$

$$u(0) = u_0. \tag{2.6}$$

Here, the right-hand side of (2.5) belongs to $\mathbb{F} := L^p(J; X)$. For optimal regularity, we will expect $\partial_t u \in L^p(J; X)$ and (consequently) $Au \in L^p(J; X)$. An even stronger assumption would include $u \in L^p(J; X)$, too, so that the "optimal" space for the solution u is given by

$$\mathbb{E} := W_p^1(J; X) \cap L^p(J; D(A)). \tag{2.7}$$

Here, for $k \in \mathbb{N}_0$ the vector-valued Sobolev space $W_p^k(J; X)$ is defined as the space of all X-valued distributions u for which $\partial^\alpha u \in L^p(J; X)$ for all $|\alpha| \le k$, see Sect. 4 (cf. also [23], Sect. 2.5).

For the initial value u_0, we define the trace space:

Definition 2.2 (a) The trace space $\gamma_t\mathbb{E}$ is defined by $\gamma_t\mathbb{E} := \{\gamma_t u : u \in \mathbb{E}\}$, where $\gamma_t u := u|_{t=0}$ stands for the time trace of the function u at time $t = 0$. We endow $\gamma_t\mathbb{E}$ with its canonical norm

$$\|x\|_{\gamma_t\mathbb{E}} := \inf\{\|u\|_{\mathbb{E}} : u \in \mathbb{E},\ \gamma_t u = x\}.$$

(b) We set ${}_0\mathbb{E} := \{u \in \mathbb{E} : \gamma_t u = 0\}$ for the space of all functions in $\mathbb{E}$ with vanishing time trace at $t = 0$.

Remark 2.3 (a) Note in the above definition that, by Sobolev's embedding theorem, one has the continuous embedding

$$W_p^1((0,T); X) \subset C([0,T], X)$$

for every finite T, where the right-hand side stands for the space of continuous X-valued functions. Therefore, the value $\gamma_t u = u(0)$ is well defined as an element of X for every $u \in \mathbb{E}$.

(b) Let $T \in (0, \infty)$ again. By (a), we obtain for $x \in \gamma_t\mathbb{E}$ and for every $u \in \mathbb{E}$ with $\gamma_t u = x$,

$$\|x\|_X = \|\gamma_t u\|_X \le \max_{t\in[0,T]} \|u(t)\|_X \le C\|u\|_{W_p^1(J;X)} \le C\|u\|_{\mathbb{E}}.$$

Therefore, $\gamma_t\mathbb{E} \subset X$ with continuous embedding. On the other hand, if $x \in D(A)$, then the function $u(t) := e^{-t}x$ belongs to $\mathbb{E}$ with $\|u\|_{\mathbb{E}} \le C\|x\|_X$ and satisfies $\gamma_t u = x$. Therefore, also the continuous embedding $D(A) \subset \gamma_t\mathbb{E}$ holds.

The following result is a deep result in the theory of interpolation of Banach spaces. Here, the real interpolation functor $(\cdot,\cdot)_{\theta,p}$ appears. We refer to [27, 32] for an introduction and survey on interpolation spaces.

Lemma 2.4 *Let A be a closed and densely defined operator, and let $\mathbb{E}$ be defined by* (2.7).

(a) The trace space $\gamma_t\mathbb{E}$ coincides with the real interpolation space with parameters $1 - \frac{1}{p}$ and p, i.e., we have

$$\gamma_t\mathbb{E} = (X, D(A))_{1-1/p,p}$$

in the sense of equivalent norms.

(b) We have the continuous embedding $\mathbb{E} \subset C([0,T]; \gamma_t\mathbb{E})$. In particular, the time trace $\gamma_t\colon \mathbb{E} \to \gamma_t\mathbb{E},\ u \mapsto u(0)$ is well defined, and $\gamma_t\mathbb{E}$ is independent of T.

(c) The norm of the continuous embedding $\mathbb{E} \subset C([0,T]; \gamma_t\mathbb{E})$ *depends, in general, on* T *and grows for decreasing* T. *On the subspace* ${}_0\mathbb{E}$, *however, this norm can be chosen independently of* $T > 0$, *i.e., there exists a constant* C_1 *independent of* T *such that*

$$\|u\|_{C([0,T];\gamma_t\mathbb{E})} \le C_1 \|u\|_{\mathbb{E}} \quad (u \in {}_0\mathbb{E}).$$

Definition 2.5 Let $T \in (0,\infty]$, $J := (0,T)$, and $p \in [1,\infty]$.

(a) We say that A has maximal L^p-regularity ($A \in \mathrm{MR}_p(J;X)$) if for each $f \in \mathbb{F}$ and $u_0 \in \gamma_t\mathbb{E}$ there exists a unique solution $u \in \mathbb{E}$ of (2.5). Here, a function $u \in \mathbb{E}$ is called a solution of (2.5)–(2.6) if equality in (2.5) holds in the space $L^p(J;X)$ (i.e., for almost all $t \in (0,T)$), and equality (2.6) holds in X.

(b) We write $A \in {}_0\mathrm{MR}_p(J;X)$ if for each $f \in \mathbb{F}$ and $u_0 \in \gamma_t\mathbb{E}$ there exists a function $u\colon [0,T] \to X$ satisfying $\partial_t u \in L^p(J;X)$ and $Au \in L^p(J;X)$ such that (2.5) holds for almost all $t \in (0,T)$ and (2.6) holds as equality in X, and if for all $f \in \mathbb{F}$ and $u_0 \in \gamma_t\mathbb{E}$ the inequality

$$\|\partial_t u\|_{L^p(J;X)} + \|Au\|_{L^p(J;X)} \le C\big(\|f\|_{L^p(J;X)} + \|u_0\|_{\gamma_t\mathbb{E}}\big) \tag{2.8}$$

holds with a constant $C = C(J)$ independent of f and u_0.

(c) We set $\mathrm{MR}_p(X) := \mathrm{MR}_p((0,\infty);X)$ and ${}_0\mathrm{MR}_p(X) := {}_0\mathrm{MR}_p((0,\infty);X)$.

Remark 2.6 (a) By the definition of the spaces, the map

$$\begin{pmatrix} \partial_t - A \\ \gamma_t \end{pmatrix} \colon \mathbb{E} \to \mathbb{F} \times \gamma_t\mathbb{E},\, u \mapsto \begin{pmatrix} \partial_t u - Au \\ \gamma_t u \end{pmatrix}$$

is continuous. If $A \in \mathrm{MR}_p(J;X)$, then, due to the definition of maximal regularity, this map is a bijection and therefore, by the open mapping theorem, an isomorphism. In particular, we obtain the a priori estimate

$$\|u\|_{L^p(J;X)} + \|\partial_t u\|_{L^p(J;X)} + \|Au\|_{L^p(J;X)} \le C\big(\|f\|_{L^p(J;X)} + \|u_0\|_{\gamma_t\mathbb{E}}\big), \tag{2.9}$$

which is stronger than (2.8).

(b) If $A \in \mathrm{MR}_p(J;X)$, then (2.5)–(2.6) with $u_0 := 0$ is uniquely solvable for all $f \in \mathbb{F}$. On the other hand, for a given $u_0 \in \gamma_t\mathbb{E}$, there exists an extension $u_1 \in \mathbb{E}$ with $\gamma_t u_1 = u_0$ by the definition of the trace space. Setting $u = u_1 + u_2$, then we see that u_2 has to satisfy

$$\begin{aligned} \partial_t u_2(t) - Au_2(t) &= \widetilde{f}(t) \quad (t > 0), \\ u_2(0) &= 0, \end{aligned} \tag{2.10}$$

where $\widetilde{f} := f - Au_1 \in \mathbb{F}$. Therefore, the operator A has maximal regularity if and only if the Cauchy problem (2.10) is uniquely solvable for all $\widetilde{f} \in \mathbb{F}$.

(c) Let the time interval J be finite, and assume $A \in {}_0\mathrm{MR}_p(J;X)$. Then the Cauchy problem (2.10) has a unique solution u for all $\widetilde{f} \in \mathbb{F}$ with $\partial_t u \in L^p(J;X)$.

As $u(0) = 0$, we can apply Poincaré's inequality in the vector-valued Sobolev space $W_p^1((0,T);X)$ (or the fundamental theorem of calculus, see [23], Proposition 2.5.9, which yields absolute continuity of u) and obtain $u \in L^p(J;X)$. This yields $u \in \mathbb{E}$, and by part (b) of this remark, we see that $A \in \mathrm{MR}_p(J;X)$. Therefore, ${}_0\mathrm{MR}_p(J;X) = \mathrm{MR}_p(J;X)$ for finite time intervals. Similarly, if $A \in {}_0\mathrm{MR}_p((0,\infty);X)$ and if A is invertible, we can estimate $\|u\|_{L^p((0,\infty);X)} \le C \|Au\|_{L^p((0,\infty);X)}$ and obtain $u \in \mathbb{E}$ again, which implies $A \in \mathrm{MR}_p((0,\infty);X)$.

It turns out that the property of maximal L^p-regularity is independent of p. For a proof of the following result, we refer to [17], Theorem 4.2.

Lemma 2.7 *If $A \in \mathrm{MR}_p(X)$ holds for some $p \in (1,\infty)$, then $A \in \mathrm{MR}_p(X)$ holds for every $p \in (1,\infty)$.*

Based on this, we write $\mathrm{MR}(X)$ instead of $\mathrm{MR}_p(X)$. Note that the constant C in (2.8) still depends on p.

By Definition 2.5 and Remark 2.6 (b), the operator A has maximal L^p-regularity in $J = (0,\infty)$ if and only if the Cauchy problem

$$\begin{aligned} \partial_t u(t) - Au(t) &= f(t) \quad (t \in (0,\infty)), \\ u(0) &= 0 \end{aligned} \tag{2.11}$$

has a unique solution $u \in W_p^1(J;X)$. We can extend f and u by zero to the whole line $t \in \mathbb{R}$ and obtain functions $f \in L^p(\mathbb{R};X)$ and $u \in W_p^1(\mathbb{R};X)$ (for this, we need $u(0) = 0$). After this, we apply the Fourier transform in t, which is defined for smooth functions by

$$(\mathscr{F}_t u)(\tau) := (2\pi)^{-1/2} \int_{\mathbb{R}} u(t) e^{-it\tau} dt.$$

For tempered distributions, we define $\mathscr{F}_t$ by duality. Note that $[\mathscr{F}_t(\partial_t u)](\tau) = i\tau(\mathscr{F}_t u)(\tau)$. Therefore, (2.11) is equivalent to

$$(i\tau - A)(\mathscr{F}_t u)(\tau) = (\mathscr{F}_t f)(\tau) \quad (\tau \in \mathbb{R}). \tag{2.12}$$

Theorem 2.8 *Let $J = (0,\infty)$ and A be a closed densely defined operator. Then $A \in {}_0\mathrm{MR}_p(J;X)$ if and only if the operator*

$$\mathscr{F}_t^{-1} i\tau(i\tau - A)^{-1}\mathscr{F}_t$$

defines a continuous operator in $L^p(\mathbb{R};X)$.

Proof By definition, $A \in {}_0\mathrm{MR}_p(J;X)$ if and only if (2.11) has a unique solution u with $\partial_t u \in L^p(\mathbb{R};X)$ (again extending the functions by zero to the whole line), and if we have an estimate of $\partial_t u$. This is equivalent to unique solvability of the Fourier transformed problem (2.12), i.e., the existence of $(i\tau - A)^{-1}$ for almost all $\tau \in \mathbb{R}$ such that the solution u satisfies

$$\partial_t u = \mathscr{F}_t^{-1} i\tau(i\tau - A)^{-1}\mathscr{F}_t f \in L^p(\mathbb{R}; X),$$

and the estimate of $\partial_t u$ is equivalent to the condition $\mathscr{F}_t^{-1} i\tau(i\tau - A)^{-1}\mathscr{F}_t \in L(L^p(\mathbb{R}; X))$. □

2.3 Maximal Regularity for Non-autonomous Problems

With respect to the nonlinear equation (2.3) and its linearization (2.4), it makes sense to define maximal regularity also for non-autonomous problems. So we consider

$$\partial_t u(t) - A(t)u(t) = f(t) \quad (t \in (0, T)), \tag{2.13}$$

$$u(0) = u_0. \tag{2.14}$$

Here we assume that all operators $A(t)$ are closed and densely defined operators in some Banach space X and have the same domain D_A. We also assume that we have a norm $\|\cdot\|_A$ on $D(A)$ which is, for every $t \in (0, T)$, equivalent to the graph norm of $A(t)$, which is given by $\|\cdot\|_X + \|A(t)\cdot\|_X$. In this way, we can identify the unbounded operator $A(t)\colon X \supset D_A \to X$ with the bounded operator $A(t) \in L(D_A, X)$. Moreover, we assume that $A \in L^\infty((0, T); L(D_A, X))$.

Analogously to the autonomous case, we consider the basic space for the right-hand side $\mathbb{F} := L^p(J; X)$ with $J := (0, T)$ and the solution space

$$\mathbb{E} := W_p^1(J; X) \cap L^p(J; D_A). \tag{2.15}$$

We identify $A\colon (0, T) \to L(D_A, X)$ with a function on $\mathbb{E}$ by setting

$$(Au)(t) := A(t)u(t) \quad (t \in (0, T),\ u \in \mathbb{E}).$$

The trace space $\gamma_t\mathbb{E}$ is defined as in Definition 2.2 a).

Definition 2.9 (a) Let $f \in \mathbb{F}$ and $u_0 \in \gamma_t\mathbb{E}$. Then a function $u\colon (0, T) \to X$ is called a strong (L^p)-solution of (2.13)–(2.14) if $u \in \mathbb{E}$ and if (2.13) holds for almost all $t \in (0, T)$ and (2.14) holds in X.

(b) We say that $A \in L^\infty((0, T); L(D_A, X))$ has maximal L^p-regularity on $(0, T)$ if for all $f \in \mathbb{F}$ and $u_0 \in \gamma_t\mathbb{E}$ there exists a unique strong solution $u \in \mathbb{E}$ of (2.13)–(2.14).

Remark 2.10 Similarly to the autonomous case, the operator $A \in L^\infty((0, T); L(D_A, X))$ has maximal regularity if and only if

$$(\partial_t - A, \gamma_t)\colon \mathbb{E} \to \mathbb{F} \times \gamma_t\mathbb{E}$$

is an isomorphism of Banach spaces. By trace results, this is equivalent to the condition that (2.13)–(2.14) with $u_0 = 0$ has a unique solution $u \in \mathbb{E}$ for every $f \in \mathbb{F}$.

The following result shows that maximal regularity for the non-autonomous operator family $(A(t))_{t\in(0,T)}$ can be reduced to maximal regularity for each $A(t)$ if the operator depends continuously on time.

Theorem 2.11 *Let* $T \in (0,\infty)$ *and* $A \in C([0,T], L(D_A, X))$. *Then A has maximal* L^p*-regularity in the sense of Definition 2.9 if and only if for every* $t \in [0,T]$ *we have* $A(t) \in \mathrm{MR}((0,T); X)$.

This is shown, using perturbation arguments, in [6], Theorem 7.1.

3 The Concept of $\mathcal{R}$-Boundedness and the Theorem of Mikhlin

In Theorem 2.8, we have seen that maximal regularity of the operator A is equivalent to the boundedness of the operator

$$\mathscr{F}_t^{-1} m \mathscr{F}_t \colon L^p(\mathbb{R}; X) \to L^p(\mathbb{R}; X),$$

where the operator-valued symbol $m\colon \mathbb{R} \to L(X)$ is given by $m(\tau) := i\tau(i\tau - A)^{-1}$. The classical theorem of Mikhlin gives sufficient conditions for a scalar-valued symbol to induce a bounded operator in $L^p(\mathbb{R}^n)$. For the operator-valued analogue, the concept of $\mathcal{R}$-boundedness can be used. Therefore, we discuss in this section the notion of an $\mathcal{R}$-bounded family and vector-valued variants of Mikhlin's theorem. As references for this section, we mention [13], Sect. 3, and [25], Sect. 2.

3.1 $\mathcal{R}$-Bounded Operator Families

Let X and Y be Banach spaces.

Definition 3.1 A family $\mathcal{T} \subset L(X,Y)$ is called $\mathcal{R}$-bounded if there exists a constant $C > 0$ and some $p \in [1,\infty)$ such that for all $N \in \mathbb{N}$, $T_j \in \mathcal{T}$, $x_j \in X$ $(j = 1,\dots,N)$ and all sequences $(\varepsilon_j)_{j\in\mathbb{N}}$ of independent and identically distributed $\{-1,1\}$-valued and symmetric random variables on a probability space $(\Omega, \mathscr{A}, \mathbb{P})$ we have

$$\Big\| \sum_{j=1}^N \varepsilon_j T_j x_j \Big\|_{L^p(\Omega,Y)} \le C \Big\| \sum_{j=1}^N \varepsilon_j x_j \Big\|_{L^p(\Omega,X)}. \tag{3.1}$$

In this case, $\mathcal{R}_p(\mathcal{T}) := \inf\{C > 0 : (3.1) \text{ holds}\}$ is called the $\mathcal{R}$-bound of $\mathcal{T}$.

Remark 3.2 (a) For the sequence of random variables as above, we have $\mathbb{P}(\{\varepsilon_j = 1\}) = \mathbb{P}(\{\varepsilon_j = -1\}) = \frac{1}{2}$. As the measure $\mathbb{P} \circ (\varepsilon_1, \ldots, \varepsilon_N)^{-1}$ is discrete, the independence of the sequence is equivalent to the condition

$$\mathbb{P}(\{\varepsilon_1 = z_1, \ldots, \varepsilon_N = z_N\}) = 2^{-N} \quad \Big((z_1, \ldots, z_N) \in \{-1, 1\}^N,\ N \in \mathbb{N}\Big).$$

Therefore, $\mathcal{R}$-boundedness is equivalent to the condition

$$\exists\, C > 0\ \forall\, N \in \mathbb{N}\ \forall\, T_1, \ldots, T_N \in \mathcal{T}\ \forall\, x_1, \ldots, x_N \in X$$
$$\bigg(\sum_{z_1,\ldots,z_N=\pm 1} \Big\| \sum_{j=1}^N z_j T_j x_j \Big\|_Y^p\bigg)^{1/p} \le C \bigg(\sum_{z_1,\ldots,z_N=\pm 1} \Big\| \sum_{j=1}^N z_j x_j \Big\|_X^p\bigg)^{1/p}. \tag{3.2}$$

However, the stochastic description is advantageous, in particular, one can choose the probability space $(\Omega, \mathscr{A}, \mathbb{P}) = ([0,1], \mathscr{B}([0,1]), \lambda)$, where $\mathscr{B}([0,1])$ stands for the Borel σ-algebra, λ for the Lebesgue measure, and the random variables ε_j are given by the Rademacher functions (see below). It seems to be unclear if the notation "$\mathcal{R}$" stands for "randomized" or for "Rademacher".

Definition 3.3 The Rademacher functions $r_n \colon [0,1] \to \{-1, 1\}$ are defined by

$$r_n(t) := \operatorname{sign} \sin(2^n \pi t) \quad (t \in [0,1]).$$

By definition, we have

$$r_1(t) = \begin{cases} 1, & t \in (0, \frac{1}{2}), \\ -1, & t \in (\frac{1}{2}, 1). \end{cases}$$

The function r_2 has value 1 on the intervals $(0, \frac{1}{4})$ and $(\frac{1}{2}, \frac{3}{4})$. An immediate calculation yields

$$\int_0^1 r_n(t) r_m(t)\, dt = \delta_{nm} \quad (n, m \in \mathbb{N}).$$

Moreover, for all $M \in \mathbb{N}$, $n_1, \ldots, n_M \in \mathbb{N}$ and $(z_1, \ldots, z_M) \in \{+1, -1\}^M$ we have

$$\lambda(\{t \in [0,1] : r_{n_1}(t) = z_1, \ldots, r_{n_M}(t) = z_M\}) = \frac{1}{2^M} = \prod_{j=1}^M \lambda(\{t \in [0,1] : r_{n_j}(t) = z_j\}).$$

Therefore, the sequence $(r_n)_{n\in\mathbb{N}}$ is independent and identically distributed on the probability space $([0,1], \mathscr{B}([0,1]), \lambda)$ as in Definition 3.1. As all properties of $(\varepsilon_j)_j$ which are needed in this definition only depend on the joint probability distribution, we can always choose $\varepsilon_n = r_n$.

Definition 3.4 Let X be a Banach space and $1 \le p < \infty$. Then $\operatorname{Rad}_p(X)$ is defined as the Banach space of all sequences $(x_n)_{n\in\mathbb{N}} \subset X$ for which the limit $\lim_{N\to\infty} \sum_{n=1}^N r_n(t)x_n =: f(t)$ exists for almost all $t \in [0, 1]$ and defines a function $f \in L^p([0, 1]; X)$. For $(x_n)_{n\in\mathbb{N}} \in \operatorname{Rad}_p(X)$, we define

$$\|(x_n)_{n\in\mathbb{N}}\|_{\operatorname{Rad}_p(X)} := \Big\| \sum_{n=1}^\infty r_n x_n \Big\|_{L^p([0,1];X)}.$$

Remark 3.5 (a) It can be shown that for any sequence $(x_n)_{n\in\mathbb{N}} \subset X$, the sequence $\big(\| \sum_{n=1}^N r_n x_n \|_{L^p([0,1];X)}\big)_{N\in\mathbb{N}}$ is increasing, and therefore $\operatorname{Rad}_p(X)$ is the space of all sequences $(x_n)_{n\in\mathbb{N}}$ such that

$$\Big\| \sum_{n=1}^\infty r_n x_n \Big\|_{L^p([0,1];X)} < \infty.$$

(b) By definition, the map $J\colon \operatorname{Rad}_p(X) \to L^p([0, 1]; X),\ (x_n)_n \mapsto \sum_{n=1}^\infty r_n x_n$ is well-defined. Assume that $J((x_n)_n) = 0$, i.e., $\sum_n r_n x_n = 0$ holds in $L^p([0, 1]; X)$. Then $\sum_n r_n f(x_n) = 0$ for all $f \in X'$. Taking the inner product in L^2 with r_{n_0} for some fixed n_0, we get, using the orthogonality, $f(x_{n_0}) = 0$ for all $f \in X'$ and therefore $x_{n_0} = 0$. As n_0 was arbitrary, we obtain $x_n = 0$ for all $n \in \mathbb{N}$, which shows that J is injective. Therefore, $\operatorname{Rad}_p(X)$ can be considered as a subspace of $L^p([0, 1]; X)$, and the norm in $\operatorname{Rad}_p(X)$ is the restriction of the norm in $L^p([0, 1]; X)$.

Theorem 3.6 (Kahane-Khintchine inequality) *The spaces* $\operatorname{Rad}_p(X)$ *are isomorphic for all* $1 \le p < \infty$, *i.e., there exist constants* $C_p > 0$ *with*

$$\frac{1}{C_p} \Big\| \sum_{n=1}^\infty r_n x_n \Big\|_{L^2([0,1];X)} \le \Big\| \sum_{n=1}^\infty r_n x_n \Big\|_{L^p([0,1];X)} \le C_p \Big\| \sum_{n=1}^\infty r_n x_n \Big\|_{L^2([0,1];X)}.$$

In the scalar case $X = \mathbb{C}$, the proof of this inequality is elementary, for arbitrary Banach spaces, however, rather complicated. In the scalar case Theorem 3.6 is known as Khintchine's inequality, in the Banach space valued case as Kahane's inequality. We omit the proof which can be found, e.g., in [23], Theorem 3.2.23. We also remark that the left inequality still holds for $p = \infty$ due to the embedding $L^\infty([0, 1]; X) \subset L^2([0, 1]; X)$, but the right inequality does not hold for $p = \infty$, as the constant C_p tends to infinity for $p \to \infty$ (see [24], Theorem 6.2.4).

Lemma 3.7 *(a) If condition (3.1) in Definition 3.1 holds for some* $p \in [1, \infty)$, *then it holds for all* $p \in [1, \infty)$. *For the corresponding* $\mathcal{R}$*-bounds* $\mathcal{R}_p(\mathcal{T})$ *the inequality*

$$\frac{1}{C_p^2}\mathcal{R}_2(\mathcal{T}) \le \mathcal{R}_p(\mathcal{T}) \le C_p^2 \mathcal{R}_2(\mathcal{T})$$

holds, where the constants C_p *are from Theorem 3.6.*

(b) A family $\mathcal{T} \subset L(X,Y)$ is $\mathcal{R}$-bounded with $\mathcal{R}_2(\mathcal{T}) \le C$ if and only if for all $N \in \mathbb{N}$ and all $T_1, \dots, T_N \in \mathcal{T}$, the map

$$\mathbf{T}((x_n)_{n\in\mathbb{N}}) := (y_n)_{n\in\mathbb{N}},\; y_n := \begin{cases} T_n x_n, & n \le N, \\ 0, & n > N \end{cases}$$

defines a bounded linear operator $\mathbf{T} \in L(\mathrm{Rad}_2(X))$ with norm $\|\mathbf{T}\| \le C$.

Proof Part (a) follows directly from Kahane's inequality, and part (b) is a reformulation of the definition of $\mathcal{R}$-boundedness and an application of the p-independence from (a). □

Remark 3.8 (a) If $\mathcal{T} \subset L(X,Y)$ is $\mathcal{R}$-bounded, then $\mathcal{T}$ is uniformly bounded with $\sup_{T\in\mathcal{T}} \|T\| \le \mathcal{R}(\mathcal{T})$. This follows immediately if we set $N = 1$ in the definition of $\mathcal{R}$-boundedness.

(b) If X and Y are Hilbert spaces, then $\mathcal{R}$-boundedness is equivalent to uniform boundedness. In fact, in this situation also the spaces $L^2([0,1];X)$ and $L^2([0,1];Y)$ are Hilbert spaces, and $(r_n x_n)_{n\in\mathbb{N}} \subset L^2([0,1];X)$ and $(r_n T_n x_n)_{n\in\mathbb{N}} \subset L^2([0,1];Y)$ are orthogonal sequences. If $\|T\| \le C_{\mathcal{T}}$ for all $T \in \mathcal{T} \subset L(X,Y)$, then

$$\begin{aligned} \Big\| \sum_{n=1}^N r_n T_n x_n \Big\|^2_{L^2[0,1];Y)} &= \sum_{n=1}^N \|r_n T_n x_n\|^2_{L^2([0,1];Y)} = \sum_{n=1}^N \|T_n x_n\|_Y^2 \le C_{\mathcal{T}}^2 \sum_{n=1}^N \|x_n\|_X^2 \\ &= C_{\mathcal{T}}^2 \Big\| \sum_{n=1}^N r_n x_n \Big\|^2_{L^2([0,1];X)} . \end{aligned}$$

Remark 3.9 Let X, Y, Z be Banach spaces, and $\mathcal{T}, \mathcal{S} \subset L(X,Y)$ and $\mathcal{U} \subset L(Y,Z)$ be $\mathcal{R}$-bounded. Then the families

$$\mathcal{T} + \mathcal{S} := \{T + S : T \in \mathcal{T},\; S \in \mathcal{S}\}$$

and

$$\mathcal{U}\mathcal{T} := \{UT : U \in \mathcal{U},\; T \in \mathcal{T}\}$$

are $\mathcal{R}$-bounded, too, with

$$\mathcal{R}(\mathcal{T} + \mathcal{S}) \le \mathcal{R}(\mathcal{T}) + \mathcal{R}(\mathcal{S}),\; \mathcal{R}(\mathcal{U}\mathcal{T}) \le \mathcal{R}(\mathcal{U})\mathcal{R}(\mathcal{T}).$$

To see this, let $S_n \in \mathcal{S}$, $T_n \in \mathcal{T}$ and $U_n \in \mathcal{U}$ for $n = 1, \dots, N$. Then the statement follows from

$$\Big\| \sum_{n=1}^N r_n (T_n + S_n) x_n \Big\|_{L^1([0,1];Y)} \le \Big\| \sum_{n=1}^N r_n T_n x_n \Big\|_{L^1([0,1];Y)} + \Big\| \sum_{n=1}^N r_n S_n x_n \Big\|_{L^1([0,1];Y)}$$

and

$$\left\| \sum_{n=1}^{N} r_n U_n T_n x_n \right\|_{L^1([0,1];Z)} \le \mathcal{R}(\mathcal{U}) \left\| \sum_{n=1}^{N} r_n T_n x_n \right\|_{L^1([0,1];Y)} .$$

The following result turns out to be useful for showing $\mathcal{R}$-boundedness.

Lemma 3.10 (Kahane's contraction principle) *Let $1 \le p < \infty$. Then for all $N \in \mathbb{N}$, for all $x_j \in X$ and all $a_j, b_j \in \mathbb{C}$ with $|a_j| \le |b_j|$, $j = 1, \ldots, N$ we have*

$$\left\| \sum_{j=1}^{N} a_j r_j x_j \right\|_{L^p([0,1];X)} \le 2 \left\| \sum_{j=1}^{N} b_j r_j x_j \right\|_{L^p([0,1];X)} . \tag{3.3}$$

Proof Considering $\widetilde{x}_j := b_j x_j$, we may assume without loss of generality that $b_j = 1$ and $|a_j| \le 1$ for all $j = 1, \ldots, N$. Treating $\operatorname{Re} a_j$ and $\operatorname{Im} a_j$ separately, we only have to show that for real a_j with $|a_j| \le 1$ the inequality

$$\left\| \sum_{j=1}^{N} a_j r_j x_j \right\|_{L^p([0,1];X)} \le \left\| \sum_{j=1}^{N} r_j x_j \right\|_{L^p([0,1];X)} \tag{3.4}$$

holds. For this, let $\{e^{(k)}\}_{k=1,\ldots,2^N}$ be a numbering of all vertices of the cube $[-1, 1]^N$. Because of $a := (a_1, \ldots, a_N)^T \in [-1, 1]^N$, the vector a can be written as a convex combination of all $e^{(k)}$, i.e., there exist $\lambda_k \in [0, 1]$ with

$$\sum_{k=1}^{2^N} \lambda_k = 1 \quad \text{and} \quad a = \sum_{k=1}^{2^N} \lambda_k e^{(k)}.$$

Therefore, for $e^{(k)} = (e_1^{(k)}, \ldots, e_N^{(k)})^T$ we see that

$$\begin{aligned} \left\| \sum_{j=1}^{N} a_j r_j x_j \right\|_{L^p([0,1];X)} &\le \sum_{k=1}^{2^N} \lambda_k \left\| \sum_{j=1}^{N} r_j e_j^{(k)} x_j \right\|_{L^p([0,1];X)} \\ &\le \max_{1 \le k \le 2^N} \left\| \sum_{j=1}^{N} r_j e_j^{(k)} x_j \right\|_{L^p([0,1];X)} = \left\| \sum_{j=1}^{N} r_j x_j \right\|_{L^p([0,1];X)} . \end{aligned}$$

In the last equality we used the fact that $\{r_j : j = 1, \ldots, N\}$ and $\{r_j e_j^{(k)} : j = 1, \ldots, N\}$ have the same joint probability distribution. □

Theorem 3.11 *Let $\mathcal{T} \subset L(X, Y)$ be $\mathcal{R}$-bounded. Then also the convex hull*

$$\operatorname{conv} \mathcal{T} := \Big\{ \sum_{k=1}^{n} \lambda_k T_k : n \in \mathbb{N}, T_k \in \mathcal{T},\ \lambda_k \in [0,1],\ \sum_{k=1}^{n} \lambda_k = 1 \Big\}$$

and the absolute convex hull

$$\operatorname{aconv} \mathcal{T} := \Big\{ \sum_{k=1}^{n} \lambda_k T_k : n \in \mathbb{N},\ T_k \in \mathcal{T},\ \lambda_k \in \mathbb{C},\ \sum_{k=1}^{n} |\lambda_k| = 1 \Big\}$$

are $\mathcal{R}$-bounded. The same holds for the closures $\overline{\operatorname{conv} \mathcal{T}}^s$ of $\operatorname{conv} \mathcal{T}$ *and* $\overline{\operatorname{aconv} \mathcal{T}}^s$ *of* $\operatorname{aconv} \mathcal{T}$ *with respect to the strong operator topology. We have $\mathcal{R}(\overline{\operatorname{conv} \mathcal{T}}^s) \le \mathcal{R}(\mathcal{T})$ and $\mathcal{R}(\overline{\operatorname{aconv} \mathcal{T}}^s) \le 2\mathcal{R}(\mathcal{T})$.*

Proof (a) Let $T_1, \dots, T_N \in \operatorname{conv}(\mathcal{T})$. Then there exist $\lambda_{k,j} \in [0,1]$ and $T_{k,j} \in \mathcal{T}$ with $\sum_{j=1}^{m_k} \lambda_{k,j} = 1$ and $T_k = \sum_{j=1}^{m_k} \lambda_{k,j} T_{k,j}$.

Define $\lambda_{k,j} := 0$ and $T_{k,j} := 0$ for $j \in \mathbb{N}$ with $j > m_k$ and $k = 1, \dots, N$. For $\ell \in \mathbb{N}^N$ we define $\lambda_\ell := \prod_{k=1}^{N} \lambda_{k,\ell_k}$ and $T_{k,\ell} := T_{k,\ell_k}$ for $k = 1, \dots, N$. Then $\lambda_\ell \in [0,1]$ as well as

$$\sum_{\ell \in \mathbb{N}^n} \lambda_\ell = \sum_{\ell_1 \in \mathbb{N}} \cdots \sum_{\ell_N \in \mathbb{N}} \lambda_{1,\ell_1} \cdot \ldots \cdot \lambda_{N,\ell_N} = 1.$$

For all $k = 1, \dots, N$ we obtain

$$\begin{aligned} \sum_{\ell \in \mathbb{N}^N} \lambda_\ell T_{k,\ell} &= \sum_{\ell \in \mathbb{N}^N} \lambda_\ell T_{k,\ell_k} = \Big(\sum_{\ell_k \in \mathbb{N}} \lambda_{k,\ell_k} T_{k,\ell_k} \Big) \prod_{j \neq \ell} \Big(\sum_{\ell_j \in \mathbb{N}} \lambda_{j,\ell_j} \Big) \\ &= \sum_{\ell_k \in \mathbb{N}} \lambda_{k,\ell_k} T_{k,\ell_k} = T_k. \end{aligned}$$

Note that these sums are finite. We get

$$\begin{aligned} \Big\| \sum_{k=1}^{N} r_k T_k x_k \Big\|_{L^p([0,1];Y)} &= \Big\| \sum_{k=1}^{N} \sum_{\ell \in \mathbb{N}^N} r_k \lambda_\ell T_{k,\ell} x_k \Big\|_{L^p([0,1];Y)} \\ &\le \sum_{\ell \in \mathbb{N}^N} \lambda_\ell \Big\| \sum_{k=1}^{N} r_k T_{k,\ell} x_k \Big\|_{L^p([0,1];Y)} \le \mathcal{R}(\mathcal{T}) \sum_{\ell \in \mathbb{N}^n} \lambda_\ell \Big\| \sum_{k=1}^{N} r_k x_k \Big\|_{L^p([0,1];X)} \\ &= \mathcal{R}(\mathcal{T}) \Big\| \sum_{k=1}^{N} r_k x_k \Big\|_{L^p([0,1];X)}. \end{aligned}$$

Consequently, $\mathcal{R}(\operatorname{conv} \mathcal{T}) \le \mathcal{R}(\mathcal{T})$.

(b) By Kahane's contraction principle, $\mathcal{R}(\mathcal{T}_0) \le 2\mathcal{R}(\mathcal{T})$, where we define

$$\mathcal{T}_0 := \{\lambda T : T \in \mathcal{T},\ \lambda \in \mathbb{C},\ |\lambda| \le 1\}.$$

Because of $\operatorname{conv} \mathcal{T}_0 = \operatorname{aconv} \mathcal{T}$, we get $\mathcal{R}(\operatorname{aconv} \mathcal{T}) \le 2\mathcal{R}(\mathcal{T})$ due to a).

(c) The closedness with respect to the strong operator topology follows directly from the definition of $\mathcal{R}$-boundedness. □

The above results are useful to prove $\mathcal{R}$-boundedness in general Banach spaces. In the special situation that X is some L^q-space, there is a helpful description of $\mathcal{R}$-boundedness:

Lemma 3.12 (Square function estimate) *Let $(G, \mathscr{A}, \mu)$ be a σ-finite measure space, $X = L^q(G)$, and let $1 \le q < \infty$. Then $\mathcal{T} \subset L(X)$ is $\mathcal{R}$-bounded if and only if there exists an $M > 0$ with*

$$\left\| \left(\sum_{j=1}^{N} |T_n f_n|^2 \right)^{1/2} \right\|_{L^q(G)} \le M \left\| \left(\sum_{j=1}^{N} |f_n|^2 \right)^{1/2} \right\|_{L^q(G)}$$

for all $N \in \mathbb{N}$, $T_n \in \mathcal{T}$ and $f_n \in L^q(G)$.

Proof We write $f \approx g$ if there are constants $C_1, C_2 > 0$ with $C_1|f| \le |g| \le C_2|f|$. To show $\mathcal{R}$-boundedness, by Kahane's inequality, we can consider the $\mathcal{R}_q$-bound. For this, we can calculate

$$\begin{aligned}
\left\| \sum_{n=1}^{N} r_n f_n \right\|^q_{L^q([0,1];L^q(G))} &= \int_0^1 \left\| \sum_{n=1}^{N} r_n(t) f_n(\cdot) \right\|^q_{L^q(G)} dt \\
&= \int_0^1 \int_G \left| \sum_{n=1}^{N} r_n(t) f_n(\omega) \right|^q d\mu(\omega)\, dt \\
&= \int_G \int_0^1 \left| \sum_{n=1}^{N} r_n(t) f_n(\omega) \right|^q dt d\mu(\omega) \\
&\approx \int_G \left(\int_0^1 \left| \sum_{n=1}^{N} r_n(t) f_n(\omega) \right|^2 dt \right)^{q/2} d\mu(\omega) \\
&= \int_G \left(\sum_{n=1}^{N} |f_n(\omega)|^2 \right)^{q/2} d\mu(\omega) = \left\| \left(\sum_{n=1}^{N} |f_n|^2 \right)^{1/2} \right\|^q_{L^q(G)}.
\end{aligned}$$

Here, Fubini's theorem and the inequality of Khintchine were used. Now the statement follows by considering the above calculation for both sides of the definition of $\mathcal{R}$-boundedness. □

Example 3.13 Using the square function estimate, it is easy to construct an example of a uniformly bounded operator family which is not $\mathcal{R}$-bounded. Let $p \in [1, \infty) \setminus \{2\}$. Then the family $\{T_n : n \in \mathbb{N}_0\} \subset L(L^p(\mathbb{R}))$, $T_n f(\cdot) := f(\cdot - n)$ of translations is not $\mathcal{R}$-bounded, as for $f_n = \chi_{[0,1]}$ we have

$$\Big\| \Big(\sum_{n=0}^{N-1} |T_n f_n|^2 \Big)^{1/2} \Big\|_{L^p(\mathbb{R})} = \|\chi_{[0,N]}\|_{L^p(\mathbb{R})} = N^{1/p},$$
$$\Big\| \Big(\sum_{n=0}^{N-1} |f_n|^2 \Big)^{1/2} \Big\|_{L^p(\mathbb{R})} = N^{1/2} \|\chi_{[0,1]}\|_{L^p(\mathbb{R})} = N^{1/2}.$$

For $1 \le p < 2$, we use the fact that $\frac{N^{1/p}}{N^{1/2}} \to \infty$ for $N \to \infty$. The proof for $p > 2$ is similar.

Lemma 3.14 *(a) Let $G \subset \mathbb{R}^n$ be open and $1 \le p < \infty$. For $\varphi \in L^\infty(G)$, define $m_\varphi \in L(L^p(G; X))$ by $(m_\varphi f)(x) := \varphi(x) f(x)$. Then for $r > 0$ one obtains*

$$\mathcal{R}_p\Big(\{ m_\varphi : \varphi \in L^\infty(G),\ \|\varphi\|_\infty \le r \} \Big) \le 2r.$$

(b) Let $1 \le p < \infty$, $G \subset \mathbb{R}^n$ be open, and $\mathcal{T} \subset L(L^p(G; X), L^p(G; Y))$ be $\mathcal{R}$-bounded. Then

$$\mathcal{R}_p\Big(\{ m_\varphi T m_\psi : T \in \mathcal{T},\ \varphi, \psi \in L^\infty(G), \|\varphi\|_\infty \le r,\ \|\psi\|_\infty \le s \} \Big) \le 4rs\mathcal{R}_p(\mathcal{T}).$$

Proof (a) By the theorem of Fubini and Kahane's contraction principle,

$$\Big\| \sum_{k=1}^N r_k m_{\varphi_k} f_k \Big\|_{L^p([0,1]; L^p(G;X))} = \Big\| \sum_{k=1}^N r_k \varphi_k f_k \Big\|_{L^p(G; L^p([0,1]; X))}$$
$$\le 2r \Big\| \sum_{k=1}^N r_k f_k \Big\|_{L^p(G; L^p([0,1]; X))} = 2r \Big\| \sum_{k=1}^N r_k f_k \Big\|_{L^p([0,1]; L^p(G;X))}.$$

(b) follows from (a) and Remark 3.9. □

In the following corollary, we consider strongly measurable function. Note that a function $N \colon G \to L(X, Y)$ is called strongly measurable if there exists a μ-zero set $A \in \mathscr{A}$ such that $N|_{G \setminus A}$ is measurable and $N(G \setminus A)$ is separable.

Corollary 3.15 *Let $(G, \mathscr{A}, \mu)$ be a σ-finite measure space and $\mathcal{T} \subset L(X, Y)$ be $\mathcal{R}$-bounded. Let*

$$\mathcal{N} := \{ N \colon G \to L(X, Y) \mid N \text{ strongly measurable with } N(G) \subset \mathcal{T} \}.$$

For $h \in L^1(G, \mu)$ and $N \in \mathcal{N}$ define

$$T_{N,h}x := \int_G h(\omega)N(\omega)x\,d\mu(\omega) \quad (x \in X).$$

Then

$$\mathcal{R}\Big(\{T_{N,h} : \|h\|_{L^1(G,\mu)} \le 1,\ N \in \mathcal{N}\}\Big) \le 2\mathcal{R}(\mathcal{T}).$$

Proof Let $\varepsilon > 0$. For $x_1, \ldots, x_N \in X$, $h \in L^1(G, \mu)$ and $N \in \mathcal{N}$ we consider the measurable map

$$M\colon G \to Y^N,\ M(\omega) := \big(N(\omega)x_j\big)_{j=1,\ldots,N}.$$

Then $M \in L^\infty(G; Y^N)$ is strongly measurable, and therefore there exist a measurable partition $G = \bigcup_{j=1}^\infty G_j$, $G_i \cap G_j = \emptyset$ for $i \neq j$, and $\omega_j \in G_j$ with

$$\|N(\omega)x_k - N(\omega_j)x_k\|_Y < \varepsilon \quad \text{for almost all } \omega \in G_j \text{and all } k = 1, \ldots, N.$$

Define

$$S := \sum_{j=1}^\infty \left(\int_{G_j} h(\omega)d\mu(\omega)\right) N(\omega_j).$$

Then $\|T_{N,h}x_k - Sx_k\|_Y < \varepsilon$ for all $k = 1, \ldots, N$. Therefore, $T_{N,h}$ is a subset of the neighbourhood of S given by $x_1, \ldots, x_N$ and ε with respect to the strong operator topology. Because of $S \in \overline{\text{aconv}\,\mathcal{T}}^s$, we obtain $T_{N,h} \in \overline{\text{aconv}\,\mathcal{T}}^s$. Now the statement follows from Theorem 3.11. □

Corollary 3.16 *Let $N\colon \Sigma_{\theta'} \to L(X, Y)$ be holomorphic and bounded, and let $N(\partial\Sigma_\theta \setminus \{0\})$ be $\mathcal{R}$-bounded for some $\theta < \theta'$. Then $N(\Sigma_\theta)$ is $\mathcal{R}$-bounded, and for every $\theta_1 < \theta$ the family $\{\lambda \frac{\partial}{\partial\lambda} N(\lambda) : \lambda \in \Sigma_{\theta_1}\}$ is $\mathcal{R}$-bounded.*

Proof Considering $M(\lambda) := N(\lambda^{2\theta/\pi})$, we may assume $\theta = \frac{\pi}{2}$. Now we use Poisson's formula

$$N(\alpha + i\beta) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\alpha}{\alpha^2 + (s-\beta)^2} N(is)ds \quad (\alpha > 0).$$

Because of $\|\frac{1}{\pi}\frac{\alpha}{\alpha^2+(\cdot-\beta)^2}\|_{L^1(\mathbb{R})} = 1$, the first assertion follows from Corollary 3.15.

By Cauchy's integral formula, we have

$$\lambda\frac{\partial}{\partial\lambda}N(\lambda) = \int_{\partial\Sigma_\theta} h_\lambda(\mu)N(\mu)d\mu \quad (\lambda \in \Sigma_{\theta_1})$$

for $h(\lambda) := \frac{1}{2\pi i} \frac{\lambda}{(\mu-\lambda)^2}$. Because of $\sup_{\lambda\in\Sigma_{\theta_1}} \|h_\lambda\|_{L^1(\partial\Sigma_\theta)} < \infty$, the second assertion follows from Corollary 3.15, too. □

Lemma 3.17 *Let $G \subset \mathbb{C}$ be open, $K \subset G$ be compact, and $H : G \to L(X, Y)$ be holomorphic. Then $H(K)$ is $\mathcal{R}$-bounded.*

Proof Let $z_0 \in K$. Then there exists an $r > 0$ with

$$H(z) = \sum_{k=0}^{\infty} H^{(k)}(z_0) \frac{(z-z_0)^k}{k!} \quad (|z-z_0| \le r).$$

Here the series converges in $L(X, Y)$ and

$$\rho_0 := \sum_{k=0}^{\infty} \|H^{(k)}(z_0)\|_{L(X,Y)} \frac{r^k}{k!} < \infty.$$

As a set with one element, $\{H^{(k)}(z_0)\}$ is $\mathcal{R}$-bounded with $\mathcal{R}$-bound $\|H^{(k)}(z_0)\|_{L(X,Y)}$. By Kahane's contraction principle, the family $\{H^{(k)}(z_0)\frac{(z-z_0)^k}{k!} : z \in B(z_0, r)\}$ is $\mathcal{R}$-bounded, too, with $\mathcal{R}$-bound not greater than $2\frac{r^k}{k!}\|H^{(k)}(z_0)\|_{L(X,Y)}$. Therefore, we obtain for all finite partial sums the $\mathcal{R}$-bound $2\rho_0$. Taking the closure with respect to the strong operator topology, the same holds for the infinite sum. By a finite covering of K, we obtain the statement of the lemma. □

Theorem 3.18 *Let $G \subset \mathbb{R}^n$ be open and $1 < p < \infty$. Let Λ be a set and $\{k_\lambda : \lambda \in \Lambda\}$ be a family of measurable kernels $k_\lambda : G \times G \to L(X, Y)$ with*

$$\mathcal{R}_p\Big(\big\{k_\lambda(z, z') : \lambda \in \Lambda\big\}\Big) \le k_0(z, z') \quad (z, z' \in G).$$

Assume that for the corresponding scalar integral operator

$$(K_0 f)(z) = \int_G k_0(z, z') f(z') dz' \quad (f \in L^p(G))$$

one has $K_0 \in L(L^p(G))$. Define

$$(K_\lambda f)(z) = \int_G k_\lambda(z, z') f(z') dz' \quad (f \in L^p(G; X)).$$

Then $K_\lambda \in L(L^p(G; X), L^p(G; Y))$ with

$$\mathcal{R}_p\big(\{K_\lambda : \lambda \in \Lambda\}\big) \le \|K_0\|_{L(L^p(G))}.$$

Proof We use the definition of $\mathcal{R}$-boundedness and get

$$\begin{aligned}
&\Big\| \sum_{j=1}^N r_j K_{\lambda_j} f_j \Big\|_{L^p([0,1];L^p(G;Y))} \\
&= \left(\int_0^1 \Big\| \sum_{j=1}^N r_j(t) \int_G k_{\lambda_j}(\cdot, z') f_j(z') dz' \Big\|_{L^p(G;Y)}^p dt \right)^{1/p} \\
&= \left(\int_0^1 \Big\| \int_G \sum_{j=1}^N r_j(t) k_{\lambda_j}(\cdot, z') f_j(z') dz' \Big\|_{L^p(G;Y)}^p dt \right)^{1/p} \\
&= \left(\int_0^1 \int_G \Big\| \int_G \sum_{j=1}^N r_j(t) k_{\lambda_j}(z, z') f_j(z') dz' \Big\|_Y^p dz\, dt \right)^{1/p} \\
&= \left(\int_G \int_0^1 \Big\| \int_G \sum_{j=1}^N r_j(t) k_{\lambda_j}(z, z') f_j(z') dz' \Big\|_Y^p dt\, dz \right)^{1/p}.
\end{aligned}$$

Setting $\varphi(t, z, z') := \sum_{j=1}^N r_j(t) k_{\lambda_j}(z, z') f_j(z')$, the integral with respect to t in the last term equals $\| \int_G \varphi(\cdot, z, z') dz' \|_{L^p([0,1])}^p$. Now we apply the inequality

$$\Big\| \int_G \varphi(\cdot, z, z') dz' \Big\|_{L^p([0,1])} \le \int_G \| \varphi(\cdot, z, z') \|_{L^p([0,1])} dz'$$

for Bochner integrals and obtain, using the assumption of $\mathcal{R}$-boundedness,

$$\begin{aligned}
&\Big\| \sum_{j=1}^N r_j K_{\lambda_j} f_j \Big\|_{L^p([0,1];L^p(G;Y))} \\
&\le \left(\int_G \left[\int_G \Big\| \sum_{j=1}^N r_j(\cdot) k_{\lambda_j}(z, z') f_j(z') \Big\|_{L^p([0,1];Y)} dz' \right]^p dz \right)^{1/p} \\
&\le \left(\int_G \left[\int_G k_0(z, z') \Big\| \sum_{j=1}^N r_j(\cdot) f_j(z') \Big\|_{L^p([0,1];X)} dz' \right]^p dz \right)^{1/p} \\
&= \left\| K_0 \left(\Big\| \sum_{j=1}^N r_j f_j(\cdot) \Big\|_{L^p([0,1];X)} \right) \right\|_{L^p(G)}
\end{aligned}$$

$$\le \|K_0\|_{L(L^p(G))} \left\| \left(\left\| \sum_{j=1}^N r_j f_j(\cdot) \right\|_{L^p([0,1];X)} \right) \right\|_{L^p(G)}$$

$$= \|K_0\|_{L(L^p(G))} \left\| \sum_{j=1}^N r_j f_j \right\|_{L^p([0,1];L^p(G;X))}.$$

3.2 Fourier Multipliers and Mikhlin's Theorem

We have already seen in Theorem 2.8 that maximal regularity is equivalent to the $L^p(\mathbb{R}; X)$-boundedness of the operator $\mathscr{F}_t^{-1} i\tau(i\tau - A)^{-1}\mathscr{F}_t$. This is a typical example of a (vector-valued) Fourier multiplier. In the analysis of partial differential equations and boundary value problems in L^p-spaces, the question of Fourier multipliers play a central role. The answer is given by the classical theorem of Mikhlin and by its Banach space valued variants.

In the following, we use the standard notation $D := -i(\partial_{x_1}, \dots, \partial_{x_n})$ as well as the standard multi-index notation $D^\alpha = (-i)^{|\alpha|}\partial_{x_1}^{\alpha_1} \dots \partial_{x_n}^{\alpha_n}$. We start with a simple example.

Example 3.19 Consider the Laplacian Δ in $L^p(\mathbb{R}^n)$ with maximal domain $D(\Delta) := \{u \in L^p(\mathbb{R}^n) : \Delta u \in L^p(\mathbb{R}^n)\}$. Obviously we have $D(\Delta) \supset W_p^2(\mathbb{R}^n)$. To show that we even have equality, we consider $u \in D(\Delta)$ and $f := u - \Delta u \in L^p(\mathbb{R}^n)$. Let $|\alpha| \le 2$. Then

$$D^\alpha u = \mathscr{F}^{-1}\xi^\alpha \mathscr{F} u = -\mathscr{F}^{-1}\frac{\xi^\alpha}{1+|\xi|^2}\mathscr{F} f$$

holds as equality in $\mathscr{S}'(\mathbb{R}^n)$, where $\mathscr{F}$ stands for the n-dimensional Fourier transform (see below). To obtain $D^\alpha u \in L^p(\mathbb{R}^n)$, we have to show $\mathscr{F}^{-1}m_\alpha \mathscr{F} f \in L^p(\mathbb{R}^n)$, where $m_\alpha(\xi) := \frac{\xi^\alpha}{1+|\xi|^2}$. So we have to prove that

$$f \mapsto \mathscr{F}^{-1}m_\alpha(\xi)\mathscr{F} f$$

defines a bounded linear operator on $L^p(\mathbb{R}^n)$. This is in fact the case, as we will see from the classical version of Mikhlin's theorem, Theorem 3.22 below.

In contrast to the above example, we will also need vector-valued versions of Mikhlin's theorem. For this, we need some preparation, starting with the vector-valued Fourier transform. Let X be a Banach space. Then the Schwartz space $\mathscr{S}(\mathbb{R}^n; X)$ is defined as the space of all infinitely smooth functions $\varphi\colon \mathbb{R}^n \to X$ for which

$$p_N(\varphi) := \sup_{x\in\mathbb{R}^n} \max_{|\alpha|\le N} (1+|x|)^N \|\partial^\alpha \varphi(x)\|_X < \infty$$

for all $N \in \mathbb{N}$. With the family of seminorms $\{p_N : N \in \mathbb{N}\}$, the Schwartz space becomes a Fréchet space. The space of all X-valued tempered distributions is defined by

$$\mathscr{S}'(\mathbb{R}^n; X) := L(\mathscr{S}(\mathbb{R}^n), X).$$

On $\mathscr{S}'(\mathbb{R}^n; X)$, we consider the family of seminorms

$$\pi_\varphi : \mathscr{S}'(\mathbb{R}^n; X) \to [0, \infty), \; u \mapsto \|u(\varphi)\|_X \quad (\varphi \in \mathscr{S}(\mathbb{R}^n)).$$

Then the family $\{\pi_\varphi : \varphi \in \mathscr{S}(\mathbb{R}^n)\}$ defines a locally convex topology on $\mathscr{S}'(\mathbb{R}^n; X)$. Note that in the scalar case $X = \mathbb{C}$, this is the weak-$*$-topology. One can see as in the scalar case that the Fourier transform, defined for $\varphi \in \mathscr{S}(\mathbb{R}^n; X)$ by

$$(\mathscr{F}\varphi)(\xi) := (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{-ix\cdot\xi}\varphi(x)dx \quad (\xi \in \mathbb{R}^n, \; \varphi \in \mathscr{S}(\mathbb{R}^n; X)),$$

can be extended by duality to an isomorphism $\mathscr{F} : \mathscr{S}'(\mathbb{R}^n; X) \to \mathscr{S}'(\mathbb{R}^n; X)$.

Definition 3.20 Let X, Y be Banach spaces, $1 \le p < \infty$, and let $m : \mathbb{R}^n \to L(X, Y)$ be a bounded and strongly measurable function. Because of $\mathscr{F}^{-1} \in L(L^1(\mathbb{R}^n; X), L^\infty(\mathbb{R}^n; Y))$, the function m induces a map $T_m : \mathscr{S}(\mathbb{R}^n; X) \to L^\infty(\mathbb{R}^n; Y)$ by

$$T_m f := \mathscr{F}^{-1} m \mathscr{F} f \quad (f \in \mathscr{S}(\mathbb{R}^n; X)).$$

The function m is called a Fourier multiplier (more precisely, an L^p-Fourier multiplier) if

$$\|T_m f\|_{L^p(\mathbb{R}^n;Y)} \le C\|f\|_{L^p(\mathbb{R}^n;X)} \quad (f \in \mathscr{S}(\mathbb{R}^n; X)).$$

As $\mathscr{S}(\mathbb{R}^n; X)$ is dense in $L^p(\mathbb{R}^n; X)$ for $p \in [1, \infty)$, this implies that T_m has a unique extension to a bounded linear operator $T_m \in L(L^p(\mathbb{R}^n; X), L^p(\mathbb{R}^n; Y))$. In this case, m is called the symbol of the operator T_m, and we write

$$\text{op}[m] := \mathscr{F} m \mathscr{F}^{-1} := T_m \tag{3.5}$$

and $\text{symb}[T_m] := m$.

We start with the scalar case $X = Y = \mathbb{C}$.

Remark 3.21 In the Hilbert space case $p = 2$, one can apply Plancherel's theorem. Therefore, we have $\text{op}[m] \in L(L^2(\mathbb{R}^n))$ if and only if the multiplication operator $g \mapsto mg$ is a bounded operator in $L^2(\mathbb{R}^n)$. This is equivalent to the condition $m \in L^\infty(\mathbb{R}^n)$.

In fact, if $m \in L^\infty(\mathbb{R}^n)$, then $\|mg\|_{L^2(\mathbb{R}^n)} \le \|m\|_{L^\infty(\mathbb{R}^n)}\|g\|_{L^2(\mathbb{R}^n)}$. On the other hand, if $m \notin L^\infty(\mathbb{R}^n)$, then there exists a sequence $(A_k)_{k\in\mathbb{N}}$ of measurable subsets of $\mathbb{R}^n$ such that $0 < \lambda(A_k) < \infty$ and $|m(x)| \ge k$ for $x \in A_k$. For the characteristic function $g_k := \chi_{A_k}$ we obtain $g_k \in L^2(\mathbb{R}^n)$ and

$$\|mg_k\|^2_{L^2(\mathbb{R}^n)} = \int |m(\xi)g_k(\xi)|^2 d\xi \geq k^2\lambda(A_k) = k^2\|g_k\|^2_{L^2(\mathbb{R}^n)}.$$

Therefore, op$[m]$ cannot be a bounded operator in $L^2(\mathbb{R}^n)$.

The following classical theorem gives a sufficient condition for a function to be a (scalar) Fourier multiplier and has many applications in the theory of partial differential equations. In the following, $[\frac{n}{2}]$ denotes the largest integer not greater than $\frac{n}{2}$. We state this result in two variants.

Theorem 3.22 (Mikhlin's multiplier theorem) *Let* $1 < p < \infty$ *and* $m\colon \mathbb{R}^n \setminus \{0\} \to \mathbb{C}$. *If one of the two conditions*

(i) $m \in C^{[\frac{n}{2}]+1}(\mathbb{R}^n \setminus \{0\})$ *and*

$$|\xi|^{|\beta|}|\partial^\beta m(\xi)| \leq C_M \quad (\xi \in \mathbb{R}^n \setminus \{0\},\ |\beta| \leq [\tfrac{n}{2}] + 1),$$

(ii) $m \in C^n(\mathbb{R}^n \setminus \{0\})$ *and*

$$\left|\xi^\beta \partial^\beta m(\xi)\right| \leq C_M \quad (\xi \in \mathbb{R}^n \setminus \{0\},\ \beta \in \{0, 1\}^n)$$

holds with a constant $C_M > 0$, *then* m *is an* L^p*-Fourier multiplier with*

$$\|\operatorname{op}[m]\|_{L(L^p(\mathbb{R}^n))} \leq c(n, p)C_M,$$

with a constant $c(n, p)$ *depending only on* n *and* p.

A proof of this theorem (which is also called Mikhlin-Hörmander theorem) can be found, e.g., in [19], Sect. 6.2.3. Condition (i) is sometimes called the Mikhlin condition, whereas condition (ii) is called the Lizorkin condition. For the L^p-continuity of singular integral operators, we also refer to [31], Sect. 6.5.

For the following result, note that a function $m\colon \mathbb{R}^n \setminus \{0\} \to \mathbb{C}$ is called (positively) homogeneous with respect to ξ of degree $d \in \mathbb{R}$ if

$$m(\rho\xi) = \rho^d m(\xi) \quad (\xi \in \mathbb{R}^n \setminus \{0\},\ \rho > 0).$$

Lemma 3.23 *Let* $m \in C^{[\frac{n}{2}]+1}(\mathbb{R}^n \setminus \{0\})$ *be homogeneous of degree* 0. *Then* m *satisfies the Mikhlin condition.*

Proof If a function $m \in C^k(\mathbb{R}^n \setminus \{0\})$ is homogeneous of degree d, then its derivative $\partial^\beta m(\xi)$ is homogeneous of degree $d - |\beta|$ for all $|\beta| \leq k$. This follows from the identities $\partial^\beta[m(\rho\xi)] = \rho^{|\beta|}(\partial^\beta m)(\rho\xi)$ and $\partial^\beta[\rho^d m(\xi)] = \rho^d(\partial^\beta m)(\xi)$.

Now let $m \in C^{[\frac{n}{2}]+1}(\mathbb{R}^n \setminus \{0\})$ be homogeneous of degree 0, and let $|\beta| \leq [\frac{n}{2}] + 1$. Then $m_\beta(\xi) := |\xi|^{|\beta|}\partial_\beta m(\xi)$ is homogeneous of degree 0 and continuous. Therefore,

$$|m_\beta(\xi)| = \left|m_\beta\Big(\frac{\xi}{|\xi|}\Big)\right| \leq \max_{|\eta|=1} |m_\beta(\eta)| < \infty \quad (\xi \in \mathbb{R}^n \setminus \{0\}).$$

□

As a first application of Mikhlin's theorem, we can now answer the question from Example 3.19.

Corollary 3.24 *Let* $1 < p < \infty$. *Then* $\{u \in L^p(\mathbb{R}^n) : \Delta u \in L^p(\mathbb{R}^n)\} = W_p^2(\mathbb{R}^n)$.

Proof As we have seen in Example 3.19, we have to show that the function $m_\alpha(\xi) := \frac{\xi^\alpha}{1+|\xi|^2}$ satisfies the Mikhlin condition for all $|\alpha| \le 2$. For this, we write $m_\alpha(\xi) = \widetilde{m}_\alpha(\xi, 1)$ where the function $\widetilde{m}_\alpha : \mathbb{R}^{n+1} \setminus \{0\} \to \mathbb{C}$ is defined by

$$\widetilde{m}_\alpha(\xi, \mu) := \frac{\xi^\alpha \mu^{2-|\alpha|}}{\mu^2 + |\xi|^2}.$$

As the function $\widetilde{m}_\alpha$ is smooth and homogeneous of degree 0, it satisfies the Mikhlin condition by Lemma 3.23. Setting $\mu = 1$, we see that also m_α satisfies the Mikhlin condition. □

As mentioned above, we also need vector-valued variants of Mikhlin's theorem. The following results assume some geometric conditions on the Banach space X. For a detailed discussion of these properties, see, e.g., [23], Chap. 4.

Definition 3.25 (a) A Banach space X is called a UMD space or a space of class HT if the symbol $m(\xi) := -i \operatorname{sgn}(\xi) \operatorname{id}_X$ yields a bounded operator $\operatorname{op}[m] \in L(L^p(\mathbb{R}; X))$. The operator $\operatorname{op}[m]$ is called the Hilbert transform.

(b) A Banach space X is said to have property (α) if there exists a constant $C > 0$ such that for all $N \in \mathbb{N}$, all i.i.d. symmetric $\{-1, 1\}$-valued random variables $\varepsilon_1, \dots, \varepsilon_N$ on Ω and $\varepsilon'_1, \dots, \varepsilon'_N$ on Ω', all $\alpha_{ij} \in \mathbb{C}$ with $|\alpha_{ij}| \le 1$, and all $x_{ij} \in X$ we have

$$\left\| \sum_{i,j=1}^N \alpha_{ij} \varepsilon_i \varepsilon'_j x_{ij} \right\|_{L^2(\Omega \times \Omega'; X)} \le C \left\| \sum_{i,j=1}^N \varepsilon_i \varepsilon'_j x_{ij} \right\|_{L^2(\Omega \times \Omega'; X)}.$$

Remark 3.26 (a) Every UMD space is reflexive, i.e., the canonical embedding into its bidual space is surjective (cf. [23], Section B.1.c), as can be seen in [23], Theorem 4.3.3. In particular, $L^1(G)$ and $L^\infty(G)$ are no UMD spaces. However, $L^1(G)$ has property (α).

(b) Every Hilbert space is a UMD space with property (α). If E is a UMD space with property (α) and if (S, σ, μ) is a σ-finite measure space, then also $L^p(S; E)$ is a UMD space with property (α) for all $p \in (1, \infty)$ (see [23], Proposition 4.2.15).

(c) More generally, if $G \subset \mathbb{R}^n$ is a domain, E is a UMD space with property (α) and $p, q \in (1, \infty)$, then the vector-valued Besov space $B^s_{pq}(G; E)$ and the vector-valued Triebel-Lizorkin space $F^s_{pq}(G; E)$ are again UMD spaces with property (α), see [23], Example 4.2.18. In particular, this holds in the scalar case $E = \mathbb{C}$.

The following result is the vector-valued analog of Mikhlin's theorem and was central in the development of the theory and application of maximal L^p-regularity.

Theorem 3.27 *Let X and Y be UMD Banach spaces, and let $1 < p < \infty$. Assume $m \in C^n(\mathbb{R}^n \setminus \{0\}; L(X, Y))$ with*

$$\mathcal{R}\Big(\big\{|\xi|^{|\alpha|}\partial^\alpha m(\xi) : \xi \in \mathbb{R}^n \setminus \{0\},\ \alpha \in \{0, 1\}^n\big\}\Big) =: \kappa < \infty.$$

Then m is a vector-valued Fourier multiplier, and for the norm of $\operatorname{op}[m]$ (*see* (3.5)) we have

$$\|\operatorname{op}[m]\|_{L(L^p(\mathbb{R}^n;X),L^p(\mathbb{R}^n;Y))} \leq C\kappa,$$

where the constant C depends only on n, p, X, and Y.

The proof of Theorem 3.27 uses Paley-Littlewood decompositions, see [25], Theorem 4.6, or [23], Theorem 5.3.18.

In the last result, we had one symbol m and the related operator $\operatorname{op}[m]$. The following theorem shows that for a family of symbols satisfying uniform Mikhlin type estimates, also the related operator family is $\mathcal{R}$-bounded.

Theorem 3.28 *Let X and Y be UMD Banach spaces with property (α). Let $\mathcal{T} \subset L(X, Y)$ be $\mathcal{R}$-bounded. Consider the set*

$$M := \Big\{m \in C^n(\mathbb{R}^n \setminus \{0\}; L(X, Y)) : \xi^\alpha D^\alpha m(\xi) \in \mathcal{T} \quad (\xi \in \mathbb{R}^n \setminus \{0\},\ \alpha \in \{0, 1\}^n)\Big\}.$$

Then $\{\operatorname{op}[m] : m \in M\} \subset L(L^p(\mathbb{R}^n; X), L^p(\mathbb{R}^n; Y))$ is $\mathcal{R}$-bounded with $\mathcal{R}_p(\{\operatorname{op}[m] : m \in M\}) \leq C\mathcal{R}_p(\mathcal{T})$, where the constant C depends only on p, m, X, and Y.

For a proof of this result, we refer to [21], Theorem 3.2. Theorem 3.28 is also the basis for an iteration process: $\mathcal{R}$-bounded symbol families yield $\mathcal{R}$-bounded operator families. For an application to pseudodifferential operators with $\mathcal{R}$-bounded symbols, we also refer to [15].

Note that Theorem 3.28 also gives a strong result in the scalar case $X = \mathbb{C}$. As $\mathbb{C}$ is a Hilbert space, boundedness in $\mathbb{C}$ equals $\mathcal{R}$-boundedness. Therefore, boundedness of a family of scalar symbols implies $\mathcal{R}$-boundedness of the corresponding operator family. The same holds if X is a general Hilbert space. We give a simple but useful example.

Corollary 3.29 *Let $\{m_\lambda : \lambda \in \Lambda\}$ be a family of matrix valued functions $m_\lambda \in C^n(\mathbb{R}^n \setminus \{0\}; \mathbb{C}^{N\times N})$ with*

$$|\xi^\alpha D^\alpha m_\lambda(\xi)|_{\mathbb{C}^{N\times N}} \leq C_0 \quad (\xi \in \mathbb{R}^n \setminus \{0\},\ \alpha \in \{0, 1\}^n,\ \lambda \in \Lambda).$$

Then $\{\operatorname{op}[m_\lambda] : \lambda \in \Lambda\} \subset L(L^p(\mathbb{R}^n; \mathbb{C}^N))$ is $\mathcal{R}$-bounded with $\mathcal{R}$-bound $C \cdot C_0$, where C only depends on p and N.

Proof As a Hilbert space, $X = \mathbb{C}^N$ is a UMD space with property (α). By assumption, we know that

$$\left\{\xi^\alpha D_\xi^\alpha m_\lambda(\xi) : \xi \in \mathbb{R}^n \setminus \{0\},\ \alpha \in \{0,1\}^n,\ \lambda \in \Lambda\right\} \subset L(X)$$

is norm bounded and consequently, as X is a Hilbert space, also $\mathcal{R}$-bounded. Choosing $\mathcal{T} := \{A \in \mathbb{C}^{N\times N} : |A| \le C_0\}$ in Theorem 3.28, we obtain the $\mathcal{R}$-boundedness of $\{\text{op}[m_\lambda] : \lambda \in \Lambda\} \subset L(L^p(\mathbb{R}^n; \mathbb{C}^N))$. □

3.3 $\mathcal{R}$-sectorial Operators

Now we come back to the question of maximal L^p-regularity. As we have seen in Theorem 2.11, maximal regularity holds if and only if the operator-valued symbol $m(\lambda) := \lambda(\lambda - A)^{-1}$ for $\lambda \in i\mathbb{R}$ induces a bounded operator in $L^p(\mathbb{R}; X)$. So we can apply the one-dimensional case of Theorem 3.27. We start with a notion from operator theory.

In the following, let

$$\Sigma_\varphi := \left\{z \in \mathbb{C} \setminus \{0\} : |\arg(z)| < \varphi\right\}$$

for $\varphi \in (0, \pi]$. We denote the spectrum and the resolvent set of an operator A by $\sigma(A)$ and $\rho(A)$, respectively.

Definition 3.30 Let $A\colon D(A) \to X$ be a linear and densely defined operator. Then A is called sectorial if there exists an angle $\varphi > 0$ such that $\rho(A) \supset \Sigma_\varphi$ and

$$\sup_{\lambda\in\Sigma_\varphi} \|\lambda(\lambda - A)^{-1}\|_{L(X)} < \infty.$$

If this is the case, we call

$$\varphi_A := \sup\{\varphi :\ \rho(A) \supset \Sigma_\varphi,\ \sup_{\lambda\in\Sigma_\varphi} \|\lambda(\lambda - A)^{-1}\|_{L(X)} < \infty\}$$

the spectral angle of A.

The following theorem is an important result from the theory of semigroups of operators (see, e.g., [18] , Theorem II.4.6).

Theorem 3.31 *Let $A\colon D(A) \mapsto X$ be linear and densely defined. Then the following statements are equivalent:*

(i) A generates a bounded holomorphic C_0-semigroup on X with angle $\vartheta \in (0, \frac{\pi}{2}]$.
(ii) A is sectorial with spectral angle $\varphi_A \ge \vartheta + \frac{\pi}{2}$.

It turns out that a similar condition characterizes operators with maximal L^p-regularity. For the following result, cf. [13], Theorem 4.4, [34], Theorem 4.2, and [25], Theorem 1.11.

Theorem 3.32 (Theorem of Weis) *Let X be a UMD Banach space, $1 < p < \infty$, and A be a sectorial operator with spectral angle $\varphi_A > \frac{\pi}{2}$. Then $A \in \mathrm{MR}((0,\infty); X)$ if the family*

$$\big\{\lambda(\lambda - A)^{-1} : \lambda \in \Sigma_\varphi\big\} \subset L(X)$$

is $\mathcal{R}$-bounded for some $\varphi > \frac{\pi}{2}$.

With respect to the last theorem, one defines $\mathcal{R}$-sectorial operators:

Definition 3.33 Let $A\colon D(A) \to X$ be a linear and densely defined operator. Then A is called $\mathcal{R}$-sectorial if there exists an angle $\varphi > 0$ with $\rho(A) \supset \Sigma_\varphi$ and

$$\mathcal{R}\big\{\lambda(\lambda - A)^{-1} : \lambda \in \Sigma_\varphi\big\} < \infty.$$

The $\mathcal{R}$-angle of A is defined as the supremum of all angles for which the above $\mathcal{R}$-bound is finite.

By Theorem 3.32, a sectorial operator has maximal regularity if it is $\mathcal{R}$-sectorial with $\mathcal{R}$-angle larger than $\frac{\pi}{2}$. In fact, one has the following equivalences.

Theorem 3.34 *Let A be the generator of a bounded holomorphic C_0-semigroup T. Then the following statements are equivalent:*

(i) There exists a $\delta > 0$ such that A is $\mathcal{R}$-sectorial with $\mathcal{R}$-angle $\varphi_{\mathcal{R}} = \frac{\pi}{2} + \delta$.
(ii) There exists an $n \in \mathbb{N}$ such that $\{t^n(it - A)^{-n} : t \in \mathbb{R} \setminus \{0\}\}$ is $\mathcal{R}$-bounded.
(iii) There exists a $\delta > 0$ such that the family $\{T_z : z \in \Sigma_\delta\}$ is $\mathcal{R}$-bounded.
(iv) The family $\{T_t, tAT_t : t > 0\}$ is $\mathcal{R}$-bounded.

Proof We only give a sketch of proof, for the full version see [25], Theorem 1.11.
(i)$\Longrightarrow$(ii) is trivial.
(ii)$\Longrightarrow$(i). We write

$$(it - A)^{-n+1} = (n-1)i \int_t^\infty (is - A)^{-n} ds$$

and obtain

$$(it)^{n-1}(it - A)^{-n+1} = \int_0^\infty h_t(s)\big[(is)^n(is - A)^{-n}\big]ds$$

for the function $h_t(s) := (n-1)t^{n-1}s^{-n}\chi_{[t,\infty)}$. We have $\int_0^\infty h_t(s)ds = 1$, and Corollary 3.15 yields (ii) for $n-1$ instead of n. Iteratively, we see that (ii) holds for $n = 1$. Now we use Corollary 3.16 to show the $\mathcal{R}$-boundedness of $\{\lambda(\lambda - A)^{-1} : \lambda \in \Sigma_{\pi/2}\}$. By considering power series expansion, one can show that $\lambda(\lambda - A)^{-1}$ is in fact $\mathcal{R}$-bounded on some larger sector.

(iii)$\Longrightarrow$(i). This follows from Corollary 3.15, too, with help of the representation

$$(\lambda - A)^{-1} = \int_0^\infty e^{-\lambda t} T_t dt.$$

(i)$\Longrightarrow$(iii) follows similarly by

$$T_z = \frac{1}{2\pi i} \int_{\Gamma_t} e^{\lambda z} (\lambda - A)^{-1} d\lambda.$$

(iii)$\Longleftrightarrow$(iv) can be shown using Corollary 3.16. □

4 L^p-Sobolev Spaces

In the definition of maximal regularity, the vector-valued Sobolev space $W_p^1(J; X)$ appears. In many cases, $X = L^p(G)$ for some domain $G \subset \mathbb{R}^n$, and it would be desirable to obtain a more explicit description of the space $\gamma_t \mathbb{E}$ of time traces in this situation. Note that Lemma 2.4 tells us that this is connected with real interpolation. A similar question arises if the operator A is a differential operator in some domain $G \subset \mathbb{R}^n$. In this case, the domain $D(A)$ is described by boundary operators, and the spaces for the boundary traces will be non-integer Sobolev spaces. For $p \neq 2$, there are different scales of non-integer Sobolev spaces: Besov spaces, Triebel-Lizorkin spaces, and Bessel potential spaces. A modern definition of these scales is based on dyadic decomposition and on the Fourier transform. A classical reference for this is the book by Triebel ([32], Sect. 2.3), where the scalar case is discussed. For a modern presentation, including the vector-valued situation, we mention the monograph by Amann ([7], Chap. VII). Note that in the vector-valued situation, the related integrals are Bochner integrals, and we refer to [23], Sect. 1, and [1], Sect. 1.1, for an introduction to vector-valued integration.

Definition 4.1 A sequence $(\varphi_k)_{k\in\mathbb{N}_0}$ of C^∞-functions $(\varphi_k)_{k\in\mathbb{N}_0}$ is called a dyadic decomposition if

(i) $\varphi_k \geq 0$, $\operatorname{supp} \varphi_0 \subset B(0, 2)$ and $\operatorname{supp} \varphi_k \subset \{\xi \in \mathbb{R}^n : 2^{k-1} < |\xi| < 2^{k+1}\}$ for all $k \in \mathbb{N}$,
(ii) $\sum_{k\in\mathbb{N}_0} \varphi_k(\xi) = 1$ for all $\xi \in \mathbb{R}^n$,
(iii) for all $\alpha \in \mathbb{N}_0^n$ there exists a $c_\alpha > 0$ with

$$|\xi|^{|\alpha|} \left|\partial^\alpha \varphi_k(\xi)\right| \leq c_\alpha \quad (\xi \in \mathbb{R}^n,\ k \in \mathbb{N}_0).$$

It is easy to define a dyadic decomposition by scaling a fixed function φ_1 (see [32], Sect. 2.3.1). The definitions of the Sobolev spaces are based on the family $(\operatorname{op}[\varphi_k])_{k\in\mathbb{N}_0}$, see (3.5). By the theorem of Paley-Wiener, for every $u \in \mathscr{S}'(\mathbb{R}^n)$ the distribution $\operatorname{op}[\varphi_k]u$ is a regular distribution and even a smooth function. Therefore, $(\operatorname{op}[\varphi_k]u)(x)$ is well defined. In the following, let X be a Banach space.

Definition 4.2 (a) For $s \in \mathbb{R}$, $p, q \in [1, \infty)$, the Besov space $B^s_{pq}(\mathbb{R}^n; X)$ is defined by $B^s_{pq}(\mathbb{R}^n; X) := \{u \in \mathscr{S}'(\mathbb{R}^n; X) : \|u\|_{B^s_{pq}(\mathbb{R}^n;X)} < \infty\}$, where

$$\|u\|_{B^s_{pq}(\mathbb{R}^n;X)} = \left[\sum_{k\in\mathbb{N}_0} 2^{skq} \left(\int_{\mathbb{R}^n} \|(\mathrm{op}[\varphi_k]u)(x)\|_X^p dx \right)^{q/p} \right]^{1/q}.$$

(b) For $s \in \mathbb{R}$ and $p, q \in [1, \infty)$ the Triebel-Lizorkin space $F^s_{pq}(\mathbb{R}^n; X)$ is defined by $F^s_{pq}(\mathbb{R}^n; X) := \{u \in \mathscr{S}'(\mathbb{R}^n; X) : \|u\|_{F^s_{pq}(\mathbb{R}^n;X)} < \infty\}$, where

$$\|u\|_{F^s_{pq}(\mathbb{R}^n;X)} = \left[\int_{\mathbb{R}^n} \left(\sum_{k\in\mathbb{N}_0} 2^{skq} \|(\mathrm{op}[\varphi_k]u)(x)\|_X^q \right)^{p/q} dx \right]^{1/p}.$$

(c) If $p = \infty$ or $q = \infty$, the above definitions hold with the standard modification.

By an application of Fubini's theorem, we immediately see that for $p = q$ the definitions of Besov spaces and Triebel-Lizorkin spaces coincide, but in general we have two different scales of Sobolev space type. For the third scale, the Bessel potential spaces, we consider the function $\langle\,\cdot\,\rangle\colon \mathbb{R}^n \to \mathbb{R},\ \xi \mapsto \langle\xi\rangle := (1 + |\xi|^2)^{1/2}$. For the following definition, we refer to [23], Definition 5.6.2.

Definition 4.3 Let $s \in \mathbb{R}$ and $p \in [1, \infty]$. Then the Bessel potential space $H^s_p(\mathbb{R}^n; X)$ is defined as the space of all $u \in \mathscr{S}'(\mathbb{R}^n; X)$ for which $\mathrm{op}[\langle\,\cdot\,\rangle^s]u \in L^p(\mathbb{R}^n; X)$. The corresponding norm is defined as

$$\|u\|_{H^s_p(\mathbb{R}^n;X)} := \|\,\mathrm{op}[\langle\,\cdot\,\rangle^s]u\|_{L^p(\mathbb{R}^n;X)}.$$

Remark 4.4 (a) Many classical Sobolev spaces can be found as special cases of the above definition.

- Let X be a UMD space, $k \in \mathbb{N}$, and $p \in (1, \infty)$, and let $W^k_p(\mathbb{R}^n; X)$ denote the classical Sobolev space,

  $$W^k_p(\mathbb{R}^n; X) := \big\{u \in L^p(\mathbb{R}^n; X) : \forall\, |\alpha| \le k : \partial^\alpha u \in L^p(\mathbb{R}^n; X)\big\}.$$

  Then $W^k_p(\mathbb{R}^n; X) = H^k_p(\mathbb{R}^n; X)$ with equivalent norms ([23], Theorem 5.6.11).

- Let $p \in (1, \infty)$ and $s \in \mathbb{R}$. Then the equality $H^s_p(\mathbb{R}^n; X) = F^s_{p2}(\mathbb{R}^n; X)$ holds if and only if X is isomorphic to a Hilbert space ([22], Theorem 1.2).
- Let $p \in [1, \infty)$ and $s \in (0, \infty) \setminus \mathbb{N}$. Then the Sobolev-Slobodeckii space $W^s_p(\mathbb{R}^n; X)$ is given as $W^s_p(\mathbb{R}^n; X) = B^s_{pp}(\mathbb{R}^n; X)$ ([7], Remark 3.6.4).
- Let $s \in (0, \infty) \setminus \mathbb{N}$. Then the classical Hölder space is given as $C^s(\mathbb{R}^n; X) = B^s_{\infty,\infty}(\mathbb{R}^n; X)$ ([7], Remark 3.6.4).

(b) Let $G \subset \mathbb{R}^n$ be a domain. Then the space $B^s_{pq}(G; X)$ is defined by restriction, i.e.

$$B^s_{pq}(G; X) := \{u \in \mathscr{D}'(G; X) : \exists \widetilde{u} \in B^s_{pq}(\mathbb{R}^n; X) : u = \widetilde{u}|_G\}$$

with canonical norm

$$\|u\|_{B^s_{pq}(G;X)} := \inf \left\{ \|\widetilde{u}\|_{B^s_{pq}(\mathbb{R}^n;X)} : u = \widetilde{u}|_G \right\}.$$

Note here that the restriction of a distribution is defined as $\widetilde{u}|_G := \widetilde{u}|_{\mathscr{D}(G)}$. In the same way, the other scales are defined on domains.

The following result can be shown with the theory of interpolation spaces and is the basis for the description of the trace spaces. We refer to [7], Theorem 2.7.4, for a proof (with $G = \mathbb{R}^n$, the case of a domain can be handled by a retraction-coretraction argument if the domain is smooth enough).

Theorem 4.5 *Let $G \subset \mathbb{R}^n$ be a sufficiently smooth domain, and let $p, q \in (1, \infty)$, $k \in \mathbb{N}$, and $s \in (0, k)$. Then*

$$B^s_{pq}(G; X) = (L^p(G; X), W^k_p(G; X))_{s/k,q}.$$

From this theorem and the description of the trace spaces as real interpolation space, one can easily obtain $\gamma_0 W^k_p(G; X) = B^{k-1/p}_{pp}(\partial G; X)$, where $\gamma_0 u := u|_{\partial G}$ stands for the trace on the boundary of the domain. This typical loss of derivatives of order $1/p$ leads to non-integer Sobolev spaces for inhomogeneous boundary data. For parabolic equations, we also have to consider time and boundary traces of the solution space:

Corollary 4.6 *Let $G \subset \mathbb{R}^n$ be a sufficiently smooth domain, $J = (0, T)$ with $T \in (0, \infty]$, $k \in \mathbb{N}$, and let $\mathbb{X} = W^1_p(J; L^p(G) \cap L^p(J; W^k_p(G))$ (the typical parabolic solution space).*

(a) For the time trace $\gamma_t : u \mapsto u|_{t=0}$, we obtain the trace space

$$\gamma_t \mathbb{X} = B^{k-k/p}_{pp}(G).$$

(b) For the boundary trace $\gamma_0 : u \mapsto u|_{\partial G}$, we obtain the trace space

$$\gamma_0 \mathbb{X} = B^{1-1/(kp)}_{pp}(J; L^p(\partial G)) \cap L^p(J; B^{k-1/p}_{pp}(\partial G)).$$

Proof We only give the main ideas for a proof and refer to [14], Sect. 3, for a complete version.

(a) By Lemma 2.4 a), we have $\gamma_t \mathbb{X} = (L^p(G), W^k_p(G))_{1-1/p,p}$ which equals $B^{k-k/p}_{pp}(G)$ due to Theorem 4.5 with $p = q$.

(b) Locally, we can choose a coordinate system such that the inner normal vector is the x_n-variable. Then we have to take the trace with respect to x_n instead of t

which gives by Lemma 2.4 a real interpolation space again. Computing the real interpolation space of the intersection then gives a Besov space both with respect to time and with respect to the other space variables. □

Remark 4.7 In the above corollary, we have considered functions which are L^p in time and L^p in space. If one considers functions which are L^p in time and L^q in space with $p \neq q$, a result similar to Corollary 4.6 holds, but now also Triebel-Lizorkin spaces appear. More precisely, for $\mathbb{X} := W_p^1(J; L^q(G)) \cap L^p(J; W_q^k(G))$ we obtain (see [14], Sect. 6, and [28], Sect. 4)

$$\begin{aligned}\gamma_t \mathbb{X} &= B_{qp}^{k-k/p}(G),\\ \gamma_0 \mathbb{X} &= F_{pq}^{1-1/(kq)}(J; L^p(\partial G)) \cap L^q(J; B_{qq}^{k-1/q}(\partial G)).\end{aligned}$$

5 Parabolic PDE Systems in the Whole Space

As a first application of the previous results, we now consider parabolic systems of partial differential equations in the whole space $\mathbb{R}^n$. In the following, let $1 < p < \infty$ and $\mathbb{C}_+ := \{z \in \mathbb{C} : \operatorname{Re} z > 0\} = \Sigma_{\pi/2}$. We assume that we have a linear differential operator $A = A(x, D)$ of the form

$$A(x, D) = \sum_{|\alpha| \le 2m} a_\alpha(x) D^\alpha$$

with $m \in \mathbb{N}$ and matrix-valued coefficients $a_\alpha : \mathbb{R}^n \to \mathbb{C}^{N\times N}$. Recall that $D := -i\partial$. The definition of parabolicity below is based on the concept of parameter-ellipticity which was developed by Agmon [4] and Agranovich-Vishik [9].

For the formal differential operator $A = A(x, D)$, we define its symbol

$$a(x, \xi) := \sum_{|\alpha| \le 2m} a_\alpha(x) \xi^\alpha$$

and the principal symbol

$$a_0(x, \xi) := \sum_{|\alpha| = 2m} a_\alpha(x) \xi^\alpha.$$

Both symbols map $\mathbb{R}^n \times \mathbb{R}^n$ into $\mathbb{C}^{N\times N}$. The L^p-realization A_p of $A(x, D)$ is defined as the unbounded linear operator $A_p : L^p(\mathbb{R}^n; \mathbb{C}^N) \supset D(A_p) \to L^p(\mathbb{R}^n; \mathbb{C}^N)$ with

$$D(A_p) := W_p^{2m}(\mathbb{R}^n; \mathbb{C}^N), \;\; A_p u := A(x, D)u \quad (u \in W_p^{2m}(\mathbb{R}^n; \mathbb{C}^N)).$$

Definition 5.1 The operator $A(x, D)$ is called parameter-elliptic with angle $\varphi \in (0, \pi]$ if

$$\big|\det(a_0(x,\xi)-\lambda)\big| \ge C_P\big(|\xi|^{2m}+|\lambda|\big)^N \quad \big(x\in\mathbb{R}^n,\ (\xi,\lambda)\in(\mathbb{R}^n\times\overline{\Sigma}_\varphi)\setminus\{0\}\big). \tag{5.1}$$

If this holds for $\varphi=\frac{\pi}{2}$ (i.e., $\Sigma_\varphi=\mathbb{C}_+$), then $\partial_t - A$ is called parabolic.

Remark 5.2 (a) For every fixed $x\in\mathbb{R}^n$, the map $(\xi,\lambda)\mapsto p(x,\xi,\lambda):=\det(a_0(x,\xi)-\lambda)$ is quasi-homogeneous in the sense that

$$p(x,r\xi,r^{2m}\lambda)=r^{2mN}p(x,\xi,\lambda) \quad \big(r>0,\ (\xi,\lambda)\in(\mathbb{R}^n\times\overline{\Sigma}_\varphi)\setminus\{0\}\big).$$

Therefore, it is sufficient to consider the compact set $\{(\xi,\lambda): |\xi|^{2m}+|\lambda|=1\}$. The operator $A(x,D)$ is parameter-elliptic if and only if

$$\inf\Big\{|\det(a_0(x,\xi)-\lambda)| : x\in\mathbb{R}^n,\ (\xi,\lambda)\in\mathbb{R}^n\times\overline{\Sigma}_\varphi \text{ with } |\xi|^{2m}+|\lambda|=1\Big\}>0.$$

(b) If $a_\alpha\in L^\infty(\mathbb{R}^n)$ for all $|\alpha|<2m$, then the lower-order terms of the symbol can be estimated uniformly in x. Therefore, $A(x,D)$ is parameter-elliptic if and only if there exist constants $C,R>0$ with

$$|\det(a(x,\xi)-\lambda)|\ge C(|\xi|^{2m}+|\lambda|)^N \quad (x\in\mathbb{R}^n,\ \lambda\in\overline{\Sigma}_\varphi,\ |\xi|\ge R).$$

This is one possible definition of parameter-elliticity and parabolicity for pseudodifferential operators. We remark that the principal symbol of a pseudodifferential operator is defined only for so-called classical symbols.

Remark 5.3 If $\partial_t - A(x,D)$ is parabolic in the sense of parameter-ellipticity in the closed sector $\overline{\mathbb{C}}_+$, then $A(x,D)$ is also parameter-elliptic in some larger sector $\overline{\Sigma}_\theta$ with $\theta>\frac{\pi}{2}$. In fact, it is easily seen that the set of all angles of rays with respect to λ, in which condition (5.1) holds, is open.

Following a standard approach in elliptic theory, we first consider the so-called model problem and then use perturbation results for variable coefficients. The remainder of this section is based on [13], Sects. 5 and 6, and [25], Sects. 6 and 7.

Theorem 5.4 *Let $A(D)=\sum_{|\alpha|=2m}a_\alpha D^\alpha$ with constant coefficients $a_\alpha\in\mathbb{C}^{N\times N}$ ($|\alpha|=2m$) and without lower-order terms. If $\partial_t - A(D)$ is parabolic with parabolicity constant C_P in (5.1), then $\rho(A_p)\supset\overline{\mathbb{C}}_+\setminus\{0\}$, and the set*

$$\big\{\lambda(\lambda-A_p)^{-1} : \lambda\in\overline{\mathbb{C}}_+\setminus\{0\}\big\}$$

is $\mathcal{R}$-bounded. Here, the $\mathcal{R}$-bound only depends on p,n,m,N,C_P and

$$M:=\sum_{|\alpha|=2m}\|a_\alpha\|_{\mathbb{C}^{N\times N}}.$$

In particular, A_p is $\mathcal{R}$-sectorial with $\mathcal{R}$-angle larger than $\frac{\pi}{2}$, and A_p has maximal L^q-regularity for all $q\in(1,\infty)$.

Proof Note that because of (5.1), for $\lambda \in \overline{\mathbb{C}}_+ \setminus \{0\}$ and $\xi \in \mathbb{R}^n$ the symbol $(\lambda - a_0(\xi))^{-1}$ is well defined. We show that the family $\{m_\lambda : \lambda \in \overline{\mathbb{C}}_+ \setminus \{0\}\}$ with $m_\lambda(\xi) := \lambda(\lambda - a_0(\xi))^{-1}$ satisfies the assumptions of Corollary 3.29.

For any $r > 0$ we have $r^{2m}\lambda - a_0(r\xi) = r^{2m}(\lambda - a_0(\xi))$. Therefore, the map $(\xi, \lambda) \mapsto \frac{1}{\lambda}(\lambda - a_0(\xi))$ is quasi-homogeneous in (ξ, λ) of degree 0, and the same holds for its inverse $(\xi, \lambda) \mapsto \lambda(\lambda - a_0(\xi))^{-1}$. By Lemma 3.23, m_λ satisfies the Mikhlin condition uniformly with respect to λ. Now we can apply Corollary 3.29 to obtain the $\mathcal{R}$-boundedness of $\{\mathrm{op}[m_\lambda] : \lambda \in \overline{\mathbb{C}}_+ \setminus \{0\}\} \subset L(L^p(\mathbb{R}^n))$. Because of $\frac{1}{\lambda}\,\mathrm{op}[m_\lambda](\lambda - A_p) = \mathrm{id}_{W_p^{2m}(\mathbb{R}^n)}$ and $\frac{1}{\lambda}(\lambda - A_p)\,\mathrm{op}[m_\lambda] = \mathrm{id}_{L^p(\mathbb{R}^n)}$, we see that $\mathrm{op}[m_\lambda] = \lambda(\lambda - A_p)^{-1}$. By Corollary 3.29, A_p is $\mathcal{R}$-sectorial with angle larger than $\frac{\pi}{2}$, and Theorem 3.32 implies that A_p has maximal L^q-regularity for all $q \in (1, \infty)$. To show the statement on the $\mathcal{R}$-bound, we have to quantify the Mikhlin constant.

For this, we write

$$(\lambda - a_0(\xi))^{-1} = \frac{1}{\det(\lambda - a_0(\xi))} b(\xi, \lambda)$$

with the adjunct matrix $b(\xi, \lambda)$. The coefficients of $b(\xi, \lambda)$ are determinants of $(N-1) \times (N-1)$-matrices which are constructed by omitting one row and one column of the matrix $\lambda - a_0(\xi)$. Therefore, we obtain

$$\|b(\xi, \lambda)\|_{\mathbb{C}^{N\times N}} \le C(m, n, M, N)(|\xi|^{2m} + |\lambda|)^{N-1}.$$

Due to (5.1), we get

$$\|\lambda(\lambda - a_0(\xi))^{-1}\|_{\mathbb{C}^{N\times N}} \le C(m, n, M, N, C_p)\frac{|\lambda|}{|\xi|^{2m} + |\lambda|} \le C(m, n, M, N, C_p).$$

For the derivatives, we note that

$$\begin{aligned}\Big\|\xi_k \frac{\partial}{\partial \xi_k} a_0(\xi)\Big\|_{\mathbb{C}^{N\times N}} &= \Big\|\xi_k \frac{\partial}{\partial \xi_k} \sum_{|\alpha|=2m} a_\alpha \xi^\alpha\Big\|_{\mathbb{C}^{N\times N}} \le \sum_{|\alpha|=2m} \|a_\alpha\|_{\mathbb{C}^{N\times N}} \Big|\xi_k \frac{\partial}{\partial \xi_k} \xi^\alpha\Big| \\ &\le 2mM|\xi|^{2m}.\end{aligned}$$

Iteratively, we obtain $\|\xi^\alpha \partial_\xi^\alpha a_0(\xi)\|_{\mathbb{C}^{N\times N}} \le C(m, n, M, N)|\xi|^{2m}$ for all $\alpha \in \{0, 1\}^n$. In the same way, the derivative of $b(\xi, \lambda)$ can be estimated. This yields

$$\|\xi^\alpha \partial_\xi^\alpha b(\xi, \lambda)\|_{\mathbb{C}^{N\times N}} \le C(m, n, M, N)(|\xi|^{2m} + |\lambda|)^{N-1}.$$

With the product rule (Leibniz rule) we have for the inverse matrix the inequality

$$\|\xi^\alpha D_\xi^\alpha (\lambda - a_0(\xi))^{-1}\|_{\mathbb{C}^{N\times N}} \le C(m, n, M, N, C_P)(|\xi|^{2m} + |\lambda|)^{-1},$$

and therefore $\|\xi^\alpha D_\xi^\alpha m_\lambda(\xi)\|_{\mathbb{C}^{N\times N}} \le C(m, n, M, N, C_P)$ for all $\alpha \in \{0, 1\}^n$, $\lambda \in \overline{\mathbb{C}}_+ \setminus \{0\}$ and all $\xi \in \mathbb{R}^n$. By Corollary 3.29, the $\mathcal{R}$-bound of $\{\lambda(\lambda - A_p)^{-1} : \lambda \in \overline{\mathbb{C}}_+ \setminus \{0\}\}$ only depends on m, n, M, N, C_P, and p. □

To generalize the above result to operators with variable coefficients, we need perturbation results for $\mathcal{R}$-boundedness. For this, we define for an $\mathcal{R}$-sectorial operator A with $\mathcal{R}$-angle $\varphi_{\mathcal{R}}(A)$ and for $\theta \in (0, \varphi_{\mathcal{R}}(A))$:

$$\begin{aligned} M_\theta(A) &:= \sup\big(\{\|\lambda(\lambda - A)^{-1}\| : \lambda \in \Sigma_\theta\}\big), \\ \widetilde{M}_\theta(A) &:= \sup\big(\{\|A(\lambda - A)^{-1}\| : \lambda \in \Sigma_\theta\}\big), \\ R_\theta(A) &:= \mathcal{R}\big(\{\lambda(\lambda - A)^{-1} : \lambda \in \Sigma_\theta\}\big), \\ \widetilde{R}_\theta(A) &:= \mathcal{R}\big(\{A(\lambda - A)^{-1} : \lambda \in \Sigma_\theta\}\big). \end{aligned}$$

Note that $\widetilde{M}_\theta(A)$ is finite because of $A(\lambda - A)^{-1} = \lambda(\lambda - A)^{-1} - 1$, and the same holds for $\widetilde{R}_\theta(A)$.

Theorem 5.5 *Let X be a Banach space and A be an $\mathcal{R}$-sectorial operator in X with angle $\varphi_{\mathcal{R}}(A) > 0$. Further, let $\theta \in (0, \varphi_{\mathcal{R}}(A))$, and let B be a linear operator in X with $D(B) \supset D(A)$ and*

$$\|Bx\| \le a\|Ax\| \quad (x \in D(A)). \tag{5.2}$$

If $a < \frac{1}{\widetilde{R}_\theta(A)}$, then $A + B$ is $\mathcal{R}$-sectorial, too, with angle larger or equal to θ and

$$R_\theta(A + B) \le \frac{R_\theta(A)}{1 - a\widetilde{R}_\theta(A)}\,.$$

Proof For $\lambda \in \overline{\Sigma_\theta} \setminus \{0\}$ one obtains

$$\|B(\lambda - A)^{-1}x\| \le a\|A(\lambda - A)^{-1}x\| \le a\widetilde{M}_\theta(A)\|x\| \quad (x \in X).$$

Because of $a < \frac{1}{\widetilde{R}_\theta(A)}$, the operator $1 + B(\lambda - A)^{-1}$ is invertible, and we get

$$\begin{aligned} (\lambda - (A + B))^{-1} &= (\lambda - A)^{-1}\big[1 + B(\lambda - A)^{-1}\big]^{-1} \\ &= (\lambda - A)^{-1}\sum_{n=0}^{\infty}(-B(\lambda - A)^{-1})^n. \end{aligned}$$

In particular, $\rho(A + B) \supset \Sigma_\theta$. By definition of $\mathcal{R}$-boundedness and due to the assumption, we get

$$\mathcal{R}(\{B(\lambda - A)^{-1} : \lambda \in \Sigma_\theta\}) \le a\mathcal{R}(\{A(\lambda - A)^{-1} : \lambda \in \Sigma_\theta\}) = a\widetilde{R}_\theta(A).$$

Inserting this into the above series yields

$$R_\theta(A+B) \le \frac{R_\theta(A)}{1 - a\widetilde{R}_\theta(A)}\,.$$

This shows that also $A + B$ is $\mathcal{R}$-sectorial with $\mathcal{R}$-angle $\ge \theta$. □

The second perturbation results deals with the case where we have an additional term $\|x\|$ on the right-hand side of (5.2). However, now the $\mathcal{R}$-sectoriality of the operator holds only with an additional shift in the operator.

Theorem 5.6 *Let A be $\mathcal{R}$-sectorial with angle $\varphi_{\mathcal{R}}(A) > 0$, and let $\theta \in (0, \varphi_{\mathcal{R}}(A))$. Let B be a linear operator satisfying $D(B) \supset D(A)$ and*

$$\|Bx\| \le a\|Ax\| + b\|x\| \quad (x \in D(A))$$

with constants $b \ge 0$ and $0 \le a < \big[\widetilde{M}_\theta(A)\widetilde{R}_\theta(A)\big]^{-1}$. Then $A + B - \mu$ is $\mathcal{R}$-sectorial for

$$\mu > \frac{bM_\theta(A)\widetilde{R}_\theta(A)}{1 - a\widetilde{M}_\theta(A)\widetilde{R}_\theta(A)}.$$

For the $\mathcal{R}$-angle, we have $\varphi_{\mathcal{R}}(A + B - \mu) \ge \theta$.

Proof For $\mu > 0$, the following inequalities hold

$$\begin{aligned}\|B(A-\mu)^{-1}x\| &\le a\|A(A-\mu)^{-1}x\| + b\|(A-\mu)^{-1}x\| \\ &\le \big(a\widetilde{M}_\theta(A) + \tfrac{b}{\mu}M_\theta(A)\big)\|x\| \quad (x \in X).\end{aligned}$$

Therefore, B satisfies the assumption of Theorem 5.5 with A being replaced by $A - \mu$. In Theorem 5.5, the condition for the constants is given by $c(\mu)\widetilde{R}_\theta(A) < 1$, where $c := a\widetilde{M}_\theta(A) + \frac{b}{\mu}M_\theta(A)$. Because of $a\widetilde{M}_\theta(A) < 1$, this is the case if

$$\mu > \frac{bM_\theta(A)\widetilde{R}_\theta(A)}{1 - a\widetilde{M}_\theta(A)\widetilde{R}_\theta(A)}.$$

□

The above perturbation results allow us to treat small perturbations in the principal part of the differential operator.

Lemma 5.7 *Let $A(x, D) = \sum_{|\alpha|=2m} a_\alpha D^\alpha$ with $a_\alpha \in \mathbb{C}^{N\times N}$ for $|\alpha| = 2m$, and assume $\partial_t - A(x, D)$ to be parabolic with constant C_P. Then there exists some $\theta > \frac{\pi}{2}$ such that $A(x, D)$ is parameter-elliptic in $\overline{\Sigma_\theta}$, and there exist $\varepsilon > 0$ and $K > 0$ such that for all operators $B(x, D) = \sum_{|\alpha|=2m} b_\alpha(x)D^\alpha$ with $b_\alpha \in L^\infty(\mathbb{R}^n; \mathbb{C}^{N\times N})$ and*

$$\sum_{|\alpha|=2m} \|b_\alpha\|_\infty < \varepsilon$$

the inequality

$$\mathcal{R}\Big(\big\{\lambda(\lambda - (A_p + B_p))^{-1} : \lambda \in \overline{\Sigma}_\theta \setminus \{0\}\big\}\Big) \le K$$

holds. Here, ε and K only depend on n, p, m, N, C_P.

Proof Let $\varepsilon > 0$ and $f \in W_p^{2m}(\mathbb{R}^n; \mathbb{C}^N)$. Then the inequality

$$\|Bf\|_{L^p(\mathbb{R}^n;\mathbb{C}^N)} \le \sum_{|\alpha|=2m} \|b_\alpha\|_\infty \|D^\alpha f\|_{L^p(\mathbb{R}^n;\mathbb{C}^N)} \le \varepsilon \max_{|\alpha|=2m} \|D^\alpha f\|_{L^p(\mathbb{R}^n;\mathbb{C}^N)},$$

holds if B satisfies the above condition. We write

$$D^\alpha f = (\mathscr{F}^{-1} m_\alpha \mathscr{F}) A(D) f$$

with

$$m_\alpha(\xi) := \xi^\alpha \left(\sum_{|\beta|=2m} a_\beta \xi^\beta \right)^{-1}.$$

Then $m_\alpha \in C^\infty(\mathbb{R}^n \setminus \{0\}; \mathbb{C}^{N\times N})$, and m_α is homogeneous of degree 0 and therefore satisfies the Mikhlin condition. Consequently, there exists some $C_1 > 0$ such that we have

$$\|\operatorname{op}[m_\alpha]\|_{L(L^p(\mathbb{R}^n;\mathbb{C}^N))} \le C_1 \quad (|\alpha| = 2m).$$

Choose $\varepsilon < \big[C_1(\widetilde{R}_\theta(A) + 1)\big]^{-1}$. Then

$$\|Bf\|_{L^p(\mathbb{R}^n;\mathbb{C}^N)} \le \varepsilon C_1 \|Af\|_{L^p(\mathbb{R}^n;\mathbb{C}^N)} \le a\|Af\|_{L^p(\mathbb{R}^n;\mathbb{C}^N)}$$

with $a = \frac{1}{\widetilde{R}_\theta(A)+1}$. By Theorem 5.5, the operator $A_p + B_p$ is $\mathcal{R}$-sectorial with angle $\ge \theta$, and

$$R_\theta(A + B) \le \frac{R_\theta(A)}{1 - a\widetilde{R}_\theta(A)} =: K.$$

□

In the next step, we consider an operator A whose coefficients in the principal part are bounded and uniformly continuous. We can reduce this situation to the small perturbation from the last lemma by introducing an infinite partition of unity. This is done in the following lemma.

Lemma 5.8 *For every $r > 0$ there exists $\varphi \in \mathscr{D}(\mathbb{R}^n)$ with $0 \le \varphi \le 1$, $\operatorname{supp}\varphi \subset (-r, r)^n$ and*

$$\sum_{\ell \in r\mathbb{Z}^n} \varphi_\ell^2(x) = 1 \quad (x \in \mathbb{R}^n).$$

Here, $\varphi_\ell(x) := \varphi(x - \ell)$.

Proof (a) We first consider the case $r = 1$ and $n = 1$. Choose some $\varphi_1 \in \mathscr{D}(\mathbb{R})$ with $\varphi_1 > 0$ in $(-\frac{3}{4}, \frac{3}{4})$, $\operatorname{supp}\varphi_1 = [-\frac{3}{4}, \frac{3}{4}]$, and $\varphi_1(x) = \varphi_1(-x)$ for all $x \in \mathbb{R}$. We set

$$\varphi(x) := \begin{cases} \sqrt{\frac{\varphi_1^2(x)}{\varphi_1^2(x)+\varphi_1^2(1-x)}} & \text{if } x \in [0, \frac{3}{4}], \\ 0, & \text{if } x \in (\frac{3}{4}, \infty), \end{cases}$$

and $\varphi(x) := \varphi(-x)$ for $x < 0$. Then $\operatorname{supp}\varphi \subset (-1, 1)$, and for $x \in [0, 1]$ we obtain

$$\sum_{\ell\in\mathbb{Z}} \varphi_\ell^2(x) = \varphi^2(x) + \varphi^2(x-1) = \varphi^2(x) + \varphi^2(1-x)$$
$$= \frac{\varphi_1^2(x)}{\varphi_1^2(x) + \varphi_1^2(1-x)} + \frac{\varphi_1^2(1-x)}{\varphi_1^2(1-x) + \varphi_1^2(x)} = 1.$$

As $\sum_{\ell\in\mathbb{Z}} \varphi_\ell^2$ is periodic with period 1, we have $\sum_{\ell\in\mathbb{Z}} \varphi_\ell^2 = 1$ in $\mathbb{R}$.

(b) In the general case, define $\varphi^{(n)}(x) := \prod_{j=1}^n \varphi(\frac{x_j}{r})$ with φ from part a). Then

$$\sum_{\ell\in r\mathbb{Z}^n} (\varphi^{(n)})^2(x-\ell) = \sum_{\ell\in r\mathbb{Z}^n} \prod_{j=1}^n \varphi^2\Big(\frac{x_j-\ell_j}{r}\Big) = \sum_{\ell\in\mathbb{Z}^n} \prod_{j=1}^n \varphi^2(y_j-\ell_j)$$
$$= \prod_{j=1}^n \sum_{\ell_j\in\mathbb{Z}} \varphi^2(y_j-\ell_j) = 1$$

for $y := \frac{x}{r}$. □

We now come to the main result of this section. Here, $\mathrm{BUC}(\mathbb{R}^n)$ stands for the space of all bounded and uniformly continuous functions.

Theorem 5.9 *Let* $A(x, D) = \sum_{|\alpha|\le 2m} a_\alpha(x) D^\alpha$ *with*

$$a_\alpha \in \mathrm{BUC}(\mathbb{R}^n; \mathbb{C}^{N\times N}) \quad (|\alpha| = 2m),$$
$$a_\alpha \in L^\infty(\mathbb{R}^n; \mathbb{C}^{N\times N}) \quad (|\alpha| < 2m).$$

Let $1 < p < \infty$. *If* $\partial_t - A(x, D)$ *is parabolic, then there exist* $\theta > \frac{\pi}{2}$ *and* $\mu > 0$ *such that* $A_p - \mu$ *is* $\mathcal{R}$*-sectorial with angle* θ. *In particular,* $A_p - \mu$ *has maximal* L^q*-regularity for all* $1 < q < \infty$.

Proof As $A(x, D)$ is parameter-elliptic in $\overline{\mathbb{C}}_+$ by assumption, there exists a $\theta > \frac{\pi}{2}$ such that $A(x, D)$ is still parameter-elliptic in $\overline{\Sigma}_\theta$ (Remark 5.3). The proof of the theorem uses localization and is done in several steps. We first explain the ideas.

(1) We fix the coefficients of A at some point $\ell \in \Gamma$, where the grid $\Gamma \subset \mathbb{R}^n$ is chosen fine enough such that in each cube the localized operator A^ℓ is a small perturbation of the model problem with frozen coefficient. Here, we apply Lemma 5.7 to see that A^ℓ is still $\mathcal{R}$-sectorial.

(2) We consider the sequence $A := (A^\ell)_{\ell\in\Gamma}$ of all localized operators and show that this defines an $\mathcal{R}$-sectorial operator in some suitably chosen sequence space X_0.
(3) The L^p-realization A_p and the operator A have the same properties up to lower-order perturbations. More precisely, we have $JA_p = AJ$ and $A_pP = PA$ modulo lower order operators, where J and P are the localization and the patching operator, respectively.
(4) With the help of the interpolation inequality for Sobolev spaces, the lower-order operators can be seen as a small perturbation, and therefore the $\mathcal{R}$-sectoriality of A implies the $\mathcal{R}$-sectoriality of A_p.

In detail, these steps can be done in the following way.

(1) Choose $\varepsilon = \varepsilon(n, p, m, N, C_p)$ as in Lemma 5.7 for the operator $\sum_{|\alpha|=2m} a_\alpha(\ell) D^\alpha$ with $\ell \in \mathbb{Z}^n$. As $a_\alpha \in \mathrm{BUC}(\mathbb{R}^n; \mathbb{C}^{N\times N})$, there exists a $\delta > 0$ with

$$\sum_{|\alpha|=2m} |a_\alpha(x) - a_\alpha(y)| < \varepsilon \quad (|x - y| \le \delta).$$

Now choose $r \in (0, \delta)$ and $\varphi \in \mathscr{D}(\mathbb{R}^n)$ as in Lemma 5.8. We write $Q := (-r, r)^n$ and $Q_\ell := Q + \ell$ for $\ell \in r\mathbb{Z}^n =: \Gamma$. Choose $\psi \in \mathscr{D}(\mathbb{R}^n)$ with $\operatorname{supp}\psi \subset Q$, $0 \le \psi \le 1$, $\psi = 1$ on $\operatorname{supp}\varphi$, and set $\psi_\ell(x) := \psi(x - \ell)$ $(\ell \in \mathbb{Z})$. Define the coefficients

$$a_\alpha^\ell(x) := \begin{cases} a_\alpha(x), & x \in Q_\ell, \\ a_\alpha(\ell), & x \notin Q_\ell \end{cases} \quad (\ell \in \Gamma,\ |\alpha| = 2m)$$

and the operator $A^\ell(x, D) := \sum_{|\alpha|=2m} a_\alpha^\ell(x) D^\alpha$. For the principal part, we obtain $A_0(x, D) = A^\ell(x, D)$ $(x \in Q_\ell)$ and therefore $A_0(x, D)u = A^\ell(x, D)u$ for all $u \in W_p^{2m}(\mathbb{R}^n; \mathbb{C}^N)$ with $\operatorname{supp} u \subset Q_\ell$.

(2) Define $X_k := \ell_p(\Gamma; W_p^k(\mathbb{R}^n; \mathbb{C}^N))$ for $k \in \mathbb{N}_0$ and the operator $A\colon X_0 \supset D(A) \to X_0$ by $D(A) := X_{2m}$ and

$$A(u_\ell)_{\ell\in\Gamma} := (A^\ell u_\ell)_{\ell\in\Gamma}.$$

By Lemma 5.7, the operator A^ℓ is $\mathcal{R}$-sectorial with $R_\theta(A^\ell) \le K$, where K does not depend on ℓ. We show that the same holds for A. For this, let $T_j = \lambda_j(A - \lambda_j)^{-1}$ with $\lambda_j \in \Sigma_\theta$ and $x_j = (f_\ell^{(j)})_{\ell\in\Gamma} \in X_0$ for $j = 1, \dots, J$. Then we obtain

$$\begin{aligned}
&\Big\| \sum_{j=1}^J r_j T_j x_j \Big\|_{L^p([0,1];X_0)} \\
&= \bigg(\int_0^1 \Big\| \sum_{j=1}^J r_j(t) T_j x_j \Big\|_{X_0}^p dt \bigg)^{1/p} \\
&= \bigg(\int_0^1 \sum_{\ell\in\Gamma} \Big\| \sum_{j=1}^J r_j(t) \lambda_j (A^\ell - \lambda_j)^{-1} f_\ell^{(j)} \Big\|_{L^p(\mathbb{R}^n;\mathbb{C}^N)}^p dt \bigg)^{1/p} \\
&= \bigg(\sum_{\ell\in\Gamma} \int_0^1 \Big\| \sum_{j=1}^J r_j(t) \lambda_j (A^\ell - \lambda_j)^{-1} f_\ell^{(j)} \Big\|_{L^p(\mathbb{R}^n;\mathbb{C}^N)}^p dt \bigg)^{1/p} \\
&= \bigg(\sum_{\ell\in\Gamma} \Big\| \sum_{j=1}^J r_j \lambda_j (A^\ell - \lambda_j)^{-1} f_\ell^{(j)} \Big\|_{L^p([0,1];L^p(\mathbb{R}^n;\mathbb{C}^N))}^p \bigg)^{1/p} \\
&\le \bigg(\sum_{\ell\in\Gamma} \big[R_\theta(A^\ell) \big]^p \Big\| \sum_{j=1}^J r_j f_\ell^{(j)} \Big\|_{L^p([0,1];L^p(\mathbb{R}^n;\mathbb{C}^N))}^p \bigg)^{1/p} \\
&\le K \Big\| \sum_{j=1}^J r_j x_j \Big\|_{L^p([0,1];X_0)},
\end{aligned}$$

i.e. $R_\theta(A) \le K$.

Now we consider the localization operator $J\colon L^p(\mathbb{R}^n;\mathbb{C}^N) \to X_0,\ f \mapsto (\varphi_\ell f)_\ell$. As we have

$$\sum_{\ell\in\Gamma} \|\varphi_\ell f\|_{L^p(\mathbb{R}^n;\mathbb{C}^N)}^p \le \sum_{\ell\in\Gamma} \|\chi_{Q_\ell} f\|_{L^p(\mathbb{R}^n;\mathbb{C}^N)}^p = 2^N \|f\|_{L^p(\mathbb{R}^n;\mathbb{C}^N)}^p,$$

the operator J is continuous. In the same way, one sees that $J \in L(W_p^{2m}(\mathbb{R}^n;\mathbb{C}^N), X_{2m})$.

Analogously, the patching operator P is defined by

$$P\colon X_0 \to L^p(\mathbb{R}^n;\mathbb{C}^N),\ (f_\ell)_{\ell\in\Gamma} \mapsto \sum_{\ell\in\Gamma} \varphi_\ell f_\ell.$$

Note here that the sum is locally finite. We obtain $P \in L(X_0, L^p(\mathbb{R}^n;\mathbb{C}^N))$ and $PJ = \mathrm{id}_{L^p(\mathbb{R}^n;\mathbb{C}^N)}$ because of $PJf = \sum_{\ell\in\Gamma} \varphi_\ell^2 f = f$.

(3) Now let A_p be the $L^p(\mathbb{R}^n;\mathbb{C}^N)$-realization of $A(x,D)$ and $A_{p,0}$ the $L^p(\mathbb{R}^n;\mathbb{C}^N)$ -realization of $A_0(x,D)$. Then for $u \in W_p^{2m}(\mathbb{R}^n;\mathbb{C}^N)$ and $\ell \in \Gamma$ the following equality holds:

$$\begin{aligned}\varphi_\ell A_p u &= A_p(\varphi_\ell u) + (\varphi_\ell A_p - A_p\varphi_\ell)u \\ &= A^\ell(\varphi_\ell u) + (A_p - A_{p,0})\psi_\ell\varphi_\ell u + \sum_{k: Q_k\cap Q_\ell\neq\emptyset}(\varphi_\ell A_p - A_p\varphi_\ell)\varphi_k^2 u.\end{aligned}$$

Thus, $JA_p = AJ + BJ$ with

$$B\big((u_\ell)_{\ell\in\Gamma}\big) := \left((A_p - A_{p,0})\psi_\ell u_\ell + \sum_{k: Q_k\cap Q_\ell\neq\emptyset}(\varphi_\ell A_p - A_p\varphi_\ell)\varphi_k u_k\right)_{\ell\in\Gamma}.$$

Writing $B((u_\ell)_\ell) = (\sum_{k\in\Gamma} B_{k\ell}u_\ell)_{k\in\Gamma}$, we see that $B_{k\ell}$ is a differential operator of order not greater than $\le 2m-1$, and the number of elements in each row of the infinite matrix $(B_{k\ell})_{k,\ell}$ is bounded. As $a_\alpha \in L^\infty(\mathbb{R}^n;\mathbb{C}^N)$, this yields $B \in L(X_{2m-1}, X_0)$.

Analogously, we obtain for $(u_\ell)_{\ell\in\Gamma} \in X_{2m}$ the equality

$$\begin{aligned}(A_pP - PA)(u_\ell)_{\ell\in\Gamma} &= A_p\left(\sum_{\ell\in\Gamma}\varphi_\ell u_\ell\right) - \sum_{\ell\in\Gamma}\varphi_\ell A^\ell u_\ell \\ &= \sum_{\ell\in\Gamma}\varphi_\ell(A_p - A_{p,0})u_\ell + \sum_{k\in\Gamma}(A_p\varphi_k - \varphi_k A_p)u_k \\ &= \sum_{\ell\in\Gamma}\varphi_\ell(A_p - A_{p,0})u_\ell + \sum_{k\in\Gamma}\sum_{\ell: Q_k\cap Q_\ell\neq\emptyset}\varphi_\ell^2(A_p\varphi_k - \varphi_k A_p)u_k \\ &= \sum_{\ell\in\Gamma}\varphi_\ell\left[(A_p - A_{p,0})u_\ell + \sum_{k: Q_k\cap Q_\ell\neq\emptyset}\varphi_\ell(A_p\varphi_k - \varphi_k A_p)u_k\right] \\ &= PD(u_\ell)_{\ell\in\Gamma}\end{aligned}$$

with

$$D(u_\ell)_{\ell\in\Gamma} := \left((A_p - A_{p,0})u_\ell + \sum_{k: Q_k\cap Q_\ell\neq\emptyset}(A_p\varphi_k - \varphi_k A_p)u_k\right)_{\ell\in\Gamma}.$$

In the same way as before, we see that $D \in L(X_{2m-1}, X_0)$.

(4) We apply the interpolation inequality for Sobolev spaces and obtain for every $\varepsilon > 0$ the inequality

$$\begin{aligned}\|B(u_\ell)_{\ell\in\Gamma}\|_{X_0} + \|D(u_\ell)_{\ell\in\Gamma}\|_{X_0} &\le C\|(u_\ell)_{\ell\in\Gamma}\|_{X_{2m-1}} \\ &\le \varepsilon\|(u_\ell)_{\ell\in\Gamma}\|_{X_{2m}} + C_\varepsilon\|(u_\ell)_{\ell\in\Gamma}\|_{X_0} \quad (u \in X_{2m}).\end{aligned}$$

Due to Theorem 5.6, there exists a $\mu > 0$ such that $A + B - \mu$ and $A + D - \mu$ are both $\mathcal{R}$-sectorial with angle $\ge \theta$.

Let $u \in W_p^{2m}(\mathbb{R}^n;\mathbb{C}^N)$ and $f := (\lambda+\mu-A_p)u \in L^p(\mathbb{R}^n;\mathbb{C}^N)$. Then

$$Jf = J(\lambda+\mu-A_p)u = (\lambda+\mu-(A+B))Ju,$$

and therefore

$$u = PJu = P(\lambda+\mu-(A+B))^{-1}Jf.$$

In particular, $\lambda+\mu-A_p$ is injective.

On the other hand, for $f \in L^p(\mathbb{R}^n;\mathbb{C}^N)$ we get

$$\begin{aligned} f = PJf &= P(\lambda+\mu-(A+D))(\lambda+\mu-(A+D))^{-1}Jf \\ &= (\lambda+\mu-A_p)P(\lambda+\mu-(A+D))^{-1}Jf \in R(\lambda+\mu-A_p), \end{aligned}$$

i.e., $\lambda+\mu-A_p$ is surjective, too. Therefore, $\lambda+\mu \in \rho(A_p)$ and

$$(\lambda+\mu-A_p)^{-1} = P(\lambda+\mu-(A+D))^{-1}J.$$

Because of $P \in L(X_0, L^p(\mathbb{R}^n;\mathbb{C}^N))$, $J \in L(L^p(\mathbb{R}^n;\mathbb{C}^N), X_{2m})$, and $R_\theta(A+D-\mu) < \infty$, it follows that $R_\theta(A_p-\mu) < \infty$, and $A_p-\mu$ is $\mathcal{R}$-sectorial with angle greater or equal to θ. □

6 Parabolic Boundary Value Problems

In the last section, we considered parabolic systems in the whole space. Now we want to show that similar results also hold for boundary value problems in sufficiently smooth domains. In addition to the parameter-ellipticity of the operator A, we now have to impose a condition on the boundary operators called Shapiro-Lopatinskii condition. For a reference for this condition, we mention, e.g., [35], Sect. 11.

6.1 The Shapiro-Lopatinksii Condition

In the following, let $p \in (1,\infty)$, and let $G \subset \mathbb{R}^n$ be a bounded domain. We consider a linear partial differential operator $A = A(x,D)$ of the form

$$A(x,D) = \sum_{|\alpha|\le 2m} a_\alpha(x)D^\alpha$$

with $m \in \mathbb{N}$, $a_\alpha\colon \overline{G} \to \mathbb{C}$ and boundary operators $B_1,\dots,B_m$ of the form

$$B_j(x', D) = \sum_{|\beta| \le m_j} b_{j\beta}(x')\gamma_0 D^\beta$$

with $m_j < 2m$, $b_{j\beta}\colon \partial G \to \mathbb{C}$. Here, γ_0 stands for the boundary trace $u \mapsto u|_{\partial G}$, which is a bounded linear map

$$\gamma_0\colon W_p^k(\Omega) \to W_p^{k-1/p}(\partial\Omega),$$

$k = 1, \ldots, 2m$ if G is a C^{2m}-domain. Note here that $W_p^{k-1/p}(\partial\Omega) = B_{pp}^{k-1/p}(\partial\Omega)$ is the Sobolev-Slobodeckii space (see Sect. 4).

The L^p-realization $A_{B,p}$ of the boundary value problem $(A, B) = (A, B_1, \ldots, B_m)$ is defined by

$$D(A_{B,p}) := \{u \in W_p^{2m}(G) : B_1(x, D)u = \cdots = B_m(x, D)u = 0\}$$

and $A_{B,p}u := A(x, D)u \quad (u \in D(A_{B,p}))$. We will assume the following smoothness:

(i) The domain Ω is bounded and of class C^{2m}.
(ii) For the coefficients a_α of $A(x, D)$ we have

$$a_\alpha \in C(\overline{G}) \quad (|\alpha| = 2m),$$
$$a_\alpha \in L^\infty(G) \quad (|\alpha| < 2m).$$

(iii) For the coefficients $b_{j\beta}$ of $B_j(x', D)$ we have

$$b_{j\beta} \in C^{2m-m_j}(\partial G) \quad (|\beta| \le m_j,\ j = 1, \ldots, m).$$

By trace results on Sobolev spaces, we immediately see the following continuity:

Lemma 6.1 *The operator*

$$(A, B)\colon W_p^{2m}(G) \to L^p(G) \times \prod_{j=1}^m W_p^{2m-m_j-1/p}(\partial G)$$

is continuous.

As usual, we define the principal symbols $a_0(x, \xi) := \sum_{|\alpha|=2m} a_\alpha(x)\xi^\alpha$ and $b_{j0}(x', \xi) := \sum_{|\beta|=m_j} b_{j\beta}(x')\xi^\beta$.

Definition 6.2 The boundary value problem (A, B) is called parameter-elliptic in the sector $\overline{\Sigma_\varphi}$ if:

(a) We have $a_0(x, \xi) - \lambda \neq 0$ for all $x \in \overline{G}$ and all $(\xi, \lambda) \in (\mathbb{R}^n \times \overline{\Sigma_\varphi}) \setminus \{0\}$.

(b) The following Shapiro-Lopatinskii condition is satisfied: for all $x' \in \partial G$ and all $(\xi', \lambda) \in (\mathbb{R}^{n-1} \times \overline{\Sigma}_\varphi) \setminus \{0\}$ the ordinary differential equation

$$\begin{aligned} (a_0(x', \xi', D_n) - \lambda)v(x_n) &= 0 \quad (x_n > 0), \\ b_{j0}(x', \xi', D_n)v(x_n)\big|_{x_n=0} &= 0 \quad (j = 1, \dots, m), \\ v(x_n) &\to 0 \quad (x_n \to \infty) \end{aligned} \tag{6.1}$$

has only the trivial solution. Here, the boundary value problem is written in coordinates corresponding to x'. These coordinates arise from the original ones by translation and rotation in such a way that the x_n-direction in the new coordinates is the direction of the inner normal at the point x'.

If this holds for the sector $\overline{\Sigma}_{\pi/2} = \{\lambda \in \mathbb{C} : \operatorname{Re} \lambda \geq 0\}$, the instationary problem $(\partial_t - A, B)$ is called parabolic.

Note that (a) implies inequality (5.1) from Definition 5.1, as $\overline{G}$ is compact and a_0 is continuous in x and homogeneous in ξ.

Definition 6.3 Assume that in the situation of Definition 6.2, (a) holds. Then $A(x, D) - \lambda$ is called proper parameter-elliptic if for all $(x', \xi', \lambda) \in \partial G \times (\mathbb{R}^{n-1} \setminus \{0\}) \times \overline{\mathbb{C}}_+$, the polynomial $a_0(x', \xi', \cdot) - \lambda$ has exactly m roots (including multiplicities) $\tau_j = \tau_j(x', \xi', \lambda)$, $j = 1, \dots, m$ with positive imaginary part. In this case, define

$$a_+(\tau) := a_+(x', \xi', \lambda, \tau) := \prod_{j=1}^{m} (\tau - \tau_j(x', \xi', \lambda)) \in \mathbb{C}[\tau].$$

We consider the equivalence class $\overline{b}_{j0} = \overline{b}_{j0}(x', \xi', \lambda, \cdot) \in \mathbb{C}[\tau]/(a_+)$ of b_{j0} modulo a_+, and write $\overline{b}_{j0}$ with respect to the canonical basis $\overline{1}, \overline{\tau}, \dots, \overline{\tau^{m-1}} \in \mathbb{C}[\tau]/(a_+)$, i.e.

$$\begin{pmatrix} \overline{b}_{10} \\ \vdots \\ \overline{b}_{m0} \end{pmatrix} = L \begin{pmatrix} \overline{1} \\ \vdots \\ \overline{\tau^{m-1}} \end{pmatrix} \quad \text{with } L = L(x', \xi', \lambda) \in \mathbb{C}^{m \times m}.$$

Then L is called the Lopatinskii matrix of (A, B) at the point x.

Lemma 6.4 *Let A be properly parameter-elliptic in $\overline{G}$. Then the Shapiro-Lopatinskii holds if and only if*

$$\det L(x', \xi', \lambda) \neq 0 \quad (x' \in \partial G,\ (\xi', \lambda) \in (\mathbb{R}^{n-1} \times \overline{\mathbb{C}}_+) \setminus \{0\}).$$

Proof Let v_j $(j = 1, \dots, m)$ be the solution of

$$\begin{aligned} \big(a_+(x', \xi, D_n) - \lambda\big)v(x_n) &= 0 \quad (x_n > 0), \\ D_n^{k-1} v(x_n)\big|_{x_n=0} &= \delta_{kj} \quad (k = 1, \dots, m). \end{aligned}$$

Then $\{v_1, \dots, v_m\}$ is a basis of the space $\mathscr{M}_+$ of all stable solutions of the ordinary differential equation $\big(a_+(x', \xi, D_n) - \lambda\big)v(x_n) = 0$. Therefore, for all $v \in \mathscr{M}_+$ we have the representation $v = \sum_{j=1}^m \lambda_j v_j$ and

$$\begin{aligned}\begin{pmatrix} b_{10}(D_n) \\ \vdots \\ b_{m0}(D_n) \end{pmatrix} v(x_n)\big|_{x_n=0} &= \begin{pmatrix} b_{10}(D_n) \\ \vdots \\ b_{m0}(D_n) \end{pmatrix} \big(v_1(x_n), \dots, v_m(x_n)\big)\big|_{x_n=0} \begin{pmatrix} \lambda_1 \\ \vdots \\ \lambda_m \end{pmatrix} \\ &= \begin{pmatrix} \overline{b}_{10}(D_n) \\ \vdots \\ \overline{b}_{m0}(D_n) \end{pmatrix} \big(v_1(x_n), \dots, v_m(x_n)\big)\big|_{x_n=0} \begin{pmatrix} \lambda_1 \\ \vdots \\ \lambda_m \end{pmatrix} \\ &= L \begin{pmatrix} D_n^0 \\ \vdots \\ D_n^{m-1} \end{pmatrix} \big(v_1(x_n), \dots, v_m(x_n)\big)\big|_{x_n=0} \begin{pmatrix} \lambda_1 \\ \vdots \\ \lambda_m \end{pmatrix} \\ &= L \begin{pmatrix} \lambda_1 \\ \vdots \\ \lambda_m \end{pmatrix}.\end{aligned}$$

Note that $b_{k0}(D_n)v_j(x_n)\big|_{x_n=0} = \overline{b}_{k0}(D_n)v_j(x_n)\big|_{x_n=0}$ holds because $a_+(D_n)v_j(x_n) = 0$. Therefore, (6.1) has only the trivial solution if and only if $\det L \neq 0$. □

Remark 6.5 (a) The condition of Lemma 6.4 can be formulated in the following way: The boundary conditions are linearly independent modulo a_+, i.e., $\overline{b}_{10}, \dots, \overline{b}_{m0}$ are linearly independent in $\mathbb{C}[\tau]/(a_+)$.

(b) The boundary conditions $B_1, \dots, B_m$ are called completely elliptic if for every proper parameter-elliptic A the boundary value problem (A, B) is parameter-elliptic. This is the case for

(i) $B_j(x', D) = \gamma_0(\frac{\partial}{\partial x_n})^{j-1}$ $(j = 1, \dots, m)$ (general Dirichlet boundary conditions),
(ii) $B_j(x', D) = \gamma_0(\frac{\partial}{\partial x_n})^{m+j-1}$ $(j = 1, \dots, m)$ (general Neumann boundary conditions).

More general, this holds for all boundary conditions of the form

$$B_j(x', D) = \gamma_0\Big(\frac{\partial}{\partial x_n}\Big)^{s+j-1} + \text{ lower order terms } \ (j = 1, \dots, m),$$

where $s \in \{0, \dots, m\}$ is fixed. To see this, we have to show that $\{\overline{\tau^{s+j-1}} : j = 1, \dots, m\}$ is linearly independent in $\mathbb{C}[\tau]/(a_+)$. If this is not the case, there exist $c_j \in \mathbb{C}$ and $p \in \mathbb{C}[\tau]$ with

$$\sum_{j=1}^{m} c_j \tau^{s+j-1} = p(\tau)a_+(\tau).$$

Because of $a_+(0) \neq 0$, it follows that τ^s is a divisor of $p(\tau)$. Therefore, $\sum_{j=1}^m c_j\tau^{j-1} = \widetilde{p}(\tau)a_+(\tau)$ with some polynomial $\widetilde{p}$, in contradiction to $\deg a_+ = m$.

(c) If the domain and the coefficients of (A, B) are infinitely smooth, then for every fixed $\lambda \in \overline{\mathbb{C}}_+$, the coefficients of $L(x', \xi', \lambda)$ are symbols of pseudodifferential operators on the closed $(n-1)$-dimensional manifold ∂G.

6.2 The Main Result on Parameter-Elliptic Boundary Value Problems

Under the condition of parameter-ellipticity, one can construct the solution operators for boundary value problems. We follow the exposition in [2], Sect. 2, and [13], Sects. 6 and 7. We start with a remark on ordinary differential equations.

Theorem 6.6 *Let* (A, B) *be parameter-elliptic in some sector* $\overline{\Sigma}_\varphi$, *and let* $(x', \xi', \lambda) \in \partial G \times ((\mathbb{R}^{n-1} \times \overline{\Sigma}_\varphi) \setminus \{0\})$. *Choose a closed curve* $\gamma = \gamma(x', \xi', \lambda)$ *in* $\{z \in \mathbb{C} : \operatorname{Im} z > 0\}$, *enclosing all roots* $\tau_1, \dots, \tau_m$ *of* a_+. *We define* p_ℓ *by*

$$a_+(x', \xi', \lambda, \tau) = \sum_{\ell=0}^{m} p_\ell(x', \xi', \lambda)\tau^{m-\ell},$$

and set $N_k(\tau) := N_k(x', \xi', \lambda, \tau) := \sum_{\ell=0}^{m-k} p_\ell(x', \xi', \lambda)\tau^{m-k-\ell}$ *and*

$$(M_1(\tau), \dots, M_m(\tau)) := \big(N_1(\tau), \dots, N_m(\tau)\big)L^{-1}.$$

Let $w_k(x_n) = w_k(x', \xi', \lambda, x_n)$ $(x_n > 0)$ *be defined by*

$$w_k(x_n) := \frac{1}{2\pi i}\int_\gamma \frac{M_k(\tau)}{a_+(\tau)} e^{ix_n\tau} d\tau \quad (k = 1, \dots, m).$$

Then $\{w_1, \dots, w_m\}$ *is a basis of the stable solution space of* $a_0(D_n)w = 0$, $w(x_n) \to 0 \quad (x_n \to \infty)$ *and satisfies the initial conditions*

$$b_{j0}(x', \xi', \lambda, D_n)w_k(x_n)\big|_{x_n=0} = \delta_{jk} \quad (j, k = 1, \dots, m).$$

Proof (i) We first show that

$$\frac{1}{2\pi i}\int_\gamma \frac{N_k(\tau)\tau^{j-1}}{a_+(\tau)} d\tau = \delta_{kj} \quad (j, k = 1, \dots, m).$$

For this, we replace γ by a large ball $\{\tau \in \mathbb{C} : |\tau| = R\}$. For $j < k$ we have $\deg\big(N_k(\tau)\tau^{j-1}\big) = m - k + j - 1 \le m - 2$. Therefore, the integrand is of order $O(R^{-2})$ for $R \to \infty$ which shows that the integral vanishes.

For $j = k$, the integrand equals $\frac{\tau^{m-1}+O(\tau^{m-2})}{(\tau-\tau_1)\cdot\ldots\cdot(\tau-\tau_m)}$. By the residue's theorem, the integral has the value 1.

For $j > k$ we consider

$$\begin{aligned} Q(\tau) &:= -a_+(\tau)\tau^{j-k-1} + N_k(\tau)\tau^{j-1} \\ &= -\sum_{\ell=0}^{m} p_\ell \tau^{m-\ell+j-k-1} + \sum_{\ell=0}^{m-k} p_\ell \tau^{m-\ell+j-k-1}. \end{aligned}$$

We obtain $\deg Q = j - 2 \le m - 2$, and therefore

$$\int_{B(0,R)} \frac{N_k(\tau)\tau^{j-1}}{a_+(\tau)}\, d\tau = \int_{B(0,R)} \frac{a_+(\tau)\tau^{j-k-1} + Q(\tau)}{a_+(\tau)}\, d\tau = \int_{B(0,R)} \frac{Q(\tau)}{a_+(\tau)}\, d\tau = 0.$$

(ii) We have modulo a_+, i.e., as equality in $\mathbb{R}[\tau]/(a_+)$:

$$\begin{aligned} &\begin{pmatrix} \overline{b}_{10}(\tau) \\ \vdots \\ \overline{b}_{m0}(\tau) \end{pmatrix} (\overline{M}_1(\tau), \ldots, \overline{M}_m(\tau)) \\ &\qquad = \begin{pmatrix} \overline{b}_{10}(\tau) \\ \vdots \\ \overline{b}_{m0}(\tau) \end{pmatrix} \big(\overline{N}_1(\tau), \ldots, \overline{N}_m(\tau)\big) L^{-1} \\ &\qquad = L \begin{pmatrix} \overline{1} \\ \vdots \\ \overline{\tau^{m-1}} \end{pmatrix} \big(\overline{N}_1(\tau), \ldots, \overline{N}_m(\tau)\big) L^{-1}. \end{aligned}$$

Therefore,

$$\begin{aligned} \left(\frac{1}{2\pi i} \int_\gamma \frac{b_{j0}(\tau) M_k(\tau)}{a_+(\tau)}\, d\tau \right)_{j,k=1,\ldots,m} &= L \cdot \left(\frac{1}{2\pi i} \int_\gamma \frac{\tau^{j-1} N_k(\tau)}{a_+(\tau)}\, d\tau \right)_{j,k=1,\ldots,m} \cdot L^{-1} \\ &= L \cdot I_m \cdot L^{-1} = I_m. \end{aligned}$$

This yields

$$\frac{1}{2\pi i} \int_\gamma \frac{b_{j0}(\tau) M_k(\tau)}{a_+(\tau)}\, d\tau = \delta_{jk} \quad (j, k = 1, \ldots, m).$$

(iii) Define w_k as in the theorem. Because of $\gamma \subset \{z \in \mathbb{C} : \operatorname{Im} z > 0\}$, we see that $w_k(x_n) \to 0$ for $x_n \to \infty$. Further,

$$a_0(D_n)w(x_n) = \frac{1}{2\pi i}\int_\gamma \frac{M_k(\tau)}{a_+(\tau)}\, a(\tau)e^{ix_n\tau}d\tau = 0,$$

as the integrand is holomorphic. Finally,

$$b_{j0}(D_n)w(x_n)\big|_{x_n=0} = \frac{1}{2\pi i}\int_\gamma \frac{b_{j0}(\tau)M_k(\tau)}{a_+(\tau)}\; e^{ix_n\tau}\big|_{x_n=0}d\tau = \delta_{jk} \quad (j,k = 1,\dots,m),$$

which finishes the proof. □

Remark 6.7 (a) With the above notation, the following expressions are quasi-homogeneous in (ξ', λ, τ), more precisely, positively homogeneous in $(\xi', \lambda^{1/2m}, \tau)$:

- $a_+(x', \xi', \lambda, \tau)$ of degree m,
- $\tau_j(\xi', \lambda)$ of degree 1,
- $p_\ell(x', \xi', \lambda)$ of degree ℓ,
- $N_k(x', \xi', \lambda, \tau)$ of degree $m - k$,
- $b_{j0}(x', \xi', \tau)$ of degree m_j $(j = 1,\dots,m)$,
- $L_{ij}(x', \xi', \lambda)$ of degree $m_i - j + 1$,
- $M_k(x', \xi', \lambda, \tau)$ of degree $m - m_k - 1$,
- $\gamma(x', \xi', \lambda)$ of degree 1,
- $\frac{M_k(\tau)}{a_+(\tau)}$ of degree $-m_k - 1$.

(b) In the following, let

$$\langle \xi' \rangle_\lambda := |\xi'| + |\lambda|^{1/2m}.$$

By (a), the length of $\gamma(x', \xi', \lambda)$ can be estimated by $C\langle\xi'\rangle_\lambda$. For $\tau \in \gamma$, one gets

$$\begin{aligned} \operatorname{Im}\tau &\ge C\langle\xi'\rangle_\lambda,\\ |\tau - \tau_j(x', \xi', \lambda)| &\ge C\langle\xi'\rangle_\lambda,\\ |e^{i\tau x_n}| &\le \exp(-C\langle\xi'\rangle_\lambda\, x_n). \end{aligned}$$

For $\gamma' \in \mathbb{N}_0^{n-1}$ and $\alpha_n \in \mathbb{N}_0$, we obtain

$$\left|D_n^{\alpha_n} D_{\xi'}^{\gamma'} w_k(x', \xi', \lambda, x_n)\right| \le C\langle\xi'\rangle_\lambda^{-m_k+\alpha_n-|\gamma'|} e^{-C\langle\xi'\rangle\, x_n}.$$

In the smooth situation, these estimates show that w_k is the symbol of a Poisson operator. Such operators belong to the pseudodifferential calculus of boundary value problems which is also known as the Boutet de Monvel calculus (see, e.g., [20]).

To show maximal regularity for parabolic boundary value problems, we again start with the model problem related to (A, B) acting in $\mathbb{R}^n_+ := \{x \in \mathbb{R}^n : x_n > 0\}$

with boundary $\partial\mathbb{R}^n_+ = \mathbb{R}^{n-1}$. For this, we fix $x_0' \in \partial G$ and choose the coordinate system corresponding to x_0'. We obtain the boundary value problem

$$\begin{aligned}(A_0(D) - \lambda)u &= f \quad \text{in } \mathbb{R}^n_+,\\ B_{j0}(D)u &= 0 \quad (j = 1, \dots, m) \text{ on } \mathbb{R}^{n-1}.\end{aligned} \tag{6.2}$$

Here we have set

$$\begin{aligned}A_0(D) &:= \sum_{|\alpha|=2m} a_\alpha(x_0')D^\alpha,\\ B_{j0}(D) &:= \sum_{|\beta|=m_j} b_{j\beta}(x_0')\gamma_0 D^\beta.\end{aligned}$$

In the following result, we construct the solution operators for the model problem.

Theorem 6.8 *Let the boundary value problem* (A, B) *be parameter-elliptic in the sector* $\overline{\Sigma}_\varphi$, *and let* $x_0' \in \partial\Omega$ *be fixed. Then the model problem* (6.2) *has for every* $f \in L^p(\mathbb{R}^n_+)$ *and* $\lambda \in \Sigma_\varphi \setminus \{0\}$ *a unique solution* $u \in W_p^{2m}(\mathbb{R}^n_+)$. *This solution is given by*

$$\begin{aligned}u = R_+R(\lambda)E_0 f &- \sum_{j=1}^m T_j(\lambda)\Lambda_{2m-m_j}(\lambda)\widetilde{B}_{j0}(D)R_+R(\lambda)E_0 f\\ &- \sum_{j=1}^m \widetilde{T}_j(\lambda)\Lambda_{2m-m_j-1}(\lambda)\partial_n\widetilde{B}_{j0}(D)R_+R(\lambda)E_0 f.\end{aligned}$$

Here, the operators are defined in the following way:

(a) $E_0\colon L^p(\mathbb{R}^n_+) \to L^p(\mathbb{R}^n),\ f \mapsto E_0 f$ *with*

$$E_0 f := \begin{cases} f, & \text{for } x_n > 0,\\ 0, & \text{for } x_n \le 0\end{cases}$$

(trivial extension by 0*).*

(b) $R(\lambda) := (A_p - \lambda)^{-1} \in L(L^p(\mathbb{R}^n))$, *where* A_p *is the* $L^p(\mathbb{R}^n)$*- realization of* $A_0(D)$.

(c) $R_+\colon L^p(\mathbb{R}^n) \to L^p(\mathbb{R}^n_+),\ u \mapsto u|_{\mathbb{R}^n_+}$, *the restriction to* $\mathbb{R}^n_+$.

(d) $\widetilde{B}_{j0}(D) := \sum_{|\beta|=m_j} b_{j0}(x_0')D^\beta$, *the boundary operators without taking the trace* γ_0 *on the boundary.*

(e) $\Lambda_s(\lambda) := (\mathscr{F}')^{-1}(\lambda + |\xi'|^{2m})^{s/2m}\mathscr{F}' \in L(W_p^s(\mathbb{R}^n_+), L^p(\mathbb{R}^n_+))$ *for* $s \in \mathbb{N}_0$, *where* $\mathscr{F}'$ *denotes the Fourier transform in the tangential variables* $x' = (x_1, \dots, x_{n-1})$.

(f) $T_j(\lambda)$ *is given by*

$$(T_j(\lambda)\varphi)(x', x_n) := \int_0^\infty (\mathscr{F}')^{-1}(\partial_n w_j)(x_0', \xi', \lambda, x_n + y_n)\mathscr{F}'(\Lambda_{-2m+m_j}(\lambda)\varphi)(\xi', y_n)dy_n$$

for $\varphi \in L^p(\mathbb{R}^n_+)$.

(g) $\widetilde{T}_j(\lambda)$ *is given by*

$$(\widetilde{T}_j(\lambda)\varphi)(x', x_n) := \int_0^\infty (\mathscr{F}')^{-1} w_j(x_0', \xi', \lambda, x_n + y_n)\mathscr{F}'(\Lambda_{-2m+m_j+1}(\lambda)\varphi)(\xi', y_n)dy_n$$

for $\varphi \in L^p(\mathbb{R}^n_+)$.

The functions $w_j(x_0', \xi', \lambda, x_n)$ *are defined in Theorem 6.6.*

Proof Here we only show the solution formula for u, as the property $u \in W_p^{2m}(\mathbb{R}^n_+)$ will be included in the proof of the $\mathcal{R}$-boundedness of the solution operators below.

Let $u_1 \in W_p^{2m}(\mathbb{R}^n_+)$ be the unique solution of

$$(A_0(D) - \lambda)u_1 = E_0 f \quad \text{in } \mathbb{R}^n,$$

which exists due to Theorem 5.4. So we have $u_1 = R(\lambda)E_0 f$. For u, we choose the ansatz $u = u_1 + u_2$. Then u is a solution of (6.2) if and only if u_2 is a solution of the boundary value problem

$$\begin{aligned}(A_0(D) - \lambda)u_2 &= 0 \quad \text{in } \mathbb{R}^n_+,\\ B_{j0}(D)u_2 &= g_j \quad (j = 1, \dots, m) \text{ on } \mathbb{R}^{n-1}\end{aligned}$$

with

$$g_j := -B_{j0}(D)R_+u_1.$$

Taking partial Fourier transform $\mathscr{F}'$ with respect to x', we obtain

$$\begin{aligned}(a_0(x_0', \xi', D_n) - \lambda)v(x_n) &= 0 \quad (x_n > 0),\\ b_{j0}(x_0', \xi', D_n)v(x_n)\big|_{x_n=0} &= h_j(\xi') \quad (j = 1, \dots, m).\end{aligned} \tag{6.3}$$

Here, $v(x_n) := v(\xi', x_n) := (\mathscr{F}'u_2(\cdot, x_n))(\xi')$ and $h_j(\xi') := (\mathscr{F}'g_j)(\xi')$. By Theorem 6.6, the unique solution of (6.3) is given by

$$v(\xi', x_n) = \sum_{j=1}^m w_j(x_0', \xi', \lambda, x_n)h_j(\xi').$$

Note that g_j is first defined only on the boundary $\mathbb{R}^{n-1}$. By

$$\widetilde{g}_j := \sum_{|\beta|=m_j} b_{j\beta}(x_0')D^\beta u_1 = \widetilde{B}_{j0}(D)u_1$$

we define an extension g_j to $\mathbb{R}^n_+$. Then $\widetilde{h}_j := \mathscr{F}'\widetilde{g}_j(\cdot, x_n)$ is an extension of h_j.

For $j = 1, \dots, m$ we write (this is sometimes called the "Volevich trick")

$$\begin{aligned} w_j(x_0', \xi', \lambda, x_n)h_j(\xi') &= -\int_0^\infty \partial_n\big[w_j(x_0', \xi', \lambda, x_n + y_n)\widetilde{h}_j(\xi', y_n)\big]dy_n \\ &= -\int_0^\infty (\partial_n w_j)(x_0', \xi', \lambda, x_n + y_n)\widetilde{h}_j(\xi', y_n)dy_n \\ &\quad - \int_0^\infty w_j(x_0', \xi', \lambda, x_n + y_n)(\partial_n\widetilde{h}_j)(\xi', y_n)dy_n. \end{aligned}$$

For $\lambda \in \mathbb{C}_+ \setminus \{0\}$ it holds that $\Lambda_{-s}(\lambda)\Lambda_s(\lambda) = \mathrm{id}_{L^p(\mathbb{R}^n)}$ for all $s \in \mathbb{R}$. Therefore, we can write $\widetilde{g}_j = \Lambda_{-2m+m_j}(\lambda)\Lambda_{2m-m_j}(\lambda)\widetilde{g}_j$ and $\partial_n\widetilde{g}_j = \Lambda_{-2m+m_j+1}(\lambda)\Lambda_{2m-m_j+1}(\lambda)$ $\partial_n\widetilde{g}_j$, respectively. This yields

$$\begin{aligned} u_2(x', x_n) &= \big((\mathscr{F}')^{-1}v(\cdot, x_n)\big)(x') \\ &= \sum_{j=1}^m \big(T_j(\lambda)\Lambda_{2m-m_j}(\lambda)\widetilde{g}_j + \widetilde{T}_j(\lambda)\Lambda_{2m-m_j+1}(\lambda)\partial_n\widetilde{g}_j\big). \end{aligned}$$

Inserting $\widetilde{g}_j = \widetilde{B}_{j0}(D)R_+u_1$ and $u = u_1 + u_2$ into this formula, the solution formula of the theorem follows. As both the whole space problem as well as (6.3) is uniquely solvable and as the Fourier transform is a bijection in $\mathscr{S}'(\mathbb{R}^{n-1})$, we obtain unique solvability with the unique solution $u = u_1 + u_2$. □

Lemma 6.9 *The one-sided Hilbert transform*

$$(Hf)(x) := \int_0^\infty \frac{f(y)}{x+y}\, dy$$

defines a bounded linear operator $H \in L(L^p(\mathbb{R}_+))$.

Proof For $\varepsilon \in (0, 1]$, let $m_\varepsilon := \mathrm{sign}(\xi)e^{-\varepsilon\xi}$ $(\xi \in \mathbb{R})$. Then $|m_\varepsilon(\xi)| \le 1$ and $|\xi| \cdot |m_\varepsilon'(\xi)| = \varepsilon|\xi|e^{-\varepsilon|\xi|} \le 1$, where we used the inequality $te^{-t} < 1$ $(t > 0)$. By Mikhlin's theorem, $\|\mathscr{F}_1^{-1}m_\varepsilon\mathscr{F}_1\|_{L(L^p(\mathbb{R}_+))} \le C$ with a constant $C > 0$ independent of ε. Here $\mathscr{F}_1$ stands for the one-dimensional Fourier transform.

For $f \in \mathscr{S}(\mathbb{R})$ we get

$$
\begin{aligned}
(\mathscr{F}_1^{-1} m_\varepsilon \mathscr{F} f)(x) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ix\xi} \operatorname{sign}(\xi) e^{-\varepsilon|\xi|} (\mathscr{F}_1)(\xi) d\xi \\
&= \frac{1}{\sqrt{2\pi}} \int_0^\infty \Big[e^{ix\xi - \varepsilon\xi} \mathscr{F}_1 f(\xi) - e^{-ix\xi-\varepsilon\xi} \mathscr{F}_1 f(-\xi) \Big] d\xi \\
&= \frac{1}{2\pi} \int_0^\infty \int_{-\infty}^{\infty} \big(e^{ix\xi-\varepsilon\xi-iy\xi} - e^{-ix\xi-\varepsilon\xi+iy\xi} \big) f(y) dy\, d\xi \\
&= \frac{1}{2\pi} \int_{-\infty}^{\infty} \left(\frac{e^{i(x-y)\xi-\varepsilon\xi}}{i(x-y)-\varepsilon}\Big|_{\xi=0}^{\infty} - \frac{e^{-i(x-y)\xi-\varepsilon\xi}}{-i(x-y)-\varepsilon}\Big|_{\xi=0}^{\infty} \right) f(y) dy \\
&= \frac{1}{2\pi} \int_{-\infty}^{\infty} \left(-\frac{1}{i(x-y)-\varepsilon} + \frac{1}{-i(x-y)-\varepsilon} \right) f(y) dy \\
&= \frac{i}{\pi} \int_{-\infty}^{\infty} \frac{x-y}{(x-y)^2+\varepsilon^2}\, f(y)\, dy.
\end{aligned}
$$

Define for $\varepsilon \in (0, 1]$

$$
(H_\varepsilon f)(x) := \int_0^\infty \frac{x+y}{(x+y)^2+\varepsilon^2}\, f(y)\, dy \quad (f \in L^p(\mathbb{R}_+)).
$$

Then $H_\varepsilon f(x) = (-\frac{\pi}{i})(\mathscr{F}_1^{-1} m_\varepsilon \mathscr{F}_1 E_0 f)(-x)$ for $x \geq 0$, where $E_0 \colon L^p(\mathbb{R}_+) \to L^p(\mathbb{R})$ again stands for the trivial extension. We obtain

$$
\|H_\varepsilon f\|_{L^p(\mathbb{R}_+)} \leq \pi \|\mathscr{F}_1^{-1} m_\varepsilon \mathscr{F} E_0 f\|_{L^p(\mathbb{R})} \leq C\|E_0 f\|_{L^p(\mathbb{R})} \leq C\|f\|_{L^p(\mathbb{R}_+)}.
$$

The sequence $H_{1/n}(|f|)$ is monotonously increasing and converges pointwise to $H(|f|)$. By monotone convergence, we see that

$$
\begin{aligned}
\|Hf\|_{L^p(\mathbb{R}_+)} \leq \|H(|f|)\|_{L^p(\mathbb{R}_+)} &= \lim_{n\to\infty} \|H_{1/n}(|f|)\|_{L^p(\mathbb{R}_+)} \\
&\leq C\| \, |f| \, \|_{L^p(\mathbb{R}_+)} = C\|f\|_{L^p(\mathbb{R}_+)}.
\end{aligned}
$$

Therefore, $H \in L(L^p(\mathbb{R}_+))$. □

The following result shows that the solution operators are indeed $\mathcal{R}$-bounded.

Theorem 6.10 *Let $\delta > 0$ be fixed. In the situation of Theorem 6.8, the following operator families in $L(L^p(\mathbb{R}^n_+))$ are $\mathcal{R}$-bounded:*

(a) $\{\Lambda_{2m-m_j}(\lambda)\widetilde{B}_{j0}(D)R_+R(\lambda)E_0 : j = 1, \ldots, m,\ \lambda \in \overline{\mathbb{C}}_+,\ |\lambda| \geq \delta\}$,

(b) $\{\Lambda_{2m-m_j-1}(\lambda)\partial_n \widetilde{B}_{j0}(D)R_+R(\lambda)E_0 : j = 1, \dots, m,\ \lambda \in \overline{\mathbb{C}}_+,\ |\lambda| \geq \delta\}$,
(c) $\{\lambda T_j(\lambda) : j = 1, \dots, m,\ \lambda \in \overline{\mathbb{C}}_+,\ |\lambda| \geq \delta\}$,
(d) $\{\lambda \widetilde{T}_j(\lambda) : j = 1, \dots, m,\ \lambda \in \overline{\mathbb{C}}_+,\ |\lambda| \geq \delta\}$.

Proof (a) We have $\Lambda_{2m-m_j}(\lambda)\widetilde{B}_{j0}(D)R_+ = R_+\Lambda_{2m-m_j}(\lambda)\widetilde{B}_{j0}(D)$. As the operators $R_+ \in L(L^p(\mathbb{R}^n), L^p(\mathbb{R}^n_+))$ and $E_0 \in L(L^p(\mathbb{R}^n_+), L^p(\mathbb{R}^n))$ are bounded, it suffices to show the $\mathcal{R}$-boundedness of

$$\{\Lambda_{2m-m_j}(\lambda)\widetilde{B}_{j0}(\lambda)R(\lambda) : j = 1, \dots, m,\ \lambda \in \overline{\mathbb{C}}_+,\ |\lambda| \geq \delta\}.$$

The corresponding family of symbols (with respect to the Fourier transform in $\mathbb{R}^n$) is given by

$$m(\xi, \lambda) := (\lambda + |\xi'|^{2m})^{\frac{2m-m_j}{2m}} b_{j0}(x_0', \xi)\big(a_0(x_0', \xi) - \lambda\big)^{-1}.$$

As $m(\xi, \lambda)$ is quasi-homogeneous of degree 0 in (ξ, λ) and bounded on $|\lambda| + |\xi|^{2m} = 1$, it follows that

$$|D^\alpha m(\xi, \lambda)| \leq C|\xi|^{-|\alpha|} \quad (\xi \in \mathbb{R}^n \setminus \{0\},\ \lambda \in \overline{\mathbb{C}}_+,\ |\lambda| \geq \delta).$$

By Corollary 3.29, the operator family in a) is $\mathcal{R}$-bounded.

(b) can be shown analogously.

(c) For $\varphi \in L^p(\mathbb{R}^n_+)$, we write

$$\lambda T_j(\lambda)\varphi = \int_0^\infty k_\lambda(x_n, y_n)\psi(y_n)dy_n$$

with $\psi \in L^p(\mathbb{R}_+; L^p(\mathbb{R}^{n-1}))$, $\psi(y_n) := \varphi(\cdot, y_n)$, and the operator valued integral kernel

$$\begin{aligned} k_\lambda(x_n, y_n) &:= (\mathscr{F}')^{-1}\widetilde{m}(\xi', \lambda, x_n + y_n)\mathscr{F}' \\ &:= (\mathscr{F}')^{-1}\lambda\partial_n w_j(x_0', \xi', \lambda, x_n + y_n)(\lambda + |\xi'|^{2m})^{-\frac{2m-m_j}{2m}}\mathscr{F}'. \end{aligned}$$

By Remark 6.7 (b), the inequalities

$$\begin{aligned} |D_{\xi'}^{\gamma'}\widetilde{m}(\xi', \lambda, x_n + y_n) &\leq C(|\xi'| + |\lambda|^{1/2m} \exp\big(-C(|\xi'| + |\lambda|^{1/2m})(x_n + y_n)\big)|\xi'|^{-|\gamma'|} \\ &\leq \frac{C}{x_n + y_n}\,|\xi'|^{-|\gamma'|} \end{aligned}$$

hold, where in the last step we again used the elementary estimate $te^{-t} < 1$ $(t > 0)$. Again by Corollary 3.29, it follows that $k_\lambda(x_n, y_n) \in L(L^p(\mathbb{R}^{n-1}))$ with

$$\mathcal{R}\big\{k_\lambda(x_n, y_n) : \lambda \in \overline{\mathbb{C}}_+,\ |\lambda| \geq \delta\big\} \leq \frac{C}{x_n + y_n}\,.$$

The scalar integral operator with kernel $k_0(x_n, y_n) := \frac{1}{x_n+y_n}$, given by

$$(K_0 g)(x_n) := \int_0^\infty \frac{g(y_n)}{x_n + y_n}\, dy_n \quad (g \in L^p(\mathbb{R}_+))$$

is the one-sided Hilbert transform in $L^p(\mathbb{R}_+)$ and, due to Lemma 6.9, a bounded linear operator $K_0 \in L(L^p(\mathbb{R}_+))$. By Theorem 3.18 we get

$$\mathcal{R}\big\{\lambda T_j(\lambda) : \lambda \in \overline{\mathbb{C}}_+,\ |\lambda| \geq \delta\big\} \leq C \|K_0\|_{L(L^p(\mathbb{R}_+))} < \infty.$$

(d) follows in the same way as (c). □

Now maximal regularity for the model problem is an immediate consequence of the previous results.

Theorem 6.11 *Let the boundary value problem* $(\partial_t - A^0, B^0)$ *be parabolic, and let* $x_0' \in \partial G$. *Choose the coordinate system corresponding to* x_0', *and consider the* L^p*-realization* $A_B^{(0)}$ *of the model problem* $(A_0(x_0', D), B(x_0', D))$. *Then* $\rho(A_B^{(0)}) \supset \overline{\mathbb{C}}_+ \setminus \{0\}$, *and for every* $\delta > 0$ *the operator family*

$$\big\{\lambda(\lambda - A_B^{(0)})^{-1} : \lambda \in \overline{\mathbb{C}}_+,\ |\lambda| \geq \delta\big\} \subset L(L^p(\mathbb{R}^n_+))$$

is $\mathcal{R}$*-bounded. In particular,* $A_B^{(0)} - \delta$ *has for every* $\delta > 0$ *maximal* L^q*-regularity for all* $1 < q < \infty$ *(and generates a bounded holomorphic* C_0*-semigroup).*

Proof Replacing in the proof of Theorem 6.10 the operators $\lambda T_j(\lambda)$ by $D^\alpha T_j(\lambda)$ (and analogously for $\widetilde{T}_j(\lambda)$) with $|\alpha| = 2m$, we see that the solution operators in fact define a solution $u \in W_p^{2m}(\mathbb{R}^n_+)$. Therefore, the solution coincides with the resolvent. Now the $\mathcal{R}$-boundedness follows directly from the resolvent description in Theorem 6.8 and the statements on $\mathcal{R}$-boundedness from Theorem 6.10. □

To deal with variable coefficients, we first study small perturbations in the principal part.

Theorem 6.12 *Let* $A^0(x, D) = \sum_{|\alpha|=2m} a_\alpha^0 D^\alpha$ *and* $B_j^0(x, D) = \sum_{|\beta|=m_j} b_{j\beta}^0 \gamma_0 D^\beta$ *with* $a_\alpha^0 \in \mathbb{C}$ *and* $b_{j\beta}^0 \in \mathbb{C}$. *Assume the boundary value problem* $(\partial_t - A^0, B^0)$ *to be parabolic in the domain* $\mathbb{R}^n_+$. *Then there exists an* $\varepsilon > 0$ *such that the following statement holds: Let* $A(x, D) = A^0(x, D) + \widetilde{A}(x, D)$ *and* $B(x, D) = B^0(x, D) + \widetilde{B}(x, D)$ *with*

$$\widetilde{A}(x, D) = \sum_{|\alpha|=2m} \widetilde{a}_\alpha(x) D^\alpha,$$
$$\widetilde{B}_j(x, D) = \sum_{|\beta|=m_j} \widetilde{b}_{j\beta}(x) D^\beta \ (j = 1, \dots, m).$$

Here, $\widetilde{a}_\alpha \in L^\infty(\mathbb{R}^n_+)$ *and* $\widetilde{b}_{j\beta} \in \mathrm{BUC}^{2m-m_j}(\mathbb{R}^{n-1})$. *Assume further that*

$$\sum_{|\alpha|=2m} \|\widetilde{a}_\alpha\|_{L^\infty(\mathbb{R}^n_+)} \le \varepsilon,$$
$$\sum_{|\beta|=m_j} \|\widetilde{b}_{j\beta}\|_{L^\infty(\mathbb{R}^{n-1})} \le \varepsilon \ (j = 1, \dots, m).$$

Let $A_{B,p}$ *be the* L^p*-realization of the boundary value problem* $(A(x, D), B(x, D))$. *Then there exists a* $\mu > 0$ *such that the operator family*

$$\big\{\lambda(A_{B,p} - \lambda)^{-1} : \lambda \in \overline{\mathbb{C}}_+, \ |\lambda| \ge \mu\big\} \subset L(L^p(\mathbb{R}^n_+))$$

is $\mathcal{R}$*-bounded. Here,* ε *and the* $\mathcal{R}$*-bound only depend on* $(A^0(x, D), B^0(x, D))$, *and* μ *additionally depends on the norms* $\|b_{j\beta}\|_{\mathrm{BUC}^{2m-m_j}}(\mathbb{R}^{n-1})$ *for* $|\beta| = m_j$, $j = 1, \dots, m$.

Proof We indicate the main steps of the proof, for a more elaborated version, see [13], Subsection 7.3.

Without loss of generality, we may assume that the coefficients of $\widetilde{B}(x, D)$ are defined on all of $\mathbb{R}^n_+$. We write the boundary value problem

$$\begin{aligned}(A(x, D) - \lambda)u &= f \quad \text{in } \mathbb{R}^n_+,\\ B_j(x, D)u &= 0 \quad (j = 1, \dots, m) \ \text{ on } \mathbb{R}^{n-1}\end{aligned}$$

in the form

$$\begin{aligned}(A^0(x, D) - \lambda)u &= f - \widetilde{A}(x, D)u \quad \text{in } \mathbb{R}^n_+,\\ B^0_j(x, D)u &= -\widetilde{B}_j(x, D)u \quad (j = 1, \dots, m) \ \text{ on } \mathbb{R}^{n-1}.\end{aligned}$$

Let $(A^0_{B,p} - \lambda)^{-1}$ be the resolvent of the L^p-realization of $(A^0(x, D), B^0(x, D))$, which exists due to Theorem 6.11. Applying the solution operators from Theorem 6.8, we obtain

$$\begin{aligned}u &= (A^0_{B,p} - \lambda)^{-1} f - (A^0_{B,p} - \lambda)^{-1} \widetilde{A}(x, D)u\\ &\quad - \sum_{j=1}^m T_j(\lambda)\Lambda_{2m-m_j}(\lambda)\widetilde{B}_j(x, D)u - \sum_{j=1}^m \widetilde{T}_j(\lambda)\Lambda_{2m-m_j-1}(\lambda)\partial_n \widetilde{B}_j(x, D)u\\ &=: (A^0_{B,p} - \lambda)^{-1} f - S(\lambda)u.\end{aligned}$$

We estimate the norm of $S(\lambda)u$. For the term $(A^0_{B,p} - \lambda)^{-1}\widetilde{A}(x, D)u$, we use

$$\|(A^0_{B,p} - \lambda)^{-1}\|_{L(L^p(\mathbb{R}^n_+), W^{2m}_p(\mathbb{R}^n_+))} \leq C_1$$

and obtain

$$\|(A^0_{B,p} - \lambda)^{-1}\widetilde{A}(x, D)u\|_{W^{2m}_p(\mathbb{R}^n_+)} \leq C_1\|\widetilde{A}(x, D)u\|_{L^p(\mathbb{R}^n_+)} \leq C_1\varepsilon\|u\|_{W^{2m}_p(\mathbb{R}^n_+)}.$$

For the other terms, we use the fact that the operator families

$$\{\lambda^{(2m-|\alpha|)/2m} D^\alpha T(\lambda) : |\alpha| \leq 2m,\ \lambda \in \overline{\mathbb{C}}_+,\ |\lambda| \geq \lambda_0\} \subset L(L^p(\mathbb{R}^n_+))$$

are $\mathcal{R}$-bounded and therefore bounded, which can be seen as in the proof of Theorem 6.10. This yields

$$\begin{aligned}\|T_j(\lambda)\Lambda_{2m-m_j}(\lambda)\widetilde{B}_j(x, D)u\|_{W^{2m}_p(\mathbb{R}^n_+)} &\leq C\|\Lambda_{2m-m_j}(\lambda)\widetilde{B}_j(x, D)u\|_{L^p(\mathbb{R}^n_+)} \\ &\leq C\|\widetilde{B}_j(x, D)u\|_{W^{2m-m_j}_p(\mathbb{R}^n_+)}.\end{aligned}$$

The terms of the form $\widetilde{b}_{j\beta}D^\beta u$ can be estimated, using the Leibniz rule, by

$$\begin{aligned}\|\widetilde{b}_{j\beta}D^\beta u\|_{W^{2m-m_j}_p(\mathbb{R}^n_+)} &\leq C \sum_{|\gamma|\leq 2m-m_j} \sum_{\delta+\delta'=\gamma} \|(D^\delta \widetilde{b}_{j\beta})(D^{\delta'+\beta}u)\|_{L^p(\mathbb{R}^n_+)} \\ &\leq C_2\varepsilon\|u\|_{W^{2m}_p(\mathbb{R}^n_+)} + C_3\|u\|_{W^{2m-1}_p(\mathbb{R}^n_+)}.\end{aligned}$$

Here, the constant C_3 depends on the norm $\|b_{j\beta}\|_{\mathrm{BUC}^{2m-m_j}(\mathbb{R}^{n-1})}$. With the interpolation inequality, we see that for some constants C_1, C_2 we have

$$\|S(\lambda)u\|_{W^{2m}_p(\mathbb{R}^n_+)} + |\lambda|\|S(\lambda)u\|_{L^p(\mathbb{R}^n_+)} \leq C_1\varepsilon\|u\|_{W^{2m}_p(\mathbb{R}^n_+)} + C_2\|u\|_{L^p(\mathbb{R}^n_+)}.$$

Now we endow $W^{2m}_p(\mathbb{R}^n_+)$ with the parameter-dependent norm $|||u||| := \|u\|_{W^{2m}_p(\mathbb{R}^n_+)} + |\lambda|\|u\|_{L^p(\mathbb{R}^n_+)}$. Note that for every fixed λ, this norm is equivalent to the standard norm. For $|\lambda| \geq 2C_2$ and $C_1\varepsilon \leq \frac{1}{2}$, it follows that

$$|||S(\lambda)u||| \leq \tfrac{1}{2}|||u|||.$$

Therefore, $(1 + S(\lambda)) \in L(W^{2m}_p(\mathbb{R}^n_+))$ is invertible (with respect to the new norm, and therefore also with respect to the standard norm). Thus, we have seen that the above boundary value problem is uniquely solvable and that the resolvent $(A_{B,p} - \lambda)^{-1}$ exists for all $\lambda \in \overline{\mathbb{C}}_+$ with $|\lambda| \geq 2C_2$.

To obtain an estimate on the $\mathcal{R}$-bounds, we can argue similarly. Starting from the identity

$$(A_{B,p} - \lambda)^{-1} = (A^0_{B,p} - \lambda)^{-1} - S(\lambda)(A_{B,p} - \lambda)^{-1},$$

one can show for sufficiently large $\mu > 0$

$$\begin{aligned}\mathcal{R}\big\{\widetilde{A}(x,D)(A_{B,p}-\lambda)^{-1} : \lambda \in \overline{\mathbb{C}}_+,\ |\lambda| \geq \mu\big\} \\ \leq \sum_{|\alpha|=2m} \|\widetilde{a}_\alpha\|_{L^\infty(\mathbb{R}^n_+)} \mathcal{R}\big\{D^\alpha (A_{B,p}-\lambda)^{-1} : \lambda \in \overline{\mathbb{C}}_+,\ |\lambda| \geq \mu\big\} \\ \leq C\varepsilon \mathcal{R}\big\{D^\alpha (A_{B,p}-\lambda)^{-1} : \lambda \in \overline{\mathbb{C}}_+,\ |\lambda| \geq \mu\big\}.\end{aligned}$$

Similarly, the other terms in $S(\lambda)(A_{B,p}-\lambda)^{-1}$ can be estimated. Consider the operator family

$$\mathcal{T} := \big\{\lambda^{2m-|\alpha|)/(2m)} D^\alpha (A_{B,p}-\lambda)^{-1} : |\alpha| \leq 2m,\ \lambda \in \overline{\mathbb{C}}_+,\ |\lambda| \geq \mu\big\}.$$

The above calculations show that for every finite subset $\mathcal{T}_0$ of $\mathcal{T}$, we get the inequality

$$\mathcal{R}(\mathcal{T}_0) \leq R_1 + (C_1\varepsilon + C_2(\mu))\mathcal{R}(\mathcal{T}_0).$$

Here,

$$C_1 := \mathcal{R}\big\{\lambda^{2m-|\alpha|)/(2m)} D^\alpha (A^0_{B,p}-\lambda)^{-1} : |\alpha| \leq 2m,\ \lambda \in \overline{\mathbb{C}}_+,\ |\lambda| \geq \mu\big\} < \infty$$

and $C_2(\mu) \to 0$ for $\mu \to \infty$. Choosing ε small enough and μ large enough, we have $C_1\varepsilon + C_2(\mu) < \frac{1}{2}$, and therefore $\mathcal{R}(\mathcal{T}_0) < 2R_1 < \infty$. As this holds for every finite subset $\mathcal{T}_0$ of $\mathcal{T}$, with R_1 being independent of $\mathcal{T}_0$, we get the same estimate for $\mathcal{T}$, i.e., $\mathcal{R}(\mathcal{T}) \leq 2R_1$. □

The last result deals with small perturbations of the top-order coefficients. As before, lower-order terms of the operators can be handled by the interpolation inequality. For a proof of maximal regularity in the situation of a bounded domain and under the above smoothness assumptions, the method of localization can be used. We mention some main ideas in the following remark.

Remark 6.13 *(Localization)* Let $(\partial_t - A, B)$ be a parabolic boundary value problem in the bounded domain G, and assume the smoothness assumptions from the beginning of this section to hold. To prove $\mathcal{R}$-sectoriality of the L^p-realization of (A, B), one can use the following steps:

(a) For every fixed $x_0 \in \partial G$, by definition of a C^{2m}-domain, there exists a neighbourhood $U(x_0) \subset \mathbb{R}^n$ and a C^{2m}-diffeomorphism $\Phi_{x_0}\colon U(x_0) \to V(x_0) := \Phi_{x_0}(U(x_0)) \subset \mathbb{R}^n$ with

$$\Phi_{x_0}(U(x_0) \cap G) = V(x_0) \cap \mathbb{R}^n_+.$$

We denote by $(\widetilde{A}, \widetilde{B})$ the transformed boundary value problem in the domain $V(x_0)$. The coefficients $\widetilde{a}_\alpha$ of $\widetilde{A}$ are defined in $V(x_0) \cap \overline{\mathbb{R}^n_+}$ and satisfy the same smoothness assumptions as a_α. In the same way, this holds for the transformed coefficients $\widetilde{b}_{j\beta}$

of $\widetilde{B}_j$. Moreover, it is possible to show that the transformed problem is parabolic in $V(x_0) \cap \mathbb{R}^n_+$.

The coefficients $\widetilde{a}_\alpha$ and $\widetilde{b}_{j\beta}$ can be extended to the half space $\overline{\mathbb{R}^n_+}$ and $\mathbb{R}^{n-1}$, respectively, in such a way that both the smoothness and the parabolicity is preserved. For $\widetilde{a}_\alpha$, we can choose an appropriate continuous extension. For the coefficients on the boundary $\widetilde{b}_{j\beta}$, we have to preserve higher smoothness. For this, one can, e.g., define

$$\widetilde{b}_{j\beta}(y) := \widetilde{b}_{j\beta}\Big(y_0 + \chi\Big(\frac{y-y_0}{r}\Big)(y-y_0)\Big) \quad (y \in \mathbb{R}^{n-1}),$$

where $\chi \in C^\infty(\mathbb{R}^{n-1})$ satisfies $\chi(x) = 1$ for $|x| \le 1$ and $\chi(x) = 0$ for $|x| \ge 2$. Here, $y_0 := \Phi_{x_0}(x_0)$, and $r > 0$ is chosen sufficiently small.

For an eventually even smaller $r = r(x_0)$, the following inequalities hold true for a given $\varepsilon > 0$:

$$\sum_{|\alpha|=2m} \|\widetilde{a}_\alpha(\,\cdot\,) - \widetilde{a}_\alpha(y_0)\|_{L^\infty(\mathbb{R}^n_+)} < \varepsilon,$$

$$\sum_{|\beta|=m_j} \|\widetilde{b}_{j\beta}(\,\cdot\,) - \widetilde{b}_{j\beta}(y_0)\|_{L^\infty(\mathbb{R}^{n-1})} < \varepsilon \quad (j = 1, \ldots, m).$$

Therefore, the localized boundary value problems satisfy the conditions of Theorem 6.12.

For fixed $\varepsilon > 0$, this construction yields an open cover of the form

$$\partial G \subset \bigcup_{x_0 \in \partial G} \Phi_{x_0}^{-1}(B(y_0, r(x_0))).$$

By compactness of ∂G, there exists a finite subcover $\partial G \subset \bigcup_{k=1}^N U_k$, where we have set $U_k := \Phi_{x_k}^{-1}(B(y_k, r(x_k)))$.

(b) In the same way, in the interior of the domain, we obtain for every $x_0 \in G$ a small neighbourhood $U(x_0) \subset \mathbb{R}^n$ and an extension $\widetilde{a}_\alpha$ of $a_\alpha|_{U(x_0)}$ such that

$$\sum_{|\alpha|=2m} \|\widetilde{a}_\alpha(\,\cdot\,) - \widetilde{a}_\alpha(x_0)\|_{L^\infty(\mathbb{R}^n)} < \varepsilon$$

holds. In this way, we obtain an open cover

$$G \setminus \bigcup_{k=1}^N U_k \subset \bigcup_{x_0 \in G} B(x_0, r(x_0)).$$

Note that no boundary operator and no diffeomorphism is involved. As $G \setminus \bigcup_{k=1}^N U_k$ is compact, there exists a finite subcover

$$G \setminus \bigcup_{k=1}^{N} U_k \subset \bigcup_{k=N+1}^{M} U_k$$

with $U_k = B(x_k, r(x_k)))$. Altogether, this yields a finite open cover $\overline{G} \subset \bigcup_{k=1}^{M} U_k$.

(c) With this construction, one obtains finitely many operators $(\widetilde{A}^{(k)}, \widetilde{B}^{(k)})$ for $k = 1, \dots, N$ and $\widetilde{A}^{(k)}$ for $k = N+1, \dots, M$, which satisfy the assumptions of Theorem 6.12 and Lemma 5.7, respectively. Now we can use the resolvents of the L^p-realization of these operators to show $\mathcal{R}$-sectoriality of $A_{B,p} - \mu$ for large μ. This can be done similarly as in the proof of Theorem 5.9, using a partition of unity and estimating the commutators with help of the interpolation inequality.

With the above techniques, it is possible to show the following main theorem on parabolic boundary value problems:

Theorem 6.14 *Assume the boundary value problem* $(\partial_t - A, B)$ *to be parabolic and to satisfy the smoothness assumptions above. Let* $1 < p < \infty$. *Then there exist* $\theta > \frac{\pi}{2}$ *and* $\mu > 0$ *such that* $\rho(A_{B,p} - \mu) \supset \overline{\mathbb{C}}_+$ *and the operator* $A_{B,p} - \mu$ *is* $\mathcal{R}$-*sectorial with angle* θ. *In particular,* $A_{B,p} - \mu$ *has maximal* L^q-*regularity for all* $q \in (1, \infty)$.

7 Quasilinear Parabolic Evolution Equations

We have seen in the previous sections that, under appropriate parabolicity and smoothness assumptions, the L^p-realization of linear boundary value problems have maximal regularity. This is the basis for the analysis of nonlinear problems, which will be described in the present section.

7.1 Well-Posedness for Quasilinear Parabolic Evolution Equations

We consider nonlinear evolution equations which can be written in the abstract form

$$\begin{aligned} \partial_t u(t) - A(t, u(t))u(t) &= F(t, u(t)) \quad \text{in } (0, T_0), \\ u(0) &= u_0. \end{aligned} \tag{7.1}$$

Here, $T_0 \in (0, \infty)$. We fix the following situation: Let $p \in (1, \infty)$, and let $X_1 \subset X_0$ be Banach spaces with X_1 being dense in X_0. With $T \in (0, T_0]$, the spaces for the right-hand side and the solution are

$$\mathbb{F} := \mathbb{F}_T := L^p((0, T); X_0) \quad \text{and} \quad \mathbb{E} := \mathbb{E}_T := H_p^1((0, T); X_0) \cap L^p((0, T); X_1),$$

respectively. The time trace space, and therefore the space for the initial value u_0, is given by $\gamma_t\mathbb{E} = (X_0, X_1)_{1-1/p,p}$ (cf. Lemma 2.4). We again set ${}_0\mathbb{E} := \{u \in \mathbb{E} : \gamma_t u = 0\}$. Here and in the following, we consider the operator A as a map $A\colon (0, T_0) \times \gamma_t\mathbb{E} \to L(X_1, X_0)$. For each $t \in (0, T_0)$ and $v \in \gamma_t\mathbb{E}$, the operator $A(t, v) \in L(X_1, X_0)$ is identified with the unbounded operator $A(t, v)$ acting in X_0 with domain X_1, and $A(t, v) \in \mathrm{MR}(X_0)$ has to be understood in this sense.

Example 7.1 We recall the example of the graphical mean curvature flow (Example 2.1), which has the form

$$\begin{aligned} \partial_t u - \Big(\Delta u - \sum_{i,j=1}^n \frac{\partial_i u \partial_j u}{1+|\nabla u|^2}\,\partial_i\partial_j u\Big) &= 0 \quad \text{in } (0, T_0), \\ u(0) &= u_0. \end{aligned} \tag{7.2}$$

This quasilinear equation can be written in the form (7.1), where

$$A(t, u(t)) = \Delta - \sum_{i,j=1}^n \frac{\partial_i u(t)\partial_j u(t)}{1+|\nabla u(t)|^2}\,\partial_i\partial_j$$

and $F = 0$. Here we have $X_0 = L^p(\mathbb{R}^n)$, $X_1 = W_p^2(\mathbb{R}^n)$, and $\gamma_t\mathbb{E} = B_{pp}^{2-2/p}(\mathbb{R}^n) = W_p^{2-2/p}(\mathbb{R}^n)$.

For the nonlinearities A and F in (7.1), we assume:

(A1) We have $A \in C([0, T_0] \times \gamma_t\mathbb{E}, L(X_1, X_0))$, and for all $R > 0$ there exists a Lipschitz constant $L(R) > 0$ with

$$\|A(t, w)v - A(t, \overline{w})v\|_{X_0} \le L(R)\|w - \overline{w}\|_{\gamma_t\mathbb{E}}\|v\|_{X_1}$$

for all $t \in [0, T_0]$, $v \in X_1$ and all $w, \overline{w} \in \gamma_t\mathbb{E}$ with $\|w\|_{\gamma_t\mathbb{E}} \le R$ and $\|\overline{w}\|_{\gamma_t\mathbb{E}} \le R$.

(A2) For the mapping $F\colon [0, T_0] \times \gamma_t\mathbb{E} \to X_0$ we assume:

(i) $F(\cdot, w)$ is measurable for every $w \in \gamma_t\mathbb{E}$,
(ii) $F(t, \cdot) \in C(\gamma_t\mathbb{E}, X_0)$ for almost all $t \in [0, T_0]$,
(iii) $f(\cdot) := F(\cdot, 0) \in L^p((0, T_0); X_0)$,
(iv) for every $R > 0$, there exists a $\varphi_R \in L^p((0, T_0))$ with

$$\|F(t, w) - F(t, \overline{w})\|_{X_0} \le \varphi_R(t)\|w - \overline{w}\|_{\gamma_t\mathbb{E}}$$

for almost all $t \in [0, T_0]$ and all $w, \overline{w} \in \gamma_t\mathbb{E}$ with $\|w\|_{\gamma_t\mathbb{E}} \le R$, $\|\overline{w}\|_{\gamma_t\mathbb{E}} \le R$.

Apart from standard conditions on measurability and continuity, the above conditions essentially mean that the functions $A(t, \cdot)v$ and $F(t, \cdot)$ are locally Lipschitz, i.e., they are Lipschitz on bounded subsets of $\gamma_t\mathbb{E}$. The following result is based on [29], Sect. 3 (see also [10]).

Theorem 7.2 *Assume (A1) and (A2) as well as* $A_0 := A(0, u_0) \in \mathrm{MR}(X_0)$. *Then there exists a* $T \in (0, T_0]$ *such that* (7.1) *has a unique solution* $u \in \mathbb{E}_T$ *in the interval* $(0, T)$.

Proof **(i)** We use the maximal regularity of $A_0 := A(0, u_0)$ in the time interval $(0, T)$ with $T \leq T_0$ to obtain estimates for the solutions of the linearized equation. For this, we first consider the equation with initial value 0,

$$\begin{aligned} \partial_t w(t) - A_0 w(t) &= g(t) \quad (t \in (0, T)), \\ w(0) &= 0. \end{aligned} \tag{7.3}$$

As $A_0 \in \mathrm{MR}(X_0)$, for every $g \in \mathbb{F}$ there exists a unique solution $w \in \mathbb{E}$, and we obtain the estimate

$$\|w\|_{\mathbb{E}} \leq C_0 \|g\|_{\mathbb{F}}$$

with a constant $C_0 > 0$ which does not depend on T or w (Lemma 4.7). By Lemma 4.4 (b), there exists a constant C_1 (again independent of $T > 0$ and w) with

$$\|w\|_{C([0,T],\gamma_t \mathbb{E})} \leq C_1 \|w\|_{\mathbb{E}}.$$

Note here that $w(0) = 0$ holds.

In the following, we consider the reference solution $u^* \in \mathbb{E}$ which is defined as the unique solution of

$$\begin{aligned} \partial_t w(t) - A_0 w(t) &= f(t) \quad (t \in (0, T)), \\ w(0) &= u_0. \end{aligned} \tag{7.4}$$

Here, $f := F(\cdot, 0) \in \mathbb{F}$ due to condition (A2) (iii).

(ii) For $r \in (0, 1]$ set

$$B_r := \{v \in \mathbb{E} : v - u^* \in {}_0\mathbb{E},\ \|v - u^*\|_{\mathbb{E}} \leq r\}.$$

For each $v \in B_r$, define $\Phi(v) := u$ as the unique solution of

$$\begin{aligned} \partial_t u(t) - A_0 u(t) &= F(t, v(t)) - (A(0, u_0) - A(t, v(t)))\, v(t) \quad (t \in (0, T)), \\ u(0) &= u_0. \end{aligned} \tag{7.5}$$

We will show that $\Phi(B_r) \subset B_r$ holds and that Φ is a contraction in B_r, given that both T and r are sufficiently small.

(iii) In this step, we show that $\Phi(B_r) \subset B_r$ holds for sufficiently small T and r. For this, we write

$$\|\Phi(v) - u^*\|_{\mathbb{E}} = \|u - u^*\|_{\mathbb{E}} \leq C_0 \big(\|F(\cdot, v) - f(\cdot)\|_{\mathbb{F}} + \|(A(0, u_0) - A(\cdot, v))v\|_{\mathbb{F}}\big). \tag{7.6}$$

Let $m_T := \sup_{t\in[0,T]} \|A(0,u_0) - A(t,u_0)\|_{L(X_1,X_0)}$. By condition (A1) with fixed $R := C_1 + \|u^*\|_{L^\infty((0,T);\gamma_t\mathbb{E})}$, we obtain

$$\begin{aligned}\|A(0,u_0)v - A(\cdot,v)v\|_{\mathbb{F}} &= \|A(0,u_0)v - A(\cdot,v)v\|_{L^p((0,T);X_0)}\\ &\le \|A(0,u_0) - A(\cdot,v)\|_{L^\infty((0,T);L(X_1,X_0))}\|v\|_{L^p((0,T);X_1)}\\ &\le \big(\|A(0,u_0) - A(\cdot,u_0)\|_{L^\infty((0,T);L(X_1,X_0))}\\ &\quad + \|A(\cdot,u_0) - A(\cdot,v(\cdot))\|_{L^\infty((0,T);L(X_1,X_0))}\big)\,\|v\|_{\mathbb{E}}\\ &\le \big(m_T + L(R)\|v-u_0\|_{L^\infty((0,T);\gamma_t\mathbb{E})}\big)\,\|v\|_{\mathbb{E}}\\ &\le (m_T + L(R)C_1\|v-u_0\|_{\mathbb{E}})\,\|v\|_{\mathbb{E}}.\end{aligned}$$

For $r \le 1$, we can estimate

$$\|v-u_0\|_{\mathbb{E}} \le \|v-u^*\|_{\mathbb{E}} + \|u^*-u_0\|_{\mathbb{E}} \le r + \|u^*-u_0\|_{\mathbb{E}}$$

and

$$\|v\|_{\mathbb{E}} \le \|v-u^*\|_{\mathbb{E}} + \|u^*\|_{\mathbb{E}} \le r + \|u^*\|_{\mathbb{E}}.$$

Therefore, we obtain

$$\|A(0,u_0)v - A(\cdot,v)v\|_{\mathbb{F}} \le \big(m_T + L(R)C_1(r + \|u^*-u_0\|_{\mathbb{E}})\big)\,(r + \|u^*\|_{\mathbb{E}}).$$

In a similar way, using (A2), we see that

$$\begin{aligned}\|F(\cdot,v) - f\|_{\mathbb{F}} &\le \|F(\cdot,v) - F(\cdot,u^*)\|_{\mathbb{F}} + \|F(\cdot,u^*) - F(\cdot,0)\|_{\mathbb{F}}\\ &\le \|\varphi_R\|_{L^p((0,T))}\Big(\|v-u^*\|_{L^\infty((0,T);\gamma_t\mathbb{E})} + \|u^*\|_{L^\infty((0,T);\gamma_t\mathbb{E})}\Big)\\ &\le \|\varphi_R\|_{L^p((0,T))}\Big(C_1\|v-u^*\|_{\mathbb{E}} + \|u^*\|_{L^\infty((0,T);\gamma_t\mathbb{E})}\Big)\\ &\le \|\varphi_R\|_{L^p((0,T))}C_1\big(r + \|u^*\|_{L^\infty((0,T);\gamma_t\mathbb{E})}\big).\end{aligned}$$

Inserting this into (7.6), we get

$$\begin{aligned}\|\Phi(v) - u^*\|_{\mathbb{E}} &\le C_0\Big[\|\varphi_R\|_{L^p((0,T))}\big(C_1 r + \|u^*\|_{L^\infty((0,T);\gamma_t\mathbb{E})}\big)\\ &\quad + \big(m_T + L(R)C_1(r + \|u^*-u_0\|_{\mathbb{E}})\big)(r + \|u^*\|_{\mathbb{E}})\Big]\\ &\le C_0(C_1 + \|u^*\|_{L^\infty((0,T);\gamma_t\mathbb{E})})\|\varphi_R\|_{L^p((0,T))}\\ &\quad + C_0(r + \|u^*\|_{\mathbb{E}})\big(m_T + L(R)C_1 r + L(R)C_1\|u^*-u_0\|_{\mathbb{E}}\big).\end{aligned} \tag{7.7}$$

In the limit $T \to 0$, we obtain the following convergences:

- $m_T \to 0$, as $A(\cdot,u_0)$ is continuous,
- $\|\varphi_R\|_{L^p((0,T))} \to 0$, as $\varphi_R \in L^p((0,T_0))$,
- $\|u^*-u_0\|_{\mathbb{E}_T} \to 0$, as $u^*-u_0 \in \mathbb{E}_{T_0}$,

- $\|u^*\|_{\mathbb{E}_T} \to 0$, as $u^* \in \mathbb{E}_{T_0}$.

First, choose $r > 0$ small enough such that

$$C_0 L(R) C_1 r < \tfrac{1}{8}$$

holds. Then, choose $T > 0$ small enough such that the following inequalities hold:

$$\begin{aligned} \|u^*\|_{\mathbb{E}} &< r, \\ C_0\big(C_1 + \|u^*\|_{L^\infty((0,T);\gamma_t\mathbb{E})}\big)\|\varphi_R\|_{L^p((0,T))} &< \tfrac{r}{2}, \\ C_0\big(m_T + L(R)C_1\|u^* - u_0\|_{\mathbb{E}}\big) &< \tfrac{1}{8}. \end{aligned}$$

Inserting this into (7.7), we obtain

$$\|\Phi(v) - u^*\|_{\mathbb{E}} \le \tfrac{r}{2} + (r + r)(\tfrac{1}{8} + \tfrac{1}{8}) = r,$$

which shows that $\Phi(B_r) \subset B_r$.

(iv) In the same way as in (iii), one sees that for sufficiently small $r > 0$ and $T > 0$ the inequality

$$\|\Phi(v) - \Phi(\overline{v})\|_{\mathbb{E}} \le \tfrac{1}{2}\|v - \overline{v}\|_{\mathbb{E}}$$

holds for all $v, \overline{v} \in B_r$. Therefore, $\Phi\colon B_r \to B_r$ is a contraction, and with the Banach fixed point theorem (contraction mapping principle), there exists a unique fixed point u of Φ. By definition of Φ, its fixed points are exactly the solutions of the nonlinear equation (7.1), which finishes the proof. □

Theorem 7.3 *Assume (A1) and (A2) to hold, and assume $A(t, v) \in \mathrm{MR}(X_0)$ for all $t \in [0, T_0)$ with $T_0 \in (0, \infty]$. Then for every $u_0 \in \gamma_t\mathbb{E}$ there exists a unique maximal solution of (7.1) with maximal existence interval $[0, T^+(u_0)) \subset [0, T_0)$. If $T^+(u_0) < T_0$ (i.e., if there is no global solution), then $T^+(u_0)$ is characterized by each of the following conditions.*

(i) $\lim_{t\nearrow T^+(u_0)} u(t)$ *does not exist in* $\gamma_t\mathbb{E}$,
(ii) $\int_0^{T^+(u_0)} \big(\|u(t)\|_{X_1}^p + \|\partial_t u(t)\|_{X_0}^p\big)dt = \infty$.

Proof Assume $u \in \mathbb{E}_T$ to be a local solution on the interval $(0, T)$. Then $u \in C([0, T]; \gamma_t\mathbb{E})$. Therefore, we can apply Theorem 7.2 in the interval (T, T_0) with initial condition $u_1 = u(T) \in \gamma_t\mathbb{E}$, and obtain an extension of u to some interval $(0, T')$ with $T' > T$. Continuing in this way, we obtain a unique maximal solution which exists in some time interval $[0, T^+(u_0))$.

If $\lim_{t\nearrow T^+(u_0)} u(t) \in \gamma_t\mathbb{E}$ exists, this can be taken as initial value at time $T^+(u_0)$. By the above arguments, we see that u can be extended to a small time interval $(T^+(u_0), T^+(u_0) + \varepsilon)$, which is a contradiction to the maximality of $T^+(u_0)$. Therefore, $T^+(u_0)$ is characterized by condition (i).

For each $T < T^+(u_0)$ we have, by definition of a solution, $\int_0^T (\|u(t)\|_{X_1}^p + \|\partial_t u(t)\|_{X_0}^p)dt < \infty$. If this also holds for $T = T^+(u_0)$, then $u \in \mathbb{E}_{T^+(u_0)}(X_1, X_0) \subset$

$C([0, T^+(u_0)]; \gamma_t\mathbb{E})$. Therefore, $\lim_{t\nearrow T^+(u_0)} u(t)$ exists in $\gamma_t\mathbb{E}$ in contradiction to (i). □

As an application of the above theorems, we obtain a result on lower-order perturbation (the map B in the following lemma) for linear non-autonomous problems.

Lemma 7.4 *Let $A \in C([0, T], L(X_1, X_0))$ with $A(t) \in \mathrm{MR}(X_0)$ $(t \in [0, T])$, and let $B \in L^p((0, T); L(\gamma_t\mathbb{E}, X_0))$. Then the initial value problem*

$$\begin{aligned} \partial_t u(t) - A(t)u(t) &= B(t)u(t) + f(t) \quad (t \in [0, T]), \\ u(0) &= u_0 \end{aligned}$$

has for each $f \in \mathbb{F}_T$ and each $u_0 \in \gamma_t\mathbb{E}$ a unique solution $u \in \mathbb{E}_T$.

Proof We set $A(t, u(t)) = A(t)$ and $F(t, u(t)) = B(t)u(t) + f(t)$. Obviously, the conditions (A1) and (A2) are satisfied with $\varphi_R(t) := \|B(t)\|_{L(\gamma_t\mathbb{E}, X_0)}$. The proof of Theorem 7.2 shows that the length of the existence interval only depends on u_0 and the constants $L(R)$, C_0, C_1 and γ_T. Because of $A \in C([0, T], L(X_1, X_0))$ and the continuity of $A \mapsto \|(\partial_t + A)^{-1}\|_{L(\mathbb{F},\mathbb{E})} = C_0(A)$, all these constants can be chosen globally in the time interval $[0, T]$. Therefore, we have global existence of the solution. □

7.2 Higher Regularity

We consider the same situation as in the last subsection and study the autonomous quasilinear differential equation

$$\begin{aligned} \partial_t u(t) - A(u(t))u(t) &= F(u(t)) \quad (t \in (0, T)), \\ u(0) &= u_0. \end{aligned} \tag{7.8}$$

Here, $T \in (0, \infty)$, $u_0 \in \gamma_t\mathbb{E}(X_1, X_0)$, $A\colon \gamma_t\mathbb{E} \to L(X_1, X_0)$ and $F\colon \gamma_t\mathbb{E} \to \mathbb{F}$.

It is well known that parabolic equations are smoothing, and the solution is even – in many applications – real analytic. We start with a definition.

Definition 7.5 Let X, Y be Banach spaces, $U \subset X$ open, and $T\colon U \to Y$ be a function. Then T is called real analytic if for all $u_0 \in U$ there exists an $r > 0$ with $B(u_0, r) \subset U$ and

$$T(u) = \sum_{k=0}^{\infty} \frac{D^k T(u_0)}{k!} \underbrace{(u - u_0, \ldots, u - u_0)}_{k-\text{times}} \quad (u \in B(u_0, r)).$$

Here, $D^k T(u_0) \in L(X \times \ldots \times X, F)$ denotes the k-th Fréchet derivative of T at u_0. In this case, we write $T \in C^\omega(U, Y)$.

The main step in the proof of smoothing properties for parabolic equations is the implicit function theorem in Banach spaces.

Theorem 7.6 (Implicit function theorem) *Let X, Y, Z be Banach spaces, $U \subset X \times Y$ be open, and $T \in C^1(U, Z)$. Further, let $(x_0, y_0) \in U$ with $T(x_0, y_0) = 0$ and $D_y T((x_0, y_0)) \in L_{\text{Isom}}(Y, Z)$, where $D_y T$ stands for the Fréchet derivative with respect to the second component. Then there exist neighbourhoods U_X of x_0 and U_Y of y_0 with $U_X \times U_Y \subset U$ and a unique function $\psi \in C^1(U_X, U_Y)$ such that*

$$T(x, \psi(x)) = 0 \quad (x \in U_X)$$

and $\psi(x_0) = y_0$. Therefore, the equation $T(x, y) = 0$ is locally solvable with respect to y. The function ψ has the same regularity as T, i.e., if $T \in C^k(U, Z)$ for $k \in \mathbb{N} \cup \{\infty, \omega\}$, then also $\psi \in C^k(U_X, U_Y)$.

With the help of the implicit function theorem, one can prove smoothing properties with respect to the time variable. As references, we mention [8], [29], Sect. 5, and [30], Sect. 5.2.

Theorem 7.7 *Let $k \in \mathbb{N} \cup \{\infty, \omega\}$, and let $A \in C^k(\gamma_t \mathbb{E}; L(X_1, X_0))$ and $F \in C^k(\gamma_t \mathbb{E}, X_0)$. Assume $u \in \mathbb{E}_T(X_1, X_0)$ to be a solution of (7.8), and assume that $A(u(t)) \in \text{MR}(X_0)$ for all $t \in [0, T]$. Then*

$$t \mapsto t^j \partial_t^j u(t) \in W_p^1(J; X_0) \cap L^p(J; X_1)$$

holds for all $j \in \mathbb{N}_0$ with $j \leq k$. In particular,

$$u \in W_p^{k+1}((\varepsilon, T); X_0) \cap W_p^k((\varepsilon, T); X_1)$$

for every $\varepsilon > 0$ as well as

$$u \in C^k((0, T); \gamma_t \mathbb{E}) \cap C^{k+1-1/p}((0, T); X_0) \cap C^{k-1/p}((0, T); X_1).$$

Here, $C^{k+1-1/p}$ and $C^{k-1/p}$ stand for the Hölder spaces of order $k + 1 - 1/p$ and $k - 1/p$, respectively. If $k = \infty$, then $u \in C^\infty((0, T); X_1)$, and if $k = \omega$, then $u \in C^\omega((0, T); X_1)$.

Proof We fix $\varepsilon \in (0, 1)$ and set $T(\varepsilon) := \frac{T}{1+\varepsilon}$. For $\lambda \in (1 - \varepsilon, 1 + \varepsilon)$ we define the function $u_\lambda \colon [0, T(\varepsilon)] \to \gamma_t \mathbb{E}$ by $u_\lambda(t) := u(\lambda t)$ $(t \in [0, T(\varepsilon)])$. Then $\partial_t u_\lambda(t) = \lambda(\partial_t u)(\lambda t)$, and therefore

$$\begin{aligned} \partial_t u_\lambda(t) - \lambda A(u_\lambda(t)) u_\lambda(t) &= \lambda F(u_\lambda(t)) \quad (t \in (0, T(\varepsilon))), \\ u_\lambda(0) &= u_0. \end{aligned}$$

Now consider the function

$$H\colon (1-\varepsilon, 1+\varepsilon) \times \mathbb{E}_{T(\varepsilon)} \to \mathbb{F}_{T(\varepsilon)} \times \gamma_t\mathbb{E}$$

defined by

$$H(\lambda, w)(t) := \begin{pmatrix} \partial_t w(t) - \lambda A(w(t))w(t) - \lambda F(w(t)) \\ w(0) - u_0 \end{pmatrix} \quad (t \in (0, T(\varepsilon)))$$

for $\lambda \in (1-\varepsilon, 1+\varepsilon)$ and $w \in \mathbb{E}_{T(\varepsilon)}$. As A and F are both of class C^k, the same holds for H. Moreover, $H(1, u) = 0$ and

$$\begin{aligned} D_\lambda H(\lambda, w) &= \begin{pmatrix} -A(w)w - F(w) \\ 0 \end{pmatrix}, \\ D_w H(\lambda, w)h &= \begin{pmatrix} \partial_t h - \lambda A(w)h - \lambda A'(w)hw - \lambda F'(w)h \\ h(0) \end{pmatrix} \end{aligned}$$

for $h \in \mathbb{E}_{T(\varepsilon)}$. Here $A'(u)$ stands for the Fréchet derivative of A at u. In particular, we obtain for $\lambda = 1$ and $w = u$

$$D_w H(1, u)h = \begin{pmatrix} \partial_t h + A(u)h + A'(u)hu - F'(u)h \\ h(0) \end{pmatrix}.$$

For $t \in [0, T(\varepsilon)]$ and $v \in \gamma_t\mathbb{E}$, we define $B(t)v := -A'(u(t))vu(t) - F'(u(t))v$. As $A \in C^1(\gamma_t\mathbb{E}, L(X_1, X_0))$ and $F \in C^1(\gamma_t\mathbb{E}, X_0)$, we get $B \in L^p((0, T); L(\gamma_t\mathbb{E}, X_0))$. Therefore, we can apply Lemma 7.4 (replacing $A(t)$ in this lemma by $A(u(t))$). Note that $t \mapsto A(u(t)) \in C([0, T(\varepsilon)], L(X_1, X_0))$ holds because of $t \mapsto u(t) \in C([0, T(\varepsilon)]; \gamma_t\mathbb{E})$. By assumption, $A(u(t)) \in \mathrm{MR}(X_0)$ for every $t \in [0, T]$, and we can apply Lemma 7.4. This yields

$$D_w H(1, u) \in L_{\mathrm{Isom}}(\mathbb{E}_{T(\varepsilon)}, \mathbb{F}_{T(\varepsilon)} \times \gamma_t\mathbb{E}).$$

Now the implicit function theorem, Theorem 7.6, tells us that there exists a $\delta > 0$ and a C^k-function $\psi\colon (1-\delta, 1+\delta) \to \mathbb{E}_{T(\varepsilon)}$ with $H(\lambda, \psi(\lambda)) = 0$ ($\lambda \in (1-\delta, 1+\delta)$) and $\psi(1) = u$.

By definition of H and the uniqueness of the solution, we obtain $\psi(\lambda) = u_\lambda$, i.e., $\lambda \mapsto u_\lambda \in C^k((1-\delta, 1+\delta), \mathbb{E}_{T(\varepsilon)})$. Because of $\mathbb{E}_{T(\varepsilon)} \subset C([0, T(\varepsilon)], \gamma_t\mathbb{E})$, we obtain $\lambda \mapsto u_\lambda(t) = u(\lambda t) \in C^k((1-\delta, 1+\delta), \gamma_t\mathbb{E})$. But this means $u \in C^k((0, T(\varepsilon)), \gamma_t\mathbb{E})$.

Now we use $\frac{\partial}{\partial\lambda} u_\lambda(t)|_{\lambda=1} = t\partial_t u(t)$ ($t \in (0, T(\varepsilon))$). As $\psi \in C^k((1-\delta, 1+\delta), \mathbb{E}_{T(\varepsilon)}$, we get $t \mapsto t\partial_t u(t) \in \mathbb{E}_{T(\varepsilon)}$. An iteration shows that $t \mapsto t^k \partial_t^k u(t) \in \mathbb{E}_{T(\varepsilon)}$, and therefore

$$u \in W_p^{k+1}((\delta, T(\varepsilon)); X_0) \cap W_p^k((\delta, T(\varepsilon)); X_1)$$

for every $\delta > 0$ and $\varepsilon > 0$. Now we apply Sobolev's embedding theorem which tells us that $W_p^k((\delta, T(\varepsilon)) \subset C^{k-1/p}([\delta, T(\varepsilon)])$. With this we obtain, as $\varepsilon > 0$ and $\delta > 0$

can be chosen arbitrary,

$$u \in C^{k+1-1/p}((0,T); X_0) \cap C^{k-1/p}((0,T); X_1).$$

In the case $k = \infty$, we get $u \in C^\infty((0,T); X_1)$. If $k = \omega$, then the function ψ is real analytic. The above embeddings are linear and therefore real analytic, too, which yields $u \in C^\omega((0,T), X_1)$. □

Remark 7.8 This method of proof is known as parameter trick or method of Angenent [8]. Note that the two main ingredients are the implicit function theorem in Banach spaces and the fact that $D_w H(1,u)$ is an isomorphism. The latter is exactly the maximal regularity of the linearization, and it can also be seen as one of the main ideas of the maximal regularity approach to show that the implicit function theorem can be applied to the nonlinear equation.

As an example, we consider the quasilinear autonomous second order equation in $\mathbb{R}^n$

$$\begin{aligned} \partial_t u(t,x) - \operatorname{tr}\big(a(u(t,x), \nabla u(t,x))\nabla^2 u(t,x)\big) &= f(u(t,x), \nabla u(t,x)) \\ &\qquad ((t,x) \in (0,T) \times \mathbb{R}^n), \\ u(0,x) &= u_0(x). \end{aligned} \tag{7.9}$$

To solve the nonlinear problem, we need the following result from the linear theory, which can be shown by the methods of Sect. 5.

Lemma 7.9 *Let $b \in \mathrm{BUC}(\mathbb{R}^n; \mathbb{R}^{n\times n}_{sym})$ with $b(x) \geq cI_n$ $(x \in \mathbb{R}^n)$ for some constant $c > 0$. Define the operator B by $D(B) := W^2_p(\mathbb{R}^n) \subset L^p(\mathbb{R}^n)$,*

$$(Bu)(x) := \operatorname{tr}\big(b(x)\nabla^2 u(x)\big) = \sum_{i,j=1}^n b_{ij}(x)\partial_i\partial_j u(x) \quad (x \in \mathbb{R}^n,\ u \in D(B)).$$

Then $B \in \mathrm{MR}(L^p(\mathbb{R}^n))$.

For the nonlinear equation, we obtain the following result (see [29], Theorem 5.1).

Theorem 7.10 *Let $p \in (n+2, \infty)$ and $k \in \mathbb{N} \cup \{\infty, \omega\}$. Assume that $a \in C^k(\mathbb{R}^{n+1}, \mathbb{R}^{n\times n}_{sym})$ and $f \in C^k(\mathbb{R}^{n+1}, \mathbb{R})$ with $f(0) = 0$, and assume that for all $(r,p) \in \mathbb{R} \times \mathbb{R}^n$ the matrix $a(r,p)$ is positive definite. Then equation (7.9) has for all $u_0 \in W^{2-2/p}_p(\mathbb{R}^n)$ a unique maximal solution $u \in L^p((0,T^+); W^2_p(\mathbb{R}^n_+)) \cap W^1_p((0,T^+); L^p(\mathbb{R}^n))$ in the existence interval $J = (0,T^+)$ with $T^+ = T^+(u_0) > 0$. Moreover,*

$$u \in C^k(J; W^{2-2/p}_p(\mathbb{R}^n)) \cap C^{k+1-1/p}(J; L^p(\mathbb{R}^n)) \cap C^{k-1/p}(J; W^2_p(\mathbb{R}^n)).$$

Proof For $X_0 := L^p(\mathbb{R}^n)$ and $X_1 := W_p^2(\mathbb{R}^n)$, the trace space is given by $\gamma_t\mathbb{E}(X_0, X_1) = (X_0, X_1)_{1-1/p,p} = W_p^{2-2/p}(\mathbb{R}^n)$. An application of Sobolev's embedding theorem yields

$$\gamma_t\mathbb{E} = W_p^{2-2/p}(\mathbb{R}^n) \subset C_0^1(\mathbb{R}^n) := \Big\{u \in C^1(\mathbb{R}^n) : \lim_{|x|\to\infty} |\partial^\alpha u(x)| = 0 \ (|\alpha| \le 1)\Big\}.$$

Now define the mappings $A\colon \gamma_t\mathbb{E} \to L(X_0, X_1)$ and $F\colon \gamma_t\mathbb{E} \to X_0$ by

$$\begin{aligned}(A(v)w)(x) &:= \operatorname{tr}\big(a(v(x), \nabla v(x))\nabla^2 w(x)\big),\\ (F(v))(x) &:= f(v(x), \nabla v(x))\end{aligned}$$

for $x \in \mathbb{R}^n$, $v \in \gamma_t\mathbb{E}$, and $w \in W_p^2(\mathbb{R}^n)$.

Let $v \in \gamma_t\mathbb{E}$. Because of $v \in C_0^1(\mathbb{R}^n)$, the set $\{(v(x), \nabla v(x)) : x \in \mathbb{R}^n\} \subset \mathbb{R}^{n+1}$ is bounded. As a is continuous by assumption, we see that

$$b_v := a(v(\cdot), \nabla v(\cdot)) \in \mathrm{BUC}(\mathbb{R}^n)$$

and $b_v(x) \ge c_v I_n$ $(x \in \mathbb{R}^n)$ with $c_v > 0$. By Lemma 7.9, we obtain $A(v) \in \mathrm{MR}(X_0)$ for all $v \in \gamma_t\mathbb{E}$.

To show that assumptions (A1) and (A2) are satisfied, we use the fact that a is a C^1-function and therefore Lipschitz on bounded sets. Therefore, we get for all $v, \overline{v} \in \gamma_t\mathbb{E}$ and $w \in X_1$ with $\|v\|_{\gamma_t\mathbb{E}} \le R$, $\|\overline{v}\|_{\gamma_t\mathbb{E}} \le R$ the inequality

$$\begin{aligned}\|A(v)w - A(\overline{v})w\|_{L^p(\mathbb{R}^n)} &= \Big\| \operatorname{tr}\big(a(v, \nabla v)w - a(\overline{v}, \nabla\overline{v})w\big)\Big\|_{L^p(\mathbb{R}^n)}\\ &\le C\big\|a(v, \nabla v) - a(\overline{v}, \nabla\overline{v})\|_{L^\infty(\mathbb{R}^n;\mathbb{R}^{n\times n})}\|\nabla^2 w\|_{L^p(\mathbb{R}^n;\mathbb{R}^{n\times n})}\\ &\le CL(R)\|v - \overline{v}\|_{C^1(\mathbb{R}^n)}\|w\|_{X_1}\\ &\le CL(R)\|v - \overline{v}\|_{\gamma_t\mathbb{E}}\|w\|_{X_1}.\end{aligned}$$

This shows assumption (A1) and, in particular, the continuity of $A\colon \gamma_t\mathbb{E} \to L(X_0, X_1)$. Similiary, assumption (A2) can be shown. Here, we have to show the continuity of $F\colon \gamma_t\mathbb{E} \to X_0$. For this we use the fact that F is a variant of the so-called Nemyckii operators, i.e.,

$$F\colon W_p^{2-2/p}(\mathbb{R}^n) \to L^p(\mathbb{R}^n),\ \ F(v) := f(v(\cdot), \nabla v(\cdot)) \quad (v \in W_p^{2-2/p}(\mathbb{R}^n)).$$

For this, we also use $f(0) = 0$. By known results on the Nemyckii operator, one obtains $A \in C^k(\gamma_t\mathbb{E}, L(X_1, X_0))$ and $F \in C^k(\gamma_t\mathbb{E}, X_0)$. Therefore, all assumptions of Theorem 7.7 are satisfied, and we obtain higher regularity for the solution u as stated in the theorem. □

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On Stability and Bifurcation in Parallel Flows of Compressible Navier-Stokes Equations

Yoshiyuki Kagei

Abstract The stability analysis of parallel flows of the compressible Navier-Stokes equations is overviewed. The asymptotic behaviour of solutions is firstly considered for small Reynolds and Mach numbers. An instability result of the plane Poiseuille flow is then given for a certain range of Reynolds and Mach numbers, together with a result of the bifurcation of wave trains from the plane Poiseuille flow.

Keywords Compressible Navier-Stokes equations · Parallel flow · stability · Asymptotic behaviour · Bifurcation

Mathematics Subject Classification 35Q30 · 76N06

1 Introduction

This article is concerned with the mathematical analysis of the stability and bifurcation problem for parallel flows of viscous compressible fluids. The governing equations of such fluids are written in the form

$$\begin{cases} \partial_t \rho + \mathrm{div}\,(\rho v) = 0, \\ \rho(\partial_t v + v \cdot \nabla v) - {}^{-}\Delta v - ({}^{-} + {}^{-\prime})\nabla \mathrm{div}\, v + \nabla P(æ) = æ g. \end{cases} \tag{1.1}$$

Here $\rho = \rho(x, t)$ is the unknown density and $v = {}^{\top}(v^1(x, t), \ldots, v^n(x, t))$ is the unknown velocity field at time $t \geq 0$ and position $x \in \mathbb{R}^n$ $(n \geq 2)$; $P = P(\rho)$ is the pressure which is assumed to be smooth in ρ satisfying

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$$P'(\rho_*) > 0$$

for a given constant $\rho_* > 0$; μ and μ' are the viscosity coefficients which are assumed to be constants satisfying

$$\mu > 0, \ \ \frac{2}{n}\mu + \mu' \geq 0;$$

and g is a given external force.

The system (1.1) is considered in an infinite layer

$$\Omega_\ell = \{x = (x', x_n); x' = (x_1, \ldots, x_{n-1}) \in \mathbb{R}^{n-1}, \ 0 < x_n < \ell\}.$$

As for the external force g, we assume that g has the form

$$g = {}^\top(g^1(x_n), 0, \ldots, 0, g^n(x_n))$$

with smooth $g^1(x_n)$ and $g^n(x_n)$.

Under Dirichlet type boundary condition on the velocity field v, one can see that the system (1.1) has a stationary solution, called a *parallel flow*, in the form $\overline{u}_s = {}^\top(\overline{\rho}_s, \overline{v}_s)$, where

$$\overline{\rho}_s = \overline{\rho}_s(x_n), \quad \frac{1}{\ell}\int_0^\ell \overline{\rho}_s(x_n)\, dx_n = \rho_*,$$

$$\overline{v}_s = {}^\top(\overline{v}_s^1(x_n), 0, \ldots, 0).$$

Typical examples are the following plane Couette flow and the plane Poiseuille flow. If $g = 0$ and the boundary condition is

$$v^1|_{x_n=\ell} = V^1, \ \ v^2|_{x_n=\ell} = \cdots = v^n|_{x_n=\ell} = 0, \ \ v|_{x_n=0} = 0,$$

then one has the plane Couette flow $\overline{u}_s = {}^\top(\overline{\rho}_s, \overline{v}_s)$ with

$$\overline{\rho}_s = \rho_*, \quad \overline{v}_s = {}^\top\left(\frac{V^1}{\ell}x_n, 0, \ldots, 0\right);$$

and if $g = {}^\top(g^1, 0, \ldots, 0)$ with g^1 satisfying $g^1 \neq 0$ and the boundary condition

$$v|_{x_n=0,\ell} = 0,$$

then one has the plane Poiseuille flow $\overline{u}_s = {}^\top(\overline{\rho}_s, \overline{v}_s)$ with

$$\overline{\rho}_s = \rho_*, \quad \overline{v}_s = {}^\top\left(\frac{g^1}{2\mu}x_n(\ell - x_n), 0, \ldots, 0\right).$$

In this article we will survey the results in [13, 15, 16] on the stability and bifurcation problem for parallel flows. In Sect. 2, we consider the stability of parallel flows under spatially localized perturbations. In Sect. 3 we give an outline of the proof of the stability result. In Sect. 4 we focus on the Poiseuille flow; and we discuss the instability of the plane Poiseuille flow and the bifurcation of wave trains from the plane Poiseiulle flow.

2 Stability of Parallel Flows

We first introduce the following non-dimensional variables:

$$x = \ell\tilde{x},\ t = \frac{\ell}{V}\tilde{t},\ v = VQ,\ \ae = \ae_*\ae,\ P = \ae_* V^2 p$$

with $V = \|v_s^1\|_{C_*^{m+1}[0,\ell]}$ for an integer $m \geq [n/2] + 1$. Here

$$\|v_s^1\|_{C_*^{m+1}[0,\ell]} = \sum_{k=0}^{m+1} \sup_{0\leq x_n\leq\ell} \ell^k |\partial_{x_n}^k v_s^1(x_n)|.$$

Under this non-dimensionalization the domain Ω_ℓ is transformed into $\Omega \equiv \Omega_1$:

$$\Omega = \{\tilde{x} = (\tilde{x}', \tilde{x}_n);\ x' = (\tilde{x}_1, \dots, \tilde{x}_{n-1}) \in \mathbb{R}^{n-1},\ 0 < \tilde{x}_n < 1\},$$

and the parallel flow u_s is transformed into $\tilde{u}_s = {}^\top(\tilde{\rho}_s, \tilde{v}_s)$ with

$$\begin{aligned} &\tilde{\rho}_s = \tilde{\rho}_s(\tilde{x}_n) > 0, \quad \int_0^1 \tilde{\rho}_s(\tilde{x}_n)\, d\tilde{x}_n = 1, \\ &\tilde{v}_s = {}^\top(\tilde{v}_s^1(\tilde{x}_n), 0, \dots, 0), \quad \|\tilde{v}_s^1\|_{C^{m+1}[0,1]} = 1. \end{aligned}$$

Hereafter we omit tildes. The perturbation $u(t) = {}^\top(\phi(t), w(t)) = {}^\top({}^2(\ae(t) - \ae_s), v(t) - v_s)$ is then governed by the system of equations

$$\begin{cases} \partial_t\phi + v_s\cdot\nabla\phi + \gamma^2 \mathrm{div}\,(\rho_s w) = f^0(u), \\ \partial_t w - \frac{\circ}{\ae_s}\Delta w - \frac{Q}{\ae_s}\nabla\mathrm{div}\, w + \nabla\left(\frac{p'(\ae_s)}{{}^2\ae_s}\OE\right) \\ \qquad + \frac{\nu\partial_{x_n}^2 v_s^1}{\gamma^2\rho_s^2}\phi e_1 + v_s\cdot\nabla w + w\cdot\nabla v_s = f(u). \end{cases} \tag{2.1}$$

Here $e_1 = {}^\top(1, 0, \dots, 0) \in \mathbb{R}^n$; ν, $\tilde{\nu}$ and γ are non-dimensional parameters defined by

$$\nu = \frac{\mu}{\rho_*\ell V},\ \tilde{\nu} = \frac{\mu + \mu'}{\rho_*\ell V},\ \gamma = \sqrt{p'(1)} = \frac{\sqrt{P'(\rho_*)}}{V};$$

and $f^0(u)$ and $f(u)$ denote the nonlinearities:

$$f^0(u) = -\mathrm{div}\,(\phi w),$$

$$\begin{aligned} f(u) = &-w \cdot \nabla w + \tfrac{\text{˚Œ}}{(\text{Œ}+{}^2\text{æ}_s)\text{æ}_s}\left(-\Delta w + \tfrac{@^2_{x_n} v^1_s}{{}^2\text{æ}_s}\text{Œ}e_1\right) \\ &-\tfrac{\tilde{\nu}\phi}{(\phi+\gamma^2\rho_s)\rho_s}\nabla \mathrm{div}\, w \\ &+\tfrac{\phi}{\gamma^2\rho_s}\nabla\left(\tfrac{p'(\rho_s)}{\gamma^2\rho_s}\phi\right) - \tfrac{1}{2\gamma^4\rho_s}\nabla\left(p''(\rho_s)\phi^2\right) \\ &+\tilde{p}_3(\rho_s, \phi, \partial_x\phi), \end{aligned}$$

where

$$\begin{aligned} \tilde{p}_3 = &\tfrac{\phi^3}{\gamma^4(\phi+\gamma^2\rho_s)\rho_s^3}\nabla p(\rho_s) - \tfrac{1}{2\gamma^6\rho_s}\nabla\left(\phi^3 p_3(\rho_s, \phi)\right) \\ &+\tfrac{\phi}{2\gamma^6\rho_s^2}\nabla\left(p''(\rho_s)\phi^2 + \tfrac{1}{\gamma^2}\phi^3 p_3(\rho, \phi)\right) \\ &-\tfrac{\phi^2}{\gamma^2(\phi+\gamma^2\rho_s)\rho_s^2}\nabla\left(\tfrac{p'(\rho_s)}{\gamma^2}\phi + \tfrac{1}{2\gamma^4}p''(\rho_s)\phi^2 + \tfrac{1}{2\gamma^6}\phi^3 p_3(\rho_s, \phi)\right) \end{aligned}$$

with

$$p_3(\rho_s, \phi) = \int_0^1 (1-\theta)^2 p''(\theta\gamma^{-2}\phi + \rho_s)\, d\theta.$$

The boundary condition is transformed into

$$w|_{@\Omega} = 0. \tag{2.2}$$

We prescribe the initial condition

$$u|_{t=0} = u_0 = {}^\top(\phi_0, w_0). \tag{2.3}$$

We note that the Reynolds number Re and the Mach number Ma are given by

$$\mathrm{Re} = \frac{1}{\nu}, \quad \mathrm{Ma} = \frac{1}{\gamma}.$$

The following result [13] states that if the Reynolds and Mach numbers are sufficiently small, the parallel flow is asymptotically stable under spatially localized small perturbations and that the asymptotic leading part of perturbations behaves purely diffusively.

Theorem 2.1 ([13]) *Let m be an integer satisfying* $m \geq [n/2] + 1$. *Then there exist constants* $\nu_0 > 0$, $\gamma_0 > 0$, $\omega_0 > 0$ *such that if*

$$\nu \geq \nu_0, \quad \frac{\gamma^2}{\nu + \tilde{\nu}} \geq \gamma_0^2, \quad \omega \equiv \|\rho_s - 1\|_{C^m[0,1]} \leq \omega_0,$$

then the following assertions hold:

If $u_0 = (\phi_0, w_0)$ *is in* $H^m(\Omega) \cap L^1(\Omega)$ *with* $\|u_0\|_{H^m \cap L^1} \ll 1$ *and satisfies suitable compatibility conditions, then there exists a unique global solution* $u(t) =$

$(\phi(t), w(t)) \in C([0,\infty); H^m(\Omega))$ of (2.1)–(2.3) and the solution $u(t)$ has the following properties.

If $n \geq 3$, then

$$\|\partial_{x'}^{\ell} u(t)\|_{L^2} = O(t^{-\frac{n-1}{4}-\frac{\ell}{2}}) \quad (t \to \infty)$$

for $\ell = 0, 1$ and

$$\|u(t) - (\sigma u^{(0)})(t)\|_{L^2} = O(t^{-\frac{n-1}{4}-\frac{1}{2}}\eta_n(t)) \quad (t \to \infty).$$

Here $u^{(0)} = u^{(0)}(x_n)$; and $\sigma = \sigma(x', t)$ is a solution of the following linear heat equation

$$\partial_t \sigma - \kappa_0 \partial_{x_1}^2 \sigma - \kappa'' \Delta'' \sigma + a_1 \partial_{x_1} \sigma = 0,$$
$$\sigma|_{t=0} = \int_0^1 \phi_0(x', x_n)\, dx_n,$$

where $\Delta'' = \partial_{x_2}^2 + \cdots + \partial_{x_{n-1}}^2$; $\kappa_0 > 0$, $\kappa'' > 0$ and a_1 are constants; and $\eta_n(t) = 1$ when $n \geq 4$ and $\eta_n(t) = \log(1+t)$ when $n = 3$.

If $n = 2$, then

$$\|\partial_{x'}^{\ell} u(t)\|_{L^2} = O(t^{-\frac{1}{4}-\frac{\ell}{2}}) \quad (t \to \infty)$$

for $\ell = 0, 1$ and

$$\|u(t) - (\sigma u^{(0)})(t)\|_{L^2} = O(t^{-\frac{3}{4}+\varepsilon}), \quad \varepsilon > 0, \quad (t \to \infty).$$

Here $u^{(0)} = u^{(0)}(x_2)$; and $\sigma = \sigma(x_1, t)$ is a solution of the following Burgers equation

$$\partial_t \sigma - \kappa_0 \partial_{x_1}^2 \sigma + a_1 \partial_{x_1} \sigma + a_2 \partial_{x_1}(\sigma^2) = 0,$$
$$\sigma|_{t=0} = \int_0^1 \phi_0(x_1, x_2)\, dx_2,$$

where $\kappa_0 > 0$ and a_j $(j = 1, 2)$ are constants.

Remark 2.2 (i) It is well known that solutions of the Burgers equation are approximated by self-similar solutions if the initial data are sufficiently small. We thus see that, when $n = 2$, in addition to the assumptions of Theorem 2.1, if $\int_\Omega |x_1||\phi_0|\, dx \ll 1$, then $\|u(t) - (\chi u^{(0)})(t)\|_{L^2} = O(t^{-\frac{3}{4}+\varepsilon})$ $(\varepsilon > 0)$ as $t \to \infty$. Here $\chi(x_1, t) = z(x_1 - a_1 t, t)$, where $z = z(x_1, t)$ is a self-similar solution of $\partial_t z - \kappa \partial_{x_1}^2 z + a_2 \partial_{x_1}(z^2) = 0$ with $\int_{\mathbb{R}} z(x_1, t)\, dx_1 = \int_\Omega \phi_0\, dx$.

(ii) Iooss and Padula [10] studied the stability of parallel flows of (1.1) in a cylindrical domain $\Omega = \{x = (x_1, x'); x_1 \in \mathbb{R},\ x' \in D\}$ under the boundary condition $v|_{@D} = v_*$ satisfying $v_* \cdot n = 0$. Here D is a smooth bounded domain of $\mathbb{R}^{n-1}$ $(n = 2, 3)$ and n denotes the unit outer vector normal to $\partial\Omega$. In [10] the linearized stability under perturbations periodic in x_1 was considered; and the following result on the spectral distribution of the linearized operator around the parallel flow was obtained. Let A_{per} denote the linearized operator under perturbations periodic in x_1. Then there exists a constant $\tilde{\Lambda} > 0$ such that

$$\sigma(-A_{per}) \cap \{\lambda; \operatorname{Re}\lambda \ge -\tilde{\Lambda}\} = \{\lambda_j\}_{j=0}^{K},$$

where λ_j $(j = 0, 1, \ldots, K)$ are eigenvalues of $-L_0$ with finite multiplicities. Furthermore, it was shown in [10] that if the Reynolds number is small in some sense, then the parallel flow is linearly stable, i.e., $\operatorname{Re}\lambda_j < 0$ for all $j = 0, 1, \ldots, K$, and therefore, the solution of the linearized problem decays exponentially as $t \to \infty$.

(iii) In the case of the cylindrical domain $\Omega = \mathbb{R} \times D$ with bounded smooth domain $D \subset \mathbb{R}^2$, Aoyama and Kagei [5] proved the stability result similar to Theorem 2.1 for $n = 2$, i.e., parallel flows are stable under spatially localized small perturbations if the Reynolds and Mach numbers are sufficiently small, and the asymptotic leading part is given by the Burgers equations.

(iv) Stability results similar to Theorem 2.1 also hold for the case of time-periodic parallel flows. If the external force g takes the form $g = {}^\top(g^1(x_n, t), 0, \ldots, 0, g^n(x_n))$ with g^1 being time-periodic as $g^1(x_n, t + T) = g^1(x_n, t)$ for some $T > 0$, then (1.1) has a time-periodic parallel flow $u_s = {}^\top(\rho_*, \boldsymbol{v}_s)$, $\boldsymbol{v}_s = {}^\top(v_s^1(x', t), \boldsymbol{0}')$ with v_s^1 being time-periodic as $v_s^1(x', t + T) = v_s^1(x', t)$. In this case the statements of Theorem 2.1 with $u^{(0)} = u^{(0)}(x')$ replaced by $u^{(0)} = u^{(0)}(x', t)$ satisfying $u^{(0)}(x', t + T) = u^{(0)}(x', t)$ hold true. Here $x' = x_2$ when $n = 2$. See [6, 7] for the stability of time-periodic parallel flows. See also [9] for the stability of spatially periodic steady states.

(v) In [12], the stability of the plane Couette flow was studied as a special case of parallel flows and similar results to Theorem 2.1 was obtained. Li and Zhang ([21]) considered the stability of the plane Couette flow of (1.1) for $n = 3$ under the Navier-slip boundary condition on the bottom

$$v^3|_{x_3=0} = 0, \quad (-\partial_{x_3} v^j + \alpha v^j)|_{x_3=0} = 0 \quad (j = 1, 2)$$

and the non-homogeneous Dirichlet boundary condition on the top

$$v^1|_{x_3=\ell} = V^1, \quad v^2|_{x_3=\ell} = v^3|_{x_3=\ell} = 0.$$

Here $\alpha > 0$ is the slip length constant. It was shown in [21] that if α is getting smaller, then the Reynolds number can be taken larger to guarantee the stability of the plane Couette flow than that given in Theorem 2.1.

(vi) In [1, 2], the stability of the motionless state $u_s = {}^\top(\rho_*, 0)$ of (1.1) with $g = 0$ was studied under the complete slip boundary condition. In this case, the asymptotic behavior of solutions are different to the one in the non-slip case (2.2) given in Theorem 2.1. More precisely, we consider (1.1) with $g = \boldsymbol{0}$, written in the form

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho \boldsymbol{v}) = 0, \\ \rho(\partial_t \boldsymbol{v} + \boldsymbol{v} \cdot \nabla \boldsymbol{v}) - \mu \operatorname{div}\mathbf{D}(\boldsymbol{v}) - \mu' \nabla \operatorname{div}\boldsymbol{v} + \nabla P(\rho) = \boldsymbol{0}. \end{cases}$$

in a cylinder $\Omega = \mathbb{R} \times D$ with $D = \{x' = (x_2, x_3) : x_2^2 + x_3^2 \le \ell^2\}$ under the complete slip condition

$$\boldsymbol{v} \cdot \boldsymbol{n}|_{\partial\Omega} = 0, \quad \mathbf{D}(\boldsymbol{v}) \cdot \boldsymbol{n} - \big(\mathbf{D}(\boldsymbol{v})\boldsymbol{n} \cdot \boldsymbol{n}\big)\boldsymbol{n}\big|_{\partial\Omega} = \mathbf{0}.$$

Here $\big(\mathbf{D}(\boldsymbol{v})_{jk}\big)_{j,k=1}^{3} = \big(\partial_{x_j} v^k + \partial_{x_k} v^j\big)_{j,k=1}^{3}$; and the viscosity constants μ and μ' satisfy $\mu > 0$ and $\frac{2}{3}\mu + \mu' > 0$. Let $u = {}^\top(\phi, \boldsymbol{w})$ be the perturbation of the motionless state ${}^\top(\rho_*, \mathbf{0})$. If the initial perturbation $u_0 = {}^\top(\phi_0, w_0)$ is sufficiently small, then it holds that

$$\|u(t) - \chi_+(t)a_+ - \chi_-(t)a_- - \chi_{\mathrm{rig}}(t)a_{\mathrm{rig}}\|_{L^2(\Omega_\ell)} \le C(1+t)^{-\frac{1}{2}},$$

where $a_\pm = {}^\top(1, \pm 1, 0, 0)$; $\chi_\pm = \chi_\pm(x_1, t) = z_\pm(x_1 \pm \gamma t, t)$ are nonlinear diffusion waves with $z_\pm = z_\pm(x_1, t)$ denoting self-similar solutions of the Burgers equations

$$\partial_t z_\pm - \frac{2\nu + \nu'}{2}\partial_{x_1}^2 z_\pm \mp c\partial_{x_1}(z_\pm^2) = 0;$$

and where $\chi_{\mathrm{rig}}(t)a_{\mathrm{rig}}$ is a diffusive rigid motion with $a_{\mathrm{rig}} = \frac{1}{\ell^2}\sqrt{\frac{2}{\beta}}\,{}^\top(0, 0, -x_3, x_2)$ and $\chi_{\mathrm{rig}} = a_{\mathrm{rig}}(4\pi\nu t)^{-1/2}e^{-x^2/(4\nu t)}$. Here $\gamma = \sqrt{P'(\rho_*)}$, $\nu = \mu/\rho_*$ and $\nu' = \mu'/\rho_*$.

In contrast to the case of the non-slip boundary condition (2.2), a hyperbolic aspect (propagation of diffusion waves) appears in the asymptotic leading part of the perturbation under the complete slip boundary condition. (Cf., [17, 22].) As for the analysis of the problem under the slip boundary condition, see also [20, 24, 27].

3 Outline of Proof of Theorem 2.1

Theorem 2.1 is proved by decomposing the problem into the low and high frequency parts. For the low frequency part, we make use of the spectral properties of the linearized semigroup, while for the high frequency part, we employ the Matsumura-Nishida energy method [23]. We here give an outline of the proof of Theorem 2.1 following the arguments in [5, 6, 13]. For simplicity we consider the case $n = 2$ only.

3.1 Notation

We first introduce notation which will be used in this section. For $1 \le p \le \infty$ we denote by $L^p(E)$ the usual Lebesgue space on a domain E and its norm is denoted by $\|\cdot\|_{L^p(E)}$. Let m be a nonnegative integer. $H^m(E)$ denotes the m th order L^2 Sobolev space on E with norm $\|\cdot\|_{H^m(E)}$. In particular, we write $L^2(E)$ for $H^0(E)$.

We denote by $C_0^m(E)$ the set of all C^m functions with compact support in E. $H_0^m(E)$ stands for the completion of $C_0^m(E)$ in $H^m(E)$.

We simply denote by $L^p(E)$ (resp., $H^m(E)$) the set of all vector fields $w = {}^\top(w^1, w^2)$ on E and its norm is denoted by $\|\cdot\|_{L^p(E)}$ (resp., $\|\cdot\|_{H^m(E)}$). For $u = {}^\top(\phi, w)$ with $\phi \in H^k(E)$ and $w = {}^\top(w^1, w^2) \in H^m(E)$, we define $\|u\|_{H^k(E)\times H^m(E)}$ by $\|u\|_{H^k(E)\times H^m(E)} = \|\phi\|_{H^k(E)} + \|w\|_{H^m(E)}$.

When $E = \Omega$ we abbreviate $L^p(\Omega)$ as L^p, and likewise, $H^m(\Omega)$ as H^m. The norm $\|\cdot\|_{L^p(\Omega)}$ is written as $\|\cdot\|_{L^p}$, and likewise, $\|\cdot\|_{H^m(\Omega)}$ as $\|\cdot\|_{H^m}$. The inner product of $L^2(\Omega)$ is denoted by

$$(f, g) = \int_\Omega f(x)\overline{g(x)}\, dx, \quad f, g \in L^2(\Omega).$$

For $u_j = {}^\top(\phi_j, w_j)$ $(j = 1, 2)$, we also define a weighted inner product $\langle u_1, u_2\rangle$ by

$$\langle u_1, u_2\rangle = \tfrac{1}{\gamma^2}\int_\Omega \phi_1\overline{\phi}_2 \tfrac{P'(\rho_s)}{\gamma^2\rho_s}\, dx + \int_\Omega w_1\cdot \overline{w}_2 æ_s\, dx,$$

where $\rho_s = \rho_s(x_2)$ is the density of the parallel flow u_s.

In the case $E = (0, 1)$ we denote the norm of $L^p(0, 1)$ by $|\cdot|_p$. The norm of $H^m(0, 1)$ is denoted by $|\cdot|_{H^m}$, respectively. The inner product of $L^2(0, 1)$ is also denoted by

$$(f, g) = \int_0^1 f(x_2)\overline{g(x_2)}dx_2, \quad f, g \in L^2(0, 1)$$

if no confusion occurs. Here $\overline{g}$ denotes the complex conjugate of g. For $u_j = {}^\top(\phi_j, w_j)$ $(j = 1, 2)$, we also denote the weighted inner product by $\langle u_1, u_2\rangle$:

$$\langle u_1, u_2\rangle = \int_0^1 \phi_1\overline{\phi}_2 \tfrac{p'(\rho_s)}{\gamma^2\rho_s}\, dx_2 + \int_0^1 w_1\cdot \overline{w}_2 æ_s\, dx_2.$$

For $f \in L^1(0, 1)$ we denote the mean value of f over $(0, 1)$ by $\langle f\rangle$:

$$\langle f\rangle = \int_0^1 f(x_2)\, dx_2.$$

We finally define the Fourier transform of f in x_1 by

$$\hat{f}(\xi) = \mathcal{F}f(\xi) = \int_{\mathbb{R}} f(x_1)e^{-i\xi x_1}\, dx_1$$

and its inverse transform is defined by

$$\mathcal{F}^{-1}f(x_1) = \frac{1}{2\pi}\int_{\mathbb{R}} f(\xi)e^{i\xi x_1}\, d\xi.$$

3.2 Spectral Properties of the Linearized Semigroup

In this subsection we consider the spectral properties of the linearized semigroup. We begin with the linearized resolvent problem associated with (2.1)–(2.3) for $n = 2$ which is written in the form

$$\lambda u + Lu = F. \tag{3.1}$$

Here $\lambda \in \mathbb{C}$ is a resolvent parameter; $F = {}^\top(f^0, f)$ with $f = {}^\top(f^1, f^2)$ is a given function in $L^2(\Omega) \times L^2(\Omega)$; and L is the operator on $L^2(\Omega) \times L^2(\Omega)$ with domain

$$D(L) = \{u = {}^\top(\phi, w) \in L^2(\Omega) \times L^2(\Omega);\ w \in H^1_0(\Omega),\ Lu \in L^2(\Omega) \times L^2(\Omega)\},$$

$$L = \begin{pmatrix} v_s^1 \partial_{x_1} & \gamma^2 \mathrm{div}(\rho_s \cdot) \\ \nabla\left(\frac{P'(\rho_s)}{\gamma^2 \rho_s} \cdot\right) & -\frac{\nu}{\rho_s}\Delta - \frac{\nu+\nu'}{\rho_s}\nabla \mathrm{div} + v_s^1 \partial_{x_1} \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ \frac{\nu \partial_{x_2}^2 v_s^1}{\gamma^2 \rho_s^2} e_1 & (\partial_{x_2} v_s^1) e_1 {}^\top e_2 \end{pmatrix},$$

where $e_1 = {}^\top(1, 0)$ and $e_2 = {}^\top(0, 1)$.

One can see that there exists a $\Lambda \gg 1$ such that $\{\lambda \in \mathbf{C};\ \mathrm{Re}\,\lambda \geq \Lambda\} \subset \rho(-L)$ and that $-L$ generates a C_0-semigroup $U(t)$. See [10] for a generation of a C_0-semigroup.

We shall decompose $U(t)$ by a projection operator associated with the spectrum of $-L$ which is obtained through the Fourier transform in x_1.

To investigate the spectrum of $-L$, let us consider the Fourier transform of (3.1) in $x_1 \in \mathbb{R}$:

$$\lambda \hat{u} + \hat{L}_\xi \hat{u} = \hat{f}, \tag{3.2}$$

with a parameter $\xi \in \mathbb{R}$. Here $\hat{L}_\xi$ is the operator on $H^1(0, 1) \times L^2(0, 1)$ with domain

$$D(\hat{L}_\xi) = H^1(0, 1) \times (H^2(0, 1) \cap H^1_0(0, 1)),$$

and

$$\hat{L}_\xi = \begin{pmatrix} i\xi v_s^1 & i\gamma^2 \rho_s \xi & \gamma^2 \partial_{x_2}(\rho_s \cdot) \\ i\xi \frac{P'(\rho_s)}{\gamma^2 \rho_s} & -\frac{\nu}{\rho_s}\partial_{x_2}^2 + \frac{\nu+\tilde{\nu}}{\rho_s}|\xi|^2 + i\xi v_s^1 & -i\frac{\tilde{\nu}}{\rho_s}\xi\partial_{x_2} \\ \partial_{x_2}\left(\frac{p(\rho_s)}{\gamma^2 \rho_s} \cdot\right) & -i\frac{\tilde{\nu}}{\rho_s}\xi\partial_{x_2} & -\frac{\nu+\tilde{\nu}}{\rho_s}\partial_{x_2}^2 + \frac{\nu}{\rho_s}|\xi|^2 + i\xi v_s^1 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ \frac{\nu \partial_{x_2}^2 v_s^1}{\gamma^2 \rho_s^2} & 0 & \partial_{x_2} v_s^1 \\ 0 & 0 & 0 \end{pmatrix}.$$

We also introduce the adjoint operator $\hat{L}_\xi^*$ with domain $D(\hat{L}_\xi^*) = D(\hat{L}_\xi)$,

$$\hat{L}_\xi^* = \begin{pmatrix} -i\xi v_s^1 & -i\gamma^2\rho_s\xi & -\gamma^2\partial_{x_2}(\rho_s\,\cdot) \\ -i\xi\frac{P'(\rho_s)}{\gamma^2\rho_s} & -\frac{\nu}{\rho_s}\partial_{x_2}^2 + \frac{\nu+\tilde{\nu}}{\rho_s}|\xi|^2 - i\xi v_s^1 & -i\frac{\tilde{\nu}}{\rho_s}\xi\partial_{x_2} \\ -\partial_{x_2}\left(\frac{p(\rho_s)}{\gamma^2\rho_s}\,\cdot\right) & -i\frac{\tilde{\nu}}{\rho_s}\xi\partial_{x_2} & -\frac{\nu+\tilde{\nu}}{\rho_s}\partial_{x_2}^2 + \frac{\nu}{\rho_s}|\xi|^2 - i\xi v_s^1 \end{pmatrix}$$
$$+\begin{pmatrix} 0 & \frac{\gamma^2\nu\partial_{x_2}^2 v_s^1}{p'(\rho_s)} & 0 \\ 0 & 0 & 0 \\ 0 & \partial_{x_2} v_s^1 & 0 \end{pmatrix}.$$

When $\xi = 0$, one can see that 0 is a simple eigenvalue of $-\hat{L}_0$ and $-\hat{L}_0^*$ and the remaining parts of spectra of $-\hat{L}_0$ and $-\hat{L}_0^*$ lie in a left-half plane strictly away from the imaginary axis. As for the eigenspaces for the eigenvalue 0, we have the following proposition.

Proposition 3.1 *0 is a simple eigenvalue of $-\hat{L}_0$ and $-\hat{L}_0^*$ and*

$$\mathrm{Ker}\,(-L_0) = \mathrm{span}\,\{u^{(0)}\}, \quad \mathrm{Ker}\,(-L_0^*) = \mathrm{span}\,\{u^{(0)*}\},$$

where the functions $u^{(0)}$ and $u^{(0)*}$ are given by

$$u^{(0)} = {}^\top(\phi^{(0)}, w^{(0)}), \quad w^{(0)} = {}^\top(w^{(0),1}, 0)$$

and

$$u^{(0)*} = {}^\top(\phi^{(0)*}, 0).$$

Here

$$\phi^{(0)}(x_2) = \alpha_0\frac{\gamma^2\rho_s(x_2)}{p'(\rho_s(x_2))}, \quad \alpha_0 = \left(\int_0^1 \frac{\gamma^2\rho_s(x_2)}{p'(\rho_s(x_2))}dx_2\right)^{-1};$$

and $w^{(0),1}$ is the solution of the following problem

$$\begin{cases} -\partial_{x_2}^2 w^{(0),1} = -\frac{1}{\gamma^2\rho_s}\partial x_2{}^2 v_s^1\phi^{(0)}, \\ w^{(0),1}\,|_{x_2=0,1} = 0; \end{cases}$$

and

$$\phi^{(0)*}(x_2) = \frac{\gamma^2}{\alpha_0}\phi^{(0)}(x_2).$$

Furthermore, it holds that

$$\langle u_0, u_0^*\rangle = 1.$$

By a perturbation argument, we have the following properties the spectrum of $-\hat{L}_\xi$ for $|\xi| \ll 1$.

Proposition 3.2 (i) *There exist positive constants c_0, ν_1, γ_1, ω_1 and r_0 such that if $\nu \geq \nu_1$, $\frac{\gamma^2}{\nu+\tilde{\nu}} \geq \gamma_1^2$ and $\omega \leq \omega_1$, then it holds that*

$$\sigma(-\hat{L}_\xi) \cap \{\lambda : |\lambda| \le \tfrac{c_0}{2}\} = \{\lambda_0(\xi)\}$$

for each ξ *with* $|\xi| \le r_0$*, where* $\lambda_0(\xi)$ *is a simple eigenvalue of* $-\hat{L}_\xi$ *that has the form*

$$\lambda_0(\xi) = -ia_1\xi - \kappa_0\xi^2 + \mathcal{O}\big(|\xi|^3\big)$$

as $|\xi| \to 0$*. Here* $a_1 \in \mathbb{R}$ *and* $\kappa_0 > 0$ *are the numbers given by*

$$a_1 = -\langle v_s^1\phi^{(0)} + \gamma^2\rho_s w^{(0),1}\rangle = O(1),$$

$$\kappa_0 = \frac{\gamma^2}{\nu}\Big\{\alpha_0\big|(-\partial_{x_2}^2)^{-\frac{1}{2}}\rho_s\big|_2^2 + O\big(\tfrac{1}{\gamma^2}\big) + \big(\tfrac{\nu}{\gamma^2} + \tfrac{1}{\nu^2}\big) \times O\big(\tfrac{\nu+\tilde{\nu}}{\gamma^2}\big)\Big\}.$$

Here $(-\partial_{x_2}^2)$ *is the operator on* $L^2(0,1)$ *under the zero Dirichlet boundary condition with domain* $D((-\partial_{x_2}^2)) = H^2(0,1) \cap H_0^1(0,1)$.

(ii) *The eigenprojections* $\hat{\Pi}(\xi)$ *and* $\hat{\Pi}^*(\xi)$ *for the eigenvalues* $\lambda_0(\xi)$ *and* $\overline{\lambda}_0(\xi)$ *of* $-\hat{L}_\xi$ *and* $-\hat{L}_\xi^*$ *are given by*

$$\hat{\Pi}(\xi)u = \langle u, u_\xi^*\rangle u_\xi, \quad \hat{\Pi}^*(\xi)u = \langle u, u_\xi\rangle u_\xi^*,$$

respectively, where u_ξ *and* u_ξ^* *are eigenfunctions for* $\lambda_0(\xi)$ *and* $\overline{\lambda}_0(\xi)$*, respectively, that satisfy* $\langle u_\xi, u_\xi^*\rangle = 1$.

Furthermore, u_ξ *and* u_ξ^* *are written in the form*

$$u_\xi(x_2) = u^{(0)}(x_2) + i\xi u^{(1)}(x_2) + |\xi|^2 u^{(2)}(x_2,\xi),$$
$$u_\xi^*(x_2) = u^{*(0)}(x_2) + i\xi u^{*(1)}(x_2) + |\xi|^2 u^{*(2)}(x_2,\xi),$$

and the following estimate holds

$$|u_\xi|_{H^k} + |u_\xi^*|_{H^k} + |u^{(1)}|_{H^k} + |u^{*(1)}|_{H^k} + |u^{(2)}|_{H^k} + |u^{*(2)}|_{H^k} \le C_{k,r_0}$$

with a constant $C_{k,r_0} > 0$.

See [4, Theorem 4.5, 4.7] and [3, Lemma 4.1] for a proof.

The asymptotic behavior of the semigroup e^{-tL} generated by $-L$ follows from Proposition 3.2. Let us introduce the characteristic function $\mathbf{1}_{\{|\eta|\le r_0\}}(\xi)$ defined by

$$\mathbf{1}_{\{|\eta|\le r_0\}}(\xi) = \begin{cases} 1, & (0 \le |\xi| \le r_0), \\ 0, & (|\xi| > r_0), \end{cases} \qquad \text{for} \quad \xi \in \mathbb{R},$$

where r_0 is the positive constant given in Proposition 3.2.

We define the projections P_0 and P_∞ by

$$P_0 = \mathcal{F}^{-1}\mathbf{1}_{\{|\eta|\le r_0\}}(\xi)\hat{\Pi}(\xi)\mathcal{F}$$

and

$$P_\infty = I - P_0.$$

Then P_0 and P_∞ satisfy

$$P_0 + P_\infty = I, \quad P^2 = P, \quad P_j L \subset L P_j, \quad P_j e^{-tL} = e^{-tL} P_j, \quad (j = 0, \infty).$$

Based on Proposition 3.2, the following decay estimates of e^{-tL} follow.

Proposition 3.3 *If $\nu \geq \nu_1$, $\frac{\gamma^2}{\nu+\widetilde{\nu}} \geq \gamma_1^2$ and $\omega \leq \omega_1$, then $e^{-tL}P_0$ and $e^{-tL}P_\infty$ satisfy the following estimates.*

(i) *If $u_0 = {}^\top(\phi_0, w_0) \in (L^1(\Omega) \times L^1(\Omega)) \cap (L^2(\Omega) \times L^2(\Omega))$, then $e^{-tL}P_0 u_0$ satisfies the following estimates*

$$\|\partial_{x_2}^k \partial_{x_1}^l e^{-tL} P_0 u_0\|_2 \leq C_{k,l}(1+t)^{-\frac{k}{4}-\frac{l}{2}}\|u_0\|_1 \tag{3.3}$$

uniformly for $t \geq 0$ and for $k = 0, 1, \ldots$, and $l = 0, 1, \ldots$.

(ii) *Let $\widetilde{H}^1(\Omega) = \{w \in L^2(\Omega); @_{x_1} w \in L^2(\Omega)\}$ with norm $\|w\|_{\widetilde{H}^1} = \|w\|_2 + \|@_{x_1} w\|_2$. If $u_0 \in H^1(\Omega) \times \widetilde{H}^1(\Omega)$, then there exists a constant $d_0 > 0$ such that $e^{-tL}P_\infty u_0$ satisfies*

$$\|e^{-tL} P_\infty u_0\|_{H^1} \leq C e^{-d_0 t}\left(\|u_0\|_{H^1 \times \widetilde{H}^1} + t^{-\frac{1}{2}}\|w_0\|_2\right) \tag{3.4}$$

uniformly for $t \geq 0$.

To treat the nonlinear problem, we need more detailed information on the structure of the P_0 part of e^{-tL}.

We have the following factorization of $e^{-tL}P_0$. See [4, Sect. 5] for the detailed argument.

Here and in what follows we assume that

$$\nu \geq \nu_1, \quad \frac{\gamma^2}{\nu + \widetilde{\nu}} \geq \gamma_1^2, \quad \omega \leq \omega_1,$$

where ν_1, γ_1 and ω_1 are the constants given in Proposition 3.2.

We define the operators

$$\mathcal{T} : L^2(\mathbb{R}) \to L^2(\Omega), \;\; \mathcal{P} : L^2(\Omega) \to L^2(\mathbb{R}), \;\; \Lambda : L^2(\mathbb{R}) \to L^2(\mathbb{R})$$

by

$$\mathcal{T}\sigma = \mathcal{F}^{-1}[\hat{\mathcal{T}}_\xi \hat{\sigma}], \qquad \hat{\mathcal{T}}_\xi \hat{\sigma} = \mathbf{1}_{\{|\eta| \leq r_0\}}(\xi) u_\xi \hat{\sigma};$$

$$\mathcal{P}u = \mathcal{F}^{-1}[\hat{\mathcal{P}}_\xi \hat{u}], \qquad \hat{\mathcal{P}}_\xi \hat{u} = \mathbf{1}_{\{|\eta| \leq r_0\}}(\xi) \langle \hat{u}, u_\xi^* \rangle;$$

$$\Lambda\sigma = \mathcal{F}^{-1}[\mathbf{1}_{\{|\eta| \leq r_0\}}(\xi) \lambda_0(\xi) \hat{\sigma}]$$

for $u \in L^2(\Omega)$ and $\sigma \in L^2(\mathbb{R})$. It then follows that

$$P_0 = \mathcal{T}\mathcal{P}, \quad \mathbf{1}_{\{|\eta|\le r_0\}}(\xi)\hat{\Pi}(\xi) = \hat{\mathcal{T}}\hat{\mathcal{P}}, \quad P_0 L \subset L P_0 = \Lambda P_0.$$

We have the following factorization of $e^{-tL}P_0$.

Proposition 3.4 *It holds that*

$$e^{-tL}P_0 = \mathcal{T}e^{t\Lambda}\mathcal{P}.$$

As for $\mathcal{T}$, we have the following estimates.

Proposition 3.5 *The operator $\mathcal{T}$ has the following properties:*

(i) $\|\mathcal{T}\sigma\|_{H^k} \le C\|\sigma\|_{L^2(\mathbb{R})}$ *for* $k = 0, 1, \dots,$ *and* $\sigma \in L^2(\mathbb{R})$.
(ii) $\mathcal{T}$ *is decomposed as* $\mathcal{T} = \mathcal{T}^{(0)} + \partial_{x_1}\mathcal{T}^{(1)} + \partial_{x_1}^2\mathcal{T}^{(2)}$, *where* $\mathcal{T}^{(j)}\sigma = \mathcal{F}^{-1}[\hat{\mathcal{T}}^{(j)}\hat{\sigma}]$ $(j = 0, 1, 2)$ *with*

$$\begin{aligned} \hat{\mathcal{T}}^{(0)}\hat{\sigma} &= \mathbf{1}_{\{|\eta|\le r_0\}}(\xi)\hat{\sigma}u^{(0)}, \\ \hat{\mathcal{T}}^{(1)}\hat{\sigma} &= \mathbf{1}_{\{|\eta|\le r_0\}}(\xi)\hat{\sigma}u^{(1)}(\cdot), \\ \hat{\mathcal{T}}^{(2)}\hat{\sigma} &= -\mathbf{1}_{\{|\eta|\le r_0\}}(\xi)\hat{\sigma}u^{(2)}(\cdot, \xi). \end{aligned}$$

Here $\mathcal{T}^{(j)}$ $(j = 0, 1, 2)$ *satisfy estimates* (i) *by replacing* $\mathcal{T}$ *with* $\mathcal{T}^{(j)}$.

Similar estimates also hold for $\mathcal{P}$.

Proposition 3.6 *The operator $\mathcal{P}$ has the following properties:*

(i) $\|\mathcal{P}u\|_{H^k(\mathbb{R})} \le C\|u\|_2$ *for* $k = 0, 1, \dots,$ *and* $u \in L^2(\Omega)$. *Furthermore,* $\|\mathcal{P}u\|_{L^2(\mathbb{R})} \le C\|u\|_1$ *for* $u \in L^1(\Omega)$.
(ii) $\mathcal{P}$ *is decomposed as* $\mathcal{P} = \mathcal{P}^{(0)} + \partial_{x_1}\mathcal{P}^{(1)} + \partial_{x_1}^2\mathcal{P}^{(2)}$, *where* $\mathcal{P}^{(j)}u = \mathcal{F}^{-1}[\hat{\mathcal{P}}^{(j)}\hat{u}]$ $(j = 0, 1, 2)$ *with*

$$\begin{aligned} \hat{\mathcal{P}}^{(0)}\hat{u} &= \mathbf{1}_{\{|\eta|\le r_0\}}(\xi)\langle \hat{u}, u^{*(0)}\rangle = \mathbf{1}_{\{|\eta|\le r_0\}}(\xi)\langle Q_0\hat{u}\rangle, \\ \hat{\mathcal{P}}^{(1)}\hat{u} &= \mathbf{1}_{\{|\eta|\le r_0\}}(\xi)\langle \hat{u}, u^{*(1)}\rangle, \\ \hat{\mathcal{P}}^{(2)}\hat{u} &= -\mathbf{1}_{\{|\eta|\le r_0\}}(\xi)\langle \hat{u}, u^{*(2)}(\xi)\rangle. \end{aligned}$$

Here $\mathcal{P}^{(j)}$ $(j = 0, 1, 2)$ *satisfy estimates* (i) *by replacing* $\mathcal{P}$.

It then follows that $e^{t\Lambda}$ satisfies the following estimates.

Proposition 3.7 *The operator $e^{t\Lambda}$ satisfies the following estimates.*

(i) $\|\partial_{x_3}^l e^{t\Lambda}\mathcal{P}u\|_{L^2(\mathbb{R})} \le C(1+t)^{-\frac{1}{4}-\frac{l}{2}}\|u\|_1$,
(ii) $\|\partial_{x_3}^l e^{t\Lambda}\mathcal{P}^{(j)}u\|_{L^2(\mathbb{R})} \le C(1+t)^{-\frac{1}{4}-\frac{l}{2}}\|u\|_1, \quad j = 0, 1, 2,$
(iii) $\|\partial_{x_3}^l(\mathcal{T} - \mathcal{T}^{(0)})e^{t\Lambda}\mathcal{P}u\|_2 \le C(1+t)^{-\frac{1}{4}-\frac{l+1}{2}}\|u\|_1,$

for $u \in L^1(\Omega)$ *and* $l = 0, 1, 2 \ldots$.

By the properties of $\mathcal{P}$ and the definition of Λ, one can see that the asymptotic behavior of $e^{t\Lambda}$ is described by $\mathcal{H}(t)$:

$$\mathcal{H}(t)\sigma = \mathcal{F}^{-1}[e^{-(ia_1\xi+\kappa_0\xi^2)t}\hat{\sigma}] \quad (\sigma \in L^2(\mathbb{R})),$$

where $\kappa_1 \in \mathbb{R}$ and $\kappa_0 > 0$ are given by Proposition 3.2. Indeed, we have the following estimates.

Proposition 3.8 *For* $u \in L^2(\Omega)$, *we set* $\sigma = \langle Q_0 u\rangle$. *If* $u \in L^1(\Omega)$, *then there holds the estimate*

$$\|\partial_{x_1}^l\big(e^{t\Lambda}\mathcal{P}u - \mathcal{H}(t)\sigma\big)\|_{L^2(\mathbb{R})} \leq Ct^{-\frac{3}{4}-\frac{l}{2}}\|u\|_1 \quad (l = 0, 1, \ldots).$$

Since

$$e^{-tL}P_0 = \mathcal{T}e^{t\Lambda}\mathcal{P} = \mathcal{T}^{(0)}e^{t\Lambda}\mathcal{P} + (\mathcal{T} - \mathcal{T}^{(0)})e^{t\Lambda}\mathcal{P},$$

one could imagine that $u^{(0)}\mathcal{H}(t)\langle\phi_0\rangle$ would appear in the asymptotic leading part of the solution of the nonlinear problem (2.1)–(2.3). It is true if $n \geq 3$, but, in the case $n = 2$, one needs to take into account the nonlinearities which leads to the nonlinear term of the Burgers equation. See [5, 6, 13] for details.

3.3 Nonlinear Problem

The problem (2.1)–(2.3) is written as

$$\frac{du}{dt} + Lu = F(u), \quad w|_{@\Omega} = 0, \quad u|_{t=0} = u_0. \tag{3.5}$$

Here $u = {}^\top(\phi, w)$; and $F(u)$ denotes the nonlinearity:

$$F(u) = {}^\top(f^0(\phi, w), f(Œ, w)).$$

We decompose the solution u into its P_0 and P_∞ parts. Let us decompose the solution $u(t)$ of (3.5) as

$$u(t) = (\sigma_1 u^{(0)})(t) + u_1(t) + u_\infty(t),$$

where

$$\sigma_1(t) = \mathcal{P}u(t), \quad u_1(t) = (\mathcal{T} - \mathcal{T}^{(0)})\mathcal{P}u(t), \quad u_\infty(t) = P_\infty u(t).$$

Observe that $P_0 u(t) = (\sigma_1 u^{(0)})(t) + u_1(t)$.

From the estimates in Sect. 3.2, one could expect that $\sigma_1(t)$ were the asymptotic leading part of $u(t)$. In fact, since $u_1(t)$ is written as

$$u_1(t) = (\mathcal{T} - \mathcal{T}^{(0)})\mathcal{P}u(t) = (\partial_{x_1}\mathcal{T}^{(1)} + \partial_{x_1}^2\mathcal{T}^{(2)})\sigma_1(t),$$

one can see from Propositions 3.5 and 3.6 that $u_1(t)$ is dominated by $\sigma_1(t)$.

Proposition 3.9 *Let $u(t)$ be a solution of* (3.5). *Then*

$$\|\partial_{x'}^k\partial_{x_3}^l\partial_t^m u_1(t)\|_2 \le C\{\|\partial_{x_3}\sigma_1(t)\|_2 + \|\partial_t\sigma_1(t)\|_2\}$$

for $1 \le k + l + 2m \le 3$.

From Proposition 3.9, it thus suffices to consider $\sigma_1(t)$ and $u_\infty(t)$.

We give an outline of the decay estimate of $u(t)$ in Theorem 2.1. By using the factorization of $e^{-tL}P_0$, we see that σ_1 satisfies

$$\sigma_1(t) = e^{t\Lambda}\mathcal{P}u_0 + \int_0^T e^{(t-\tau)\Lambda}\mathcal{P}F(\tau)d\tau; \tag{3.6}$$

We employ this formula to estimate $\sigma_1(t)$.

On the other hand, $u_\infty(t)$ satisfies

$$\partial_t u_\infty + Lu_\infty = F_\infty, \quad w_\infty\,|_{@\Omega}= 0, \quad u_\infty\,|_{t=0}= u_{\infty,0}, \tag{3.7}$$

where $F_\infty = P_\infty F$ and $u_{\infty,0} = P_\infty u_0$. To estimate $u_\infty(t)$ we use the estimate (3.4) of $e^{-tL}P_\infty$ and the Matsumura-Nishida energy method.

We introduce the quantity $M_1(t)$ defined by

$$M_1(t) = \sup_{0\le\tau\le t}(1+\tau)^{\frac{1}{4}}\|\sigma_1(\tau)\|_2 + \sup_{0\le\tau\le t}(1+\tau)^{\frac{3}{4}}\{\|\partial_{x_1}\sigma_1(\tau)\|_2 + \|\partial_\tau\sigma_1(\tau)\|_2\};$$

and we define the quantity $M(t) \ge 0$ by

$$M(t)^2 = M_1(t)^2 + \sup_{0\le\tau\le t}(1+\tau)^{\frac{3}{2}}E_\infty(\tau) \quad (t \in [0, T])$$

with

$$E_\infty(t) = \|u_\infty(t)\|_{H^2}^2 + \|\partial_t u_\infty(t)\|_2^2.$$

We also introduce a quantity $D_\infty(t)$ for $u_\infty = {}^\top(\phi_\infty, w_\infty)$ defined by

$$D_\infty(t) = \|\partial_x\phi_\infty(t)\|_{H^1}^2 + \|\partial_t\phi_\infty(t)\|_{H^1}^2 + \|\partial_x w_\infty(t)\|_{H^2}^2 + \|@_t w_\infty(t)\|_{H^1}^2.$$

By using the estimates in Sect. 3.2, we can show that if $M(t) \le 1$ for $t \in [0, T]$, then

$$M_1(t) \le C\{\|u_0\|_{L^1} + M(t)^2\}. \tag{3.8}$$

Furthermore, using the estimate (3.4) of $e^{-tL}P_\infty$ and the Matsumura-Nishida energy method, we can obtain the estimate

$$\begin{aligned} &E_\infty(t) + \int_0^\infty e^{-a(t-\tau)} D_\infty(\tau)d\tau \\ &\le C\{e^{-at}E_\infty(0) + (1+t)^{-\frac{3}{2}}M(t)^4 + \int_0^t e^{-a(t-\tau)}\mathcal{R}(\tau)d\tau\}. \end{aligned} \tag{3.9}$$

Here $a = a(\nu, \widetilde{\nu}, \gamma)$ is a positive constant; and $\mathcal{R}(t)$ is a function satisfying the estimate

$$\mathcal{R}(t) \le C\{(1+t)^{-\frac{3}{2}}M(t)^3 + (1+t)^{-\frac{1}{4}}M(t)D_\infty(t)\}. \tag{3.10}$$

Combining these estimates with the local existence of solutions, one can prove that if $\|u_0\|_{H^2\cap L^1}$ is sufficiently small, then

$$M(t) \le C\|u_0\|_{H^2\cap L^1}$$

uniformly for $t \ge 0$, which proves the decay estimate of $u(t)$ in Theorem 2.1.

4 Instability and Bifurcation in Poiseuille Flows

We have seen that parallel flows are stable if the Reynolds and Mach numbers are small enough. In this section we consider what happens if the Reynolds and Mach numbers are not necessarily small. It is expected that parallel flows become unstable when the Reynolds number increases. We shall see that, for a certain range of Mach numbers, the plane Poiseuille flow becomes unstable for Reynolds numbers beyond a critical value which is much smaller than that for the case of the incompressible fluids. After the instability of the plane Poiseuille flow, a bifurcation of wave trains (spatio-temporal traveling waves) occurs. These results were proved in [15, 16]. In this section we review the instability and bifurcation results given in [15, 16]. See also [14] where the proof of the bifurcation of the wave trains is outlined.

Bifurcation problems for equations in fluid mechanics have been paid much attention and, in fact, have been extensively studied. The mathematical analysis of such problems were mainly done for the incompressible Navier-Stokes equations since 1960s; see, e.g., [11, 18, 19, 28], and so on. Since the incompressible Navier-Stokes equations are classified in semilinear parabolic systems, classical bifurcation theories for elliptic equations can be directly applied to bifurcation problems for the incompressible Navier-Stokes equations. See e.g., Crandall and Rabinowitz [8].

On the other hand, the compressible Navier-Stokes equations are classified in quasilinear hyperbolic-parabolic systems, and bifurcation theory applicable to the incompressible problems does not work well for the compressible Navier-Stokes equations.

The first result for the multi-dimensional compressible bifurcation problems was given by Nishida et al. [25] who proved the existence of bifurcating compressible convection solutions for thermal convection problem. The main difficulty in the proof of the bifurcation for the compressible system arises from the convection term $v \cdot \nabla æ$ in (1.1). This term causes the derivative-loss in a standard setting, and therefore, it is not Frechét differentiable if one would try to handle by a classical bifurcation analysis. In [25], the effective viscous flux is used to overcome this difficulty and close the estimates for the proof of the bifurcation of stationary convective patterns. On the other hand, the effective viscous flux is not used in the analysis of the bifurcation of wave trains from the plane Poiseuille flow in [16]. Instead of it, the convection term $v \cdot \nabla æ$ in (1.1) is regarded as a part of the principal part as in the proof of the local solvability of the time evolution problem and an iterative argument based on the method of characteristics is employed. See [14].

4.1 Notation

We first formulate the problem in a non-dimensional form and then introduce notation used for the functional setting in this section.

We transform the problem into the non-dimensional form under the following variable transformations: $x = \ell\tilde{x}$, $t = \frac{\ell}{V}\tilde{t}$, $v = VQ$, $\rho = \rho_*\tilde{\rho}$, $P = \rho_* P'(\rho_*)p$, where $V = \frac{\rho_* g \ell^2}{\mu}$.

In terms of these new non-dimensional variables, the system of equations (1.1) is transformed into the one which takes the following form after omitting tildes:

$$\partial_t \rho + \mathrm{div}\,(\rho v) = 0, \tag{4.1}$$

$$\rho(\partial_t v + v \cdot \nabla v) - {}^\circ\Delta v - ({}^\circ + {}^{\circ\prime})\nabla \mathrm{div}\, v + {}^2\nabla p(æ) = {}^\circ æ e_1, \tag{4.2}$$

where ν, ν' and γ are the non-dimensional parameters given by

$$\nu = \frac{\mu}{\rho_* \ell V}, \quad \nu' = \frac{\mu'}{\rho_* \ell V}, \quad \gamma = \frac{\sqrt{P'(\rho_*)}}{V}.$$

The assumption $P'(\rho_*) > 0$ is reduced to the form $p'(1) = 1$. Here we also used the relation $\frac{\ell g}{V^2} = \nu$.

The system (4.1)–(4.2) is then considered on the two-dimensional infinite layer:

$$\{x = (x_1, x_2)\,;\; x_1 \in \mathbb{R},\; 0 < x_2 < 1\}.$$

Under the above non-dimensionalization, the plane Poiseuille flow is transformed into $u_s = {}^\top(\rho_s, v_s)$, where

$$\rho_s = 1, \quad v_s = {}^\top(v_s^1(x_2), 0), \quad v_s^1(x_2) = \frac{1}{2}x_2(1 - x_2).$$

Let $u(t) = {}^\top(\phi(t), w(t)) = {}^\top({}^2(\text{æ}(t) - \text{æ}_s), v(t) - v_s)$ be the perturbation. Noting that $-\Delta v_s = e_1$, we have the system of equations for the perturbation u:

$$\partial_t\phi + v_s^1\partial_{x_1}\phi + \gamma^2 \operatorname{div} w = f^0(u), \tag{4.3}$$

$$\partial_t w - {}^\circ\Delta w - \text{Q}\nabla \operatorname{div} w + \nabla\text{Œ} - \frac{\circ}{2}\text{Œ}e_1 + v_s^1 @_{x_1} w + (@_{x_2}v_s^1)w^2 e_1 = f(u). \tag{4.4}$$

Here $\tilde{\nu} = \nu + \nu'$; and $f^0(u)$ and $f(u) = {}^\top(f^1, f^2)$ are the nonlinear terms:

$$f^0(u) = -\operatorname{div}(\phi w),$$

$$f(u) = -w\cdot\nabla w - \frac{\text{Œ}}{{}^2 + \text{Œ}}\left({}^\circ\Delta w + \frac{\circ}{2}\text{Œ}e_1 + \text{Q}\nabla\operatorname{div} w\right) + P^{(1)}(\text{Œ})\text{Œ}\nabla\text{Œ}$$

with

$$P^{(1)}(\phi) = \frac{1}{\gamma^2 + \phi}\left(1 - \int_0^1 p''(1 + \theta\gamma^{-2}\phi)\,d\theta\right).$$

The boundary conditions on $\{x_2 = 0, 1\}$ is the non-slip one, and a periodic boundary condition is imposed in x_1 direction:

$$w|_{x_2=0,1} = 0, \quad \text{Œ},\ w : \frac{2\text{ß}}{}\text{-periodic in } x_1, \tag{4.5}$$

where α is a given positive number. We note that the Reynolds number Re and the Mach number Ma are given by

$$\mathrm{Re} = \frac{1}{16\nu}, \quad \mathrm{Ma} = \frac{1}{8\gamma}.$$

We next introduce notation used in this section. Since we consider the system of equations (4.3)–(4.4) under periodic boundary condition in x_1, we introduce the basic period cell $\Omega_\alpha = \mathbb{T}_\alpha \times (0, 1)$, where $\mathbb{T}_\alpha = \mathbb{R}/\frac{2\pi}{\alpha}\mathbb{Z}$ and $\alpha > 0$ is a given constant.

We denote by $L^2(\Omega_\alpha)$ the usual L^2 space on Ω_α with norm $\|\cdot\|_2$, and likewise, by $H^k(\Omega_\alpha)$ the k th order L^2 Sobolev space on Ω_α with norm $\|\cdot\|_{H^k}$. We also denote by $C_0^\infty(\Omega_\alpha)$ the space of functions in $C^\infty(\Omega_\alpha)$ which vanish near $x_2 = 0, 1$. We define $H_0^1(\Omega_\alpha)$ by the $H^1(\Omega_\alpha)$-closure of $C_0^\infty(\Omega_\alpha)$.

The inner product of $f_j \in L^2(\Omega_\alpha)$ $(j = 1, 2)$ is denoted by

$$(f_1, f_2) = \int_{\Omega_\alpha} f_1(x)\overline{f_2(x)}\,dx.$$

Here $\overline{z}$ denotes the complex conjugate of z.

We define the average $\langle\phi\rangle$ of ϕ over Ω_α by

$$\langle\phi\rangle = \frac{1}{|\Omega_\alpha|}\int_{\Omega_\alpha}\phi(x)\,dx.$$

We also define $L^2_*(\Omega_\alpha)$ by

$$L^2_*(\Omega_\alpha) = \{\phi \in L^2(\Omega_\alpha);\ \langle\phi\rangle = 0\}.$$

Furthermore, we set

$$H^k_*(\Omega_\alpha) = H^k(\Omega_\alpha) \cap L^2_*(\Omega_\alpha).$$

The inner product of $u_j = {}^\top(\phi_j, w_j) \in L^2(\Omega)$ $(j = 1, 2)$ is defined by

$$\langle u_1, u_2\rangle = \frac{1}{\gamma^2}\int_{\Omega_\alpha}\phi_1(x)\overline{\phi_2(x)}\,dx + \int_{\Omega_\alpha} w_1(x)\cdot\overline{w_2(x)}\,dx.$$

In the following we omit Ω_α in $L^2(\Omega_\alpha)$, $H^k(\Omega_\alpha)$, …, and etc., and simply write them as L^2, H^k, …, and etc.

4.2 Instability of Plane Poiseuille Flow

In this section we state the instability result on the plane Poiseuille flow obtained in [15].

We first introduce the linearized operator L. We define the operator L on $L^2_* \times (L^2)^2$ by

$$D(L) = \left\{u = {}^\top(\phi, w) \in L^2_* \times (L^2)^2;\ w \in (H^1_0)^2,\ Lu \in L^2_* \times (L^2)^2\right\},$$

$$L = \begin{pmatrix} v_s^1\partial_{x_1} & \gamma^2\mathrm{div} \\ \nabla & -\nu\Delta - \tilde{\nu}\nabla\mathrm{div}\end{pmatrix} + \begin{pmatrix} 0 & 0 \\ -\frac{\nu}{\gamma^2}e_1 & v_s^1\partial_{x_1} + (\partial_{x_2}v_s^1)e_1{}^\top e_2\end{pmatrix}.$$

The argument in [10] applies to see that $-L$ generates a C_0-semigroup in $L^2_* \times (L^2)^2$.

The following result gives an instability criterion for the plane Poiseuille flow in terms of the Reynolds and Mach numbers.

Theorem 4.1 ([15]) *There exist positive constants r_0 and η_0 such that if $\alpha \le r_0$, then*

$$\sigma(-L) \cap \left\{\lambda \in \mathbb{C};\ |\lambda| \le \eta_0\right\} = \{\lambda_{\alpha k};\ |k| = 1, \dots, n_0\}$$

for some $n_0 \in \mathbb{N}$, where $\lambda_{\alpha k}$ are simple eigenvalues of $-L$ that satisfy

$$\lambda_{\alpha k} = -\frac{i}{6}(\alpha k) + \kappa_0(\alpha k)^2 + O(|\alpha k|^3)$$

as $\alpha k \to 0$*. Here* κ_0 *is the number given by*

$$\kappa_0 = \frac{1}{12\nu}\left[\left(\frac{1}{280} - \gamma^2\right) - \frac{\nu}{30\gamma^2}\left(3\nu + \nu'\right)\right].$$

As a consequence, if $\gamma^2 < \frac{1}{280}$ *and* $\nu(3\nu + \nu') < 30\gamma^2\left(\frac{1}{280} - \gamma^2\right)$*, then* $\kappa_0 > 0$ *and the plane Poiseuille flow* $u_s = {}^\top(\phi_s, v_s)$ *is linearly unstable.*

Remark 4.2 In terms of the Reynolds and Mach numbers, the instability condition given in Theorem 4.1 is restated as

$$\mathrm{Ma} > \sqrt{\frac{35}{8}} \sim 2.09, \quad \frac{1}{35} - \frac{1}{8\mathrm{Ma}^2} > \frac{\mathrm{Ma}^2}{15\mathrm{Re}}\left(\frac{3}{\mathrm{Re}} + \frac{1}{\mathrm{Re}'}\right), \tag{4.6}$$

where $\mathrm{Re}' = \frac{1}{16\nu'}$. Therefore, Reynolds and Mach numbers are not small when (4.6) is satisfied. For example, if $\mathrm{Ma} = 2.5$, $\mathrm{Re} = \frac{173}{16} \sim 10.81$ and $\frac{1}{\mathrm{Re}'} = -\frac{2}{3\mathrm{Re}}$ (i.e., $\nu' = -\frac{2\nu}{3}$), then instability condition (4.6) is satisfied. In the case of the incompressible flows, Orszag [26] numerically obtained a critical value $\mathrm{Re}_c \sim 5772$ such that if $\mathrm{Re} < \mathrm{Re}_c$, then the plane Poiseuille flow is linearly stable, while if $\mathrm{Re} > \mathrm{Re}_c$, then the plane Poiseuille flow is linearly unstable. We thus see that in the case of the compressible flows, the plane Poiseuille flow becomes linearly unstable for much smaller values of Reynolds numbers.

The proof of Theorem 4.1 is given by an analytic perturbation method. See [15] for details.

Remark 4.3 The eigenspace for $\lambda_{\alpha k}$ is spanned by a function of the form $u(x_2)e^{i\alpha k x_1}$ with an eigenfunction $u(x_2)$ for the eigenvalue $\lambda_{\alpha k}$ of $-\hat{L}_{\alpha k}$, where $\hat{L}_{\alpha k}$ is the operator $\hat{L}_\xi$ given in (3.2) with $\xi = \alpha k$. See [15, Sects. 4–6].

4.3 Bifurcation of Wave Trains

We have seen that the plane Poiseuille flow becomes unstable beyond a certain value of ν if $\gamma^2 < 1/280$. We shall see that after the instability occurs, a wave train bifurcates from the plane Poiseuille flow.

We fix γ in such a way that $\frac{1}{280} - \gamma^2 > 0$; and we regard ν as a bifurcation parameter. We denote the eigenvalue $\lambda_{\alpha k}$ by $\lambda_{\alpha k}(\nu)$:

$$\lambda_{\alpha k} = \lambda_{\alpha k}(\nu),$$

and the linearized operator L by L_ν:

$$L = L_\nu.$$

Let $\tilde{\nu}_0 > 0$ be taken in such a way that $\kappa_0 = 0$, where κ_0 is the coefficient of $(\alpha k)^2$ of $\lambda_{\alpha k}(\nu)$ described in Theorem 4.1. A perturbation argument then applies to see that, for each $0 < \alpha \ll 1$, there exists $\nu_0 > 0$ such that $\mathrm{Re}\,\lambda_{\pm\alpha}(\nu_0) = 0$, $\mathrm{Re}\,\lambda_{\pm\alpha}(\nu) < 0$ iff $\nu > \nu_0$ and $\mathrm{Re}\,\lambda_{\pm\alpha}(\nu) > 0$ iff $\nu < \nu_0$; if $\alpha \ll 1$, then $\lambda_{\pm\alpha}(\nu)$ cross the imaginary axis from left to right at $\nu = \nu_0$ when ν is decreased. See [16, Sect. 6].

We make the following assumption:

$$\sigma(-L_{\nu_0}) \cap \{\lambda; \mathrm{Re}\,\lambda = 0\} = \{\lambda_\alpha(\nu_0), \lambda_{-\alpha}(\nu_0)\}. \tag{4.7}$$

The bifurcation of wave trains is stated as follows.

Theorem 4.4 ([16]) *Assume that* Assumption *4.7 holds true. Then there is a solution branch* $\{\nu, u\} = \{\nu_\varepsilon, u_\varepsilon\}$ $(|\varepsilon| \ll 1)$ *such that*

$$\begin{aligned}
&\nu_\varepsilon = \nu_0 + O(\varepsilon),\\
&u_\varepsilon = u_\varepsilon(x_1 - c_\varepsilon t, x_2), \quad u_\varepsilon(x_1 + \tfrac{2\pi}{\alpha}, x_2) = u_\varepsilon(x_1, x_2),\\
&u_\varepsilon(x_1, x_2) = \varepsilon \begin{pmatrix} 1 \\ \frac{1}{2\gamma^2}(-x_2^2 + x_2) \\ 0 \end{pmatrix} \frac{\sqrt{2}}{2} \cos \alpha x_1 (1 + O(\alpha)) + O(\varepsilon^2),\\
&c_\varepsilon = \tfrac{1}{6} + O(\varepsilon).
\end{aligned}$$

To prove Theorem 4.4, we employ the Lyapunov-Schmidt reduction. We decompose the problem into the finite dimensional part and its complementary (infinite dimensional) part. In a standard bifurcation theory, the nonlinearity is regarded as a perturbation of the linearized part. This does not work well for the problem under consideration, since the term $w \cdot \nabla Œ$ on the right-hand side of (4.3) causes derivative loss in a standard setting. We thus regard this term as a part of the principal part in the equation of the infinite dimensional part, as in the proof of the local solvability of the time quasilinear evolution problem. This is the main difference to the case of the incompressible problem, where a standard bifurcation theory is applicable. See [16] for details. See also [14] where an outline of the proof of Theorem 4.4 is given.

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Uniform Regularity for a Compressible Gross-Pitaevskii-Navier-Stokes System

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Abstract Uniform regularity estimates are proved for a compressible Gross-Pitaevskii-Navier-Stokes system in $\mathbb{T}^n$ with $n \geq 3$.

Keywords Gross-Pitaevskii · Navier-Stokes · Euler · Uniform regularity

Mathematics Subject Classification 35Q30 · 35Q55 · 35B40 · 76D03

1 Introduction

In this paper we consider the following compressible Gross-Pitaevskii-Navier-Stokes system in superfluidity of Bose-Einstein condensates [1]:

$$\partial_t \rho + \mathrm{div}\,(\rho u) = 0, \tag{1.1}$$

$$\rho \partial_t u + \rho u \cdot \nabla u + \nabla p - \mu \Delta u - (\lambda + \mu)\nabla \mathrm{div}\, u = |\psi|^2 \nabla h, \tag{1.2}$$

$$\partial_t \psi + u \cdot \nabla \psi = \frac{i}{2}\epsilon \Delta \psi + i(1 - |\psi|^2)\psi, \tag{1.3}$$

$$(\rho, u, \psi)(\cdot, 0) = (\rho_0, u_0, \psi_0)(\cdot) \ \text{ in } \ \mathbb{T}^n \ (n \geq 3). \tag{1.4}$$

Here ψ is a complex-valued function, $|\psi|^2 := \psi\overline{\psi}$ is the mass density, ρ denotes the density, u is the velocity, $i := \sqrt{-1}$, $p := a\rho^\gamma$ is the pressure with the constants $a > 0$ and $\gamma > 1$, $h := h(x)$ is a given real potential with sufficient smoothness, λ

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and μ are two viscosity constants satisfying

$$\mu > 0 \text{ and } \lambda + \frac{2}{n}\mu \geq 0,$$

and $\epsilon > 0$ is a constant.

When $u = 0$, (1.3) is the well-known Gross-Pitaevskii system. Lin and Zhang [2] (see also [3]) considered the semiclassical limit. When u is not identically zero, (1.3) takes the form of a general quasilinear Schrödinger equation, which was studied in [4–7].

When $h = 0$, (1.1) and (1.2) reduce to the compressible Navier-Stokes equations. Zajaczkowski [8] studied the well-posedness of strong solutions.

By the "artificial viscosity method" in [4, Chap. 10] and the method in [8], it is straightforward to show the local well-posedness of smooth solutions to the problem, and therefore we omit the details here. The aim of this paper is to show regularity estimates which are uniform in (ϵ, λ, μ). We will prove

Theorem 1.1 *Let $0 < \epsilon, \mu < 1$ and let $0 < \lambda + \mu < 1$. Let $s > 1 + \frac{n}{2}$ and let $\rho_0, u_0, \psi_0 \in H^s(\mathbb{T}^n)$ satisfy $0 < \frac{1}{C_0} \leq \rho_0 \leq C_0$. Let (ρ, u, ψ) be the unique local smooth solutions to the problem (1.1)–(1.4) on the time interval $[0, T]$. Then the estimate*

$$\|(\rho, u, \psi)(\cdot, t)\|_{H^s} \leq C \text{ in } [0, T_0] \tag{1.5}$$

holds for some positive constants C and T_0 $(\leq T)$ independent of ϵ, λ and μ.

Remark 1.1 Here $T > 0$ is the local existence time of solution. We can prove a similar result when $\Omega := \mathbb{R}^n$.

To prove Theorem 1.1, we will rewrite (1.1) as follows:

$$\frac{1}{\gamma p}\partial_t p + \frac{1}{\gamma p} u \cdot \nabla p + \operatorname{div} u = 0. \tag{1.6}$$

We define

$$M(t) := 1 + \sup_{0 \leq t' \leq t} \Big\{ \|(\rho, u, \psi, p)(\cdot, t')\|_{H^s} + \|\partial_t u(\cdot, t')\|_{L^2} + \Big\|\frac{1}{\rho}(\cdot, t')\Big\|_{L^\infty} + \Big\|\frac{1}{p}(\cdot, t')\Big\|_{L^\infty} \Big\}. \tag{1.7}$$

We can prove

Theorem 1.2 *There exist nondecreasing continuous functions $C_0(\cdot)$ and $C(\cdot)$ and $T_0 \in (0, 1]$ such that for any $t \in [0, T_0)$,*

$$M(t) \leq C_0(M(0))\exp(tC(M(t))). \tag{1.8}$$

It follows from (1.8) that [9–11]:

$$\sup_{t\in[0,T_0]} M(t) < +\infty. \tag{1.9}$$

In the following proofs, we will use the bilinear commutator and product estimates due to Kato-Ponce [12]:

$$\|\Lambda^s(fg) - f\Lambda^s g\|_{L^p} \leq C(\|\nabla f\|_{L^{p_1}}\|\Lambda^{s-1}g\|_{L^{q_1}} + \|g\|_{L^{p_2}}\|\Lambda^s f\|_{L^{q_2}}), \tag{1.10}$$

$$\|\Lambda^s(fg)\|_{L^p} \leq C(\|f\|_{L^{p_1}}\|\Lambda^s g\|_{L^{q_1}} + \|\Lambda^s f\|_{L^{p_2}}\|g\|_{L^{q_2}}) \tag{1.11}$$

with $s > 0$, $\Lambda := (-\Delta)^{\frac{1}{2}}$ and $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{q_1} = \frac{1}{p_2} + \frac{1}{q_2}$.

We only need to show Theorem 1.2.

2 Proof of Theorem 1.2

First, testing (1.1) by ρ^{q-1}, we see that

$$\frac{1}{q}\frac{\mathrm{d}}{\mathrm{d}t}\int \rho^q \mathrm{d}x = \left(-1 + \frac{1}{q}\right)\int \rho^q \operatorname{div} u \mathrm{d}x \leq \left(1 - \frac{1}{q}\right)\|\operatorname{div} u\|_{L^\infty}\int \rho^q \mathrm{d}x,$$

and thus

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\rho\|_{L^q}^q \leq (q-1)\|\operatorname{div} u\|_{L^\infty}\|\rho\|_{L^q}^q,$$

which gives

$$\|\rho\|_{L^q} \leq \|\rho_0\|_{L^q}\exp\left(\left(1 - \frac{1}{q}\right)\int_0^t \|\operatorname{div} u\|_{L^\infty}\mathrm{d}\tau\right). \tag{2.1}$$

Taking $q \to +\infty$, we get

$$\|\rho\|_{L^\infty} \leq \|\rho_0\|_{L^\infty}\exp(tC(M(t))). \tag{2.2}$$

It follows from (1.1) that

$$\partial_t\left(\frac{1}{\rho}\right) + u\cdot\nabla\left(\frac{1}{\rho}\right) - \frac{1}{\rho}\operatorname{div} u = 0. \tag{2.3}$$

Testing (2.3) by $\left(\frac{1}{\rho}\right)^{q-1}$, we find that

$$\frac{1}{q}\frac{\mathrm{d}}{\mathrm{d}t}\int\left(\frac{1}{\rho}\right)^q\mathrm{d}x=\left(1+\frac{1}{q}\right)\int\left(\frac{1}{\rho}\right)^q\operatorname{div}u\mathrm{d}x\le\left(1+\frac{1}{q}\right)\left\|\frac{1}{\rho}\right\|_{L^q}^q\|\operatorname{div}u\|_{L^\infty},$$

and therefore

$$\frac{\mathrm{d}}{\mathrm{d}t}\left\|\frac{1}{\rho}\right\|_{L^q}^q\le(q+1)\left\|\frac{1}{\rho}\right\|_{L^q}^q\|\operatorname{div}u\|_{L^\infty},$$

which gives

$$\left\|\frac{1}{\rho}\right\|_{L^q}\le\left\|\frac{1}{\rho_0}\right\|_{L^q}\exp\left(\left(1+\frac{1}{q}\right)\int_0^t\|\operatorname{div}u\|_{L^\infty}\mathrm{d}\tau\right)$$

and we have

$$\left\|\frac{1}{\rho}\right\|_{L^\infty}\le\left\|\frac{1}{\rho_0}\right\|_{L^\infty}\exp(tC(M(t)))\tag{2.4}$$

by sending $q\to+\infty$.

(2.2) and (2.4) give

$$\|p\|_{L^\infty}+\left\|\frac{1}{p}\right\|_{L^\infty}\le C_0(M(0))\exp(tC(M(t))).\tag{2.5}$$

It is easy to verify that

$$\frac{\mathrm{d}}{\mathrm{d}t}\int|u|^2\mathrm{d}x=2\int u\partial_t u\mathrm{d}x\le2\|u\|_{L^2}\|\partial_t u\|_{L^2}\le C(M(t)),$$

which implies

$$\|u\|_{L^2}\le C_0(M(0))\exp(tC(M(t))).\tag{2.6}$$

Testing (1.3) by $\overline{\psi}$ and taking the real parts, we derive

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int|\psi|^2\mathrm{d}x&=-\frac{1}{2}\int|\psi|^2\operatorname{div}u\mathrm{d}x\\&\le\frac{1}{2}\|\psi\|_{L^2}^2\|\operatorname{div}u\|_{L^\infty}\le C(M(t)),\end{aligned}$$

which gives

$$\|\psi\|_{L^2}\le C_0(M(0))\exp(tC(M(t))).\tag{2.7}$$

Applying Λ^s to (1.3), testing by $\Lambda^s\overline{\psi}$, taking the real parts, and using (1.10) and (1.11), we compute

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int|\Lambda^s\psi|^2\mathrm{d}x &= -\mathrm{Re}\int(\Lambda^s(u\cdot\nabla\psi)-u\cdot\nabla\Lambda^s\psi)\Lambda^s\overline{\psi}\mathrm{d}x-\frac{1}{2}\int|\Lambda^s\psi|^2\mathrm{div}\,u\mathrm{d}x\\ &\quad -\mathrm{Re}i\int\Lambda^s(\psi^2\overline{\psi})\Lambda^s\overline{\psi}\mathrm{d}x\\ &\le C\|\nabla u\|_{L^\infty}\|\Lambda^s\psi\|_{L^2}^2+C\|\nabla\psi\|_{L^\infty}\|\Lambda^s u\|_{L^2}\|\Lambda^s\psi\|_{L^2}+\frac{1}{2}\|\mathrm{div}\,u\|_{L^\infty}\|\Lambda^s\psi\|_{L^2}^2\\ &\quad +C\|\psi\|_{L^\infty}^2\|\Lambda^s\psi\|_{L^2}^2\le C(M(t)),\end{aligned}$$

which leads to

$$\|\Lambda^s\psi\|_{L^2}\le C_0(M(0))\exp(tC(M(t))). \tag{2.8}$$

Applying Λ^s to (1.6), testing by $\Lambda^s p$, and using (1.6), (1.10) and (1.11), we compute

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int\frac{1}{\gamma p}(\Lambda^s p)^2\mathrm{d}x+\int\Lambda^s p\Lambda^s\mathrm{div}\,u\mathrm{d}x\\ &=\frac{1}{2}\int(\Lambda^s p)^2\left[\mathrm{div}\left(\frac{u}{\gamma p}\right)-\frac{1}{\gamma p^2}\partial_t p\right]\mathrm{d}x-\int\left(\Lambda^s\left(\frac{1}{\gamma p}\partial_t p\right)-\frac{1}{\gamma p}\Lambda^s\partial_t p\right)\Lambda^s p\mathrm{d}x\\ &\quad-\int\left(\Lambda^s\left(\frac{u}{\gamma p}\cdot\nabla p\right)-\frac{u}{\gamma p}\cdot\nabla\Lambda^s p\right)\Lambda^s p\mathrm{d}x\\ &\le C\|\Lambda^s p\|_{L^2}^2\left\|\mathrm{div}\left(\frac{u}{\gamma p}\right)-\frac{1}{\gamma p^2}\partial_t p\right\|_{L^\infty}\\ &\quad+C\|\partial_t p\|_{L^\infty}\left\|\Lambda^s\left(\frac{1}{\gamma p}\right)\right\|_{L^2}\|\Lambda^s p\|_{L^2}+C\left\|\nabla\frac{1}{\gamma p}\right\|_{L^\infty}\|\Lambda^{s-1}\partial_t p\|_{L^2}\|\Lambda^s p\|_{L^2}\\ &\quad+C\|\nabla p\|_{L^\infty}\left\|\Lambda^s\left(\frac{u}{\gamma p}\right)\right\|_{L^2}\|\Lambda^s p\|_{L^2}+C\left\|\nabla\frac{u}{\gamma p}\right\|_{L^\infty}\|\Lambda^s p\|_{L^2}^2\\ &\le C(M(t))+C(M(t))\|\partial_t p\|_{L^\infty}+C(M(t))\|\Lambda^{s-1}\partial_t p\|_{L^2}\\ &\le C(M(t))+C(M(t))\|u\cdot\nabla p+\gamma p\mathrm{div}\,u\|_{L^\infty}+C(M(t))\|\Lambda^{s-1}(u\cdot\nabla p+\gamma p\mathrm{div}\,u)\|_{L^2}\\ &\le C(M(t)).\end{aligned} \tag{2.9}$$

Here we have used

$$\left\|\Lambda^s\frac{1}{p}\right\|_{L^2}\le C(M(t))\|\Lambda^s p\|_{L^2}\le C(M(t)), \tag{2.10}$$

which follows from the estimate [13, Proposition 2.1, p. 43]: Assume $g(u)$ is a smooth vector-valued function and $u\in L^\infty\cap H^s$. Then for $s\ge 1$,

$$\|\Lambda^s g(u)\|_{L^2}\le C\left\|\frac{\partial g}{\partial u}\right\|_{H^{s-1}}\|u\|_{L^\infty}^{s-1}\|\Lambda^s u\|_{L^2}. \tag{2.11}$$

Applying Λ^{s-1} to (1.2), testing by $\Lambda^{s-1}\partial_t u$, and using (1.10) and (1.11), we obtain

$$
\begin{aligned}
&\frac{\mu}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int|\Lambda^s u|^2\mathrm{d}x+\frac{\lambda+\mu}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int(\Lambda^{s-1}\mathrm{div}\,u)^2\mathrm{d}x+\int\rho|\Lambda^{s-1}\partial_t u|^2\mathrm{d}x\\
&=\int\Lambda^{s-1}(\psi\overline{\psi}\nabla h)\Lambda^{s-1}\partial_t u\mathrm{d}x-\int\Lambda^{s-1}\nabla p\cdot\Lambda^{s-1}\partial_t u\mathrm{d}x\\
&\quad-\int\Lambda^{s-1}(\rho u\cdot\nabla u)\cdot\Lambda^{s-1}\partial_t u\mathrm{d}x-\int[\Lambda^{s-1}(\rho\partial_t u)-\rho\Lambda^{s-1}\partial_t u]\Lambda^{s-1}\partial_t u\mathrm{d}x\\
&\le C\|\Lambda^{s-1}(\psi\overline{\psi}\nabla h)\|_{L^2}\|\Lambda^{s-1}\partial_t u\|_{L^2}+C\|\Lambda^s p\|_{L^2}\|\Lambda^{s-1}\partial_t u\|_{L^2}\\
&\quad+C\|\rho\|_{H^{s-1}}\|u\|_{H^s}^2\|\Lambda^{s-1}\partial_t u\|_{L^2}+C(\|\nabla\rho\|_{L^\infty}\|\Lambda^{s-2}\partial_t u\|_{L^2}+\|\partial_t u\|_{L^\infty}\|\Lambda^{s-1}\rho\|_{L^2})\|\Lambda^{s-1}\partial_t u\|_{L^2}\\
&\le C(M(t))\|\Lambda^{s-1}\partial_t u\|_{L^2}+C(M(t))(\|\Lambda^{s-2}\partial_t u\|_{L^2}+\|\partial_t u\|_{L^\infty})\|\Lambda^{s-1}\partial_t u\|_{L^2}\\
&\le C(M(t))\|\Lambda^{s-1}\partial_t u\|_{L^2}\\
&\quad+C(M(t))(\|\partial_t u\|_{L^2}^{\frac{1}{s-1}}\|\Lambda^{s-1}\partial_t u\|_{L^2}^{\frac{s-2}{s-1}}+\|\partial_t u\|_{L^2}+\|\partial_t u\|_{L^2}^{\frac{s-1-\frac{n}{2}}{s-1}}\|\Lambda^{s-1}\partial_t u\|_{L^2}^{\frac{n}{2(s-1)}})\|\Lambda^{s-1}\partial_t u\|_{L^2}\\
&\qquad\left(\text{with } s-1>\frac{n}{2}\right)\\
&\le C(M(t))\|\Lambda^{s-1}\partial_t u\|_{L^2}+C(M(t))(\|\Lambda^{s-1}\partial_t u\|_{L^2}^{\frac{s-2}{s-1}}+\|\Lambda^{s-1}\partial_t u\|_{L^2}^{\frac{n}{2(s-1)}})\|\Lambda^{s-1}\partial_t u\|_{L^2}\\
&\le\frac{1}{2}\int\rho|\Lambda^{s-1}\partial_t u|^2\mathrm{d}x+C(M(t)),
\end{aligned}
$$

which gives

$$
\int_0^t\int|\Lambda^{s-1}\partial_t u|^2\mathrm{d}x\mathrm{d}\tau\le C_0(M(0))\exp(tC(M(t))). \tag{2.12}
$$

Applying Λ^s to (1.2), testing by $\Lambda^s u$, and using (1.1), (1.10) and (1.11), we have

$$
\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int\rho|\Lambda^s u|^2\mathrm{d}x+\mu\int|\Lambda^{s+1}u|^2\mathrm{d}x+(\lambda+\mu)\int(\Lambda^s\mathrm{div}\,u)^2\mathrm{d}x+\int\Lambda^s\nabla p\cdot\Lambda^s u\mathrm{d}x\\
&=\int\Lambda^s(\psi\overline{\psi}\nabla h)\cdot\Lambda^s u\mathrm{d}x-\int(\Lambda^s(\rho\partial_t u)-\rho\Lambda^s\partial_t u)\Lambda^s u\mathrm{d}x\\
&\quad-\int(\Lambda^s(\rho u\cdot\nabla u)-\rho u\cdot\nabla\Lambda^s u)\Lambda^s u\mathrm{d}x\\
&\le C\|\Lambda^s(\psi\overline{\psi}\nabla h\|_{L^2}\|\Lambda^s u\|_{L^2}+C(\|\nabla\rho\|_{L^\infty}\|\Lambda^{s-1}\partial_t u\|_{L^2}+\|\partial_t u\|_{L^\infty}\|\Lambda^s\rho\|_{L^2})\|\Lambda^s u\|_{L^2}\\
&\quad+C(\|\nabla u\|_{L^\infty}\|\Lambda^s(\rho u)\|_{L^2}+\|\nabla(\rho u)\|_{L^\infty}\|\Lambda^s u\|_{L^2})\|\Lambda^s u\|_{L^2}\\
&\le C(M(t))+C(M(t))(\|\Lambda^{s-1}\partial_t u\|_{L^2}+\|\partial_t u\|_{L^\infty})\\
&\le C(M(t))+\|\Lambda^{s-1}\partial_t u\|_{L^2}^2.
\end{aligned} \tag{2.13}
$$

Summing up (2.9) and (2.13), we have

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int\left(\frac{1}{\gamma p}(\Lambda^s p)^2+\rho|\Lambda^s u|^2\right)\mathrm{d}x$$
$$+\mu\int|\Lambda^{s+1}u|^2\mathrm{d}x+(\lambda+\mu)\int(\Lambda^s\operatorname{div}u)^2\mathrm{d}x$$
$$+\int(\Lambda^s p\Lambda^s\operatorname{div}u+\Lambda^s\nabla p\cdot\Lambda^s u)\mathrm{d}x$$
$$\le C(M(t))+\|\Lambda^{s-1}\partial_t u\|_{L^2}^2. \tag{2.14}$$

Notice that the last term of the LHS in (2.14) is zero. Then using (2.12), we have

$$\|\Lambda^s(p,u)\|_{L^2}\le C_0(M(0))\exp(tC(M(t))). \tag{2.15}$$

On the other hand, it follows from (1.2) that

$$\|\partial_t u\|_{L^2}=\left\|\frac{1}{\rho}\left(|\psi|^2\nabla h+\mu\Delta u+(\lambda+\mu)\nabla\operatorname{div}u-\nabla p-\rho u\cdot\nabla u\right)\right\|_{L^2}$$
$$\le C_0(M(0))\exp(tC(M(t))). \tag{2.16}$$

Using the following estimate [13]:

$$\|\Lambda^s\rho\|_{L^2}\le C(1+\|p\|_{L^\infty})^\sigma\|f\|_{W^{\sigma,\infty}(I)}\|\Lambda^s p\|_{L^2} \tag{2.17}$$

with $\rho=f(p):=\left(\frac{p}{a}\right)^{\frac{1}{\gamma}}$ and

$$I\subset\left(\frac{1}{C_0(M(0))}\exp(-tC(M(t))),C_0(M(0))\exp(tC(M(t)))\right),$$

and σ is an integer satisfying $\sigma\ge s$, we have

$$\|\Lambda^s\rho\|_{L^2}\le C_0(M(0))\exp(tC(M(t))). \tag{2.18}$$

Combining (2.4), (2.5), (2.6), (2.7), (2.8), (2.15), (2.16), and (2.18), we conclude that (1.8) holds true.

This completes the proof. □

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Singular Limit Problem to the Keller-Segel System in Critical Spaces and Related Medical Problems—An Application of Maximal Regularity

Takayoshi Ogawa

Abstract We consider singular limit problems of the Cauchy problem for the Patlak-Keller-Segel equation and related problems appeared in the theory of medical and bio-chemical dynamics. It is shown that the solution to the Patlak-Keller-Segel equation in a scaling critical function class converges strongly to a solution of the drift-diffusion system of parabolic-elliptic equations as the relaxation time parameter $\tau \to \infty$. Analogous problem related to the Chaplain-Anderson model for cancer growth model is also presented as well as Arzhimer's model that involves the multi-component drift-diffusion system. For the proof, we use generalized maximal regularity for the heat equations and systematically apply embeddings between the interpolation spaces shown in [40, 41]. The argument requires generalized version of maximal regularity developed in [40, 61], for the Cauchy problem of the heat equation.

Keywords Keller-Segel equation · Drift-diffusion system · Singular limit problem · Maximal regularity · Critical space · Global well-posedness · Scaling invariance · Bounded mean oscillation

AMS Subject Classification Primary 35K45 · Secondary 35K58 · 35Q70 · 35Q81 · 35Q92 · 92C50

1 Introduction—The Singular Limit Problem

A mathematical model describing an interactive dynamics for behaviors of a chemical-biology reaction is called as chemotaxis. One of a simplest model of chemotaxis was derived by Patlak [62] and Keller-Segel [32] is named as the Patlak-Keller-Segel model that describes a spatial dynamics of the mucus mold and reaction with the chemical substance that attract themselves. The model is involved the density of

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mucus molds and the density of a chemical substance. More the mucus attracted then more the chemical substance is created by them and then, the total density increase until they can produce their sprout.

In fact, such kind of dynamics appears in various fields of mathematical science. For instance, the behavior of the interstellar material governed by the gravity shows dissipative movements. Once the gravity exists then the gravity object starts to gather each other. Such a model is a simplest case for the dynamics of gravity material in astronomy. An analogous model does surprisingly appear in the model of semi-conductor in the earlier work of the devise simulations (Mock [46]). Though this model is too simple to work with the latest devises since they are involved with the quantum mechanics approach that requires more delicate setting, the basic dissipative nature stems from their original form. One important remark for those models is that the electric forth works in an opposite direction so that the mathematical system for the semiconductor has a repulsive nature.

Those models are typically given by a balance between a dissipative nature and attractive driving force. Our main purpose of this survey is to make a mathematical connection between such a two similar models and we make a bridge between them by a method of mathematical analysis, namely singular limit problem.

Several mathematical problems arose from medical science that describes the tumor growth or Arzheimer's disease. Those models exhibit very similar nature and the model shows how the disease grows under the very similar condition. Applying the dimension analysis, the critical setting is beyond our realistic spacial dimension 3 and hence the uniform boundedness for the solution is the most important question on that problems. The uniform boundedness of the solution ensure that the disease can be control by a medical or natural treatment. We give two different but very similar models which is unstable in the critical spacial dimension 4 but not in 3.

1.1 Keller-Segel System in the Scaling Invariant Spaces

We consider the Cauchy problem of the Patlak-Keller-Segel system in n-dimensional Euclidian space $\mathbb{R}^n$:

$$\begin{cases} \partial_t u_\tau - \Delta u_\tau + \nabla \cdot (u_\tau \nabla \psi_\tau) = 0, & t > 0,\ x \in \mathbb{R}^n, \\ \dfrac{1}{\tau}\partial_t \psi_\tau - \Delta \psi_\tau = u_\tau, & t > 0,\ x \in \mathbb{R}^n, \\ \qquad u_\tau(0, x) = u_0(x), \quad \psi_\tau(0, x) = \psi_0(x), & t = 0,\ x \in \mathbb{R}^n, \end{cases} \tag{1.1}$$

where $u_\tau = u_\tau(t, x)$ and $\psi_\tau(t) = \psi_\tau(t, x)$ denotes the unknown density of mucus molds and the distribution of the chemical substance and (u_0, ψ_0) is given pair of the initial data. The constant $\lambda \geq 0$ is a parameter often chosen as $\lambda = 0$. The notation ∂_t describes the partial derivative with respect to the time variable and $\nabla = (\partial_{x_1}, \partial_{x_2}, \dots \partial_{x_n})$ stands for the gradient, $\Delta \equiv \sum_{k=1}^n \partial_{x_k}^2$ denotes the Laplacian for the spatial variables. The problem (1.1) originally introduced as the chemical-biological reaction by the chemotaxis, the chemical substance produced by mucus molds attracts other mucus molds. Such a reaction often observe other situation. The original model is considered in a bounded domain $\Omega \subset \mathbb{R}^n$ for $n = 1, 2, 3$.

$$
\begin{cases}
\partial_t u - \Delta u + \kappa \nabla \cdot (u \nabla \psi) = 0, & t > 0,\ x \in \Omega, \\
\partial_t \psi - \Delta \psi + \lambda \psi = u, & t > 0,\ x \in \Omega, \\
\dfrac{\partial u}{\partial \nu} = \dfrac{\partial \psi}{\partial \nu} = 0, & t > 0,\ x \in \partial\Omega, \\
u(0, x) = u_0(x), \quad \psi(0, x) = \psi_0(x), & t = 0,\ x \in \Omega,
\end{cases}
\tag{1.2}
$$

where ν denotes the outer normal vector at the boundary point and $\kappa \in \mathbb{R} \setminus \{0\}$ is a coupling constant. The system (1.2) is called the Patlak-Keller-Segel system introduced by Patlak [62], Keller-Segel [32] for a model of chemotaxis dynamics. Since the problem has a non-local property, it is interesting to consider the domain as the whole space $\Omega = \mathbb{R}^n$ and consider it as the Cauchy problem in $\mathbb{R}^n$. In such a case, the effect of the drift nonlinear term in the first equation is strengthened and the non-local property is visible. To see the scaling invariant property, we focus on the most unstable setting $\lambda = 0$.

When the dynamics of the chemical substance is relatively slow, the dynamics of ψ can be subordinate to the dynamics of mucus molds. Then introducing the *relaxation time* parameter $\tau^{-1} > 0$, the limiting process can be considered as $\tau \to \infty$. Then the set of the limiting functions

$$
\begin{cases}
\lim\limits_{\tau \to \infty} u_\tau(t, x) = u(t, x), \\
\lim\limits_{\tau \to \infty} \psi_\tau(t, x) = \psi(t, x)
\end{cases}
\tag{1.3}
$$

formally solves the initial value problem of the drift-diffusion system:

$$
\begin{cases}
\partial_t u - \Delta u + \nabla \cdot (u \nabla \psi) = 0, & t > 0,\ x \in \mathbb{R}^n, \\
\quad - \Delta \psi = u, & t > 0,\ x \in \mathbb{R}^n, \\
\quad u(0, x) = u_0(x), & t = 0,\ x \in \mathbb{R}^n.
\end{cases}
\tag{1.4}
$$

The system (1.4) is called the drift-diffusion equation and originally appeared in the theory of semiconductor (Mock [46]) and the formation of stars in astronomy (Chandrasekhar [11]). Such a simplified Patlak-Keller-Segel system was introduced by Jäger-Luckhaus [29] (see also [4–7, 13–15, 27–30, 36–39, 47–52, 64–66, 69, 73] and one-dimensional case [9, 69]). For the role of the constant $\lambda > 0$ see for instance Bedrossian [2].

The singular limit problem has been considered by Biler-Brandolese [5], Raczyński [63] and Lemarié-Rieusset [44]. They considered the case of the vanishing initial condition $\psi_0 \equiv 0$ and showed the strong convergence (1.3) of the small global solutions in an elegant way in the scaling invariant spaces such as pseudo-measure or $L^1(\mathbb{R}^2)$ for small initial data. Lemarié-Rieusset [44] showed the singular limit in the scaling invariant Morrey space. The existence and well-posedness of both of two Cauchy problems (1.1) and (1.4) can be seen in the scaling critical function spaces. Both of the systems (1.1) and (1.4) are invariant under the following scaling transform: For $\mu > 0$,

$$\begin{cases} u_\mu(t,x) = \mu^2 u(\mu^2 t, \mu x), \\ \psi_\mu(t,x) = \psi(\mu^2 t, \mu x), \end{cases}$$

and the invariant class of the sense of Fujita-Kato [24] is now identified in the Bochner-Lebesgue class as

$$\begin{cases} u \in L^\theta\big(\mathbb{R}_+; L^q(\mathbb{R}^n)\big), & \dfrac{2}{\theta} + \dfrac{n}{q} = 2, \quad \dfrac{n}{2} \le q, \quad 2 \le \theta, \\ \psi \in L^\sigma\big(\mathbb{R}_+; L^r(\mathbb{R}^n)\big), & \dfrac{2}{\sigma} + \dfrac{n}{r} = 0, \quad \infty \le q, \quad \infty \le \sigma, \end{cases}$$

where $L^\theta\big(I; L^q(\mathbb{R}^n)\big)$ stands for the Bochner-Lebesgue space equipped with the norm:

$$\|f\|_{L^\theta(I;L^q)} \equiv \left(\int_I \|f(t,\cdot)\|_q^\theta dt\right)^{1/\theta} < \infty, \quad I = (0,T).$$

It is natural to choose $\theta = \sigma = \infty$ in order to find a solution in the same Lebesgue class as the initial data for all the time, then we find that $p = \frac{n}{2}$, $r = \infty$. On the other hand, in the view of the expecting limiting problem (1.4), it is difficult to choose as $r = \infty$ as the regularity theory of elliptic partial differential equation for the critical external force $u \in L^{\frac{n}{2}}(\mathbb{R}^n)$, unless the solution of the first component has better regularity. Kurokiba-Ogawa [40, 41] considered the same problem in the scaling critical function space and obtained the singular limit indeed converges to the limiting problem in the strong sense, in both small global solution and large local solution, in the unified way by applying the Fujita-Kato principle [24].

We should emphasize a remarkable property of the solution to both the system (1.1) and (1.4) as $t \to \infty$. First, the solution is non-negative if the initial data is non-negative which follows from weak maximum principle. Secondly under the positivity setting, the solution preserves the total mass $m = \|u(t)\|_1$ and has the free energy bound: For simplicity, we set $\tau = 1$ and omit the suffix τ.

$$H[u(t)] + \int_0^t \int_{\mathbb{R}^n} u(s)\big|\nabla\big(\log u(s) - \psi(s)\big)\big|^2 dxds \le H[u_0], \quad t > 0,$$

where $H[u]$ denotes *the Helmholtz free energy* consisting of the entropy of the system and the inner energy as follows:

$$H[u(t)] = \begin{cases} \int_{\mathbb{R}^n} u(t)\log u(t)dx + \frac{1}{2}\|\psi(t)\|_{H^1}^2 - \frac{1}{2}\int_{\mathbb{R}^n} u(t)(-\Delta)^{-1}u(t)dx & \text{for (1.1)} \\ \int_{\mathbb{R}^n} u(t)\log u(t)dx - \frac{1}{2}\int_{\mathbb{R}^n} u(t)(-\Delta)^{-1}u(t)dx & \text{for (1.4).} \end{cases}$$

Then using those quantities, one can show that a solution of (1.1) and (1.4) blows up in a finite time under suitable assumptions on the data (see [4, 6, 37, 47, 48, 64, 65]). In particular for $n = 2$, the threshold for the global existence and finite time blow up is clarified [2, 4, 47–49, 52, 71] namely for the solution with $\|u_0\|_1 \le 8\pi$ then the corresponding solution exists globally in time and if $\|u_0\|_1 > 8\pi$, then the finite moment solution blows up in a finite time. This fact can be derived by the virial identity: For the non-negative solution to (1.4), it holds that

$$\int_{\mathbb{R}^2} |x - x_0|^2 u(t,x)dx = \int_{\mathbb{R}^2} |x - x_0|^2 u_0(x)dx + 4\|u_0\|_1\left(1 - \frac{\|u_0\|_1}{8\pi}\right)t. \quad (1.5)$$

Besides the total mass $\|u(t)\|_1$ preserves and hence if the initial data is non-negative with $\|u_0\|_1 > 8\pi$ the positive solution can not exists globally in time since the right hand side of (1.5) reaches negative value within a finite time. The corresponding results were shown by Nagai-Ogawa [49], Mizoguchi [45] for (1.1) with $\tau = 1$. The higher dimensional cases are more unstable, namely the L^1-norm of the solution is not the scaling critical quantity and hence it does not control a solution for whole time. It is shown that if the initial critical quantity $\|u_0\|_{\frac{n}{2}}$ is small enough then the solution exists globally in time, while if the initial data is non-negative and large enough so that

$$H[u_0] < \frac{n}{b}\|u_0\|_1 \log\left(\frac{\|u_0\|_1^{1+\frac{b}{n}}}{C_n \int_{\mathbb{R}^n} |x - \bar{x}|^b u_0 dx}\right),$$

for some $b > 0$ and $C_n > 0$, then the solution is unstable and blows up in a finite time under the assumption either $b \ge 2$ or the data is radially symmetric (cf. Biler [4], Calvez et al. [10], Ogawa-Wakui [60]). The proof of such an instability relies on how to control the entropy part of the Helmholtz free energy $H[u]$. Namely the Shannon inequality;

$$-\int_{\mathbb{R}^n} f(x)\log f(x)dx \le \frac{n}{2}\|f\|_1 \log\left(\frac{2\pi e \int_{\mathbb{R}^n} \left|x - x_0\right|^2 f(x)dx}{\|f\|_1^{1+\frac{2}{n}}}\right) \quad (1.6)$$

and its generalization works well. Indeed, the inequality (1.6) plays as a role of the Fourier dual of the well-known logarithmic Sobolev inequality due to Stam [67] and Gross [25]:

$$\int_{\mathbb{R}^n} f(x)\log\big(f(x)\big)dx \le \frac{n}{2}\|f\|_1 \log\left(\frac{1}{2n\pi e\|f\|_1^{1-\frac{2}{n}}}\int_{\mathbb{R}^n}\big|\nabla \log\big(f(x)\big)\big|^2 f(x)dx\right). \tag{1.7}$$

It is remarkable that the inequalities (1.6) and (1.7) reproduce the Heisenberg uncertainty inequality intermediated by the entropy functional: For any $1 < p < n$,

$$\|f\|_1 \le \frac{1}{n}\left(\int_{\mathbb{R}^n} |x-\bar{x}|^{p'} f(x)dx\right)^{1/p'}\left(\int_{\mathbb{R}^n} |\nabla \log f(x)|^p f(x)dx\right)^{1/p},$$

See for the above generalization Ogawa-Seraku [56] and further inequality involving the logarithmic weight, Kubo-Ogawa-Suguro [35].

1.2 The Chaplain-Anderson Model and the Fujie-Senba Equation

The problem (1.1) is related to the tumor growth model considered by Chaplain-Anderson [12] which consists of multi-component nonlinear ordinary differential equations of various stage of the bio-chemical reactions. Then introducing the additional chemical stage Fujie-Ito-Yokota [21] proposed the following variant from the Chaplain-Anderson model as a tumor invasion. We introduce the following simplified version of their system: Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with smooth boundary $\partial\Omega$.

$$\begin{cases} \partial_t u_\tau - \Delta u_\tau + \nabla\cdot(u_\tau \nabla\psi_\tau) = 0, & t>0, x\in\Omega,\\ \partial_t f_{\tau,\alpha} = -\alpha\phi_\tau f_{\tau,\alpha} & t>0, x\in\Omega,\\ \frac{1}{\tau}\partial_t \phi_\tau - \Delta\phi_\tau = u_\tau, & t>0, x\in\Omega,\\ \frac{1}{\tau}\partial_t \psi_\tau - \Delta\psi_\tau = \alpha\phi_\tau f_{\alpha,\tau}, & t>0, x\in\Omega,\\ \quad u_\tau(0,x) = u_0(x), \quad f_\alpha(0,x) = f_0(x), & \\ \quad \phi_\tau(0,x) = \phi_0(x), \quad \psi_\tau(0,x) = \psi_0(x), & t=0, x\in\Omega \end{cases} \tag{1.8}$$

with suitable boundary conditions on $\partial\Omega$, where the unknown functions $u_\tau = u_\tau(t,x) : \mathbb{R}_+ \times \Omega \to \mathbb{R}$, $f_{\tau,\alpha} = f_{\tau,\alpha}(t,x) : \mathbb{R}_+ \times \Omega \to \mathbb{R}$, $\phi_\tau = \phi_\tau(t,x) : \mathbb{R}_+ \times \Omega \to \mathbb{R}$ and $\psi_\tau = \psi_\tau(t,x) : \mathbb{R}_+ \times \Omega \to \mathbb{R}$ denote the density of mucus mold, the density of chemical substance of the first stage and the one of chemical substance of the second stage, respectively. $\tau > 0$ and $\alpha > 0$ are constants. The system (1.8) was introduced by Fujie-Ito-Yokota [21] observing that the drift-diffusion process is taking into account on the original Chaplain-Anderson model [12] (cf. Fujie-Ito-Winkler-Yokota [22]). If the third component of the system (1.8) is a given function, then the solution of the second equation is given by the third component ϕ_τ as

$$\alpha f_{\alpha,\tau}(t,x) = \alpha f_0(x)\exp\left(-\alpha\int_0^t \phi_\tau(s,x)ds\right).$$

T. Senba considered the critical setting of the system (1.8) as the global behavior of solution in the whole space $\Omega = \mathbb{R}^n$ and consider the case when

$$\alpha f_0(x)\exp\left(-\alpha\int_0^t \phi_\tau(s,x)ds\right) \to \alpha_* \quad \text{(constant)}.$$

Then the system (1.8) is reduced into the following slightly simplified Cauchy problem:

$$\begin{cases} \partial_t u_\tau - \Delta u_\tau + \nabla\cdot(u_\tau \nabla\psi_\tau) = 0, & t>0, x\in\mathbb{R}^n,\\ \frac{1}{\tau}\partial_t\phi_\tau - \Delta\phi_\tau = u_\tau, & t>0, x\in\mathbb{R}^n,\\ \frac{1}{\tau}\partial_t\psi_\tau - \Delta\psi_\tau = \phi_\tau, & t>0, x\in\mathbb{R}^n,\\ \quad u_\tau(0,x) = u_0(x), & \\ \quad \phi_\tau(0,x) = \phi_0(x), \quad \psi_\tau(0,x) = \psi_0(x), & t=0, x\in\mathbb{R}^n, \end{cases} \tag{1.9}$$

where α_* is chosen as 1 for simplicity. The mathematical structure of the simplified Chaplain-Anderson equation is interesting since it has double staged potentials and it makes the system critical when the spatial dimension $n = 4$ comparing with the case of Patlak-Keller-Segel system (1.1). Passing to the limiting problem:

$$\begin{cases} u_\tau(t,x) \to u(t,x),\\ \phi_\tau(t,x) \to \phi(t,x),\\ \psi_\tau(t,x) \to \psi(t,x) \end{cases}$$

as $\tau \to \infty$, the limiting functions formally solve the following version of drift-diffusion system studied by Fujie-Senba [23]:

$$\begin{cases} \partial_t u - \Delta u + \nabla\cdot(u\nabla\psi) = 0, & t>0, x\in\mathbb{R}^n,\\ -\Delta\phi = u, & t>0, x\in\mathbb{R}^n,\\ -\Delta\psi = \phi, & t>0, x\in\mathbb{R}^n,\\ \ u(0,x) = u_0(x), & t=0, x\in\mathbb{R}^n. \end{cases} \tag{1.10}$$

Then the system (1.9) is observed that the solution blows up in a finite time for the four dimensional case in Fujie-Senba [23] analogously to the case of drift-diffusion system (1.4) under the assumption $\|u_0\|_1 > (8\pi)^2$.

The setting in the four space dimension is a natural extension from the case to the problem (1.1) and (1.4) in two space dimension. We consider such a large initial data case, the solution of the full-parabolic system of Chaplain-Anderson type equation converges to the solution to Fujie-Senba equation (1.10) of the drift-diffusion type in a critical function space.

In the medical science model, there is another problem closely related to the above problem (1.10). For a positive constant $\beta > 0$ we consider the Cauchy problem of the multi-component chemical attraction model:

$$\begin{cases} \partial_t u - \Delta u + \nabla \cdot (u \nabla(\psi - \beta\phi)) = 0, & t > 0, x \in \mathbb{R}^n, \\ \frac{1}{\tau}\partial_t \phi - \Delta\phi + \lambda_1 \phi = u, & t > 0, x \in \mathbb{R}^n, \\ \frac{1}{\tau}\partial_t \psi - \Delta\psi + \lambda_2 \psi = u, & t > 0, x \in \mathbb{R}^n, \\ \quad u(0,x) = u_0(x), & t = 0, x \in \mathbb{R}^n, \\ \quad \phi(0,x) = \phi_0(x), \quad \psi(0,x) = \psi_0(x), & t = 0, x \in \mathbb{R}^n. \end{cases} \tag{1.11}$$

The original system of the model was introduced for describing the aggregation of microglia in Alzheimer's disease and ψ and ϕ are the concentration of chemoattractant and chemo-repellent, respectively. By passing a limit $\tau \to \infty$, one can derive a similar problem of drift-diffusion system;

$$\begin{cases} \partial_t u - \Delta u + \nabla \cdot (u \nabla(\psi - \beta\phi)) = 0, & t > 0, x \in \mathbb{R}^n, \\ -\Delta\phi + \lambda_1 \phi = u, & t > 0, x \in \mathbb{R}^n, \\ -\Delta\psi + \lambda_2 \psi = u, & t > 0, x \in \mathbb{R}^n, \\ \quad u(0,x) = u_0(x), & t = 0, x \in \mathbb{R}^n. \end{cases} \tag{1.12}$$

Under the assumption $\lambda_1 \neq \lambda_2 > 0$, the nature of the solution (u, ψ, ϕ) to (1.12) is related to the problem (1.4) if $0 \leq \beta < 1$, namely instability and finite time blowing-up happen for $n = 2$ while it is rather closer to (1.10) when $\beta = 1$. The solution remains bounded for the lower space dimensions $n = 2, 3$ but may occur finite time blow-up for $n = 4$ as the model (1.10). The global well-posedness for the non-negative solution is obtained by Jin-Liu [29], Shi-Wang [66], Nagai-Yamada [53–55] for $\beta \neq 1$.

Analogous theory for the Cauchy problems from (1.9) to (1.10) and (1.11) to (1.12) can be available in the scaling critical function spaces (cf. Kurokiba-Ogawa [40, 41]). Both of the systems (1.9) and (1.10) are invariant under the following scaling transform:

$$\begin{cases} u_\mu(t,x) = \mu^4 u(\mu^2 t, \mu x), \\ \phi_\mu(t,x) = \mu^2 \phi(\mu^2 t, \mu x), \\ \psi_\mu(t,x) = \psi(\mu^2 t, \mu x) \end{cases}$$

for $\mu > 0$, and the invariant class of the sense of Fujita-Kato is now identified in the Lebesgue-Bochner class as

$$\begin{cases} u \in L^{\theta}\big(\mathbb{R}_+; L^q(\mathbb{R}^n)\big), & \dfrac{2}{\theta}+\dfrac{n}{p}=4, \quad \dfrac{n}{4}\le p\le\theta, \\ \phi \in L^{\rho}\big(\mathbb{R}_+; L^s(\mathbb{R}^n)\big), & \dfrac{2}{\rho}+\dfrac{n}{q}=2, \quad \dfrac{n}{2}\le q\le\rho, \\ \psi \in L^{\sigma}\big(\mathbb{R}_+; L^r(\mathbb{R}^n)\big), & \dfrac{2}{\sigma}+\dfrac{n}{r}=0, \quad \infty\le r\le\sigma. \end{cases}$$

It is natural to choose $\theta=\sigma=\infty$ in order to find a solution in the same Lebesgue class as the initial data for all the time, then we find that $p=\frac{n}{4}$, $q=\frac{n}{2}$, $r=\infty$. As is mentioned above, there is a difficulty associated with regularity of the third component if $n=4$. Namely under $r=\infty$, it is generally difficult to obtain $\psi\in L^{\infty}(\mathbb{R}^n)$ from $u\in L^{\frac{n}{2}}$ in four spatial dimensions. In stead of that, it is natural to choose the class for ψ in the class of bounded mean oscillation, $BMO(\mathbb{R}^n)$ [34].

Definition For a measurable function $f=f(x)$ with $x\in\mathbb{R}^n$,

$$BMO(\mathbb{R}^n)=\left\{f\in L^1_{loc}(\mathbb{R}^n);\ \|f\|_{BMO}<\infty\right\},$$

where

$$\|f\|_{BMO}\equiv\sup_{x\in\mathbb{R}^n,R>0}\frac{1}{|B_R|}\int_{B_R(x)}|f(y)-\overline{f_{B_R}}|dy,\qquad \overline{f_{B_R}}=\frac{1}{|B_R|}\int_{B_r(x)}f(y)dy$$

and $B_R(x)$ denotes n-dimensional ball centered at x with radius $R>0$. For $s\ge 0$, let

$$B\dot{M}O^s=B\dot{M}O^s(\mathbb{R}^n)\equiv\Big\{f\in L^1_{loc}(\mathbb{R}^n);\ |\nabla|^s f\in BMO(\mathbb{R}^n)\Big\}.$$

Let $C_0(\mathbb{R}^n)$ be a set of all continuous functions over $\mathbb{R}^n$ with vanishing at $|x|\to\infty$. We set $VMO=VMO(\mathbb{R}^n)$ (vanishing mean oscillation) by the completion of $C_0(\mathbb{R}^n)$ by BMO semi-norm, i.e., $VMO(\mathbb{R}^n)=\overline{C_0(\mathbb{R}^n)}^{BMO}$.

The class BMO and VMO are quasi-Banach spaces and if we identify the elements of BMO up to constants, it is regarded as the Banach space. We also introduce the space-time space $\widetilde{L^2(I;BMO)}$ that is introduce by Koch-Tataru [33] for solving the incompressible Navier-Stokes equation in the limiting scaling invariant class.

Definition For $I=(0,T)$ with $T\le\infty$,

$$\widetilde{L^2(I;BMO(\mathbb{R}^n))}\equiv\Big\{f=f(t,x);\ I\times\mathbb{R}^n\to\mathbb{R};\ \|f\|_{\widetilde{L^2(I;BMO)}}<\infty\Big\},$$

with

$$\begin{aligned}\|f\|^2_{\widetilde{L^2(I;BMO)}}&\equiv\sup_{x_0\in\mathbb{R}^n,R>0}\int_{I\cap(0,R^2)}\frac{1}{|B_R|^2}\iint_{B_R(x_0)\times B_R(x_0)}\big|f(t,x)-f(t,y)\big|^2dxdydt\\ &\simeq\sup_{x_0\in\mathbb{R}^n,R>0}\int_{I\cap(0,R^2)}\frac{1}{|B_R|}\int_{B_R(x_0)}\big|f(t,x)-\overline{f_{B_R(x_0)}}(t)\big|^2dxdydt,\end{aligned}$$

where

$$\overline{f_{B_R}}(t) = \frac{1}{|B_R|}\int_{B_R(x_0)} f(t,x)dx.$$

The class $\widetilde{L^2(I; VMO(\mathbb{R}^n))}$ is similarly introduced as above.

The equivalent norm in the above can be identify from this definition of mean average immediately.

The class BMO, $B\dot{M}O^s$ and VMO are quasi-Banach spaces and if we identify the elements of those spaces up to constants, they are regarded as Banach spaces.

Definition We call the set of the exponents (θ, q), (σ, r) as the admissible for the problem (1.1), if they satisfy

$$\begin{cases} u \in L^\theta\big(\mathbb{R}_+; L^q(\mathbb{R}^n)\big), & \dfrac{2}{\theta} + \dfrac{n}{q} = 2, \quad \dfrac{n}{2} < q < 2 < \theta, \\ \nabla\psi \in L^\sigma\big(\mathbb{R}_+; L^r(\mathbb{R}^n)\big), & \dfrac{2}{\sigma} + \dfrac{n}{r} = 1, \quad n < r < \sigma. \end{cases} \tag{1.13}$$

Definition We call the set of the exponents (θ, p), (ρ, q), (σ, r) as the admissible for the problem (1.9), if they satisfy $\frac{n}{4} < p < 2$, $\frac{n}{3} < q < n$, $n < r$ and

$$\begin{cases} u \in L^\theta\big(\mathbb{R}_+; L^p(\mathbb{R}^n)\big), & \dfrac{2}{\theta} + \dfrac{n}{p} = 4, \quad \dfrac{n}{4} < p < \theta, \\ \phi \in L^\rho\big(\mathbb{R}_+; L^q(\mathbb{R}^n)\big), & \dfrac{2}{\rho} + \dfrac{n}{q} = 2, \quad \dfrac{n}{2} < q < \rho, \\ \nabla\psi \in L^\sigma\big(\mathbb{R}_+; L^r(\mathbb{R}^n)\big), & \dfrac{2}{\sigma} + \dfrac{n}{r} = 1, \quad n < r < \sigma. \end{cases} \tag{1.14}$$

We should note that the limiting case $\theta = \rho = \sigma = \infty$ and $p = 1$, $q = 2$ and $r = \infty$ is the class where we consider the solution of both the systems (1.9) and (1.10).

2 Well-Posedness Issue in the Critical Setting

In what follows, we consider the solvability of the Cauchy problems (1.1), (1.4) and (1.9), (1.10) in the scaling invariant class, namely the admissible class defined in the above and consider the singular limit problem $\tau \to \infty$. To this end, we apply the method of generalized maximal regularity for the heat equation developed in [61].

First we consider the well-posedness issue for the Cauchy problems (1.1) and (1.4) (see [40, 41]).

Definition Let $\tau > 0$ and $1 \le p, r \le \infty$. For initial data $(u_0, \psi_0) \in L^p(\mathbb{R}^n) \times \dot{W}^{1,r}(\mathbb{R}^n)$, (u_τ, ψ_τ) is a (mild) solution to (1.1) if the following integral equation is solved:

$$\begin{cases} u_\tau(t) = e^{t\Delta}u_0 + \displaystyle\int_0^t e^{(t-s)\Delta}\nabla\cdot\big(u_\tau(s)\nabla\psi_\tau(s)\big)ds, \\ \psi_\tau(t) = e^{t\tau\Delta}\psi_0 + \displaystyle\int_0^t e^{(t-s)\tau\Delta}\tau u_\tau(s)ds \end{cases}$$

in $C(I; L^p(\mathbb{R}^n)) \times C(I; \dot{W}^{1,r}(\mathbb{R}^n))$.

Definition Let $1 \le p, r \le \infty$. For initial data $u_0 \in L^p(\mathbb{R}^n)$, (u, ψ) is a (mild) solution to (1.4) if the following integral equation is solved:

$$\begin{cases} u(t) = e^{t\Delta}u_0 + \displaystyle\int_0^t e^{(t-s)\Delta}\nabla\cdot\big(u(s)\nabla\psi(s)\big)ds, \\ \psi(t) = (-\Delta+\lambda)^{-1}u(t) = \displaystyle\lim_{t\to\infty}\int_0^t e^{s\Delta}u(t)ds \end{cases}$$

in $C(I; L^p(\mathbb{R}^n)) \times C(I; \dot{W}^{1,r}(\mathbb{R}^n))$.

Here we give the definition of the mild solution as follows:

Definition Let $\tau > 0$. For initial data $(u_0, \phi_0, \psi_0) \in L^1(\mathbb{R}^4)\times L^2(\mathbb{R}^4)\times VMO(\mathbb{R}^4)$, $(u_\tau, \phi_\tau, \psi_\tau)$ is a (mild) solution to (1.9) if following integral equation is solved:

$$\begin{cases} u_\tau(t) = e^{t\Delta}u_0 + \displaystyle\int_0^t e^{(t-s)\Delta}\nabla\cdot\big(u_\tau(s)\nabla\psi_\tau(s)\big)ds, \\ \phi_\tau(t) = e^{t\tau\Delta}\phi_0 + \displaystyle\int_0^t e^{(t-s)\tau\Delta}\tau u_\tau(s)ds, \\ \psi_\tau(t) = e^{t\tau\Delta}\psi_0 + \displaystyle\int_0^t e^{(t-s)\tau\Delta}\tau \phi_\tau(s)ds. \end{cases}$$

in $C(I; L^1(\mathbb{R}^4)) \times C(I; L^2(\mathbb{R}^4)) \times C(I; VMO(\mathbb{R}^4))$.

One can relax the condition on the initial data and solution into $\psi_0 \in BMO(\mathbb{R}^4)$ and $\psi_\tau \in C_w(I; BMO(\mathbb{R}^4))$ in the above definition. For initial data $u_0 \in L^1(\mathbb{R}^4)$, (u, ϕ, ψ) is a (mild) solution to (1.10) if the following integral equation is solved:

$$\begin{cases} u(t) = e^{t\Delta}u_0 + \displaystyle\int_0^t e^{(t-s)\Delta}\nabla\cdot\big(u(s)\nabla\psi(s)\big)ds, \\ \phi(t) = (-\Delta)^{-1}u(t) = \displaystyle\lim_{t\to\infty}\int_0^t e^{s\Delta}\phi(t)ds, \\ \psi(t) = (-\Delta)^{-1}u(t) = \displaystyle\lim_{t\to\infty}\int_0^t e^{s\Delta}\psi(t)ds \end{cases}$$

in $C(I; L^1(\mathbb{R}^4)) \times C(I; L^2(\mathbb{R}^4)) \times C(I; BMO(\mathbb{R}^4))$, where $(-\Delta)^{-1}u \equiv -\frac{1}{2\pi}\log|x| * u(x)$.

2.1 Well-Posedness of the Full System

We first state the existence and well-posedness in both time local and global with small data as follows: One of the difficulty of the problem is regularity for the solution ϕ_τ and ϕ since ϕ is involving the four-dimensional bi-Poisson equations. To avoid such a difficulty, we introduce a class of functions with *bounded mean oscillation* (BMO).

We define a mild (strong) solution of system (1.1) and (1.4). Let $e^{t\Delta}$ denote the heat evolution operator given by

$$e^{t\Delta}u_0 \equiv \int_{\mathbb{R}^n} G_t(x-y)u_0(y)dy \quad \text{for } u_0 \in C_0(\mathbb{R}^n), \tag{2.1}$$

where $G_t(x) = \frac{1}{(4\pi t)^{n/2}} \exp\left(-\frac{|x|^2}{4t}\right)$ is the Gauss kernel for $t > 0$.

We choose pair of exponents for the solution class as (θ, q) and (θ, r) defined in (1.14). Then a natural class for the common initial data is indeed given by the sharp trace estimate from the semi-group representation in the real interpolation theory such as

$$\left.\begin{aligned} &\|e^{t\Delta}u_0\|_{L^\theta(I;L^p)} < \infty, \\ &\|\nabla e^{t\Delta}\psi_0\|_{L^\sigma(I;L^r)} < \infty, \end{aligned}\right\} \Longrightarrow (u_0, \psi_0) \in \dot{B}^{-\frac{2}{\theta}}_{p,\theta}(\mathbb{R}^n) \times \dot{B}^{1-\frac{2}{\sigma}}_{r,\sigma}(\mathbb{R}^n).$$

Definition The homogeneous Besov spaces. Let $s \in \mathbb{R}$, $1 \le p, \sigma \le \infty$. The homogeneous Besov space denoted by $\dot{B}^s_{p,\sigma} = \dot{B}^s_{p,\sigma}(\mathbb{R}^n)$ is given by defined by

$$\|f\|_{\dot{B}^s_{p,\sigma}} = \Big(\sum_{j\in\mathbb{Z}} 2^{s\sigma j}\|\phi_j * f\|_p^\sigma\Big)^{1/\sigma},$$

where $\{\phi_j\}$ stands for the Littlewood-Paley dyadic decomposition of unity in the Fourier space $\xi \in \mathbb{R}^n$

$$\begin{cases} \widehat{\phi}(\xi) = \widehat{\varphi}(|\xi|), \quad \xi \in \mathbb{R}^n, \\ \widehat{\phi}(\xi) \in C_0^\infty(B_2(0) \setminus \overline{B_{1/2}}(0)). \\ \widehat{\phi}_j(\xi) \equiv \widehat{\phi}(2^j\xi) \quad \text{with} \displaystyle\sum_{j\in\mathbb{Z}} \widehat{\phi}(\xi) \equiv 1, \quad \xi \neq 0 \end{cases}$$

(cf. Triebel [70]).

2.2 Well-Posedness of the Keller-Segel System

We first state the existence and well-posedness of the problem (1.1) in both time local and global with small data as follows: We assume $\kappa = 1$ throughout the paper.

Proposition 2.1 *Let $n \geq 2$, and (θ, q), (σ, r) be the admissible pairs defined in (1.13). For $\frac{n}{2} \leq p < n$, $\lambda \geq 0$ and fix $\tau > 0$. For $n = 2$, we further assume that $\lambda > 0$ and $u_0 \in L^1(\mathbb{R}^2) \cap \dot{B}^0_{1,4}(\mathbb{R}^2)$.*

(1) For $\frac{n}{2} < p < n$, let $(u_0, \psi_0) \in L^p(\mathbb{R}^n) \times \dot{W}^{1,\frac{np}{n-p}}(\mathbb{R}^n)$. Then there exist $T = T(\|u_0\|_p, \|\nabla\psi_0\|_n) > 0$ and the unique strong solution (u_τ, ψ_τ) to (1.1) in

$$u_\tau \in C\big([0,T); L^p(\mathbb{R}^n)\big) \cap L^2\big(0,T; \dot{W}^{1,p}(\mathbb{R}^n)\big),$$
$$\psi_\tau \in C\big([0,T); \dot{W}^{1,\frac{np}{n-p}}(\mathbb{R}^n)\big) \cap L^2\big(0,T; \dot{W}^{1,\frac{np}{n-p}}(\mathbb{R}^n)\big).$$

(2) Let $(u_0, \psi_0) \in L^{\frac{n}{2}}(\mathbb{R}^n) \times \dot{W}^{1,n}(\mathbb{R}^n)$. Then there exist $T = T(u_0, \psi_0) > 0$ and the unique strong solution (u_τ, ψ_τ) to (1.1) in

$$u_\tau \in C\big([0,T); L^{\frac{n}{2}}(\mathbb{R}^n)\big) \cap L^\theta\big(0,T; L^q(\mathbb{R}^n)\big),$$
$$\psi_\tau \in C\big([0,T); \dot{W}^{1,n}(\mathbb{R}^n)\big) \cap L^\sigma\big(0,T; \dot{W}^{1,r}(\mathbb{R}^n)\big).$$

(3) There exists $\varepsilon_0 > 0$ such that for any $(u_0, \psi_0) \in L^{\frac{n}{2}}(\mathbb{R}^n) \times \dot{W}^{1,n}(\mathbb{R}^n)$ with

$$\|u_0\|_{L^{\frac{n}{2}}} + \|\nabla\psi_0\|_{L^n} < \varepsilon_0, \tag{2.2}$$

there exists a unique global solution (u_τ, ψ_τ) to (1.1) such that

$$u_\tau \in BUC\big(\mathbb{R}_+; L^{\frac{n}{2}}(\mathbb{R}^n)\big) \cap L^\theta\big(\mathbb{R}_+; L^q(\mathbb{R}^n)\big),$$
$$\psi_\tau \in BUC\big(\mathbb{R}_+; \dot{W}^{1,n}(\mathbb{R}^n)\big) \cap L^\sigma\big(\mathbb{R}_+; \dot{W}^{1,r}(\mathbb{R}^n)\big),$$

where (θ, q) and (σ, r) are admissible pairs and $BUC(I; X)$ denotes the set of bounded continuous functions on X. Furthermore, the solution satisfies the a priori estimate: For admissible pairs (θ, q) and (σ, r),

$$\sup_{t>0}\|u_\tau(t)\|_{L^{\frac{n}{2}}} + \|u_\tau\|_{L^\theta(\mathbb{R}_+;L^q)} + \sup_{t>0}\|\nabla\psi_\tau(t)\|_{L^n} + \|\nabla\psi_\tau\|_{L^\sigma(\mathbb{R}_+;L^r)} \leq \tilde{\varepsilon}_0,$$

where $\tilde{\varepsilon}_0$ is independent of $\tau > 0$.

Our statement also assures that the existence and the a priori bound for the solution is independent of $\tau > 0$. The extra assumption $u_0 \in \dot{B}^0_{1,4}(\mathbb{R}^n)$ on the initial data for the two dimensional case is required for treating the solution in Bochner spaces.

On the other hand, the solvability of the initial value problem (1.4) is shown in non-critical spaces (Kurokiba-Ogawa [37, 38]), and the critical space (Kozono-

Sugiyama-Yahagi [34], Corrias-Escobedo-Matos [14]). Biler-Brandolese [5] constructed a strong solution in a weaker scaling invariant class and Lemarié-Rieusset generalize it into the Morrey class [44]. Comparing with Proposition 2.1, we restrict the choice of the critical exponent (θ, q) and (σ, r) with $\theta = \sigma$ (and naturally $\frac{n}{q} - \frac{n}{r} = 1$), since the system (1.4) is of parabolic-elliptic type and the function class for the solution has to have a common time integrability.

Proposition 2.2 *Let $n \geq 2$ and let (θ, q) and (σ, r) be admissible pairs defined in (1.13) with restricting $\theta = \sigma$. For $\frac{n}{2} \leq p < n$ and $\lambda \geq 0$, assume that $\lambda > 0$ if $n = 2$.*

(1) For $\frac{n}{2} < p < n$ assume $u_0 \in L^p(\mathbb{R}^n)$. Then there exists $T = T(\|u_0\|_p) > 0$ and the strong solution (u, ψ) to (1.4) uniquely exists and

$$u \in C\big([0, T); L^p(\mathbb{R}^n)\big) \cap L^2\big(0, T; \dot{W}^{1,p}(\mathbb{R}^n)\big),$$
$$\psi \in C\big([0, T); \dot{W}^{1,\frac{np}{n-p}}(\mathbb{R}^n)\big) \cap L^2\big(0, T; \dot{W}^{1,p}(\mathbb{R}^n)\big).$$

(2) Let $\lambda > 0$ and $u_0 \in L^{\frac{n}{2}}(\mathbb{R}^n)$. Assume further $u_0 \in L^1(\mathbb{R}^2) \cap \dot{B}^0_{1,4}(\mathbb{R}^2)$ if $n = 2$. Then there exists $T = T(u_0) > 0$ such that the unique strong solution (u, ψ) to (1.4) exists and

$$u \in C\big([0, T); L^{\frac{n}{2}}(\mathbb{R}^n)\big) \cap L^\theta\big(0, T; L^q(\mathbb{R}^n)\big),$$
$$\psi \in C\big([0, T); \dot{W}^{1,n}(\mathbb{R}^n)\big) \cap L^\theta\big(0, T; \dot{W}^{1,r}(\mathbb{R}^n)\big).$$

(3) There exists $\varepsilon_0 > 0$ such that for any $u_0 \in L^{\frac{n}{2}}(\mathbb{R}^n)$ with

$$\|u_0\|_{L^{\frac{n}{2}}} < \varepsilon_0,$$

there exists a unique global solution (u_τ, ψ_τ) to (1.4) such that

$$u \in BUC\big(\mathbb{R}_+; L^{\frac{n}{2}}(\mathbb{R}^n)\big) \cap L^\theta\big(\mathbb{R}_+; L^q(\mathbb{R}^n)\big),$$
$$\psi \in BUC\big(\mathbb{R}_+; \dot{W}^{1,n}(\mathbb{R}^n)\big) \cap L^\theta\big(\mathbb{R}_+; \dot{W}^{1,r}(\mathbb{R}^n)\big).$$

Furthermore, the solution satisfies the a priori estimate: For admissible pairs (θ, q) and (σ, r),

$$\sup_{t>0} \|u(t)\|_{L^{\frac{n}{2}}} + \|u\|_{L^\theta(\mathbb{R}_+; L^q)} + \sup_{t>0} \|\nabla\psi(t)\|_{L^n} + \|\nabla\psi\|_{L^\theta(\mathbb{R}_+; L^r)} \leq \tilde{\varepsilon}_0.$$

The threshold constant ε_0 is known to be 8π for $n = 2$ but not known for higher dimensional cases $n \geq 3$. The best known result states that $\varepsilon_0 = \frac{8}{nS_b^2}$ and S_b is the best constant of the Sobolev inequality (see [15, 60]). One conjecture is that when $\lambda = 0$, $\varepsilon_0 = \left(\frac{2n}{n-2}\right) C_{HLS}^{-1}$, where C_{HLS} is the best constant of the Hardy-Littlewood-Sobolev inequality. One can find regularity of the solution ψ in Proposition 2.2 as $\psi \in BMO(\mathbb{R}^n)$ for all $n \geq 2$ (cf. [34]). On the other hand the solution to (1.1), it is not clear if the similar regularity can be obtained because of maximal regularity is not clear for ψ.

In both propositions, the limiting case $n = 2$ and $\lambda = 0$ is excluded since the limiting function ψ does not belong to $\dot{W}^{1,2}(\mathbb{R}^2)$. Biler-Brandolese [5] and Raczyński [63] treated this case with an elegant method of functional analysis with a choice of a suitable class where one can treat the solution of the Poisson equation and the limiting process with both solutions to (1.1) and (1.4) for small data case. We treat this limiting case in the other place since we need more delicate treatment on the regularity of the solutions (cf. [40]).

As in stated in Proposition 2.2, the existence and the uniqueness of the solution to (1.1) for each $\tau > 0$ is known (Kozono-Sugiyama-Yahagi [34]).

2.3 Two-Dimensional Critical Case for Keller-Segel System

Let $n = 2$ and $\lambda = 0$. In this case we need a slight modification of the statement of the well-posedness in Propositions 2.1 and 2.2. We re-define the mild solution of system (1.1) and (1.4). Let $e^{t\Delta}$ denote the heat evolution operator given by (2.1).

Definition Let $\tau > 0$. For initial data $(u_0, \psi_0) \in L^1(\mathbb{R}^2) \times BMO(\mathbb{R}^2)$, (u_τ, ψ_τ) is a (mild) solution to (1.1) if the following integral equation is solved:

$$\begin{cases} u_\tau(t) = e^{t\Delta}u_0 + \displaystyle\int_0^t e^{(t-s)\Delta}\nabla\cdot\big(u_\tau(s)\nabla\psi_\tau(s)\big)ds, \\ \psi_\tau(t) = e^{t\tau\Delta}\psi_0 + \displaystyle\int_0^t e^{(t-s)\tau\Delta}\tau u_\tau(s)ds \end{cases}$$

in $C(I; L^1(\mathbb{R}^2)) \times C(I; VMO(\mathbb{R}^2))$.

For initial data $u_0 \in L^1(\mathbb{R}^2)$, (u, ψ) is a (mild) solution to (1.4) if the following integral equation is solved:

$$\begin{cases} u(t) = e^{t\Delta}u_0 + \displaystyle\int_0^t e^{(t-s)\Delta}\nabla\cdot\big(u(s)\nabla\psi(s)\big)ds, \\ \psi(t) = (-\Delta)^{-1}u(t) = \displaystyle\lim_{t\to\infty}\int_0^t e^{s\Delta}u(t)ds \end{cases}$$

in $C(I; L^1(\mathbb{R}^2)) \times C(I; BMO(\mathbb{R}^2))$, where $(-\Delta)^{-1}u \equiv -\frac{1}{2\pi}\log|x| * u(x)$.

We choose pair of exponents for the solution class as (q, θ) and (θ, r) defined in (1.13). Then a natural class for the common initial data is indeed given by the sharp trace estimate from the semi-group representation in the real interpolation theory such as

$$\left.\begin{aligned}\|e^{t\Delta}u_0\|_{L^\theta(I;L^q)} &< \infty,\\ \|\nabla e^{t\Delta}\psi_0\|_{L^\sigma(I;L^r)} &< \infty,\end{aligned}\right\} \Longrightarrow (u_0,\psi_0)\in \dot{B}^{-\frac{2}{\theta}}_{q,\theta}(\mathbb{R}^2)\times \dot{B}^{1-\frac{2}{\theta}}_{r,\theta}(\mathbb{R}^2).$$

By the embedding theorem, the limiting case $q\to 2$, $r\to\infty$ is realized by

$$\begin{aligned} u_0 &\in \dot{B}^0_{1,\theta}(\mathbb{R}^2)\subset \dot{B}^{-\frac{2}{\theta}}_{q,\theta}(\mathbb{R}^2), \quad \frac{1}{q}=1-\frac{1}{\theta},\\ \psi_0 &\in \dot{B}^{1-\frac{2}{\theta}}_{r,\theta}(\mathbb{R}^2)\subset \dot{B}^0_{\infty,\theta}(\mathbb{R}^2), \quad \frac{1}{r}=\frac{1}{2}-\frac{1}{\theta}.\end{aligned}$$

Or one can restrict the class of ψ_0 itself by choosing at $\theta=2$ and $r=\infty$ to have

$$\|\psi_0\|_{\dot{B}^0_{\infty,2}(\mathbb{R}^2)} \simeq \|\nabla\psi_0\|_{\dot{B}^{-\frac{2}{\theta}}_{r,\theta}}.$$

Hence we introduce a common class for the initial data and consider the equation in the class

$$(u_\tau(t),\psi_\tau(t))\in C(I;L^1\cap \dot{B}^{-\frac{2}{\theta}}_{q,\theta})\times C(I;BMO\cap \dot{B}^{1-\frac{2}{\theta}}_{r,\theta}).$$

One can find regularity of the solution ψ in Proposition 2.2 as $\psi\in BMO(\mathbb{R}^n)$ (cf. [34]). On the other hand the solution to (1.1), it is not clear if the similar regularity can be obtained. We first illustrate that such a common space is possible for both of system (1.1) and (1.4).

Theorem 2.3 *Let $n=2$ and $\lambda=0$. For admissible pairs (θ,q) and (θ,r) defined in (1.13), assume* $(u_0,\psi_0)\in \big(L^1(\mathbb{R}^2)\cap \dot{B}^{-\frac{2}{\theta}}_{q,\theta}(\mathbb{R}^2)\big)\times\big(VMO(\mathbb{R}^2)\cap \dot{B}^{1-\frac{2}{\theta}}_{r,\theta}(\mathbb{R}^2)\big)$.

(1) Then there exist $T=T(u_0,\psi_0)>0$ and the unique strong solution (u_τ,ψ_τ) to (1.1) in

$$\begin{aligned} u_\tau &\in C\big([0,T);L^1(\mathbb{R}^2)\big)\cap L^\theta\big(0,T;L^q(\mathbb{R}^2)\big),\\ \psi_\tau &\in C\big([0,T);VMO(\mathbb{R}^2)\big)\cap L^\theta\big(0,T;L^r(\mathbb{R}^2)\big).\end{aligned}$$

Furthermore, the solution satisfies the regularity estimates: For any admissible pairs (θ,q) and (θ,r),

$$\begin{aligned}\sup_{t\in[0,T)}\|u_\tau(t)\|_{L^1} &+ \sup_{1<q\le 2}\|u_\tau\|_{L^\theta(0,T;L^q)}\\ &+\sup_{t\in[0,T)}\|\psi_\tau\|_{VMO}+\sup_{2<r<\infty}\|\nabla\psi_\tau\|_{L^\theta(0,T;L^r)}<\infty.\end{aligned}$$

(2) Assume further that for some $\varepsilon_0>0$,

$$\|u_0\|_{L^1}+\|\psi_0\|_{BMO}\le\varepsilon_0,$$

then there exists a unique global solution (u_τ,ψ_τ) to (1.1) such that

$$u_\tau \in BUC\big(\mathbb{R}_+; L^1(\mathbb{R}^2)\big) \cap L^\theta\big(\mathbb{R}_+; L^q(\mathbb{R}^2)\big) \cap L^2\big(\mathbb{R}_+; L^2(\mathbb{R}^2)\big),$$
$$\psi_\tau \in L^\infty\big(\mathbb{R}_+; BMO(\mathbb{R}^2)\big) \cap L^\theta\big(\mathbb{R}_+; L^r(\mathbb{R}^2)\big) \cap \widetilde{L^2(\mathbb{R}_+}; B\dot{M}O^1(\mathbb{R}^2)).$$

Furthermore, the solution satisfies the a priori estimate: For any admissible pairs (θ, q) *and* (θ, r)*,*

$$\begin{aligned}&\sup_{t>0}\|u_\tau(t)\|_{L^1} + \sup_{1<q\le 2} \|u_\tau\|_{L^\theta(\mathbb{R}_+;L^q)}\\ &\quad + \sup_{t>0}\|\psi_\tau(t)\|_{BMO} + \sup_{2<r<\infty}\|\nabla\psi_\tau\|_{L^\theta(\mathbb{R}_+;L^r)} + \|\nabla\psi_\tau\|_{\widetilde{L^2(\mathbb{R}_+};BMO)} \le \tilde{\varepsilon}_0,\end{aligned}$$

where $\tilde{\varepsilon}_0$ *is independent of* $\tau > 0$.

Remark Our statement also assures that the existence and the a priori bound for the solution is independent of $\tau > 0$. The extra assumption $(u_0, \psi_0) \in \dot{B}^0_{1,\theta}(\mathbb{R}^2) \times \dot{B}^{1-\frac{2}{\theta}}_{r,\theta}(\mathbb{R}^2)$ on the initial data is required for estimate involving maximal regularity. The possible weakest assumption is choosing $\theta = 4$ in our setting. Indeed, such an extra assumption on the initial data can be removed if we employ the weaker topology for constructing the solution: For instance

$$\begin{aligned}&\sup_{t>0}\|u_\tau(t)\|_1 + \sup_{t>0} t^{\frac{1}{\theta}}\|u_\tau(t)\|_{L^1} < \infty,\\ &\sup_{t>0}\|\psi_\tau(t)\|_{VMO} + \|\nabla\psi_\tau\|_{\widetilde{L^2(I;VMO)}} < \infty.\end{aligned}$$

Then the assumption on the initial data can be relaxed into the simplest way $(u_0, \psi_0) \in L^1(\mathbb{R}^2) \times VMO(\mathbb{R}^2)$. Such a function space is not suitable for proving the singular limit problem as we see below (see Sect. 1.3). Then we modify the existence class such as following: For any small $\eta_0 > 0$

$$u_\tau \in C\big([0,T); L^1(\mathbb{R}^2)\big) \cap C\big(0,T; L^2(\mathbb{R}^2)\big) \cap L^\theta\big(\eta_0, T; L^q(\mathbb{R}^2)\big),$$
$$\psi_\tau \in C\big([0,T); VMO(\mathbb{R}^2)\big) \cap \widetilde{L^2([0,T)}; V\dot{M}O^1(\mathbb{R}^2)) \cap L^\theta\big(\eta_0, T; L^r(\mathbb{R}^2)\big).$$

This is possible because the solution is getting smoother after $t > 0$ and $(u_\tau, \psi_\tau) \in \dot{B}^0_{1,\theta}(\mathbb{R}^2) \times \dot{B}^{1-\frac{2}{\theta}}_{r,\theta}(\mathbb{R}^2)$. We notice that the smallness assumption on the initial condition on (u_0, ψ_0) can be relaxed into $\varepsilon_0 = 4\pi$ if the both initial data are non-negative (cf. [49]).

The corresponding solvability of the initial value problem (1.4) has already known in non-critical space (Kurokiba-Ogawa [37, 38]), and the critical space (Nagai-Ogawa [50]). Here we show the result as a summary:

Theorem 2.4 ([37, 50]) *Let* $n = 2$*,* $\lambda = 0$ *and assume that* (θ, q) *and* (θ, r) *be admissible pairs defined in* (1.13)*. Let* $u_0 \in L^1(\mathbb{R}^2) \cap \dot{B}^0_{1,\theta}(\mathbb{R}^2)$.

(1) There exists $T = T(u_0) > 0$ *such that the unique strong solution* (u, ψ) *to* (1.4) *exists and*

$$u \in C\big([0,T); L^1(\mathbb{R}^2)\big) \cap L^\theta\big(0,T; L^q(\mathbb{R}^2)\big),$$
$$\psi \in C\big([0,T); VMO(\mathbb{R}^2)\big) \cap L^\theta\big([0,T); L^r(\mathbb{R}^2)\big).$$

(2) There exists $\varepsilon_0 > 0$ *such that for any* $u_0 \in L^1(\mathbb{R}^2)$ *with*

$$\|u_0\|_{L^1} < M_*, \tag{2.3}$$

there exists a unique global solution (u_τ, ψ_τ) *to* (1.4) *such that*

$$u \in BUC\big(\mathbb{R}_+; L^1(\mathbb{R}^2)\big) \cap L^\theta\big(\mathbb{R}_+; L^q(\mathbb{R}^2)\big),$$
$$\psi \in L^\infty\big(\mathbb{R}_+; BMO(\mathbb{R}^2)\big) \cap L^\theta\big(\mathbb{R}_+; L^r(\mathbb{R}^2)\big) \cap \widetilde{L^2(\mathbb{R}_+}; B\dot{M}O^1(\mathbb{R}^2)).$$

Furthermore, the solution satisfies the a priori estimate: If $\|u_0\|_1 \le \varepsilon_0$*,*

$$\sup_{t>0}\|u(t)\|_{L^1} + \|u\|_{L^\theta(\mathbb{R}_+;L^q)} + \sup_{t>0}\|\psi(t)\|_{BMO} + \|\nabla\psi\|_{\widetilde{L^2(\mathbb{R}_+;BMO)}} \le \tilde{\varepsilon}_0.$$

Remark As is well-known, the threshold mass M_* is known as 8π if the initial data u_0 is non-negative function [4, 6, 50, 52]. The existence and the uniqueness of the solution to (1.4) for $u_0 \in L^1(\mathbb{R}^2)$ is considered in Kozono-Sugiyama-Yahagi [34], where $\psi \in C(I; BMO(\mathbb{R}^2))$.

The main difference from our previous result [40] is that ψ_τ nor ψ never belongs to $\dot{W}^{1,2}(\mathbb{R}^2)$ in two spatial dimension. Therefore we avoid to choose function spaces such as $L^\theta(0,T; \dot{W}^{1,r}(\mathbb{R}^2))$ since it naturally requires that $\psi_\tau \in \dot{B}^1_{r,\theta}(\mathbb{R}^2)$ which may not be true for $\tau = 0$.

2.4 Singular Limit for the Keller-Segel System

One can find that the singular limit problem accompanies with the initial layer if we consider the presence of the initial data ψ_0. Since the system (1.1) and (1.4) have a common structure in the equation for u_τ and u, the main issue is how to formulate for the equation of ψ_τ and ψ. Indeed, noticing

$$\begin{aligned}\lim_{\tau\to\infty}\int_0^t e^{s\Delta}u(t)ds =&\Big[\Delta^{-1}e^{s\Delta}\Big]_{s=0}^{\infty} u(t)\\ =&(-\Delta+\lambda)^{-1}u(t),\end{aligned}$$

we compare the equations

$$\psi_\tau(t) = e^{t\tau\Delta}\psi_0 + \int_0^t e^{(t-s)\tau\Delta}\tau u(s)ds,$$

$$\psi(t) = (-\Delta+\lambda)^{-1}u(t) = \lim_{t\to\infty}\int_0^t e^{s\Delta}u(t)ds.$$

Then we show by employing analogous argument for proving the existence of solution via the contraction mapping theorem, we show the difference of those equation converges to 0 as $\tau \to \infty$ except the initial layer. We then consider the singular limit problem as $\tau \to \infty$ in the scaling invariant space.

Theorem 2.5 ([40]) *Let $n \geq 3$ and $\tau > 0$ and assume that $(u_0, \psi_0) \in L^{\frac{n}{2}}(\mathbb{R}^n) \times \dot{W}^{1,n}(\mathbb{R}^n)$.*[1] *Let (u_τ, ψ_τ) be a unique strong solution to (1.1) in $\big(C(I; L^{\frac{n}{2}}(\mathbb{R}^n)) \cap L^\theta(I; L^q(\mathbb{R}^n))\big) \times \big(C\big(I; \dot{W}^{1,n}(\mathbb{R}^n)\big) \cap L^\theta(I; \dot{W}^{1,r}(\mathbb{R}^n))\big)$, where (θ, q) and (θ, r) are admissible pairs defined in (1.13) and $I = (0, T)$ with $T \leq \infty$. For $T = \infty$, the smallness of the data (2.2) is assumed.*

(1) (Existence of the limit solution) Then for the same initial data u_0, there exists a unique strong solution (u, ψ) to (1.4) as

$$(u,\psi) \in \big(C(I; L^{\frac{n}{2}}(\mathbb{R}^n)) \cap L^\theta(I; L^q(\mathbb{R}^n))\big) \times \big(C(I; \dot{W}^{1,n}(\mathbb{R}^n)) \cap L^\theta(I; \dot{W}^{1,r}(\mathbb{R}^n)\big).$$

(2) (Singular limit) For any admissible pairs (θ, q) and (σ, r) defined in (1.13) with $\theta = \sigma$,

$$\lim_{\tau\to\infty}\Big(\|u_\tau - u\|_{L^\theta(I;L^q)} + \|\nabla\psi_\tau - \nabla\psi\|_{L^\theta(I;L^r)}\Big) = 0.$$

(3) (Initial layer) For any $t_0 > 0$, setting $I_{t_0} = (t_0, \infty) \cap I$,

$$\sup_{t\in I_{t_0}}\|u_\tau(t) - u(t)\|_{L^{\frac{n}{2}}} + \sup_{t\in I_{t_0}}\|\nabla\psi_\tau(t) - \nabla\psi(t)\|_{L^n} \to 0. \qquad \tau\to\infty, \tag{2.4}$$

On the other hand, for some small $t_1 > 0$, let

$$\eta_\tau(t) = \chi_{[0,t_1\tau^{-1}]}(t)\big(\psi_0 - (-\Delta)^{-1}u_0\big)$$

and $\chi_{[a,b]}(t)$ be the characteristic function on $[a, b]$. Then

$$\sup_{t\in[0,t_1\tau^{-1}]}\|u_\tau(t) - u(t)\|_{L^{\frac{n}{2}}} + \sup_{t\in[0,t_1\tau^{-1}]}\|\nabla\psi_\tau(t) - \nabla\psi(t) - \nabla\eta_\tau(t)\|_{L^n} \to 0. \qquad \tau\to\infty, \tag{2.5}$$

Namely $\psi_\tau(t)$ shows the initial layer $\psi_0 - (-\Delta)^{-1}u_0$ as $\tau \to \infty$.

Two dimensional case is stated in a different way. We consider the Cauchy problem (1.1) with $\lambda = 0$ in $\mathbb{R}^2$. We restrict ourselves in the case of the small data global solution (for general setting see Kurokiba-Ogawa [41]).

[1] It is also valid for $n = 2$. Assuming further $\lambda > 0$ and $u_0 \in \dot{B}^0_{1,4}(\mathbb{R}^n)$.

Theorem 2.6 ([41]) *Let $n = 2$. For admissible pairs (θ, q) and (θ, r) defined in (1.13), assume that $(u_0, \psi_0) \in L^1(\mathbb{R}^2) \cap \dot{B}^{-\frac{2}{\theta}}_{q,\theta}(\mathbb{R}^2) \times VMO(\mathbb{R}^2) \cap \dot{B}^{1-\frac{2}{\theta}}_{r,\theta}(\mathbb{R}^2)$ with the smallness assumption (2.3). For $\tau > 0$, let (u_τ, ψ_τ) be a unique strong solution to (1.1) in $\big(C(I; L^1(\mathbb{R}^2)) \cap L^\theta(I; L^q(\mathbb{R}^2))\big) \times \big(C\big(I; BMO(\mathbb{R}^2)\big) \cap \widetilde{L^2(I}; B\dot{M}O^1$ $(\mathbb{R}^2))\big)$, where $I = (0, \infty)$.*

(1) (Existence of the limit solution) Then for the same initial data u_0, there exists a unique strong solution (u, ψ) to (1.4) as

$$(u, \psi) \in \big(C(I; L^1(\mathbb{R}^2)) \cap L^\theta(I; L^q(\mathbb{R}^2))\big) \times \big(C(I; BMO(\mathbb{R}^2)) \cap \widetilde{L^2(I}; B\dot{M}O^1(\mathbb{R}^2))\big).$$

If $T < \infty$, then $\psi \in \big(BUC(I; VMO(\mathbb{R}^2)) \cap L^\theta(I; \dot{W}^{1,r}(\mathbb{R}^2))\big)$.

(2) (Singular limit) For any admissible pairs (θ, q) and (θ, r) defined in (1.13),

$$\lim_{\tau\to\infty} \Big(\|u_\tau - u\|_{L^\theta(I;L^q)} + \|\nabla(\psi_\tau - \psi)\|_{L^\theta(I;L^r)} + \|\nabla(\psi_\tau - \psi)\|_{\widetilde{L^2(I;BMO)}} \Big) = 0.$$

(3) (Initial layer) For any $\eta_0 > 0$,

$$\sup_{t\in[\eta_0,\infty)\cap I} \|u_\tau(t) - u(t)\|_{L^1} + \sup_{t\in[\eta_0,\infty)\cap I} \|\psi_\tau(t) - \psi(t)\|_{BMO} \to 0, \qquad \tau \to \infty.$$

Besides, $\psi_\tau(t)$ has the initial layer as $\tau \to \infty$. Namely it never converges to $\psi = (-\Delta)^{-1}u(0) = (-\Delta)^{-1}u_0$ for $t < \eta_0$.

We should like to emphasize that the solution to drift-diffusion equation (1.1) is in $C(I; L^1(\mathbb{R}^2)) \times C(I; VMO(\mathbb{R}^2))$ and the solution for (1.4) is in $C(I; L^1(\mathbb{R}^2)) \times C(I; BMO(\mathbb{R}^2))$. Namely the class of the solutions of two system are different for ψ_τ and ψ each other (cf. [5]). Nevertheless, the convergence is shown in the topology in wider topology $C(I_{t_0}; L^1(\mathbb{R}^2)) \times C(I_{t_0}; BMO(\mathbb{R}^2))$ except the initial layer, where $I_{t_0} = (t_0, \infty) \cap I$. It is interesting to consider the limiting case $\theta = q = 2$ and $r = \infty$. In this case, $(u_0, \psi_0) \in \dot{B}^{-1}_{2,2}(\mathbb{R}^2) \times \dot{B}^0_{\infty,2}(\mathbb{R}^2)$.

Some generalization of the singular limit observed above can be derived. For instance, we may consider the Cauchy problem of the Keller-Segel type equation with a fractional dissipative system [9]. Let $1 \le \alpha \le 2$.

$$\begin{cases} \partial_t u_\tau + (-\Delta)^{\alpha/2} u_\tau + \nabla \cdot (u_\tau \nabla \psi_\tau) = 0, & t > 0,\ x \in \mathbb{R}^n, \\ \dfrac{1}{\tau}\partial_t \psi_\tau + (-\Delta)^{\alpha/2}\psi_\tau = u_\tau, & t > 0,\ x \in \mathbb{R}^n, \\ \quad u_\tau(0,x) = u_0(x), \quad \psi_\tau(0,x) = \psi_0(x), & t = 0,\ x \in \mathbb{R}^n, \end{cases}$$

Then the singular limit problem in a critical function class

$$\begin{cases} u \in L^\theta\big(\mathbb{R}_+; L^q(\mathbb{R}^n)\big), & \dfrac{\alpha}{\theta} + \dfrac{n}{q} = 2\alpha - 2, \quad \dfrac{n}{2(\alpha-1)} \le q, \quad \alpha \le \theta, \\ \nabla\psi \in L^\sigma\big(\mathbb{R}_+; L^r(\mathbb{R}^n)\big), & \dfrac{\alpha}{\sigma} + \dfrac{n}{r} = \alpha - 1, \quad \dfrac{n}{\alpha-1} \le q, \quad q \le \sigma, \end{cases}$$

and the limit function (u, ψ) by $\tau \to \infty$ solves

$$\begin{cases} \partial_t u + (-\Delta)^{\alpha/2} u + \nabla \cdot (u \nabla \psi) = 0, & t > 0,\ x \in \mathbb{R}^n, \\ (-\Delta)^{\alpha/2} \psi = u, & t > 0,\ x \in \mathbb{R}^n, \\ u(0, x) = u_0(x), & t = 0,\ x \in \mathbb{R}^n. \end{cases}$$

We discuss such a problem in [42].

2.5 Formal Observation for the Singular Limit

Let us consider the formulation how to prove the singular limit problem (2.4). The external term of the ϕ_τ-equation in (1.9) can be regarded by changing variable $\tau t - \tau s = s'$ $(s = t - \frac{1}{\tau} s')$

$$\begin{aligned} \psi_\tau(t) =& e^{t\tau\Delta}\psi_0 + \int_0^t e^{(t-s)\tau\Delta} \tau u_\tau(s) ds \\ =& e^{t\tau\Delta}\psi_0 + \int_0^{\tau t} e^{s'\Delta} u_\tau(t - \tau^{-1}s') ds' \end{aligned} \tag{2.6}$$

and thus

$$\begin{aligned} \psi_\tau(t) - \psi(t) =& e^{\tau t\Delta}\psi_0 + \int_0^t e^{\tau(t-s)\Delta} \tau u_\tau(s) ds - (-\Delta)^{-1} u(t) \\ =& e^{\tau t\Delta}\psi_0 + \int_0^{\tau t} e^{s\Delta} \Big(u_\tau(t - \tau^{-1}s) - u(t - \tau^{-1}s) \Big) ds \\ &+ \int_0^{\tau t} e^{s\Delta} \Big(u(t - \tau^{-1}s) - u(t) \Big) ds - \int_{\tau t}^\infty e^{s\Delta} u(t) ds \\ \equiv& I_0 + I_1 + I_2 + I_3. \end{aligned} \tag{2.7}$$

Then by using the dissipative estimates for the heat equation of u, we show that

$$\Big\| u(t - \tau^{-1}t) - u(t) \Big\|_1 \to 0 \qquad \tau \to \infty \qquad \text{a.a. } t.$$

Such a formal computation can be justified by employing the similar argument to construct the solution. In particular, in order to justify the above procedure, one may introduce a typical metric induced from the norm such as

$$|||u|||_{\theta,q} \equiv \sup_{t>0} t^{1/\theta} \|u(t)\|_p, \qquad |||\psi|||_{\theta,r} \equiv \sup_{t>0} t^{1/\theta} \|\psi\|_q, \qquad |||\nabla\psi|||_{\theta,r} \equiv \sup_{t>0} t^{1/\theta} \|\nabla\psi\|_r,$$

where (θ, p), (θ, q) and (θ, r) are the admissible exponents given in (1.14). However such a choice of metric does not work well for the critical cases since the integrability conditions and the stability conditions are not consistent in the following estimate: For the first term of the right hand side of (2.7),

$$\begin{aligned}
\|I_1\|_{\dot{W}^{1,r}} \le & C\int_0^{\tau t} |s|^{-\left(\frac{1}{q}-\frac{1}{r}\right)-\frac{1}{2}} \left\|u_\tau\left(t-\frac{s}{\tau}\right)-u\left(t-\frac{s}{\tau}\right)\right\|_q ds \\
\le & C\int_0^{\tau t} |s|^{-\left(\frac{1}{q}-\frac{1}{r}\right)-\frac{1}{2}} \left|t-\frac{s}{\tau}\right|^{-\frac{1}{\theta}} ds\, \|\!|u_\tau(\cdot)-u(\cdot)\|\!|_{\theta,q} \\
= & Ct^{-\left(\frac{1}{q}-\frac{1}{r}\right)-\frac{1}{2}-\frac{1}{\theta}} \int_0^{\tau t} \left|\frac{s}{t}\right|^{-\left(\frac{1}{q}-\frac{1}{r}\right)-\frac{1}{2}} \left|1-\frac{s}{\tau t}\right|^{-\frac{1}{\theta}} ds\, \|\!|u_\tau(\cdot)-u(\cdot)\|\!|_{\theta,q} \\
\le & C\tau^{\frac{1}{2}-\left(\frac{1}{q}-\frac{1}{r}\right)} \cdot t^{\frac{1}{2}-\left(\frac{1}{q}-\frac{1}{r}\right)-\frac{1}{\theta}} B\left(\frac{1}{2}-\left(\frac{1}{q}-\frac{1}{r}\right), 1-\frac{1}{\theta}\right) \|\!|u_\tau(\cdot)-u(\cdot)\|\!|_{\theta,q},
\end{aligned}$$

where $B(p, q)$ denotes the Beta function given by

$$B(p, q) = \int_0^1 (1-s)^{p-1} s^{q-1} ds.$$

Now we notice that the condition on the convergence of the Beta function requires

$$\frac{1}{2} - \left(\frac{1}{q} - \frac{1}{r}\right) > 0$$

while the stable condition on the exponent τ should be given by

$$\frac{1}{2} - \left(\frac{1}{q} - \frac{1}{r}\right) \le 0$$

to justify the singular limit $\tau \to \infty$. Unfortunately those conditions do not hold simultaneously. To avoid such a difficulty, Biler-Brandolese [5] and Raczyński [63] used a smoothing property of the solution. Kurokiba-Ogawa [41] used a generalized version of maximal regularity, where the convergences are stated both in the large data local case and small data global case.

3 The Singular Limit Problem for the Chaplain-Anderson Systems

We developed the similar method for the simplified version of the Chaplain-Anderson system (1.9) and the limit function solves the Fujie-Senba system (1.10). We summarized the result in the following.

3.1 The Well-Posedness

By the embedding theorem, the limiting case $p, q, r \to \infty$ is realized by

$$\begin{aligned} u_0 &\in \dot{B}^{-\frac{2}{\theta}}_{p,\theta}(\mathbb{R}^4) \subset \dot{B}^{-4+\frac{4}{p}}_{p,\theta}(\mathbb{R}^4), \quad \frac{1}{p} = 1 - \frac{1}{2\theta}, \\ \phi_0 &\in \dot{B}^{-\frac{2}{\rho}}_{q,\rho}(\mathbb{R}^4) \subset \dot{B}^{-2+\frac{4}{q}}_{q,\rho}(\mathbb{R}^4), \quad \frac{1}{q} = \frac{1}{2} - \frac{1}{2\rho}, \\ \psi_0 &\in \dot{B}^{1-\frac{2}{\sigma}}_{r,\sigma}(\mathbb{R}^4) \subset \dot{B}^{\frac{4}{r}}_{r,\sigma}(\mathbb{R}^4), \quad \frac{1}{r} = \frac{1}{4} - \frac{1}{2\sigma}. \end{aligned}$$

Or one can restrict the class of ϕ_0 itself by choosing at $\sigma = 2$ and $r = \infty$ to have

$$\|\psi_0\|_{\dot{B}^{1-\frac{2}{\sigma}}_{r,\theta}(\mathbb{R}^2)} \simeq \|\psi_0\|_{\dot{B}^0_{\infty,2}},$$

where $\dot{B}^0_{\infty,2}(\mathbb{R}^4) \subsetneq BMO(\mathbb{R}^4)$. Hence we introduce a common class for the initial data and consider the equation in the class

$$(u_\tau(t), \phi_\tau(t), \psi_\tau(t)) \in C(I; L^1 \cap \dot{B}^{-\frac{2}{\theta}}_{p,\theta}) \times C(I; L^2 \cap \dot{B}^{-\frac{2}{\rho}}_{q,\theta}) \times C(I; VMO \cap \dot{B}^{1-\frac{2}{\sigma}}_{r,\theta}).$$

It is known that regularity of the solution ψ in Proposition 2.2 as $\psi \in BMO(\mathbb{R}^n)$ (cf. [34]). On the other hand the solution to (1.1), it is not clear if the similar regularity can be obtained. We first illustrate that such a common space is available for treating both of system (1.1) and (1.4).

Theorem 3.1 *Let $n = 4$, and (θ, p), (ρ, q) and (σ, r) are admissible pairs defined in (1.14). Assume $(u_0, \phi_0, \psi_0) \in \big(L^1(\mathbb{R}^4) \cap \dot{B}^{-\frac{2}{\theta}}_{p,\theta}(\mathbb{R}^4)\big) \times \big(L^2(\mathbb{R}^4) \cap \dot{B}^{-\frac{2}{\rho}}_{q,\rho}(\mathbb{R}^4)\big) \times \big(VMO(\mathbb{R}^4) \cap \dot{B}^{1-\frac{2}{\sigma}}_{r,\sigma}(\mathbb{R}^4)\big)$.*

(1) Then there exist $T = T(u_0, \psi_0) > 0$ and the unique strong solution $(u_\tau, \phi_\tau, \psi_\tau)$ to (1.9) in

$$\begin{aligned} u_\tau &\in C\big([0, T); L^1(\mathbb{R}^4)\big) \cap L^\theta\big(0, T; L^p(\mathbb{R}^4)\big), \\ \phi_\tau &\in C\big([0, T); L^2(\mathbb{R}^4)\big) \cap L^\rho\big(0, T; L^q(\mathbb{R}^4)\big), \\ \psi_\tau &\in C\big([0, T); VMO(\mathbb{R}^4)\big) \cap L^\sigma\big(0, T; \dot{W}^{1,r}(\mathbb{R}^4)\big). \end{aligned}$$

Furthermore, the solution satisfies the regularity estimates: For any admissible pairs (θ, q), (ρ, s) and (σ, r), there exists $M > 0$ independent of τ such that

$$\sup_{t\in[0,T)} \|u_\tau(t)\|_1 + \|u_\tau\|_{L^\theta(0,T;L^p)} + \sup_{t\in[0,T)} \|\phi_\tau\|_2 + \|\phi_\tau\|_{L^\rho(0,T;L^q)} + \sup_{t\in[0,T)} \|\psi_\tau\|_{VMO} + \|\nabla\psi_\tau\|_{L^\sigma(0,T;L^r)} \le M. \tag{3.1}$$

(2) Assume further that for some $\varepsilon_0 > 0$,

$$\|u_0\|_1 + \|u_0\|_{\dot B^{-\frac{2}{\theta}}_{p,\theta}} + \|\phi_0\|_2 + \|\phi_0\|_{\dot B^{-\frac{2}{\rho}}_{q,\rho}} + \|\psi_0\|_{BMO} + \|\psi_0\|_{\dot B^{1-\frac{2}{\sigma}}_{r,\sigma}} \le \varepsilon_0, \tag{3.2}$$

then there exists a unique global solution $(u_\tau, \phi_\tau, \psi_\tau)$ *to* (1.9) *such that*

$$\begin{aligned} &u_\tau \in BUC\big(\mathbb{R}_+; L^1(\mathbb{R}^4)\big) \cap L^\theta\big(\mathbb{R}_+; L^p(\mathbb{R}^4)\big),\\ &\phi_\tau \in BUC\big(\mathbb{R}_+; L^2(\mathbb{R}^4)\big) \cap L^\rho\big(\mathbb{R}_+; L^q(\mathbb{R}^4)\big),\\ &\psi_\tau \in BUC\big(\mathbb{R}_+; BMO(\mathbb{R}^4)\big) \cap L^\sigma\big(\mathbb{R}_+; \dot W^{1,r}(\mathbb{R}^4)\big). \end{aligned}$$

Furthermore, the solution satisfies the a priori estimate: For any admissible pairs (θ, p), (ρ, q) *and* (σ, r),

$$\sup_{t>0}\|u_\tau(t)\|_1 + \|u_\tau\|_{L^\theta(\mathbb{R}_+;L^p)} + \sup_{t>0} \|\phi_\tau(t)\|_2 + \|\phi_\tau\|_{L^\rho(\mathbb{R}_+;L^q)} + \sup_{t>0} \|\psi_\tau(t)\|_{BMO} + \|\nabla\psi_\tau\|_{L^\sigma(\mathbb{R}_+;L^r)} \le \tilde\varepsilon_0,$$

where $\tilde\varepsilon_0$ *is independent of* $\tau > 0$.

Our statement also assures that the existence and the a priori bound for the solution is independent of $\tau > 0$. The extra assumption

$$(u_0, \phi_0, \psi_0) \in \dot B^{-\frac{2}{\theta}}_{p,\theta}(\mathbb{R}^4) \times \dot B^{-\frac{2}{\rho}}_{q,\rho}(\mathbb{R}^4) \times \dot B^{1-\frac{2}{\sigma}}_{r,\sigma}(\mathbb{R}^4) = \dot B^{-4+\frac{4}{p}}_{p,\theta}(\mathbb{R}^4) \times \dot B^{-2+\frac{4}{q}}_{q,\rho}(\mathbb{R}^4) \times \dot B^{\frac{4}{r}}_{r,\sigma}(\mathbb{R}^4)$$

on the initial data is required for estimates involving maximal regularity ($p = 1$, $q = 2$, $r = \infty$ and $\theta = \rho = \sigma = 2$ is the best possible choice). One can relax this condition by employing another kind of function space with satisfying

$$\sup_{t>0} t^{\frac{1}{\theta}} \|u_\tau(t)\|_p + \sup_{t>0} t^{\frac{1}{\rho}} \|\phi_\tau(t)\|_q + \sup_{t>0} t^{\frac{1}{\sigma}} \|\nabla\psi_\tau(t)\|_r < \infty.$$

Within such a function class, one may construct a local or global solution to (1.1) for

$$\begin{aligned}(u_0, \phi_0, \psi_0) \in &\big(L^1(\mathbb{R}^2) \cap \dot B^{-\frac{2}{\theta}}_{p,\infty}(\mathbb{R}^4)\big) \times \big(L^2(\mathbb{R}^4) \cap \dot B^{-\frac{2}{\rho}}_{q,\infty}(\mathbb{R}^4)\big) \times \big(VMO(\mathbb{R}^4) \cap \dot B^{1-\frac{2}{\sigma}}_{r,\infty}(\mathbb{R}^4)\big)\\ &= (L^1(\mathbb{R}^4) \times L^2(\mathbb{R}^4) \times VMO(\mathbb{R}^4)).\end{aligned}$$

However such a function space is not suitable for the singular limit problem with the initial trace as we see below (see Sect. 1.3). Indeed, such an extra-assumption in the initial data can be removed by using the fact that the solution of Keller-Segel equation has a higher regularity after $t > 0$ and such a smoothing effect gives a better regularity assumption as in Theorem 3.1.

We notice that the smallness assumption on the initial condition on (u_0, ψ_0) can be relaxed into $\varepsilon_0 = 4\pi$ if the both initial data are non-negative (cf. [49]).

The corresponding solvability of the initial value problem (1.4) has already known in non-critical space (Kurokiba-Ogawa [38]), and the critical space (Nagai-Ogawa [50]). Here we show the result for the system (1.10).

Proposition 3.2 ([23], cf. [50]) *Let (θ, p), (ρ, q) and (σ, r) be admissible pair defined in* (1.14) *and let $u_0 \in L^1(\mathbb{R}^4) \cap \dot{B}^{-\frac{2}{\theta}}_{p,\theta}(\mathbb{R}^4)$.*

(1) There exists $T = T(u_0) > 0$ such that the unique strong solution (u, ϕ, ψ) to (1.10) *satisfying $-\Delta\phi = \psi$ exists and*

$$\begin{aligned} &u \in C\big([0, T); L^1(\mathbb{R}^4)\big) \cap L^\theta\big(0, T; L^p(\mathbb{R}^4)\big),\\ &\phi \in C\big([0, T); L^2(\mathbb{R}^4)\big) \cap L^\rho\big([0, T); L^q(\mathbb{R}^4)\big),\\ &\psi \in C\big([0, T); BMO(\mathbb{R}^4)\big) \cap L^\sigma\big([0, T); \dot{W}^{1,r}(\mathbb{R}^4)\big). \end{aligned}$$

Besides the solution satisfies the bound similar to (3.1).

(2) There exists $\varepsilon_0 > 0$ such that for any $u_0 \in L^1(\mathbb{R}^2)$ with

$$\|u_0\|_1 < M_*,$$

there exists a unique global solution (u, ϕ, ψ) to (1.10) *such that*

$$\begin{aligned} &u \in BUC\big(\mathbb{R}_+; L^1(\mathbb{R}^4)\big) \cap L^\theta\big(\mathbb{R}_+; L^p(\mathbb{R}^4)\big),\\ &\phi \in BUC\big(\mathbb{R}_+; L^2(\mathbb{R}^4)\big) \cap L^\rho\big(\mathbb{R}_+; L^q(\mathbb{R}^4)\big),\\ &\psi \in BUC\big(\mathbb{R}_+; BMO(\mathbb{R}^4)\big) \cap L^\sigma\big(\mathbb{R}_+; \dot{W}^{1,r}(\mathbb{R}^4)\big). \end{aligned}$$

Furthermore, the solution satisfies the a priori estimate: If $\|u_0\|_1 \le \varepsilon_0$,

$$\begin{aligned} \sup_{t>0} \|u(t)\|_1 + \|u\|_{L^\theta(\mathbb{R}_+;L^q)} + \sup_{t>0} \|\psi(t)\|_2 + \|\psi\|_{L^\rho(\mathbb{R}_+;L^q)}&\\ + \sup_{t>0} \|\phi(t)\|_{BMO} + \|\nabla\phi\|_{L^\sigma(\mathbb{R}_+;L^r)} \le \tilde{\varepsilon}_0.& \end{aligned}$$

As is observed in [23], the threshold mass M_* is given by $(8\pi)^2$ if the initial data u_0 is non-negative function (cf. [4, 6, 50, 52] for two dimensional case of (1.4)). The existence and the uniqueness of the solution to (1.4) for $u_0 \in L^1(\mathbb{R}^2)$ is considered in Kozono-Sugiyama-Yahagi [34], where $\psi(t) \in C(I; BMO)$.

3.2 Singular Limit Problem

The solution to (1.9) is formulated by the Duhamel formula as follows:

$$\begin{cases} u_\tau(t) = e^{t\Delta}u_0 + \int_0^t e^{(t-s)\Delta}\nabla\cdot\big(u_\tau(s)\nabla\psi_\tau(s)\big)ds, \\ \phi_\tau(t) = e^{t\tau\Delta}\phi_0 + \int_0^t e^{(t-s)\tau\Delta}\tau u_\tau(s)ds, \\ \psi_\tau(t) = e^{t\tau\Delta}\psi_0 + \int_0^t e^{(t-s)\tau\Delta}\tau\phi_\tau(s)ds. \end{cases} \tag{3.3}$$

One can find that the singular limit problem accompanies with the initial layer because of the presence of the initial data ψ_0. Since the system (1.9) and (1.10) have the common structure in the equation for u_τ and u, the main issue is how to formulate for the equation of (ϕ_τ, ψ_τ) and (ϕ, ψ). By employing analogous argument for proving the singular limit problem from (1.1) to (1.4) in Kurokiba-Ogawa [40, 41], we show the singular limit of the simplified Chaplain-Anderson system to the Fujie-Senba system

$$\begin{cases} u(t) = e^{t\Delta}u_0 + \int_0^t e^{(t-s)\Delta}\nabla\cdot\big(u(s)\nabla\psi(s)\big)ds, \\ \phi(t) = (-\Delta)^{-1}\psi(t) = \lim_{t\to\infty}\int_0^t e^{s\Delta}u(t)ds, \\ \psi(t) = (-\Delta)^{-1}u(t) = \lim_{t\to\infty}\int_0^t e^{s\Delta}\phi(t)ds \end{cases} \tag{3.4}$$

as follows:

Theorem 3.3 *Let $n = 4$ and let (θ, p), (θ, q) and (θ, r) be admissible pairs defined in (1.14) and assume that $(u_0, \phi_0, \psi_0) \in \big(L^1(\mathbb{R}^4)\cap \dot{B}^{-\frac{2}{\theta}}_{p,\theta}(\mathbb{R}^4)\big) \times \Big(L^2(\mathbb{R}^4)\cap \dot{B}^{-\frac{2}{\theta}}_{q,\theta}(\mathbb{R}^4)\Big) \times \Big(VMO(\mathbb{R}^4)\cap \dot{B}^{1-\frac{2}{\theta}}_{r,\theta}(\mathbb{R}^4)\Big)$. For $\tau > 0$, let $(u_\tau, \phi_\tau, \psi_\tau)$ be a unique strong solution to (1.9) in $\big(C(I; L^1(\mathbb{R}^4))\cap L^\theta(I; L^p(\mathbb{R}^4))\big) \times \big(C(I; L^2(\mathbb{R}^4))\cap L^\theta(I; L^q(\mathbb{R}^4))\big) \times \big(C\big(I; VMO(\mathbb{R}^4)\big)\cap L^\theta(I; \dot{W}^{1,r}(\mathbb{R}^4))\big)$, where $I = (0, T)$ with $T \le \infty$. For the global interval $T = \infty$, the smallness of the initial data (3.2) with $\theta = \rho = \sigma$ is assumed.*

(1) (Existence of the limit solution) Then for the same initial data u_0, there exists a unique strong solution (u, ϕ, ψ) to (1.10) as

$$\begin{aligned}(u,\phi,\psi) \in &\big(C(I; L^1(\mathbb{R}^4))\cap L^\theta(I; L^p(\mathbb{R}^4))\big) \times \big(C(I; L^2(\mathbb{R}^4))\cap L^\theta(I; L^q(\mathbb{R}^4))\big) \\ &\times \big(L^\infty(I; BMO(\mathbb{R}^4))\cap L^\theta(I; \dot{W}^{1,r}(\mathbb{R}^4))\big).\end{aligned}$$

If $T < \infty$, then $\psi \in \big(BUC(I; VMO(\mathbb{R}^4))\cap L^\theta(I; \dot{W}^{1,r}(\mathbb{R}^4))\big)$.

(2) *(Singular limit) For any admissible pairs* (θ, p), (θ, q) *and* (θ, r) *defined in* (1.14),

$$\lim_{\tau\to\infty}\Big(\|u_\tau - u\|_{L^\theta(I;L^p)} + \|\phi_\tau - \phi\|_{L^\theta(I;L^q)} + \|\nabla(\psi_\tau - \psi)\|_{L^\theta(I;L^r)}\Big) = 0.$$

(3) *(Initial layer) For any* $t_0 > 0$,

$$\begin{aligned}\sup_{t\in[t_0,\infty)\cap I}\|u_\tau(t) - u(t)\|_{L^1} + \sup_{t\in[t_0,\infty)\cap I}\|\psi_\tau(t) - \psi(t)\|_{L^2}\\ + \sup_{t\in[t_0,\infty)\cap I}\|\phi_\tau(t) - \phi(t)\|_{BMO} \to 0,\end{aligned} \tag{3.5}$$

as $\tau \to \infty$. *Besides,* $(\phi_\tau(t), \psi_\tau(t))$ *has the initial layer as* $\tau \to \infty$. *Namely it never converges to neither* $\phi = (-\Delta)^{-1}u(0) = (-\Delta)^{-1}u_0$ *nor* $\psi = (-\Delta)^{-2}u(0) = (-\Delta)^{-2}u_0$ *for* $t < \eta_0$.

We should like to emphasize that the solution to the Chaplain-Anderson type equation (1.9) is in $C(I; L^1(\mathbb{R}^4)) \times C(I; L^2(\mathbb{R}^4)) \times C(I; VMO(\mathbb{R}^4))$ and the solution for the Fujie-Senba equation (1.10) is in $C(I; L^1(\mathbb{R}^4)) \times C(I; L^2(\mathbb{R}^4)) \times C(I; BMO(\mathbb{R}^4))$. Namely the class of the solutions of two systems are different from each other (cf. [5, 41]). Nevertheless, the convergence is shown in the weaker topology $C(I_{t_0}; L^1(\mathbb{R}^4)) \times C(I_{t_0}; L^2(\mathbb{R}^4)) \times C(I_{t_0}; BMO(\mathbb{R}^4))$ except the initial layer, where $I_{t_0} = (t_0, \infty) \cap I$.

Remark We should also mention that the very similar result for the multi-component parabolic equation of Keller-Segel type (1.11) holds. In such a case, the global behavior of solutions for $\beta_1 \neq \beta_2$ is very close to the simpler model (1.1) and one for (1.12) is close to (1.4). However the global behavior for the equi-coefficient case $\beta_1 = \beta_2$, the behavior is closer to the case of simplified Chaplain-Anderson model and Fujie-Senba model and the global behavior of solutions are stable in two dimension but not the case in four dimension that was observed for the solution for (1.10) in [23]. One can derive very similar setting of the function class as the above theorem.

In what follows, for $1 \le p, r, \theta \le \infty$, let $L^p(\mathbb{R}^n)$ be the Lebesgue space in the variable x, let $L^\theta(I; X)$ be a Bochner class on the Banach space X over the time interval $I = (0, T)$ $(T \le \infty)$, $\dot{W}^{1,r}(\mathbb{R}^n)$ denotes the homogeneous Sobolev space with $\|\nabla f\|_r < \infty$. For $s \in \mathbb{R}$ and $1 \le \sigma \le \infty$, let $\dot{B}^s_{p,\sigma} = \dot{B}^s_{p,\sigma}(\mathbb{R}^n)$ and $\dot{F}^s_{p,\sigma} = \dot{F}^s_{p,\sigma}(\mathbb{R}^n)$ the homogeneous Besov and Lizorkin-Triebel spaces, respectively and the norms of those spaces are given by the following: For $1 \le p, \sigma \le \infty$ and $s \in \mathbb{R}$,

$$\begin{aligned}\|f\|_{\dot{B}^s_{p,\sigma}} &= \Big(\sum_{j\in\mathbb{Z}} 2^{s\sigma j}\|\phi_j * f\|_p^\sigma\Big)^{1/\sigma},\\ \|f\|_{\dot{F}^s_{p,\sigma}} &= \Big\|\Big(\sum_{j\in\mathbb{Z}} 2^{s\sigma j}|\phi_j * f|^\sigma\Big)^{1/\sigma}\Big\|_p,\end{aligned}$$

where $\{\phi_j\}$ denotes the Littlewood-Paley dyadic decomposition of unity. In particular, we notice that $\dot{F}^s_{p,2} \simeq \dot{W}^{s,p}(\mathbb{R}^n)$ for $s \in \mathbb{R}$ and $1 < p < \infty$ by the well-known Littlewood-Paley theorem. We should notice that the inclusion of the sequence space $\ell_\theta \subset \ell_\sigma$directly gives that $\dot{B}^s_{p,\theta}(\mathbb{R}^n) \subset \dot{B}^s_{p,\sigma}(\mathbb{R}^n)$ and $\dot{F}^s_{p,\theta}(\mathbb{R}^n) \subset \dot{F}^s_{p,\sigma}(\mathbb{R}^n)$ if $\theta \le \sigma$.

4 Preliminary Estimates

4.1 Inequalities and Embeddings in Four Space Dimensions

In general, the function inequality in two dimensional Euclidian space is different from higher dimensions. Here we summarized the Sobolev type inequalities in two dimensions.

Lemma 4.1 *Let $n = 2$ and let $f = f(x)$ be a measurable function on $\mathbb{R}^2$. There exists a constant $C > 0$ such that the following inequality hold:*

$$\|f\|_2 \le C\|\nabla f\|_1, \tag{4.1}$$

$$\|f\|_{\dot{B}^{-1}_{2,\sigma}} \le C\|f\|_{\dot{B}^0_{1,\sigma}}, \qquad 1 \le \sigma \le \infty. \tag{4.2}$$

$$\|\nabla g\|_{\dot{B}^{-1}_{2,2}} \le C\|\nabla g\|_1, \tag{4.3}$$

$$\||\nabla|^{-1}\nabla g\|_2 \le C\|\nabla g\|_1, \tag{4.4}$$

$$\|f\|_{BMO} \le C\|\nabla f\|_2. \tag{4.5}$$

$$\||\nabla|^{-1} f\|_{BMO} \le C\|f\|_2. \tag{4.6}$$

$$\||\nabla|^{-1} f\|_{\dot{B}^0_{\infty,2}} \le C\|f\|_2. \tag{4.7}$$

***Proof of Lemma* 4.1** The inequality (4.1) is due to Gagliardo and Nash, and is obtained by a straightforward computation (see for instance [8]). Indeed, by integrating the both sides of the following inequality in x-$y \in \mathbb{R}^2$,

$$|f(x,y)|^2 = \left(\int_{-\infty}^{x} \partial_x f(z,y)dz\right)\left(\int_{-\infty}^{y} \partial_y f(x,w)dw\right),$$

we obtain (4.1). The embedding (4.2) and (4.7) are direct consequences of the Bernstein type lemma and Hausdorf-Young's inequality. Equations (4.2), (4.3) and (4.4) follow from (4.1) and the boundedness of the singular integral operator in $L^2(\mathbb{R}^n)$. The inequality (4.5) follows from the Poincaré inequality in two dimensions. The inequality (4.6) is a consequence from (4.5) and the boundedness of the singular integral operators in BMO. □

We notice that the following inequalities generally fail to hold in $n = 2$.

$$\begin{aligned}\|f\|_\infty \le& C\|\nabla f\|_2.\\ \|f\|_{\dot W^{-1,\infty}} \le& C\|f\|_2.\end{aligned}$$

Lemma 4.2 *Let* (θ, q) *and* (θ, r) *be admissible pairs defined in* (1.14) *with* $1 \le q \le 2 \le \theta < \infty$ *and* $2 < \theta$. *Then the following continuous embeddings hold:*

$$L^{\frac{pr}{p+r}}(\mathbb{R}^4) \subset \dot W^{-1+\frac{2}{\sigma},p}(\mathbb{R}^4) \simeq \dot F^{-1+\frac{2}{\sigma}}_{p,2}(\mathbb{R}^4) \subset \dot F^{-1+\frac{2}{\sigma}}_{p,\theta}(\mathbb{R}^4) \subset \dot B^{-1+\frac{2}{\sigma}}_{p,\theta}(\mathbb{R}^4). \quad (4.8)$$

Proof of Lemma 4.2 Noticing the relations

$$\frac{1}{2\theta}+\frac{1}{p}=1, \qquad \frac{1}{2\rho}+\frac{1}{q}=\frac{3}{4}, \qquad \frac{1}{2\sigma}+\frac{1}{r}=\frac{1}{4},$$

the first embedding is due to the Sobolev inequality

$$\|f\|_{\dot W^{-1+\frac{2}{\sigma},p}} \le S_b\|f\|_{\frac{pr}{p+r}}$$

with

$$\frac{1}{p}-\frac{1}{4}\left(-1+\frac{2}{\sigma}\right)=\frac{1}{p}+\frac{1}{r}.$$

The second relation is due to the well-known theorem by Littlewood-Paley: $\dot F^0_{q,2}(\mathbb{R}^n) \simeq L^q(\mathbb{R}^n)$ for any $1 < q < \infty$ (see Stein [68]). The third embedding is due to the property of the sequence spaces $\ell_2 \subset \ell_\theta$ under $2 \le \theta$. The last embedding is given by the Minkowski inequality such as

$$\|f\|_{\dot B^{-1+\frac{2}{\sigma}}_{p,\theta}} = \Big(\sum_{j\in\mathbb{Z}} 2^{(-1+\frac{2}{\sigma})j\theta}\|\phi_j * f\|_p^\theta\Big)^{1/\theta} \le \Big\|\Big(\sum_{j\in\mathbb{Z}} 2^{(-1+\frac{2}{\sigma})j\theta}|\phi_j * f|^\theta\Big)^{1/\theta}\Big\|_p = \|f\|_{\dot F^{-1+\frac{2}{\sigma}}_{p,\theta}}$$

under the restriction $q \le \theta < \infty$. □

Here we recall the embedding results between the function spaces involving the real interpolation spaces:

Lemma 4.3 *Let* (θ, q) *and* (σ, r) *be admissible pairs defined in* (1.13) *with* $1 \le q \le 2 \le \theta < \infty$ *and* $2 \le n < \sigma$. *Then the following continuous embeddings hold:*

$$L^{\frac{qr}{q+r}}(\mathbb{R}^n) \subset \dot W^{-1+\frac{2}{\sigma},q}(\mathbb{R}^n) \simeq \dot F^{-1+\frac{2}{\sigma}}_{q,2}(\mathbb{R}^n) \subset \dot F^{-1+\frac{2}{\sigma}}_{q,\theta}(\mathbb{R}^n) \subset \dot B^{-1+\frac{2}{\sigma}}_{q,\theta}(\mathbb{R}^n). \quad (4.9)$$

Proof of Lemma 4.3 Noticing the relations

$$\frac{2}{\theta}+\frac{n}{q}=2, \qquad \frac{2}{\sigma}+\frac{n}{r}=1,$$

the first embedding is due to the Sobolev inequality

$$\|f\|_{\dot W^{-1+\frac{2}{\sigma},q}} \le S_b \|f\|_{\frac{rq}{r+q}}$$

with

$$\frac{1}{q} - \frac{1}{n}\left(-1+\frac{2}{\sigma}\right) = \frac{1}{r} + \frac{1}{q}.$$

The second relation is due to the well-known theorem by Littlewood-Paley: $\dot F^0_{q,2}(\mathbb{R}^n) \simeq L^q(\mathbb{R}^n)$ for any $1 < q < \infty$ (see Stein [68]). The third embedding is due to the property of the sequence spaces $\ell_2 \subset \ell_\theta$ under $2 \le \theta$. The last embedding is given by the Minkowski inequality such as

$$\|f\|_{\dot B^{-1+\frac{2}{\sigma}}_{q,\theta}} = \Big(\sum_{j\in\mathbb{Z}} 2^{(-1+\frac{2}{\sigma})j\theta}\|\phi_j * f\|_q^\theta\Big)^{1/\theta} \le \Big\|\Big(\sum_{j\in\mathbb{Z}} 2^{(-1+\frac{2}{\sigma})j\theta}|\phi_j * f|^\theta\Big)^{1/\theta}\Big\|_q = \|f\|_{\dot F^{-1+\frac{2}{\sigma}}_{q,\theta}}$$

under the restriction $q \le \theta < \infty$. □

4.2 Heat Evolution on VMO

It is well-known that the heat kernel has a dissipative estimate of L^p-L^q type:

$$\|e^{t\Delta}u_0\|_p \le Ct^{-\frac{n}{2}(\frac{1}{q}-\frac{1}{p})}\|u_0\|_q \tag{4.10}$$

for any $1 \le q \le p \le \infty$. Here is a dissipative estimate for the heat kernel on BMO and VMO. Besides by the density, the heat evolution $\{e^{t\Delta}\}_{t\ge 0}$ generates a C_0-semigroup over VMO but not over BMO.

Lemma 4.4 *The heat evolution operator $e^{t\Delta}$ is a bounded operator from $BMO(\mathbb{R}^n)$ to $BMO(\mathbb{R}^n)$. If $u_0 \in VMO(\mathbb{R}^n)$, then*

$$\|e^{t\Delta}u_0\|_{VMO} \to 0 \quad \text{as } t \to \infty. \tag{4.11}$$

***Proof of Lemma* 4.4** Let $A_k = B_{k+1} \setminus B_k$ be annulus with $B_k = B_{2^k R}(x_0)$. Since $e^{t\Delta}$ is a bounded operator on $L^2(\mathbb{R}^n)$, we see that for $u_0 \in BMO(\mathbb{R}^n)$

$$\begin{aligned}
&\|e^{t\Delta}u_0\|_{BMO} \\
&\le C \sup_{x_0,R>0}\left(\frac{1}{|B_R|^2}\iint_{B_R(x_0)\times B_R(x_0)} |e^{t\Delta_x}u_0(x) - e^{t\Delta_y}u_0(y)|^2 dxdy\right)^{1/2} \\
&\le C \sup_{x_0,R>0}\left(\frac{1}{|B_R|^2}\iint_{\mathbb{R}^n\times\mathbb{R}^n} |e^{t\Delta_x}e^{t\Delta_y}\chi_{B_R\times B_R}(x,y)(u_0(x)-u_0(y))|^2 dxdy\right)^{1/2} \\
&\quad + C \sup_{x_0,R>0}\sum_{k\ge 1}\left(\frac{1}{|B_R|^2}\iint_{B_R(x_0)\times B_R(x_0)} |e^{t\Delta_x}e^{t\Delta_y}\chi_{A_k\times A_k}(x,y)(u_0(x)-u_0(y))|^2 dxdy\right)^{1/2}
\end{aligned}$$

$$
\begin{aligned}
&\le C \sup_{x_0, R>0} \left(\frac{1}{|B_R|^2} \iint_{B_R(x_0)\times B_R(x_0)} |u_0(x) - u_0(y)|^2 dxdy \right)^{1/2} \\
&\quad + C \sup_{x_0, R>0} \sum_{k\ge 1} \left(\frac{1}{|B_R|^2} \iint_{B_k\times B_k} \right. \\
&\quad \times \left. \left(\iint_{A_k\times A_k} \frac{t^{1/2}}{(t^{1/2}+|x-x_1|)^{n+1}} \frac{t^{1/2}}{(t^{1/2}+|y-y_1|)^{n+1}} |u_0(x_1) - u_0(y_1)|^2 dx_1 dy_1 \right)^2 dxdy \right)^{1/2} \\
&\le C\|u_0\|_{BMO} + C \sum_{k\ge 1} 2^{-k} \sup_{x_0, R>0} \left(\frac{1}{|B_k|^2} \iint_{B_k\times B_k} |u_0(x_1) - u_0(y_1)|^2 dx_1 dy_1 \right)^{1/2} \\
&\le C\|u_0\|_{BMO}.
\end{aligned} \tag{4.12}
$$

See for the details Stein [68, p.159]. Besides for any $u_0 \in VMO(\mathbb{R}^n)$ there exists a sequence $\{u_{0,n}\}_{n=1}^{\infty} \subset C_0(\mathbb{R}^n)$ such that for any $\varepsilon > 0$ there exists $n >> 1$ such that

$$
\|u_{0,n} - u_0\|_{VMO} < \varepsilon.
$$

Then from the dissipative estimate (4.10) and (4.12),

$$
\begin{aligned}
\|e^{t\Delta} u_0\|_{VMO} &\le \|e^{t\Delta} u_{0,n}\|_{VMO} + \|e^{t\Delta}(u_{0,n} - u_0)\|_{VMO} \\
&\le 2\|e^{t\Delta} u_{0,n}\|_{\infty} + 2\|u_{0,n} - u_0\|_{VMO} \\
&\le C t^{-\frac{n}{2}} \|u_{0,n}\|_1 + \varepsilon.
\end{aligned}
$$

By passing $t \to \infty$, we obtain (4.11). □

5 Generalized Maximal Regularity

Let X be a proper Banach space, and we regard $A = (-\Delta)^{\alpha/2}$ as a closed linear operator in X with a dense domain $\mathcal{D}(A)$. Given $u_0 \in X$ and $f \in L^\rho(0, T; X)$ $(1 \le \rho \le \infty)$, we consider the abstract Cauchy problem

$$
\begin{cases}
\dfrac{d}{dt} u + Au = f, & t > 0, \\
\quad u(0) = u_0.
\end{cases}
$$

Then it is called that A has maximal L^ρ-regularity if there exists a unique solution $u \in C([0, T); X)$ such that $\dfrac{d}{dt} u$, $Au \in L^\rho(0, T; X)$ and it satisfies the estimate

$$
\left\| \frac{d}{dt} u \right\|_{L^\rho(0,T;X)} + \|Au\|_{L^\rho(0,T;X)} \le C \Big(\|u_0\|_{(X, \mathcal{D}(A))_{1-\frac{1}{\rho}, \rho}} + \|f\|_{L^\rho(0,T;X)} \Big),
$$

under the restriction $u_0 \in (X, \mathcal{D}(A))_{1-\frac{1}{\rho},\rho}$, where $(X, \mathcal{D}(A))_{1-\frac{1}{\rho},\rho}$ denotes the real interpolation space between X and $\mathcal{D}(A)$ and C is a positive constant independent of u_0 and f. Maximal regularity for parabolic equations is well established within the general framework on Banach spaces X that satisfy the unconditional martingale differences (called as UMD). For details, see [1, 3, 16–20, 31, 43, 72]. On the other hand, maximal regularity on Banach spaces which is not UMD, for instance non-reflexive Banach space such as L^1 or L^∞ requires a different treatment. For example, we explicitly proved maximal regularity on the homogenous Banach spaces in [57].

We consider the Cauchy problem of the heat equation: For $\nu > 0$,

$$\begin{cases} \partial_t v - \nu\Delta v = f, & t > 0, x \in \mathbb{R}^n, \\ v(0,x) = v_0(x), & t > 0, x \in \mathbb{R}^n. \end{cases} \tag{5.1}$$

Then maximal regularity is given by the following way:

(1) ([57]) For any $1 < \rho, p \le \infty$, there exists a constant $C > 0$ independent of u and T such that

$$\|\partial_t v\|_{L^\rho(I;L^p)} + \nu\|\nabla^2 v\|_{L^\rho(I;L^p)} \le C\big(\|v_0\|_{\dot{B}^{\frac{2}{\rho'}}_{p,\sigma}} + \|f\|_{L^\rho(I;L^p)}\big).$$

(2) ([17, 58]) For any $1 \le p \le \infty$,

$$\|\partial_t v\|_{L^1(I;\dot{B}^0_{p,\rho})} + \nu\|\nabla^2 v\|_{L^1(I;\dot{B}^0_{p,\rho})} \le C\big(\|v_0\|_{\dot{B}^0_{p,1}} + \|f\|_{L^\rho(I;\dot{B}^0_{p,1})}\big). \tag{5.2}$$

The first estimate is well-known result from the general framework [3, 18, 20, 26] and the case $\sigma = \infty$ is generally excluded since such spaces are not UMD (unconditional martingale difference) and it is not covered by the general theory of UMD. The remarkable feature of the latter estimates is that the estimate (5.2) allows the case $\sigma = \infty$ and it is useful to estimate for applying the integral equation. On the other hand the latter estimate involves the homogeneous Besov spaces and it is not easy to make clear the relation between the Legesgue spaces since

$$\dot{B}^0_{p,2}(\mathbb{R}^n) \subset \dot{F}^0_{p,2}(\mathbb{R}^n) \simeq L^p(\mathbb{R}^n) \subset \dot{B}^0_{p,\infty}$$

if $2 \le p < \infty$, and

$$\dot{B}^0_{p,1}(\mathbb{R}^n) \subset \dot{F}^0_{p,2}(\mathbb{R}^n) \simeq L^p(\mathbb{R}^n) \subset \dot{B}^0_{p,2}$$

if $1 \le p \le 2$.

We state the following general version is useful to apply the semi-linear parabolic equations [39–42, 61].

Theorem 5.1 (Generalized maximal regularity [40]) *Let* $1 \le \nu \le \rho \le \infty$, $1 \le p \le \infty$, $s \in \mathbb{R}$, $\mu > 0$ *and let* $I = (0, T) \subset \mathbb{R}_+$ *be an interval (possibly* $I = \mathbb{R}_+$*). Given* *initial data* $v_0 \in \dot{B}^{s+2-2/\rho}_{p,\rho}(\mathbb{R}^n)$ *and the external force* $f \in L^\nu(I;\dot{B}^{s+2/\nu-2/\rho}_{p,\nu}(\mathbb{R}^n))$, *the solution of the Cauchy problem of the heat equation* (5.1) *fulfills following estimates:*

(1) *Suppose that* $f \equiv 0$. *Then for any* $1 \le \sigma \le \infty$, *there exists a constant* $C > 0$ *independent of* T *such that*

$$\left\|\partial_t v\right\|_{L^\rho(I;\dot{B}^s_{p,\sigma})} + \nu\left\|\nabla^2 v\right\|_{L^\rho(I;\dot{B}^s_{p,\sigma})} \le C\|v_0\|_{\dot{B}^{s+2-\frac{2}{\rho}}_{p,\rho}}, \tag{5.3}$$

where $\nabla^2 = \partial_{x_i}\partial_{x_j}$.

(2) *Suppose that* $u_0 = 0$, *then for any* $\nu \le \sigma \le \rho$, *there exists a constant* $C > 0$ *independent of* $T > 0$ *such that*

$$\left\|\partial_t v\right\|_{L^\rho(I;\dot{B}^s_{p,\sigma})} + \nu\left\|\nabla^2 v\right\|_{L^\rho(I;\dot{B}^s_{p,\sigma})} \le C\|f\|_{L^\nu(I;\dot{B}^{s+\frac{2}{\nu}-\frac{2}{\rho}}_{p,\sigma})}. \tag{5.4}$$

We notice that the above estimates remain valid for the problem:

$$\begin{cases} \partial_t v - \nu\Delta v + \lambda u = f, & t > 0,\ x \in \mathbb{R}^n, \\ v(0,x) = v_0(x), & x \in \mathbb{R}^n, \end{cases}$$

where $\lambda > 0$. Indeed, the case $\lambda > 0$ can be reduced into the case $\lambda = 0$ by the estimate

$$\|e^{t\Delta}\phi_j * u_0\|_p \le \|e^{t\Delta}\phi_j * u_0\|_p$$

for all $1 \le p \le \infty$, where $\{\phi_j\}_j$ is the Littlewood-Paley partition of unity.

The proof of Theorem 5.1 is separated into a homogeneous estimate and an inhomogeneous estimate (cf. [57]): For the homogeneous term, the following proposition directly shows the result (5.3). For simplicity, we assume that $\nu = 1$. The general case can be obtained by using the scaling transformation $t' = \sqrt{\nu}t$ and $f'(t',x) = \nu^{-1}f(t,x)$.

Proposition 5.2 *For* $0 < T \le \infty$, *we set* $I = [0,T)$. *Let* $1 \le p,\sigma \le \infty$, $1 \le \rho \le \infty$ *and* $s \in \mathbb{R}$. *For* $v_0 \in \dot{B}^{s-2/\rho}_{p,\rho}$, *the solution of the heat equation* $e^{t\Delta}v_0$ *satisfies the following estimate: we have for any* $0 < T \le \infty$ *and* $s \in \mathbb{R}$, *that*

$$\left(\int_0^T \|e^{t\Delta}v_0\|^\rho_{\dot{B}^s_{p,\sigma}}\,dt\right)^{1/\rho} \le C\|v_0\|_{\dot{B}^{s-\frac{2}{\rho}}_{p,\rho}}. \tag{5.5}$$

For $v_0 \in \dot{\mathcal{B}}^{s-\frac{2}{\rho}}_{\infty,\rho} = \overline{C_0^\infty}^{\dot{B}^{s-\frac{2}{\rho}}_{\infty,\rho}}$, *then for any* $0 < T \le \infty$ *and* $1 \le \sigma \le \infty$, $1 \le \rho \le \infty$,

$$\left(\int_0^T \|e^{t\Delta}v_0\|^\rho_{\dot{\mathcal{B}}^s_{\infty,\sigma}}\,dt\right)^{1/\rho} \le C\|v_0\|_{\dot{\mathcal{B}}^{s-\frac{2}{\rho}}_{\infty,\rho}}, \tag{5.6}$$

where C is independent of T.

Those estimates (5.5), (5.6) are known as a characteristic definition of the Besov space by the heat semi-group. Indeed the reversed inequalities also hold if $T=\infty$. For the proof, see for instance [40, 58, 61].

Remark It is known by the evolutional Besov formulation that for any $1 \le p \le \infty$ with $1/\rho+\gamma>0$. The above estimate (5.6) is an extension of such an expression.

We next consider the solution of the inhomogeneous heat equation with 0-initial data:

$$\begin{cases} \partial_t v - \Delta v = f, & t>0,\ x\in\mathbb{R}^n, \\ v(0,x)=0, & x\in\mathbb{R}^n. \end{cases}$$

The following proposition is the key to proving Theorem 5.1 (cf. Ogawa-Shimizu [57]).

Proposition 5.3 ([40]) *Let $\{e^{t\Delta}\}_{t\ge 0}$ be a heat semigroup in $\mathbb{R}^n$ and $I=(0,T)$ for any $0<T\le\infty$. Then for $1\le p,\gamma\le\infty$ and $1<\nu,\sigma\le\infty$ with $1<\nu\le\sigma\le\rho<\infty$, we have*

$$\left\| \int_0^t e^{(t-s)\Delta} f(s)ds \right\|_{L^\rho(I;(\dot B^{s+2}_{p,\gamma},\dot B^s_{p,\gamma})_{1-1/\rho,\sigma})} \le C\|f\|_{L^\nu(I;(\dot B^{s-2}_{p,\gamma},\dot B^s_{p,\gamma})_{1/\nu,\sigma})}. \tag{5.7}$$

***Proof of Proposition* 5.3** Since we show the result by using the duality argument, we show only the case for $1<\rho<\infty$ and $1<p\le\infty$. For the case $p=1$, the proof requires a similar treatment involving the Hardy space $\mathcal{H}^1(\mathbb{R}^n)$. This is because the base space $L^1(\mathbb{R}^n)$ is not the dual of $L^\infty(\mathbb{R}^n)$ (see for the detail [57]). The end-point case $\rho=1$ required another treatment, too (see [58]). We also show the inequality for the case $T=\infty$. The other case is obtained by letting $f(t)$ by $\chi_{(0,T)}(t)f(t)$. Let

$$\chi_j(r)=\begin{cases} 1, & 2^j<r\le 2^{j+1}, \\ 0, & \text{otherwise} \end{cases}$$

and $g(t)\in C^\infty(I;\mathcal{S}(\mathbb{R}^n))$. We consider the dual coupling: For $j\in\mathbb{Z}$,

$$\begin{aligned} &\left| \int_0^\infty \left(\int_0^t e^{(t-s)\Delta} f(s)ds, g(t) \right)_{L^2} dt \right| \\ &\quad \le \sum_{j\in\mathbb{Z}} \iint_{t>s>0} \chi_j(t-s) \left| \left(e^{(t-s)\Delta} f(s), g(t)\right)_{L^2} \right| dsdt. \end{aligned}$$

Introducing

$$T_j(f,g) \equiv \iint_{t>s>0} \chi_j(t-s) \left| (e^{(t-s)\Delta} f(s), g(t)) \right| dsdt,$$

we claim that $T : (f \times g) \to \{T_j(f, g)\}_j$ is a bilinear bounded form:

$$L^{\nu}\big(I; (\dot{B}^{s-2}_{p,\gamma}, \dot{B}^{s}_{p,\gamma})_{1/\nu,\rho}\big) \times L^{\rho'}\big(I; (\dot{B}^{-s-2}_{p',\gamma'}, \dot{B}^{-s}_{p',\gamma'})_{1-1/\rho,\rho'}\big) \to \ell_1,$$

where $1 \le p, \gamma \le \infty$, $1 < \rho \le \infty$, $1 \le \nu < \infty$ with $\nu \le \rho$. From the dissipative estimate for the heat evolution operator,

$$\|e^{t\Delta}u_0\|_{\dot{B}^{s}_{p,\gamma}} \le Ct^{-\frac{n}{2}\left(\frac{1}{q}-\frac{1}{p}\right)-\frac{s-s'}{2}}\|u_0\|_{\dot{B}^{s'}_{q,\gamma}},$$

we have for $l = 0, 2$ and $m = 0, 2$ that

$$\begin{aligned} T_j(f, g) &\le \iint_{t>s>0} \chi_j(t-s)\|e^{(t-s)\Delta}f(s)\|_{\dot{B}^{s+l}_{p,\gamma}}\|g(t)\|_{\dot{B}^{-s-l}_{p',\gamma'}}dsdt \\ &\le C\iint_{t>s>0} \chi_j(t-s)|t-s|^{-(l+m)/2}\|f(s)\|_{\dot{B}^{s-m}_{p,\gamma}}\|g(t)\|_{\dot{B}^{-s-l}_{p',\gamma'}}dsdt \qquad (5.8) \\ &\le C2^{-(l+m)j/2}\iint_{t>s>0} \chi_j(t-s)\|f(s)\|_{\dot{B}^{s-m}_{p,\gamma}}\|g(t)\|_{\dot{B}^{-s-l}_{p',\gamma'}}dsdt. \end{aligned}$$

We decompose that $f = f_0 + f_1$ with $f_0 \in \dot{B}^{s-2}_{p,\gamma}$ and $f_1 \in \dot{B}^{s}_{p,\gamma}$, and $g = g_0 + g_1$ with $g_0 \in \dot{B}^{-s-2}_{p',\gamma'}$ and $g_1 \in \dot{B}^{-s}_{p',\gamma'}$ and taking the infimum over all representations of $g \in \dot{B}^{-s-2}_{p',\gamma'} + \dot{B}^{-s}_{p',\gamma'}$,

$$\begin{aligned} |T_j(f_0, g)| &\le \inf_{g=g_0+g_1}(|T_j(f_0, g_0)| + |T_j(f_0, g_1)|), \\ |T_j(f_1, g)| &\le \inf_{g=g_0+g_1}(|T_j(f_1, g_0)| + |T_j(f_1, g_1)|). \end{aligned}$$

Adding both sides and taking the infimum over all representations $f \in \dot{B}^{s-2}_{p,\gamma} + \dot{B}^{s}_{p,\gamma}$,

$$\begin{aligned} |T_j(f, g)| \le \inf_{f=f_0+f_1}\Big\{&\inf_{g=g_0+g_1}(|T_j(f_0, g_0)| + |T_j(f_0, g_1)|) \\ &+ \inf_{g=g_0+g_1}(|T_j(f_1, g_0)| + |T_j(f_1, g_1)|)\Big\}. \end{aligned}$$

Adding in j and applying the estimates (5.8), we have

$$\begin{aligned} &\sum_{j\in\mathbb{N}}|T_j(f, g)| \\ &\le C\sum_{j\in\mathbb{Z}} 2^{\frac{j}{\nu}-\frac{j}{\rho}-j}\iint_{t>s>0}\chi_j(t-s)\inf_{f=f_0+f_1}\big(2^{-\frac{j}{\nu}}\|f_0(s)\|_{\dot{B}^{s-2}_{p,\gamma}} + 2^{j(1-\frac{1}{\nu})}\|f_1(s)\|_{\dot{B}^{s}_{p,\gamma}}\big) \\ &\qquad\qquad \times \inf_{g=g_0+g_1}\big(2^{-j(1-\frac{1}{\rho})}\|g_0(t)\|_{\dot{B}^{-s-2}_{p',\gamma'}} + 2^{\frac{j}{\rho}}\|g_1(t)\|_{\dot{B}^{-s}_{p',\gamma'}}\big)dsdt. \end{aligned}$$

We then let

$$\begin{cases} F_j(s) \equiv \|f(s)\|_{\{\dot B^{s-2}_{p,\gamma},\dot B^s_{p,\gamma}\}_{1/\nu,j}} \equiv \inf_{f=f_0+f_1} \big(2^{-\frac{j}{\nu}}\|f_0(s)\|_{\dot B^{s-2}_{p,\gamma}} + 2^{j(1-\frac{1}{\nu})}\|f_1(s)\|_{\dot B^s_{p,\gamma}}\big), \\ G_j(t) \equiv \|g(t)\|_{\{\dot B^{-s-2}_{p',\gamma'},\dot B^{-s}_{p',\gamma'}\}_{1-1/\rho,j}} \equiv \inf_{g=g_0+g_1} \big(2^{-j(1-\frac{1}{\rho})}\|g_0(t)\|_{\dot B^{-s-2}_{p',\gamma'}} + 2^{\frac{j}{\rho}}\|g_1(t)\|_{\dot B^{-s}_{p',\gamma'}}\big). \end{cases}$$

It follows by letting $\frac{1}{\mu}+\frac{1}{\rho'}+\frac{1}{\nu}=2$ and using the Hausdorff-Young inequality (including the case $\nu=1$) that

$$\begin{aligned} &\sum_{j\in\mathbb{Z}}|T_j(f,g)| \\ &\le C\sum_{j\in\mathbb{Z}} 2^{j(\frac{1}{\nu}-\frac{1}{\rho}-1)}\left(\int_0^\infty \chi_j(r)^\mu dr\right)^{1/\mu}\left(\int_0^\infty F_j(s)^\nu ds\right)^{1/\nu}\left(\int_0^\infty G_j(t)^{\rho'}dt\right)^{1/\rho'} \\ &\le C\sum_{j\in\mathbb{Z}}\left(\int_0^\infty F_j(s)^\nu ds\right)^{\frac{1}{\nu}}\left(\int_0^\infty G_j(t)^{\rho'}dt\right)^{\frac{1}{\rho'}}. \end{aligned}$$

Here we used the fact that $\frac{1}{\mu}=2-\frac{1}{\nu}-\frac{1}{\rho'}=1-\frac{1}{\nu}+\frac{1}{\rho}$. By using the fact that $\int_{2^j<\lambda\le 2^{j+1}}\frac{d\lambda}{\lambda}=\int_{\mathbb{R}_+}\chi_j(\lambda)\frac{d\lambda}{\lambda}=\log 2$, we apply the Hölder inequality for j in $\frac{1}{\sigma}+\frac{1}{\sigma'}=1$ and noting $\sigma'\ge\rho'$ and the Minkowski inequality, $\nu\le\sigma$, we obtain

$$\begin{aligned} \sum_{j\in\mathbb{Z}}|T_j(f,g)| &\le C\sum_{j\in\mathbb{Z}}\big\|F_j(s)\big\|_{L^\nu(I)}\big\|G_j(t)\big\|_{L^{\rho'}(I)} \\ &\le C\left(\sum_{j\in\mathbb{Z}}\big\|F_j(s)\big\|^\sigma_{L^\nu(I)}\right)^{1/\sigma}\left(\sum_{j\in\mathbb{Z}}\big\|G_j(t)\big\|^{\sigma'}_{L^{\rho'}(I)}\right)^{1/\sigma'} \\ &\quad\text{(Minkowski's inequality by} \nu\le\sigma and \rho'\le\sigma') \\ &\le C\left(\int_0^\infty\Big(\sum_{j\in\mathbb{Z}}F_j(s)^\sigma\Big)^{\frac{\nu}{\sigma}}ds\right)^{1/\nu}\left(\int_0^\infty\Big(\sum_{j\in\mathbb{Z}}G_j(t)^{\sigma'}\Big)^{\rho'/\sigma'}dt\right)^{1/\rho'} \\ &\le C\left(\int_0^\infty\Big(\sum_{j\in\mathbb{Z}}\int_{2^j<\lambda\le 2^{j+1}}\|f(s)\|^\sigma_{\{\dot B^{s-2}_{p,\gamma},\dot B^s_{p,\gamma}\}_{1/\nu,j}}\frac{d\lambda}{\lambda}\Big)^{\nu/\sigma}ds\right)^{1/\nu} \\ &\qquad\times\left(\int_0^\infty\Big(\sum_{j\in\mathbb{Z}}\int_{2^j<\lambda\le 2^{j+1}}\|g(t)\|^{\sigma'}_{\{\dot B^{-s-2}_{p',\gamma'},\dot B^{-s}_{p',\gamma'}\}_{1-1/\nu,j}}\frac{d\lambda}{\lambda}\Big)^{\rho'/\sigma'}dt\right)^{1/\rho'} \\ &= C\|f\|_{L^\nu(I;(\dot B^{s-2}_{p,\gamma},\dot B^s_{p,\gamma})_{1/\nu,\sigma})}\|g\|_{L^{\rho'}(I;(\dot B^{-s-2}_{p',\gamma'},\dot B^{-s}_{p',\gamma'})_{1-1/\rho,\sigma'})}. \end{aligned}$$

We conclude that

$$\|\{T_j(f,g)\}_j\|_{\ell_1} \le C\|f\|_{L^\nu(I;(\dot B^{s-2}_{p,\gamma},\dot B^{s}_{p,\gamma})_{1/\nu,\sigma})}\|g\|_{L^{\rho'}(I;(\dot B^{-s-2}_{p',\gamma'},\dot B^{-s}_{p',\gamma'})_{1-1/\rho,\sigma'})}.$$

Noting for $1<\nu<\infty$, $1\le\gamma'<\infty$ and $1\le p'<\infty$,

$$\left((\dot B^{-s-2}_{p',\gamma'},\dot B^{-s}_{p',\gamma'})_{1-1/\nu,\sigma'}\right)^* = (\dot B^{s+2}_{p,\gamma},\dot B^{s}_{p,\gamma})_{1-1/\nu,\sigma},$$

we obtain the desired estimate (5.7) by a duality argument. □

Proof of Theorem 5.1 For simplicity we show the proof for the case $\mu=1$. The other cases are shown by a simple scaling argument of time-variable $t\to t'/\mu$. Since the homogeneous estimate (5.3) can be obtained from Proposition 5.2, we only show the estimate (5.4). For $1<\nu\le\sigma\le\rho<\infty$, we have shown that

$$\left\|\int_0^t e^{(t-s)\Delta}f(s)ds\right\|_{L^\rho(I;\dot B^{s+\frac{2}{\rho}}_{p,\sigma})} \le C\|f\|_{L^\nu(I;\dot B^{s-2+\frac{2}{\nu}}_{p,\sigma})} \tag{5.9}$$

from the interpolation result. Then the estimate (5.4) follows from (5.9), since $\partial_t u = \Delta u + f$.

Finally we treat the case $\rho=\infty$. In this case, we modify the above argument in the different interpolation parameter. Namely, we have from (5.8) that

$$\sum_{j\in\mathbb{Z}}|T_j(f,g)| \le C\sum_{j\in\mathbb{Z}}2^{\frac{j}{\nu}-j}\iint_{t>s>0}\chi_j(t-s)F_j(s)G_j(t)dsdt,$$

where

$$\begin{cases} F_j(s) \equiv \|f(s)\|_{\{\dot B^{s-2}_{p,\gamma},\dot B^{s}_{p,\gamma}\}_{\frac12+\frac1\nu,j}} \equiv \inf\limits_{f=f_0+f_1}\left(2^{-j(\frac12+\frac1\nu)}\|f_0(s)\|_{\dot B^{s-2}_{p,\gamma}} + 2^{j(\frac12-\frac1\nu)}\|f_1(s)\|_{\dot B^{s}_{p,\gamma}}\right),\\ G_j(t) \equiv \|g(t)\|_{\{\dot B^{-s-2}_{p',\gamma'},\dot B^{-s}_{p',\gamma'}\}_{\frac12,j}} \equiv \inf\limits_{g=g_0+g_1}\left(2^{-\frac12 j}\|g_0(t)\|_{\dot B^{-s-2}_{p',\gamma'}} + 2^{\frac12 j}\|g_1(t)\|_{\dot B^{-s}_{p',\gamma'}}\right). \end{cases}$$

It follows by letting $\frac1\mu+\frac1\nu+1=2$ and using the Hausdorff-Young inequality with $\frac1\mu = 2-1-\frac1\nu = 1-\frac1\nu$,

$$\begin{aligned}\sum_{j\in\mathbb{Z}}|T_j(f,g)| &\le C\sum_{j\in\mathbb{Z}}2^{j(\frac1\nu-1)}\|\chi_j(\cdot)\|_\mu\|F_j(\cdot)\|_{L^\nu(I)}\|G_j(\cdot)\|_{L^1(I)}\\ &= C\sum_{j\in\mathbb{Z}}\|F_j(\cdot)\|_{L^\nu(I)}\|G_j(\cdot)\|_{L^1(I)}\\ &\le C\left(\sum_{j\in\mathbb{Z}}\|F_j(\cdot)\|^\sigma_{L^\nu(I)}\right)^{1/\sigma}\left(\sum_{j\in\mathbb{Z}}\|G_j(\cdot)\|^{\sigma'}_{L^1(I)}\right)^{1/\sigma'}.\end{aligned}$$

By using the fact that $\int_{2^j<\lambda\le 2^{j+1}}\frac{d\lambda}{\lambda} = \int_{\mathbb{R}_+}\chi_j(\lambda)\frac{d\lambda}{\lambda} = \log 2$, we apply the Hölder and the Minkowski inequalities for j to obtain

$$\sum_{j\in\mathbb{Z}} |T_j(f,g)| \le C\|f\|_{L^\nu(I;(\dot B^{s-2}_{p,\gamma},\dot B^s_{p,\gamma})_{\frac12+\frac1\nu,\sigma})}\|g\|_{L^1(I;(\dot B^{-s-2}_{p',\gamma'},\dot B^{-s}_{p',\gamma'})_{\frac12,\sigma'})}.$$

We conclude that

$$\|\{T_j(f,g)\}_j\|_{\ell_1} \le C\|f\|_{L^\nu(I;(\dot B^{s-2}_{p,\gamma},\dot B^s_{p,\gamma})_{\frac12+\frac1\nu,\sigma})}\|g\|_{L^1(I;(\dot B^{-s-2}_{p',\gamma'},\dot B^{-s}_{p',\gamma'})_{\frac12,\sigma'})}.$$

Noting for $2<\nu\le\rho=\infty$, $1\le\gamma'<\infty$ and $1\le p'<\infty$,

$$\big((\dot B^{-s-2}_{p',\gamma'},\dot B^{-s}_{p',\gamma'})_{\frac12-\frac1\rho,\sigma'}\big)^* = (\dot B^{s+2}_{p,\gamma},\dot B^s_{p,\gamma})_{\frac12-\frac1\rho,\sigma},$$

we obtain the desired estimate (5.7) by letting $\rho=\infty$ and the duality argument. □

Theorem 5.4 (Maximal regularity in BMO [33, 59]) *There exists $C_M>0$ such that for all $f\in\widetilde{L^2(\mathbb{R}_+;BMO(\mathbb{R}^n))}$ and $\nabla u_0\in BMO(\mathbb{R}^n)$, then the solution of the Cauchy problem (5.1) admits a unique solution $v\in\dot W^{1,2}(\mathbb{R}_+;BMO(\mathbb{R}^n))\cap\widetilde{L^2(\mathbb{R}_+;B\dot MO^2(\mathbb{R}^n))}$ which satisfies the following estimate:*

$$\|\partial_t v\|_{\widetilde{L^2(\mathbb{R}_+;BMO)}}+\nu\|\Delta v\|_{\widetilde{L^2(\mathbb{R}_+;BMO)}}\le C_M\big(\|\nabla u_0\|_{BMO}+\|f\|_{\widetilde{L^2(\mathbb{R}_+;BMO)}}\big).$$

The proof of Theorem 5.4 can be seen in [59].

6 Proof of Well-Posedness for Keller-Segel System

In this section we show the local and global well-posedness of the solution to (1.1) stated in Proposition 2.1. The proof of Proposition 2.2 for the limiting equation (1.4) is very similar to the case for (1.1) and it is indeed simpler than Proposition 2.1 and we do not show the case for (1.4) (cf. [34, 38]).

Proof of Proposition 2.1 Since the case $p>\frac n2$ is not the end-point case, the proof is easier and the result is more or less known. Hence we only show the critical case: $p=\frac n2$. We show first the local existence of solution for $I=(0,T)$. Consider the mild solution to the corresponding integral equation:

$$\begin{cases} u_\tau(t)=e^{t\Delta}u_0+\displaystyle\int_0^t e^{(t-s)\Delta}\nabla\cdot\big(u_\tau(s)\nabla\psi_\tau(s)\big)ds,\\ \psi_\tau(t)=e^{t\tau\Delta}\psi_0+\displaystyle\int_0^{\tau t} e^{(\tau t-s)\Delta}u_\tau(\tau^{-1}s)ds. \end{cases} \tag{6.1}$$

⟨Step 1⟩ (The local wellposedness): Let $n\ge3$, $\lambda\ge0$ and $\tau>0$. We show the local in time existence and well-posedness of the solutions for the large initial data $(u_0,\psi_0)\in L^{\frac n2}(\mathbb{R}^n)\times\dot W^{1,n}(\mathbb{R}^n)$. Let (q,θ) and (r,θ) satisfy the Serrin admissible conditions:

$$\begin{cases} \frac{2}{\theta} + \frac{n}{q} = 2, \quad \frac{n}{2} < q \le \theta, \quad 2 \le \theta, \\ \frac{2}{\sigma} + \frac{n}{r} = 1, \quad 2 \le n < r \le \sigma \end{cases} \tag{6.2}$$

and set $I = (0, T)$ for some $0 < T < \infty$ chosen later and let

$$X_M = \Big\{(u, \psi) \in \Big(C(I; L^{\frac{n}{2}}) \cap L^\theta(I; L^q)\Big) \times \Big(C(I; \dot{W}^{1,n}) \cap L^\sigma(I; \dot{W}^{1,r})\Big);$$
$$\|u\|_{L^\infty(I;L^{\frac{n}{2}})} + \|\nabla\psi\|_{L^\infty(I;L^n)} \le M,$$
$$\|u\|_{L^\theta(I;L^q)} + \|\nabla\psi\|_{L^\sigma(I;L^r)} \le N\Big\},$$

where

$$M = 4C_0\big(\|u_0\|_{\frac{n}{2}} + \|\nabla\psi_0\|_n^2\big)$$

and $N > 0$ is chosen small later. Introducing the metric on X_M by

$$|||(u, \psi) - (\tilde{u}, \tilde{\psi})|||_T \equiv \|u - \tilde{u}\|_{L^\theta(I;L^q)} + \|\nabla(\psi - \tilde{\psi})\|_{L^\sigma(I;L^r)},$$

one can show that X_M is a complete metric space.

We then introduce a pair of the solution maps $(\Xi[u_\tau, \psi_\tau], \Psi[u_\tau, \psi_\tau])$ as follows: For $(u_0, \psi_0) \in L^{\frac{n}{2}} \times \dot{W}^{1,n}$ and $(u_\tau, \psi_\tau) \in X_M$, let

$$\begin{cases} \Xi[u_\tau, \psi_\tau](t) \equiv e^{t\Delta}u_0 + \displaystyle\int_0^t e^{(t-s)\Delta}\nabla\cdot\big(u_\tau(s)\nabla\psi_\tau(s)\big)ds, \\ \Psi[u_\tau, \psi_\tau](t) \equiv e^{\tau t\Delta}\psi_0 + \displaystyle\int_0^{\tau t} e^{(\tau t-s)\Delta}\Phi[u_\tau, \psi_\tau](\tau^{-1}s)ds \end{cases} \tag{6.3}$$

and claim that the map (Ξ, Ψ) is contraction in the critical space X_M.

Then by maximal regularity (5.3) in Theorem 5.1 with $s = -2$, $\sigma = 1$ with the embeddings $\dot{B}^0_{q,1}(\mathbb{R}^n) \subset L^q(\mathbb{R}^n)$ and $\dot{W}^{-\frac{2}{\theta},q}(\mathbb{R}^n) \simeq \dot{F}^{-\frac{2}{\theta}}_{q,2}(\mathbb{R}^n) \subset \dot{F}^{-\frac{2}{\theta}}_{q,\theta}(\mathbb{R}^n) \subset \dot{B}^{-\frac{2}{\theta}}_{q,\theta}(\mathbb{R}^n)$ (note that we assume $q < \theta$ and $2 \le \theta$) to see

$$\begin{aligned} \|e^{t\Delta}u_0\|_{L^\theta(I;L^q)} \le& \|e^{t\Delta}u_0\|_{L^\theta(I;\dot{B}^0_{q,1})} \\ \le& C\|u_0\|_{\dot{B}^{-2+\frac{2}{\theta'}}_{q,\theta}} \le C\|u_0\|_{\dot{F}^{-\frac{2}{\theta}}_{q,\theta}} \\ \le& C\|u_0\|_{\dot{F}^{-\frac{2}{\theta}}_{q,2}} \le C\|u_0\|_{\dot{W}^{-\frac{2}{\theta},q}} \\ \le& C_0 S_b\|u_0\|_{\frac{n}{2}}, \end{aligned} \tag{6.4}$$

where we used the Sobolev type inequality with the relation of the critical exponents (6.2);

$$\|f\|_{\dot W^{-\frac{2}{\theta},q}} \le S_b\|f\|_{\frac{n}{2}} \text{ with } \quad \frac{1}{q}+\frac{2}{n\theta}=\frac{2}{n}.$$

Also it follows similarly for $\nabla\psi_0$ that

$$\begin{aligned}\|\nabla e^{\tau t\Delta}\psi_0\|_{L^\sigma(I;L^r)} \le & C\|\nabla\psi_0\|_{\dot B^{-2+\frac{2}{\sigma'}}_{r,\sigma}} \le C\|\nabla\psi_0\|_{\dot F^{-\frac{2}{\sigma}}_{r,\sigma}} \\ \le & C_0 S_b\|\nabla\psi_0\|_n,\end{aligned} \tag{6.5}$$

where we used the Sobolev type inequality with (6.2);

$$\|f\|_{\dot W^{-\frac{2}{\sigma},r}} \le S_b\|f\|_n \text{ with } \quad \frac{1}{r}+\frac{2}{n\sigma}=\frac{1}{n}.$$

Hence from (6.4) and (6.5), we can choose the time interval $|I|\le T$ sufficiently[2] small such that for some small $\varepsilon_0>0$,

$$\begin{aligned}&\|e^{t\Delta}u_0\|_{L^\theta(I;L^q)}<\varepsilon_0, \\ &\|e^{\tau t\Delta}\nabla\psi_0\|_{L^\sigma(I;L^r)} \le \|e^{t\Delta}\nabla\psi_0\|_{L^\sigma(I;L^r)}<\varepsilon_0\end{aligned} \tag{6.6}$$

for any $\tau>1$ and the choice of T is independent of $\tau>1$.

Noticing $\frac{n}{2}<q<\theta<\infty$, we apply (6.6), maximal regularity (5.4) and the embedding (4.9) to have

$$\begin{aligned}\big\|\Xi[u_\tau,\psi_\tau]\big\|_{L^\theta(I;L^q)} \le & \|e^{t\Delta}u_0\|_{L^\theta(I;L^q)} + \Big\|\int_0^t e^{(t-s)\Delta}\nabla\cdot\big(u_\tau(s)\nabla\psi_\tau(s)\big)ds\Big\|_{L^\theta(I;\dot B^{-2}_{q,1})} \\ \le & \varepsilon_0 + C\Big\|u_\tau(s)\nabla\psi_\tau(s)\Big\|_{L^{\frac{\theta\sigma}{\theta+\sigma}}(I;\dot B^{-1+\frac{2}{\sigma}}_{q,\theta})} \\ \le & \varepsilon_0 + C\Big\|u_\tau(s)\nabla\psi_\tau(s)\Big\|_{L^{\frac{\theta\sigma}{\theta+\sigma}}(I;L^{\frac{rq}{r+q}})} \\ \le & \varepsilon_0 + C\big\|u_\tau(\cdot)\|_{L^\theta(I;L^q)}\big\|\nabla\psi_\tau(\cdot)\big\|_{L^\sigma(I;L^r)}.\end{aligned} \tag{6.7}$$

Meanwhile by the embedding

$$L^q(\mathbb{R}^n)\subset \dot W^{-n(\frac{1}{q}-\frac{1}{r}),r}(\mathbb{R}^n), \qquad \frac{1}{r}=\frac{1}{q}-\frac{1}{n}\left(\frac{n}{q}-\frac{n}{r}\right) \tag{6.8}$$

and noting the relations

$$\frac{2}{\theta}-\frac{2}{\sigma}=1-\frac{n}{q}+\frac{n}{r}, \quad \theta\le\sigma, \quad 2\le n<r<\sigma,$$

[2]The choice of T is independent of $\tau>1$.

$$\begin{aligned}\big\|\nabla\Psi[u_\tau,\psi_\tau]\big\|_{L^\sigma(I;L^r)} \le& \|e^{\tau t\Delta}\nabla\psi_0\|_{L^\sigma(I;L^r)} + \Big\|\nabla\int_0^{\tau t} e^{(\tau t-s)\Delta}u_\tau(\tau^{-1}s)ds\Big\|_{L^\sigma(I;L^r)}\\ \le&\varepsilon_0 + C\Big\|u_\tau(\tau^{-1}s)\Big\|_{L^\theta(I;\dot B_{r,\sigma}^{-1+\frac{2}{\theta}-\frac{2}{\sigma}})}\\ \le&\varepsilon_0 + C\Big\|u_\tau(\tau^{-1}s)|_{s=\tau t}\Big\|_{L^\theta(I;\dot F_{r,2}^{-1+\frac{2}{\theta}-\frac{2}{\sigma}})}\\ \le&\varepsilon_0 + C\big\|u_\tau(t)\big\|_{L^\theta(I;\dot W^{-n(\frac{1}{q}-\frac{1}{r}),r})}\\ \le&\varepsilon_0 + C\big\|u_\tau(\cdot)\big\|_{L^\theta(I;L^q)}.\end{aligned} \tag{6.9}$$

From (6.7) and (6.9),

$$\big\|\Xi[u_\tau,\psi_\tau]\big\|_{L^\theta(I;L^q)} + \big\|\nabla\Psi[u_\tau,\psi_\tau]\big\|_{L^\sigma(I;L^r)} \le 2\varepsilon_0 + C_1\Big(\big\|u_\tau\big\|_{L^\theta(I;L^q)}\Big)^2. \tag{6.10}$$

Choosing $\varepsilon_0 > 0$ small enough in (6.10) and we conclude that by choosing (θ, q) and (σ, r)

$$\big\|\Xi[u_\tau,\psi_\tau]\big\|_{L^\theta(I;L^q)} + \big\|\nabla\Psi[u_\tau,\psi_\tau]\big\|_{L^\sigma(I;L^r)} \le 4\varepsilon_0 \equiv N. \tag{6.11}$$

Similarly by Lemma 4.3, we proceed the estimate similar to (6.9) to see that

$$\begin{aligned}\big\|\Xi[u_\tau,\psi_\tau]\big\|_{L^\infty(I;L^{\frac{n}{2}})} \le& \|e^{t\Delta}u_0\|_{L^\infty(I;L^{\frac{n}{2}})} + \Big\|\int_0^t e^{(t-s)\Delta}\nabla\cdot\big(u_\tau(s)\nabla\psi_\tau(s)\big)ds\Big\|_{L^\infty(I;\dot B^{-2}_{\frac{n}{2},1})}\\ \le& C_0\|u_0\|_{\frac{n}{2}} + C\Big\|u_\tau(s)\nabla\psi_\tau(s)\Big\|_{L^\sigma(I;L^{\frac{nr}{2r+n}})}\\ \le&\frac{1}{4}M + C\big\|u_\tau(\cdot)\|_{L^\infty(I;L^{\frac{n}{2}})}\big\|\nabla\psi_\tau(\cdot)\big\|_{L^\sigma(I;L^r)} \le \frac{1}{4}M + C\varepsilon_0 M \le \frac{1}{2}M\end{aligned} \tag{6.12}$$

and

$$\begin{aligned}\big\|\nabla\Psi[u_\tau,\psi_\tau]\big\|_{L^\infty(I;L^n)} \le& \|e^{\tau t\Delta}\nabla\psi_0\|_{L^\infty(I;L^n)} + \Big\|\nabla\int_0^{\tau t} e^{(\tau t-s)\Delta}u_\tau(\tau^{-1}s)ds\Big\|_{L^\infty(I;L^n)}\\ \le& C_0\|\nabla\psi_0\|_n + C\Big\|u_\tau(\tau^{-1}s)\Big\|_{L^\theta(I;\dot B_{n,\infty}^{-1+\frac{2}{\theta}})}\\ \le&\frac{1}{4}M + C\big\|u_\tau(t)\big\|_{L^\theta(I;\dot W^{-1+\frac{2}{\theta},n})}\\ &\Big(\text{since}\,\frac{1}{n} = \frac{1}{q} - \frac{1}{n}\Big(1-\frac{2}{\theta}\Big)\Big)\\ \le&\frac{1}{4}M + C\big\|u_\tau(\cdot)\big\|_{L^\theta(I;L^q)} \le \frac{1}{2}M.\end{aligned} \tag{6.13}$$

The estimates (6.12) and (6.13) implies

$$\big\|\Xi[u_\tau,\psi_\tau]\big\|_{L^\infty(I;L^{\frac{n}{2}})} + \big\|\nabla\Psi[u_\tau,\psi_\tau]\big\|_{L^\infty(I;L^n)} \le M. \tag{6.14}$$

Combining (6.11) and (6.14), we obtain that $(\Phi[u_\tau, \psi_\tau], \Psi[u_\tau, \psi_\tau]) \in X_M$. Analogously from (6.7) for the difference of solutions

$$\begin{aligned}
&\Big\| \Xi[u_\tau, \psi_\tau] - \Xi[\tilde{u}_\tau, \tilde{\psi}_\tau] \Big\|_{L^\theta(I;L^q)} \\
&\quad \le \Big\| \int_0^t e^{(t-s)\Delta} \nabla \cdot \big(u_\tau(s) \nabla \psi_\tau(s) - \tilde{u}_\tau(s) \nabla \tilde{\psi}_\tau(s) \big) ds \Big\|_{L^\theta(I;\dot{B}^{-2}_{q,1})} \\
&\quad \le C \Big\| u_\tau(s) \nabla \psi_\tau(s) - \tilde{u}_\tau(s) \nabla \tilde{\psi}_\tau(s) \Big\|_{L^{\frac{\theta\sigma}{\theta+\sigma}}(I;\dot{B}^{-1+\frac{2}{\sigma}}_{q,\theta})} \\
&\quad \le C \Big\| u_\tau(s) \nabla (\psi_\tau(s) - \tilde{\psi}_\tau(s)) \Big\|_{L^{\frac{\theta\sigma}{\theta+\sigma}}(I;L^{\frac{rq}{r+q}})} \\
&\qquad\qquad + C \Big\| \big(u_\tau(s) - \tilde{u}_\tau(s) \big) \nabla \tilde{\psi}_\tau(s) \Big\|_{L^{\frac{\theta\sigma}{\theta+\sigma}}(I;L^{\frac{rq}{r+q}})} \\
&\quad \le C \big\| u_\tau(s) \big\|_{L^\theta(I;L^q)} \big\| \nabla (\psi_\tau(s) - \tilde{\psi}_\tau(s)) \big\|_{L^\sigma(I;L^r)} \\
&\qquad + C \big\| u_\tau(s) - \tilde{u}_\tau(s) \|_{L^\theta(I;L^q)} \big\| \nabla \tilde{\psi}_\tau(s) \big\|_{L^\sigma(I;L^r)} \\
&\quad \le C N \| (u_\tau, \psi_\tau) - (\tilde{u}_\tau, \tilde{\psi}_\tau) \|_M .
\end{aligned} \tag{6.15}$$

Analogously from (6.9) and (6.15), we have

$$\begin{aligned}
&\Big\| \nabla \Big(\Psi[u_\tau, \psi_\tau] - \Psi[\tilde{u}_\tau, \tilde{\psi}_\tau] \Big) \Big\|_{L^\sigma(I;L^r)} \\
&\quad \le \Big\| \int_0^{\tau t} \nabla e^{(\tau t - s)\Delta} \Big(\Phi[u_\tau, \psi_\tau](\tau^{-1}s) - \Phi[\tilde{u}_\tau, \tilde{\psi}_\tau](\tau^{-1}s) \Big) ds \Big\|_{L^\sigma(I;L^r)} \\
&\quad \le C \Big\| \Phi[u_\tau, \psi_\tau](\tau^{-1}s)|_{\tau t} - \Phi[\tilde{u}_\tau, \tilde{\psi}_\tau](\tau^{-1}s)|_{\tau t} \Big\|_{L^\theta(I;\dot{F}^{-1+\frac{2}{\theta}-\frac{2}{\sigma}}_{r,2})} \\
&\quad \le C \big\| \Phi[u_\tau, \psi_\tau](t) - \Phi[\tilde{u}_\tau, \tilde{\psi}_\tau](t) \big\|_{L^\theta(I;\dot{W}^{-n(\frac{1}{q}-\frac{1}{r}),r})} \\
&\quad \le C \big\| \Phi[u_\tau, \psi_\tau](t) - \Phi[\tilde{u}_\tau, \tilde{\psi}_\tau](t) \big\|_{L^\theta(I;L^q)} \\
&\quad \le C N \| (u_\tau, \psi_\tau) - (\tilde{u}_\tau, \tilde{\psi}_\tau) \|_M .
\end{aligned} \tag{6.16}$$

Choosing N smaller as

$$CN \le \frac{1}{4}, \tag{6.17}$$

if necessary, (6.15) and (6.16) with (6.17) yield that

$$\begin{aligned}
&\| (\Xi[u_\tau, \psi_\tau], \Psi[u_\tau, \psi_\tau]) - (\Xi[\tilde{u}_\tau, \tilde{\psi}_\tau], \Psi[\tilde{u}_\tau, \tilde{\psi}_\tau]) \|_T \\
&\quad \le \frac{1}{2} \| (u_\tau, \psi_\tau) - (\tilde{u}_\tau, \tilde{\psi}_\tau) \|_T
\end{aligned}$$

under the smallness assumption (6.6) on the interval. Thus the map (Φ, Ψ) is contraction onto X_M and the Banach fixed point theorem implies that there exists a unique fixed point $(u_\tau, \psi_\tau) \in X_M$ that solves the Eq. (1.1) in the critical space. In particular, from (6.12) and (6.14), the a priori estimate

$$\|u_\tau\|_{L^\infty(I;L^{\frac{n}{2}})} + \|u_\tau\|_{L^\theta(I;L^q)} + \|\nabla\psi_\tau\|_{L^\infty(I;L^n)} + \|\nabla\psi_\tau\|_{L^\sigma(I;L^r)} \le M$$

does not depend on the parameter $\tau > 0$.

⟨Step 2⟩ (Global existence for small data). We show the case of global existence with the critical small data. Let $n \ge 3$, $\lambda \ge 0$ and $\tau > 0$ or $n = 2$, $\lambda > 0$ and $\tau > 0$, and let (θ, q), (σ, r) be Serrin-admissible as is given by

$$\begin{cases} \dfrac{2}{\theta} + \dfrac{n}{q} = 2, & \dfrac{n}{2} < q < n, \quad n < \theta < \infty, \\ \dfrac{2}{\sigma} + \dfrac{n}{r} = 1, & n < r \le \sigma < \infty. \end{cases}$$

Also we call

$$(\theta, q) = \Big(\infty, \frac{n}{2}\Big), \quad (\sigma, r) = (\infty, n)$$

the end-point admissible pairs. Fixing the admissible pair for $I = \mathbb{R}_+$ as (θ, q), (σ, r), we introduce the complete metric space:

$$X_M = \Big\{(u,\psi) \in \Big(C(I; L^{\frac{n}{2}}) \cap L^\theta(I; L^q)\Big) \times \Big(C(I; \dot{W}^{1,n}) \cap L^\sigma(I; \dot{W}^{1,r})\Big); \\ |\!|\!|(u,\psi)|\!|\!|_M \equiv \|u\|_{L^\infty(I;L^{\frac{n}{2}})} + \|u\|_{L^\theta(I;L^q)} + \|\nabla\psi\|_{L^\infty(I;L^n)} + \|\nabla\psi\|_{L^\sigma(I;L^r)} \le M\Big\},$$

where

$$M = 4C_0\big(\|u_0\|_{\frac{n}{2}} + \|\nabla\psi_0\|_n^2\big)$$

is chosen small later. For any admissible exponents (θ, q) and (σ, r) (not the end-point exponents), we define the metric on X_M by

$$|\!|\!|(u,\psi) - (\tilde{u}, \tilde{\psi})|\!|\!|_M \equiv \|u - \tilde{u}\|_{L^\theta(I;L^q)} + \|\nabla(\psi - \tilde{\psi})\|_{L^\sigma(I;L^r)}.$$

By this metric, X_M is a complete metric space. For $(u_0, \psi_0) \in L^{\frac{n}{2}} \times \dot{W}^{1,n}$ and $(u_\tau, \psi_\tau) \in X_M$, we define a pair of the solution maps $(\Phi[u_\tau, \psi_\tau], \Psi[u_\tau, \psi_\tau])$ by (6.3) and claim that the map (Φ, Ψ) is contraction in the critical space X_M. Let $n \ge 3$. Noticing $\frac{n}{2} < q < \theta < \infty$ and the embedding (4.9), we apply maximal regularity and the embedding (6.4) to have

$$\begin{aligned} \big\|\Xi[u_\tau, \psi_\tau]\big\|_{L^\theta(I;L^q)} \le& \|e^{t\Delta}u_0\|_{L^\theta(I;\dot{B}^0_{q,1})} + \Big\|\int_0^t e^{(t-s)\Delta}\nabla\cdot\big(u_\tau(s)\nabla\psi_\tau(s)\big)ds\Big\|_{L^\theta(I;\dot{B}^0_{q,1})} \\ \le& C_0\|u_0\|_{\dot{B}^{-\frac{2}{\theta}}_{q,\theta}} + C\Big\|u_\tau(s)\nabla\psi_\tau(s)\Big\|_{L^{\frac{\theta\sigma}{\theta+\sigma}}(I;\dot{B}^{-1+\frac{2}{\sigma}}_{q,\theta})} \\ \le& C_0\|u_0\|_{\frac{n}{2}} + C\big\|u_\tau(\cdot)\|_{L^\theta(I;L^q)}\big\|\nabla\psi_\tau(\cdot)\big\|_{L^\sigma(I;L^r)}. \end{aligned} \tag{6.18}$$

Meanwhile by the embedding

$$L^q(\mathbb{R}^n) \subset \dot{W}^{-n(\frac{1}{q}-\frac{1}{r}),r}(\mathbb{R}^n), \qquad \frac{1}{r} = \frac{1}{q} - \frac{1}{n}\left(\frac{n}{q} - \frac{n}{r}\right)$$

and noting the relations

$$\frac{2}{\theta} - \frac{2}{\sigma} = 1 - \frac{n}{q} + \frac{n}{r}, \quad \theta \le \sigma, \quad 2 \le n < r < \sigma,$$

we have from maximal regularity and the embeddings (6.5) and (6.8) to obtain

$$\begin{aligned} \big\|\nabla\Psi[u_\tau,\psi_\tau]\big\|_{L^\sigma(I;L^r)} \le & C_0\|e^{\tau t\Delta}\nabla\psi_0\|_{L^\sigma(I;L^r)} + \Big\|\nabla\int_0^{\tau t} e^{(\tau t-s)\Delta}u_\tau(\tau^{-1}s)ds\Big\|_{L^\sigma(I;L^r)} \\ \le & C_0\|\nabla\psi_0\|_{\dot{B}^{-\frac{2}{\sigma}}_{r,\sigma}} + C\Big\|u_\tau(\tau^{-1}s)\Big\|_{L^\theta(I;\dot{B}^{-1+\frac{2}{\theta}-\frac{2}{\sigma}}_{r,\sigma})} \\ \le & C_0\|\nabla\psi_0\|_n + C\big\|u_\tau(\cdot)\big\|_{L^\theta(I;L^q)}. \end{aligned} \tag{6.19}$$

From (6.18) and (6.19),

$$\big\|\Xi[u_\tau,\psi_\tau]\big\|_{L^\theta(I;L^q)} + \big\|\nabla\Psi[u_\tau,\psi_\tau]\big\|_{L^\sigma(I;L^r)} \le C_0\Big(\|u_0\|_{\frac{n}{2}} + \|\nabla\psi_0\|_n^2\Big) + C_1\Big(\big\|u_\tau\big\|_{L^\theta(I;L^q)}\Big)^2. \tag{6.20}$$

Choosing $\varepsilon_0 > 0$ small enough and

$$C_0\big(\|u_0\|_{\frac{n}{2}} + \|\nabla\psi_0\|_n\big) \equiv \frac{M}{4} \le \varepsilon_0, \quad C_1 M \le \frac{1}{4} \tag{6.21}$$

in (6.20), we conclude that by choosing (θ, q) and (σ, r)

$$\big\|\Xi[u_\tau,\psi_\tau]\big\|_{L^\theta(I;L^q)} + \big\|\nabla\Psi[u_\tau,\psi_\tau]\big\|_{L^\sigma(I;L^r)} \le \frac{1}{2}M. \tag{6.22}$$

Similar estimates of (6.12) and (6.13) imply that

$$\big\|\Xi[u_\tau,\psi_\tau]\big\|_{L^\infty(I;L^{\frac{n}{2}})} + \big\|\nabla\Psi[u_\tau,\psi_\tau]\big\|_{L^\infty(I;L^n)} \le \frac{1}{2}M. \tag{6.23}$$

Combining (6.22) and (6.23), we obtain that $(\Xi[u_\tau,\psi_\tau], \Psi[u_\tau,\psi_\tau]) \in X_M$. Analogously from (6.18) for the difference of solutions

$$\begin{aligned}
&\left\| \Xi[u_\tau, \psi_\tau] - \Xi[\tilde{u}_\tau, \tilde{\psi}_\tau] \right\|_{L^\theta(I;L^q)} \\
&\quad \le \left\| \int_0^t e^{(t-s)\Delta} \nabla \cdot \big(u_\tau(s) \nabla \psi_\tau(s) - \tilde{u}_\tau(s) \nabla \tilde{\psi}_\tau(s) \big) ds \right\|_{L^\theta(I;\dot{B}^{-2}_{q,1})} \\
&\quad \le C \left\| u_\tau(s) \nabla \psi_\tau(s) - \tilde{u}_\tau(s) \nabla \tilde{\psi}_\tau(s) \right\|_{L^{\frac{\theta\sigma}{\theta+\sigma}}(I;\dot{B}^{-1+\frac{2}{\sigma}}_{q,\theta})} \\
&\quad \le C \left\| u_\tau(s) \nabla (\psi_\tau(s) - \tilde{\psi}_\tau(s)) \right\|_{L^{\frac{\theta\sigma}{\theta+\sigma}}(I;L^{\frac{rq}{r+q}})} \\
&\qquad\qquad + C \left\| \big(u_\tau(s) - \tilde{u}_\tau(s) \big) \nabla \tilde{\psi}_\tau(s) \right\|_{L^{\frac{\theta\sigma}{\theta+\sigma}}(I;L^{\frac{rq}{r+q}})} \\
&\quad \le C \left\| u_\tau(s) \right\|_{L^\theta(I;L^q)} \left\| \nabla (\psi_\tau(s) - \tilde{\psi}_\tau(s)) \right\|_{L^\sigma(I;L^r)} \\
&\qquad + C \left\| u_\tau(s) - \tilde{u}_\tau(s) \right\|_{L^\theta(I;L^q)} \left\| \nabla \tilde{\psi}_\tau(s) \right\|_{L^\sigma(I;L^r)} \\
&\quad \le CM \| (u_\tau, \psi_\tau) - (\tilde{u}_\tau, \tilde{\psi}_\tau) \|_M .
\end{aligned} \tag{6.24}$$

Analogously from (6.16) and using (6.24), we have

$$\begin{aligned}
&\left\| \nabla \Big(\Psi[u_\tau, \psi_\tau] - \Psi[\tilde{u}_\tau, \tilde{\psi}_\tau] \Big) \right\|_{L^\sigma(I;L^r)} \\
&\quad \le \left\| \int_0^{\tau t} \nabla e^{(\tau t - s)\Delta} \Big(\Xi[u_\tau, \psi_\tau](\tau^{-1} s) - \Xi[\tilde{u}_\tau, \tilde{\psi}_\tau](\tau^{-1} s) \Big) ds \right\|_{L^\sigma(I;L^r)} \\
&\quad \le C \left\| \Xi[u_\tau, \psi_\tau](\tau^{-1} s)|_{\tau t} - \Xi[\tilde{u}_\tau, \tilde{\psi}_\tau](\tau^{-1} s)|_{\tau t} \right\|_{L^\theta(I;\dot{F}^{-1+\frac{2}{\theta}-\frac{2}{\sigma}}_{r,2})} \\
&\quad \le C \left\| \Xi[u_\tau, \psi_\tau](t) - \Xi[\tilde{u}_\tau, \tilde{\psi}_\tau](t) \right\|_{L^\theta(I;\dot{W}^{-n(\frac{1}{q}-\frac{1}{r}),r})} \\
&\quad \le C \left\| \Xi[u_\tau, \psi_\tau](t) - \Xi[\tilde{u}_\tau, \tilde{\psi}_\tau](t) \right\|_{L^\theta(I;L^q)} \\
&\quad \le CM \| (u_\tau, \psi_\tau) - (\tilde{u}_\tau, \tilde{\psi}_\tau) \|_M .
\end{aligned} \tag{6.25}$$

Choosing M smaller as

$$CM \le \frac{1}{4},$$

if necessary, we have from (6.24), (6.25), that

$$\begin{aligned}
&\| (\Xi[u_\tau, \psi_\tau], \Psi[u_\tau, \psi_\tau]) - (\Xi[\tilde{u}_\tau, \tilde{\psi}_\tau], \Psi[\tilde{u}_\tau, \tilde{\psi}_\tau]) \|_M \\
&\quad \le \frac{1}{2} \| (u_\tau, \psi_\tau) - (\tilde{u}_\tau, \tilde{\psi}_\tau) \|_M
\end{aligned}$$

under the smallness assumption (6.21) on the initial data. Thus the map (Φ, Ψ) is contraction onto X_M and the Banach fixed point theorem implies that there exists a unique fixed point $(u_\tau, \psi_\tau) \in X_M$ that solves the Eq. (1.1) in the critical space. In particular, from (6.22) and (6.23), the a priori estimate

$$\|u_\tau\|_{L^\infty(I;L^{\frac{n}{2}})} + \|u_\tau\|_{L^\theta(I;L^q)} + \|\nabla\psi_\tau\|_{L^\infty(I;L^n)} + \|\nabla\psi_\tau\|_{L^\sigma(I;L^r)} \le M \tag{6.26}$$

does not depend on the parameter $\tau > 0$. □

For two dimensional case, we need to involve the class of bounded mean oscillations ($BMO(\mathbb{R}^2)$) for the small data global existence and the vanishing mean oscillation ($VMO(\mathbb{R}^2)$) for the local existence for large data. The proof is entirely close to the higher dimensional case but the role of those limiting function class is subtle. See for the detailed proof of Theorem 2.3, Kurokiba-Ogawa [41].

7 Proof for the Singular Limit

In this section, we recall the proof of the convergence of the singular limit problem for the higher dimensional Patlak-Keller-Segel equation (1.1). The key part is to introduce the critical Bochner- Lebesgue spaces with the admissible exponents defined in (1.13). We only show the proof for Theorem 2.5. See for the proof of Theorem 2.6 [41].

The following lemma is useful for proving the strong convergence in the critical Bochner spaces (cf. [40]).

Lemma 7.1 *Let* $1 \le \theta, p \le \infty$ *and* $f \in W^{1,\theta}(I; L^p(\mathbb{R}^n)) \cap L^\theta(I; \dot{W}^{2,p}(\mathbb{R}^n))$, *where* $t \in I = (0, T)$ *with* $T \le \infty$. *Then for any* $\tau > 0$,

$$\Big\| \int_0^{\tau t} \Delta e^{s\Delta}\Big(f\big(t - \frac{s}{\tau}\big) - f(t)\Big) ds \Big\|_{L^\theta(I;L^p)} \le \Big\| \int_0^{\tau t} e^{r\Delta} \frac{\partial}{\partial r} f\big(t - \frac{r}{\tau}\big) dr \Big\|_{L^\theta(I;L^p)} + \big\| e^{\tau t\Delta} f(0) \big\|_{L^\theta(I;L^p)} + \big\| e^{\tau t\Delta} f(t) \big\|_{L^\theta(I;L^p)}.$$

Proof of Lemma 7.1 Since f is absolute continuous in $L^p(\mathbb{R}^n)$, by the mean value theorem and change of order of integration, we see

$$\begin{aligned}
&\Big\| \int_0^{\tau t} \Delta e^{s\Delta}\Big(f\big(t - \frac{s}{\tau}\big) - f(t)\Big) ds \Big\|_{L^\theta(I;L^p)} \\
=&\Big\| \int_0^{\tau t} \Delta e^{s\Delta}\Big(\int_0^{s} \frac{\partial}{\partial r} f\big(t - \frac{r}{\tau}\big) dr\Big) ds \Big\|_{L^\theta(I;L^p)} \\
=&\Big\| \int_0^{\tau t} \Big(\int_r^{\tau t} \Delta e^{s\Delta} ds\Big) \frac{\partial}{\partial r} f\big(t - \frac{r}{\tau}\big) dr \Big\|_{L^\theta(I;L^p)} \\
=&\Big\| \int_0^{\tau t} e^{r\Delta} \frac{\partial}{\partial r} f\big(t - \frac{r}{\tau}\big) dr - e^{\tau t\Delta}\Big[f\big(t - \frac{r}{\tau}\big)\Big]_{r=0}^{\tau t} \Big\|_{L^\theta(I;L^p)} \\
\le&\Big\| \int_0^{\tau t} e^{r\Delta} \frac{\partial}{\partial r} f\big(t - \frac{r}{\tau}\big) dr \Big\|_{L^\theta(I;L^p)} + \big\| e^{\tau t\Delta} f(0) \big\|_{L^\theta(I;L^p)} + \big\| e^{\tau t\Delta} f(t) \big\|_{L^\theta(I;L^p)}.
\end{aligned}$$

□

Proof of Theorem 2.5 We first show Theorem 2.5 for the small data case: For the large data case, the proof is simply changed by $T < \infty$. Note that we restrict ourselves as $\theta = \sigma$. For the small initial data, one can obtain the a priori estimate for u_τ in

$BUC(\mathbb{R}_+; L^{\frac{n}{2}}) \cap L^2(\mathbb{R}_+; L^{\frac{n}{2}})$ and the bound is independent of $\tau > 0$ since it is determined by the initial data. Namely by (6.26),

$$\|u_\tau(t)\|_{L^\infty(\mathbb{R}_+; L^{\frac{n}{2}})} + \|\nabla\psi_\tau(t)\|_{L^\infty(\mathbb{R}_+; L^n)} \le C(\|u_0\|_{\frac{n}{2}} + \|\nabla\psi_0\|_n^2).$$

⟨Step 1⟩: Let $I = (0, \infty)$ and we consider the difference of solutions between (6.1) and the following:

$$\begin{cases} u(t) = e^{t\Delta}u_0 + \displaystyle\int_0^t e^{(t-s)\Delta}\nabla\cdot\big(u(s)\nabla\psi(s)\big)ds, \\ \psi(t) = \displaystyle\int_0^{\tau t} e^{s\Delta}u_\tau(\tau^{-1}s)ds + \int_{\tau t}^\infty e^{s\Delta}u(t)ds, \end{cases} \tag{7.1}$$

as

$$\begin{cases} u_\tau(t) - u(t) = \displaystyle\int_0^t e^{(t-s)\Delta}\nabla\cdot\Big(\big(u_\tau(s)-u(s)\big)\nabla\psi_\tau(s)\Big)ds \\ \qquad\qquad + \displaystyle\int_0^t e^{(t-s)\Delta}\nabla\cdot\Big(u(s)\big(\nabla\psi_\tau(s)-\nabla\psi(s)\big)\Big)ds, & t\in I, \\ \psi_\tau(t)-\psi(t) = e^{\tau t\Delta}\psi_0 + \displaystyle\int_0^{\tau t} e^{s\Delta}\big(u_\tau(t-\tau^{-1}s) - u(t-\tau^{-1}s)\big)ds \\ \qquad\qquad + \displaystyle\int_0^{\tau t} e^{s\Delta}\big(u(t-\tau^{-1}s)-u(t)\big)ds - \int_{\tau t}^\infty e^{s\Delta}u(t)ds, & t\in I. \end{cases} \tag{7.2}$$

Choose admissible exponents (θ, q), (σ, r) with $\theta = \sigma$ such as

$$\begin{cases} \dfrac{2}{\theta} + \dfrac{n}{q} = 2, \quad \dfrac{n}{2} < q \le \theta, \quad 2 \le \theta, \\ \dfrac{2}{\theta} + \dfrac{n}{r} = 1, \quad 2 \le n < r \le \sigma = \theta. \end{cases}$$

Let the time interval $I = (0, \infty)$. We apply the similar estimate in (6.24), it follows from Lemma 4.3 that

$$\begin{aligned} \|u_\tau - u\|_{L^\theta(I;L^q)} \le & C\|(u_\tau-u)\nabla\psi_\tau\|_{L^{\frac{\theta}{2}}(I;\dot B^{-1+\frac{2}{\theta}}_{q,\theta})} + C\big\|u(t)\nabla(\psi_\tau-\psi)\big\|_{L^{\frac{\theta}{2}}(I;\dot B^{-1+\frac{2}{\theta}}_{q,\theta})} \\ \le & C\big\|(u_\tau-u)\nabla\psi_\tau\big\|_{L^{\frac{\theta}{2}}(I;L^{\frac{rq}{r+q}})} + C\big\|u\nabla(\psi_\tau-\psi)\big\|_{L^{\frac{\theta}{2}}(I;L^{\frac{rq}{r+q}})} \\ \le & C\|u_\tau-u\|_{L^\theta(I;L^q)}\big\|\nabla\psi_\tau\big\|_{L^\theta(I;L^r)} + C\|u\|_{L^\theta(\mathbb{R}_+;L^q)}\big\|\nabla(\psi_\tau-\psi)\big\|_{L^\theta(I;L^r)} \\ \le & CM\Big(\|u_\tau-u\|_{L^\theta(I;L^q)} + \big\|\nabla\psi_\tau-\nabla\psi\big\|_{L^\theta(I;L^r)}\Big). \end{aligned} \tag{7.3}$$

Similarly for the difference for w_ψ,

$$
\begin{aligned}
\|\nabla\psi_\tau-\nabla\psi\|_{L^\theta(I;L^r)} =&\|\nabla e^{\tau t\Delta}\psi_0\|_{L^\theta(I;L^r)}\\
&+\left\|\int_0^{\tau t}\nabla e^{s\Delta}\big(u_\tau(t-\tau^{-1}s)-u(t-\tau^{-1}s)\big)ds\right\|_{L^\theta(I;L^r)}\\
&+\left\|\int_0^{\tau t}\nabla e^{s\Delta}\big(u(t-\tau^{-1}s)-u(t)\big)ds\right\|_{L^\theta(I;L^r)}\\
&+\left\|\int_{\tau t}^{\infty}\nabla e^{s\Delta}u(t)ds\right\|_{L^\theta(I;L^r)}\\
\equiv& I_0+I_1+I_2+I_3.
\end{aligned}
\tag{7.4}
$$

We see from $\theta<\infty$ that

$$
\begin{aligned}
I_0=&\left(\int_0^\infty\left\|e^{\tau t\Delta}\nabla\psi_0\right\|_{L^r}^\theta dt\right)^{1/\theta}\\
=&\tau^{-1/\theta}\left(\int_0^\infty\left\|e^{s\Delta}\nabla\psi_0\right\|_{L^r}^\theta ds\right)^{1/\theta}\\
&\to 0\quad\text{as}\quad\tau\to\infty.
\end{aligned}
\tag{7.5}
$$

Since (θ,r) is the admissible, i.e., $2\le n<r\le\theta$, we have from (2.6) and the generalized maximal regularity (5.4) in Theorem 5.1 that

$$
\begin{aligned}
I_1=&\left\|\int_0^{\tau t}\nabla e^{s\Delta}\big(u_\tau(t-\tau^{-1}s)-u(t-\tau^{-1}s)\big)ds\right\|_{L^\theta(I;L^r)}\\
=&\left\|\int_0^{\tau t}\nabla e^{(\tau t-s)\Delta}\big(u_\tau(\tau^{-1}s)-u(\tau^{-1}s)\big)ds\right\|_{L^\theta(I;L^r)}\\
\le& C\left\|u_\tau(\tau^{-1}s)-u(\tau^{-1}s)|_{s=\tau t}\right\|_{L^\theta(I;\dot B^{-1}_{r,\theta})}\\
\le& C\left\|u_\tau-u\right\|_{L^\theta(I;\dot W^{-1,r})}\\
\le& C\left\|u_\tau-u\right\|_{L^\theta(I;L^q)}
\end{aligned}
\tag{7.6}
$$

for all $\tau\ge 2$, where $C>0$ is independent of $\tau>0$ and

$$
L^q(\mathbb{R}^n)\subset\dot W^{-1.r}(\mathbb{R}^n),\qquad\frac{1}{q}=\frac{1}{r}+\frac{1}{n}.
$$

The third term of (7.4), we apply the Sobolev inequality and Lemma 7.1 to see that

$$
\begin{aligned}
I_2 \le & S_b \Big\| \int_0^{\tau t} \Delta e^{s\Delta}\Big(u\big(t-\frac{s}{\tau}\big)-u(t)\Big)ds\Big\|_{L^\theta(I;L^{\frac{nr}{n+r}})} \\
= & S_b \Big\| \int_0^{\tau t} \Delta e^{s\Delta}\Big(\int_0^s \frac{\partial}{\partial r}u\big(t-\frac{r}{\tau}\big)dr\Big)ds\Big\|_{L^\theta(I;L^q)} \\
\le & S_b \Big\| \int_0^{\tau t} e^{r\Delta}\frac{\partial}{\partial r}u\big(t-\frac{r}{\tau}\big)dr\Big\|_{L^\theta(I;L^q)} \\
& + S_b\big\|e^{\tau t\Delta}u_0\big\|_{L^\theta(I;L^q)} + S_b\big\|e^{\tau t\Delta}u(t)\big\|_{L^\theta(I;L^q)} \\
= & I_{2,1}+I_{2,2}+I_{2,3}.
\end{aligned}
\tag{7.7}
$$

For treating the first term of the right hand side of (7.7), we proceed by changing the variable $r = \tau s$

$$
\begin{aligned}
I_{2,1} = & \Big\| \int_0^t e^{\tau s\Delta}\frac{\partial}{\partial s}u\big(t-s\big)ds\Big\|_{L^\theta(I;L^q)} \\
\le & \Big\| \int_0^t e^{\tau s\Delta}\Delta u\big(t-s\big)ds\Big\|_{L^\theta(I;L^q)} + \Big\| \int_0^t e^{\tau s\Delta}\nabla\cdot\big(u(t-s)\nabla\psi(t-s)\big)ds\Big\|_{L^\theta(I;L^q)} \\
\equiv & J_1+J_2.
\end{aligned}
\tag{7.8}
$$

Then applying (5.4) in Theorem 5.1 and the remark after the statement to the equation;

$$
\begin{cases}
\partial_t v - \tau(\Delta v - \lambda v) = u, & t>0,\ x\in\mathbb{R}^n, \\
v(0,x)=0, & x\in\mathbb{R}^n,
\end{cases}
$$

we see by regarding $\nu \to \tau$, that

$$
J_1 = \Big\| \int_0^t e^{\tau(t-r)\Delta}\Delta u\big(r\big)dr\Big\|_{L^\theta(I;L^q)} \le C\tau^{-1}\|u\|_{L^\theta(I;L^q)}. \tag{7.9}
$$

Similarly analogous estimate (6.18), it follows that

$$
\begin{aligned}
J_2 = & \Big\| \int_0^t e^{\tau(t-r)\Delta}\nabla\cdot(u\nabla\psi)\big(r\big)dr\Big\|_{L^\theta(I;L^q)} \\
\le & C\tau^{-1}\|u\nabla\psi\|_{L^{\frac{\theta}{2}}(I;L^{\frac{rq}{r+q}})} \\
\le & C\tau^{-1}\|u\|_{L^\theta(I;L^q)}\|\nabla\psi\|_{L^{\frac{\theta}{2}}(I;L^r)}.
\end{aligned}
\tag{7.10}
$$

On the other hand, since u_0 and $u(t)\in L^q$ for almost everywhere,

$$
\lim_{\tau\to\infty}\|e^{\tau t\Delta}\{u_0-u(t)\}\|_q = 0
$$

and $e^{\tau t\Delta}$ is a bounded operator from L^q to itself. Since

$$\|e^{t\Delta}u_0\|_{L^\theta(I;L^q)} \le C\|u_0\|_{\dot{B}^{-\frac{2}{\theta}}_{q,\theta}} \le C\|u_0\|_{\frac{n}{2}},$$
$$\|e^{t\Delta}u(t)\|_{L^\theta(I;L^q)} \le C\|u\|_{L^\theta(I;L^q)},$$

we see for all $\tau > 1$ that

$$\|e^{\tau t\Delta}u_0\|_q^\theta \le C\|e^{t\Delta}u_0\|_q^\theta \in L^1(I),$$
$$\|e^{\tau t\Delta}u(t)\|_q^\theta \le C\|e^{t\Delta}u(t)\|_q^\theta \in L^1(I).$$

Hence by the Lebesgue dominated convergence theorem, for any $\varepsilon > 0$, we may choose sufficiently large $\tau > 0$ such that

$$I_{2,2} + I_{2,3} < \varepsilon \tag{7.11}$$

Combining the estimates (7.7)–(7.10) and (7.11), we obtain that for any $\varepsilon > 0$, there exists a large $\tau > 0$ such that

$$I_2 \le \varepsilon. \tag{7.12}$$

Lastly for the forth term, setting $I = (\eta_0\tau^{-1}, \infty)$, employing the Sobolev embedding:

$$\|f\|_r \le S_b\|\nabla f\|_{\frac{nr}{n+r}}, \quad r < \infty,$$

we see

$$\begin{aligned} I_3 &= \left\|\nabla(-\Delta)^{-1}e^{\tau t\Delta}\, u(t)\right\|_{L^\theta(I;L^r)} \\ &\le S_b\left\|\nabla^2(-\Delta)^{-1}e^{\tau t\Delta}\, u(t)\right\|_{L^\theta(I;L^{\frac{nr}{r+n}})} \\ &= S_b\left(\int_0^\infty \left\|e^{\tau t\Delta}\, u(t)\right\|_{\frac{nr}{r+n}}^\theta dt\right)^{1/\theta} \\ &= S_b\left(\int_0^\infty \left\|u(s)\right\|_{\frac{nr}{r+n}}^\theta ds\right)^{1/\theta}. \end{aligned} \tag{7.13}$$

For the admissible (θ, q), the limiting solution u is integrable in $L^\theta(0,\infty; L^q)$ and especially, by

$$\frac{2}{\theta} + \frac{n(r+n)}{rn} = \frac{2}{\sigma} + \frac{n}{r} + 1 = 2,$$

we find that $(\sigma, \frac{rn}{r+n}) = (\theta, q)$ and is also the admissible exponent for u and

$$\left(\int_0^\infty \|u(s)\|_{\frac{rn}{r+n}}^\sigma ds\right)^{1/\sigma} = \left(\int_0^\infty \|u(s)\|_q^\theta ds\right)^{1/\theta} < \infty. \tag{7.14}$$

Hence from the fourth line of (7.13), the integrant

$$\left\| e^{\tau t\Delta} u(t) \right\|_{\frac{nr}{r+n}}^{\theta}$$

is $L^1(\mathbb{R}_+)$ and it is dominated by the integrable function $\left\| u(s) \right\|_{\frac{nr}{r+n}}^{\theta}$ as

$$\left\| e^{\tau t\Delta} u(t) \right\|_{\frac{nr}{r+n}}^{\theta} \leq C \left\| u(t) \right\|_{\frac{nr}{r+n}}^{\theta}.$$

Besides for almost all $t > 0$,

$$\lim_{\tau \to \infty} \| e^{\tau t\Delta} u(t) \|_{\frac{nr}{r+n}}^{\theta} = 0. \tag{7.15}$$

Applying the Lebesgue dominated convergence theorem, it follows from (7.13), (7.14) and (7.15) that for any $\varepsilon > 0$

$$\begin{aligned} I_3 =& \left\| \int_{\tau t}^{\infty} \nabla e^{s\Delta} ds\, u(t) \right\|_{L^\theta(I;L^r)} \\ \leq& S_b \left(\int_0^{\infty} \left\| e^{\tau t\Delta} u(t) \right\|_{\frac{nr}{r+n}}^{\theta} dt \right)^{1/\theta} < \varepsilon, \end{aligned} \tag{7.16}$$

as $\tau \to \infty$. Combining all the estimates (7.4), (7.5), (7.6), (7.12) and (7.16), we obtain

$$\|\nabla \psi_\tau(t) - \nabla \psi(t)\|_{L^\theta(I;L^r)} \leq C \|u_\tau - u\|_{L^\theta(I;L^q)} + \varepsilon. \tag{7.17}$$

Gathering (7.3) and (7.17), we see that for any $\varepsilon > 0$, choosing τ sufficiently large such that

$$\|u_\tau - u\|_{L^\theta(I;L^q)} \leq CM \Big(\|u_\tau - u\|_{L^\theta(I;L^q)} + \varepsilon \Big). \tag{7.18}$$

In particular, from (7.17) and (7.18), for small M, we see by choosing $\tau > 0$ small that

$$\|u_\tau - u\|_{L^\theta(I;L^q)} + \|\nabla \psi_\tau - \nabla \psi\|_{L^\theta(I;L^r)} \leq \varepsilon. \tag{7.19}$$

⟨Step 2⟩: From maximal regularity (5.4) in Theorem 5.1, we estimate the first component of the integral equation (7.2) as follows:

$$
\begin{aligned}
&\|u_\tau-u\|_{L^\infty(I;L^{\frac{n}{2}})}\\
&\le\Big\|\int_0^t\nabla e^{(t-s)\Delta}\Big(u_\tau(s)\nabla\psi_\tau(s)-u(s)\nabla\psi(s)\Big)ds\Big\|_{L^\infty(I;L^{\frac{n}{2}})}\\
&\le C\big\|u_\tau(s)\nabla\psi_\tau(s)-u(s)\nabla\psi(s)\big\|_{L^{\frac{\theta}{2}}(I;\dot B^{-1+\frac{4}{\theta}}_{\frac{n}{2},\infty})}\\
&\le C\Big(\big\|\big(u_\tau(s)-u(s)\big)\nabla\psi_\tau(s)\big\|_{L^{\frac{\theta}{2}}(I;\dot W^{-1+\frac{4}{\theta},\frac{n}{2}})}+\big\|u(s)\big(\nabla\psi_\tau(s)-\nabla\psi(s)\big)\big\|_{L^{\frac{\theta}{2}}(I;\dot W^{-1+\frac{4}{\theta},\frac{n}{2}})}\Big)\\
&\le C\Big(\big\|\big(u_\tau(s)-u(s)\big)\nabla\psi_\tau(s)\big\|_{L^{\frac{\theta}{2}}(I;L^{\frac{rq}{r+q}})}+\big\|u(s)\big(\nabla\psi_\tau(s)-\nabla\psi(s)\big)\big\|_{L^{\frac{\theta}{2}}(I;L^{\frac{rq}{r+q}})}\Big)\\
&\le C\Big(\|u_\tau-u\|_{L^\theta(I;L^q)}\|\nabla\psi_\tau\|_{L^\theta(I;L^r)}+\|\nabla\psi_\tau-\nabla\psi\|_{L^\theta(I;L^r)}\|u\|_{L^\theta(I;L^q)}\Big),
\end{aligned}
\tag{7.20}
$$

where

$$
\|f\|_{\dot W^{-1+\frac{4}{\theta},\frac{n}{2}}}\le C\|f\|_{\frac{rq}{r+q}},\qquad \frac{2}{n}-\frac{1}{n}\left(-1+\frac{4}{\theta}\right)=\frac{1}{r}+\frac{1}{q}.
$$

For any $t_0>0$ set $I_{t_0}=I\cap(t_0,\infty)$ with $I=(0,T)$. Let $\eta_\tau(t)\equiv\chi_{[0,t_1\tau^{-1}]}(t)(e^{\tau t\Delta}\psi_0-(-\Delta+\lambda)^{-1}u_0)$. From (7.2),

$$
\psi_\tau(t)-\psi(t)-\eta_\tau(t)=
\begin{cases}
e^{\tau t\Delta}\psi_0+\displaystyle\int_0^{\tau t}e^{s\Delta}\big(u_\tau(t-\tau^{-1}s)-u(t-\tau^{-1}s)\big)ds\\
\quad+\displaystyle\int_0^{\tau t}e^{s\Delta}\big(u(t-\tau^{-1}s)-u(t)\big)ds-\int_{\tau t}^\infty e^{s\Delta}u(t)ds, & t\in I_{t_0},\\
e^{\tau t\Delta}\psi_0-\psi_0+(-\Delta+\lambda)^{-1}u_0-\displaystyle\int_0^\infty e^{s\Delta}u(t)ds\\
\quad+\displaystyle\int_0^{\tau t}e^{(\tau t-s)\Delta}u_\tau(\tau^{-1}s)ds, & t\in(0,\tau^{-1}t_1).
\end{cases}
\tag{7.21}
$$

For any $\varepsilon>0$, choose $t_0>0$ small enough such that applying the similar estimate from (7.5), (7.6), (7.12), (7.13) and (7.16) we see for large $\tau>0$ that

$$
\begin{aligned}
&\|\nabla\psi_\tau-\nabla\psi-\nabla\eta_\tau\|_{L^\infty(I_{t_0};L^n)}=\|\nabla\psi_\tau-\nabla\psi\|_{L^\infty(I_{t_0};L^n)}\\
&\le\big\|\nabla e^{\tau t\Delta}\psi_0\big\|_{L^\infty(I_{t_0};L^n)}+\Big\|\nabla\int_0^{\tau t}e^{s\Delta}\big(u_\tau(t-\tau^{-1}s)-u(t-\tau^{-1}s)\big)ds\Big\|_{L^\infty(I_{t_0};L^n)}\\
&\qquad+\Big\|\nabla\int_0^{\tau t}e^{s\Delta}\big(u(t-\tau^{-1}s)-u(t)\big)ds\Big\|_{L^\infty(I_{t_0};L^n)}+\Big\|\nabla\int_{\tau t}^\infty e^{s\Delta}u(t)ds\Big\|_{L^\infty(I_{t_0};L^n)}\\
&\le\big\|\nabla e^{\tau t_0\Delta}\psi_0\big\|_n+C\Big\|u_\tau(t-\tau^{-1}t)-u(t-\tau^{-1}t)\Big\|_{L^\theta(I_{t_0};\dot B^{-1+\frac{2}{\theta}}_{r,\infty})}\\
&\quad+\Big\|u\big(t-\tau^{-1}t\big)-u(t)\Big\|_{L^\theta(I_{t_0};L^q)}+S_b\big\|e^{\tau t_0\Delta}u(t)\big\|_{L^\theta(I_{t_0};L^{\frac{nr}{r+n}})}\\
&\le\big\|\nabla e^{\tau t_0\Delta}\psi_0\big\|_n+C\big\|u_\tau-u\big\|_{L^\theta(I_{t_0};L^q)}+2\varepsilon\le4\varepsilon.
\end{aligned}
\tag{7.22}
$$

Hence $\nabla\psi_\tau(t)$ converges to $\nabla\psi(t)$ locally uniformly in $L^n(\mathbb{R}^n)$ over I. On the other hand, from the second expression for $\psi_\tau-\psi-\eta_\tau$ in (7.21), we choose $t_1>0$ small enough so that

$$
\begin{aligned}
&\|\nabla\psi_\tau - \nabla\psi - \nabla\eta_\tau\|_{L^\infty(0,\tau^{-1}t_1;L^n)} \\
&\le \big\|\nabla e^{\tau t\Delta}\psi_0 - \nabla\psi_0\big\|_{L^\infty(0,\tau^{-1}t_1;L^n)} + \left\|\nabla(-\Delta+\lambda)^{-1}u_0 - \nabla\int_0^\infty e^{s\Delta}u(t)ds\right\|_{L^\infty(0,\tau^{-1}t_1;L^n)} \\
&\qquad + \left\|\nabla\int_0^{\tau t} e^{(\tau t-s)\Delta}u_\tau(\tau^{-1}s)ds\right\|_{L^\infty(0,\tau^{-1}t_1;L^n)} \\
&\le \sup_{t\le[0,\tau^{-1}t_1]}\big\|(e^{\tau t\Delta}-I)\nabla\psi_0\big\|_n + C\left\|\nabla(-\Delta+\lambda)^{-1}u_0 - \nabla(-\Delta+\lambda)^{-1}u(t)\right\|_{L^\infty(0,\tau^{-1}t_1;L^n)} \\
&\quad + C\left\|u_\tau(t-\tau^{-1}s)|_{s=\tau t}\right\|_{L^\theta(0,\tau^{-1}t_1;\dot B^{-1+\frac{2}{\theta}}_{n,\infty})} \\
&\le \big\|(e^{t_1\Delta}-I)\nabla\psi_0\big\|_n + C\big\|u(t)-u_0\big\|_{L^\infty(0,\tau^{-1}t_1;L^{\frac{n}{2}})} + C\big\|u_\tau(t)\big\|_{L^\theta(0,\tau^{-1}t_1;L^q)} \\
&\le 3\varepsilon,
\end{aligned}
\tag{7.23}
$$

because of the strong continuity of solution $u_\tau(t)$ in $L^{\frac{n}{2}}(\mathbb{R}^n)$ and uniform bound for $u_\tau \in L^\theta(I; L^q)$.

Hence by passing $\tau \to \infty$ in (7.20), (7.22) and (7.23) we conclude from (7.19) that the convergence (2.4) and (2.5) hold. This completes the proof. □

8 Proof for the Well-Posedness of Chaplain-Anderson and Fujie-Senba System

The proof for the well-posedness of the simplified Chaplain-Anderson system (1.9) is very much similar to the case of the Keller-Segel system (1.1). The only minor difference is how to treat the second component ϕ_τ.

Proof of Theorem 3.1 ⟨Step 1⟩ (The local wellposedness): Let $\tau > 0$. We show the local in time existence and well-posedness of the solutions for the large initial data $(u_0, \phi_0, \psi_0) \in \big(L^1(\mathbb{R}^4) \cap \dot B^0_{1,\theta}(\mathbb{R}^4)\big) \times \big(L^2(\mathbb{R}^4) \cap \dot B^{1-\frac{2}{\rho}}_{1,\rho}(\mathbb{R}^4)\big) \times \big(VMO(\mathbb{R}^4) \cap \dot B^{1-\frac{2}{\sigma}}_{r,\sigma}(\mathbb{R}^4)\big)$. Let (θ, p), (ρ, q) and (σ, r) satisfy admissible conditions:

$$
\begin{cases}
\dfrac{1}{2\theta} + \dfrac{1}{p} = 1, \quad 1 < p \le \theta, \quad 2 < \theta, \\
\dfrac{1}{2\rho} + \dfrac{1}{q} = \dfrac{1}{2}, \quad 2 < q \le \rho, \quad \theta \le \rho, \\
\dfrac{1}{2\sigma} + \dfrac{1}{r} = \dfrac{1}{4}, \quad 4 < r \le \sigma, \quad \max\left(\rho, \dfrac{pq}{p+q}\right) < \sigma, \qquad \dfrac{1}{2} - \dfrac{1}{\theta} + 1 - \dfrac{1}{\theta} = \dfrac{1}{r} + \dfrac{1}{q} > \dfrac{1}{\theta}
\end{cases}
\tag{8.1}
$$

and set $I = (0, T)$ for some $0 < T < \infty$ chosen later and let

$$
X_T = \left\{
\begin{aligned}
&u \in L^\theta(I; L^q(\mathbb{R}^4)), \\
&\phi \in C(I; L^2(\mathbb{R}^4)) \cap L^\rho(I; \dot W^{1,q}(\mathbb{R}^4)) \\
&\psi \in C(I; BMO(\mathbb{R}^2)) \cap L^\sigma(I; \dot W^{1,r}(\mathbb{R}^4)); \\
&\qquad \|\phi\|_{L^\infty(I;L^2)} + \|\psi\|_{L^\infty(I;VMO)} \le M, \\
&\qquad \max\big(\|u\|_{L^\theta(I;L^p(\mathbb{R}^4))}, \|\phi\|_{L^\rho(I;L^q(\mathbb{R}^4))}, \|\nabla\psi\|_{L^\sigma(I;L^r(\mathbb{R}^4))}\big) \le N
\end{aligned}
\right\},
$$

where

$$M = 4C_0\big(\|u_0\|_1 + \|\phi_0\|_2 + \|\psi_0\|_{BMO}^2\big)$$

and $N > 0$ will be determined later. Introducing the metric on X_M by

$$|||(u, \phi, \psi) - (\tilde{u}, \tilde{\phi}, \tilde{\psi})|||_T \equiv \|u - \tilde{u}\|_{L^\theta(I;L^p)} + \|\phi - \tilde{\phi}\|_{L^\rho(I;L^q)} + \|\nabla(\psi - \tilde{\psi})\|_{L^\sigma(I;L^r)},$$

we can show that X_M is a complete metric space. Indeed, since

$$C([0, T]; VMO(\mathbb{R}^4)) \subset L^\infty(0, T; BMO(\mathbb{R}^4)),$$

the pre-dual of $L^\infty(0, T; BMO(\mathbb{R}^4))$ coincides $L^1(0, T; \mathcal{H}^1(\mathbb{R}^4))$, where $\mathcal{H}^1$ denotes the Hardy space with absolute integrable, which is separable. Hence the Banach-Alaoglu theorem implies that the weak-$*$ sequence compactness holds in $L^\infty(0, T; BMO(\mathbb{R}^4))$ (cf. Brezis [8, Cor. III.26.]). By this fact, any Cauchy sequence converges a limit (u, ϕ, ψ) in $L^\theta(I; L^p(\mathbb{R}^4)) \times L^\rho(I; L^q(\mathbb{R}^4)) \times L^\sigma(I; \dot{W}^{1,r}(\mathbb{R}^4))$ and weak-$*$ compactness ensures that the limit indeed belongs to X_T.

We then introduce a pair of solution map $\big(\Xi[u_\tau, \psi_\tau], \Phi[u_\tau, \psi_\tau], \Psi[u_\tau, \psi_\tau]\big)$ as follows: For $(u_0, \phi_0, \psi_\tau) \in \big(L^1(\mathbb{R}^4) \cap \dot{B}^{-\frac{2}{\theta}}_{p,\theta}(\mathbb{R}^4)\big) \times \big(L^2(\mathbb{R}^4) \cap \dot{B}^{-\frac{2}{\rho}}_{q,\rho}(\mathbb{R}^4)\big) \times \big(BMO(\mathbb{R}^4) \cap \dot{B}^{1-\frac{2}{\sigma}}_{r,\sigma}(\mathbb{R}^4)\big)$ and $(u_\tau, \phi_\tau, \psi_\tau) \in X_M$, let

$$\begin{cases} \Xi[u_\tau, \psi_\tau](t) \equiv e^{t\Delta}u_0 + \displaystyle\int_0^t e^{(t-s)\Delta}\nabla\cdot\big(u_\tau(s)\nabla\psi_\tau(s)\big)ds, \\ \Phi[u_\tau, \psi_\tau](t) \equiv e^{\tau t\Delta}\phi_0 + \displaystyle\int_0^{\tau t} e^{(\tau t-s)\Delta}\Xi[u_\tau, \psi_\tau](\tau^{-1}s)ds, \\ \Psi[u_\tau, \psi_\tau](t) \equiv e^{\tau t\Delta}\psi_0 + \displaystyle\int_0^{\tau t} e^{(\tau t-s)\Delta}\Phi[u_\tau, \psi_\tau](\tau^{-1}s)ds \end{cases} \tag{8.2}$$

and claim that the map $\big(\Xi, \Psi, \Phi\big)$ is contraction in the critical space X_T.

First we claim that (Ξ, Ψ, Φ) is onto X_T. By maximal regularity (5.3) in Theorem 5.1 with $s = -2$, $\sigma = 1$ to see by the Littlewood-Paley theorem that

$$\begin{aligned} \|e^{t\Delta}u_0\|_{L^\theta(I;L^p)} \le& C\|e^{t\Delta}u_0\|_{L^\theta(I;\dot{F}^0_{p,2})} \\ \le& C\|u_0\|_{\dot{B}^{-2+\frac{2}{\theta'}}_{p,\theta}} \le C\|u_0\|_{\dot{B}^{-\frac{2}{\theta}}_{p,\theta}}, \end{aligned} \tag{8.3}$$

$$\|e^{\tau t\Delta}\phi_0\|_{L^\rho(I;L^q)} \le \|e^{\tau t\Delta}\phi_0\|_{L^\rho(I;\dot{B}^0_{q,2})} \le C\|\phi_0\|_{\dot{B}^{-\frac{2}{\rho}}_{q,\rho}} \tag{8.4}$$

and

$$\|\nabla e^{\tau t\Delta}\psi_0\|_{L^\sigma(I;L^r)} \le \|\nabla e^{\tau t\Delta}\psi_0\|_{L^\sigma(I;\dot{B}^0_{r,2})} \le C\|\psi_0\|_{\dot{B}^{1-\frac{2}{\sigma}}_{r,\sigma}}. \tag{8.5}$$

Hence from (8.3)–(8.5), we can choose the time interval $|I| \le T$ sufficiently small such that for some small $\varepsilon_0 > 0$,[3]

$$\begin{aligned} &\|e^{t\Delta}u_0\|_{L^\theta(I;L^p)} \le \varepsilon_0, \\ &\|e^{\tau t\Delta}\phi_0\|_{L^\rho(I;L^q)} \le \varepsilon_0, \\ &\|\nabla e^{\tau t\Delta}\psi_0\|_{L^\sigma(I;L^r)} \le \varepsilon_0 \end{aligned} \tag{8.6}$$

for any $\tau > 1$ and the choice of T is independent of $\tau > 1$. Since (θ, p), (ρ, q) and (σ, r) are admissible pairs, it follows from (8.1) that

$$\frac{1}{2\theta} + \frac{1}{p} = 1, \qquad \frac{1}{2\sigma} + \frac{1}{r} = \frac{1}{4},$$

we apply (8.6), maximal regularity (5.4), the bound for the initial data (8.3) and the embedding (4.8) in Lemma 4.3 to have

$$\begin{aligned} \big\|\Xi[u_\tau, \psi_\tau]\big\|_{L^\theta(I;L^p)} \le& \|e^{t\Delta}u_0\|_{L^\theta(I;L^p)} + \Big\|\int_0^t e^{(t-s)\Delta}\nabla\cdot\big(u_\tau(s)\nabla\psi_\tau(s)\big)ds\Big\|_{L^\theta(I;\dot B^0_{p,1})} \\ \le& \varepsilon_0 + C\Big\|u_\tau(s)\nabla\psi_\tau(s)\Big\|_{L^{\frac{\theta\sigma}{\theta+\sigma}}(I;\dot B^{-1+\frac{2}{\sigma}}_{p,\theta})} \\ \le& \varepsilon_0 + C\Big\|u_\tau(s)\nabla\psi_\tau(s)\Big\|_{L^{\frac{\theta\sigma}{\theta+\sigma}}(I;\dot F^{-1+\frac{2}{\sigma}}_{p,\theta})} \\ &\left(\text{since } L^{\frac{pr}{p+r}} \simeq \dot F^0_{\frac{pr}{r+p},2} \subset \dot F^{-1+\frac{2}{\sigma}}_{p,2} \subset \dot F^{-1+\frac{2}{\sigma}}_{p,\theta} \text{ by } \frac{1}{p} - \frac{1}{4}\left(-1+\frac{2}{\sigma}\right) = \frac{1}{p}+\frac{1}{r}\right) \\ \le& \varepsilon_0 + C\Big\|u_\tau(s)\nabla\psi_\tau(s)\Big\|_{L^{\frac{\theta\sigma}{\theta+\sigma}}(I;L^{\frac{pr}{p+r}})} \\ \le& \varepsilon_0 + C\big\|u_\tau(\cdot)\big\|_{L^\theta(I;L^p)}\big\|\nabla\psi_\tau(\cdot)\big\|_{L^\sigma(I;L^r)} \\ \le& \varepsilon_0 + CN^2 \le N \end{aligned} \tag{8.7}$$

for any $2 < \theta \le 4$ and $1 < q < 2$, and for some small ε_0, $CN^2 \le \frac{1}{2}N$.

From (8.1) in particular, $1/2\theta + 1/p = 1$, $1/2\rho + 1/q = 1/2$, we see that

$$\frac{1}{q} - \frac{1}{4}\left(-2 + \frac{2}{\theta} - \frac{2}{\rho}\right) = \frac{1}{p}$$

and from (5.4) in Theorem 5.1 and (6.6), it follows

[3]The choice of T is independent of $\tau > 1$.

$$\begin{aligned}\big\|\Phi[u_\tau,\psi_\tau]\big\|_{L^\rho(I;L^q)} \le&\|e^{\tau t\Delta}\psi_0\|_{L^\rho(I;L^q)} + \Big\|\int_0^{\tau t} e^{(\tau t-s)\Delta}\,\Xi[u_\tau,\psi_\tau](\tau^{-1}s)ds\Big\|_{L^\rho(I;L^q)}\\
\le&\varepsilon_0 + C\Big\|\,\Xi[u_\tau,\psi_\tau](\tau^{-1}s)|_{s=\tau t}\Big\|_{L^\theta(I;\dot B_{q,\rho}^{-2+\frac{2}{\theta}-\frac{2}{\rho}})}\\
\le&\varepsilon_0 + C\Big\|\,\Xi[u_\tau,\psi_\tau](\tau^{-1}s)|_{s=\tau t}\Big\|_{L^\theta(I;\dot F_{q,2}^{-2+\frac{2}{\theta}-\frac{2}{\rho}})}\\
\le&\varepsilon_0 + C\big\|\,\Xi[u_\tau,\psi_\tau](t)\big\|_{L^\theta(I;\dot W^{-2+\frac{2}{\theta}-\frac{2}{\rho},q})}\\
\le&\varepsilon_0 + C\big\|\,\Xi[u_\tau,\psi_\tau](\cdot)\big\|_{L^\theta(I;L^p)}\\
\le&\varepsilon_0 + C\big(\varepsilon_0 + CN^2\big) \le N.
\end{aligned} \tag{8.8}$$

Similarly by

$$\frac{2}{\rho}+\frac{4}{q}=2, \qquad \frac{2}{\sigma}+\frac{4}{r}=1,$$

we see that

$$\frac{1}{r}-\frac{1}{4}\left(-1+\frac{2}{\rho}-\frac{2}{\sigma}\right)=\frac{1}{q}$$

and it follows that

$$\begin{aligned}\big\|\nabla\Psi[u_\tau,\psi_\tau]\big\|_{L^\sigma(I;L^r)} \le&\|e^{\tau t\Delta}\nabla\psi_0\|_{L^\sigma(I;L^r)} + \Big\|\nabla\int_0^{\tau t} e^{(\tau t-s)\Delta}\Phi[u_\tau,\psi_\tau](\tau^{-1}s)ds\Big\|_{L^\sigma(I;L^r)}\\
\le&\varepsilon_0 + C\Big\|\Phi[u_\tau,\psi_\tau](\tau^{-1}s)|_{s=\tau t}\Big\|_{L^\rho(I;\dot B_{r,\theta}^{-1+\frac{2}{\rho}-\frac{2}{\sigma}})}\\
\le&\varepsilon_0 + C\Big\|\Phi[u_\tau,\psi_\tau](\tau^{-1}s)|_{s=\tau t}\Big\|_{L^\rho(I;\dot F_{r,2}^{-1+\frac{2}{\rho}-\frac{2}{\sigma}})}\\
\le&\varepsilon_0 + C\big\|\Phi[u_\tau,\psi_\tau](t)\big\|_{L^\rho(I;\dot W^{-1+\frac{2}{\rho}-\frac{2}{\sigma},r})}\\
\le&\varepsilon_0 + C\big\|\Phi[u_\tau,\psi_\tau](\cdot)\big\|_{L^\rho(I;L^q)}\\
\le&\varepsilon_0 + ((1+C)\varepsilon_0 + CN^2) \le N
\end{aligned} \tag{8.9}$$

under the conditions $(2+C)\varepsilon_0 < \frac{1}{2}N$ and $CN \le \frac{1}{2}$.

From (8.9), and choosing $N > 0$ small enough so that for some small $\varepsilon_0 > 0$,

$$\max\Big(\big\|\,\Xi[u_\tau,\psi_\tau]\big\|_{L^\theta(I;L^p)},\ \big\|\Phi[u_\tau,\psi_\tau]\big\|_{L^\rho(I;L^q)},\ \big\|\nabla\Psi[u_\tau,\psi_\tau]\big\|_{L^\sigma(I;L^r)}\Big) \le N. \tag{8.10}$$

On the other hand, since the heat kernel is a bounded operator in $VMO(\mathbb{R}^2)$, we again use generalized maximal regularity (5.4) and the embeddings $\dot{B}^0_{\infty,2}(\mathbb{R}^4) \subset BMO(\mathbb{R}^4)$ and $\dot{B}^{-3+\frac{2}{\rho}}_{\infty,2}(\mathbb{R}^4) \subset \dot{B}^0_{q,2}(\mathbb{R}^4)$ to see that

$$\begin{aligned}
\big\|\Psi[u_\tau,\psi_\tau]\big\|_{L^\infty(I;VMO)} \le& \|e^{\tau t\Delta}\psi_0\|_{L^\infty(I;VMO)} + \Big\|\int_0^{\tau t} e^{(\tau t-s)\Delta}\Phi[u_\tau,\psi_\tau](\tau^{-1}s)ds\Big\|_{L^\infty(I;VMO)}\\
\le& C_0\|\psi_0\|_{VMO} + \Big\|\int_0^{\tau t} e^{(\tau t-s)\Delta}\Phi[u_\tau,\psi_\tau](\tau^{-1}s)ds\Big\|_{L^\infty(I;\dot{F}^0_{\infty,2})}\\
\le& \frac{1}{4}M + C\Big\|\Phi[u_\tau,\psi_\tau](\tau^{-1}s)|_{s=\tau t}\Big\|_{L^\rho(I;\dot{F}^{-2+\frac{2}{\rho}}_{\infty,2})}\\
\le& \frac{1}{4}M + C\Big\|\Phi[u_\tau,\psi_\tau](\tau^{-1}s)|_{s=\tau t}\Big\|_{L^\rho(I;\dot{F}^0_{q,2})}\\
&(\text{since} q<2,\ L^q(\mathbb{R}^4)\simeq \dot{F}^0_{q,2}(\mathbb{R}^4)\subset \dot{F}^{-2+\frac{2}{\rho}}_{\infty,2}(\mathbb{R}^4))\\
\le& \frac{1}{8}M + C\Big\|\Phi[u_\tau,\psi_\tau](t)\Big\|_{L^\rho(I;L^q)} \le \frac{1}{4}M
\end{aligned} \tag{8.11}$$

under the assumption

$$CN \le \frac{1}{8}M$$

and (8.10).

Combining (8.10) and (8.11), we obtain $\big(\Xi[u_\tau,\psi_\tau],\Psi[u_\tau,\psi_\tau],\Phi[u_\tau,\psi_\tau]\big) \in X_M$. Analogously from (6.15), we have for the difference of solutions that

$$\begin{aligned}
&\Big\|\Xi[u_\tau,\psi_\tau]-\Xi[\tilde{u}_\tau,\tilde{\psi}_\tau]\Big\|_{L^\theta(I;L^p)}\\
&\le\Big\|\int_0^t e^{(t-s)\Delta}\nabla\cdot\big(u_\tau(s)\nabla\psi_\tau(s)-\tilde{u}_\tau(s)\nabla\tilde{\psi}_\tau(s)\big)ds\Big\|_{L^\theta(I;\dot{B}^0_{p,1})}\\
&\le C\Big\|u_\tau(s)\nabla\psi_\tau(s)-\tilde{u}_\tau(s)\nabla\tilde{\psi}_\tau(s)\Big\|_{L^{\frac{\theta\sigma}{\theta+\sigma}}(I;\dot{B}^{-1+\frac{2}{\sigma}}_{q,\theta})}\\
&\le C\Big\|u_\tau(s)\nabla(\psi_\tau(s)-\tilde{\psi}_\tau(s))\Big\|_{L^{\frac{\theta\sigma}{\theta+\sigma}}(I;L^{\frac{rq}{r+q}})}\\
&\qquad + C\Big\|\big(u_\tau(s)-\tilde{u}_\tau(s)\big)\nabla\tilde{\psi}_\tau(s)\Big\|_{L^{\frac{\theta\sigma}{\theta+\sigma}}(I;L^{\frac{rq}{r+q}})}\\
&\le C\big\|u_\tau(s)\big\|_{L^\theta(I;L^p)}\big\|\nabla(\psi_\tau(s)-\tilde{\psi}_\tau(s))\big\|_{L^\sigma(I;L^r)}\\
&\quad + C\big\|u_\tau(s)-\tilde{u}_\tau(s)\|_{L^\theta(I;L^p)}\big\|\nabla\tilde{\psi}_\tau(s)\big\|_{L^\sigma(I;L^r)}\\
&\le C_1N\|(u_\tau,\psi_\tau)-(\tilde{u}_\tau,\tilde{\psi}_\tau)\|_T,
\end{aligned} \tag{8.12}$$

$$
\begin{aligned}
\big\|\Phi[u_\tau,\psi_\tau]-&\Phi[\tilde u_\tau,\tilde\psi_\tau]\big\|_{L^\rho(I;L^q)}\\
\le&\Big\|\int_0^t e^{(t-s)\Delta}\,\Xi[u_\tau(s),\psi_\tau(s)]-\Xi[\tilde u_\tau(s),\tilde\psi_\tau(s)]\big)ds\Big\|_{L^\rho(I;\dot B^0_{q,1})}\\
\le&C\Big\|\Xi[u_\tau(s),\psi_\tau(s)]-\Xi[\tilde u_\tau(s),\tilde\psi_\tau(s)]\Big\|_{L^\theta(I;\dot B_{q,\theta}^{-2+\frac{2}{\theta}-\frac{2}{\rho}})}\\
\le&C\Big\|\Xi[u_\tau(s),\psi_\tau(s)]-\Xi[\tilde u_\tau(s),\tilde\psi_\tau(s)]\Big\|_{L^\theta(I;L^p)}\\
\le&CC_1N|||(u_\tau,\psi_\tau)-(\tilde u_\tau,\tilde\psi_\tau)|||_T.
\end{aligned}
\tag{8.13}
$$

Again from (8.7) and (8.13), we have

$$
\begin{aligned}
\Big\|\nabla\Big(\Psi[u_\tau,\psi_\tau]-&\Psi[\tilde u_\tau,\tilde\psi_\tau]\Big)\Big\|_{L^\sigma(I;L^r)}\\
\le&\Big\|\int_0^{\tau t}\nabla e^{(\tau t-s)\Delta}\big(\Phi[u_\tau,\psi_\tau](\tau^{-1}s)-\Phi[\tilde u_\tau,\tilde\psi_\tau](\tau^{-1}s)\big)ds\Big\|_{L^\sigma(I;L^r)}\\
\le&C\Big\|\Phi[u_\tau,\psi_\tau](\tau^{-1}s)|_{s=\tau t}-\Phi[\tilde u_\tau,\tilde\psi_\tau](\tau^{-1}s)|_{s=\tau t}\Big\|_{L^\rho(I;\dot F_{r,2}^{-1+\frac{2}{\rho}-\frac{2}{\sigma}})}\\
\le&C\big\|\nabla\Phi[u_\tau,\psi_\tau](t)-\nabla\Phi[\tilde u_\tau,\tilde\psi_\tau](t)\big\|_{L^\rho(I;\dot W^{-2+\frac{2}{\rho}-\frac{2}{\sigma},r})}\\
\le&C\big\|\nabla\Phi[u_\tau,\psi_\tau](t)-\nabla\Phi[\tilde u_\tau,\tilde\psi_\tau](t)\big\|_{L^\rho(I;L^q)}\\
\le&C^2C_1N|||(u_\tau,\psi_\tau)-(\tilde u_\tau,\tilde\psi_\tau)|||_T.
\end{aligned}
\tag{8.14}
$$

Choosing ep_0 and hence N small enough

$$
(1+C+C^2)C_1N\le\frac{1}{2},
\tag{8.15}
$$

if necessary, (8.12) and (8.13) with (8.15) yield that

$$
\begin{aligned}
&|||(\Xi[u_\tau,\psi_\tau],\Phi[u_\tau,\psi_\tau],\Psi[u_\tau,\psi_\tau])-(\Xi[\tilde u_\tau,\tilde\psi_\tau],\Phi[\tilde u_\tau,\tilde\psi_\tau],\Psi[\tilde u_\tau,\tilde\psi_\tau])|||_T\\
&\le\frac{1}{2}|||(u_\tau,\phi_\tau,\psi_\tau)-(\tilde u_\tau,\tilde\phi_\tau,\tilde\psi_\tau)|||_T
\end{aligned}
$$

under the smallness assumption (8.6) on the interval. Thus the map (Ξ,Φ,Ψ) is contraction onto X_M and the Banach fixed point theorem implies that there exists a unique fixed point $(u_\tau,\phi_\tau,\psi_\tau)\in X_M$ that solves the Eq. (1.1) in the critical space. In particular by (8.11), it follows that for $u_0\in L^1(\mathbb{R}^4)\cap\dot B_{p,\theta}^{-\frac{2}{\theta}}(\mathbb{R}^4)$, $\phi_0\in L^2(\mathbb{R}^4)\cap\dot B_{q,\rho}^{-\frac{2}{\rho}}(\mathbb{R}^4)$ and $\psi_0\in BMO(\mathbb{R}^4)\cap\dot B_{r,\sigma}^{1-\frac{2}{\sigma}}(\mathbb{R}^4)$,

$$
\|u_\tau\|_{L^2(I;L^2)}+\|\psi_\tau\|_{L^\infty(I;BMO)}\le 2M
$$

does not depend on the parameter $\tau > 0$. Next, we show the continuous dependence of the initial data in $L^\theta(I; L^p) \times L^\rho(I; \dot{W}^{1,q}) \times \big(L^\infty(I; BMO) \cap L^\sigma(I; \dot{W}^{1,r})\big)$. Let $(u_\tau, \phi_\tau, \psi_\tau)$ and $(\tilde{u}_\tau, \tilde{\phi}_\tau, \tilde{\psi}_\tau)$ be a solution of (1.9) corresponding to the initial data (u_0, ϕ_τ, ψ_0) and $(\tilde{u}_0, \tilde{\phi}_0, \tilde{\psi}_0)$, respectively. Then very much similar estimate of (6.15) and (6.16), we obtain from

$$\frac{2}{\theta} + \frac{4}{p} = 4, \qquad \frac{1}{2\sigma} + \frac{1}{r} = \frac{1}{4},$$

we see that

$$\frac{1}{p} - \frac{1}{4}\left(-\frac{2}{\theta}\right) = 1, \qquad \frac{1}{2\sigma} + \frac{1}{r} = \frac{1}{4},$$

and hence

$$\begin{aligned}
&\|u_\tau - \tilde{u}_\tau\|_{L^\theta(I;L^p)} \\
&\le \|e^{t\Delta}u_0 - e^{t\Delta}\tilde{u}_0\|_{L^\theta(I;L^p)} + \left\| \int_0^t e^{(t-s)\Delta}\nabla\cdot\big(u_\tau(s)\nabla\psi_\tau(s) - \tilde{u}_\tau(s)\nabla\tilde{\psi}_\tau(s)\big)ds \right\|_{L^\theta(I;\dot{B}^0_{p,1})} \\
&\le C_0\|u_0 - \tilde{u}_0\|_{\dot{B}^{-\frac{2}{\theta}}_{p,\theta}} + C\left\|u_\tau(s)\nabla\psi_\tau(s) - \tilde{u}_\tau(s)\nabla\tilde{\psi}_\tau(s)\right\|_{L^{\frac{\theta\sigma}{\theta+\sigma}}(I;\dot{B}^{-1+\frac{2}{\sigma}-\frac{2}{\theta}}_{p,\theta})} \\
&\le C_0\|u_0 - \tilde{u}_0\|_{\dot{B}^{-\frac{2}{\theta}}_{p,\theta}} + C\left\|u_\tau(s)\right\|_{L^\theta(I;L^q)}\left\|\nabla(\psi_\tau(s) - \tilde{\psi}_\tau(s))\right\|_{L^\sigma(I;L^r)} \\
&\quad + C\left\|u_\tau(s) - \tilde{u}_\tau(s)\right\|_{L^\theta(I;L^q)}\left\|\nabla\tilde{\psi}_\tau(s)\right\|_{L^\sigma(I;L^q)} \\
&\le C_0\|u_0 - \tilde{u}_0\|_{\dot{B}^{-\frac{2}{\theta}}_{p,\theta}} + C_1N|||(u_\tau, \phi_\tau, \psi_\tau) - (\tilde{u}_\tau, \tilde{\phi}_\tau, \tilde{\psi}_\tau)|||_T.
\end{aligned} \tag{8.16}$$

On the other hand, since

$$\frac{1}{4}\left(\frac{2}{\rho} + \frac{4}{q}\right) = \frac{1}{2} \quad\Rightarrow\quad \frac{1}{q} - \frac{1}{4}\left(1 - \frac{2}{\rho}\right) = \frac{1}{2}$$

and

$$\frac{1}{4}\left(\frac{2}{\theta} + \frac{4}{p}\right) = \frac{1}{4}, \quad \frac{1}{4}\left(\frac{2}{\rho} + \frac{4}{q}\right) = \frac{1}{2} \quad\Rightarrow\quad \frac{1}{q} - \frac{1}{4}\left(-\frac{2}{\rho} + \frac{2}{\theta}\right) = \frac{1}{p}$$

the embedding $\dot{W}^{1-\frac{2}{\rho},q}(\mathbb{R}^4) \subset L^2(\mathbb{R}^4)$ and $\dot{B}^{-\frac{2}{\rho}+\frac{2}{\theta}}_{q,\infty}(\mathbb{R}^4) \subset \dot{B}^0_{p,\infty}(\mathbb{R}^4)$ hold and Along the similar way to (8.8) and (8.9), we see that

$$
\begin{aligned}
&\big\|\phi_\tau - \tilde\phi_\tau\big\|_{L^\rho(I;L^q)} \\
&\quad \le \|e^{\tau t\Delta}\phi_0 - e^{\tau t\Delta}\tilde\phi_0\|_{L^\rho(I;L^q)} + \Big\|\int_0^{\tau t} e^{(\tau t-s)\Delta}\big(u_\tau(\tau^{-1}s) - \tilde u_\tau(\tau^{-1}s)\big)ds\Big\|_{L^\rho(I;L^q)} \\
&\quad \le C_0\|\phi_0 - \tilde\phi_0\|_{\dot B^{-\frac{2}{\rho}}_{q,\rho}} + C\big\|u_\tau(t) - \tilde u_\tau(t)\big\|_{L^\theta(I;L^p)} \\
&\quad \le C_0\|\psi_0 - \tilde\psi_0\|_{\dot B^{-\frac{2}{\rho}}_{q,\rho}} + C_1 |\!|\!|(u_\tau, \phi_\tau, \psi_\tau) - (\tilde u_\tau, \tilde\phi_\tau, \tilde\psi_\tau)|\!|\!|_T.
\end{aligned}
\tag{8.17}
$$

and

$$
\begin{aligned}
&\big\|\nabla\big(\psi_\tau - \tilde\psi_\tau\big)\big\|_{L^\sigma(I;L^r)} \\
&\quad \le \|\nabla(e^{\tau t\Delta}\psi_0 - e^{\tau t\Delta}\tilde\psi_0)\|_{L^\sigma(I;L^r)} + \Big\|\int_0^{\tau t} \nabla e^{(\tau t-s)\Delta}\big(\phi_\tau(\tau^{-1}s) - \tilde\phi_\tau(\tau^{-1}s)\big)ds\Big\|_{L^\sigma(I;L^r)} \\
&\quad \le C_0\|\psi_0 - \tilde\psi_0\|_{\dot B^{1-\frac{2}{\sigma}}_{r,\sigma}} + C\big\|\phi_\tau(t) - \tilde\phi_\tau(t)\big\|_{L^\rho(I;L^q)} \\
&\quad \le C_0\|\psi_0 - \tilde\psi_0\|_{\dot B^{1-\frac{2}{\sigma}}_{r,\sigma}} + C_1 |\!|\!|(u_\tau, \phi_\tau, \psi_\tau) - (\tilde u_\tau, \tilde\phi_\tau, \tilde\psi_\tau)|\!|\!|_T.
\end{aligned}
\tag{8.18}
$$

Finally likewise in (8.11), and

$$
\begin{aligned}
\big\|\psi_\tau - \tilde\psi_\tau\big\|_{L^\infty(I;VMO)} \le& \|e^{\tau t\Delta}\psi_0 - e^{\tau t\Delta}\tilde\psi_0\|_{VMO} \\
&+ \Big\|\int_0^{\tau t} e^{(\tau t-s)\Delta}\big(\phi_\tau(\tau^{-1}s) - \tilde\phi_\tau(\tau^{-1}s)\big)ds\Big\|_{L^\infty(I;VMO)} \\
\le& C_0\|\psi_0 - \tilde\psi_0\|_{VMO} + \Big\|\int_0^{\tau t} e^{(\tau t-s)\Delta}\big(\phi_\tau(\tau^{-1}s) - \tilde\phi_\tau(\tau^{-1}s)\big)ds\Big\|_{L^\infty(I;\dot B^0_{\infty,2})} \\
\le& C_0\|\psi_0 - \tilde\psi_0\|_{VMO} + C\Big\|\big(\phi_\tau(\tau^{-1}s) - \tilde\phi_\tau(\tau^{-1}s)\big)|_{s=\tau t}\Big\|_{L^\rho(I;\dot B^{-2+\frac{2}{\rho}}_{\infty,\infty})} \\
\le& C_0\|\psi_0 - \tilde\psi_0\|_{VMO} + C\Big\|\phi_\tau - \tilde\phi_\tau\Big\|_{L^\rho(I;\dot B^0_{q,\infty})} \\
\le& C_0\|\psi_0 - \tilde\psi_0\|_{VMO} + C|\!|\!|(u_\tau, \phi_\tau, \psi_\tau) - (\tilde u_\tau, \tilde\phi_\tau, \tilde\psi_\tau)|\!|\!|_T.
\end{aligned}
\tag{8.19}
$$

Those estimates (8.16)–(8.19) yield that the solution $(u_\tau, \phi_\tau, \psi_\tau)$ converges to $(\tilde u_\tau, \tilde\phi_\tau, \tilde\psi_\tau)$ in $L^\theta(I; L^p) \times L^\rho(I; \dot W^{1,q}) \times \big(C(I; VMO) \cap L^\sigma(I; \dot W^{1,r})\big)$ as $(u_0, \phi_0, \psi_0) \to (\tilde u_0, \tilde\phi, \tilde\psi_0)$ in $\dot B^{-\frac{2}{\theta}}_{p,\theta} \times \dot B^{-\frac{2}{\rho}}_{q,\rho} \times (\dot B^{1-\frac{2}{\sigma}}_{r,\sigma} \cap VMO)$.

Finally we show that the solution $(u_\tau, \phi_\tau, \psi_\tau)$ obtained above is in

$$
L^\infty\big(I; L^1(\mathbb{R}^4)\big) \times L^\infty\big(I; L^2(\mathbb{R}^4)\big) \times L^\infty\big(I; VMO(\mathbb{R}^4)\big),
$$

where $I = (0, T)$. Choosing $\theta = \sigma = 4$, the choice $(4, \frac{8}{7})$ and $(4, 8)$ corresponds the admissible for (θ, p) and (σ, r) with

$$
\frac{2}{\theta} + \frac{4}{p} = 4, \quad \frac{2}{\sigma} + \frac{4}{r} = 1,
$$

we use generalized maximal regularity (5.4) in Theorem 5.1 and the embedding $L^1(\mathbb{R}^n) \subset \dot B^0_{1,\infty}(\mathbb{R}^n)$ to have

$$
\begin{aligned}
\big\|u_\tau\big\|_{L^\infty(I;L^1)} \le & C_0\|u_0\|_{\dot{B}^0_{1,\infty}} + \Big\| \int_0^t e^{(t-s)\Delta}\nabla\cdot\big(u_\tau(s)\nabla\psi_\tau(s)\big)ds\Big\|_{L^\infty(I;\dot{B}^0_{1,1})} \\
\le & C_0\|u_0\|_1 + C\Big\|u_\tau(s)\nabla\psi_\tau(s)\Big\|_{L^{\frac{\theta}{2}}(I;\dot{B}^{-1+\frac{4}{\theta}}_{1,\infty})} \\
\le & C_0\|u_0\|_1 + C\Big\|u_\tau(s)\nabla\psi_\tau(s)\Big\|_{L^{\frac{4}{2}}(I;L^1)} \\
\le & \frac{1}{4}M + C\big\|u_\tau(s)\big\|_{L^4(I;L^{\frac{8}{7}})}\big\|\nabla\psi_\tau(s)\big\|_{L^4(I;L^4)} \le M.
\end{aligned}
\tag{8.20}
$$

Besides let $(u_\tau, \phi_\tau, \psi_\tau)$ and $(\tilde{u}_\tau, \tilde{\phi}_\tau, \tilde{\psi}_\tau)$ are two solutions of (1.9) corresponding the initial data (u_0, ϕ_0, ψ_0) and $(\tilde{u}_0, \tilde{\phi}_0, \tilde{\psi}_0)$, respectively. Then,

$$
\begin{aligned}
&\big\|u_\tau - \tilde{u}_\tau\big\|_{L^\infty(I;L^1)} \\
&\le C_0\|u_0 - \tilde{u}_0\|_{\dot{B}^0_{1,\infty}} + \Big\| \int_0^t e^{(t-s)\Delta}\nabla\cdot\big(u_\tau(s)\nabla\psi_\tau(s) - \tilde{u}_\tau(s)\nabla\tilde{\psi}_\tau(s)\big)ds\Big\|_{L^\infty(I;\dot{B}^0_{1,1})} \\
&\le C_0\|u_0 - \tilde{u}_0\|_1 + C\big\|u_\tau(s) - u_\tau(s)\big\|_{L^4(I;L^{\frac{8}{7}})}\big\|\nabla\psi_\tau(s)\big\|_{L^4(I;L^8)} \\
&\qquad + C\big\|u_\tau(s)\big\|_{L^4(I;L^{\frac{8}{7}})}\big\|\nabla\psi_\tau(s) - \nabla\tilde{\psi}_\tau(s)\big\|_{L^4(I;L^8)} \\
&\le C_0\|u_0 - \tilde{u}_0\|_1 + C|||(u_\tau, \phi_\tau, \psi_\tau) - (\tilde{u}_\tau, \tilde{\phi}_\tau, \tilde{\psi}_\tau)|||_T.
\end{aligned}
\tag{8.21}
$$

From (8.20) and (8.21), the continuous dependence of the solution u_τ in $L^\infty(I; L^1)$ is also shown. Choosing $\theta = 2$, the choice $(2, \frac{4}{3})$ and $(4, 8)$ corresponds the admissible for (θ, p) and (σ, r) with

$$
\frac{2}{\theta} + \frac{4}{p} = 4, \quad \frac{2}{\sigma} + \frac{4}{r} = 1,
$$

we use generalized maximal regularity (5.4) in Theorem 5.1 and the embedding $L^1(\mathbb{R}^n) \subset \dot{B}^0_{1,\infty}(\mathbb{R}^n)$ to have

$$
\begin{aligned}
\big\|\phi_\tau\big\|_{L^\infty(I;L^2)} \le & C_0\|\phi_0\|_{\dot{B}^0_{2,\infty}} + \Big\| \int_0^{\tau t} e^{(\tau t - s)\Delta}u_\tau(\tau^{-1}s)ds\Big\|_{L^\infty(I;\dot{B}^0_{2,1})} \\
\le & C_0\|\phi_0\|_2 + C\Big\|u_\tau(\tau^{-1}s)|_{s=\tau t}\Big\|_{L^\theta(I;\dot{B}^{-2+\frac{2}{\theta}}_{2,\infty})} \\
\le & \frac{1}{4}M + C\big\|u_\tau\big\|_{L^2(I;L^{\frac{4}{3}})} \le M.
\end{aligned}
\tag{8.22}
$$

For the convergence of ϕ_τ in $L^\infty(I;L^2)$ is also shown in a similar way.

$$
\begin{aligned}
\|\phi_\tau - \tilde{\phi}_\tau\|_{L^\infty(I;L^2)} \le & C_0\|\phi_0 - \tilde{\phi}_0\|_2 + C\|u_\tau(s) - \widetilde{u_\tau}(s)\|_{L^2(I;L^{\frac{4}{3}})} \\
\le & C_0\|u_0 - \tilde{u}_0\|_2 + C|\!|\!|(u_\tau,\phi_\tau,\psi_\tau) - (\tilde{u}_\tau,\tilde{\phi}_\tau,\tilde{\psi}_\tau)|\!|\!|_T.
\end{aligned}
\tag{8.23}
$$

Those estimates (8.20)–(8.23) and (8.19) conclude the continuous dependence of the solution on the initial data in (8). This show the local well-posedness of (1.1).

⟨Step 2⟩ (Global existence for small data). Since our function space in the previous step is scaling invariant, the global existence for (1.1) also follows almost similar (but somewhat simpler) way to the case for local well-posedness. Let (θ, p), (ρ, q), (σ, r) be admissible pairs given by (6.2). Fixing the admissible pair for $I = \mathbb{R}_+$ as (θ, p), (ρ, q), (σ, r), we introduce the complete metric space:

$$
\begin{aligned}
X_M = \Big\{ & u \in BUC(I;L^1(\mathbb{R}^4)) \cap L^\theta(I;L^q(\mathbb{R}^4)), \\
& \phi \in BUC(I;L^2(\mathbb{R}^4)) \cap L^\rho(I;L^q(\mathbb{R}^4)), \\
& \psi \in BUC(I;BMO(\mathbb{R}^4)) \cap L^\sigma(I;\dot{W}^{1,r}(\mathbb{R}^4)); \\
& \|u\|_{L^\infty(I;L^1)}, \|u\|_{L^\theta(I;L^p)}, \|\phi\|_{L^\infty(I;L^2)}, \|\nabla\phi\|_{L^\rho(I;L^q)}, \|\psi\|_{L^\infty(I;BMO)}, \|\nabla\psi\|_{L^\sigma(I;L^r)} \le M \Big\},
\end{aligned}
$$

where

$$
M = 4C_0\big(\|u_0\|_1 + \|u_0\|_{\dot{B}^0_{1,\theta}} + \|\phi_0\|_2 + \|\phi_0\|_{\dot{B}^{1-\frac{2}{\rho}}_{q,\rho}} + \|\psi_0\|_{BMO} + \|\nabla\psi_0\|_{\dot{B}^{1-\frac{2}{\sigma}}_{r,\sigma}}\big)
$$

is chosen small later. For any admissible exponents (θ, p), (ρ, q) and (σ, r) (note that they are not the end-point exponents), we define the metric on X_M by

$$
|\!|\!|(u,\phi,\psi) - (\tilde{u},\tilde{\phi},\tilde{\psi})|\!|\!|_M \equiv \|u - \tilde{u}\|_{L^\theta(I;L^p)} + \|\phi - \tilde{\phi}\|_{L^\rho(I;L^q)} + \|\nabla(\psi - \tilde{\psi})\|_{L^\sigma(I;L^r)}.
$$

By this metric, X_M is a complete metric space. For $(u_0, \psi_0, \phi_0) \in \big(L^1(\mathbb{R}^4) \cap \dot{B}^{-\frac{2}{\theta}}_{p,\theta}(\mathbb{R}^4)\big) \times \big(L^2(\mathbb{R}^4) \cap \dot{B}^{1-\frac{2}{\theta}}_{q,\theta}(\mathbb{R}^4)\big) \times \big(BMO(\mathbb{R}^4) \cap \dot{B}^{1-\frac{2}{\theta}}_{r,\theta}(\mathbb{R}^4)\big)$ and $(u_\tau, \psi_\tau, \phi_\tau) \in X_M$, we define a pair of the solution map $\big(\Xi[u_\tau,\psi_\tau], \Psi[u_\tau,\psi_\tau], \Phi[u_\tau,\psi_\tau]\big)$ by (8.2) and claim that the map (Ξ, Φ, Ψ) is contraction in the critical space X_M. Noticing $1 < p < 2 < q \le r$, $2 < \theta < \infty$ and the embedding (4.8), we apply maximal regularity and the embedding

$$
\begin{aligned}
\big\| \Xi[u_\tau, \psi_\tau] \big\|_{L^\infty(I;L^1)} \le & C_0 \|u_0\|_{\dot B^0_{1,\infty}} + \Big\| \int_0^t e^{(t-s)\Delta} \nabla \cdot \big(u_\tau(s) \nabla \psi_\tau(s)\big) dss \Big\|_{L^\infty(I;\dot B^0_{1,1})} \\
\le & C_0 \|u_0\|_1 + C \Big\| u_\tau(s) \nabla \psi_\tau(s) \Big\|_{L^{\frac{\theta\rho}{\theta+\rho}}(I;\dot B^{-1+\frac{2}{\sigma}-\frac{2}{\theta}}_{1,\infty})} \\
\le & C_0 \|u_0\|_1 + C \Big\| u_\tau(s) \nabla \psi_\tau(s) \Big\|_{L^{\frac{\theta\rho}{\theta+\rho}}(I;\dot B^0_{\frac{pq}{p+q},\infty})} \\
\le & C_0 \|u_0\|_1 + C \Big\| u_\tau(s) \nabla \psi_\tau(s) \Big\|_{L^{\frac{\theta\rho}{\theta+\rho}}(I;L^1)} \\
\le & \frac{1}{4} M + C \big\| u_\tau(s) \|_{L^\theta(I;L^p)} \big\| \nabla \psi_\tau(s) \big\|_{L^\rho(I;L^q)} \\
\le & \frac{1}{4} M + C M^2 \le M
\end{aligned}
\tag{8.24}
$$

under $CM \le \frac{1}{2}$. For any admissible exponents (θ, p) (ρ, q) and (σ, r),

$$
\begin{aligned}
\big\| \Phi[u_\tau, \psi_\tau] \big\|_{L^\rho(I;L^q)} \le & \|e^{t\Delta} u_0\|_{L^\rho(I;L^q)} + \Big\| \int_0^t e^{(t-s)\Delta} u_\tau(s) ds \Big\|_{L^\rho(I;\dot B^0_{q,1})} \\
\le & C_0 \|u_0\|_{\dot B^{-\frac{2}{\rho}}_{q,\rho}} + C \big\| u_\tau(s) \big\|_{L^\theta(I;\dot B^{-2+\frac{2}{\theta}-\frac{2}{\rho}}_{q,\theta})} \\
\le & C_0 \|u_0\|_{\dot B^{-\frac{2}{\rho}}_{q,\rho}} + C \big\| u_\tau(s) \big\|_{L^\theta(I;\dot F^{-2+\frac{2}{\theta}-\frac{2}{\rho}}_{q,\theta})} \\
\le & C_0 \|u_0\|_{\dot B^{-\frac{2}{\rho}}_{q,\rho}} + C \big\| u_\tau(s) \big\|_{L^\theta(I;L^p)} \\
\le & \frac{1}{4} M + C M^2 \le M
\end{aligned}
\tag{8.25}
$$

for $CM \le \frac{1}{2}$.

$$
\begin{aligned}
\big\| \nabla \Psi[u_\tau, \psi_\tau] \big\|_{L^\sigma(I;L^r)} \le & \|\nabla e^{t\Delta} \psi_0\|_{L^\sigma(I;L^r)} + \Big\| \nabla \int_0^t e^{(t-s)\Delta} \phi_\tau(s) ds \Big\|_{L^\sigma(I;\dot B^0_{r,1})} \\
\le & C_0 \|\psi_0\|_{\dot B^{1-\frac{2}{\sigma}}_{r,\sigma}} + C \big\| \phi_\tau(s) \big\|_{L^\rho(I;\dot B^{-1-\frac{2}{\sigma}+\frac{2}{\rho}}_{r,\sigma})} \\
\le & C_0 \|\psi_0\|_{\dot B^{1-\frac{2}{\sigma}}_{r,\sigma}} + C \big\| \phi_\tau(s) \big\|_{L^\rho(I;\dot F^0_{q,2})} \\
\le & C_0 \|\psi_0\|_{\dot B^{1-\frac{2}{\sigma}}_{r,\sigma}} + C \big\| \nabla \phi_\tau(s) \big\|_{L^\rho(I;L^q)} \\
\le & \frac{1}{4} M + C M^2 \le M
\end{aligned}
\tag{8.26}
$$

for $CM \le \frac{1}{2}$.

It follows from (8.24), (8.25) and (8.26) that

$$
\max \Big\{ \big\| \Xi[u_\tau, \psi_\tau] \big\|_{L^\infty(I;L^1)}, \big\| \Phi[u_\tau, \psi_\tau] \big\|_{L^\rho(I;L^q)}, \big\| \Psi[u_\tau, \psi_\tau] \big\|_{L^\theta(I;L^r)} \Big\} \le M. \tag{8.27}
$$

Meanwhile by embedding $\dot{B}^0_{\infty,2}(\mathbb{R}^n) \subset BMO(\mathbb{R}^n)$ and maximal regularity

$$\begin{aligned}
\left\|\Psi[u_\tau,\psi_\tau]\right\|_{L^\infty(I;BMO)} \le& \|e^{\tau t\Delta}\psi_0\|_{BMO} + \left\|\int_0^{\tau t} e^{(\tau t-s)\Delta}\Phi[u_\tau,\psi_\tau](\tau^{-1}s)ds\right\|_{L^\infty(I;BMO)}\\
\le& C_0\|\psi_0\|_{BMO} + \left\|\int_0^{\tau t} e^{(\tau t-s)\Delta}\Phi[u_\tau,\psi_\tau](\tau^{-1}s)ds\right\|_{L^\infty(I;\dot{B}^0_{\infty,2})}\\
\le& \frac{1}{4}M + C\left\|\Phi[u_\tau,\psi_\tau](\tau^{-1}s)|_{s=\tau t}\right\|_{L^4(I;\dot{B}^{-2+\frac{2}{4}}_{\infty,2})}\\
\le& \frac{1}{4}M + C\left\|\Phi[u_\tau,\psi_\tau](\tau^{-1}s)|_{s=\tau t}\right\|_{L^4(I;\dot{B}^{-\frac{3}{2}}_{\infty,2})}\\
\le& \frac{1}{4}M + C\left\|\Phi[u_\tau,\psi_\tau](t)\right\|_{L^4(I;\dot{B}^0_{\frac{4}{3},2})}\\
&(\text{since} q<2,\ L^{\frac{4}{3}}(\mathbb{R}^2) \simeq \dot{F}^0_{\frac{4}{3},2}(\mathbb{R}^2) \subset \dot{B}^0_{\frac{4}{3},2}(\mathbb{R}^2))\\
\le& \frac{1}{4}M + C\left\|\Phi[u_\tau,\psi_\tau](t)\right\|_{L^4(I;L^{\frac{4}{3}})}\\
\le& \frac{1}{4}M + (\frac{1}{4} + CM)M \le M
\end{aligned} \tag{8.28}$$

for small $CM \le \frac{1}{4}$.

$$\begin{aligned}
\left\|\Psi[u_\tau,\psi_\tau]\right\|_{L^\theta(I;L^r)} \le& \|e^{\tau t\Delta}\psi_0\|_{L^\theta(I;L^r)} + \left\|\int_0^{\tau t} e^{(\tau t-s)\Delta}\Phi[u_\tau,\psi_\tau](\tau^{-1}s)ds\right\|_{L^\theta(I;L^r)}\\
\le& C_0\|\psi_0\|_{\dot{B}^{1-\frac{2}{\theta}}_{r,\theta}} + C\left\|\Phi[u_\tau,\psi_\tau](\tau^{-1}s)\right\|_{L^\theta(I;\dot{B}^{-2+\frac{2}{\theta}}_{r,\theta})}\\
\le& \frac{1}{4}M + C\left\|\Phi[u_\tau,\psi_\tau](\tau^{-1}s)|_{s=\tau t}\right\|_{L^\theta(I;\dot{B}^{-2+\frac{2}{\theta}}_{r,\theta})}\\
\le& \frac{1}{4}M + C\left\|\Phi[u_\tau,\psi_\tau](t)\right\|_{L^\theta(I;\dot{F}^{-2+\frac{2}{\theta}}_{r,\theta})}\\
\le& \frac{1}{4}M + C\left\|\Phi[u_\tau,\psi_\tau]\right\|_{L^\theta(I;L^q)}\\
\le& \frac{1}{4}M + C\left(\frac{1}{4} + CM\right)M \le M.
\end{aligned} \tag{8.29}$$

From (8.28) and (8.29),

$$\max\left\{\left\|\Psi[u_\tau,\psi_\tau]\right\|_{L^\infty(I;BMO)}, \left\|\nabla\Psi[u_\tau,\psi_\tau]\right\|_{L^\theta(I;L^r)}\right\} \le M. \tag{8.30}$$

Combining (8.27) and (8.30), we obtain that $(\Phi[u_\tau,\psi_\tau], \Psi[\Phi[u_\tau,\psi_\tau]]) \in X_M$. Analogously from (8.12)–(8.14) for the difference of solutions

$$\begin{aligned}
&\left\| \Xi[u_\tau, \psi_\tau] - \Xi[\tilde{u}_\tau, \tilde{\psi}_\tau] \right\|_{L^\theta(I;L^q)} \le CM \|(u_\tau, \phi_\tau, \psi_\tau) - (\tilde{u}_\tau, \tilde{\phi}_\tau, \tilde{\psi}_\tau)\|_M, \\
&\left\| \Phi[u_\tau, \psi_\tau] - \Phi[\tilde{u}_\tau, \tilde{\psi}_\tau] \right\|_{L^\rho(I;L^q)} \le CM \|(u_\tau, \phi_\tau, \psi_\tau) - (\tilde{u}_\tau, \tilde{\phi}_\tau, \tilde{\psi}_\tau)\|_M, \\
&\left\| \nabla\big(\Psi[u_\tau, \psi_\tau] - \Psi[\tilde{u}_\tau, \tilde{\psi}_\tau] \big) \right\|_{L^\sigma(I;L^r)} \le CM \|(u_\tau, \phi_\tau, \psi_\tau) - (\tilde{u}_\tau, \tilde{\phi}_\tau, \tilde{\psi}_\tau)\|_M .
\end{aligned} \tag{8.31}$$

Choosing M smaller as

$$CM \le \frac{1}{4},$$

if necessary, we have from (8.31) that

$$\|(\Xi[u_\tau, \psi_\tau], \Phi[u_\tau, \psi_\tau], \Psi[u_\tau, \psi_\tau]) - (\Xi[\tilde{u}_\tau, \tilde{\psi}_\tau], \Phi[\tilde{u}_\tau, \tilde{\psi}_\tau], \Psi[\tilde{u}_\tau, \tilde{\psi}_\tau])\|_M \le \frac{1}{2} \|(u_\tau, \psi_\tau) - (\tilde{u}_\tau, \tilde{\psi}_\tau)\|_M$$

under the smallness assumption on the initial data. Thus the map (Ξ, Φ, Ψ) is contraction onto X_M and the Banach fixed point theorem implies that there exists a unique fixed point $(u_\tau, \psi_\tau) \in X_M$ that solves the Eq. (1.9) in the critical space. In particular, from (8.27) and (8.30), the a priori estimate

$$\max\Big(\|u_\tau\|_{L^\infty(I;L^1)}, \|u_\tau\|_{L^\theta(I;L^p)}, \|\phi_\tau\|_{L^\infty(I;L^2)}, \|\nabla\psi_\tau\|_{L^\rho(I;L^q)}, \|\psi_\tau\|_{L^\infty(I;BMO)}, \|\nabla\psi_\tau\|_{L^\theta(I;L^r)} \Big) \le M$$

does not depend on the parameter $\tau > 0$. □

9 Proof for the Singular Limit for Chaplain-Anderson Model

***Proof of Theorem* 3.3** We first show Theorem 3.3 for the small data case:
⟨Step 1⟩: For any $\eta_0 > 0$, let $I = (0, \infty)$ and we consider the difference of solutions between (3.3) and (3.4) as the following:

$$\begin{cases}
u_\tau(t) - u(t) = \displaystyle\int_0^t e^{(t-s)\Delta} \nabla \cdot \Big(\big(u_\tau(s) - u(s)\big) \nabla\psi_\tau(s) \Big) ds \\
\qquad\qquad + \displaystyle\int_0^t e^{(t-s)\Delta} \nabla \cdot \Big(u(s) \big(\nabla\psi_\tau(s) - \nabla\psi(s) \big) \Big) ds, & t \in I, \\
\phi_\tau(t) - \phi(t) = e^{\tau t \Delta}\phi_0 + \displaystyle\int_0^{\tau t} e^{s\Delta} \big(u_\tau(t - \tau^{-1}s) - u(t - \tau^{-1}s) \big) ds \\
\qquad\qquad + \displaystyle\int_0^{\tau t} e^{s\Delta} \big(u(t - \tau^{-1}s) - u(t) \big) ds + \int_{\tau t}^{\infty} e^{s\Delta} u(t) ds, & t \in I, \\
\psi_\tau(t) - \psi(t) = e^{\tau t \Delta}\psi_0 + \displaystyle\int_0^{\tau t} e^{s\Delta} \big(\phi_\tau(t - \tau^{-1}s) - \phi(t - \tau^{-1}s) \big) ds \\
\qquad\qquad + \displaystyle\int_0^{\tau t} e^{s\Delta} \big(\phi(t - \tau^{-1}s) - \phi(t) \big) ds + \int_{\tau t}^{\infty} e^{s\Delta} \phi(t) ds, & t \in I.
\end{cases} \tag{9.1}$$

As we have seen in (1.14), we choose the admissible exponents (θ, p), (θ, q) (θ, r) for $n = 4$ as

$$\begin{cases} \dfrac{1}{2\theta} + \dfrac{1}{p} = 1, \quad 1 < p \le 2 \le \theta, \\ \dfrac{1}{2\theta} + \dfrac{1}{q} = \dfrac{1}{2}, \quad 2 < q \le \theta, \\ \dfrac{1}{2\theta} + \dfrac{1}{r} = \dfrac{1}{4}, \quad 4 < r \le \theta. \end{cases}$$

Let the time interval $I = (\eta_0 \tau^{-1}, \infty)$ for any $\eta_0 > 0$. We apply the similar estimate in (6.15), it follows from Lemma 4.3 that

$$\begin{aligned} &\|u_\tau - u\|_{L^\theta(I;L^p)} \\ &\quad \le C \left\| \int_0^t e^{(t-s)\Delta} \nabla \cdot \big(u_\tau(s) - u(s)\big) \nabla \psi_\tau(s) ds \right\|_{L^\theta(I;L^p)} \\ &\qquad + C \left\| \int_0^t e^{(t-s)\Delta} \nabla \cdot u(s) \big(\nabla \psi_\tau(t) - \nabla \psi_\tau(s)\big) ds \right\|_{L^\theta(I;L^p)} \\ &\quad \le C \| \big(u_\tau - u\big) \nabla \psi_\tau \|_{L^{\frac{\theta}{2}}(I;\dot{B}^{-1+\frac{2}{\theta}}_{p,\theta})} + C \big\| u \nabla \big(\psi_\tau - \psi\big) \big\|_{L^{\frac{\theta}{2}}(I;\dot{B}^{-1+\frac{2}{\theta}}_{p,\theta})} \\ &\quad \le C \big\| \big(u_\tau - u\big) \nabla \psi_\tau \big\|_{L^{\frac{\theta}{2}}(I;L^{\frac{pr}{p+r}})} + C \big\| u \nabla \big(\psi_\tau - \psi\big) \big\|_{L^{\frac{\theta}{2}}(I;L^{\frac{pr}{p+r}})} \\ &\quad \le C \|u_\tau - u\|_{L^\theta(I;L^p)} \big\| \nabla \psi_\tau \big\|_{L^\theta(I;L^r)} + C \|u\|_{L^\theta(\mathbb{R}_+;L^p)} \big\| \nabla \big(\psi_\tau - \psi\big) \big\|_{L^\theta(I;L^r)} \\ &\quad \le C M \Big(\|u_\tau - u\|_{L^\theta(I;L^p)} + \big\| \nabla \psi_\tau - \nabla \psi(t) \big\|_{L^\theta(I;L^q)} \Big). \end{aligned} \tag{9.2}$$

For the second equation of (9.1),

$$\begin{aligned} \|\phi_\tau - \phi\|_{L^\theta(I;L^q)} = &\|e^{\tau t \Delta} \phi_0\|_{L^\theta(I;L^q)} \\ &+ \left\| \int_0^{\tau t} e^{s\Delta} \big(u_\tau(t - \tau^{-1}s) - u(t - \tau^{-1}s)\big) ds \right\|_{L^\theta(I;L^q)} \\ &\quad + \left\| \int_0^{\tau t} e^{s\Delta} \big(u(t - \tau^{-1}s) - u(t)\big) ds \right\|_{L^\theta(I;L^q)} \\ &\qquad + \left\| \int_{\tau t}^{\infty} e^{s\Delta} u(t) ds \right\|_{L^\theta(I;L^q)} \\ \equiv &I_0 + I_1 + I_2 + I_3. \end{aligned} \tag{9.3}$$

Let $t \in (0, \infty)$, we see under the assumption $\phi_0 \in \dot{B}^{-\frac{2}{\theta}}_{q,\theta}(\mathbb{R}^4)$ that

$$
\begin{aligned}
I_0 =&\left(\int_0^\infty \left\|e^{\tau t\Delta}\phi_0\right\|_{L^q}^\theta dt\right)^{1/\theta}\\
=&\tau^{-1/\theta}\left(\int_0^\infty \left\|e^{s\Delta}\phi_0\right\|_{L^q}^\theta ds\right)^{1/\theta}\\
\le&\tau^{-1/\theta}\left(\int_0^\infty \left\|e^{s\Delta}\phi_0\right\|_{L^q}^\theta ds\right)^{1/\theta} \to 0 \quad \text{as} \quad \tau\to\infty.
\end{aligned}
\tag{9.4}
$$

We note that (θ, q) is the admissible, i.e., $2 < q \le \theta$ that

$$
\frac{2}{\theta}+\frac{4}{p}=4, \quad \frac{2}{\theta}+\frac{4}{q}=2, \Rightarrow \frac{1}{q}=\frac{1}{p}-\frac{1}{4}\cdot 2 \tag{9.5}
$$

and hence from (5.4) in Theorem 5.1, the second term of the right hand side of (9.3) is estimated as follows:

$$
\begin{aligned}
I_1 =&\Big\| \int_0^{\tau t} e^{s\Delta}\big(u_\tau(t-\tau^{-1}s)-u(t-\tau^{-1}s)\big)ds\Big\|_{L^\theta(I;L^q)}\\
\le& C\left\|u_\tau(t-\tau^{-1}t)-u(t-\tau^{-1}t)\right\|_{L^\theta(I;\dot B^{-2}_{q,\theta})}\\
\le& C\left\|u_\tau\big((1-\tau^{-1})t\big)-u\big((1-\tau^{-1})t\big)\right\|_{L^\theta(I;\dot F^{-2}_{q,\theta})}\\
\le& C(1-\tau^{-1})^{-1/\theta}\left\|u_\tau(t)-u(t)\right\|_{L^\theta(I;\dot F^{-2}_{q,2})} \le C\left\|u_\tau(t)-u(t))\right\|_{L^\theta(I;L^p)}\\
\le& C\left\|u_\tau-u\right\|_{L^\theta(I;L^p)}
\end{aligned}
\tag{9.6}
$$

for all $\tau \ge 2$, where $C > 0$ is independent of $\tau > 0$. The third term of (9.3), we apply the relation (9.5) and the Sobolev inequality

$$
\|f\|_q \le C\|\Delta f\|_p, \quad \frac{1}{q}=\frac{1}{p}-\frac{2}{4}
$$

and Lemma 7.1 to see that

$$
\begin{aligned}
I_2 =&\Big\| \int_0^{\tau t} e^{s\Delta}\Big(u\big(t-\frac{s}{\tau}\big)-u(t)\Big)ds\Big\|_{L^\theta(I;L^q)}\\
\le& S_b\Big\| \int_0^{\tau t} \Delta e^{s\Delta}\Big(u\big(t-\frac{s}{\tau}\big)-u(t)\Big)ds\Big\|_{L^\theta(I;L^p)}\\
\le& S_b\Big\| \int_0^{\tau t} e^{r\Delta}\frac{\partial}{\partial r}u\left(t-\frac{r}{\tau}\right)dr\Big\|_{L^\theta(I;L^p)}\\
&+S_b\left\|e^{\tau t\Delta}u_0\right\|_{L^\theta(I;L^p)}+S_b\left\|e^{\tau t\Delta}u(t)\right\|_{L^\theta(I;L^p)}\\
=&I_{2,1}+I_{2,2}+I_{2,3}.
\end{aligned}
\tag{9.7}
$$

For treating the first term of the right hand side of (9.7), we proceed by changing the variable $r = \tau s$

$$
\begin{aligned}
I_{2,1} =& \left\| \int_0^t e^{\tau s\Delta} \frac{\partial}{\partial s} u\big(t-s\big) ds \right\|_{L^\theta(I;L^p)} \\
\le& \left\| \int_0^t e^{\tau s\Delta} \Delta u\big(t-s\big) ds \right\|_{L^\theta(I;L^p)} + \left\| \int_0^t e^{\tau s\Delta} \nabla \cdot \big(u(t-s)\nabla\psi(t-s)\big) ds \right\|_{L^\theta(I;L^p)} \\
\equiv& J_1 + J_2.
\end{aligned}
\tag{9.8}
$$

Then applying (5.4) in Theorem 5.1, we see by regarding $\mu \to \tau$, that

$$
J_1 = \left\| \int_0^t e^{\tau(t-r)\Delta} \Delta u\big(r\big) dr \right\|_{L^\theta(I;L^p)} \le C\tau^{-1} \|u\|_{L^\theta(I;L^p)}. \tag{9.9}
$$

Similarly by an analogous estimate of (6.7), it follows that

$$
\begin{aligned}
J_2 =& \left\| \int_0^t e^{\tau(t-r)\Delta} \nabla \cdot (u\nabla\psi)\big(r\big) dr \right\|_{L^\theta(I;L^p)} \\
\le& C\tau^{-1} \|u\nabla\psi\|_{L^{\frac{\theta}{2}}(I;L^{\frac{pr}{p+r}})} \\
\le& C\tau^{-1} \|u\|_{L^\theta(I;L^p)} \|\nabla\psi\|_{L^\theta(I;L^r)}.
\end{aligned}
\tag{9.10}
$$

On the other hand, since u_0 and $u(t) \in L^q$ for almost everywhere,

$$
\lim_{\tau\to\infty} \|e^{\tau t\Delta}\{u_0 - u(t)\}\|_p = 0
$$

and $e^{\tau t\Delta}$ is a bounded operator from L^p to itself. Since

$$
\begin{aligned}
\|e^{t\Delta}u_0\|_{L^\theta(I;L^p)} &\le C\|u_0\|_{\dot{B}^{-\frac{2}{\theta}}_{p,\theta}}, \\
\|e^{t\Delta}u(t)\|_{L^\theta(I;L^p)} &\le C\|u\|_{L^\theta(I;L^p)},
\end{aligned}
$$

we see for all $\tau > 1$ that

$$
\begin{aligned}
\|e^{\tau t\Delta}u_0\|_p^\theta &\le C\|e^{t\Delta}u_0\|_p^\theta \in L^1(I), \\
\|e^{\tau t\Delta}u(t)\|_p^\theta &\le C\|e^{t\Delta}u(t)\|_p^\theta \in L^1(I).
\end{aligned}
$$

Hence by the Lebesgue dominated convergence theorem, for any $\varepsilon > 0$, we may choose sufficiently large $\tau > 0$ such that

$$
I_{2,2} + I_{2,3} < \varepsilon \tag{9.11}
$$

Combining the estimates (9.7)–(9.11), we obtain that for any $\varepsilon > 0$, there exists a large $\tau > 0$ such that

$$
I_2 \le \varepsilon. \tag{9.12}
$$

Lastly for the forth term, setting $I = (\eta_0 \tau^{-1}, \infty)$, employing the Sobolev embedding, we see

$$
\begin{aligned}
I_3 =& \Big\| \int_{\tau t}^{\infty} e^{s\Delta} ds\, u(t) \Big\|_{L^\theta(I;L^q)} \\
=& \Big\| (-\Delta)^{-1} e^{\tau t \Delta}\, u(t) \Big\|_{L^\theta(I;L^q)} \\
\le& S_b \Big\| \nabla^2 (-\Delta)^{-1} e^{\tau t \Delta}\, u(t) \Big\|_{L^\theta(I;L^p)} \\
=& S_b \left(\int_0^\infty \big\| e^{\tau t \Delta}\, u(t) \big\|_p^\theta dt \right)^{1/\theta} \\
=& S_b \left(\int_0^\infty \big\| u(s) \big\|_p^\theta ds \right)^{1/\theta} .
\end{aligned}
\tag{9.13}
$$

Since

$$
\left(\int_0^\infty \big\| u(s) \big\|_p^\theta ds \right)^{1/\theta} < \infty.
$$

Hence from the fourth line of (9.13), the integrant

$$
\big\| e^{\tau t \Delta}\, u(t) \big\|_p^\theta
$$

is $L^1(\mathbb{R}_+)$ and it is dominated by the integrable function $\big\| u(s) \big\|_p^\theta$ as;

$$
\big\| e^{\tau t \Delta}\, u(t) \big\|_p^\theta \le \big\| u(t) \big\|_p^\theta .
$$

Besides for almost all $t > 0$,

$$
\lim_{\tau \to \infty} \| e^{\tau t \Delta}\, u(t) \|_p^\theta = 0. \tag{9.14}
$$

Applying the Lebesgue dominated convergence theorem, it follows from (9.14) that for any $\varepsilon > 0$

$$
\begin{aligned}
I_3 =& \Big\| \int_{\tau t}^{\infty} \nabla e^{s\Delta} ds\, u(t) \Big\|_{L^\theta(I;L^q)} \\
\le& S_b \left(\int_0^\infty \big\| e^{\tau t \Delta}\, u(t) \big\|_p^\theta dt \right)^{1/\theta} < \varepsilon,
\end{aligned}
\tag{9.15}
$$

as $\tau \to \infty$. Combining all the estimates (9.3), (9.4), (9.6), (9.12) and (9.15), we obtain

$$
\big\| \nabla \big(\phi_\tau(t) - \phi(t) \big) \big\|_{L^\theta(I;L^r)} \le C \| u_\tau - u \|_{L^\theta(I;L^q)} + \varepsilon. \tag{9.16}
$$

For the third component of the system, we proceed in a similar way to the case of second component:

$$
\begin{aligned}
\|\nabla\psi_\tau-\nabla\psi\|_{L^\theta(I;L^r)} =&\|\nabla e^{\tau t\Delta}\psi_0\|_{L^\theta(I;L^r)}\\
&+\left\|\int_0^{\tau t}\nabla e^{s\Delta}\big(\phi_\tau(t-\tau^{-1}s)-\phi(t-\tau^{-1}s)\big)ds\right\|_{L^\theta(I;L^r)}\\
&+\left\|\int_0^{\tau t}\nabla e^{s\Delta}\big(\phi_\infty(t-\tau^{-1}s)-\phi(t)\big)ds\right\|_{L^\theta(I;L^r)}\\
&+\left\|\int_{\tau t}^{\infty}\nabla e^{s\Delta}\phi(t)ds\right\|_{L^\theta(I;L^r)}\\
\equiv& K_0+K_1+K_2+K_3.
\end{aligned}
\tag{9.17}
$$

Let $t\in(0,\infty)$, we see under the assumption $\psi_0\in\dot B^{1-\frac{2}{\theta}}_{r,\theta}(\mathbb{R}^2)$ that

$$
\begin{aligned}
K_0=&\left(\int_0^\infty\left\|e^{\tau t\Delta}\nabla\psi_0\right\|_{L^r}^\theta dt\right)^{1/\theta}\\
=&\tau^{-1/\theta}\left(\int_0^\infty\left\|e^{s\Delta}\nabla\psi_0\right\|_{L^r}^\theta ds\right)^{1/\theta}\\
\le&\tau^{-1/\theta}\left(\int_0^\infty\left\|e^{s\Delta}\nabla\psi_0\right\|_{L^r}^\theta ds\right)^{1/\theta}\\
&\to 0 \quad\text{as}\quad \tau\to\infty.
\end{aligned}
\tag{9.18}
$$

We have from the second equation of (9.1), (5.4) in Theorem 5.1 and (θ,r) is the admissible, i.e., $2<r\le\theta$ that

$$
\begin{aligned}
K_1=&\left\|\int_0^{\tau t}\nabla e^{s\Delta}\big(\phi_\tau(t-\tau^{-1}s)-\phi(t-\tau^{-1}s)\big)ds\right\|_{L^\theta(I;L^r)}\\
\le& C\left\|\phi_\tau(t-\tau^{-1}t)-\phi(t-\tau^{-1}t)\right\|_{L^\theta(I;\dot B^{-1}_{r,\theta})}\\
\le& C\left\|\phi_\tau\big((1-\tau^{-1})t\big)-\phi\big((1-\tau^{-1})t\big)\right\|_{L^\theta(I;\dot F^{-1}_{r,\theta})}\\
\le& C(1-\tau^{-1})^{-1/\theta}\left\|\phi_\tau(t)-\phi(t)\right\|_{L^\theta(I;\dot F^{-1}_{r,2})}\le C\left\|\phi_\tau(t)-\phi(t)\right\|_{L^\theta(I;\dot W^{-1,r})}\\
\le& C\left\|\phi_\tau-\phi\right\|_{L^\theta(I;L^q)}
\end{aligned}
\tag{9.19}
$$

for all $\tau\ge 2$, where $C>0$ is independent of $\tau>0$.

The third term of (9.17), we apply the Sobolev inequality and Lemma 7.1 to see that

$$
\begin{aligned}
K_2 =& \Big\| \int_0^{\tau t} \nabla e^{s\Delta} \Big(\phi\big(t - \frac{s}{\tau}\big) - \phi(t) \Big) ds \Big\|_{L^\theta(I;L^r)} \\
\le& S_b \Big\| \int_0^{\tau t} \Delta e^{s\Delta} \Big(\phi\big(t - \frac{s}{\tau}\big) - \phi(t) \Big) ds \Big\|_{L^\theta(I;L^q)} \\
\le& S_b \Big\| \int_0^{\tau t} e^{r\Delta} \frac{\partial}{\partial r} \phi\big(t - \frac{r}{\tau}\big) dr \Big\|_{L^\theta(I;L^q)} \\
&+ S_b \big\| e^{\tau t\Delta} \phi_0 \big\|_{L^\theta(I;L^q)} + S_b \big\| e^{\tau t\Delta} \phi(t) \big\|_{L^\theta(I;L^q)} \\
=& L_{2,1} + L_{2,2} + L_{2,3}.
\end{aligned}
\tag{9.20}
$$

For treating the first term of the right hand side of (9.20), we proceed by changing the variable $r = \tau s$

$$
\begin{aligned}
L_{2,1} =& \Big\| \int_0^{t} e^{\tau s\Delta} \frac{\partial}{\partial s} \phi\big(t - s\big) ds \Big\|_{L^\theta(I;L^q)} \\
\le& C\tau^{-1} \|\phi\|_{L^\theta(I;L^q)}.
\end{aligned}
\tag{9.21}
$$

On the other hand, since ϕ_0 and $\phi(t) \in L^q$ for almost everywhere,

$$
\lim_{\tau\to\infty} \|e^{\tau t\Delta}(\phi_0 - \phi(t))\|_q = 0
$$

and $e^{\tau t\Delta}$ is a bounded operator from L^q to itself. Since

$$
\begin{aligned}
&\|e^{t\Delta}\phi_0\|_{L^\theta(I;L^q)} \le C\|\phi_0\|_{\dot{B}^{-\frac{2}{\theta}}_{q,\theta}}, \\
&\|e^{t\Delta}\phi(t)\|_{L^\theta(I;L^q)} \le C\|\phi\|_{L^\theta(I;L^q)},
\end{aligned}
$$

we see for all $\tau > 1$ that

$$
\begin{aligned}
&\|e^{\tau t\Delta}\phi_0\|_q^\theta \le C\|e^{t\Delta}\phi_0\|_q^\theta \in L^1(I), \\
&\|e^{\tau t\Delta}\phi(t)\|_q^\theta \le C\|e^{t\Delta}\phi(t)\|_q^\theta \in L^1(I).
\end{aligned}
$$

Hence by the Lebesgue dominated convergence theorem, for any $\varepsilon > 0$, we may choose sufficiently large $\tau > 0$ such that

$$
L_{2,2} + L_{2,3} < \varepsilon \tag{9.22}
$$

Combining the estimates (9.20), (9.21) and (9.22), we obtain that for any $\varepsilon > 0$, there exists a large $\tau > 0$ such that

$$
K_2 \le \varepsilon. \tag{9.23}
$$

For the forth term, setting $I = (\eta_0\tau^{-1}, \infty)$, employing the Sobolev embedding:

$$\|f\|_r \le S_b \|\nabla f\|_q, \quad \frac{1}{r} = \frac{1}{q} - \frac{1}{4}$$

and boundedness of the semi-group $e^{t\Delta} : L^q \to L^q$, we see

$$\begin{aligned} K_3 =& \left\| \nabla(-\Delta)^{-1} e^{\tau t\Delta}\, \phi(t) \right\|_{L^\theta(I;L^r)} \\ \le& S_b \left\| \nabla^2(-\Delta)^{-1} e^{\tau t\Delta}\, \phi(t) \right\|_{L^\theta(I;L^q)} \\ =& S_b \left(\int_0^\infty \left\| e^{\tau t\Delta}\, \phi(t) \right\|_q^\theta dt \right)^{1/\theta} \\ \le& C S_b \left(\int_0^\infty \left\| \phi(s) \right\|_q^\theta ds \right)^{1/\theta}. \end{aligned} \tag{9.24}$$

Since

$$\left(\int_0^\infty \left\| \phi(s) \right\|_q^\theta ds \right)^{1/\theta} < \infty,$$

the integrant in the fourth line of (9.24);

$$\left\| e^{\tau t\Delta}\, \phi(t) \right\|_q^\theta$$

is $L^1(\mathbb{R}_+)$ and it is dominated by the integrable function $\left\| u(s) \right\|_{\frac{4q}{q+4}}^\theta$ as;

$$\left\| e^{\tau t\Delta}\, u(t) \right\|_q^\theta \le \left\| u(t) \right\|_q^\theta.$$

Besides for almost all $t > 0$,

$$\lim_{\tau \to \infty} \| e^{\tau t\Delta}\, u(t) \|_q^\theta = 0. \tag{9.25}$$

Applying the Lebesgue dominated convergence theorem, it follows from (9.25) that for any $\varepsilon > 0$

$$\begin{aligned} K_3 =& \left\| \int_{\tau t}^\infty \nabla e^{s\Delta} ds\, u(t) \right\|_{L^\theta(I;L^r)} \\ \le& S_b \left(\int_0^\infty \left\| e^{\tau t\Delta}\, u(t) \right\|_q^\theta dt \right)^{1/\theta} < \varepsilon, \end{aligned} \tag{9.26}$$

as $\tau \to \infty$. Combining all the estimates (9.17), (9.18), (9.19), (9.23) and (9.26), we obtain

$$\left\| \nabla\big(\psi_\tau - \psi\big) \right\|_{L^\theta(I;L^r)} \le C \|\phi_\tau - \phi\|_{L^\theta(I;L^q)} + \varepsilon. \tag{9.27}$$

Gathering (9.2), (9.16) and (9.27), we see that for any $\varepsilon > 0$, choosing τ sufficiently large such that

$$\|u_\tau - u\|_{L^\theta(I;L^q)} \le CM\Big(\|u_\tau - u\|_{L^\theta(I;L^q)} + \varepsilon\Big). \tag{9.28}$$

In particular, from (9.16), (9.27) and choosing $M > 0$ small enough in (9.28), for small $\varepsilon > 0$,

$$\|u_\tau - u\|_{L^\theta(I;L^q)} + \|\nabla\phi_\tau - \nabla\phi\|_{L^\theta(I;L^q)} + \|\nabla\psi_\tau - \nabla\psi\|_{L^\theta(I;L^r)} \le \varepsilon. \tag{9.29}$$

$\langle$Step 2$\rangle$: For any $\eta_0 > 0$, let $I = (0, T) \cap (\eta_0, \infty)$. From maximal regularity (5.4) in Theorem 5.1, we estimate the first component of the integral equation (9.1) as follows: Recalling

$$\frac{2}{\theta} + \frac{4}{p} = 4, \quad \frac{2}{\theta} + \frac{4}{r} = 1,$$

we proceed in very similar way in (8.21) by the Hölder inequality with for $\theta = 4$, $p = \frac{8}{7}$, $r = 8$ that

$$\begin{aligned}
&\|u_\tau - u\|_{L^\infty(I;L^1)} \\
&\le \Big\| \int_0^t \nabla e^{(t-s)\Delta}\Big(u_\tau(s)\nabla\psi_\tau(s) - u(s)\nabla\psi(s)\Big)ds\Big\|_{L^\infty(I;\dot B^0_{1,1})} \\
&\le C\big\|u_\tau(s)\nabla\psi_\tau(s) - u(s)\nabla\psi(s)\big\|_{L^{\frac{\theta}{2}}(I;\dot B^{-1+\frac{4}{\theta}}_{1,\infty})} \\
&\le C\Big(\big\|\big(u_\tau(s) - u(s)\big)\nabla\psi_\tau(s)\big\|_{L^2(I;L^1)} + \big\|u(s)\big(\nabla\psi_\tau(s) - \nabla\psi(s)\big)\big\|_{L^2(I;L^1)}\Big) \\
&\le C\Big(\|u_\tau - u\|_{L^4(I;L^{\frac{8}{7}})}\big\|\nabla\psi_\tau\big\|_{L^4(I;L^8)} + \|\nabla\psi_\tau - \nabla\psi\|_{L^4(I;L^8)}\big\|u\big\|_{L^4(I;L^{\frac{8}{7}})}\Big).
\end{aligned} \tag{9.30}$$

Then the limiting process (9.29) implies the convergence for this case, too.

On the other hand, for the second component, we choose $\theta = 2$ $p = \frac{4}{3}$, that

$$\begin{aligned}
&\|\phi_\tau - \phi\|_{L^\infty(I;L^2)} \\
&\le \big\|e^{\tau t\Delta}\phi_0\big\|_{L^\infty(I;L^2)} + \Big\|\int_0^{\tau t} e^{s\Delta}\big(u_\tau(t-\tau^{-1}s) - u(t-\tau^{-1}s)\big)ds\Big\|_{L^\infty(I;L^2)} \\
&\qquad + \Big\|\int_0^{\tau t} e^{s\Delta}\big(u(t-\tau^{-1}s) - u(t)\big)ds\Big\|_{L^\infty(I;L^2)} + \Big\|\int_{\tau t}^{\infty} e^{s\Delta}u(t)ds\Big\|_{L^\infty(I;L^2)} \\
&\le \big\|e^{\tau\eta_0\Delta}\psi_0\big\|_2 + C\big\|u_\tau(t-\tau^{-1}t) - u(t-\tau^{-1}t)\big\|_{L^2(I;\dot B^{-1}_{2,\infty})} \\
&\quad + CS_b\Big\|\int_0^{\tau t} e^{s\Delta}\big(u(t-\tau^{-1}s) - u(t)\big)ds\Big\|_{L^\infty(I;L^2)} + C\big\|e^{\tau\eta_0\Delta}u(t)\big\|_{L^2(I;\dot B^{-1}_{2,\infty})} \\
&\le \big\|e^{\tau\eta_0\Delta}\psi_0\big\|_2 + CS_b\big\|u_\tau - u\big\|_{L^2(I;L^{\frac{4}{3}})} + \widetilde{I}_2 + \widetilde{I}_3.
\end{aligned} \tag{9.31}$$

As in a similar manner, the last two terms in the right hand side of (9.31) can be treated as follows: For the third term, we apply Lemma 7.1 to obtain

$$\begin{aligned}
\widetilde{I_2} =& \Big\| \int_0^{\tau t} e^{s\Delta}\Big(u\big(t-\frac{s}{\tau}\big)-u(t)\Big)ds\Big\|_{L^\infty(I;L^2)} \\
\le& S_b \Big\| \int_0^{\tau t} \Delta e^{s\Delta}\Big(u\big(t-\frac{s}{\tau}\big)-u(t)\Big)ds\Big\|_{L^\infty(I;L^1)} \\
\le& S_b \Big\| \int_0^{\tau t} e^{r\Delta}\frac{\partial}{\partial r}u\big(t-\frac{r}{\tau}\big)dr\Big\|_{L^\infty(I;L^1)} \\
&+ S_b\big\|e^{\tau t\Delta}u_0\big\|_{L^\infty(I;L^1)} + S_b\big\|e^{\tau t\Delta}u(t)\big\|_{L^\infty(I;L^1)} \\
=& \widetilde{I_{2,1}} + \widetilde{I_{2,2}} + \widetilde{I_{2,3}}.
\end{aligned} \tag{9.32}$$

For treating the first term of the right hand side of (9.32), we proceed by changing the variable $r = \tau s$

$$\begin{aligned}
\widetilde{I_{2,1}} =& \left\| \int_0^t e^{\tau s\Delta}\frac{\partial}{\partial s}u\big(t-s\big)ds\right\|_{L^\infty(I;L^1)} \\
\le& \left\| \int_0^t e^{\tau s\Delta}\Delta u\big(t-s\big)ds\right\|_{L^\infty(I;\dot{B}^0_{1,\infty})} + \left\| \int_0^t e^{\tau s\Delta}\nabla\cdot\big(u(t-s)\nabla\psi(t-s)\big)ds\right\|_{L^\infty(I;\dot{B}^0_{1,\infty})} \\
\equiv& J_1 + J_2.
\end{aligned} \tag{9.33}$$

Then applying (5.4) in Theorem 5.1, we see by regarding $\mu \to \tau$, that

$$J_1 = \left\| \int_0^t e^{\tau(t-r)\Delta}\Delta u\big(r\big)dr\right\|_{L^\infty(I;\dot{B}^0_{1,\infty})} \le C\tau^{-1}\|u\|_{L^\infty(I;\dot{B}^0_{1,\infty})} \le C\tau^{-1}\|u\|_{L^\infty(I;L^1)}. \tag{9.34}$$

Similarly by the analogous estimate of (8.24), choosing $\theta = 4$, $p = \frac{8}{7}$ and $r = 8$, it follows that

$$\begin{aligned}
J_2 =& \left\| \int_0^t e^{\tau(t-r)\Delta}\nabla\cdot(u\nabla\psi)\big(r\big)dr\right\|_{L^\infty(I;\dot{B}^0_{1,\infty})} \\
\le& C\tau^{-1}\|u\nabla\psi\|_{L^{\frac{\theta}{2}}(I;\dot{B}^{-1+\frac{4}{\theta}}_{1,\infty})} \\
\le& C\tau^{-1}\|u\nabla\psi\|_{L^2(I;L^1)} \\
\le& C\tau^{-1}\|u\|_{L^4(I;L^{\frac{8}{7}})}\|\nabla\psi\|_{L^4(I;L^8)}.
\end{aligned} \tag{9.35}$$

On the other hand, since u_0 and $u(t) \in L^q$ for almost everywhere, $e^{\tau t\Delta}$ is a bounded operator from L^p to itself. Since

$$\begin{aligned}
&\|e^{t\Delta}u_0\|_{L^\infty(I;L^1)} \le C\|u_0\|_1, \\
&\|e^{t\Delta}u(t)\|_{L^\infty(I;L^1)} \le C\|u\|_1,
\end{aligned}$$

we see for all $\tau > 1$ that

$$\|e^{\tau t\Delta}u_0\|_1 \le C\|e^{t\Delta}u_0\|_1 \in L^1(I),$$
$$\|e^{\tau t\Delta}u(t)\|_1 \le C\|e^{t\Delta}u(t)\|_1 \in L^1(I).$$

Hence by the Lebesgue dominated convergence theorem, for any $\varepsilon > 0$, we may choose sufficiently large $\tau > 0$ such that

$$\widetilde{I_{2,2}} + \widetilde{I_{2,3}} < \varepsilon \tag{9.36}$$

Combining the estimates (9.32)–(9.36), we obtain that for any $\varepsilon > 0$, there exists a large $\tau > 0$ such that

$$\widetilde{I_2} \le \varepsilon. \tag{9.37}$$

The last term in (9.39),

$$\widetilde{I_3} \le \left\| e^{\tau\eta_0\Delta}\, u(t) \right\|_{L^2(I;\dot{B}^{-1}_{2,\infty})} \le \varepsilon \tag{9.38}$$

since

$$\left\| e^{\tau\eta_0\Delta}\, u(t) \right\|^2_{\dot{B}^{-1}_{2,\infty}} \to 0, \qquad \tau \to \infty,$$
$$\left\| e^{\tau\eta_0\Delta}\, u(t) \right\|_{L^2(I;\dot{B}^{-1}_{2,\infty})} \le \left\| e^{\eta_0\Delta}\, u(t) \right\|_{L^2(I;\dot{B}^{-1}_{2,\infty})}$$

and the Lebesgue dominated convergence theorem. From (9.31), (9.37) and (9.38), we conclude that

$$\|\phi_\tau - \phi\|_{L^\infty(I;L^2)} \to 0 \quad \tau \to \infty. \tag{9.39}$$

For the third component, we see in a similar way that

$$\begin{aligned}
&\|\psi_\tau - \psi\|_{L^\infty(I;VMO)} \\
\le &\left\| e^{\tau t\Delta}\psi_0 \right\|_{L^\infty(I;VMO)} + \left\| \int_0^{\tau t} e^{s\Delta}\big(\phi_\tau(t-\tau^{-1}s) - (\phi(t-\tau^{-1}s)\big)ds \right\|_{L^\infty(I;VMO)} \\
&\quad + \left\| \int_0^{\tau t} e^{s\Delta}\big(\phi(t-\tau^{-1}s) - \phi(t)\big)ds \right\|_{L^\infty(I;VMO)} + \left\| \int_{\tau t}^{\infty} e^{s\Delta}\phi(t)ds \right\|_{L^\infty(I;VMO)} \\
\le &\left\| e^{\tau\eta_0\Delta}\psi_0 \right\|_{VMO} + C\left\| \phi_\tau(t-\tau^{-1}t) - \phi(t-\tau^{-1}t) \right\|_{L^4(I;\dot{B}^{-2+\frac{2}{4}}_{2,\infty})} \\
&+ CS_b \left\| \int_0^{\tau t} e^{s\Delta}\big(\phi(t-\tau^{-1}s) - \phi(t)\big)ds \right\|_{L^\theta(I;VMO)} + C\left\| e^{\tau\eta_0\Delta}\,\phi(t) \right\|_{L^\theta(I;VMO)} \\
\le &\left\| e^{\tau\eta_0\Delta}\psi_0 \right\|_{VMO} + C\left\| \phi_\tau - \phi \right\|_{L^4(I;L^{\frac{4}{3}})} + S_b\left\| \int_0^{\tau t} e^{r\Delta}\frac{\partial}{\partial r}\phi\big(t - \frac{r}{\tau}\big)dr \right\|_{L^\infty(I;VMO)} \\
&+ S_b\left\| e^{\tau t\Delta}\phi_0 \right\|_{L^\infty(I;L^1)} + S_b\left\| e^{\tau t\Delta}\phi(t) \right\|_{L^\infty(I;L^1)} + C\left\| e^{\tau\eta_0\Delta}\,\phi(t) \right\|_{L^\theta(I;VMO)}.
\end{aligned} \tag{9.40}$$

The first, second and the last three terms in the right hand side of (9.40) are converging to 0 as $\tau \to \infty$ as before. The remaining third term treated as follows:

$$\Big\| \int_0^{\tau t} e^{r\Delta} \frac{\partial}{\partial r} \phi\Big(t - \frac{r}{\tau}\Big) dr \Big\|_{L^\infty(I;VMO)} = \Big\| \int_0^{t} e^{\tau s\Delta} \frac{\partial}{\partial s} \phi\big(t - s\big) ds \Big\|_{L^\infty(I;VMO)}$$
$$\le C\tau^{-1} \Big\| \phi\big(t-s\big)|_{s=t} \Big\|_{L^\infty(I;VMO)} \to 0 \quad \tau \to \infty. \tag{9.41}$$

Combining (9.40) and (9.41), we conclude the convergence

$$\lim_{\tau\to\infty} \|\psi_\tau - \psi\|_{L^\infty(I;VMO)} = 0 \tag{9.42}$$

and hence (3.5) in Theorem 3.3 has proven by gathering (9.30), (9.39) and (9.42).

⟨Step 3⟩ For the interval near $t = 0$, we treat the initial layer for the second and third equations. Since the argument are very similar, we only treat the second equation: Let $\eta(t) \equiv \chi_{[0,t_0]}(t)(\phi_0 - (-\Delta)^{-1}u_0)$. From (9.1),

$$\phi_\tau(t) - \phi(t) - \eta(t) = e^{\tau t\Delta}\phi_0 - \phi_0 + (-\Delta)^{-1}u_0$$
$$+ \int_0^{\tau t} e^{(\tau t - s)\Delta} u_\tau(\tau^{-1}s) ds - \int_0^\infty e^{s\Delta} u(t) ds \quad t \in (0, t_0). \tag{9.43}$$

Then for any $\varepsilon > 0$, choose $t_0 > 0$ small enough so that applying the Sobolev embedding

$$L^q(\mathbb{R}^2) \simeq \dot{F}^0_{q,2}(\mathbb{R}^2) \subset \dot{B}^0_{q,\infty}(\mathbb{R}^2) \subset \dot{B}^{-2+\frac{2}{\theta}}_{\infty,\infty}(\mathbb{R}^2),$$

we estimate (9.43) to obtain

$$\begin{aligned}
&\big\|\phi_\tau - \phi - \eta\big\|_{L^\infty(0,t_0;L^2)} \\
\le &\big\|e^{\tau t\Delta}\phi_0\big\|_{L^\infty(0,t_0;L^2)} + \Big\|(-\Delta)^{-1}u_0 - \int_0^\infty e^{s\Delta}u(t)ds\Big\|_{L^\infty(0,t_0;L^2)} \\
&\qquad + \Big\|\int_0^{\tau t} e^{(\tau t-s)\Delta}u_\tau(\tau^{-1}s)ds\Big\|_{L^\infty(0,t_0;L^2)} \\
\le &\sup_{t\le t_0}\big\|e^{\tau t\Delta}\phi_0\big\|_2 + C\Big\|(-\Delta)^{-1}u_0 - (-\Delta)^{-1}u(t)\Big\|_{L^\infty(0,t_0;L^2)} \\
&+ C\Big\|u_\tau(t-\tau^{-1}s)|_{s=\tau t}\Big\|_{L^\theta(0,t_0;\dot{F}^{-2+\frac{2}{\theta}}_{2,\infty})} \\
\le &\sup_{t\le t_0}\big\|e^{\tau t\Delta}\phi_0\big\|_2 + C\big\|u(t) - u_0\big\|_{L^\infty(0,t_0;L^1)} + C\big\|u_\tau(\tau^{-1}s)|_{s=\tau t}\big\|_{L^\theta(0,t_0;L^p)} \\
\le &\sup_{t\le t_0}\big\|e^{\tau t\Delta}\phi_0\big\|_2 + C\big\|u(t) - u_0\big\|_{L^\infty(0,t_0;L^1)} + C\big\|u_\tau(t)\big\|_{L^\theta(0,t_0;L^p)} \\
\le &\sup_{t\le t_0}\big\|e^{\tau t\Delta}\phi_0\big\|_2 + 2\varepsilon,
\end{aligned} \tag{9.44}$$

for small $t_0 > 0$ because of the uniform bound for $u_\tau \in L^\theta(I; L^q)$. Since $\phi_0 \in L^2(\mathbb{R}^4)$, we have for any $\eta_0 > 0$ that

$$\|e^{\tau\eta_0\Delta}\phi_0\|_2 \to 0 \tag{9.45}$$

as $\tau \to \infty$ for and by passing $\tau \to \infty$, we conclude from (9.44) and (9.45) that the locally uniform convergence (3.5) holds. This completes the proof. □

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HJB Equation, Dynamic Programming Principle, and Stochastic Optimal Control

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Abstract The paper is an extended version of lecture notes from a mini-course given by the author in the workshop Optimal Control and PDE in Tohoku University in 2017. The main objective of the lecture notes is to give a short but rigorous introduction to the dynamic programming approach to stochastic optimal control problems. The manuscript discusses, among other things, the classical necessary and sufficient conditions for optimality, properties of the value function, and it contains a proof of the dynamic programming principle, and a proof that the value function is a unique viscosity solution of the associated Hamilton-Jacobi-Bellman equation.

Keywords Stochastic optimal control · Dynamic programming principle · Viscosity solution · Hamilton-Jacobi-Bellman equation · Value function

Mathematics Subject Classification 49L12 · 49L25 · 93E20

1 Introduction

This expository paper is an extended version of lecture notes from a mini-course lectures given by the author in the workshop Optimal Control and PDE in the Thematic Program 'Nonlinear Partial Differential Equations for Future Applications', Tohoku University, July 17–21, 2017. The main objective of the lecture notes is to give a short but rigorous introduction to the dynamic programming approach to stochastic optimal control problems with a proof of the dynamic programming principle (DPP) and the derivation of the associated Hamilton-Jacobi-Bellman equation.

The central theme of the manuscript is the DPP which links the stochastic optimal control problem to a Hamilton-Jacobi-Bellman partial differential equation. Contrary

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to the deterministic case where its proof is rather elementary, the proof of the DPP in the stochastic case is very difficult and technical and cannot be taken for granted. Thus its good understanding is essential and very important. The setup of the stochastic optimal control problem presented in this paper and the proof of the DPP follows the more abstract presentation in [9], which was based on the approach of [24]. The reader should consult [9] for more details on various concepts used here and missing proofs of some results used in the manuscripts. Other proofs of the DPP for various setups and control problems can be found in [3–5, 8, 11–13, 16–19, 22, 23, 25]. The reader can read Sect. 2.7 (Bibliographical Notes) of [9] for more information about these proofs and different approaches. Also, there are many books available [2, 3, 7, 10, 11, 13–16, 18, 19, 21, 24], where the reader can learn more about various aspects of stochastic optimal control and dynamic programming. Regarding viscosity solutions, their basic theory can be found for instance in [1, 6, 11], also [9] can be useful even though it deals with infinite dimensional problems.

2 Stochastic Optimal Control Problem

Throughout the paper, for $x, y \in \mathbb{R}^n$, we will write $|x|$ for the Euclidean norm of x and $\langle x, y\rangle$ for the inner product of x and y. If $r \in \mathbb{R}$, $|r|$ will also mean the absolute value of r. For a matrix X, we will write $\|X\|$ to denote the operator norm of X.

Let $T > 0$ be a fixed constant. For any initial time $t \in [0, T]$ and $x \in \mathbb{R}^n$, the state equation of the problem is given by a stochastic differential equation (SDE)

$$\begin{cases} dX(s) = b(s, X(s), a(s))ds + \sigma(s, X(s), a(s))dW(s), & s \in (t, T] \\ X(t) = x, \end{cases} \tag{2.1}$$

where W is a standar m-dimensional Brownian motion, and $a(\cdot) : [0, T] \to \Lambda$ is a control process. We make the following assumptions throughout this paper.

- The control space Λ is a Polish space (a complete separable metric space).
- The functions $b : [0, T] \times \mathbb{R}^n \times \Lambda \to \mathbb{R}^n, \sigma : [0, T] \times \mathbb{R}^n \times \Lambda \to \mathbb{R}^{n\times m}$ are continuous, $b(\cdot, \cdot, a)$, $\sigma(\cdot, \cdot, a)$ are uniformly continuous on bounded subsets of $[0, T] \times \mathbb{R}^n$, uniformly for $a \in \Lambda$.
- There exists $C \geq 0$ such that

$$|b(s, x, a) - b(s, y, a)| \leq C|x - y|$$

$$\|\sigma(s, x, a) - \sigma(s, y, a)\| \leq C|x - y|$$

$$|b(s, x, a)| + \|\sigma(s, x, a)\| \leq C(1 + |x|),$$

for all $s \in [0, T]$, $x, y \in \mathbb{R}^n$, $a \in \Lambda$.

The goal is to minimize, over all $a(\cdot)$ which will be specified later, the cost functional

$$J(t,x;a(\cdot)) = \mathbb{E}\bigg[\int_t^T e^{-\int_t^s c(X(r))dr} L(s, X(s), a(s))ds \\ + e^{-\int_t^T c(X(r))dr} g(X(T))\bigg],$$

where $L : [t,T] \times \mathbb{R}^n \times \Lambda \to \mathbb{R}$, $c, g : \mathbb{R}^n \to \mathbb{R}$ are continuous functions and c is nonnegative. Here, c is a function responsible for discounting, L is the so-called running cost, and g is the terminal cost.

We assume the following throughout the paper.

- The functions $L : [0,T] \times \mathbb{R}^n \times \Lambda \to \mathbb{R}$, $g : \mathbb{R}^n \to \mathbb{R}$ are continuous, and $L(\cdot,\cdot,a)$ is uniformly continuous on bounded subsets of $[0,T] \times \mathbb{R}^n$, uniformly for $a \in \Lambda$.
- There exist $C, N \geq 0$ such that

  $$|L(t,x,a)| + |g(x)| \leq C(1 + |x|^N)$$

  for all $(t,x,a) \in [0,T] \times \mathbb{R}^n \times \Lambda$.

We do not give precise assumptions on the function c since later we will assume that $c \equiv 0$.

Definition 2.1 (*generalized reference probability space*) Let $t \in [0,T)$. The 5-tuple $\mu = (\Omega, \mathscr{F}, \mathscr{F}_s^t, \mathbb{P}, W)$ is called a generalized reference probability space if:

- $(\Omega, \mathscr{F}, \mathbb{P})$ is a complete probability space;
- $\{\mathscr{F}_s^t\}_{t \leq s \leq T}$ is a right-continuous complete filtration in $\mathscr{F}$, i.e. it is a family of σ-fields such that $\mathscr{F}_{s_1}^t \subset \mathscr{F}_{s_2}^t$ for $t \leq s_1 \leq s_2 \leq T$,

  $$\mathscr{F}_s^t = \bigcap_{r>s} \mathscr{F}_r^t \quad \text{for every } s,$$

  and $\mathscr{F}_s^t$ contains all $\mathbb{P}$-null sets of $\mathscr{F}$ for every s;
- W is a standard $\mathscr{F}_s^t$-Brownian motion in $\mathbb{R}^m$, i.e. it is a process adapted to $\mathscr{F}_s^t$ such that $W(t_2) - W(t_1)$ is independent of $\mathscr{F}_{t_1}^t$ for $t \leq t_1 < t_2 \leq T$, $W(t_2) - W(t_1) \sim \mathcal{N}(0, (t_2 - t_1)I)$ for $t \leq t_1 < t_2 \leq T$, and W has continuous trajectories $\mathbb{P}$-almost surely.

Definition 2.2 (*reference probability space*) We say that a generalized reference probability space μ is a reference probability space (RPS) if $\mathscr{F}_s^t = \sigma(\mathscr{F}_s^{t,0}, \mathcal{N})$, where $\mathscr{F}_s^{t,0} = \sigma(W(r) : t \leq r \leq s)$ is the filtration generated by W, and $\mathcal{N}$ is the collection of all the $\mathbb{P}$-null sets in $\mathscr{F}$, and moreover if $W(t) = 0$, $\mathbb{P}$-almost surely.

2.1 Strong Formulation of Optimal Control Problem

For a generalized reference probability space $\mu = (\Omega, \mathscr{F}, \mathscr{F}_s^t, \mathbb{P}, W)$, we consider the set of admissible controls

$$\mathcal{U}_t^\mu = \{a(\cdot) : [t, T] \times \Omega \to \Lambda : a(\cdot) \text{ is } \mathscr{F}_s^t\text{-progressively measurable}\}.$$

A process $a(\cdot)$ is progressively measurable if for every $s \in (t, T]$, $a(\cdot) : [t, s] \times \Omega \to \Lambda$ is $\mathcal{B}([t, s]) \otimes \mathscr{F}_s^t / \mathcal{B}(\Lambda)$ measurable.

The goal in the strong formulation of optimal control problem is to minimize $J(t, x; a(\cdot))$ over all $a(\cdot) \in \mathcal{U}_t^\mu$. In this formulation the generalized reference probability space μ is fixed.

2.2 Weak Formulation of Optimal Control Problem

In the weak formulation of optimal control problem, we set

$$\mathcal{U}_t = \bigcup_\mu \mathcal{U}_t^\mu,$$

where the union is taken over all generalized reference probability spaces μ. The goal then is to minimize $J(t, x; a(\cdot))$ over all $a(\cdot) \in \mathcal{U}_t$.

We will consider here a special weak formulation of our optimal control problem, where

$$\mathcal{U}_t = \left\{ \bigcup_\mu \mathcal{U}_t^\mu : \mu \text{ is a reference probability space} \right\}. \tag{2.2}$$

In the rest of the paper, unless stated otherwise, $\mathcal{U}_t$ will always be defined by (2.2).

2.3 State Equation

We say that $X(\cdot)$ is a solution of the state equation (2.1) if $X(\cdot)$ is progressively measurable and for every $s \geq t$

$$X(s) = x + \int_t^s b(r, X(r), a(r))dr + \int_t^s \sigma(r, X(s), a(r))dW(r), \quad \mathbb{P}\text{-a.s.}$$

The theorem below collects basic properties of the solutions of (2.1) (see e.g. [13], Chap. 2, Sect. 5, also [9], Theorem 1.130).

Theorem 2.3 *Let $t \in [0, T]$, let $\mu = (\Omega, \mathscr{F}, \mathscr{F}_s^t, \mathbb{P}, W)$ be a generalized reference probability space, and $a(\cdot) \in \mathcal{U}_t^\mu$. Then for any $\mathbb{R}^n$-valued $\xi \in L^p(\Omega, \mathscr{F}_t^t, \mathbb{P})$, $p \geq 2$, the SDE (2.1) has unique solution $X(s; t, \xi, a(\cdot))$ such that:*

- *$X(\cdot)$ has continuous trajectories;*
- $$\mathbb{E}\left[\max_{t \leq s \leq T} |X(s; t, \xi, a(\cdot))|^p\right] \leq C_p(1 + \mathbb{E}[|\xi|^p]), \quad ;$$
- $$\mathbb{E}\left[\max_{t \leq s \leq T} |X(s) - Y(s)|^2\right] \leq C|x - y|^2,$$

  where $X(s) = X(s; t, x, a(\cdot))$, $Y(s) = Y(s; t, y, a(\cdot))$ are solutions of (2.1) with initial conditions $X(t) = x \in \mathbb{R}^n$, $Y(t) = y \in \mathbb{R}^n$;
- $$\mathbb{E}\left[\max_{t \leq r \leq s} |X(r) - x|^2\right] \leq C_R(s - t),$$

  if $x \in \mathbb{R}^n$, $|x| \leq R$, where $X(s) = X(s; t, x, a(\cdot))$.

The solution of (2.1) can be obtained as the fixed point of the map

$$K[Y](s) = \xi + \int_t^s b(r, Y(r), a(r))dr + \int_t^s \sigma(r, Y(r), a(r))dW(r)$$

in the space of continuous, progressively measurable processes such that $\|Y\| := \left(\mathbb{E}\left[\max_{t \leq s \leq T} |Y(s)|^p\right]\right)^{\frac{1}{p}} < +\infty$.

3 Dynamic Programming Principle and HJB Equation

The value function for (2.1) in the weak formulation with initial time t is defined as

$$V(t, x) = \inf_{a(\cdot) \in \mathcal{U}_t} J(t, x; a(\cdot)),$$

where $\mathcal{U}_t$ is defined by (2.2).

The central part of the theory is the dynamic programming principle, which states that if $0 \leq t < \eta < T$, then

$$V(t, x) = \inf_{a(\cdot) \in \mathcal{U}_t} \mathbb{E}\left[\int_t^\eta e^{-\int_t^s c(X(r))dr} L(r, X(r), a(r))dr + e^{-\int_t^\eta c(X(r))dr} V(\eta, X(\eta))\right],$$

where $X(r) = X(r; t, x, a(\cdot))$.

The dynamic programming principle is a functional equation for the value function. It connects the stochastic optimal control problem with a partial differential equation (PDE) called the Hamilton-Jacobi-Bellman (HJB) equation which can be used to prove verification theorems, obtain conditions for optimality, construct optimal feedback controls, etc. We remark that the statement of the DPP implies that the functions $V(\eta, X(\eta))$ are measurable, i.e. it requires some apriori knowledge about V. Obviously $V(\eta, X(\eta))$ is measurable if $V(\eta, \cdot)$ is Borel measurable, in particular if $V(\eta, \cdot)$ is continuous. We will prove the DPP in Sect. 4.6 assuming for simplicity that $c \equiv 0$. The proof contains all essential difficulties and the proof in the general case can be easily deduced from it.

We define

$$T_{t,r}(\psi) = \inf_{a(\cdot)\in\mathcal{U}_t} \mathbb{E}\left[\int_t^r e^{-\int_t^s c(X(\tau))d\tau} L(s, X(s), a(s))ds + e^{-\int_t^r c(X(\tau))d\tau} \psi(X(r)) \right].$$

If the DPP holds, then we have $T_{t,T}(\psi) = T_{t,r}(T_{r,T}(\psi))$ for $t < r < T$. Thus the DPP defines a two parameter evolution system and the HJB equation is the PDE associated to this evolution system, the generator equation. The HJB equation has the form

$$\begin{cases} u_t + \inf_{a\in\Lambda} \left\{ \frac{1}{2}\mathrm{Tr}[\sigma(t,x,a)\sigma^*(t,x,a)D^2u] + \langle b(t,x,a)Du\rangle \right. \\ \qquad \left. - c(x)u + L(t,x,a) \right\} = 0, \quad (t,x) \in (0,T)\times\mathbb{R}^n, \\ u(T,x) = g(x), \quad x \in \mathbb{R}^n. \end{cases} \tag{3.1}$$

For $(t, x, r, p, S, a) \in [0, T] \times \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n \times S(n) \times \Lambda$, where $S(n)$ is the set of all symmetric $n \times n$ matrices, we denote

$$F_{CV}(t,x,r,p,S,a) := \frac{1}{2}\mathrm{Tr}\left[\sigma(t,x,a)\sigma^*(t,x,a)S\right] + \langle b(t,x,a), p\rangle \\ - c(x)r + L(t,x,a),$$

and call it the current value Hamiltonian of the system. Its infimum over $a \in \Lambda$,

$$F(t,x,r,p,S) := \inf_{a\in\Lambda} \left\{ \frac{1}{2}\mathrm{Tr}\left[\sigma(t,x,a)\sigma^*(t,x,a)S\right] + \langle b(t,x,a), p\rangle \right. \\ \left. - c(x)r + L(t,x,a) \right\} \tag{3.2}$$

is called the Hamiltonian. Using this notation, the HJB equation (3.1) can be rewritten as

$$\begin{cases} u_t + F(t, x, u, Du, D^2u) = 0, & (t, x) \in (0, T) \times \mathbb{R}^n, \\ u(T, x) = g(x), & x \in \mathbb{R}^n. \end{cases} \tag{3.3}$$

In the rest of the paper we will always assume that

- $c \equiv 0$.

3.1 Verification Theorem, Necessary and Sufficient Conditions for Optimality

The HJB equation can be used to characterize optimal controls. We call $(X(\cdot; t, x, a(\cdot)), a(\cdot))$ an admissible pair if $X(\cdot; t, x, a(\cdot)) : [t, T] \to \mathbb{R}^n$ is the unique solution of the state equation (2.1).

Theorem 3.1 (Verification Theorem, Sufficient Condition for Optimality) *Let $u : [0, T] \times \mathbb{R}^n \to \mathbb{R}$ be a classical solution of (2.1) such that there exist C, $p \geq 0$ such that*

$$|u(t, x)|, |u_t(t, x)|, |Du(t, x)|, \|D^2u(t, x)\| \leq C(1 + |x|^p) \tag{3.4}$$

for all $(t, x) \in [0, T] \times \mathbb{R}^n$. Let $(X^*(\cdot), a^*(\cdot))$ be an admissible pair at (t, x) such that

$$a^*(s) \in \arg\min_{a\in\Lambda} F_{CV}(s, X^*(s), Du(s, X^*(s)), D^2u(s, X^*(s)), a) \tag{3.5}$$

for almost every $s \in [t, T]$ and $\mathbb{P}$-a.s. Then the pair $(X^*(\cdot), a^*(\cdot))$ is optimal at (t, x) and $u(t, x) = V(t, x)$.

Proof If $a(\cdot) \in \mathcal{U}_t$, then by Ito's formula,

$$\begin{aligned} u(t, x) &= \mathbb{E}\Big[u(T, X(T)) - \int_t^T \Big\{u_t(s, X(s)) + \langle b(s, X(s), a(s)), Du(s, X(s))\rangle \\ &\qquad + \frac{1}{2}\mathrm{Tr}(\sigma(s, X(s), a(s))\sigma^*(s, X(s), a(s))D^2u(s, X(s)))\Big\}ds\Big] \\ &= \mathbb{E}\Big[g(X(T)) - \int_t^T \big[u_t(s, X(s)) - L(s, X(s), a(s)) \\ &\qquad + F_{CV}(s, X(s), Du(s, X(s)), D^2u(s, X(s), a(s)))\big] \\ &\leq J(t, x; a(\cdot)), \end{aligned} \tag{3.6}$$

where the last inequality follows since $F_{CV} - F \geq 0$. The equality above holds if and only if $F_{CV} = F$. Therefore, we have $u \leq V$ by taking the infimum over $a(\cdot) \in \mathcal{U}_t$ in the right hand side of (3.6).

Now, let $(X^*(\cdot), a^*(\cdot))$ be an admissible pair at (t, x) satisfying (3.5). If $a(\cdot) = a^*(\cdot)$, then the above gives $u(t, x) = J(t, x; a^*(\cdot)) \geq V(t, x)$. Thus, we have $u(t, x) = V(t, x) = J(t, x; a^*(\cdot))$, which implies that $a^*(\cdot)$ is optimal. □

If we know from the beginning that the solution u in Theorem 3.1 is the value function V, then (3.5) also becomes a necessary condition for optimality.

Corollary 3.2 (Verification Theorem, Necessary Condition for Optimality) *Let $u = V$ in Theorem 3.1 (i.e. the value function V is a smooth solution of (3.1) satisfying (3.4)). If $(X^*(\cdot), a^*(\cdot))$ is an optimal pair at (t, x), then we must have*

$$a^*(s) \in \arg\min_{a \in \Lambda} F_{CV}(s, X^*(s), Du(s, X^*(s)), D^2u(s, X^*(s)), a)$$

for almost every $s \in [t, T]$ and $\mathbb{P}$-a.s.

Proof Since $(X^*(\cdot), a^*(\cdot))$ is an optimal pair at (t, x), we have $V(t, x) = J(t, x; a^*(\cdot))$. Thus inequality (3.6) for $a(\cdot) = a^*(\cdot)$ becomes equality and thus we must have

$$\begin{aligned} F_{CV}(s, X^*(s), &Du(s, X^*(s)), D^2u(s, X^*(s)), a^*(s)) \\ &= F(s, X^*(s), Du(s, X^*(s)), D^2u(s, X^*(s))) \end{aligned}$$

for almost every $s \in [t, T]$ and $\mathbb{P}$-a.s. Hence the claim follows. □

3.2 Construction of Optimal Feedback Controls

We define the multivalued function

$$\begin{cases} \phi : (0, T) \times \mathbb{R}^n \to \mathcal{P}(\Lambda) \\ \phi : (t, x) \to \arg\min_{a \in \Lambda} F_{CV}(t, x, DV(t, x), D^2V(t, x), a) \end{cases} \tag{3.7}$$

and consider the Closed Loop Equation

$$\begin{cases} dX(s) \in b(s, X(s), \phi(s, X(s)))ds + \sigma(s, X(s), \phi(s, X(s)))dW(s) \\ X(t) = x. \end{cases} \tag{3.8}$$

Corollary 3.3 *Suppose that ϕ admits a measurable selection $\psi_t : (t, T) \times \mathbb{R}^n \to \Lambda$ such that the Closed Loop Equation*

$$\begin{cases} dX(s) = b(s, X(s), \psi_t(s, X(s)))ds + \sigma(s, X(s), \psi_t(s, X(s)))dW(s) \\ X(t) = x. \end{cases} \tag{3.9}$$

has a solution $X_{\psi_t}(\cdot)$ in some generalized reference probability space μ. Then the pair $(X_{\psi_t}(\cdot), a_{\psi_t}(\cdot))$ is optimal at (t, x), where $a_{\psi_t}(\cdot) = \psi_t(\cdot, X_{\psi_t}(\cdot))$.

3.3 Uniqueness in Law

Definition 3.4 Let $X_i(s) : (\Omega_i, \mathscr{F}_i, \mathbb{P}_i) \to (\Omega, \mathscr{F})$ be two processes for $i = 1, 2$. $X_1(\cdot)$ and $X_2(\cdot)$ have the same finite-dimensional distributions on $[t, T]$ if there is a set D of full measure on $[t, T]$ such that for any $n \geq 1$, $t \leq t_1 < t_2 < \cdots < t_n \leq T$, $t_j \in D$ $(1 \leq j \leq n)$ and $A \in \underbrace{\mathscr{F} \otimes \mathscr{F} \otimes \ldots \otimes \mathscr{F}}_{n-\text{times}}$,

$$\mathbb{P}_1(w_1 : (X_1(t_1), \ldots, X_1(t_n))(\omega_1) \in A) = \mathbb{P}_2(w_2 : (X_2(t_1), \ldots, X_2(t_n))(\omega_2) \in A).$$

In this case we write $\mathcal{L}_{\mathbb{P}_1}(X_1(\cdot)) = \mathcal{L}_{\mathbb{P}_2}(X_2(\cdot))$.

Let $(\Omega_1, \mathscr{F}_1, \mathscr{F}_s^{1,t}, \mathbb{P}_1, W_1)$ and $(\Omega_2, \mathscr{F}_2, \mathscr{F}_s^{2,t}, \mathbb{P}_2, W_2)$ be two generalized reference probability spaces. Let $(\widetilde{\Omega}, \widetilde{\mathscr{F}})$ be a measurable space and $\Theta_i : \Omega_i \to \widetilde{\Omega}$, $i = 1, 2$, be two random variables. Let $f_i : [t, T] \times \Omega_i \to \mathbb{R}^n$, $i = 1, 2$, be two processes satisfying

$$\mathbb{E}\left\{\int_t^T |f_i(s)|ds\right\} < +\infty, \quad i = 1, 2,$$

and let $\phi_i : [t, T] \times \Omega_i \to \mathbb{R}^{n\times m}$, $i = 1, 2$, be two $\mathscr{F}_s^{i,t}$-progressively measurable processes satisfying

$$\mathbb{E}\left\{\int_t^T \|\phi_i(s)\|^2 ds\right\} < +\infty, \quad i = 1, 2.$$

The following facts can be proved (see [20], Theorems 8.3 and 8.6, where they were proved for more general Banach space-valued processes).

- If $\mathcal{L}_{\mathbb{P}_1}(f_1(\cdot), \Theta_1) = \mathcal{L}_{\mathbb{P}_2}(f_2(\cdot), \Theta_2)$ on $[t, T]$, then

$$\mathcal{L}_{\mathbb{P}_1}\left(\int_t^\cdot f_1(s)ds, \Theta_1\right) = \mathcal{L}_{\mathbb{P}_2}\left(\int_t^\cdot f_2(s)ds, \Theta_2\right) \quad \text{on } [t, T].$$

- If $\mathcal{L}_{\mathbb{P}_1}(\phi_1(\cdot), W_1(\cdot), \Theta_1) = \mathcal{L}_{\mathbb{P}_2}(\phi_2(\cdot), W_2(\cdot), \Theta_2)$, then

$$\mathcal{L}_{\mathbb{P}_1}\left(\int_t^\cdot \phi_1(s)dW_1(s), \Theta_1\right) = \mathcal{L}_{\mathbb{P}_2}\left(\int_t^\cdot \phi_2(s)dW_2(s), \Theta_2\right) \quad \text{on } [t, T].$$

Theorem 3.5 *Let $\mu_i = (\Omega_i, \mathscr{F}_i, \mathscr{F}_s^{i,t}, \mathbb{P}_i, W_i)$, $i = 1, 2$, be two generalized reference probability spaces, $a_i(\cdot) \in \mathcal{U}_t^{\mu_i}$, $i = 1, 2$, and let $\xi_i \in L^2(\Omega_i, \mathscr{F}_t^{i,t}, \mathbb{P}_i)$, $i = 1, 2$, be two $\mathbb{R}^n$-valued random variables. Let $X_i(\cdot)$ be the unique solution of the state equation with control $a_i(\cdot)$ and such that $X_i(t) = \xi_i$. If $\mathcal{L}_{\mathbb{P}_1}(a_1(\cdot), W_1(\cdot), \xi_1) = \mathcal{L}_{\mathbb{P}_2}(a_2(\cdot), W_2(\cdot), \xi_2)$ on $[t, T]$, then $\mathcal{L}_{\mathbb{P}_1}(X_1(\cdot), a_1(\cdot)) = \mathcal{L}_{\mathbb{P}_2}(X_2(\cdot), a_2(\cdot))$ on $[t, T]$.*

Proof The solutions $X_i(\cdot)$ are obtained as the limits of maps

$$K_i[Z_i(\cdot)](s) = \xi_i + \int_t^s b(r, Z_i(r), a_i(r))dr + \int_t^s \sigma(r, Z_i(r), a_i(r))dW_i(r),$$

i.e. $Z_i^1(s) = \xi_i$, $Z_i^{k+1}(s) = K_i[Z_i^k](s)$. Using previous result we have

$$\mathcal{L}_{\mathbb{P}_1}(Z_1^k(\cdot), W_1(\cdot), a_1(\cdot)) = \mathcal{L}_{\mathbb{P}_2}(Z_2^k(\cdot), W_2(\cdot), a_2(\cdot))$$

so passing to the limit as $k \to +\infty$ gives the result. We refer to the proofs of Lemma 1.136 and Proposition 1.137 of [9] for the full details of the proof. □

4 Value Function and Proof of Dynamic Programming Principle

We first need to introduce and develop more technical tools. From now on we will always assume without loss of generality that W has everywhere continuous trajectories.

4.1 Predictable Processes

Definition 4.1 Let $\mu = (\Omega, \mathscr{F}, \mathscr{F}_s^t, \mathbb{P}, W)$ be a reference probability space. The σ-field of $\mathscr{F}_s^{t,0}$-predictable sets $\mathcal{P}_{[t,T]}^{\Omega}$ is the σ-field generated by all sets of the form $(s, r] \times A$, $t \le s < r \le T$, $A \in \mathscr{F}_s^{t,0}$ and $\{t\} \times A$, $A \in \mathscr{F}_t^{t,0}$. The process $a(\cdot)$ with values in Λ is $\mathscr{F}_s^t$-predictable if it is $\mathcal{P}_{[t,T]}^{\Omega}/\mathcal{B}(\Lambda)$ measurable.

Lemma 4.2 *Assume $a(\cdot) \in \mathcal{U}_t^{\mu}$. Then there exists $\mathscr{F}_s^{t,0}$-predictable process $\widetilde{a}(\cdot)$ such that $\widetilde{a}(\cdot) = a(\cdot)$, $dt \otimes \mathbb{P}$-a.e.*

The proof of Lemma 4.2 is in [9], Lemma 1.99. The idea is to approximate $a(\cdot)$ by simple processes.

It is easy to see that $X(\cdot; t, x, a(\cdot))$ is indistinguishable with $X(\cdot; t, x, \widetilde{a}(\cdot))$, i.e., there is a set $\Omega_1 \subset \Omega$, $\mathbb{P}(\Omega_1) = 1$ such that $X(\cdot; t, x, a(\cdot))(\omega) = X(\cdot; t, x, \widetilde{a}(\cdot))(\omega)$ on $[t, T]$ for any $\omega \in \Omega_1$. Therefore, without loss of generality, we can assume that all controls in $\mathcal{U}_t^{\mu}$ are $\mathscr{F}_s^{t,0}$-predictable.

4.2 Canonical Reference Probability Space

Set $\mathbf{W} := \{\omega \in C([t, T], \mathbb{R}^m) : \omega(t) = 0\}$, equipped with the usual sup-norm. Let $\mathcal{B}(\mathbf{W})$ be the Borel σ-field and $\mathbb{P}_*$ be the Wiener measure on $(\mathbf{W}, \mathcal{B}(\mathbf{W}))$, i.e. the unique probability measure on $\mathbf{W}$ that makes the mapping $\mathcal{W} : [t, T] \times \mathbf{W} \to \mathbb{R}^m$, $\mathcal{W}(s, \omega) = \omega(s)$, a Wiener process on $\mathbf{W}$ with values in $\mathbb{R}^m$. Denote by $\mathscr{F}_*$ the completion of $\mathcal{B}(\mathbf{W})$, set $\mathcal{B}_s^{t,0} = \sigma(\mathcal{W}(\tau); t \le \tau \le s)$, $\mathcal{B}_s^t = \sigma(\mathcal{B}_s^{t,0}, \mathcal{N}_*)$, where $\mathcal{N}_*$ are the $\mathbb{P}_*$-null sets, and let $\mathcal{P}_{[t,T]}^{\mathbf{W}}$ be the σ-field of $\mathcal{B}_s^{t,0}$-predictable sets. The 5-tuple $\nu_W := (\mathbf{W}, \mathscr{F}_*, \mathcal{B}_s^t, \mathbb{P}_*, \mathcal{W})$ is called the canonical reference probability space.

4.3 Independence of Value Function of Reference Probability Spaces

The following lemma gives a representation of control processes and its proof can be found in [9], Lemma 2.20.

Lemma 4.3 *Let μ be a reference probability space and $a(\cdot) \in \mathcal{U}_t^\mu$ be $\mathscr{F}_s^{t,0}$-predictable. Then there exists a $\mathcal{P}_{[t,T]}^{\mathbf{W}}/\mathcal{B}(\Lambda)$-measurable function $f : [t, T] \times \mathbf{W} \to \Lambda$ such that $a(s, \omega) = f(s, W(\cdot, \omega))$ for $\omega \in \Omega$, $s \in [t, T]$.*

Let now $a(\cdot) \in \mathcal{U}_t^\mu$ and f be from Lemma 4.3. Suppose that $\mu_1 = (\Omega_1, \mathscr{F}_1, \mathscr{F}_s^{1,t}, \mathbb{P}, W_1)$ is another reference probability space. Then $a_1(s, \omega) = f(s, W_1(\cdot, \omega))$ is $\mathscr{F}_s^{1,t,0}$-predictable and $\mathcal{L}_{\mathbb{P}}(a(\cdot), W(\cdot)) = \mathcal{L}_{\mathbb{P}_1}(a_1(\cdot), W_1(\cdot))$. Thus it follows that $V^{\mu_1}(t, x) \le V^\mu(t, x)$, and by the same argument we can also obtain the reverse inequality. Thus the value function is independent of the choice of a reference probability space and we have the following theorem.

Theorem 4.4 *For every reference probability space μ, we have*

$$V^\mu(t, x) = \inf_{a(\cdot) \in \mathcal{U}_t^\mu} J(t, x; a(\cdot)) = V(t, x).$$

4.4 Standard Reference Probability Spaces

Definition 4.5 A measurable space $(\Omega', \mathscr{F}')$ is standard if it is Borel isomorphic to one of the following: $(\{1, \dots, n\}, \mathcal{B}(\{1, \dots, n\}))$, $(\mathbb{N}, \mathcal{B}(\mathbb{N}))$ or $(\{0, 1\}^{\mathbb{N}}, \mathcal{B}(\{0, 1\}^{\mathbb{N}}))$.

If S is a Polish space, then $(S, \mathcal{B}(S))$ is standard. Also if $(\Omega', \mathscr{F}')$ is standard then it is Borel isomorphic to $[0, 1]$ with the Borel σ-field.

Definition 4.6 A reference probability space μ is standard if there exists σ-field $\mathscr{F}'$ such that $\mathscr{F}_T^{t,0} \subset \mathscr{F}' \subset \mathscr{F}$, $\mathscr{F}$ is the completion of $\mathscr{F}'$, and $(\Omega, \mathscr{F}')$ is standard.

The canonical reference probability space μ_W is standard.

The following is very important. If $(\Omega', \mathscr{F}', \mathbb{P})$ is a probability space such that $(\Omega', \mathscr{F}')$ is standard, then for any σ-field $\mathcal{G} \subset \mathscr{F}'$ there exists a regular conditional probability $p : \Omega' \times \mathscr{F}' \to [0, 1]$ given $\mathcal{G}$ (see e.g. [9], Sect. 1.1.5 for definitions and details). So if μ is a standard reference probability space then there exists regular conditional probability given $\mathscr{F}_s^{t,0}$. We will write $\mathbb{P}_{\omega_0} := p(\omega_0, \cdot)$ or, with abuse of notation, when we want to emphasize the σ-field, we will write $\mathbb{P}(\cdot|\mathscr{F}_s^{t,0})(\omega_0)$. We will write $\mathbb{E}_{\omega_0}$ to denote the expectation with respect to the measure $\mathbb{P}_{\omega_0}$.

We have that for every $\mathscr{F}_s^{t,0}/\mathcal{B}(\mathbb{R}^n)$ measurable random variable Y, for $\mathbb{P}$-a.s. ω_0, $\mathbb{P}_{\omega_0}(Y(\omega) = Y(\omega_0)) = 1$ (see [9], Theorem 1.45, and references there). Also (see e.g. [9], page 102), if $Y \in L^1(\Omega, \mathscr{F}, \mathbb{P})$, then $\mathbb{E}_{\omega_0}$ (as a function of ω_0) belongs to $L^1(\Omega, \mathscr{F}, \mathbb{P})$ and

$$\mathbb{E}[Y] = \mathbb{E}[\mathbb{E}[Y|\mathscr{F}_s^t]] = \mathbb{E}[\mathbb{E}_{\omega_0}[Y]].$$

4.5 "Conditioned" Reference Probability Spaces

Suppose that $0 \le t < \eta < T$, and $\mu = (\Omega, \mathscr{F}, \mathscr{F}_s^t, \mathbb{P}, W)$ is a standard reference probability space. We set $W_\eta(s) = W(s) - W(\eta)$. We will use conditional expectations in the proof of the dynamic programming principle so we need to make sure that we stay within the framework of the reference probability spaces. The following two lemmas guarantee this. They correspond to Lemmas 2.25 and 2.26 of [9] and their proofs can be found there.

Lemma 4.7 *For $\mathbb{P}$-a.s. ω_0, $\mu^{\omega_0} = (\Omega, \mathscr{F}_{\omega_0}, \mathscr{F}^\eta_{\omega_0,s}, \mathbb{P}_{\omega_0})$ is a reference probability space on $[\eta, T]$, where $\mathscr{F}_{\omega_0}$ is the completion of $\mathscr{F}'$ by $\mathbb{P}_{\omega_0} = \mathbb{P}(\cdot|\mathscr{F}_s^{t,0})(\omega_0)$ null sets, and $\mathscr{F}^\eta_{\omega_0,s}$ is the augmented filtration generated by W_η.*

Lemma 4.8 *Let μ, μ^{ω_0} be as above, and let $a(\cdot) \in \mathcal{U}_t^\mu$ be $\mathscr{F}_s^{t,0}$-predictable. Then $a_{|[\eta,T]}(\cdot) \in \mathcal{U}_\eta^{\mu^{\omega_0}}$ for $\mathbb{P}$-a.e. ω_0.*

We just remark here that Lemma 4.8 follows easily after we show that $\mathscr{F}_s^{t,0} \subset \mathscr{F}^\eta_{\omega_0,s}$ for $\eta \le s \le T$.

Finally we need to ensure that the solution of (2.1) in the reference probability space μ is also the solution in the reference probability spaces μ^{ω_0}.

Lemma 4.9 *Let μ, μ^{ω_0} and $a(\cdot)$ be as in Lemma 4.8, and let $X^\mu(\cdot; t, x, a(\cdot))$ be the solution of (2.1) in the reference probability space μ. Then, up to an indistinguishable modification, for $\mathbb{P}$-a.e. ω_0, $X^\mu(\cdot; t, x, a(\cdot))$ is the solution of (2.1) on $[\eta, T]$, with initial condition $X^\mu(\eta)$, in the reference probability space μ^{ω_0}, i.e. for $\mathbb{P}$-a.e. ω_0, $X^\mu(\cdot; t, x, a(\cdot)) = X^{\mu^{\omega_0}}(\cdot; \eta, X^\mu(\eta), a(\cdot))$.*

Proof We first observe that, using the continuity of trajectories of $X^\mu(\cdot) := X^\mu(\cdot; t, x, a(\cdot))$, one can show that, up to an indistinguishable modification, $X^\mu(\cdot)$ can be considered to be $\mathscr{F}^\eta_{\omega_0,s}$-progressively measurable on $[\eta, T]$ for $\mathbb{P}$-a.e. ω_0 (see [9], the proof of Proposition 2.26, page 103, for a precise argument). In fact one obtains that $X^\mu(\cdot)$ is indistinguishable with a process which is $\sigma(\mathscr{F}_s^{t,0}, \widetilde{\Omega})$-progressively measurable for some $\widetilde{\Omega}$ such that $\mathbb{P}(\widetilde{\Omega}) = 1$.

It is easy to see that $\mathbb{P}$ a.s. we have

$$X^\mu(s) = X^\mu(\eta) + \int_\eta^s b(s, X^\mu(s), a(s))ds + \int_\eta^s \sigma(s, X^\mu(s), a(s))dW_\eta(s) \quad \text{on } [\eta, T].$$

Since for every set Ω_1 such that $\mathbb{P}(\Omega_1) = 1$, we have $\mathbb{P}_{\omega_0}(\Omega_1) = 1$ for $\mathbb{P}$-a.e. ω_0, the above identity is satisfied $\mathbb{P}_{\omega_0}$-a.s. for $\mathbb{P}$ a.e. ω_0. Thus to prove that $X^\mu(\cdot)$ is the solution in the reference probability spaces μ^{ω_0}, it is enough to show that for $\mathbb{P}$-a.e. ω_0, the stochastic integral

$$I_\mu(s) = \int_\eta^s \sigma(s, X^\mu(s), a(s))dW_\eta(s)$$

in the reference probability space μ is $\mathbb{P}_{\omega_0}$-a.e. equal on $[\eta, T]$ to the same stochastic integral in the reference probability space μ^{ω_0}. We denote this integral by $I_{\mu^{\omega_0}}(s)$. Since the stochastic integrals have continuous paths it is enough to show it for a single s. We note that $I_{\mu^{\omega_0}}(s)$ is well defined since $\sigma(s, X^\mu(s), a(s))$ is $\mathscr{F}^\eta_{\omega_0,s}$-progressively measurable on $[\eta, T]$ for $\mathbb{P}$-a.e. ω_0. We also note that since

$$\mathbb{E}\left[\int_\eta^s |X^\mu(r)|^2 dr\right] = \mathbb{E}\left[\mathbb{E}_{\omega_0}\left[\int_\eta^s |X^\mu(r)|^2 dr\right]\right],$$

we have $E_{\omega_0}\left[\int_\eta^s |X^\mu(r)|^2 dr\right] < \infty$ for $\mathbb{P}$-a.e. ω_0.

Denote $\Phi(s) = \sigma(s, X^\mu(s), a(s))$. There exist a sequence of elementary and $\mathscr{F}_r^{t,0}$-progressively measurable processes Φ_n, such that

$$\mathbb{E}\int_\eta^s |\Phi(r) - \Phi_n(r)|^2 dr \to 0.$$

The processes Φ_n are also $\mathscr{F}^\eta_{\omega_0,r}$-progressively measurable. Since $\mathbb{E}|\int_\eta^s [\Phi(r) - \Phi_n(r)]dW_\eta(r)|^2 \to 0$, passing to a subsequence if necessary, we can assume that

$$\int_\eta^s \Phi_n(r)dW_\eta(r) \to I_\mu(s), \quad \text{on } \Omega_2, \tag{4.1}$$

where Ω_2 is a set such that $\mathbb{P}(\Omega_2) = 1$ and hence $\mathbb{P}_{\omega_0}(\Omega_2) = 1$ for $\mathbb{P}$-a.e. ω_0. Since

$$\mathbb{E}\left[E_{\omega_0}\left[\int_\eta^s |\Phi(r) - \Phi_n(r)|^2 dr\right]\right] = \mathbb{E}\left[\int_\eta^s |\Phi(r) - \Phi_n(r)|^2 dr\right] \to 0$$

as $n \to \infty$, up to a subsequence, for $\mathbb{P}$-a.e. ω_0, we have

$$\mathbb{E}_{\omega_0}\left[\int_\eta^s |\Phi(r)-\Phi_n(r)|^2 dr\right] \to 0$$

as $n \to \infty$. So, for $\mathbb{P}$-a.e. ω_0, there exists a subsequence of Φ_n such that

$$\int_\eta^s \Phi_n(r)dW_\eta(r) \to I_{\mu^{\omega_0}}(s), \quad \mathbb{P}_{\omega_0}\text{-a.e.} \tag{4.2}$$

Thus (4.1) and (4.2) imply that, for $\mathbb{P}$-a.e. ω_0, $I_\mu(s) = I_{\mu^{\omega_0}}(s)$, $\mathbb{P}_{\omega_0}$-a.e.

□

4.6 Proof of the Dynamic Programming Principle

We first show the uniform continuity in x of the cost functionals.

Lemma 4.10 *For every $R > 0$, there is a modulus ρ_R such that*

$$|J(t,x;a(\cdot)) - J(t,y;a(\cdot))| + |V(t,x) - V(t,y)| \le \rho_R(|x-y|)$$

for all $t \in [0,T]$, $x, y \in \mathbb{R}^n$, $|x|, |y| \le R$, $a(\cdot) \in \mathcal{U}_t$. Moreover,

$$|J(t,x;a(\cdot))| + |V(t,x)| \le C(1+|x|^N)$$

for all $(t,x) \in [0,T] \times \mathbb{R}^n$, $a(\cdot) \in \mathcal{U}_t$.

Proof The result follows easily from the assumptions about L and g, and the estimates of Theorem 2.3. □

We can now prove the DPP. We remind that we assume that $c \equiv 0$.

Theorem 4.11 (Dynamic Programming Principle) *Let $0 \le t < \eta \le T$, $x \in \mathbb{R}^n$. Then*

$$V(t,x) = \inf_{a(\cdot)\in\mathcal{U}_t} \mathbb{E}\left[\int_t^\eta L(r, X(r), a(r))dr + V(\eta, X(\eta))\right],$$

where $X(r) = X(r; t, x, a(\cdot))$.

Proof Denote

$$\widetilde{\mathcal{U}}_t = \left\{\bigcup_\mu \mathcal{U}_t^\mu : \mu \text{ is a standard RPS}\right\}.$$

It is enough to show DPP with $\mathcal{U}_t$ replaced by $\widetilde{\mathcal{U}}_t$ since V^η is the same for every reference probability space μ and we have joint uniqueness in law. Of course it is enough to assume $t < \eta < T$. Thus we will show

$$V(t,x) = \inf_{a(\cdot)\in\widetilde{\mathcal{U}}_t} \mathbb{E}\left[\int_t^\eta L(s,X(s),a(s))ds + V(\eta,X(\eta))\right] \tag{4.3}$$

for $t < \eta < T$. In fact it would be enough to replace $\mathcal{U}_t$ by $\mathcal{U}_t^{\nu_W}$, where ν_W is the canonical RPS.

Part 1. (Inequality $\geq$ in (4.3)) Let μ be standard reference probability space and $a(\cdot) \in \mathcal{U}_t^\mu$. We can assume that $a(\cdot)$ is $\mathscr{F}_s^{t,0}$-predictable and hence, by Lemma 4.8, $a_{|[\eta,T]}(\cdot) \in \mathcal{U}_\eta^{\mu^{\omega_0}}$ for $\mathbb{P}$-a.e. ω_0. By Lemma 4.9, for $\mathbb{P}$-a.e. ω_0, $X^\mu(\cdot) = X^{\mu^{\omega_0}}(\cdot;\eta, X^\mu(\eta), a(\cdot))$ on $[\eta,T]$, where $\mu^{\omega_0} = (\Omega, \mathscr{F}_{\omega_0}, \mathscr{F}^\eta_{\omega_0,s}, \mathbb{P}_{\omega_0}, \mathcal{W}_\eta)$. We also recall that for $\mathbb{P}$-a.e. ω_0, $\mathbb{P}_{\omega_0}(\{\omega : X(\eta,\omega) = X(\eta,\omega_0)\}) = 1$. Therefore,

$$\begin{aligned}
J(t,x,a(\cdot)) &= \mathbb{E}\left[\int_t^\eta L(s,X(s),a(s))ds\right] \\
&\qquad + \mathbb{E}\left[\int_\eta^T L(s,X(s),a(s))ds + g(X(T))\right] \\
&= \mathbb{E}\left[\int_t^\eta L(s,X(s),a(s))ds\right] + \mathbb{E}\left[\mathbb{E}_{\omega_0}\left[\int_\eta^T L(s,X(s),a(s))ds + g(X(T))\right]\right] \\
&= \mathbb{E}\left[\int_t^\eta L(s,X(s),a(s))ds\right] + \mathbb{E}\left[J^{\mu^{\omega_0}}(\eta, X(\eta,\omega_0); a(\cdot))\right] \\
&\geq \mathbb{E}\left[\int_t^\eta L(s,X(s),a(s))ds + V(\eta,X(\eta))\right].
\end{aligned}$$

Taking infimum over all $a(\cdot) \in \widetilde{\mathcal{U}}_t$ above implies

$$V(t,x) \geq \inf_{a(\cdot)\in\widetilde{\mathcal{U}}_t} \mathbb{E}\left[\int_t^\eta L(s,X(s),a(s))ds + V(\eta,X(\eta))\right].$$

Part 2. (Inequality $\leq$ in (4.3)) Let $\varepsilon > 0$ and let $a(\cdot) \in \mathcal{U}_t^\mu$ for some standard reference probability space $\mu = (\Omega, \mathscr{F}, \mathscr{F}_s^t, \mathbb{P}, W)$. Using continuity of J and V in x from Lemma 4.10, we can find a partition $\{D_j\}$ of $\mathbb{R}^n$ into disjoint Borel sets D_j, $j = 1, 2, \cdots$ such that if $x, y \in D_j$ and $\widetilde{a}(\cdot) \in \mathcal{U}_t$ then

$$|J(\eta,x;\widetilde{a}(\cdot)) - J(\eta,y;\widetilde{a}(\cdot))| + |V(\eta,x) - V(\eta,y)| < \varepsilon.$$

For each j, we choose $x_j \in D_j$ and $a_j(\cdot) \in \mathcal{U}_t^{\mu_j}$ for some reference probability space $\mu_j = (\Omega_j, \mathscr{F}_j, \mathscr{F}^\eta_{j,s}, \mathbb{P}_j, W_j)$ such that

$$J(\eta, x_j; a_j(\cdot)) < V(\eta, x_j) + \varepsilon.$$

We can assume that $a_j(\cdot)$ are $\mathscr{F}^{\eta,0}_{j,s}$-predictable.

Now let $f_j : [\eta,T] \times C([\eta,T],\mathbb{R}^n) \to \Lambda$ be functions such that

$$a_j(s,\omega) = f_j(s, W_j(\cdot,\omega)).$$

Then the processes

$$\widetilde{a}_j(s,\omega) = f_j(s, W_\eta(\cdot,\omega))$$

are $\mathscr{F}_s^{t,0}$-progressively measurable and for $\mathbb{P}$-a.e. ω_0 are $\mathscr{F}_{\omega_0,s}^{\eta}$-progressively measurable in the reference probability spaces $\mu^{\omega_0} = (\Omega, \mathscr{F}_{\omega_0}, \mathscr{F}_{\omega_0,s}^{\eta}, \mathbb{P}_{\omega_0}, W_\eta)$. Moreover,

$$\mathcal{L}_{\mathbb{P}_{\omega_0}}(\widetilde{a}_j(\cdot), W_\eta) = \mathcal{L}_{\mathbb{P}_j}(a_j(\cdot), W_j).$$

We define new control

$$a^\eta(s,\omega) = a(s,\omega)\mathbf{1}_{\{t\le s<\eta\}} + \mathbf{1}_{\{s\ge\eta\}}\sum_{j\in\mathbb{N}} \widetilde{a}_j(s,\omega)\mathbf{1}_{\{X(\eta;t,x,a(\cdot))\in D_j\}}.$$

Denote $O_j = \{w : X(\eta; t, x, a(\cdot)) \in D_j\}$. We notice that $a^\eta(\cdot) \in \mathcal{U}_t^\mu$. Denote $X(s) = X(s; t, x, a^\eta(\cdot))$. Then $X(s) = X(s; t, x, a(\cdot))$ on $[t,\eta]$, $\mathbb{P}$ a.s. Since for $\mathbb{P}$-a.e. ω_0, $X(\eta,\omega) = X(\eta,\omega_0)$, $\mathbb{P}_{\omega_0}$-a.s., if $\omega_0 \in O_j$ then $a^\eta(\cdot) = \widetilde{a}_j(\cdot)$ on $[\eta, T]$, $\mathbb{P}_{\omega_0}$-a.s., and thus for $\mathbb{P}$-a.s. ω_0, $a^\eta_{|[\eta,T]} \in \mathcal{U}_\eta^{\mu^{\omega_0}}$, and

$$\mathcal{L}_{\mathbb{P}_{\omega_0}}(a^\eta(\cdot), W_\eta(\cdot)) = \mathcal{L}_{\mathbb{P}_j}(a_j(\cdot), W_j(\cdot)).$$

Moreover, by Lemma 4.9, we can assume that for $\mathbb{P}$-a.e. ω_0, $X(\cdot) = X^{\mu^{\omega_0}}(\cdot; \eta, X(\eta), a^\eta(\cdot))$ on $[\eta, T]$, $\mathbb{P}_{\omega_0}$ a.s. Thus, by Theorem 3.5,

$$\mathcal{L}_{\mathbb{P}_{\omega_0}}(X(\cdot), a^\eta(\cdot)) = \mathcal{L}_{\mathbb{P}_j}(X^{\mu_j}(\cdot), a_j(\cdot)), \tag{4.4}$$

where $X^{\mu_j} = X(s; \eta, X(\eta; t, x, a(\cdot))(\omega_0), a_j(\cdot))$. Therefore,

$$\begin{aligned}
&\mathbb{E}\left[\int_\eta^T L(s, X(s), a^\eta(s))ds + g(X(T))\right]\\
&= \mathbb{E}\left[\mathbb{E}_{\omega_0}\left[\int_\eta^T L(s, X(s), a^\eta(s))ds + g(X(T))\right]\right]\\
&= \sum_{j=1}^\infty \int_{O_j} \mathbb{E}_{\omega_0}\left[\int_\eta^T L(s, X(s), a^\eta(s))ds + g(X(T))\right] d\mathbb{P}(\omega_0)\\
&= \sum_{j=1}^\infty \int_{O_j} J_{\mathbb{P}_{\omega_0}}(\eta, X(\eta; t, x, a(\cdot))(\omega_0); a^\eta(\cdot)) d\mathbb{P}(\omega_0)\\
&= \sum_{j=1}^\infty \int_{O_j} J_{\mathbb{P}_j}(\eta, X(\eta; t, x, a(\cdot))(\omega_0); a_j(\cdot)) d\mathbb{P}(\omega_0),
\end{aligned}$$

where we used (4.4) to get the last equality. Now, for $\omega_0 \in O_j$,

$$\begin{aligned} J_{\mathbb{P}_j}(\eta, X(\eta; t, x, a(\cdot))(\omega_0); a_j(\cdot)) &< J_{\mathbb{P}_j}(\eta, x_j; a_j(\cdot)) + \varepsilon \\ &< V(\eta, x_j) + 2\varepsilon \\ &< V(\eta, X(\eta; t, x, a(\cdot))(\omega_0)) + 3\varepsilon. \end{aligned}$$

Therefore,

$$\mathbb{E}\left[\int_\eta^T L(s, X(s), a^\eta(s))ds + g(X(T))\right] < \mathbb{E}[V(\eta, X(\eta))] + 3\varepsilon,$$

so we obtain

$$J(t, x; a^\eta(\cdot)) \leq \mathbb{E}\left[\int_t^\eta L(s, X(s), a(s))ds + V(\eta, X(\eta))\right] + 3\varepsilon.$$

Since $a(\cdot)$ was arbitrary, the above inequality implies

$$V(t, x) \leq \inf_{a(\cdot)\in\widetilde{\mathcal{U}}_t} \mathbb{E}\left[\int_t^\eta L(s, X(s), a(s))ds + V(\eta, X(\eta))\right] + 3\varepsilon.$$

It now remains to send $\varepsilon \to 0$. □

4.7 Continuity of the Value Function in t

Having the dynamic programming principle we can now easily prove the continuity of the value function in t.

Corollary 4.12 *For every $R > 0$ there exists a modulus $\bar{\rho}_R$ such that*

$$|V(t, x) - V(s, x)| \leq \bar{\rho}_R(|t - s|) \quad \textit{for all} \ \ t, s \in [0, T], x \in \mathbb{R}^n, |x| \leq R.$$

Proof Suppose $s > t$, and $|x| \leq R$, $R \geq 1$. Using the dynamic programming principle, and estimates of Theorem 2.3 and Lemma 4.10, we have

$$
\begin{aligned}
|V(t,x)-V(s,x)| &\le \sup_{a(\cdot)\in\mathcal{U}_t}\mathbb{E}\left|\int_t^s L(r,X(r),a(r))dr+V(s,X(s))-V(s,x)\right| \\
&\le \sup_{a(\cdot)\in\mathcal{U}_t}\mathbb{E}\int_t^s C(1+|X(r)|^N)dr+\sup_{a(\cdot)\in\mathcal{U}_t}\mathbb{E}|V(s,X(s))-V(s,X)| \\
&\le C(1+|x|^N)(s-t)+\sup_{a(\cdot)\in\mathcal{U}_t}\mathbb{E}\left[\rho_{2R}\left(|X(s)-x|\mathbf{1}_{\{|X(s)|\le 2R\}}\right)\right] \\
&\qquad\qquad +\sup_{a(\cdot)\in\mathcal{U}_t}\mathbb{E}\left[C(1+|X(s)|^N+|x|^N)\mathbf{1}_{\{|X(s)|>2R\}}\right] \\
&\le C_R(s-t)+\sup_{a(\cdot)\in\mathcal{U}_t}\rho_{2R}\left(\mathbb{E}|X(s)-x|\right)+\sup_{a(\cdot)\in\mathcal{U}_t}C_R\left(\mathbb{P}\left(\{|X(s)|>2R\}\right)\right)^{\frac{1}{2}} \\
&\le C_R(s-t)+\rho_{2R}\left(C_R\sqrt{s-t}\right)+C_R\sqrt{s-t}.
\end{aligned}
$$

Above we also used Jensen's inequality and the fact that we can assume that the moduli ρ_{2R} are concave. □

4.8 Dynamic Programming Principle with Stopping Times

The knowledge that V is continuous allows us to formulate the dynamic programming principle in a stopping time version.

For every $a(\cdot)\in\mathcal{U}_t^{\mu}$ for some reference probability space $\mu=(\Omega,\mathscr{F},\mathscr{F}_s^t,\mathbb{P},W)$, we choose an $\mathscr{F}_s^t$-stopping time $\tau_{a(\cdot)}$ with values in $[t,T]$. We define $\mathcal{V}_t$ to be the set of such pairs $(a(\cdot),\tau_{a(\cdot)})$, over all $a(\cdot)\in\mathcal{U}_t$. The set of all $\mathscr{F}_s^t$-stopping times with values in $[t,T]$ will be denoted by $\mathcal{A}^{\mu}$.

We recall that τ is an $\mathscr{F}_s^t$-stopping time if for all $s\ge t$, $\{\tau\le s\}=\{\omega\in\Omega:\tau(\omega)\le s\}\in\mathscr{F}_s^t$. For instance if A is an open or a closed subset of $\mathbb{R}^n$, then the exit time of $X(\cdot)$ from A is a stopping time.

Theorem 4.13 (DPP-Stopping Time Formulation I)*) Let* $0\le t\le T$, $x\in\mathbb{R}^n$. *Then*

$$
V(t,x)=\inf_{(a(\cdot),\tau_{a(\cdot)})\in\mathcal{V}_t}\mathbb{E}\left[\int_t^{\tau_{a(\cdot)}}L(s,X(s),a(s))ds+V(\tau_{a(\cdot)},X(\tau_{a(\cdot)}))\right].
$$

The proof of Theorem 4.13 in its Hilbert space version can be found in [9], page 241 (Theorem 3.70 there). Theorem 4.13 in turn easily implies another formulation of the dynamic programming principle with stopping times, given below in Theorem 4.14. In a slightly different formulation such a version of the dynamic programming principle can be found for instance in [11], page 176.

Theorem 4.14 (DPP-Stopping Time Formulation II) *Let* $0\le t\le T$, $x\in\mathbb{R}^n$. *Then*

$$V(t,x) = \inf_{\mu \in \mathcal{R}, a(\cdot) \in \mathcal{U}_t^\mu} \inf_{\tau \in \mathcal{A}_t^\mu} \mathbb{E}\left[\int_t^\tau L(s, X(s), a(s))ds + V(\tau, X(\tau))\right]$$
$$= \inf_{\mu \in \mathcal{R}, a(\cdot) \in \mathcal{U}_t^\mu} \sup_{\tau \in \mathcal{A}_t^\mu} \mathbb{E}\left[\int_t^\tau L(s, X(s), a(s))ds + V(\tau, X(\tau))\right],$$

where $\mathcal{R}$ is the set of all reference probability spaces.

5 Value Function Solves the HJB Equation

We can now prove that the value function V is the viscosity solution of the HJB equation (3.3). The stopping time formulation of the dynamic programming principle is very useful for this purpose.

Definition 5.1 An upper-semicontinuous function $u : (0, T] \times \mathbb{R}^n \to \mathbb{R}$ is a viscosity subsolution of the terminal value problem (3.3) if $u(T, x) \leq g(x)$ for all $x \in \mathbb{R}^n$, and whenever $u - \phi$ has a local maximum at $(t, x) \in (0, T) \times \mathbb{R}^n$ for some $\phi \in C^{1,2}((0, T) \times \mathbb{R}^n)$, then

$$\phi_t(t,x) + F(t, x, D\phi(t,x), D^2\phi(t,x)) \geq 0.$$

A lower-semicontinuous function $u : (0, T] \times \mathbb{R}^n \to \mathbb{R}$ is a viscosity supersolution of the terminal value problem (3.3) if $u(T, x) \geq g(x)$ for all $x \in \mathbb{R}^n$, and whenever $u - \phi$ has a local minimum at $(t, x) \in (0, T) \times \mathbb{R}^n$ for some $\phi \in C^{1,2}((0, T) \times \mathbb{R}^n)$, then

$$\phi_t(t,x) + F(t, x, D\phi(t,x), D^2\phi(t,x)) \leq 0.$$

A function u is a viscosity solution of (3.3) if it is a viscosity subsolution and a viscosity supersolution of (3.3).

Theorem 5.2 *Value function V is the unique viscosity solution of the HJB equation (3.1). The uniqueness holds within the class of continuous functions $u : (0, T] \times \mathbb{R}^n \to \mathbb{R}$ such that there exist $C, a > 0$ such that*

$$|u(t,x)| \leq Ce^{a(\log(1+|x|))^2} \quad \textit{for all } (t,x) \in (0,T] \times \mathbb{R}^n.$$

Proof We will only prove the existence part. The proof of uniqueness can be found for instance in [9], Theorem 3.50, pages 206–212.

V is a viscosity supersolution: Suppose that $V - \phi$ has a local minimum at $(t, x) \in (0, T) \times \mathbb{R}^n$ for some $\phi \in C^{1,2}((0, T) \times \mathbb{R}^n)$. Let $0 < \varepsilon < 1$. For $a(\cdot) \in \mathcal{U}_t$ we take $\tau_{a(\cdot)} = \min(t + \varepsilon, \tau^\varepsilon_{a(\cdot)})$, where $\tau^\varepsilon_{a(\cdot)}$ is the exit time of $X(s) = X(s; t, x, a(\cdot))$ from $B_{\varepsilon^{\frac{1}{4}}}(x)$. Recall that we have

$$\mathbb{E}\left[\sup_{t\le r\le t+\varepsilon}|X(r)-x|^2\right]\le C\varepsilon,$$

which implies

$$\mathbb{P}(\tau^{\varepsilon}_{a(\cdot)}<t+\varepsilon)<C\sqrt{\varepsilon}. \tag{5.1}$$

Now, for (s, y) in a neighborhood of (t, x),

$$V(s,y)-V(t,x)\ge \phi(s,y)-\phi(t,x). \tag{5.2}$$

By Theorem 4.13, there exists $a_\varepsilon(\cdot)\in\mathcal{U}_t$ such that, denoting $\tau_\varepsilon := \tau_{a_\varepsilon(\cdot)}$ and $X_\varepsilon(s) := X(s; t, x, a_\varepsilon(\cdot))$,

$$V(t,x)+\varepsilon^2\ge \mathbb{E}\left[\int_t^{\tau_\varepsilon}L(s,X_\varepsilon(s),a_\varepsilon(s))ds+V(\tau_\varepsilon,X_\varepsilon(\tau_\varepsilon))\right].$$

Therefore, by (5.2) and Itô's formula,

$$\begin{aligned}
\varepsilon^2 &\ge \mathbb{E}\left[\int_t^{\tau_\varepsilon}L(s,X_\varepsilon(s),a_\varepsilon(s))ds\right]+\mathbb{E}[\phi(\tau_\varepsilon,X_\varepsilon(\tau_\varepsilon))-\phi(t,x)] && (5.3)\\
&=\mathbb{E}\Bigg[\int_t^{\tau_\varepsilon}\Bigg[L(s,X_\varepsilon(s),a_\varepsilon(s))+\phi_t(s,X_\varepsilon(s))\\
&\quad+\frac{1}{2}\mathrm{Tr}(\sigma(s,X_\varepsilon(s),a_\varepsilon(s))\sigma^*(s,X_\varepsilon(s),a_\varepsilon(s))D^2\phi(s,X_\varepsilon(s)))\\
&\quad+\langle D\phi(s,X_\varepsilon(s)),b(s,X_\varepsilon(s),a_\varepsilon(s))\rangle\Bigg]ds\Bigg]\\
&\ge\mathbb{E}\Bigg[\int_t^{\tau_\varepsilon}\big[\phi_t(t,x)+\frac{1}{2}\mathrm{Tr}(\sigma(t,x,a_\varepsilon(s))\sigma^*(t,x,a_\varepsilon(s))D^2\phi(t,x))\\
&\quad+\langle D\phi(t,x),b(t,x,a_\varepsilon(s))\rangle+L(t,x,a_\varepsilon(s))\big]ds\Bigg]-\varepsilon\widetilde{\rho}(\varepsilon), && (5.4)
\end{aligned}$$

where $\widetilde{\rho}(\varepsilon)\to 0$ as $\varepsilon\to 0$. Therefore, taking the infimum inside the integral and using (5.1), we obtain

$$\begin{aligned}
\varepsilon^2&\ge\mathbb{E}\Bigg[\int_t^{t+\varepsilon}\inf_{a\in\Lambda}\big[\phi_t(t,x)+\frac{1}{2}\mathrm{Tr}(\sigma(t,x,a)\sigma^*(t,x,a)D^2\phi(t,x))\\
&\quad+\langle b(t,x,a),D\phi(t,x)\rangle+L(t,x,a)\big]ds\Bigg]-\varepsilon\widetilde{\rho}(\varepsilon)-C\varepsilon\sqrt{\varepsilon}\\
&=\varepsilon\left[\phi_t(t,x)+F(t,x,D\phi(t,x),D^2\phi(t,x))\right]-\varepsilon\widetilde{\rho}(\varepsilon)-C\varepsilon\sqrt{\varepsilon}.
\end{aligned}$$

We now divide both sides of the above inequality by ε and send $\varepsilon\to 0$ to obtain

$$\phi_t(t,x) + F(t,x,D\phi(t,x),D^2\phi(t,x)) \le 0.$$

V is a viscosity subsolution: We fix $a \in \Lambda$ and take the constant control $a(s) \equiv a$. Let now $V - \phi$ have a local maximum at $(t,x) \in (0,T) \times \mathbb{R}^n$ for some $\phi \in C^{1,2}((0,T) \times \mathbb{R}^n)$. Let $0<\varepsilon<1$. By Theorem 4.13, denoting $X(s):=X(s;t,x,a(\cdot))$,

$$V(t,x) \le \mathbb{E}\left[\int_t^{\tau_\varepsilon} L(s,X(s),a)ds + V(\tau_\varepsilon, X(\tau_\varepsilon))\right],$$

where $\tau_\varepsilon = \min(t+\varepsilon, \tau_a^\varepsilon)$, where as before τ_a^ε is the exit time of $X(s)$ from $B_{\varepsilon^{\frac{1}{4}}}(x)$. The rest of the proof follows the proof of the supersolution property and the arguments are even easier as neither the process $X(\cdot)$ nor the control changes with ε. We obtain in place of (5.3),

$$\mathbb{E}\Bigg[\int_t^{\tau_\varepsilon} \big[\phi_t(t,x) + \frac{1}{2}\mathrm{Tr}(\sigma(t,x,a)\sigma^*(t,x,a)D^2\phi(t,x)) + \langle D\phi(t,x), b(t,x,a)\rangle + L(t,x,a)\big]ds\Bigg] \ge -\varepsilon\widetilde{\rho}(\varepsilon)$$

which, using (5.1), produces

$$\phi_t(t,x) + \frac{1}{2}\mathrm{Tr}(\sigma(t,x,a)\sigma^*(t,x,a)D^2\phi(t,x)) + \langle b(t,x,a), D\phi(t,x)\rangle + L(t,x,a) \ge -\widetilde{\rho}(\varepsilon) - C\sqrt{\varepsilon}.$$

We then send $\varepsilon \to 0$ and take the infimum over all $a \in \Lambda$.

We remark that, alternatively, one can also argue using the dominated convergence theorem here. □

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Regularity of Solutions of Obstacle Problems –Old & New–

Shigeaki Koike

Abstract Two kinds of machinery to show regularity of solutions of bilateral/unilateral obstacle problems are presented. Some generalizations of known results in the literature are included. Several important open problems in the topics are given.

Keywords Bilateral/Unilateral obstacle problem · Regularity of solutions · Bernstein method · Bellman-Isaacs equation · Penalization · Fully nonlinear elliptic equation · Weak Harnack inequality · L^p viscosity solution

Mathematics Subject Classification 49J40 · 35J86 · 35J87

1 Introduction

In this survey, we overview regularity of solutions of obstacle problems associated with second-order uniformly elliptic partial differential equations (PDE for short). Particularly, we show two different arguments to obtain estimates on solutions of obstacle problems due to maximum principles. On the other hand, there have appeared a huge amount of results concerning on regularity of solutions of variational inequalities, whose typical example is the obstacle problem. However, our methods here do not rely on integration by parts.

One of techniques here is the so-called Bernstein method, which is relatively old, while the other is quite a new one. Inspired by an idea in [20], we have found an interesting argument in [42], which can be applied to fully nonlinear PDE with unbounded coefficients and inhomogeneous terms.

According to [52], it seems that Fichera [24, 25] first studied the Signorini problem as a variational inequality, where a free boundary arises on the boundary of domains.

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Stampacchia in [54] announced variational inequalities in Hilbert spaces as a modification of Lax-Milgram theorem. Later, Lions-Stampacchia in [46] introduced unilateral obstacle problems in the whole domain as an example of minimization problems associated with energy functionals over closed convex sets.

Afterwards, several regularity results on solutions of variational inequalities appeared in [6, 7, 27, 44].

We shall first consider a minimizing problem of given energies under restrictions. Fix a bounded domain $\Omega \subset \mathbb{R}^n$ with smooth boundary $\partial\Omega$. For a given $\psi \in C(\overline{\Omega})$, which is called an upper obstacle, we set a closed convex set

$$K^\psi := \{u \in H_0^1(\Omega) \mid u \leq \psi \text{ a.e. in } \Omega\},$$

where $H_0^1(\Omega)$ is the closure of $C_0^\infty(\Omega)$ with respect to $H^1(\Omega)$ norm.

For any fixed $f \in L^2(\Omega)$, by setting our energy

$$E[u] := \int_\Omega \left(\frac{1}{2}|Du|^2 - fu\right) dx$$

for $u \in K^\psi$, it is known that there is a unique $u \in K^\psi$ such that

$$E[u] = \min_{v \in K^\psi} E[v].$$

Formally, we observe that

$$\begin{cases} -\Delta u \leq f \text{ in } \Omega, \\ \quad u \leq \psi \text{ in } \Omega, \\ -\Delta u = f \text{ in } \{x \in \Omega \mid u(x) < \psi(x)\}. \end{cases}$$

Hence, we may write down this problem as a Bellman equation

$$\max\{-\Delta u - f, u - \psi\} = 0 \quad \text{in } \Omega \tag{1.1}$$

under the Dirichlet condition $u = 0$ on $\partial\Omega$.

Obstacle problems arise in various settings both from purely mathematical interests and from their rich applications. For later topics, we only refer to some text books [3, 26, 34, 45, 53, 56] because it is too wide for this article to mention these issues. We will concentrate on regularity of solutions of obstacle problems but not on regularity of the free boundary, which may be more interesting subject. See [11, 14, 28] and references therein for this topics.

It is worth mentioning that for (1.1), we can only expect solutions to belong to $W^{2,\infty}(\Omega)$ in general even if ψ and f are smooth enough. The first example is a simple one.

Example 1.1 Let $\Omega := (-\frac{5}{4}, \frac{5}{4})$ for $n = 1$, and $\psi(x) = x^2 - 1$. We easily see that

$$u(x) := \begin{cases} |x| - \frac{5}{4} \ (\frac{1}{2} < |x| \le \frac{5}{4}), \\ x^2 - 1 \ \ (|x| \le \frac{1}{2}), \end{cases}$$

satisfies

$$\max\left\{-\frac{d^2u}{dx^2}, u - \psi\right\} = 0 \quad \text{a.e. in } \Omega$$

under the Dirichlet condition $u(\pm\frac{5}{4}) = 0$. We notice that this u is not twice differentiable at $x = \pm\frac{1}{2}$.

We next show the other example when there is a 0th order term of unknown functions.

Example 1.2 Let Ω and ψ be the same ones as in Example 1.1. For the inhomogeneous term $f \in C^2(\overline{\Omega})$, we choose

$$f(x) = \begin{cases} |x| - \frac{5}{4} & (\frac{1}{4} < |x| \le \frac{5}{4}), \\ -8x^4 + 3x^2 - \frac{37}{32} & (|x| \le \frac{1}{4}). \end{cases}$$

It is easy to verify that the same function u in Example 1.1 satisfies

$$\max\left\{-\frac{d^2u}{dx^2} + u - f, u - \psi\right\} = 0 \quad \text{a.e. in } \Omega.$$

We next consider a minimizing problem under the other kind of restriction. Given two obstacles $\varphi, \psi \in C(\overline{\Omega})$ satisfying the compatibility condition

$$\varphi \le \psi \quad \text{in } \Omega, \quad \text{and} \quad \varphi \le 0 \le \psi \quad \text{on } \partial\Omega, \tag{1.2}$$

we introduce the closed convex set

$$K_\varphi^\psi := \{u \in H_0^1(\Omega) \mid \varphi \le u \le \psi \text{ a.e. in } \Omega\}.$$

Again, it is known that there is a unique $u \in K_\varphi^\psi$ such that

$$E[u] = \min_{v \in K_\varphi^\psi} E[v].$$

We observe that u satisfies at least formally

$$\min\{\max\{-\Delta u - f, u - \psi\}, u - \varphi\} = 0 \quad \text{in } \Omega. \tag{1.3}$$

This is a bilateral obstacle problem, which is an Isaacs equation while (1.1) is called a Bellman equation for unilateral obstacle problems.

Because of (1.2), it is easy to see formally that (1.3) is equivalent to the following PDE:

$$\max\{\min\{-\Delta u - f, u - \varphi\}, u - \psi\} = 0 \quad \text{in } \Omega.$$

Using the standard Euclidean inner product $\langle \cdot, \cdot \rangle$, we consider the energy

$$E[u] := \int_\Omega \left(\frac{1}{2} \langle ADu, Du \rangle + \frac{1}{2} cu^2 - fu \right) dx,$$

where $A := (a_{ij}) : \Omega \to S^n$ is positively definite; $\exists \theta > 0$ such that

$$\langle A(x)\xi, \xi \rangle \geq \theta |\xi|^2 \quad \text{for any } \xi \in \mathbb{R}^n \text{ and } x \in \Omega. \tag{1.4}$$

Here and later S^n denotes the set of real-valued symmetric matrices of order n.

When $a_{ij} \in C^1(\overline{\Omega})$ for simplicity, the minimizer of $E[\cdot]$ over $H_0^1(\Omega)$ formally satisfies

$$Lu = f \quad \text{in } \Omega,$$

where

$$Lu := -\text{Tr}(AD^2 u) + \langle b, Du \rangle + cu.$$

Here, we set

$$b := (b_1, \ldots, b_n) = -\left(\sum_{j=1}^n \frac{\partial a_{1j}}{\partial x_j}, \ldots, \sum_{j=1}^n \frac{\partial a_{nj}}{\partial x_j} \right).$$

Hence, as before, we derive the Bellman equation associated with the minimization of $E[\cdot]$ over K^ψ:

$$\max\{Lu - f, u - \psi\} = 0 \quad \text{in } \Omega.$$

Throughout this paper, we shall suppose that there is $M_c > 0$ such that

$$0 \leq c(x) \leq M_c \quad \text{for } x \in \overline{\Omega}. \tag{1.5}$$

If we suppose that c is positive in $\overline{\Omega}$, then particularly, L^∞ estimates become easier to prove. In fact, under (1.5), we need a perturbation function such as w in Proposition 2.1. We choose $R_0 > 0$ such that

$$\Omega \subset B_{R_0}. \tag{1.6}$$

Here and later, we set $B_r := \{y \in \mathbb{R}^n \mid |x| < r\}$, and $B_r(x) := x + B_r$ for $x \in \mathbb{R}^n$.

In this survey, we are concerned with regularity of solutions for obstacle problems, where the PDE part is given by the above linear second-order uniformly elliptic operator L or Bellman-Isaacs ones. We will always assume that the existence of (approximate) solutions of each obstacle problem. In Sects. 2 and 3, using Bernstein method, we obtain (local) $W^{2,\infty}(\Omega)$ estimates on solutions of approximate equations via penalization. We consider the case when the PDE part is linear with bilateral obstacles in Sect. 2 while we deal with Bellman equations with bi- and unilateral obstacles in Sect. 3. In Sect. 4, to show the Hölder continuity of the first derivative, we apply the weak Harnack inequality to solutions of bilateral obstacle problems, where the main PDE part can be of Isaacs type, and moreover, coefficients and inhomogeneous terms can be unbounded. Since fully nonlinear PDE contain 0th order terms in Sect. 4, we need to modify basic tools such as the Aleksandrov-Bakelman-Pucci (ABP for short) maximum principle, weak Harnack inequality and local maximum principle to PDE with 0th order terms. In Appendix, we present those for the reader's convenience.

2 A Linear Operator Case

Although some results in this section will be generalized in Sect. 3, we will present those to clarify our basic argument.

In this section, for coefficients in the linear operator L, and obstacles, we impose that

$$a_{ij}, b_i, f, c, \varphi, \psi \in C^2(\overline{\Omega}). \tag{2.1}$$

To introduce penalty equations, we need $\beta \in C^2(\mathbb{R})$ such that

$$\begin{cases} (i) & \beta(t) = 0 \text{ for } t \leq 0, \\ (ii) & \beta(t) \text{ grows linearly } t \gg 1, \\ (iii) & \beta' \geq 0 \text{ and } \beta'' \geq 0 \text{ in } \mathbb{R}. \end{cases} \tag{2.2}$$

For instance, it is easy to verify that $\beta \in C^2(\mathbb{R})$ defined by

$$\beta(t) := \begin{cases} 0 & \text{for } t \leq 0, \\ -t^4 + 4t^3 & \text{for } t \in (0, 2), \\ 16(t-1) & \text{for } t \geq 2 \end{cases}$$

satisfies all the properties in (2.2).

For $\varepsilon \in (0, 1)$, we will use $\beta_\varepsilon(t) := \beta(t/\varepsilon)$ for $t \in \mathbb{R}$. Furthermore, we easily observe that

$$\text{there is } \hat{C} > 0 \text{ such that } -\hat{C} \leq \beta_\varepsilon(t) - t\beta_\varepsilon'(t) \leq 0. \tag{2.3}$$

We shall consider approximate equations with penalized terms:

$$Lu + \beta_\varepsilon(u - \psi) - \beta_\varepsilon(\varphi - u) = f \quad \text{in } \Omega \tag{2.4}$$

under the Dirichlet condition

$$u = 0 \quad \text{on } \partial\Omega. \tag{2.5}$$

Hereafter, we will use the notations: for $t, s \in \mathbb{R}$,

$$t \vee s := \max\{t, s\} \quad \text{and} \quad t \wedge s := \min\{t, s\}.$$

For simplicity, we will write

$$u_{x_i},\ u_{x_i x_j},\ etc. \quad \text{for} \quad \frac{\partial u}{\partial x_i},\ \frac{\partial^2 u}{\partial x_i \partial x_j},\ etc.,\ respectively.$$

We also use the summation convention for repeated indices, e.g..

$$a_{ij} u_{x_i x_j} = \sum_{i,j=1}^{n} a_{ij} u_{x_i x_j}.$$

Proposition 2.1 (L^∞ estimates) *Assume* (1.2), (1.4), (1.5) *and* (2.1). *Let* $u^\varepsilon \in C(\overline{\Omega}) \cap C^2(\Omega)$ *be solutions of* (2.4) *satisfying* (2.5). *Then, there is* $\hat{C} > 0$ *such that*

$$-\hat{C} \max_{\overline{\Omega}} f^- - \max_{\overline{\Omega}} \psi^- \le u^\varepsilon \le \max_{\overline{\Omega}} \varphi^+ + \hat{C} \max_{\overline{\Omega}} f^+ \quad in\ \overline{\Omega} \quad for\ \varepsilon \in (0, 1).$$

Proof We shall only prove the second inequality since the first one can be shown similarly. We shall write u for u^ε for simplicity.

Setting $C_0 := \max_{\overline{\Omega}} \varphi^+ \ge 0$ and $C_1 := \max_{\overline{\Omega}} f^+$, we shall suppose

$$\Theta := \max_{\overline{\Omega}} \{u - C_0 - \mu(C_1 + \delta) w\} > 0.$$

Here $\mu > 0, \delta \in (0, 1)$ and $w(x) := e^{2\gamma R_0} - e^{\gamma(x_1 + R_0)} > 0$ for $x = (x_1, \ldots, x_n) \in \Omega$, where $\gamma \ge 1$, and $R_0 > 0$ is from (1.6).

By letting $\hat{x} \in \overline{\Omega}$ satisfy $\Theta = u(\hat{x}) - C_0 - \mu(C_1 + \delta) w(\hat{x})$, (2.5) yields $\hat{x} \in \Omega$. Hence, at $\hat{x} = (\hat{x}_1, \ldots, \hat{x}_n) \in \Omega$, the weak maximum principle implies

$$\begin{aligned} 0 &\le -a_{ij} u_{x_i x_j} + b_i u_{x_i} + \mu(C_1 + \delta)\gamma e^{\gamma(\hat{x}_1 + R_0)}(-a_{11}\gamma + b_1) \\ &\le f - cu - \overline{\beta_\varepsilon} + \underline{\beta_\varepsilon} + \mu(C_1 + \delta)\gamma e^{\gamma(\hat{x}_1 + R_0)}(-\theta\gamma + |b_1|). \end{aligned} \tag{2.6}$$

Here and later, to distinguish composite functions $\beta_\varepsilon(u - \psi)$ and $\beta_\varepsilon(\varphi - u)$, we use the following notation:

$$\overline{\beta_\varepsilon}(\cdot) := \beta_\varepsilon(u(\cdot) - \psi(\cdot)) \quad \text{and} \quad \underline{\beta_\varepsilon}(\cdot) := \beta_\varepsilon(\varphi(\cdot) - u(\cdot)).$$

Thus, for a fixed $\gamma := (\max_{\overline{\Omega}} |b_1| + \theta)/\theta$, (2.6) together with (1.5) implies

$$\theta\mu(C_1+\delta)\gamma \le f - c\{C_0+\mu(C_1+\delta)w\} + \underline{\beta_\varepsilon} \le f + \underline{\beta_\varepsilon} \quad \text{at } \hat{x}.$$

Since $\varphi - u \le \varphi - C_0 - \mu(C_1+\delta)w \le \varphi - C_0 \le 0$ at $\hat{x}$, this inequality yields

$$\theta\mu(C_1+\delta)\gamma \le f(\hat{x}),$$

which is a contradiction for $\mu > 1/(\theta\gamma)$. Therefore, for fixed $\mu, \gamma > 0$ in the above, we have $\Theta \le 0$, which concludes the proof. □

We notice that in the above proof, we do not need the whole of (2.1) but we do not present "minimal" hypotheses on regularity of given functions for the sake of presentations.

Proposition 2.2 ($W^{2,p}$ estimates) *Assume* (1.2), (1.4), (1.5) *and* (2.1). *Let* $u^\varepsilon \in C^2(\overline{\Omega})$ *be solutions of* (2.4) *satisfying* (2.5). *Then, there is* $\tilde{C} > 0$ *such that for* $\varepsilon \in (0,1)$,

$$\begin{cases} \|\overline{\beta_\varepsilon}(u^\varepsilon - \psi)\|_{L^\infty(\Omega)} \le \max\limits_{\overline{\Omega}} f^+ + M_c \max\limits_{\overline{\Omega}} \psi^- + \tilde{C}\|D\psi\|_{W^{1,\infty}(\Omega)}, \\ \|\underline{\beta_\varepsilon}(\varphi - u^\varepsilon)\|_{L^\infty(\Omega)} \le \max\limits_{\overline{\Omega}} f^- + M_c \max\limits_{\overline{\Omega}} \varphi^+ + \tilde{C}\|D\varphi\|_{W^{1,\infty}(\Omega)}. \end{cases} \tag{2.7}$$

In particular, for each $p \in (1,\infty)$, *there is* $\tilde{C}_p > 0$ *such that*

$$\|u^\varepsilon\|_{W^{2,p}(\Omega)} \le \tilde{C}_p \quad \textit{for } \varepsilon \in (0,1). \tag{2.8}$$

Proof We shall only show the bound for $\overline{\beta_\varepsilon}$ since we can prove the other one similarly. We shall simply write u for u^ε again.

Suppose that $\Theta := \max_{\overline{\Omega}} \overline{\beta_\varepsilon} > 0$. In view of the second inequality of (1.2), we can choose $\hat{x} \in \Omega$ such that $\Theta = \overline{\beta_\varepsilon}(u(\hat{x}) - \psi(\hat{x}))$. Since $\overline{\beta_\varepsilon}$ is nondecreasing, we see that $u - \psi$ attains its maximum at $\hat{x} \in \Omega$. Hence, we have at $\hat{x}$,

$$\begin{aligned} 0 &\le -a_{ij}(u-\psi)_{x_i x_j} + b_i(u-\psi)_{x_i} \\ &= f - cu - \overline{\beta_\varepsilon} + \underline{\beta_\varepsilon} + a_{ij}\psi_{x_i x_j} - b_i \psi_{x_i} \\ &\le f - c\psi - \overline{\beta_\varepsilon} + \underline{\beta_\varepsilon} + C\|D\psi\|_{W^{1,\infty}(\Omega)}. \end{aligned}$$

Here and later, C denotes the various positive constant depending only on known quantities.

Note that the first inequality of (1.2) yields

$$(\varphi - u)(\hat{x}) \le (\psi - u)(\hat{x}) < 0.$$

Therefore, we have $0 \le \overline{\beta_\varepsilon} \le \overline{\beta_\varepsilon}(\hat{x}) \le \max_{\overline{\Omega}} f^+ + M_c \max_{\overline{\Omega}} \psi^- + C\|D\psi\|_{W^{1,\infty}(\Omega)}$ in $\overline{\Omega}$, where $M_c > 0$ is the constant in (1.5) □

Remark 2.3 When we consider Bellman operators in Sect. 3, the L^∞ estimate on the penalty terms for obstacles does not imply (2.8) because we will have one more penalty term, which cannot be evaluated by the above argument.

Now, we show local $W^{2,\infty}$ estimates on solutions of (2.4). Our argument is more or less standard though we do not know if the next proposition has appeared somewhere to our knowledge.

Proposition 2.4 (Local $W^{2,\infty}$ estimates) *Assume* (1.2), (1.4), (1.5) *and* (2.1). *Let* $u^\varepsilon \in C^4(\Omega) \cap C^1(\overline{\Omega})$ *be solutions of* (2.4). *Then, for each compact set* $K \Subset \Omega$, *there is* $\tilde{C}_K > 0$ *independent of* $\varepsilon \in (0, 1)$ *such that*

$$\max_K |D^2 u^\varepsilon| \le \tilde{C}_K.$$

Proof Choose $\zeta \in C_0^\infty(\Omega)$ such that

$$0 \le \zeta \le 1 \quad \text{in } \Omega, \quad \text{and} \quad \zeta = 1 \quad \text{on } K.$$

Putting $M := \max_\Omega \zeta |D^2 u^\varepsilon|$, we may suppose $M \ge 1$.

Writing u and β for u^ε and β_ε, respectively, we set

$$V := \zeta^2 |D^2 u|^2 + \gamma M \{\beta(u - \psi) + \beta(\varphi - u)\} + \gamma M |Du|^2.$$

We shall write $\overline{\beta} := \beta(u - \psi)$ and $\underline{\beta} := \beta(\varphi - u)$ again for simplicity. In the proceeding calculations, we shall more simply write u_{ij}, u_{ijk}, $a_{ij,k}$ etc. for $u_{x_i x_j}$, $u_{x_i x_j x_k}$, $(a_{ij})_{x_k}$ etc., respectively.

We may suppose that $\max_{\overline{\Omega}} V = V(\hat{x}) > 0$ for some $\hat{x} \in \Omega$. By setting $L_0 \xi := -a_{ij}\xi_{ij} + b_i \xi_i$, since $L_0 V(\hat{x}) \ge 0$ by the weak maximum principle, at $\hat{x}$, we have

$$\begin{aligned}
0 \le -a_{ij} & \left\{ \begin{array}{l} 2\zeta\zeta_{ij}|D^2u|^2 + 2\zeta_i\zeta_j|D^2u|^2 + 8\zeta\zeta_i u_{k\ell}u_{k\ell j} + 2\zeta^2 u_{k\ell}u_{k\ell ij} \\ +2\zeta^2 u_{k\ell i}u_{k\ell j} + \gamma M \overline{\beta}''(u-\psi)_i(u-\psi)_j + \gamma M \overline{\beta}'(u-\psi)_{ij} \\ +\gamma M \underline{\beta}''(\varphi-u)_i(\varphi-u)_j + \gamma M \underline{\beta}'(\varphi-u)_{ij} + 2\gamma M u_k u_{kij} \\ +2\gamma M u_{ki}u_{kj} \end{array} \right\} \\
& +b_i \left\{ \begin{array}{l} 2\zeta\zeta_i|D^2u|^2 + 2\zeta^2 u_{k\ell}u_{k\ell i} + \gamma M \overline{\beta}'(u-\psi)_i + \gamma M \underline{\beta}'(\varphi-u)_i \\ +2\gamma M u_k u_{ki} \end{array} \right\} \\
\le & -2\theta(\zeta^2|D^3u|^2 + \gamma M|D^2u|^2) - \gamma M\theta(\overline{\beta}''|D(u-\psi)|^2 + \underline{\beta}''|D(\varphi-u)|^2) \\
& +C(|D^2u|^2 + \zeta|D^2u||D^3u|) + \gamma M\overline{\beta}' L_0(u-\psi) + \gamma M \underline{\beta}' L_0(\varphi-u) \\
& +2\zeta^2 u_{k\ell}L_0 u_{k\ell} + 2\gamma M u_k L_0 u_k.
\end{aligned}$$

By Young's inequality, at $\hat{x}$, we have

$$\begin{aligned}
I_0 &:= \theta\zeta^2|D^3u|^2 + \theta\gamma M\{|D^2u|^2 + \overline{\beta}''|D(u-\psi)|^2 + \underline{\beta}''|D(\varphi-u)|^2)\} \\
&\le \gamma M\{\overline{\beta}' L_0(u-\psi) + \underline{\beta}' L_0(\varphi-u)\} + 2\zeta^2 u_{k\ell}L_0u_{k\ell} + 2\gamma M u_k L_0 u_k \\
&=: I_1 + I_2 + I_3
\end{aligned}$$

for large $\gamma > 1$.

Since (2.8) for $p > n$ implies the $W^{1,\infty}$ estimates on u, we will not mention the dependence on $\|u\|_{W^{1,\infty}(\Omega)}$ in the calculations below. In order to estimate I_3, we differentiate (2.4) with respect to x_k to obtain

$$L_0 u_k = f_k + a_{ij,k}u_{ij} - b_{i,k}u_i - cu_k - c_k u - \overline{\beta}'(u-\psi)_k + \underline{\beta}'(\varphi - u)_k.$$

Thus, we have

$$\begin{aligned} I_0 \le\, & C\gamma M(1+|D^2u|) + I_1 + I_2 \\ & +\gamma M\{\overline{\beta}'(-|Du|^2+|D\psi|^2) + \underline{\beta}'(-|Du|^2+|D\varphi|^2)\}. \end{aligned} \tag{2.9}$$

To estimate I_2, we differentiate (2.4) with respect to x_k and x_ℓ to obtain

$$\begin{aligned} L_0 u_{k\ell} =\, & f_{k\ell} + a_{ij,k\ell}u_{ij} + a_{ij,k}u_{ij\ell} + a_{ij,\ell}u_{ijk} - b_{i,k\ell}u_i - b_{i,k}u_{i\ell} - b_{i,\ell}u_{ik} \\ & -\overline{\beta}'(u-\psi)_{k\ell} - \overline{\beta}''(u-\psi)_k(u-\psi)_\ell + \underline{\beta}'(\varphi-u)_{k\ell} \\ & +\underline{\beta}''(\varphi-u)_k(\varphi-u)_\ell. \end{aligned}$$

Hence, we have

$$\begin{aligned} I_2 \le\, & \theta\zeta^2|D^3u|^2 + C(1+|D^2u|^2) + 2M\{\overline{\beta}''|D(u-\psi)|^2 + \underline{\beta}''|D(\varphi-u)|^2\} \\ & +\zeta^2\{\overline{\beta}'(-|D^2u|^2+|D^2\psi|^2) + \underline{\beta}'(-|D^2u|^2+|D\varphi|^2)\}. \end{aligned}$$

Thus, inserting this in (2.9) with $\gamma \ge 2/\theta$, we have

$$\begin{aligned} \theta\gamma M|D^2u|^2 \le\, & C\gamma M(1+|D^2u|) + C(1+|D^2u|^2) \\ & +\overline{\beta}'\left\{\begin{array}{c} -\zeta^2(|D^2u|^2-|D^2\psi|^2) - M(|Du|^2-|D\psi|^2) \\ +\gamma M(f - u - \overline{\beta} + \underline{\beta} - L_0\psi) \end{array}\right\} \\ & +\underline{\beta}'\left\{\begin{array}{c} -\zeta^2(|D^2u|^2-|D^2\varphi|^2) - M(|Du|^2-|D\varphi|^2) \\ +\gamma M(-f + u + \overline{\beta} - \underline{\beta} + L_0\varphi) \end{array}\right\} \\ =:\, & J_1 + J_2 + J_3. \end{aligned}$$

Case 1 : $J_2 \le 0$ and $J_3 \le 0$: In this case, for a largely fixed $\gamma \gg 2/\theta$, we immediately have

$$|D^2u|^2(\hat{x}) \le C,$$

which together with Propositions 2.1 and 2.2 implies

$$M^2 \le V(\hat{x}) \le C(1+M).$$

Case 2 : $J_2 > 0$ or $J_3 > 0$: We shall only consider the case of $J_2 > 0$ since the other one can be shown similarly. In view of (2.7), we see that

$$\zeta^2 |D^2 u|^2(\hat{x}) \le C(1+M),$$

which yields

$$M^2 \le V(\hat{x}) \le C(1+M).$$

Therefore, M is bounded independently from $\varepsilon \in (0,1)$. □

Remark 2.5 We note that our choice of auxiliary functions V does not work for Bellman operators in Sect. 3. Instead, we will barrow a different one from [23], which can be applied only to unilateral obstacle problems.

As mentioned in Sect. 1, Jensen in [32] showed $W^{2,\infty}(\Omega)$ estimates under additional assumptions on the coefficients. Here, in order to simplify the argument, we shall obtain the $W^{2,\infty}$ bound near the flat boundary under additional assumptions. Setting $x' = (x_1, \ldots, x_{n-1}) \in \mathbb{R}^{n-1}$, we suppose that Ω satisfies

$$\begin{cases} \Omega \cap B_1 = \{x = (x', x_n) \mid |x| < 1, x_n > 0\}, \\ \partial\Omega \cap B_1 = \{(x', 0) \mid |x'| < 1\}. \end{cases} \tag{2.10}$$

To show $W^{2,\infty}$ estimates near $\partial\Omega$ for bilateral obstacle problems, we follow the argument in [31].

Theorem 2.6 *Assume* (1.2), (1.4), (1.5), (2.1) *and* (2.10). *Assume also that*

$$a_{in} = 0 \quad \text{on } \partial\Omega \cap B_1. \tag{2.11}$$

Let $u^\varepsilon \in C^4(\overline{\Omega})$ *be solutions of* (2.4). *Then, there is* $\hat{C} > 0$ *such that*

$$|D^2 u^\varepsilon| \le \hat{C} \quad \text{in } \overline{\Omega} \cap B_{\frac{1}{2}}.$$

Remark 2.7 Under hypothesis (2.11), we note that

$$-a_{nn} u^\varepsilon_{nn} + b_n u^\varepsilon_n = f \quad \text{on } \partial\Omega \cap B_1 \tag{2.12}$$

since $u^\varepsilon_i = u^\varepsilon_{ij} = 0$ for $1 \le i, j \le n-1$ on $\partial\Omega \cap B_1$ by (2.5).

Proof As before, we shall write u for u^ε, and use other simplified notations.

We choose $\eta \in C_0^\infty(B_1)$ such that

$$\begin{cases} 0 \le \eta \le 1 & \text{in } B_1, \\ \eta = 1 & \text{in } B_{\frac{1}{2}}, \\ \eta_{x_n} = 0 & \text{on } \partial\Omega \cap B_1. \end{cases} \tag{2.13}$$

Setting

$$v_{ij} := \begin{cases} u_{ij} & \text{for } (i,j) \ne (n,n), \\ u_{nn} - \hat{b}_n u_n + \hat{f} & \text{for } (i,j) = (n,n), \end{cases}$$

where $\hat{b}_n = b_n/a_{nn}$ and $\hat{f} = f/a_{nn}$, we define

$$|D^2 v|^2 := \sum_{i,j=1}^n v_{ij}^2 = \sum_{(i,j)\ne(n,n)} u_{ij}^2 + (u_{nn} - \hat{b}_n u_n + \hat{f})^2.$$

Consider W defined by

$$W := e^{Ax_n}\eta^2 |D^2 v|^2 + \gamma M(\overline{\beta} + \underline{\beta}) + \gamma M |Du|^2,$$

where $M := \max_{\overline{\Omega}} \eta |D^2 u|$, and $A, \gamma > 1$ will be fixed. We may suppose $M \ge 1$.

Let $\hat{x} = (\hat{x}_1, \hat{x}_2, \ldots, \hat{x}_n) \in \overline{\Omega} \cap \overline{B}_1$ be a point such that $\max_{\overline{\Omega}\cap\overline{B}_1} W = W(\hat{x}) > 0$. Because of $W(\hat{x}) > 0$, we may also assume that $\hat{x} \in \overline{\Omega} \cap B_1$.

Since the argument in the proof of Proposition 2.4 can be applied to the case when $\hat{x} \in \Omega \cap B_1$ with some minor changes, we may suppose $\hat{x} \in \partial\Omega \cap B_1$, and we will obtain a contradiction. Since $|D^2 v|^2 = 2\sum_{i=1}^{n-1} u_{in}^2$ at $\hat{x}$, (2.5) implies

$$W_n = 2e^{A\hat{x}_n}\eta^2 \sum_{i=1}^{n-1}(Au_{in}^2 + 2u_{in}u_{inn}) + 2\gamma M u_n(\hat{b}_n u_n - \hat{f}).$$

By noting $u_{inn} = (\hat{b}_n u_n - \hat{f})_i$ at $\hat{x}$, this equality implies

$$\begin{aligned} W_n &\ge 2e^{A\hat{x}_n}\eta^2 \left\{ (A - C)\sum_{i=1}^{n-1} u_{in}^2 - C \right\} - CM \\ &\ge 2e^{A\hat{x}_n} \left\{ \eta^2 (A - C)\sum_{i=1}^{n-1} u_{in}^2 - CM \right\} \\ &\ge 2e^{A\hat{x}_n}(\eta^2 |D^2 v|^2 - CM) \end{aligned}$$

for a fixed $A > 1$. If the right hand side of the above is non-positive, then we have

$$\eta^2 |D^2 v|^2(\hat{x}) \le CM,$$

which implies the uniform bound of M independent of $\varepsilon \in (0, 1)$. Therefore, we have $W_n(\hat{x}) > 0$ but this implies that $\hat{x}$ is not the maximum of W, which is a contradiction. □

Following [31], we give a sufficient condition to derive (2.11). We use the following notation:

$$B_r^+ := \{x = (x_1, \dots, x_n) \in B_r \mid x_n > 0\}.$$

Although B_1^+ is not a smooth domain, considering an appropriate smooth domain $\Omega \supset B_1^+$, we may assume ∂B_1^+ is smooth. The next proposition yields (2.11).

Proposition 2.8 *Suppose that there is* $\alpha \in (0, 1)$ *such that*

$$a_{ij} \in C^{3,\alpha}(\overline{B}_1^+) \quad \textit{for } 1 \le i, j \le n.$$

There is a C^4*-diffeomorphism* $T = (T_1, \dots, T_n) : \overline{B}_1^+ \to T(\overline{B}_1^+)$ *such that* $T_k \in C^{4,\alpha}(\overline{B}_1^+)$ *such that*

$$\hat{a}_{kl}(y) = \sum_{i,j=1}^{n} a_{ij}(T^{-1}(y)) \frac{\partial T_k}{\partial x_i}(x) \frac{\partial T_\ell}{\partial x_j}(x)$$

and

$$\hat{a}_{kn}(y', 0) = 0 \quad (1 \le k \le n-1), \quad \textit{for } T^{-1}(y', 0) \in \overline{B}_1^+.$$

Proof. We begin with considering the following PDE

$$-a_{ij}(x)u_{x_i x_j} + b_i(x)u_{x_i} + c(x)u + \beta_\varepsilon(u - \psi) - \beta_\varepsilon(\varphi - u) = f(x) \quad \text{in } B_1^+$$

such that $u(x) = 0$ for $x = (x_1, \dots, x_{n-1}, 0) \in \overline{B}_1^+$. Consider the change of variable $\hat{T} = (\hat{T}^1, \dots, \hat{T}^n) : \overline{B}_+^1 \to \mathbb{R}^n$ defined by

$$y_k = \hat{T}^k(x) = \begin{cases} x_k + T^k(x) - T^k(x', 0) & \text{for } x = (x', x_n) \in \overline{B}_1^+, \ 1 \le k \le n-1, \\ x_n & \text{for } x = (x', x_n) \in \overline{B}_1^+, \ k = n. \end{cases}$$

Here, $T = (T^1, \dots, T^n) \in C^{4,\alpha}(\overline{B}_1^+; \mathbb{R}^n)$ is the solution of

$$\begin{cases} -\Delta T^k + T^k = 0 & \text{in } B_1^+, \\ \langle DT^k, \nu \rangle = \dfrac{a_{kn}}{a_{nn}} & \text{on } \partial B_1^+, \end{cases} \tag{2.14}$$

where ν is the outward unit normal of ∂B_1^+.

It is easy to rewrite the equation for $v(y) := u(x)$ with this new variable $y = \hat{T}(x)$:

$$-\hat{a}_{ij}(y)v_{y_i y_j} + \hat{b}_i(y)u_{y_i} + \hat{c}(y)v + \beta_\varepsilon(v - \hat{\psi}) - \beta_\varepsilon(\hat{\varphi} - v) = \hat{f}(y),$$

where $\hat{c}(y) = c(x)$, $\hat{f}(y) = f(x)$, $\hat{\psi}(y) = \psi(x)$, $\hat{\varphi}(y) = \varphi(x)$,

$$\hat{a}_{ij}(y) = \sum_{k,\ell=1}^{n} a_{k\ell}(x)\hat{T}^i_{x_k}(x)\hat{T}^j_{x_\ell}(x),$$

and

$$\hat{b}_i(y) = \sum_{k=1}^{n} b_i(x)\hat{T}^i_{x_k}(x) - \sum_{k,\ell=1}^{n} a_{k\ell}(x)\hat{T}^i_{x_k x_\ell}(x).$$

In view of the boundary condition of (2.5), it is immediate to verify that for $1 \le i \le n-1$,

$$\begin{aligned}\hat{a}_{in}(y', 0) &= \sum_{k,\ell=1}^{n} a_{k\ell}(x', 0)\hat{T}^i_{x_k}(x', 0)\hat{T}^n_{x_\ell}(x', 0)\\ &= \sum_{k=1}^{n} a_{kn}(x', 0)\hat{T}^i_{x_k}(x', 0)\\ &= a_{in}(x', 0) + a_{nn}(x', 0)\hat{T}^i_{x_n}(x', 0) = 0. \qquad \square\end{aligned}$$

Open question 1: Is it possible to obtain $W^{2,\infty}(\Omega)$ estimates with no extra assumption (2.11) on a_{ij}?

3 A Bellman Type Operator Case

In this section, we obtain $W^{1,\infty}$ bounds for solutions of bilateral obstacle problems when the PDE part is of Bellman type. However, we do not know if we can show further estimates on the second derivative of solutions of penalized systems below for bilateral obstacle problems. Thus, following [43], we will discuss local $W^{2,\infty}$ estimates on solutions of unilateral obstacle problems for Bellman equations.

3.1 Bilateral Obstacles

We first consider the following bilateral obstacle problems

$$\min\{\max\{F(x, u, Du, D^2u), u - \psi\}, u - \varphi\} = 0 \quad \text{in } \Omega, \tag{3.1}$$

where $F : \overline{\Omega} \times \mathbb{R} \times \mathbb{R}^n \times S^n \to \mathbb{R}$ is defined by

$$F(x, r, \xi, X) := \max_{k \in \mathcal{N}} \{-\mathrm{Tr}(A^k(x)X) + \langle b^k(x), \xi \rangle + c^k(x)r - f^k(x)\}. \tag{3.2}$$

Here, by letting $N \geq 2$ be a fixed integer, for $k \in \mathcal{N} := \{1, 2, \ldots, N\}$, functions $A^k = (a^k_{ij}) : \overline{\Omega} \to S^n, b^k = (b^k_i) : \overline{\Omega} \to \mathbb{R}^n, c^k : \overline{\Omega} \to \mathbb{R}$ and $f^k : \Omega \to \mathbb{R}$ are given. We will use linear operators

$$L^k u := -\mathrm{Tr}(A^k(x)D^2 u) + \langle b^k(x), Du \rangle + c^k(x)u.$$

As in Sect. 2, we suppose that there is $\theta > 0$ such that

$$\langle A^k(x)\xi, \xi \rangle \geq \theta |\xi|^2 \quad \text{for any } \xi \in \mathbb{R}^n \text{ and } (x, k) \in \Omega \times \mathcal{N}, \tag{3.3}$$

and there is $M_c > 0$ such that

$$0 \leq c^k \leq M_c \quad \text{in } \overline{\Omega} \quad \text{for } k \in \mathcal{N}. \tag{3.4}$$

Following [22], we introduce a system of PDE via penalization: for $k \in \mathcal{N}$,

$$\begin{cases} L^k u^k + \beta_\varepsilon(u^k - u^{k+1}) + \beta_\varepsilon(u^k - \psi) - \beta_\varepsilon(\varphi - u^k) = f^k \text{ in } \Omega, \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad u^k = 0 \quad \text{on } \partial\Omega, \end{cases} \tag{3.5}$$

where $u^{N+1} := u^1$ and β_ε is given in Sect. 2. In order to distinguish three β_ε in (3.5), we will simply write

$$\begin{cases} \beta^k(x) := \beta^k_\varepsilon(x) = \beta_\varepsilon(u^k(x) - u^{k+1}(x)), \\ \overline{\beta}^k(x) := \overline{\beta}^k_\varepsilon(x) = \beta_\varepsilon(u^k(x) - \psi(x)), \\ \underline{\beta}^k(x) := \underline{\beta}^k_\varepsilon(x) = \beta_\varepsilon(\varphi(x) - u^k(x)). \end{cases}$$

For given functions, we suppose that

$$a^k_{ij}, b^k_i, f^k, c^k, \psi, \varphi \in C^2(\overline{\Omega}) \quad \text{for } 1 \leq i, j \leq n, \text{ and } k \in \mathcal{N}. \tag{3.6}$$

Setting

$$\overline{f} := \max_{k \in \mathcal{N}} f^{k,+}, \quad \text{and} \quad \underline{f} := \max_{k \in \mathcal{N}} f^{k,-},$$

we have the L^∞ estimates on $u^{k,\varepsilon}$ independent of $(\varepsilon, k) \in (0, 1) \times \mathcal{N}$.

Proposition 3.1 (L^∞ estimates) *Assume* (1.2)*,* (3.3) *and* (3.6)*. Let* $u^\varepsilon := (u^{k,\varepsilon}) \in C^2(\overline{\Omega}; \mathbb{R}^N)$ *be solutions of* (3.5)*. Then, there is* $\hat{C} > 0$ *such that*

$$-\hat{C} \max_{\overline{\Omega}} \underline{f} - \max_{\overline{\Omega}} \psi^- \leq u^{k,\varepsilon} \leq \max_{\overline{\Omega}} \varphi^+ + \hat{C} \max_{\overline{\Omega}} \overline{f} \quad \text{in } \overline{\Omega} \quad \text{for } (\varepsilon, k) \in (0, 1) \times \mathcal{N}.$$

Proof Setting $C_0 := \max_{\overline{\Omega}} \psi^-$ and $C_1 := \max_{\overline{\Omega}} \underline{f}$, we suppose

$$\min_{k\in\mathcal{N},x\in\overline{\Omega}} u^{k,\varepsilon}(x) + C_0 + \mu(C_1+\delta)w(x) < 0.$$

Here, $\delta > 0$ will be sent to 0 in the end, and w is the function in the proof of Proposition 2.1; $w(x) := e^{2\gamma R_0} - e^{\gamma(x_1+R_0)} > 0$ in $\Omega \subset B_{R_0}$, where $\gamma > 0$ will be fixed later. Dropping $\varepsilon > 0$ from $u^{k,\varepsilon}$ and β_ε, we may assume that there is $\hat{x} \in \Omega$ such that

$$u^1(\hat{x}) + C_0 + \mu(C_1+\delta)w(\hat{x}) = \min_{k\in\mathcal{N},x\in\overline{\Omega}} \{u^k(x) + C_0 + \mu(C_1+\delta)w(x)\} < 0.$$

By setting $\gamma := \left(\max_{k\in\mathcal{N},x\in\overline{\Omega}} |b_1^k| + \theta\right)/\theta$, the weak maximum principle implies that at $\hat{x} \in \Omega$,

$$\begin{aligned}0 &\geq -a_{ij}^1 u_{ij}^1 + b_i^1 u_i^1 + \mu(C_1+\delta)e^{\gamma(\hat{x}_1+R_0)}\gamma(\gamma a_{11}^1 - b_1^1)\\ &\geq f^1 - c^1 u - \beta(u^1-u^2) - \beta(u^1-\psi) + \theta\mu(C_1+\delta)\gamma\\ &\geq -\underline{f} + c^1\{C_0 + \mu(C_1+\delta)w\} - \beta(u^1-u^2) - \beta(u^1-\psi) + \mu(C_1+\delta)\gamma.\end{aligned}$$

Since $u^1 \leq u^2$ and $u^1 - \psi \leq 0$ at $\hat{x}$, these observation yield

$$\underline{f}(\hat{x}) \geq \theta\mu(C_1+\delta)\gamma,$$

which gives a contradiction when $\mu > 1/(\theta\gamma)$. Therefore, we conclude the proof of the first inequality.

The second inequality can be shown more easily since we may avoid the penalty term $\beta_\varepsilon(u^k - u^{k+1})$ in the opposite inequalities. □

Next, we show L^∞ estimates on $\beta_\varepsilon(u^k - \psi)$ and $\beta_\varepsilon(\varphi - u^k)$ independent of $(\varepsilon, k) \in (0,1) \times \mathcal{N}$.

Proposition 3.2 (L^∞ estimates on penalty terms) *Assume* (1.2), (3.3) *and* (3.6). *Let* $u^\varepsilon := (u^{k,\varepsilon}) \in C^2(\overline{\Omega}; \mathbb{R}^N)$ *be solutions of* (3.5). *Then, there exists* $\tilde{C}_1 > 0$ *such that for* $\varepsilon \in (0,1)$ *and* $k \in \mathcal{N}$,

$$\begin{cases} \|\beta_\varepsilon(u^{k,\varepsilon} - \psi)\|_{L^\infty(\Omega)} \leq \max\limits_{\overline{\Omega}} \overline{f} + M_c \max\limits_{\overline{\Omega}} \psi^- + \tilde{C}_1\|D\psi\|_{W^{1,\infty}(\Omega)},\\ \|\beta_\varepsilon(\varphi - u^{k,\varepsilon})\|_{L^\infty(\Omega)} \leq \max\limits_{\overline{\Omega}} \underline{f} + M_c \max\limits_{\overline{\Omega}} \varphi^+ + \tilde{C}_1\|D\varphi\|_{W^{1,\infty}(\Omega)}.\end{cases}$$

Proof We shall write u^k for $u^{k,\varepsilon}$ as before. By the same reason in the proof of Proposition 3.1, we shall only show the estimates on $\beta_\varepsilon(\varphi - u^k)$.

Suppose $\max_{\overline{\Omega},\mathcal{N}} \underline{\beta}^k = \underline{\beta}^1(x_0) > 0$ for some $x_0 \in \Omega$. Thus, we may assume $\max_{\overline{\Omega},\mathcal{N}}(\varphi - u^k) = (\varphi - u^1)(x_0) > 0$. Hence, at $x_0 \in \Omega$, we have

$$0 \leq -a_{ij}^1(\varphi - u^1)_{ij} + b_i^1(\varphi - u^1)_i \leq -f^1 + c^1u^1 + \beta^1 + \overline{\beta}^1 - \underline{\beta}^1 + C\|D\varphi\|_{W^{1,\infty}(\Omega)}.$$

Since $u^1 - u^2 \le 0$, and $\varphi - u^1 > 0$ at x_0, we have

$$\underline{\beta}^1 \le -f^1 + c^1\varphi + C\|D\varphi\|_{W^{1,\infty}(\Omega)},$$

which concludes the assertion as in the proof of Proposition 2.2. □

Remark 3.3 Notice that we cannot apply the above argument to obtain L^∞-estimates on $\beta_\varepsilon(u^{k,\varepsilon} - u^{k+1,\varepsilon})$. Therefore, unlike Proposition 2.2, we cannot obtain $W^{2,p}$ estimates on $u^{k,\varepsilon}$.

For further regularity, we first obtain the estimate of first derivatives on $\partial\Omega$ in Proposition 3.4 below. To this end, we shall use $W^{1,\infty}$ estimates on approximate solutions of the associated unilateral obstacle problems via penalization.

Proposition 3.4 (Gradient estimates on $\partial\Omega$) *Assume* (1.2), (3.3) *and* (3.6). *Let* $u^\varepsilon := (u^{k,\varepsilon}) \in C^1(\overline{\Omega}; \mathbb{R}^n) \cap C^2(\Omega; \mathbb{R}^n)$ *be solutions of* (3.5). *Then, there exists* $\tilde{C}_2 > 0$ *such that for* $\varepsilon \in (0, 1)$ *and* $k \in \mathcal{N}$,

$$\|Du^{k,\varepsilon}\|_{L^\infty(\partial\Omega)} \le \tilde{C}_2.$$

Proof Because $u^{k,\varepsilon} = 0$ on $\partial\Omega$, we only need the estimate

$$\left|\frac{\partial u^{k,\varepsilon}}{\partial n}(z)\right| \le C \quad \text{for any } z \in \overline{\Omega},$$

where $n = n(z) \in \partial B_1$ denotes the outward unit vector at $z \in \partial\Omega$.

Let $v^\varepsilon = (v^{k,\varepsilon}) : \overline{\Omega} \to \mathbb{R}^N$ be the unique solution of the penalized system of the following unilateral obstacle problem.

$$\begin{cases} L^k v^k + \beta_\varepsilon(v^k - v^{k+1}) + \beta_\varepsilon(v^k - \psi) = f^k \text{ in } \Omega, \\ \qquad\qquad\qquad\qquad\qquad\qquad\quad v^k = 0 \quad \text{on } \partial\Omega. \end{cases} \tag{3.7}$$

Due to Lemmas 2.1, 2.2 and 3.1 in [43], we find $\hat{C}_1 > 0$, and for each compact $K \Subset \Omega$, $\hat{C}_1(K) > 0$ such that

$$\|v^{k,\varepsilon}\|_{W^{1,\infty}(\Omega)} \le \hat{C}_1, \quad \text{and} \quad \|D^2 v^{k,\varepsilon}\|_{L^\infty(K)} \le \hat{C}_1(K). \tag{3.8}$$

We claim that

$$v^{k,\varepsilon} \le u^{k,\varepsilon} \quad \text{in } \overline{\Omega} \quad \text{for } (\varepsilon, k) \in (0, 1) \times \mathcal{N}.$$

Indeed, if we suppose $\Theta := \max_{\overline{\Omega},\mathcal{N}}(v^{k,\varepsilon} - u^{k,\varepsilon} - \delta w) > 0$, where $\delta > 0$ will be sent to 0, and w is the function in Proposition 2.1, then we may suppose $\Theta = (v^{1,\varepsilon} - u^{1,\varepsilon} - \delta w)(\hat{x})$ for some $\hat{x} \in \Omega$. Hence, at $\hat{x}$, we have

$$
\begin{aligned}
0 &\le -a^1_{ij}(v^{1,\varepsilon}-u^{1,\varepsilon})_{ij} + b^1_i(v^{1,\varepsilon}-u^{1,\varepsilon})_i + \delta\gamma e^{\gamma\hat{x}_1}(-\theta\gamma + |b^1_1|) \\
&\le -c^1(v^{1,\varepsilon}-u^{1,\varepsilon}) - \beta_\varepsilon(v^{1,\varepsilon}-v^{2,\varepsilon}) - \beta_\varepsilon(v^{1,\varepsilon}-\psi) + \beta_\varepsilon(u^{1,\varepsilon}-u^{2,\varepsilon}) \\
&\quad +\beta_\varepsilon(u^{1,\varepsilon}-\psi) - \beta_\varepsilon(\varphi - u^{1,\varepsilon}) - \theta\delta\gamma
\end{aligned}
$$

provided $\gamma \ge (\max_{\mathcal{N},\overline{\Omega}} |b^k_1| + \theta)/\theta$. Since $v^{1,\varepsilon} > u^{1,\varepsilon}$ and $v^{1,\varepsilon} - v^{2,\varepsilon} \ge u^{1,\varepsilon} - u^{2,\varepsilon}$ at $\hat{x}$, we immediately obtain a contradiction. Therefore, we have

$$
v^{k,\varepsilon} \le u^{k,\varepsilon} + \delta w \quad \text{in } \overline{\Omega},
$$

which concludes the claim by sending $\delta \to 0$. Therefore, we have

$$
\frac{\partial u^{k,\varepsilon}}{\partial n}(z) \le \frac{\partial v^{k,\varepsilon}}{\partial n}(z) \le \hat{C}_1 \quad \text{for any } z \in \partial\Omega. \tag{3.9}
$$

On the other hand, for each $k \in \mathcal{N}$, we next let $w^{k,\varepsilon}$ be solutions of

$$
\begin{cases} L^k u - \beta_\varepsilon(\varphi - u) = f^k & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}
$$

We claim that for $(\varepsilon, k) \in (0,1) \times \mathcal{N}$,

$$
u^{k,\varepsilon} \le w^{k,\varepsilon} \quad \text{in } \overline{\Omega}.
$$

Indeed, assuming $\max_{\overline{\Omega},\mathcal{N}}(u^{k,\varepsilon} - w^{k,\varepsilon} - \delta w) = (u^{1,\varepsilon} - w^{1,\varepsilon} - \delta w)(\hat{x}) > 0$ for some $\hat{x} \in \Omega$, at $\hat{x}$, we have

$$
\begin{aligned}
0 &\le -a^1_{ij}(u^{1,\varepsilon}-w^{1,\varepsilon})_{ij} + b^1_i(u^{1,\varepsilon}-w^{1,\varepsilon})_i + \delta\gamma e^{\gamma\hat{x}_1}(-\theta\gamma + |b^1_1|) \\
&< -\beta^1_\varepsilon(u^{1,\varepsilon}-u^{2,\varepsilon}) - \beta^1_\varepsilon(u^{1,\varepsilon}-\psi) + \beta_\varepsilon(\varphi - u^{1,\varepsilon}) - \beta_\varepsilon(\varphi - w^{1,\varepsilon}) - \theta\delta\gamma \\
&< 0
\end{aligned}
$$

for large $\gamma > 1$ as before. Hence, the same argument to obtain (3.9) implies

$$
\frac{\partial u^{k,\varepsilon}}{\partial n}(z) \ge \frac{\partial w^{k,\varepsilon}}{\partial n}(z) \quad \text{for any } z \in \partial\Omega. \tag{3.10}
$$

By the same argument as in the proof of Proposition 2.2, we find $\tilde{C} > 0$ such that

$$
0 \le \beta_\varepsilon(\varphi - w^{k,\varepsilon}) \le \tilde{C} \quad \text{in } \overline{\Omega} \text{ and for } (\varepsilon, k) \in (0,1) \times \mathcal{N},
$$

which implies

$$
\max_{k\in\mathcal{N}} \|Dw^{k,\varepsilon}\|_{L^\infty(\Omega)} \le C \quad \text{for any } \varepsilon \in (0,1).
$$

This together with (3.9) and (3.10) concludes the assertion. □

Now, we shall use Bernstein method to derive $W^{1,\infty}(\Omega)$ estimates on $u^{k,\varepsilon}$.

Proposition 3.5 *Assume* (1.2), (3.3) *and* (3.6). *Let* $u^\varepsilon = (u^{k,\varepsilon}) \in C^1(\overline{\Omega}; \mathbb{R}^N) \cap C^3(\Omega; \mathbb{R}^N)$ *be solutions of* (3.5). *There exists* $\tilde{C}_3 > 0$ *such that*

$$\max_{k\in\mathcal{N}} \|u^{k,\varepsilon}\|_{W^{1,\infty}(\Omega)} \le \tilde{C}_3 \quad \text{for } \varepsilon \in (0,1).$$

Proof We shall drop ε from $u^{k,\varepsilon}$. Set

$$V^k(x) := |Du^k|^2 + \gamma(u^k)^2.$$

In view of Proposition 3.4, we may suppose that

$$\max_{\mathcal{N},\overline{\Omega}} V^k = V^1(\hat{x}) > 0$$

for some $\hat{x} \in \Omega$. We shall write u and v for u^1 and u^2, respectively. Furthermore, we shall write β, $\overline{\beta}$ and $\underline{\beta}$ for β^1, $\overline{\beta}^1$ and $\underline{\beta}^1$, respectively.

We then have at $\hat{x} \in \Omega$,

$$\begin{aligned}
0 &\le -2a^1_{ij}(u_{ki}u_{kj} + u_k u_{kij} + \gamma u_i u_j + \gamma u u_{ij}) + 2b^1_i(u_k u_{ki} + u u_i) \\
&\le -2\theta(|D^2u|^2 + \gamma|Du|^2) + 2\gamma u(f^1 - c^1 u - \beta - \overline{\beta} + \underline{\beta}) \\
&\quad + 2u_k \left\{ \begin{array}{l} f^1_k + a^1_{ij,k}u_{ij} - b^1_{i,k}u_i - c^1_k u - c^1 u_k \\ -\beta'(u-v)_k - \overline{\beta}'(u-\psi)_k + \underline{\beta}'(\varphi - u)_k \end{array} \right\} \\
&\le -\gamma\theta|Du|^2 + C + \beta'(-|Du|^2 + |Dv|^2 - \gamma u^2 + \gamma v^2) \\
&\quad + \overline{\beta}'(-|Du|^2 + |D\psi|^2 - \gamma u^2 + \gamma\psi^2) \\
&\quad + \underline{\beta}'(-|Du|^2 + |D\varphi|^2 - \gamma u^2 + \gamma\varphi^2)
\end{aligned}$$

for large $\gamma > 1$. We use (2.3) to obtain the last inequality in the above.

Since we may suppose the last two terms are non-positive and $V^1 \ge V^2$ at $\hat{x}$, we have $\gamma\theta|Du(\hat{x})|^2 \le C$, which concludes the assertion. □

Since we do not know L^∞ estimates on $\beta_\varepsilon(u^k - u^{k+1})$, it seems difficult to find a weak (or viscosity) solution of (3.1) only with $W^{1,\infty}$ estimates. Thus, we shall switch to unilateral obstacle problems.

3.2 Unilateral Obstacles

In order to show local $W^{2,\infty}$ estimates on solutions of obstacle problems, we shall restrict ourselves to consider unilateral obstacle ones;

$$\left\{ \begin{array}{r} \max\{F(x,u,Du,D^2u), u - \psi\} = 0 \text{ in } \Omega, \\ u = 0 \text{ on } \partial\Omega, \end{array} \right. \tag{3.11}$$

where F is of Bellman type defined in (3.2).

Lenhart in [43] showed the $W^{2,\infty}_{loc}(\Omega)$ estimates on solutions of (3.11). We will recall the argument here.

We notice that proceeding arguments for $W^{2,\infty}$ estimates can not be applied to the following unilateral obstacle problem with the same F because the PDE below is of Isaacs type:

$$\min\{F(x, u, Du, D^2u), u - \varphi\} = 0 \quad \text{in } \Omega. \tag{3.12}$$

Open question 2: Is it possible to obtain (local) $W^{2,\infty}$ estimates on solutions of (3.12)?

In place of (1.2), we only need to suppose

$$\psi \geq 0 \quad \text{on } \partial\Omega. \tag{3.13}$$

The penalized system of (3.11) is as follows: for $u^\varepsilon = (u^{k,\varepsilon})$,

$$\begin{cases} L^k u^{k,\varepsilon} + \beta_\varepsilon(u^{k,\varepsilon} - u^{k+1,\varepsilon}) + \beta_\varepsilon(u^{k,\varepsilon} - \psi) = f^k \text{ in } \Omega, \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad u^{k,\varepsilon} = 0 \quad \text{on } \partial\Omega, \end{cases} \tag{3.14}$$

where $u^{N+1,\varepsilon} := u^{1,\varepsilon}$.

It is easy to establish the next lemma by following the proofs of Propositions 3.1, 3.2 and 3.5. We note that the Bernstein method with the standard barrier argument can also work for the Bellman equation with unilateral obstacles. We refer to Lemma 2.1 in [43] for the details.

Lemma 3.6 *There exists $\hat{C} > 0$ such that*

$$\|u^{k,\varepsilon}\|_{W^{1,\infty}(\Omega)} + \|\beta_\varepsilon(u^{k,\varepsilon} - \psi)\|_{L^\infty(\Omega)} \leq \hat{C} \quad \textit{for } (\varepsilon, k) \in (0, 1) \times \mathcal{N}.$$

Following the argument in [43] with a bit simpler auxilialy function V below than that there, we establish $W^{2,\infty}_{loc}(\Omega)$ estimates.

Theorem 3.7 (Local $W^{2,\infty}$ estimates) *Assume* (3.3), (3.6) *and* (3.13). *Let $u^\varepsilon = (u^{k,\varepsilon}) \in C^4(\Omega : \mathbb{R}^N) \cap C^1(\overline{\Omega} : \mathbb{R}^N)$ be solutions of* (3.14). *Then, for each compact $K \Subset \Omega$, there is $C_K > 0$ such that*

$$\max_{x \in K, k \in \mathcal{N}} |D^2 u^{k,\varepsilon}(x)| \leq C_K \quad \textit{for } \varepsilon \in (0, 1).$$

Proof We shall simply write u^k for $u^{k,\varepsilon}$ again.

Let $\zeta \in C^\infty_0(\Omega)$ be the same function as in the proof of Proposition 2.4. Putting $M^k = \max_{\overline{\Omega}} \zeta |D^2 u^k|$, we may suppose $M = \max_{\mathcal{N}} M^k = \zeta(\hat{z})|D^2 u^1(\hat{z})| \geq 1$ for some $\hat{z} \in \Omega$. By change of variables using the orthogonal matrix B such that $BA^1(\hat{z})^t B = (\alpha_k \delta_{k\ell})$, we may suppose that

$$L^1u^1(\hat{z}) = -\alpha_k u^1_{kk}(\hat{z}) + b^1_k(\hat{z})u^1_k(\hat{z}) + c^1(\hat{z})u^1(\hat{z})$$

for some $\alpha_k \geq \theta$. For each $i \in \mathcal{N}$, setting

$$V^i := \zeta^2|D^2u^i|^2 + \gamma M\zeta^2\alpha_k u^i_{kk} + \gamma M|Du^i|^2,$$

we may suppose that $\max_{\mathcal{N},\overline{\Omega}} V^i = V^{i_0}(\hat{x}) > 0$ for some $\hat{x} \in \Omega$ and $i_0 \in \mathcal{N}$.

We note that

$$\begin{aligned} M^2 = \zeta^2|D^2u^1|^2(\hat{z}) &\leq V^{i_0}(\hat{x}) - \gamma M\zeta^2\alpha_k u^1_{kk}(\hat{z}) \\ &\leq V^{i_0}(\hat{x}) + \gamma M\zeta^2(f^1 - b^1_i u^1_i - c^1u^1)(\hat{z}). \end{aligned}$$

Thus, for a fixed $\gamma > 1$, once we obtain

$$|D^2u^{i_0}|^2(\hat{x}) \leq CM, \tag{3.15}$$

then we have

$$M^2 \leq V^{i_0}(\hat{x}) + CM \leq CM(1 + \sqrt{M}),$$

which concludes the assertion.

We shall write a_{ij}, b_i, c, V, u and v for $a^{i_0}_{ij}, b^{i_0}_i, c^{i_0}, V^{i_0}, u^{i_0}$ and u^{i_0+1}, respectively, for simplicity. The weak maximum principle yields, at $\hat{x}$,

$$\begin{aligned} 0 &\leq -a_{ij}V_{ij} + b_iV_i \\ &= -a_{ij}\left\{\begin{array}{l} 2\zeta\zeta_{ij}|D^2u|^2 + 2\zeta_i\zeta_j|D^2u|^2 + 8\zeta\zeta_i u_{k\ell}u_{k\ell j} + 2\zeta^2 u_{k\ell}u_{k\ell ij} \\ +2\zeta^2 u_{k\ell i}u_{k\ell j} + 2\gamma M\zeta\zeta_{ij}\alpha_k u_{kk} + 2\gamma M\zeta_i\zeta_j\alpha_k u_{kk} \\ +4\gamma M\zeta\zeta_i\alpha_k u_{kkj} + \gamma M\zeta^2\alpha_k u_{kkij} + 2\gamma M u_k u_{kij} + 2\gamma M u_{ki}u_{kj} \end{array}\right\} \\ &\quad +b_i\left\{\begin{array}{l} 2\zeta\zeta_i|D^2u|^2 + 2\zeta^2 u_{k\ell}u_{k\ell i} + 2\gamma M\zeta\zeta_i\alpha_k u_{kk} + \gamma M\zeta^2\alpha_k u_{kki} \\ +2\gamma M u_k u_{ki} \end{array}\right\}. \end{aligned}$$

Hence, setting $L_0v := -a_{ij}v_{ij} + b_iv_i$, at $\hat{x}$, we have

$$\begin{aligned} &2\theta(\zeta^2|D^3u|^2 + \gamma M|D^2u|^2) \\ &\leq C(|D^2u|^2 + \zeta|D^2u||D^3u| + \gamma M|D^2u| + \gamma M\zeta|D^3u|) \\ &\quad +\gamma M\zeta^2\alpha_k L_0u_{kk} + 2\zeta^2 u_{k\ell}L_0u_{k\ell} + 2\gamma M u_k L_0u_k \\ &=: I_1 + I_2 + I_3 + I_4. \end{aligned}$$

By the definition of I_2 and I_3, we have

$$
\begin{aligned}
I_2+I_3 = \gamma M\zeta^2\alpha_k &\left\{\begin{array}{l} f_{kk}+a_{ij,kk}u_{ij}+2a_{ij,k}u_{ijk}-b_{i,kk}u_i-2b_{i,k}u_{ik} \\ -c_{kk}u-2c_k u_k - cu_{kk}-\beta''(u-v)_k^2 \\ -\beta'(u-v)_{kk}-\overline{\beta}''(u-\psi)_k^2-\overline{\beta}'(u-\psi)_{kk} \end{array}\right\} \\
+2\zeta^2 u_{k\ell} &\left\{\begin{array}{l} f_{k\ell}+a_{ij,k\ell}u_{ij}+2a_{ij,k}u_{ij\ell}-b_{i,k\ell}u_i-2b_{i,k}u_{i\ell} \\ -c_{k\ell}u-c_k u_\ell-c_\ell u_k-cu_{k\ell}-\beta''(u-v)_k(u-v)_\ell \\ -\beta'(u-v)_{k\ell}-\overline{\beta}''(u-\psi)_k(u-\psi)_\ell-\overline{\beta}'(u-\psi)_{k\ell} \end{array}\right\} \\
\le \gamma M\zeta^2 &\left\{\begin{array}{l} C(1+|D^2u|+|D^3u|)-\theta\beta''|D(u-v)|^2 \\ -\alpha_k\beta'(u-v)_{kk}-\theta\overline{\beta}''|D(u-\psi)|^2-\alpha_k\overline{\beta}'(u-\psi)_{kk} \end{array}\right\} \\
+\zeta^2 &\left\{\begin{array}{l} C|D^2u|(1+|D^2u|+|D^3u|)+2\beta''|D(u-v)|^2|D^2u| \\ +\beta'(-|D^2u|^2+|D^2v|^2)+2\overline{\beta}''|D(u-\psi)|^2|D^2u| \\ +\overline{\beta}'(-|D^2u|^2+|D^2v|^2) \end{array}\right\}.
\end{aligned}
$$

Moreover, I_4 is estimated by

$$
\begin{aligned}
I_4 &\le 2\gamma M u_k\{f_k+a_{ij,k}u_{ij}-b_{i,k}u_i-c_k u-cu_k-\beta'(u-v)_k-\overline{\beta}'(u-\psi)_k\} \\
&\le \gamma M\{C(1+|D^2u|)+\beta'(-|Du|^2+|Dv|^2)+\overline{\beta}'(-|Du|^2+|D\psi|^2)\}.
\end{aligned}
$$

Hence, these inequalities together with Young's inequality give

$$
\begin{aligned}
&\theta(\zeta^2|D^3u|^2+\gamma M|D^2u|^2) \\
\le\ & I_1+C\gamma M(\gamma M+|D^2u|)+M(2-\gamma\theta)\zeta^2\beta''|D(u-v)|^2 \\
&+M(2-\gamma\theta)\zeta^2\overline{\beta}''|D(u-\psi)|^2+\beta'(-V^{i_0}+V^{i_0+1}) \\
&+\overline{\beta}'(-V^{i_0}+\zeta^2|D^2\psi|^2+\gamma M\zeta^2\alpha_k\psi_{kk}+\gamma M|D\psi|^2).
\end{aligned}
$$

Note $V^{i_0}\ge V^{i_0+1}$ at $\hat{x}$. Furthermore, we may suppose $0\ge -V^{i_0}+\zeta^2|D^2\psi|^2+\gamma M\zeta^2\alpha_k\psi_{kk}+\gamma M|D\psi|^2$ at $\hat{x}$. Thus, taking $\gamma\ge 2/\theta$, we have

$$
\begin{aligned}
\theta(\zeta^2|D^3u|^2+\gamma M|D^2u|^2) &\le I_1+C\gamma M(\gamma M+|D^2u|) \\
&\le C(1+|D^2u|^2+\gamma^2M^2)+\theta\zeta^2|D^3u|^2.
\end{aligned}
$$

Remembering $M,\gamma\ge 1$, we have

$$
(\theta\gamma M-C)|D^2u|^2(\hat{x})\le C(1+\gamma^2M^2),
$$

which implies

$$
\theta\gamma M|D^2u|^2(\hat{x})\le C\gamma^2M^2
$$

provided $\theta\gamma M\ge 2C$. This yields (3.15). □

Open question 3: Is it possible to obtain $W^{2,\infty}(\Omega)$ or $W^{2,p}(\Omega)$ estimates for Bellman equations with unilateral obstacles under additional conditions if necessary?
Open question 4: Is it possible to obtain local $W^{2,\infty}_{loc}(\Omega)$ estimates for Bellman equations with bilateral obstacles?

4 A Fully Nonlinear Operator Case

In Sects. 2 and 3, thanks to Bernstein method, we establish estimates on solutions of approximate PDE (or systems of PDE), which present the existence of (strong) solutions belonging to the associated function spaces (i.e. $W^{2,\infty}(\Omega)$ or $W^{2,\infty}_{loc}(\Omega)$). See [43, 47] for the details. We also refer to [2] for a modern version of Bernstein method.

We note that there is a fully nonlinear uniformly elliptic equation which does not have classical solutions. See [48]. Furthermore, in [49], it is shown that there exists a viscosity solution of a fully nonlinear uniformly elliptic PDE whose second derivative is not bounded. On the other hand, we also know there is a classical solution of a special Isaacs equation consisting of three linear operators in [9].

In this section, we study more general PDE such as Isaacs equations with bilateral obstacles, and with unbounded, possibly discontinuous coefficients and inhomogeneous terms. In fact, to our knowledge, we do not know any regularity results for obstacle problems of Isaacs equations via penalization. In order to see a difficulty in the study of Isaacs equations via penalization, let us consider approximate Isaacs equations with no obstacles via penalization:

$$L^{k,\ell}u^{k,\ell} + \beta_\varepsilon(u^{k,\ell} - u^{k+1,\ell}) - \beta_\varepsilon(u^{k,\ell+1} - u^{k,\ell}) = f^{k,\ell} \quad \text{in } \Omega, \tag{4.1}$$

where $u^{M+1,\ell} = u^{1,\ell}$ for $\ell \in \mathcal{N}$ and $u^{k,N+1} = u^{k,1}$ for $k \in \mathcal{M}$. Here, by setting $\mathcal{M} := \{1, \ldots, M\}$ and $\mathcal{N} := \{1, \ldots, N\}$, $u^{k,\ell} : \overline{\Omega} \to \mathbb{R}$ for $(k, \ell) \in \mathcal{M} \times \mathcal{N}$ are unknown functions, and linear operators are defined by

$$L^{k,\ell}\zeta := -\mathrm{Tr}(A^{k,\ell}(x)D^2\zeta) + \langle b^{k,\ell}(x), D\zeta\rangle + c^{k,\ell}(x)\zeta,$$

where given functions $A^{k,\ell} : \overline{\Omega} \to S^n$, $b^{k,\ell} : \overline{\Omega} \to \mathbb{R}^n$ and $c^{k,\ell} : \overline{\Omega} \to [0, \infty)$ satisfy enough regularity.

If we obtain L^∞_{loc} estimates on $\beta_\varepsilon(u^{k,\ell} - u^{k+1,\ell})$ and $\beta_\varepsilon(u^{k,\ell+1} - u^{k,\ell})$, then it is easy to verify that $u^{k,\ell}_\varepsilon$ converge to a single limit u as $\varepsilon \to 0$ (along a subsequence if necessary), which is a solution of

$$\min_{\ell\in\mathcal{N}} \max_{k\in\mathcal{M}} \{L^{k,\ell}u - f^{k,\ell}\} = 0 \quad \text{in } \Omega. \tag{4.2}$$

However, it is difficult to show L^∞ estimates on the first and second penalty terms. In fact, in a pioneering work [47], we first derive $W^{2,\infty}$ estimates on solutions of penalized problems for Bellman equations (i.e. $N = 1$), and then this gives L^∞ bounds for the penalty term. Moreover, Bernstein method does not work to obtain $W^{2,\infty}$ estimates on solutions of (4.1). Furthermore, even if we establish $W^{2,\infty}$ estimates on approximate solutions, since we have two penalty terms with opposite signs in (4.1), we still do not know if solutions of the system (4.1) converge to a single solution of (4.2).

Open question 5: Is it possible to obtain a weak/viscosity solution of (4.2) satisfying (2.5) via penalization?

If we restrict ourselves to try to establish $C^{1,\gamma}$ estimates on solutions of bilateral obstacle problems for $\gamma \in (0, 1)$, then we can accomplish such estimates even when F is of Isaacs type;

$$G(x, r, \xi, X) := \min_{\ell \in \mathcal{N}} \max_{k \in \mathcal{M}} \left\{ -\mathrm{Tr}(A^{k,\ell}(x)X) + \langle b^{k,\ell}(x), \xi \rangle + c^{k,\ell}(x)r \right\}.$$

Moreover, since we do not need systems of PDE via penalization, we may deal with compact sets $\mathcal{M}, \mathcal{N}$ in $\mathbb{R}^m$ for some $m \in \mathbb{N}$. Furthermore, since we will not differentiate PDE (because it is impossible!), it is possible to treat discontinuous coefficients and inhomogeneous terms. In this procedure, we need to show the existence of weak/viscosity solutions of Isaacs equations with obstacles by a different method. We only refer to [16] and [42] for the existence issue.

This section is based on a recent work by the author and Tateyama in [42].

4.1 Equi-Continuity

Modifying arguments by Duque in [20], we present an idea to apply the weak Harnack inequality to obtain estimates on solutions of obstacle problems when the PDE part may be fully nonlinear. Here, the terminology fully nonlinear means that the mapping $(\xi, X) \in \mathbb{R}^n \times S^n \to G(x, r, \xi, X) \in \mathbb{R}$ is neither convex nor concave for each $(x, r) \in \Omega \times \mathbb{R}$.

In what follows, we suppose that

$$\begin{cases} (i) & G(x, 0, 0, O) = 0 \text{ for } x \in \Omega, \\ (ii) & \mathcal{P}^-(X - Y) \le G(x, r, \xi, X) - G(x, r, \xi, Y) \le \mathcal{P}^+(X - Y) \\ & \text{for } x \in \Omega, r \in \mathbb{R}, \xi \in \mathbb{R}^n, X, Y \in S^n, \\ (iii) & \text{there is } \mu \in L^q(\Omega) \text{ such that } q > n, \text{ and} \\ & |G(x, r, \xi, X) - G(x, r, \eta, X)| \le \mu(x)|\xi - \eta| \\ & \text{for } x \in \Omega, r \in \mathbb{R}, \xi, \eta \in \mathbb{R}^n, X \in S^n, \\ (iv) & \text{there is } c_0 \in C(\overline{\Omega}) \text{ such that } c_0 \ge 0 \text{ in } \Omega, \text{ and} \\ & G(x, r, \xi, X) - G(x, s, \xi, X) \ge c_0(x)(r - s) \\ & \text{for } x \in \Omega, r, s \in \mathbb{R}, \xi \in \mathbb{R}^n, X \in S^n, \\ (v) & f \in L^p(\Omega) \text{ for } q \ge p > p_0. \end{cases} \tag{4.3}$$

Here, $p_0 \in [\frac{n}{2}, n)$ is the so-called Escauriaza's constant in [21], and for a fixed $\theta \in (0, 1]$, Pucci operators $\mathcal{P}^\pm : S^n \to \mathbb{R}$ are defined as follows:

$$\mathcal{P}^+(X) := \max\{-\mathrm{Tr}(AX) \mid A \in S^n_\theta\} \text{ and } \mathcal{P}^-(X) := \min\{-\mathrm{Tr}(AX) \mid A \in S^n_\theta\},$$

where $S^n_\theta := \{X \in S^n \mid \theta I \le X \le \theta^{-1} I\}$. Under hypotheses $(i)-(iv)$ in (4.3), we easily verify that

$$\mathcal{P}^-(X) - \mu(x)|\xi| + c_0(x)r \le G(x, r, \xi, X) \le \mathcal{P}^+(X) + \mu(x)|\xi| + c_0(x)r$$

for $x \in \Omega$, $r \in \mathbb{R}$, $\xi \in \mathbb{R}^n$ and $X \in S^n$.

In a celebrated paper [10] by Caffarelli, it has turned out that to establish the regularity of viscosity solutions of fully nonlinear uniformly elliptic PDE

$$G(x, u, Du, D^2u) = f(x) \quad \text{in } \Omega,$$

instead of this equation, it is essential to study extremal inequalities:

$$G^-(x, u, Du, D^2u) \le f^+(x) \quad \text{and} \quad G^+(x, u, Du, D^2u) \ge -f^-(x),$$

where $G^\pm(x, r, \xi, X) := \mathcal{P}^\pm(X) \pm \mu(x)|\xi| \pm c_0(x)r^\pm$.

Furthermore, according to [10] again, the key for the regularity theory is the weak Harnack inequality for supersolutions.

We recall the definition of L^p-viscosity solutions of

$$H(x, u, Du, D^2u) = 0 \quad \text{in } \Omega, \tag{4.4}$$

where $H : \Omega \times \mathbb{R} \times \mathbb{R}^n \times S^n \to \mathbb{R}$ is given (not necessarily continuous).

Definition 4.1 We say that $u \in C(\Omega)$ is an L^p viscosity subsolution (resp., supersolution) of (4.4) if it follows that

$$\lim_{r\to 0} ess. \inf_{B_r(x)} H(y, u(y), D\zeta(y), D^2\zeta(y)) \le 0$$

$$\left(\text{resp., } \lim_{r\to 0} ess. \sup_{B_r(x)} H(y, u(y), D\zeta(y), D^2\zeta(y)) \ge 0\right)$$

whenever for any $\zeta \in W^{2,p}_{loc}(\Omega)$, $u - \zeta$ attains its local maximum (resp., minimum) at $x \in \Omega$. Finally, we say that $u \in C(\Omega)$ is an L^p viscosity solution of (4.4) if it is both of an L^p viscosity subsolution and an L^p viscosity supersolution of (4.4).

Throughout this section, we at least suppose that

$$\varphi, \psi \in C(\overline{\Omega}) \tag{4.5}$$

satisfy (1.2). Under hypotheses (4.3), (4.5) and (1.2), we consider

$$\min\{\max\{G(x, u, Du, D^2u) - f(x), u - \psi(x)\}, u - \varphi(x)\} = 0 \quad \text{in } \Omega \tag{4.6}$$

under the Dirichlet condition (2.5). It is immediate to see that if $u \in C(\Omega)$ is an L^p viscosity subsolution (resp., supersolution) of (4.6) with this G, then it is an L^p viscosity subsolution (resp., supersolution) of

$$\min\{\max\{G^-(x,u,Du,D^2u) - f^+(x), u - \psi(x)\}, u - \varphi(x)\} = 0$$

$$\big(\text{resp.,}\quad \min\{\max\{G^+(x,u,Du,D^2u) + f^-(x), u - \psi(x)\}, u - \varphi(x)\} = 0\big)$$

in Ω. We will only use these information in the argument below.

We recall a reasonable result without proof.

Proposition 4.2 (Proposition 2.9 in *[42]*) *Under the same hypotheses as in Theorem 4.3, if $u \in C(\Omega)$ is an L^p viscosity subsolution (resp., supersolution) of* (4.6)*, then it follows that*

$$u \le \psi \quad (\textit{resp., } u \ge \varphi) \quad \textit{in } \Omega.$$

In what follows, we call ω a modulus of continuity of functions if

$$\omega \in C([0,\infty)) \text{ is nondecreasing, and } \omega(0) = 0.$$

We also use the notation A^i for the set of interior points of $A \subset \mathbb{R}^n$.

Theorem 4.3 (Theorem 2.10 in *[42]*) *Assume* (4.3), (4.5) *and* (1.2)*. Then, there exists a modulus of continuity ω such that for any L^p viscosity solution of* (4.6) *satisfying* (2.5)*, it follows that*

$$|u(x) - u(y)| \le \omega(|x - y|) \quad \textit{for any } x, y \in \overline{\Omega}.$$

Moreover, if we suppose $\varphi, \psi \in C^\alpha(\overline{\Omega})$ for some $\alpha \in (0,1)$, then there are $\hat{C} > 0$ and $\hat{\alpha} \in (0, \alpha]$ such that for any L^p viscosity solution of (4.6) *satisfying* (2.5)*, it follows that*

$$|u(x) - u(y)| \le \hat{C}|x - y|^{\hat{\alpha}} \quad \textit{for any } x, y \in \overline{\Omega}.$$

For $r > 0$ and $x \in \mathbb{R}^n$, we define closed cubes as follows:

$$Q_r := \left[-\frac{r}{2}, \frac{r}{2}\right]^n, \quad Q_r(x) := x + Q_r.$$

Proof We shall only give a proof for local estimates since we can modify the argument below by using the weak Harnack inequality near $\partial\Omega$. See [29] for its usage.

Fix any $K \Subset \Omega$. We shall divide K by

$$K^+ := \{x \in K \mid u(x) = \psi(x)\}, \quad K^- := \{x \in K \mid u(x) = \varphi(x)\},$$

and $K_0 := K \setminus (K^+ \cup K^-)$.

It is standard to show the assertion when $x, y \in K_0^i$. See [42] for the details.

Let ω_0 be the modulus of continuity of obstacles;

$$|\psi(x)-\psi(y)| \vee |\varphi(x)-\varphi(y)| \le \omega_0(|x-y|) \quad \text{for } x, y \in \overline{\Omega}.$$

Fix any $\hat{x} \in K$. We may suppose $\hat{x}=0$ by translation. For $r \in (0, d_0/(2\sqrt{n}))$, where $d_0 := \mathrm{dist}(\partial\Omega, K)$, we set

$$\overline{u} := u \vee (\varphi(0)+\omega_0(2\sqrt{n}r)) \quad \text{and} \quad \underline{u} := u \wedge (\psi(0)-\omega_0(2\sqrt{n}r)).$$

Notice that $\varphi(0)+\omega_0(2\sqrt{n}r) \ge \varphi$ and $\psi \ge \psi(0)-\omega_0(2\sqrt{n}r)$ in $Q_{4r} \subset \Omega$. It is standard to see that $\overline{u}$ and $\underline{u}$ are, respectively, an L^p viscosity subsolution and supersolution of

$$G^-(x,u,Du,D^2u) - f^+(x) = 0 \quad \text{and} \quad G^+(x,u,Du,D^2u) + f^-(x) = 0 \quad \text{in} Q_{4r}.$$

For $s \in (0, d_0)$, set

$$M_s := \sup_{Q_s} \overline{u} \quad \text{and} \quad m_s := \inf_{Q_s} \underline{u}.$$

We then define

$$\overline{U} := M_{4r} - \overline{u} \quad \text{and} \quad \underline{U} := \underline{u} - m_{4r} \quad \text{for } r \in (0, d_0/(2\sqrt{n})).$$

It is immediate to see that $\overline{U}$ and $\underline{U}$ are nonnegative L^p viscosity supersolutions of

$$\mathcal{P}^+(D^2u) + \mu|Du| + c_0 u + f^{\pm} = 0 \quad \text{in } Q_{4r}.$$

Since $\|\mu\|_{L^n(Q_{4r})} \le \|\mu\|_{L^q(Q_{4r})}(2\sqrt{n}r)^{1-\frac{n}{q}}$, we can apply Proposition 5.7 in Appendix with the standard scaling to have

$$\left(\int_{Q_r} \overline{U}^{\varepsilon_0} dx\right)^{\frac{1}{\varepsilon_0}} \le C r^{\frac{n}{\varepsilon_0}} \left(\inf_{Q_r} \overline{U} + r^{\alpha_0}\|f^+\|_{L^{p\wedge n}(Q_{4r})}\right),$$
$$\left(\int_{Q_r} \underline{U}^{\varepsilon_0} dx\right)^{\frac{1}{\varepsilon_0}} \le C r^{\frac{n}{\varepsilon_0}} \left(\inf_{Q_r} \underline{U} + r^{\alpha_0}\|f^-\|_{L^{p\wedge n}(Q_{4r})}\right),$$

where $\alpha_0 := 2 - \frac{n}{p\wedge n} \in (0,1]$. By noting $M_{4r} - m_{4r} = \overline{U} + (\overline{u}-u) + (u-\underline{u}) + \underline{U} \le \overline{U} + 4\omega_0(2\sqrt{n}r) + \underline{U}$, the above inequalities imply

$$M_{4r} - m_{4r} \le C\left(\inf_{Q_r} \overline{U} + \inf_{Q_r} \underline{U} + r^{\alpha_0} + \omega_0(2\sqrt{n}r)\right),$$

which gives a decay estimate of oscillations:

$$M_r - m_r \le \theta_0(M_{4r} - m_{4r}) + r^{\alpha_0} + \omega_0(2\sqrt{n}r).$$

Since $u(x) - u(y) \le \overline{u}(x) - \underline{u}(y)$, it is standard (e.g. in [29]) to obtain equi-continuity of u.

If φ and ψ are Hölder continuous, then the above estimate implies the Hölder continuity with some exponent. □

4.2 $C^{1,\gamma}$ Estimates

Now, assuming that there is $\hat{\gamma} \in (0, 1)$ such that

$$\varphi, \psi \in C^{1,\hat{\gamma}}(\Omega), \tag{4.7}$$

we will suppose (4.3) but $q \ge p > n$ in (v). Under this assumption, we will use the Hölder exponent

$$\gamma_0 := \min\left\{1 - \frac{n}{p}, \hat{\gamma}\right\} \in (0, 1).$$

For simplicity, we will also suppose

$$\varphi < \psi \quad \text{in } \Omega. \tag{4.8}$$

For G in (4.3), we use the notation:

$$\theta(x, y) := \sup_{X \in S^n} \frac{|G(x, 0, 0, X) - G(y, 0, 0, X)|}{1 + \|X\|} \quad \text{for } x, y \in \Omega.$$

Theorem 4.4 *Assume (4.3) replaced by $q \ge p > n$ in (v), (4.7) and (4.8). For any $K \Subset \Omega$, there exist $\hat{C}_K > 0$, $\gamma \in (0, \gamma_0]$, $r_0 \in (0, dist(K, \partial\Omega))$, and $\delta_0 > 0$ such that if $u \in C(\Omega)$ is an L^p viscosity solution of (4.6), and if*

$$r^{-1}\|\theta(y, \cdot)\|_{L^n(B_r(y))} \le \delta_0 \quad \textit{for } r \in (0, r_0) \textit{ and } y \in N_K, \tag{4.9}$$

where by setting $C_K[u] := \{x \in K \mid u(x) = \varphi(x) \text{ or } u(x) = \psi(x)\}$, we define the non-coincidence set by $N_K[u] := \{x \in K \mid dist(x, C_K[u]) > 0\}$, then it follows that

$$|Du(x) - Du(y)| \le \hat{C}_K|x - y|^{\gamma} \quad \textit{for } x, y \in K.$$

Proof Following the argument in the proof of Proposition 5.1 in [42], we can find $\gamma_1 \in (0, 1)$ such that

$$|Du(x) - Du(y)| \le C|x - y|^{\gamma_1} \quad B_r(x) \subset N_K[u] \text{ for some } r > 0. \tag{4.10}$$

In fact, we need some modification of the standard argument in [10] since our PDE contains unbounded ingredients. See Sect. 5.1 in [42] for the details. We only need (4.9) to prove this fact.

We shall show the assertion near the coincidence set. Thus, we shall fix $z \in K$ such that $u(z) = \varphi(z)$. Again, we may suppose $z = 0$ by translation. We will show that

$$|u(x) - u(0) - \langle D\varphi(0), x\rangle| \le Cr^{1+\gamma_0} \quad x \in Q_{\frac{r}{4}},$$

which implies that u is differentiable at 0, $Du(0) = D\varphi(0)$, and moreover,

$$|Du(x) - Du(0)| \le C|x|^{\gamma_0} \quad \text{for } x \in Q_{\frac{r}{4}}.$$

We refer to [1] for its readable proof.

Setting $v := u - \varphi(0) - \langle D\varphi(0), x\rangle + Ar^{1+\hat{\gamma}}$ for large $A > 0$, we claim that v is a nonnegative L^p viscosity supersolution of

$$\mathcal{P}^+(D^2u) + \mu|Du| + c_0 v + g^- = 0 \quad \text{in } Q_{4r},$$

where $g^-(x) := f^-(x) + |D\varphi(0)|\mu(x) + c_0\{\varphi(0) + \langle D\varphi(0), x\rangle\}$. Considering $\hat{v} := v\left(\inf_{Q_r} u + \delta_0^{-1}\|g^-\|_{L^{p\wedge n}(Q_{4r})}\right)^{-1}$, we note that we may apply Proposition 5.7 to find $\varepsilon_0 > 0$ such that

$$\begin{aligned} r^{-\frac{n}{\varepsilon_0}}\|v\|_{L^{\varepsilon_0}(Q_r)} &\le C\left(\inf_{Q_r} v + r^{2-\frac{n}{p}}\|g^-\|_{L^n(Q_{4r})}\right) \\ &\le C(v(0) + r^{2-\frac{n}{p}}) \\ &\le Cr^{1+\gamma_0}. \end{aligned} \tag{4.11}$$

For large $\nu > 1$, it is easy to verify that $w := v \vee (\nu Ar^{1+\hat{\gamma}})$ is an L^p viscosity subsolution of

$$\mathcal{P}^-(D^2u) - \mu|Du| - g^+ = 0 \quad \text{in } Q_{4r},$$

where $g^+ = f^+ + |D\varphi(0)|\mu - c_0\{\varphi(0) + \langle D\varphi(0), x\rangle - Ar^{1+\hat{\gamma}}\}$. In view of Proposition 5.8, we have

$$\sup_{Q_{\frac{r}{4}}} v \le \tilde{C}\left\{r^{-\frac{n}{\varepsilon_0}}\left(\int_{Q_r} w^{\varepsilon_0}dx\right)^{\frac{1}{\varepsilon_0}} + r^{2-\frac{n}{p}}\|f^+ + \mu\|_{L^n(Q_{4r})}\right\},$$

where $\tilde{C} = \tilde{C}(\varepsilon_0) > 0$. Hence, by (4.11), we have

$$v \le Cr^{1+\gamma_0} \quad \text{in } Q_{\frac{r}{4}}.$$

The opposite inequality is trivial because Proposition 4.2 yields

$$u(x) - \varphi(0) - \langle D\varphi(0), x\rangle \geq \varphi(x) - \varphi(0) - \langle D\varphi(0), x\rangle \geq -Cr^{1+\hat{\gamma}} \geq -Cr^{1+\gamma_0}$$

for $|x| \leq r$.

Now, we shall combine two cases to establish the estimate. For $x, y \in N_K$, we may assume $0 < \mathrm{dist}(y, C_K[u]) \leq \mathrm{dist}(x, C_K[u])$. Choose $\hat{x}, \hat{y} \in C_K[u]$ such that

$$|x - \hat{x}| = \mathrm{dist}(x, C_K[u]) \geq |y - \hat{y}| = \mathrm{dist}(y, C_K[u]).$$

Case 1 : $|x - y| < \frac{1}{2}|x - \hat{x}|$. In this case, by (4.10), we have

$$|Du(x) - Du(y)| \leq C|x - y|^{\gamma_1}.$$

Case 2 : $|x - y| \geq \frac{1}{2}|x - \hat{x}| \geq \frac{1}{2}|y - \hat{y}|$. We may suppose that $(u - \varphi)(\hat{x}) = (u - \varphi)(\hat{y})$ (or $(u - \psi)(\hat{x}) = (u - \psi)(\hat{y})$) because $\psi(x) - \varphi(y) \geq \tau_0 > 0$ for $y \in B_r(x) \cap K$ with small $r > 0$.

Thus, due to the above observation, we have

$$\begin{aligned}
&|Du(x) - Du(y)| \\
&\leq |Du(x) - Du(\hat{x})| + |Du(\hat{x}) - Du(\hat{y})| + |Du(\hat{y}) - Du(y)| \\
&\leq C|x - \hat{x}|^{\gamma_0} + |D\varphi(\hat{x}) - D\varphi(\hat{y})| + C|y - \hat{y}|^{\gamma_0} \\
&\leq C|x - y|^{\gamma_0} + C|\hat{x} - \hat{y}|^{\hat{\gamma}} \\
&\leq C|x - y|^{\gamma_0}
\end{aligned}$$

because $|\hat{x} - \hat{y}|^{\hat{\gamma}} \leq |\hat{x} - x|^{\hat{\gamma}} + |x - y|^{\hat{\gamma}} + |y - \hat{y}|^{\hat{\gamma}}$ and $\gamma_0 \leq \hat{\gamma}$. □

Open question 6: What is a sufficient condition to obtain $W^{2,\infty}_{loc}(\Omega)$ or $W^{2,p}_{loc}(\Omega)$ estimates on solutions of Isaacs equations with obstacles?

5 Appendix

In [38, 39], we established the ABP maximum principle and weak Harnack inequality for L^p viscosity solutions only when the PDE does not contain 0th order terms for the sake of simplicity. Since in Sect. 4 we obtain the results assuming (4.3), which allows the PDE to admit 0th order terms, we shall give the ABP maximum principle and weak Harnack inequality for those.

The ABP maximum principle can be proved immediately due to known results.

Proposition 5.1 *Assume $\mu \in L^q(\Omega)$, $f \in L^p(\Omega)$ for $q > n$ and $q \geq p > p_0$. Assume also that $c_0 \in C(\overline{\Omega})$ is nonnegative in $\overline{\Omega}$. Then, there exists a universal constant $C_0 > 0$ (depending on $\|\mu\|_{L^q(\Omega)}$) such that if $u \in C(\overline{\Omega})$ is an L^p viscosity subsolution (resp., supersolution) of*

$$\mathcal{P}^-(D^2u) - \mu(x)|Du| - c_0(x)u^- - f^+(x) = 0 \quad \text{in } \Omega \tag{5.1}$$

$$\left(\mathcal{P}^+(D^2u) + \mu(x)|Du| + c_0(x)u^+ + f^-(x) = 0 \quad in\ \Omega\right),$$

then it follows that

$$\max_{\Omega} u \le \max_{\partial\Omega} u^+ + C_0 d_\Omega^{2-\frac{n}{p}} \|f^+\|_{L^{p\wedge n}(\Omega^+[u])} \tag{5.2}$$

$$\left(resp., \quad \min_{\Omega} u \ge -\max_{\partial\Omega} u^- - C_0 d_\Omega^{2-\frac{n}{p}} \|f^-\|_{L^{p\wedge n}(\Omega^-[u])}\right),$$

where $\Omega^\pm[u] := \{x \in \Omega \mid \pm u(x) > \max_{\partial\Omega} u^\pm\}$ *and* $d_\Omega := \sup\{|x-y| \mid x, y \in \Omega\}$.

Proof We shall only show the first assertion. It is immediate to verify that u is an L^p viscosity subsolution of

$$\mathcal{P}^-(D^2u) - \mu(x)|Du| - f^+(x) = 0 \quad \text{in } \Omega^+[u].$$

Hence, we can apply Proposition 2.8 and Theorem 2.9 in [38] to conclude our proof. □

We next show the weak Harnack inequality. We first present a decay of distribution functions of L^p viscosity supersolutions.

Lemma 5.2 (cf. Theorem 2.3 in [41]) *Assume the same hypotheses in Proposition 5.1. There are* $r_0, \delta_0 > 0$ *and* $A \ge 1$ *such that for any nonnegative* L^p *viscosity supersolution of*

$$\mathcal{P}^+(D^2u) + \mu(x)|Du| + c_0(x)u - f(x) = 0 \quad in\ Q_4,$$

if $\inf_{Q_1} u \le 1$ *and* $\|\mu\|_{L^{p\wedge n}(Q_4)} \vee \|f^-\|_{L^{p\wedge n}(Q_4)} \le \delta_0$, *then we have*

$$|\{x \in Q_1 \mid u(x) > t\}| \le \frac{A}{t^{r_0}} \quad for\ t > 1.$$

Remark 5.3 It is trivial that the conclusion holds true for any $t > 0$ since $A \ge 1$.

Remark 5.4 The assertion is known in [39] when $c_0 \equiv 0$. In fact, in our case, we do not know if the strong maximum principle holds when the coefficient to the first derivative (i.e. μ) is unbounded. Therefore, we will use an auxiliary function φ_0, which is a strong solution of PDE with no first derivative terms. We notice that if we add $\mu|D\varphi_0|$ in the left hand side of (5.3), then we cannot show (5.4) below. We will then have μ in the inhomogeneous term which is small in L^n norm.

Proof In view of Proposition 2.4 in [40] with some modifications as in the proof of Lemma 4.2 in [39], there exists $\varphi_0 \in W^{2,p'}(Q_4 \setminus Q_1) \cap C(Q_4 \setminus Q_1^i)$ for any $p' > n$ such that

$$\begin{cases} \mathcal{P}^-(D^2u) + c_0(x)u = 0 & \text{in } Q_4^i \setminus Q_1, \\ u = 0 & \text{on } \partial Q_4, \\ u = -1 & \text{on } \partial Q_1. \end{cases} \tag{5.3}$$

Since φ is also an $L^{p'}$ viscosity solution of the PDE in the above, if we suppose $\sup_{Q_4 \setminus Q_1} \varphi_0 > 0$ or $\inf_{Q_4 \setminus Q_1} \varphi_0 < -1$, then this contradicts to the definition of $L^{p'}$ viscosity solution. Thus, we have $-1 \leq \varphi_0 \leq 0$ in $Q_4 \setminus Q_1$.

Furthermore, we claim that there is $\theta_0 > 0$ such that

$$\varphi_0 \leq -\theta_0 \quad \text{in } Q_3 \setminus Q_1. \tag{5.4}$$

Although the proof of (5.4) is known in [33] for instance, we will give a proof of this claim for the reader's convenience in the end.

Extending φ_0 appropriately in Q_1, for large $\lambda > 1$, we may suppose that $\varphi := \lambda\varphi_0 \in W^{2,p'}(Q_4)$ is an $L^{p'}$ strong solution of

$$\mathcal{P}^-(D^2u) + c_0u = \xi \quad \text{in } Q_4$$

such that $\varphi \leq -2$ in Q_3, where $\xi \in L^q(Q_1)$ satisfies $\xi = 0$ in $Q_4 \setminus Q_1$.

We observe that $w := u + \varphi$ is an L^p viscosity supersolution of

$$\mathcal{P}^+(D^2w) + \mu|Dw| + c_0w^+ = -\mu|D\varphi| - f^- + \xi \quad \text{in } Q_4.$$

Hence, setting $\Omega := \{x \in Q_4^i \mid w(x) < 0\}$, by Proposition 5.1, we have

$$\begin{aligned} -1 \geq \inf_{Q_1} w \geq \inf_{Q_4} w = \inf_{\Omega} w & \\ & \geq -C\|\mu|D\varphi| + f^- - \xi\|_{L^{p\wedge n}(\Omega)} \\ & \geq -C\left(\delta_0 + |\{x \in Q_1 \mid w(x) < 0\}|^{\frac{1}{p\wedge n}}\right). \end{aligned}$$

Therefore, for a fixed $\delta_0 > 0$, we can find $\theta_1 \in (0, 1)$ such that

$$\theta_1 \leq |\{x \in Q_1 \mid u(x) \leq M\}|,$$

where $M := \max_{Q_4}(-\varphi) > 1$. It is now standard by an induction argument to see that

$$|\{x \in Q_1 \mid u(x) > M^k\}| \leq (1 - \theta_1)^k \quad k \in \mathbb{N},$$

which implies the decay of distribution function of u. Therefore, we conclude the assertion by the standard argument. See [39] for the details.

Proof of claim (5.4) (cf. Theorem 1 in [33]) It is enough to show that $\varphi_0(x) < 0$ for $x \in Q_4^i \setminus Q_1$. Setting $K_0 := \{x \in Q_4^i \setminus Q_1 \mid \varphi_0(x) = 0\}$, we may suppose $K_0 \neq \emptyset$. We can choose $R > 0$, $z \in K_0$ and $\hat{z} \in \Omega_0 := (Q_4^i \setminus Q_1) \cap K_0^c$ such that

$$\overline{B}_R(\hat{z}) \setminus \{z\} \subset \Omega_0, \quad \text{and} \quad \partial B_R(\hat{z}) \cap K_0 = \{z\}.$$

Setting an open annulus $A_0 := \{x \in \mathbb{R}^n \mid R/2 < |x - \hat{z}| < R\}$, we introduce $\zeta(x) := \varepsilon(e^{-\beta R^2/2} - e^{-\beta|x-\hat{z}|^2/2}) \le 0$, where $\beta > 1$ and $\varepsilon \in (0, 1)$ will be chosen later. Furthermore, we have

$$M_1 := \max_{x \in \overline{A}_0} (\varphi_0 - \zeta)(x) \ge (\varphi_0 - \zeta)(z) = 0.$$

We also note that $(\varphi_0 - \zeta)(x) < 0$ if $x \in \partial B_R(\hat{z}) \setminus \{z\}$. Now, setting $\theta_0 := \min_{x \in \partial B_{R/2}(\hat{z})}(-\varphi_0(x)) > 0$ and $\varepsilon := \theta_0/2$, we observe that

$$\max_{x \in \partial B_{R/2}(\hat{z})} (\varphi_0 - \zeta)(x) \le -\theta_0 + \varepsilon e^{-\frac{\beta R^2}{8}} \le -\frac{\theta_0}{2} < 0.$$

Next, assume that $\varphi_0 - \zeta$ attains its maximum at $\hat{x} \in A_0$. Since φ_0 is a viscosity subsolution of

$$\mathcal{P}^-(D^2 u) + c_0 u = 0 \quad \text{in } Q_4^i \setminus Q_1,$$

we have

$$0 \ge e^{-\frac{\beta|\hat{x}-\hat{z}|^2}{2}} \left\{\beta \mathcal{P}^-(I - \beta(\hat{x} - \hat{z}) \otimes (\hat{x} - \hat{z}))\right\} + c_0(\hat{x})\varphi_0(\hat{x}).$$

Following an argument in p. 20 of [12], since $\mathcal{P}^-(I - \beta(\hat{x} - \hat{z}) \otimes (\hat{x} - \hat{z})) \ge -\frac{n-1}{\theta} + \left(\frac{\beta R^2}{4} - 1\right)\theta \ge 1$ provided $\beta \ge \beta_0$ for some $\beta_0 > 1$, we have

$$0 \ge e^{-\frac{\beta|\hat{x}-\hat{z}|^2}{2}} (\beta - c_0(\hat{x})),$$

which yields a contradiction when $\beta > \beta_0 + \max_{x \in \overline{\Omega}} c_0$. Therefore, because $(\varphi_0 - \zeta)(z - he) \le (\varphi_0 - \zeta)(z) = 0$ for small $h > 0$, where $e := (z - \hat{z})/|z - \hat{z}|$, we have

$$\frac{\varphi_0(z - he) - \varphi_0(z)}{-h} \ge \varepsilon \frac{e^{-\frac{\beta|z-he-\hat{z}|^2}{2}} - e^{-\frac{\beta|z-\hat{z}|^2}{2}}}{h}.$$

Sending $h \to 0+$, we have $\langle D\varphi_0(z), e\rangle = 0 \ge \varepsilon e^{-\frac{\beta R^2}{2}} \beta R > 0$, which is a contradiction. Hence, we have $K_0 = \emptyset$. □

Remark 5.5 It is possible to give precise functions φ_0 by considering larger ball $B_{2\sqrt{n}} \supset Q_4$. See [30] for such a function.

Remark 5.6 Concerning the strong maximum principle for PDE of divergence type with 0th order terms, we refer to [51] and references therein.

Now, we present our weak Harnack inequality.

Proposition 5.7 (cf. Theorem 3.1 in [39]) *Assume the same hypotheses in Proposition 5.1. There are* $\varepsilon_0 > 0$, $\delta_0 > 0$ *and* $\hat{C} > 0$ *such that for any nonnegative* L^p *viscosity supersolution of*

$$\mathcal{P}^+(D^2u) + \mu|Du| + c_0 u - f = 0 \quad \text{in } Q_4,$$

if $\|\mu\|_{L^{p\wedge n}(Q_4)} \leq \delta_0$, *then we have*

$$\left(\int_{Q_1} u^{\varepsilon_0} dx\right)^{\frac{1}{\varepsilon_0}} \leq \hat{C}\left(\inf_{Q_1} u + \|f^-\|_{L^{p\wedge n}(Q_4)}\right).$$

Proof In place of u, considering

$$V := \frac{u}{\inf_{Q_1} u + \delta_0^{-1}\|f^-\|_{L^{p\wedge n}(Q_4)} + \varepsilon},$$

where $\varepsilon > 0$ will be sent to 0 in the end, and $\delta_0 > 0$ will be fixed later, we may suppose $\|f^-\|_{L^{p\wedge n}(Q_4)} \leq \delta_0$ and $\inf_{Q_1} u \leq 1$.

In view of Lemma 5.2, we easily verify that for any $\varepsilon_0 \in (0, r_0)$, there is $\hat{C} = \hat{C}(\varepsilon_0) > 0$ such that

$$\left(\int_{Q_1} V^{\varepsilon_0} dx\right)^{\frac{1}{\varepsilon_0}} \leq \hat{C},$$

which implies the conclusion by sending $\varepsilon \to 0$. □

In order to establish the Harnack inequality, we combine the weak Harnack inequality with the next local maximum principle.

Proposition 5.8 (Theorem 3.1 in *[41]*) *Assume the same hypotheses in Proposition 5.1. For any* $\varepsilon > 0$, *there is* $\hat{C}_\varepsilon > 0$ *such that for any* L^p *viscosity subsolution of*

$$\mathcal{P}^-(D^2u) - \mu|Du| - c_0 u^- - f = 0 \quad \text{in } Q_4, \tag{5.5}$$

we have

$$\sup_{Q_{\frac{1}{4}}} u \leq \hat{C}_\varepsilon \left\{\left(\int_{Q_1} (u^+)^\varepsilon dx\right)^{\frac{1}{\varepsilon}} + \|f^+\|_{L^p(Q_4)}\right\}.$$

Since we have unbounded coefficient μ, we cannot use the standard argument as in [29]. We follow the idea of the proof of Lemma 4.4 in [12] with some modifications. We first prepare the following lemma:

Lemma 5.9 (*cf. Theorem 2.3 in [41]*) *For* $q \geq p > p_0$ *and* $q > n$, *let* $f \in L^p(Q_4)$ *and* $\mu \in L^q(Q_4)$ *be nonnegative. Assume that* $u \in C(Q_4)$ *is an* L^p *viscosity subsolution of* (5.5) *satisfying*

$$|\{x \in Q_1 \mid u(x) \geq t\}| \leq \frac{A}{t^{r_0}} \quad \mathit{for}\ \forall t > 1, \tag{5.6}$$

where the constants $A \geq 1$ *and* $r_0 > 0$ *are from Lemma 5.2. Then, there are an integer* J, $\nu > 1$ *and* $\ell_j > 0$ $(j \geq J)$ *such that* $\sum_{j=J}^{\infty} \ell_j < \infty$, *and if* $u(x_0) \geq \nu^{j-1}$ *for* $j \geq J$ *and* $x_0 \in Q_{\frac{1}{2}}$, *then* $\sup_{Q_{\ell_j}(x_0)} u \geq \nu^j$.

Proof We will fix $\nu > 1$, $J \in \mathbb{N}$ and $\ell_j \in (0, 1)$ for $j \geq J$. Suppose

$$\sup_{Q_{\ell_j}(x_0)} u \leq \nu^j,$$

then we will obtain a contradiction.

Setting $x = x_0 + \frac{\ell_j}{4} y$ for $y \in Q_4$, we define

$$v(y) := \alpha \left(1 - \frac{1}{\nu^j} u(x_0 + 4^{-1} \ell_j y) \right),$$

where $\alpha := \nu(\nu - 1)^{-1}$ (or $\nu = \alpha(\alpha - 1)^{-1}$). Thus, we immediately verify that $v \geq 0$ in Q_4, and $\inf_{Q_3} v \leq v(0) \leq \alpha(1 - \nu^{-1}) = 1$.

We next set

$$\alpha := 2(2A)^{\frac{1}{r_0}} > 1 \quad \left(i.e.\ \nu = 2(2A)^{\frac{1}{r_0}} \{2(2A)^{\frac{1}{r_0}} - 1\}^{-1} > 1 \right),$$

and

$$\ell_j := \left(\frac{2^{2n+2r_0+1} A}{\nu^{j r_0}} \right)^{\frac{1}{n}}.$$

Choose $J_0 \in \mathbb{N}$ such that

$$\alpha < (2^{2n+2r_0+1} A)^{\frac{1}{r_0}} < \nu^{J_0}.$$

Notice that $\ell_j < 1$ for $j \geq J_0$. We next choose $J_1 \geq J_0$ such that

$$\frac{\alpha}{\nu^j} \left(\frac{\ell_j}{4} \right)^{2 - \frac{n}{p \wedge n}} < 1 \quad \text{for } j \geq J_1.$$

We then see that v is a nonnegative L^p viscosity supersolution of

$$\mathcal{P}^+(D^2 u) + \hat{\mu}|Du| + \hat{c}_0 u + \hat{f} = 0 \quad \text{in } Q_4,$$

where

$$\hat{\mu}(y) = \frac{\ell_j}{4} \mu \left(x_0 + \frac{\ell_j}{4} y \right), \ \hat{c}_0 = \frac{\ell_j^2}{16} \nu^j c_0 \text{ and } \hat{f}(y) = \frac{\alpha \ell_j^2}{16 \nu^j} f^+ \left(x_0 + \frac{\ell_j}{4} y \right).$$

Because of our choice of $\alpha > 1$, ℓ_j and $J_1 \in \mathbb{N}$, we verify that for $j \geq J_1$,

$$\|\hat{\mu}\|_{L^{p\wedge n}(Q_4)} = \left(\frac{\ell_j}{4}\right)^{1-\frac{n}{p\wedge n}} \|\mu\|_{L^{p\wedge n}(Q_{\ell_j}(x_0))} \leq \left(\frac{\ell_j}{4}\right)^{1-\frac{p\wedge n}{q}} \|\mu\|_{L^q(Q_4)},$$

and

$$\|\hat{f}\|_{L^{p\wedge n}(Q_4)} = \frac{\alpha}{\nu^j}\left(\frac{\ell_j}{4}\right)^{2-\frac{n}{n\wedge p}} \|f^+\|_{L^{p\wedge n}(Q_{\ell_j}(x_0))} \leq \|f^+\|_{L^{p\wedge n}(Q_{\ell_j}(x_0))}.$$

Finally, we choose $J_2 \geq J_1$ such that $\|\hat{\mu}\|_{L^{p\wedge n}(Q_4)} \leq \delta_0$, where $\delta_0 > 0$ is the constant in Lemma 5.2.

In view of Lemma 5.2, we have

$$|\{x \in Q_1 \mid v(x) > \alpha/2\}| \leq A\left(\frac{2}{\alpha}\right)^{r_0},$$

which yields

$$\left|\left\{x \in Q_{\frac{\ell_j}{4}}(x_0) \;\middle|\; u(x) < \frac{\nu^j}{2}\right\}\right| \leq A\left(\frac{2}{\alpha}\right)^{r_0}\left(\frac{\ell_j}{4}\right)^n \leq \frac{1}{2}\left(\frac{\ell_j}{4}\right)^n.$$

However, (5.6) implies

$$\left|\left\{x \in Q_{\frac{\ell_j}{4}}(x_0) \;\middle|\; u(x) \geq \frac{\nu^j}{2}\right\}\right| \leq \left|\left\{x \in Q_1 \;\middle|\; u(x) \geq \frac{\nu^j}{2}\right\}\right| \leq A\left(\frac{2}{\nu^j}\right)^{r_0}.$$

Hence, we have

$$\frac{\ell_j^n}{2^{2n+1}} \leq A\left(\frac{2}{\nu^j}\right)^{r_0},$$

which implies a contradiction to the definition of ℓ_j. □

Proof of Proposition 5.8. We first consider the case of $\varepsilon = r_0$, where $r_0 > 0$ is the constant from Lemma 5.2.

Choose $z \in Q_{\frac{1}{4}}$ such that $u(z) = \max_{Q_{\frac{1}{4}}} u$. Setting $v(y) := u(z + sy)$ for $s > 0$, we observe that v is an L^p viscosity subsolution of

$$\mathcal{P}^-(D^2u) - \hat{\mu}|Du| - c_0u^- - \hat{f} = 0 \quad \text{in } Q_4,$$

where $\hat{\mu}(y) := s\mu(z + sy)$ and $\hat{f}(y) := s^2 f^+(z + sy)$.

Since we may suppose $\int_{Q_1} (v^+)^{r_0} dy > 0$, by setting

$$w(y) := v(y)\left\{A^{-\frac{1}{r_0}}\left(\int_{Q_1}(v^+)^{r_0}dy\right)^{\frac{1}{r_0}} + \delta_0^{-1}\|\hat{f}\|_{L^{p\wedge n}(Q_4)}\right\}^{-1},$$

it is immediate to see that

$$|\{y \in Q_1 \mid w(y) \geq t\}| \leq \frac{1}{t^{r_0}}\int_{Q_1} w^{r_0} \leq \frac{A}{t^{r_0}}.$$

Furthermore, we verify that w is an L^p viscosity subsolution of

$$\mathcal{P}^-(D^2u) - \mu|Du| - g = 0 \quad \text{in } Q_4,$$

where $g(y) := \delta_0 \hat{f}(y)\|\hat{f}\|^{-1}_{L^{p\wedge n}(Q_4)}$.

Let $\nu > 1$, $J \in \mathbb{N}$ and $\ell_j \in (0, 1)$ be from Lemma 5.9. There is $\hat{J} \geq J$ such that

$$\sum_{j=\hat{J}}^{\infty} \ell_j \leq \frac{1}{8}.$$

We claim that $\sup_{Q_{\frac{1}{4}}} w \leq \nu^{\hat{J}-1}$. Indeed, if $w(x_0) \geq \nu^{\hat{J}-1}$ for some $x_0 \in Q_{\frac{1}{4}}$, then thanks to Lemma 5.9, we can choose $x_j \in Q_{\ell_j}(x_0)$ (for $j \geq \hat{J}$) such that

$$w(x_j) \geq \nu^j.$$

Since $x_j \in Q_{\frac{1}{2}}$ for $j \geq \hat{J}$, this contradicts to the continuity of $w \in C(Q_4)$. Hence, we have

$$\begin{aligned}\sup_{Q_{\frac{1}{4}}} u \leq \sup_{Q_{\frac{1}{4}}} v &\leq C\left\{\left(\int_{Q_1}(v^+)^{r_0}dx\right)^{\frac{1}{r_0}} + \|\hat{f}\|_{L^{p\wedge n}(Q_4)}\right\}\\ &\leq C\left\{\left(\int_{Q_1}(u^+)^{r_0}dx\right)^{\frac{1}{r_0}} + \|f\|_{L^{p\wedge n}(Q_4)}\right\}.\end{aligned}$$

In case when $\varepsilon > r_0$, instead of the above w, consider

$$\hat{w}(y) := v(y)\left\{A^{-\frac{1}{\varepsilon}}\left(\int_{Q_1}(v^+)^{\varepsilon}dy\right)^{\frac{1}{\varepsilon}} + \delta_0^{-1}\|\hat{f}\|_{L^{p\wedge n}(Q_4)}\right\}^{-1}.$$

Thus, we have

$$|\{y \in Q_1 \mid \hat{w}(y) \geq t\}| \leq \frac{A}{t^{\varepsilon}} \leq \frac{A}{t^{r_0}} \quad \text{for } t > 1.$$

Therefore, Lemma 5.9 implies the conclusion.

On the other hand, if $0 < \varepsilon < r_0$, then considering

$$\tilde{w}(y) := v(y)\left\{A^{-\frac{1}{r_0}}\left(\int_{Q_1}(v^+)^{\varepsilon}dy\right)^{\frac{1}{\varepsilon}} + \delta_0^{-1}\|\hat{f}\|_{L^{p\wedge n}(Q_4)}\right\}^{-1},$$

we have

$$|\{y \in Q_1 \mid \tilde{w}(y) \geq t\}| \leq \frac{A}{t^{r_0}}\int_{Q_1}(v^+)^{r_0}dy\left(\int_{Q_1}(v^+)^{\varepsilon}dy\right)^{-\frac{r_0}{\varepsilon}} \leq \frac{A}{t^{r_0}} \quad \text{for } t > 1.$$

Hence, Lemma 5.9 concludes the proof in this case. □

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High-Energy Eigenfunctions of the Laplacian on the Torus and the Sphere with Nodal Sets of Complicated Topology

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Abstract Let Σ be an oriented compact hypersurface in the round sphere $\mathbb{S}^n$ or in the flat torus $\mathbb{T}^n$, $n \geq 3$. In the case of the torus, Σ is further assumed to be contained in a contractible subset of $\mathbb{T}^n$. We show that for any sufficiently large enough odd integer N there exists an eigenfunctions ψ of the Laplacian on $\mathbb{S}^n$ or $\mathbb{T}^n$ satisfying $\Delta\psi = -\lambda\psi$ (with $\lambda = N(N + n - 1)$ or N^2 on $\mathbb{S}^n$ or $\mathbb{T}^n$, respectively), and with a connected component of the nodal set of ψ given by Σ, up to an ambient diffeomorphism.

Keywords Eigenfunctions of the Laplacian · Nodal sets · Isotopy type · Inverse localization

Mathematics Subject Classification 58J50

1 Introduction

Let M be a closed manifold of dimension $n \geq 3$ endowed with a smooth Riemannian metric g. The Laplace eigenfunctions of M satisfy the equation

$$\Delta u_k = -\lambda_k u_k \,,$$

where $0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \dots$ are the eigenvalues of the Laplacian. The zero set $u_k^{-1}(0)$ is called the nodal set of the eigenfunction.

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The study of the nodal sets of the eigenfunctions of the Laplacian in a compact Riemannian manifold is a classical topic in geometric analysis with a number of important open problems [18, 19]. When the Riemannian metric is not fixed, the nodal set is quite flexible. Indeed, it has been recently shown that [9], given a separating hypersurface Σ in M, there is a metric g on the manifold for which the nodal set $u_1^{-1}(0)$ of the first eigenfunction is precisely Σ. This result has been extended to the class of metrics conformal to a metric g_0 prescribed a priori [10], and to higher codimension submanifolds arising as the joint nodal set of several eigenfunctions corresponding to a degenerate eigenvalue [5].

For a fixed Riemannian metric, the problem is much more rigid than when one can freely choose a metric adapted to the geometry of the hypersurface that one aims to recover from the nodal set of the eigenfunctions. In this case, the techniques developed in [5, 9, 10] do not work. Nevertheless, since the Hausdorff measure of the nodal sets of the eigenfunctions grows as the eigenvalue tends to infinity [13, 14], one expects that the nodal set may become topologically complicated for high-energy eigenfunctions.

Our goal in this paper is to establish the existence of high-energy eigenfunctions of the Laplacian on the round sphere $\mathbb{S}^n$ and the flat torus $\mathbb{T}^n$ with nodal sets diffeomorphic to a given submanifold. All along this paper, $\mathbb{S}^n$ denotes the unit sphere in $\mathbb{R}^{n+1}$ and $\mathbb{T}^n$ is the standard flat n-torus, $(\mathbb{R}/2\pi\mathbb{Z})^n$.

More precisely, our main theorem shows that for a sequence of high enough eigenvalues, there exist m eigenfunctions of the Laplacian on $\mathbb{S}^n$ or $\mathbb{T}^n$ with a joint nodal set diffeomorphic to a given codimension m submanifold Σ. For the construction we need to assume that the normal bundle of Σ is trivial. This means that a small tubular neighborhood of the submanifold Σ must be diffeomorphic to $\Sigma \times \mathbb{R}^m$. In the statement, structural stability means that any small enough perturbation of the corresponding eigenfunction (in the C^k norm with $k \geq 1$) still has a union of connected components of the nodal set that is diffeomorphic to the submanifold Σ under consideration. Throughout, diffeomorphisms are of class C^∞ and submanifolds are C^∞ and without boundary.

Theorem 1.1 *Let Σ be a finite union of (disjoint, possibly knotted or linked) codimension $m \geq 1$ compact submanifolds of $\mathbb{S}^n$ or $\mathbb{T}^n$, $n \geq 3$, with trivial normal bundle. In the case of the torus, we further assume that Σ is contained in a contractible subset. If $m = 1$, we also assume that Σ is connected. Then for any large enough odd integer N there are m eigenfunctions $\psi_1, \dots, \psi_m$ of the Laplacian with eigenvalue $\lambda = N(N + n - 1)$ (in $\mathbb{S}^n$) or $\lambda = N^2$ (in $\mathbb{T}^n$), and a diffeomorphism Φ such that $\Phi(\Sigma)$ is the union of connected components of the joint nodal set $\psi_1^{-1}(0) \cap \dots \cap \psi_m^{-1}(0)$. Furthermore, $\Phi(\Sigma)$ is structurally stable.*

An important observation is that the proof of this theorem yields a reasonably complete understanding of the behavior of the diffeomorphism Φ, which is, in particular, connected with the identity. Oversimplifying a little, the effect of Φ is to uniformly rescale a contractible subset of the manifold that contains Σ to have a diameter of order $1/N$. In particular, the control that we have over the diffeomorphism Φ allows us to prove an analog of this result for quotients of the sphere by finite groups of isometries (lens spaces). Notice that $\Phi(\Sigma)$ is not guaranteed to contain all the components of the nodal set of the eigenfunction.

The proof of the main theorem involves an interplay between rigid and flexible properties of high-energy eigenfunctions of the Laplacian. Indeed, rigidity appears because high-energy eigenfunctions in any Riemannian n-manifold behave, locally in sets of diameter $1/\sqrt{\lambda}$, as monochromatic waves in $\mathbb{R}^n$ do in balls of diameter 1. We recall that a monochromatic wave is any solution to the Helmholtz equation $\Delta\phi + \phi = 0$. The catch here is that, in general, one cannot check whether a given monochromatic wave in $\mathbb{R}^n$ actually corresponds to a high-energy eigenfunction on the compact manifold.

To prove the converse implication, what we call an inverse localization theorem (see Sects. 2 and 3), it is key to exploit some flexibility that arises in the problem as a consequence of the fact that large eigenvalues of the Laplacian in the torus or in the sphere have increasingly high multiplicities (for this reason the proof does not work in a general Riemannian manifold). The inverse localization is a powerful tool to ensure that any monochromatic wave in a compact set of $\mathbb{R}^n$ can be reproduced in a small ball of the manifold by a high-energy eigenfunction. This allows us to transfer any structurally stable nodal set that can be realized in Euclidean space to high-energy eigenfunctions on $\mathbb{S}^n$ and $\mathbb{T}^n$. The inverse localization was first introduced in [11] to construct high-energy Beltrami fields on the torus and the sphere with topologically complicated vortex structures, and was also exploited in [6, 7] to solve a problem of Berry [2] on knotted nodal lines of high-energy eigenfunctions of the harmonic oscillator and the hydrogen atom, and in [17] to analyze the nodal sets of the eigenfunctions of the Dirac operator.

One should notice that the techniques introduced in [8] to prove the existence of solutions to second-order elliptic PDEs in $\mathbb{R}^n$ (including the monochromatic waves) with a prescribed nodal set Σ do not work for compact manifolds. The reason is that the proof is based on the construction of a local solution in a neighborhood of Σ, which is then approximated by a global solution in $\mathbb{R}^n$ using a Runge-type global approximation theorem. For compact manifolds the complement of the set Σ is precompact, so we cannot apply the global approximation theorem obtained in [8]. In fact, as is well known, this is not just a technical issue, but a fundamental obstruction in any approximation theorem of this sort. This invalidates the whole strategy followed in [8] and makes it apparent that new tools are needed to prove the existence of Laplace eigenfunctions with geometrically complex nodal sets in compact manifolds.

We finish this introduction with two corollaries. It is known that an oriented codimension one or two submanifold in $\mathbb{S}^n$ or $\mathbb{T}^n$ has trivial normal bundle [15], therefore the main theorem implies the following:

Corollary 1.2 *Let Σ be an oriented, compact, connected hypersurface in $\mathbb{S}^n$ or $\mathbb{T}^n$, $n \geq 3$. In the case of the torus, we further assume that Σ is contained in a contractible subset. Then for any large enough odd integer N there is an eigenfunction ψ of the Laplacian with eigenvalue $\lambda = N(N+n-1)$ (in $\mathbb{S}^n$) or $\lambda = N^2$ (in $\mathbb{T}^n$), and a diffeomorphism Φ such that $\Phi(\Sigma)$ is a structurally stable connected component of the nodal set $\psi^{-1}(0)$.*

Corollary 1.3 *Let Σ be a finite union of (disjoint, possibly knotted or linked) codimension two compact submanifolds in $\mathbb{S}^n$ or $\mathbb{T}^n$, $n \geq 3$. In the case of the torus, we further assume that Σ is contained in a contractible subset. Then for any large enough odd integer N there is a complex-valued eigenfunction ψ of the Laplacian with eigenvalue $\lambda = N(N+n-1)$ (in $\mathbb{S}^n$) or $\lambda = N^2$ (in $\mathbb{T}^n$), and a diffeomorphism Φ such that $\Phi(\Sigma)$ is a union of structurally stable connected components of the nodal set $\psi^{-1}(0)$.*

The paper is organized as follows. In Sects. 2 and 3 we prove an inverse localization theorem for the eigenfunctions of the Laplacian on $\mathbb{S}^n$ and $\mathbb{T}^n$, respectively. Theorem 1.1 is then proved in Sect. 4. Finally, in Sect. 5, we prove a refinement of the inverse localization Theorem on $\mathbb{S}^n$ that allows us to approximate several given monochromatic waves by a single eigenfunction of the Laplacian in different small regions of $\mathbb{S}^n$.

2 An Inverse Localization Theorem on the Sphere

In this section we prove an inverse localization theorem for eigenfunctions of the Laplacian on $\mathbb{S}^n$ for $n \geq 2$. We recall that the eigenvalues of the Laplacian on the n-sphere are of the form $N(N+n-1)$, where N is a nonnegative integer, and the corresponding multiplicity is given by

$$d(N,n) := \binom{N+n-1}{N} \frac{2N+n-1}{N+n-1}\,.$$

For the precise statement of the theorem, let us fix an arbitrary point $p_0 \in \mathbb{S}^n$ and take a patch of normal geodesic coordinates $\Psi : \mathbb{B} \to B$ centered at p_0. Here and in what follows, B_ρ (resp. $\mathbb{B}_\rho$) denotes the ball in $\mathbb{R}^n$ (resp. the geodesic ball in $\mathbb{S}^n$) centered at the origin (resp. at p_0) and of radius ρ, and we shall drop the subscript when $\rho = 1$. For the ease of notation, we will use the $\mathbb{R}^m$-valued functions $\phi := (\phi_1, \dots, \phi_m)$ and $\psi := (\psi_1, \dots, \psi_m)$, and the action of the Laplacian on such functions is understood componentwise.

Theorem 2.1 *Let ϕ be an $\mathbb{R}^m$-valued monochromatic wave in $\mathbb{R}^n$, satisfying $\Delta\phi + \phi = 0$. Fix a positive integer r and a positive constant δ'. For any large enough integer N, there is an $\mathbb{R}^m$-valued eigenfunction ψ of the Laplacian on $\mathbb{S}^n$ with eigenvalue $N(N+n-1)$ such that*

$$\left\| \phi - \psi \circ \Psi^{-1}\Big(\frac{\cdot}{N}\Big) \right\|_{C^r(B)} \leq \delta'\,.$$

To prove Theorem 2.1, we will proceed in two successive approximation steps. First, we will approximate the function ϕ in B by an $\mathbb{R}^m$-valued function φ that can be written as a finite sum of terms of the form

$$\frac{c_j}{|x-x_j|^{\frac{n}{2}-1}} J_{\frac{n}{2}-1}(|x-x_j|)$$

with $c_j \in \mathbb{R}^m$ and $x_j \in \mathbb{R}^n$, $j = 1, \dots, N'$, for N' large enough (Proposition 2.2 below). Here $J_{\frac{n}{2}-1}$ denotes the Bessel function of the first kind of order $\frac{n}{2} - 1$. Notice that any function of this form is a monochromatic wave. In the second step, we show that there is a collection of m eigenfunctions $(\psi_1, \dots, \psi_m) =: \psi$ in $\mathbb{S}^n$ with eigenvalue $N(N+n-1)$ such that, when considered in a ball of radius N^{-1}, they approximate $\varphi := (\varphi_1, \dots, \varphi_m)$ in the unit ball, provided that N is large enough.

Proposition 2.2 *Given any $\delta > 0$, there is a constant $R > 0$ and finitely many constant vectors $\{c_j\}_{j=1}^{N'} \subset \mathbb{R}^m$ and points $\{x_j\}_{j=1}^{N'} \subset B_R$ such that the function*

$$\varphi := \sum_{j=1}^{N'} \frac{c_j}{|x-x_j|^{\frac{n}{2}-1}} J_{\frac{n}{2}-1}(|x-x_j|)$$

approximates the function ϕ in the unit ball as

$$\|\phi - \varphi\|_{C^r(B)} < \delta\,.$$

Proof It is more convenient to work with complex-valued functions, so we set $\widetilde{\phi} := \phi + i\phi$. First, we notice that, since $\widetilde{\phi}$ is also a solution of the Helmholtz equation, it can be written in the ball B_2 as an expansion

$$\widetilde{\phi} = \sum_{l=0}^{\infty} \sum_{k=1}^{d(l,n-1)} b_{lk}\, j_l(r)\, Y_{lk}(\omega), \tag{2.1}$$

where $r := |x| \in \mathbb{R}^+$ and $\omega := x/r \in \mathbb{S}^{n-1}$ are spherical coordinates in $\mathbb{R}^n$, Y_{lk} is a basis of spherical harmonics of eigenvalue $l(l+n-2)$, j_l are n-dimensional hyperspherical Bessel functions and $b_{lk} \in \mathbb{C}^m$ are constant coefficients.

The series in (2.1) is convergent in the L^2 sense, so for any $\delta' > 0$, we can truncate the sum at some integer L

$$\phi_1 := \sum_{l=0}^{L} \sum_{k=1}^{d(l,n-1)} b_{lk}\, j_l(r)\, Y_{lk}(\omega) \tag{2.2}$$

so that it approximates $\widetilde{\phi}$ as

$$\|\phi_1 - \widetilde{\phi}\|_{L^2(B_2)} < \delta'\,. \tag{2.3}$$

The $\mathbb{C}^m$-valued function ϕ_1 decays as $|\phi_1(x)| \leq C/|x|^{\frac{n-1}{2}}$ for large enough $|x|$ (because of the decay properties of the spherical Bessel functions). Hence, Herglotz's theorem (see e.g. [12, Theorem 7.1.27]) ensures that we can write

$$\phi_1(x) = \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{S}^{n-1}} f_1(\xi)\, e^{ix\cdot\xi}\, d\sigma(\xi)\,, \tag{2.4}$$

where $d\sigma$ is the area measure on $\mathbb{S}^{n-1} := \{\xi \in \mathbb{R}^n : |\xi| = 1\}$ and f_1 is a $\mathbb{C}^m$-valued function in $L^2(\mathbb{S}^{n-1})$.

We now choose a smooth $\mathbb{C}^m$-valued function f_2 approximating f_1 as

$$\|f_1 - f_2\|_{L^2(\mathbb{S}^{n-1})} < \delta'\,,$$

which is always possible since smooth functions are dense in $L^2(\mathbb{S}^{n-1})$. The function defined as the inverse Fourier transform of f_2

$$\phi_2(x) := \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{S}^{n-1}} f_2(\xi)\, e^{ix\cdot\xi}\, d\sigma(\xi)\,, \tag{2.5}$$

approximates ϕ_1 uniformly: by the Cauchy–Schwarz inequality, we get

$$|\phi_2(x) - \phi_1(x)| = \left| \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{S}^{n-1}} (f_2(\xi) - f_1(\xi))\, e^{ix\cdot\xi}\, d\sigma(\xi) \right| \leq C\|f_2 - f_1\|_{L^2(\mathbb{S}^{n-1})} < C\delta' \tag{2.6}$$

for any $x \in \mathbb{R}^n$.

Our next objective is to approximate the function f_2 by a trigonometric polynomial: for any given δ', we will find a constant $R > 0$, finitely many points $\{x_j\}_{j=1}^{N'} \subset B_R$ and constants $\{c_j\}_{j=1}^{N'} \subset \mathbb{C}^m$ such that the smooth function in $\mathbb{R}^n$

$$f(\xi) := \sum_{j=1}^{N'} c_j\, e^{-ix_j\cdot\xi}\,,$$

when restricted to the unit sphere, approximates f_2 in the C^0 norm,

$$\|f - f_2\|_{C^0(\mathbb{S}^{n-1})} < \delta'\,. \tag{2.7}$$

In order to do so, we begin by extending f_2 to a smooth function $g : \mathbb{R}^n \to \mathbb{C}^m$ with compact support,

$$g(\xi) := \chi(|\xi|)\, f_2\left(\frac{\xi}{|\xi|}\right),$$

where $\chi(s)$ is a real-valued smooth bump function, being 1 when, for example, $|s - 1| < \frac{1}{4}$, and vanishing for $|s - 1| > \frac{1}{2}$. The inverse Fourier transform $\widehat{g}$ of g is Schwartz, so it is easy to see that, outside some ball B_R, the L^1 norm of $\widehat{g}$ is very small,

$$\int_{\mathbb{R}^n\setminus B_R} |\widehat{g}(x)|\, dx < \delta'\,,$$

and therefore we get a very good approximation of g by just considering its Fourier representation with frequencies within the ball B_R, that is,

$$\sup_{\xi\in\mathbb{R}^n}\left|g(\xi)-\frac{1}{(2\pi)^{\frac{n}{2}}}\int_{B_R}\widehat{g}(x)\,e^{-ix\cdot\xi}\,dx\right|<\delta'/2\,. \tag{2.8}$$

Next, let us show that we can approximate the integral

$$\int_{B_R}\widehat{g}(x)\,e^{-ix\cdot\xi}\,dx$$

by the sum

$$f(\xi):=\sum_{j=1}^{N'}c_j\,e^{-ix_j\cdot\xi} \tag{2.9}$$

with constants $c_j\in\mathbb{C}^m$ and points $x_j\in B_R$, so that we have the bound

$$\sup_{\xi\in\mathbb{S}^{n-1}}\left|\int_{B_R}\widehat{g}(x)\,e^{-ix\cdot\xi}\,dx-f(\xi)\right|<\delta'/2\,. \tag{2.10}$$

Indeed, consider a covering of the ball B_R by closed sets $\{U_j\}_{j=1}^{N'}$, with piecewise smooth boundaries, pairwise disjoint interiors, and diameters not exceeding δ''. Since the function $e^{-ix\cdot\xi}\,\widehat{g}(x)$ is smooth, we have that for each $x, y\in U_j$

$$\sup_{\xi\in\mathbb{S}^{n-1}}\left|\widehat{g}(x)\,e^{-ix\cdot\xi}-\widehat{g}(y)\,e^{-iy\cdot\xi}\right|<C\delta''\,,$$

with the constant C depending on $\widehat{g}$ (and therefore on δ') but not on δ''. If x_j is any point in U_j and we set $c_j:=\widehat{g}(x_j)\,|U_j|$ in (2.9), we get

$$\begin{aligned}\sup_{\xi\in\mathbb{S}^{n-1}}\left|\int_{B_R}\widehat{g}(x)\,e^{-ix\cdot\xi}\,dx-f(\xi)\right|&\leq\sum_{j=1}^{N'}\int_{U_j}\sup_{\xi\in\mathbb{S}^{n-1}}\left|\widehat{g}(x)\,e^{-ix\cdot\xi}-\widehat{g}(x_j)\,e^{-ix_j\cdot\xi}\right|dx\\&\leq C\delta''\,,\end{aligned}$$

with C depending on δ' and R but not on δ'' nor N'. By taking δ'' so that $C\delta''<\delta'/2$, the estimate (2.10) follows.

Now, in view of (2.8) and (2.10), one has

$$\|f-g\|_{C^0(\mathbb{S}^{n-1})}<\delta'\,,$$

so the estimate (2.7) follows upon noticing that the function f_2 is the restriction to $\mathbb{S}^{n-1}$ of the function g.

To conclude, set

$$\widetilde{\varphi}(x) := \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{S}^{n-1}} f(\xi)\, e^{ix\cdot\xi}\, d\sigma(\xi) = \sum_{j=1}^{N'} \frac{c_j}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{S}^{n-1}} e^{i(x-x_j)\cdot\xi}\, d\sigma(\xi) = \\ = \sum_{j=1}^{N'} \frac{c_j}{|x-x_j|^{\frac{n}{2}-1}} J_{\frac{n}{2}-1}(|x-x_j|)\,,$$

then from Eq. (2.7) we infer that

$$\|\widetilde{\varphi} - \phi_2\|_{C^0(\mathbb{R}^n)} \leq \int_{\mathbb{S}^{n-1}} |f(\xi) - f_2(\xi)|\, d\sigma(\xi) < C\delta'\,,$$

and from Eqs. (2.3) and (2.6) we get the L^2 estimate

$$\|\widetilde{\phi} - \widetilde{\varphi}\|_{L^2(B_2)} \leq C\|\widetilde{\varphi} - \phi_2\|_{C^0(\mathbb{R}^n)} + C\|\phi_2 - \phi_1\|_{C^0(\mathbb{R}^n)} + \\ + \|\phi_1 - \widetilde{\phi}\|_{L^2(B_2)} < C\delta'\,. \tag{2.11}$$

Furthermore, both $\widetilde{\varphi}$ and $\widetilde{\phi}$ are $\mathbb{C}^m$-valued functions satisfying the Helmholtz equation in $\mathbb{R}^n$ (note that the Fourier transform of $\widetilde{\varphi}$ is supported on $\mathbb{S}^{n-1}$), so by standard elliptic regularity estimates we have

$$\|\widetilde{\phi} - \widetilde{\varphi}\|_{C^r(B)} \leq C\|\widetilde{\phi} - \widetilde{\varphi}\|_{L^2(B_2)} < C\delta'\,.$$

This in particular implies that

$$\|\phi - \operatorname{Re}\widetilde{\varphi}\|_{C^r(B)} < C\delta'\,,$$

and taking δ' small enough so that $C\delta' < \delta$, resetting $c_j := \operatorname{Re} c_j$, and defining $\varphi := \operatorname{Re}\widetilde{\varphi}$, the proposition follows. □

The second step consists in showing that, for any large enough integer N, we can find an $\mathbb{R}^m$-valued eigenfunction ψ of the Laplacian on $\mathbb{S}^n$ with eigenvalue $N(N+n-1)$ that approximates, in the ball $\mathbb{B}_{1/N}$, when appropriately rescaled, the function φ in the unit ball. The proof is based on asymptotic expansions of ultraspherical polynomials, and uses the representation of φ as a sum of shifted Bessel functions which we obtained in the previous proposition as a key ingredient. It is then straightforward that Theorem 2.1 follows from Propositions 2.2 and 2.3, provided that N is large enough and δ is chosen so that $2\delta < \delta'$.

Proposition 2.3 *Given a constant $\delta > 0$, for any large enough positive integer N there is an $\mathbb{R}^m$-valued eigenfunction ψ of the Laplacian on $\mathbb{S}^n$ with eigenvalue $N(N+n-1)$ satisfying*

$$\left\|\varphi - \psi\circ\Psi^{-1}\Big(\frac{\cdot}{N}\Big)\right\|_{C^r(B)} < \delta\,.$$

Proof Consider the ultraspherical polynomial of dimension $n+1$ and degree N, $C_N^n(t)$, which is defined as

$$C_N^n(t) := \frac{\Gamma(N+1)\Gamma(\frac{n}{2})}{\Gamma(N+\frac{n}{2})}\, P_N^{(\frac{n}{2}-1,\, \frac{n}{2}-1)}(t)\,, \tag{2.12}$$

where $\Gamma(t)$ is the gamma function and $P_N^{(\alpha,\beta)}(t)$ are the Jacobi polynomials (see e.g. [16, Chap. IV, Sect. 4.7]). We have included a normalizing factor so that $C_N^n(1) = 1$ for all N.

Let p, q be two points in $\mathbb{S}^n$, considered as vectors in $\mathbb{R}^{n+1}$ with $|p| = |q| = 1$. The addition theorem for ultraspherical polynomials ensures that $C_N^n(p\cdot q)$ (where $p\cdot q$ denotes the scalar product in $\mathbb{R}^{n+1}$ of the vectors p and q) can be written as

$$C_N^n(p\cdot q) = \frac{2\pi^{\frac{n+1}{2}}}{\Gamma(\frac{n+1}{2})}\frac{1}{d(N,n)}\sum_{k=1}^{d(N,n)} Y_{Nk}(p)\, Y_{Nk}(q)\,, \tag{2.13}$$

with $\{Y_{Nk}\}_{k=1}^{d(N,n)}$ being an arbitrary orthonormal basis of eigenfunctions of the Laplacian on $\mathbb{S}^n$ (spherical harmonics) with eigenvalue $N(N+n-1)$.

The function φ is written as the finite sum

$$\varphi(x) = \sum_{j=1}^{N'} \frac{c_j}{|x-x_j|^{\frac{n}{2}-1}} J_{\frac{n}{2}-1}(|x-x_j|)\,,$$

with coefficients $c_j \in \mathbb{R}^m$ and points $x_j \in B_R$. With these c_j and x_j we define, for any point $p \in \mathbb{S}^n$, the function

$$\psi(p) := \sum_{j=1}^{N'} \frac{c_j}{2^{\frac{n}{2}-1}\Gamma(\frac{n}{2})} C_N^n(p\cdot p_j)\,,$$

where $p_j := \Psi^{-1}(\frac{x_j}{N})$. As long as $N > R$, p_j is well defined. In view of Eq. (2.13) it is clear that ψ is an $\mathbb{R}^m$-valued eigenfunction of the Laplacian on $\mathbb{S}^n$ with eigenvalue $N(N+n-1)$.

Our aim is to study the asymptotic properties of the eigenfunction ψ. To begin with, note that if we consider points $p := \Psi^{-1}(\frac{x}{N})$ with $N > R$ and $x \in B_R$, we have

$$p\cdot p_j = \cos\big(\mathrm{dist}_{\mathbb{S}^n}(p, p_j)\big) = \cos\left(\frac{|x-x_j| + O(N^{-1})}{N}\right), \tag{2.14}$$

as $N\to\infty$. The last equality comes from $\Psi : \mathbb{B} \to B$ being a patch of normal geodesic coordinates (by $\mathrm{dist}_{\mathbb{S}^n}(p, p_j)$ we mean the distance between p and p_j considered on the sphere $\mathbb{S}^n$). From now on we set

$$\widetilde{\psi}(x) := \psi \circ \Psi^{-1}\left(\frac{x}{N}\right). \tag{2.15}$$

When N is large, one has

$$\frac{\Gamma(N+1)}{\Gamma(N+\frac{n}{2})} = N^{1-\frac{n}{2}} + O(N^{-\frac{n}{2}}),$$

so from Eq. (2.14) we infer

$$C_N^n(p \cdot p_j) = \left(\Gamma\left(\frac{n}{2}\right)N^{1-\frac{n}{2}} + O(N^{-\frac{n}{2}})\right) P_N^{(\frac{n}{2}-1,\, \frac{n}{2}-1)}\left(\cos\left(\frac{|x-x_j| + O(N^{-1})}{N}\right)\right). \tag{2.16}$$

By virtue of Darboux's formula for the Jacobi polynomials [16, Theorem 8.1.1], we have the estimate

$$\frac{1}{N^{\frac{n}{2}-1}} P_N^{(\frac{n}{2}-1,\, \frac{n}{2}-1)}\left(\cos\frac{t}{N}\right) = 2^{\frac{n}{2}-1}\frac{J_{\frac{n}{2}-1}(t)}{t^{\frac{n}{2}-1}} + O(N^{-1}),$$

uniformly in compact sets (e.g., for $|t| \leq 2R$). Hence, in view of Eq. (2.16), the function $\widetilde{\psi}$ can be written as

$$\begin{aligned}\widetilde{\psi}(x) &= \sum_{j=1}^{N'} \frac{c_j}{2^{\frac{n}{2}-1}\Gamma(\frac{n}{2})} C_N^n\left(\cos\left(\frac{|x-x_j| + O(N^{-1})}{N}\right)\right) \\ &= \sum_{j=1}^{N'} \frac{c_j}{|x-x_j|^{\frac{n}{2}-1}} J_{\frac{n}{2}-1}(|x-x_j|) + O(N^{-1}),\end{aligned}$$

for N big enough and $x, x_j \in B_R$. From this we get the uniform bound

$$\|\varphi - \widetilde{\psi}\|_{C^0(B_2)} < \delta' \tag{2.17}$$

for any $\delta' > 0$ and all N large enough.

It remains to promote this bound to a C^r estimate. For this, note that, since the eigenfunction ψ has eigenvalue $N(N+n-1)$, the rescaled function $\widetilde{\psi}$ verifies on B the equation

$$\Delta\widetilde{\psi} + \widetilde{\psi} = \frac{1}{N} A\widetilde{\psi},$$

with

$$A\widetilde{\psi} := -(n-1)\widetilde{\psi} + G_1\,\partial\widetilde{\psi} + G_2\,\partial^2\widetilde{\psi},$$

where $\partial^k\widetilde{\psi}$ is a matrix whose entries are k-th order derivatives of $\widetilde{\psi}$, and $G_k(x, N)$ are smooth matrix-valued functions with uniformly bounded derivatives, i.e.,

$$\sup_{x\in B} |\partial_x^\alpha G_k(x, N)| \le C_\alpha\,, \tag{2.18}$$

with constants C_α independent of N.

Since φ satisfies the Helmholtz equation $\Delta\varphi + \varphi = 0$, the difference $\varphi - \widetilde{\psi}$ satisfies

$$\Delta(\varphi - \widetilde{\psi}) + (\varphi - \widetilde{\psi}) = \frac{1}{N} A\widetilde{\psi}\,,$$

and, considering the estimates (2.17) and (2.18), by standard elliptic estimates (applied to the uniformly elliptic operator $\Delta + 1 - N^{-1}A$) we get

$$\begin{aligned}\|\varphi - \widetilde{\psi}\|_{C^{r,\alpha}(B)} &< C\|\varphi - \widetilde{\psi}\|_{C^0(B_2)} + \frac{C}{N}\|A\varphi\|_{C^{r-2,\alpha}(B_2)}\\ &< C\delta' + \frac{C}{N}\|\varphi\|_{C^{r,\alpha}(B_2)}\,,\end{aligned}$$

so we conclude that, for N big enough and δ' small enough,

$$\|\varphi - \widetilde{\psi}\|_{C^r(B)} \le C\delta' + \frac{C\|\varphi\|_{C^{r,\alpha}}}{N} < \delta\,.$$

The proposition then follows. □

3 An Inverse Localization Theorem on the Torus

In this section we prove an inverse localization theorem for eigenfunctions of the Laplacian on $\mathbb{T}^n$ for $n \ge 3$. We recall that the eigenvalues of the Laplacian on the n-torus are the integers of the form

$$\lambda = |k|^2$$

for some $k \in \mathbb{Z}^n$. In particular, the spectrum of the Laplacian in $\mathbb{T}^n$ contains the set of the squares of integers.

As in the previous section, we fix an arbitrary point $p_0 \in \mathbb{T}^n$ and take a patch of normal geodesic coordinates $\Psi : \mathbb{B} \to B$ centered at p_0.

Theorem 3.1 *Let ϕ be an $\mathbb{R}^m$-valued function in $\mathbb{R}^n$, satisfying $\Delta\phi + \phi = 0$. Fix a positive integer r and a positive constant δ'. For any large enough odd integer N, there is an $\mathbb{R}^m$-valued eigenfunction ψ of the Laplacian on $\mathbb{T}^n$ with eigenvalue N^2 such that*

$$\left\| \phi - \psi \circ \Psi^{-1}\Big(\frac{\cdot}{N}\Big) \right\|_{C^r(B)} \le \delta'\,.$$

Proof Arguing as in the proof of Proposition 2.2 we can readily show that for any $\delta > 0$, there exists an $\mathbb{R}^m$-valued function ϕ_1 on $\mathbb{R}^n$ that approximates the function ϕ in the ball B as

$$\|\phi_1 - \phi\|_{C^0(B_2)} < \delta\,, \tag{3.1}$$

and that can be represented as the inverse Fourier transform of a distribution supported on the unit sphere of the form

$$\phi_1(x) = \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{S}^{n-1}} f(\xi)\, e^{i\xi\cdot x}\, d\sigma(\xi)\,.$$

Again $\mathbb{S}^{n-1}$ denotes the unit sphere $\{\xi \in \mathbb{R}^n : |\xi| = 1\}$ and f is a smooth $\mathbb{C}^m$-valued function on $\mathbb{S}^{n-1}$ satisfying $f(\xi) = \bar{f}(-\xi)$.

Let us now cover the sphere $\mathbb{S}^{n-1}$ by finitely many closed sets $\{U_k\}_{k=1}^{N'}$ with piecewise smooth boundaries and pairwise disjoint interiors such that the diameter of each set is at most ϵ. We can then repeat the argument used in the proof of Proposition 2.2 to infer that, if ξ_k is any point in U_k and we set

$$c_k := f(\xi_k)\,|U_k|\,,$$

the function

$$\widetilde{\psi}(x) := \frac{1}{(2\pi)^{\frac{n}{2}}} \sum_{k=1}^{N'} c_k\, e^{i\xi_k\cdot x}$$

approximates the function ϕ_1 uniformly with an error proportional to ϵ:

$$\|\widetilde{\psi} - \phi_1\|_{C^0(B_2)} < C\epsilon\,.$$

The constant C depends on δ but not on ϵ nor N', so one can choose the maximal diameter ϵ small enough so that

$$\|\widetilde{\psi} - \phi_1\|_{C^0(B_2)} < \delta\,. \tag{3.2}$$

In turn, the uniform estimate

$$\|\widetilde{\psi} - \phi\|_{C^0(B_2)} \leq \|\widetilde{\psi} - \phi_1\|_{C^0(B_2)} + \|\phi - \phi_1\|_{C^0(B_2)} < 2\delta$$

can be readily promoted to the C^r bound

$$\|\widetilde{\psi} - \phi\|_{C^r(B)} < C\delta\,. \tag{3.3}$$

This follows from standard elliptic estimates as both $\widetilde{\psi}$ (whose Fourier transform is supported on $\mathbb{S}^{n-1}$) and ϕ satisfy the Helmholtz equation:

$$\Delta\widetilde{\psi} + \widetilde{\psi} = 0\,, \qquad \Delta\phi + \phi = 0\,.$$

Furthermore, replacing $\widetilde{\psi}$ by its real part if necessary, we can safely assume that the function $\widetilde{\psi}$ is $\mathbb{R}^m$-valued.

Let us now observe that for any large enough odd integer N one can choose the points $\xi_k \in U_k \subset \mathbb{S}^{n-1}$ so that they have rational components (i.e., $\xi_k \in \mathbb{Q}^n$) and the rescalings $N\xi_k$ are integer vectors (i.e., $N\xi_k \in \mathbb{Z}^n$). This is because for $n \geq 3$, rational points $\xi \in \mathbb{S}^{n-1} \cap \mathbb{Q}^n$ of height N (and so with $N\xi \in \mathbb{Z}^n$) are uniformly distributed on the unit sphere as $N \to \infty$ through odd values [4] (in fact, the requirement for N to be odd can be dropped for $n \geq 4$ [4]).

Choosing ξ_k as above, we are now ready to prove the inverse localization theorem in the torus. Without loss of generality, we will take the origin as the base point p_0, so that we can identify the ball $\mathbb{B}$ with B through the canonical 2π-periodic coordinates on the torus. In particular, the diffeomorphism $\Psi : \mathbb{B} \to B$ that appears in the statement of the theorem can be understood to be the identity.

Since $N\xi_k \in \mathbb{Z}^n$, it follows that the function

$$\psi(x) := \frac{1}{(2\pi)^{\frac{n}{2}}} \sum_{k=1}^{N'} c_k e^{iN\xi_k\cdot x}$$

is 2π-periodic (that is, invariant under the translation $x \to x + 2\pi\, a$ for any vector $a \in \mathbb{Z}^n$). Therefore it defines a well-defined function on the torus, which we will still denote by ψ.

Since the Fourier transform of ψ is now supported on the sphere of radius N, ψ is an eigenfunction of the Laplacian on the torus $\mathbb{T}^n$ with eigenvalue N^2,

$$\Delta\psi + N^2\psi = 0\,.$$

The theorem then follows provided that δ is chosen small enough for $C\delta < \delta'$. □

We conclude this section noticing that the statement of Theorem 3.1 does not hold for $\mathbb{T}^2$. The reason is that rational points $\xi \in \mathbb{S}^1 \cap \mathbb{Q}^2$ with $N\xi \in \mathbb{Z}^2$ are no longer uniformly distributed on the unit circle (not even dense) as $N \to \infty$ through any sequence of odd values, counterexamples can be found in [3]. Nevertheless, a slightly different statement can be proved using [3]:

Theorem 3.2 *Let ϕ be an $\mathbb{R}^m$-valued function in $\mathbb{R}^2$, satisfying $\Delta\phi + \phi = 0$. Fix a positive integer r and a positive constant δ'. Then there exists a sequence of integers $\{N_l\}_{l=1}^{\infty} \nearrow \infty$, and $\mathbb{R}^m$-valued eigenfunctions ψ_l of the Laplacian on $\mathbb{T}^2$ with eigenvalues N_l^2 such that*

$$\left\| \phi - \psi_l \circ \Psi^{-1}\Big(\frac{\cdot}{N_l}\Big) \right\|_{C^r(B)} \leq \delta'$$

for l large enough.

4 Proof of the Main Theorem

For the ease of notation, we shall write $\mathbb{M}^n$ to denote either $\mathbb{T}^n$ or $\mathbb{S}^n$. Let Φ' be a diffeomorphism of $\mathbb{M}^n$ mapping the codimension m submanifold Σ into the ball $\mathbb{B}_{1/2} \subset \mathbb{M}^n$, and the ball $\mathbb{B}_{1/2}$ into itself. In $\mathbb{S}^n$, the existence of such a diffeomorphism is trivial, while in the case of $\mathbb{T}^n$ it follows from the assumption that Σ is contained in a contractible set.

Consider the submanifold Σ' in $B_{1/2} \subset \mathbb{R}^n$ defined as $\Phi'(\Sigma)$ in the patch of normal geodesic coordinates:

$$\Sigma' := (\Psi \circ \Phi')(\Sigma)\,.$$

It is shown in [8, Theorem 1.3] if $m \geq 2$ and [8, Remark A.2] if $m = 1$, that there is an $\mathbb{R}^m$-valued monochromatic wave $\phi = (\phi_1, \ldots, \phi_m)$, satisfying $\Delta\phi + \phi = 0$ in $\mathbb{R}^n$, and a diffeomorphism Φ_1 (close to the identity, and different from the identity only on $B_{1/2}$ when $m > 1$) such that $\Phi_1(\Sigma') \subset B_{1/2}$ is a union of connected components of the joint nodal set $\phi_1^{-1}(0) \cap \cdots \cap \phi_m^{-1}(0)$. In addition, the construction in [8] ensures that the regularity condition $rk(\nabla\phi_1, \ldots, \nabla\phi_m) = m$ holds at any point of $\Phi_1(\Sigma')$, so it is a structurally stable nodal set of ϕ by Thom's isotopy theorem [1].

Now, the inverse localization theorem (Theorem 2.1 in the case of $\mathbb{S}^n$ and Theorem 3.1 for $\mathbb{T}^n$) allows us to find, for any large enough odd integer N, an $\mathbb{R}^m$-valued function $\psi = (\psi_1, \ldots, \psi_m)$ in $\mathbb{M}^n$ satisfying $\Delta\psi = -\lambda\psi$ (with $\lambda := N(N+n-1)$ or $\lambda := N^2$ in the sphere or the torus, respectively) and such that $\psi \circ \Psi^{-1}(\frac{\cdot}{N})$ approximates ϕ in the $C^r(B)$ norm as much as we want.

The structural stability property ensures the existence of a second diffeomorphism $\Phi_2 : \mathbb{R}^n \to \mathbb{R}^n$ close to the identity, and different from the identity only on $B_{1/2}$, such that $\Phi_2(\Phi_1(\Sigma'))$ is a union of connected components of the joint nodal set of the $\mathbb{R}^m$-valued function $\psi \circ \Psi^{-1}(\frac{\cdot}{N})$. Therefore, the corresponding submanifold

$$\Phi(\Sigma) := \Psi^{-1}\Big(\frac{1}{N}\Phi_2(\Phi_1((\Psi \circ \Phi')(\Sigma)))\Big)$$

is a union of connected components of the nodal set of ψ. The map $\Phi : \mathbb{M}^n \to \mathbb{B}_{\frac{1}{2N}}$ thus defined is easily extended to a diffeomorphism of the whole manifold $\mathbb{M}^n$. Finally, we have by the construction that $\Phi(\Sigma)$ is structurally stable, and hence Theorem 1.1 follows.

5 Final Remark: Inverse Localization on the Sphere in Multiple Regions

Theorem 2.1 in Sect. 2 can be refined to include inverse localization at different points of the sphere. This way, we get an eigenfunction of the Laplacian that approximates several given solutions of the Helmholtz equation in different regions. The fast decay of ultraspherical polynomials of high degree outside the domains where they behave as shifted Bessel functions is behind this multiple localization. Notice that, in contrast, trigonometric polynomials do not exhibit this decay, hence the lack of an analog of the following result in the case of the torus. All along this section we assume that $n \geq 2$.

Let $\{p_\alpha\}_{\alpha=1}^{N'}$ be a set of points in $\mathbb{S}^n$, with N' an arbitrarily large (but fixed throughout) integer. We denote by $\Psi_\alpha : \mathbb{B}_\rho(p_\alpha) \to B_\rho$ the corresponding geodesic patches on balls of radius ρ centered at the points p_α. We fix a radius ρ such that no two balls intersect, for example by setting

$$\rho := \frac{1}{2} \min_{\alpha \neq \beta} dist_{\mathbb{S}^n}(p_\alpha, p_\beta) .$$

We further choose the points $\{p_\alpha\}_{\alpha=1}^{N'}$ so that no pair of points are antipodal in $\mathbb{S}^n \subset \mathbb{R}^{n+1}$, i.e. $p_\alpha \neq -p_\beta$ for all α, β. The reason is that the eigenfunctions of the Laplacian on the sphere with eigenvalue $N(N+n-1)$ have parity $(-1)^N$:

$$\psi(p_\alpha) = (-1)^N \psi(-p_\alpha)$$

(they are the restriction to the sphere of homogenous harmonic polynomials of degree N); so that prescribing the behavior of an eigenfunction in a ball around the point p_α automatically determines its behavior in the antipodal ball.

Proposition 5.1 *Let $\{\phi_\alpha\}_{\alpha=1}^{N'}$ be a set of N' $\mathbb{R}^m$-valued monochromatic waves in $\mathbb{R}^n$, $1 \leq m \leq n$, satisfying $\Delta\phi_\alpha + \phi_\alpha = 0$. Fix a positive integer r and a positive constant δ. For any large enough integer N, there is an $\mathbb{R}^m$-valued eigenfunction ψ of the Laplacian on $\mathbb{S}^n$ with eigenvalue $N(N+n-1)$ such that*

$$\left\| \phi_\alpha - \psi \circ \Psi_\alpha^{-1}\Big(\frac{\cdot}{N}\Big) \right\|_{C^r(B)} < \delta$$

for all $1 \leq \alpha \leq N'$.

Proof We use the notation introduced in the proof of Proposition 2.3 without further mention. Applying Theorem 2.1 to each ϕ_α we obtain, for high enough N, $\mathbb{R}^m$-valued eigenfunctions of the Laplacian $\{\psi_\alpha\}_{\alpha=1}^{N'}$ satisfying the bound

$$\left\| \phi_\alpha - \psi_\alpha \circ \Psi_\alpha^{-1}\Big(\frac{\cdot}{N}\Big) \right\|_{C^r(B)} < \delta' .$$

For each α, the $\mathbb{R}^m$-valued eigenfunction $\psi_\alpha(p)$ is a linear combination (with coefficients in $\mathbb{R}^m$) of ultraspherical polynomials $C_N^n(p \cdot q_j)$, where $\{q_j\}$ is a finite set of points such that $dist_{\mathbb{S}^n}(p_\alpha, q_j)$ is proportional to N^{-1}, for all j. Recall that the ultraspherical polynomials satisfy the asymptotic formula

$$C_N^n(p\cdot q)=\frac{\Gamma(\frac{n}{2})}{N^{\frac{n}{2}-1}}\,P_N^{(\frac{n}{2}-1,\,\frac{n}{2}-1)}(\cos(dist_{\mathbb{S}^n}(p,q)))+O(N^{-\frac{n}{2}})\,,$$

so considering the fact that the Jacobi polynomials behave as (see [16, Theorem 7.32.2])

$$N^{1-\frac{n}{2}}\,P_N^{(\frac{n}{2}-1,\frac{n}{2}-1)}(\cos t)=\frac{O(N^{\frac{1-n}{2}})}{t^{\frac{n-1}{2}}},$$

uniformly for $N^{-1}<t<\pi-N^{-1}$, we can conclude that the functions $C_N^n(p\cdot q_j)$ are uniformly bounded as

$$|C_N^n(p\cdot q_j)|\leq\frac{C_\rho}{N^{\frac{n-1}{2}}}$$

for any point p satisfying

$$\min_j dist_{\mathbb{S}^n}(p,q_j)\geq\rho\quad\text{and}\quad\min_j dist_{\mathbb{S}^n}(p,-q_j)\geq\rho\,,$$

and where C_ρ is a constant depending only on ρ. The same decay is thus also exhibited by the eigenfunction ψ_α,

$$\|\psi_\alpha\|_{C^0(\mathbb{S}^n\setminus(\mathbb{B}(p_\alpha,\rho)\cup\mathbb{B}(-p_\alpha,\rho))}\leq\frac{C}{N^{\frac{n-1}{2}}}$$

since it is just a normalized linear combination of ultraspherical polynomials (here the constant C depends on ρ and on the particular coefficients in the expansion of ψ_α, that is, on ϕ_α and δ').

Now, if we define the $\mathbb{R}^m$-valued eigenfunction

$$\psi:=\sum_{\alpha=1}^{N'}\psi_\alpha$$

and we choose N large enough, the statement of the proposition follows for $r=0$. By standard elliptic estimates, the C^0 bound can be easily promoted to a C^r bound, so we are done. □

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