Local Weighted Average Sampling Over Multiply Generated Spline Spaces of Polynomial Growth

S. Yugesh and P. Devaraj

Abstract We analyse the following average sampling problem over multiply generated spline spaces of polynomial growth: Let h be a nonnegative integrable function supported in $\left[-\frac{1}{2}, \frac{1}{2}\right]$. Given a sequence of samples $\{y_n\}_{n\in\mathbb{Z}}$, of polynomial growth, find a spline f having polynomial growth such that $\int_{-\frac{1}{2}}^{\frac{1}{2}} f(n-y)h(y)dy = y_n, n \in \mathbb{Z}$. It is shown that this problem has a unique solution for certain subspaces of the multiply generated spline spaces of polynomial growth.

Keywords Multiply generated spline · Multiply generated spline space · Average sampling

1 Introduction and Preliminaries

In signal and image processing, continuous signals need to be represented by their discrete samples. A significant problem in signal processing is how to represent a continuous signal in terms of its discrete samples. One of the important themes of sampling theory is, to recover a continuous function from its discrete sample values. The sampling theorems provide the reconstruction formulas and hence such theorems become the most useful tool in the field of signal and image processing. The famous Shannon sampling theorem states that finite energy bandlimited signals are completely characterized by their sample values [2–5, 10]. Moreover, Shannon gave the following reconstruction formula

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$$f(x) = \sum_{k\in\mathbb{Z}} f\left(\frac{k\pi}{\Omega}\right)\frac{\sin(\Omega x - k\pi)}{\Omega x - k\pi}.$$

In the Shannon reconstruction formula, *sinc* function is used and is in fact a scaling function of a multiresolution analysis used in wavelet theory, since the bandlimited condition forces the signal to be of infinite duration which is not always realistic. Further, the *sinc* function itself is not very suitable for rapid communications. Some of these constraints necessitate the need for investigating other function classes for which the sampling theorem holds. Many generalizations of the classical Shannon sampling theorem have been proposed. Moreover, in [2–5, 10, 14], the sampling procedure in shift-invariant spaces as well as spline spaces has been analysed. The requirement is that the signal to be bandlimited can be avoided by considering signals in spaces like the wavelet subspaces, shift-invariant spaces and spline subspaces.

Recently, the sampling and reconstruction technique was investigated for multiply generated shift-invariant spaces and spline subspaces in [1, 6, 9, 11–13]. In these literatures, they consider finite energy signals. In this paper, we consider the space of functions having polynomial growth. In [11, 12], the multiply generated spline space is defined as

$$\mathcal{S} = \left\{ f : f = \sum_{i=1}^{r}\sum_{n\in\mathbb{Z}} a_i(n)\beta_{d_i}(t-n)\right\}$$

with suitable coefficients $a_i(n)$, where β_{d_i} is the cardinal central B-spline of degree d_i and is defined by

$$\beta_{d_i} = \chi_{[-\frac{1}{2},\frac{1}{2}]} \star \chi_{[-\frac{1}{2},\frac{1}{2}]} \star \cdots \star \chi_{[-\frac{1}{2},\frac{1}{2}]} \;\; (d_i + 1 \text{ terms}).$$

We consider the following subspace of the multiply generated spline space:

$$\mathcal{S}_N := \left\{ f : f = \sum_{n\in\mathbb{Z}} a_n \sum_{i=1}^{r} \beta_{d_i}(t-n)\right\}$$

If $M = max\{d_1, d_2, \ldots, d_r\}$ and $m = min\{d_1, d_2, \ldots, d_r\}$, then $f \in \mathcal{S}_N$ provided that $f(x) \in \mathbb{C}^{m-1}$ and that the restriction of $f(x)$ to any interval between consecutive knots is identical with a polynomial of degree not exceeding M. If d_i's are distinct, then $\sum_{i=1}^{r}\beta_{d_i}(.-n)$, $n \in \mathbb{Z}$, are globally linearly independent (Lemma 1).

Let

$$\mathcal{S}_{\mathcal{N},\gamma} = \{f(t) \in \mathcal{S}_N : f(t) = O(|t|^{\gamma}) \; as \; t \to \pm\infty\}$$

and

$$\mathcal{D}_{\gamma} = \{\{y_n\} : y_n = O(|n|^{\gamma}) \; as \; n \to \pm\infty\}.$$

For $\gamma \geq 0$, a given sequence of numbers $\{y_n\}_{n\in\mathbb{Z}} \in \mathcal{D}_\gamma$ the following problem: Find a spline $f \in \mathcal{S}_{\mathcal{N},\gamma}$, satisfying $f(n) = y_n$, $n \in \mathbb{Z}$, has a unique solution. However, in many real applications, sampling points are not always measured exactly. Sometimes, the sampling steps need to be fluctuated according to the signals so as to reduce the number of samples and computational complexity. Therefore average sampling have been analysed in the literature [1, 2, 5, 7, 10–14]. In most of these studies, the sequence of samples $\{y_n\}$ is assumed to be in ℓ^2 or ℓ^p. In [8], we considered the average samples $\{f \star h(n)\}$ of polynomial growth and analysed the following problem.

Problem:

Given a sequence of numbers $\{y_n\}_{n\in\mathbb{Z}} \in \mathcal{D}_\gamma$, find a multiply generated spline $f \in \mathcal{S}_{\mathcal{N},\gamma}$, such that

$$f \star h(n) = y_n, \quad n \in \mathbb{Z},$$

where h satisfies

$$supp(h) \subseteq \left[-\frac{1}{2}, \frac{1}{2}\right], \qquad h(t) \geq 0, t \in \mathbb{R} \tag{1}$$

$$0 < \int_{-\frac{1}{2}}^{0} h(t)dt < \infty \quad \text{and} \quad 0 < \int_{0}^{\frac{1}{2}} h(t)dt < \infty. \tag{2}$$

In [8], it is shown that this problem has a unique solution for $d_1 = 1$, $d_2 = 2$, $d_3 = 3$ and $d_4 = 4$. In this paper, we analyse all the possible cases for $d_i \leq 4$ and show that the solution to the local average sampling problem is unique.

Lemma 1 *Let $d_i \in \mathbb{N}$ be distinct. Then the set $\left\{\sum_{i=1}^{r} \beta_{d_i}(. - k) : k \in \mathbb{Z}\right\}$ of functions is globally linearly independent on $\mathbb{R}$.*

Proof For $N \in \mathbb{N}$, consider $S_N = \left\{f|_{[-N,N]} : f \in S\right\}$. That is the restriction to $[-N, N]$ of the functions in S. We shall show that the set

$$\left\{\sum_{i=1}^{r} \beta_{d_i}(. - k) : k = -(N-M), -(N-M)+1, \ldots, N-M\right\}$$

is linearly independent on $[-N, N]$, where $M = Max(d_1, d_2, \ldots, d_r)$. Without loss of generality, we may assume that $d_1 < d_2 < \ldots < d_r$. Let

$$\sum_{k=-(N-M)}^{N-M} c_k \sum_{i=1}^{r} \beta_{d_i}(x-k) = 0 \tag{3}$$

for $x \in [-N, N]$. For $x = N - \frac{1}{2}$ and $k = N - M$, $-d_r < x - k < d_r$ and

$$\sum_{i=1}^{r} \beta_{d_i}(x-k) = 0$$

for $-(N-M) \leq k \leq N-M-1$ and hence by substituting this x in Eq. (3) we get $c_{N-M} = 0$. Similarly, by choosing suitable x and substituting in (3), we get $c_k = 0$ for $-(N-M) \leq k \leq N-M$. As $S = \bigcup_{N \in \mathbb{N}} S_N$, we get that $\left\{\sum_{i=1}^{r} \beta_{d_i}(.-k) : k \in \mathbb{Z}\right\}$ is linearly independent on $\mathbb{R}$. □

2 Average Sampling Theorem for Multiply Generated Spline Space

Theorem 1 (Main Theorem) *Let $d_i \leq 4$ and $h(t)$ be an integrable function satisfying conditions 1, 2. Then for a given sequence of numbers $\{y_n\}_{n\in\mathbb{Z}} \in \mathcal{D}_\gamma$, there exists a unique $f \in \mathcal{S}_{\mathcal{N},\gamma}$, such that*

$$f \star h(n) = y_n, n \in \mathbb{Z}. \tag{4}$$

The Generalized Euler-Frobenius Laurent polynomial is defined as

$$G_h(z) = \sum_{i=1}^{r} G_{h,d_i}(z) = \sum_{i=1}^{r}\sum_{n\in\mathbb{Z}} h \star \beta_{d_i}(n) z^n.$$

It is easy to see that

$$G_{h,d_i}(z) = \int_{-\frac{1}{2}}^{\frac{1}{2}} h(t)\Upsilon_{z,d_i}(t)dt, \tag{5}$$

where $\Upsilon_{z,d_i}(t)$ is the exponential Euler spline and is defined as

$$\Upsilon_{z,d_i}(t) := \sum_{n\in\mathbb{Z}} z^n \beta_{d_i}(n-t).$$

Therefore

$$G_h(z) = \int_{-\frac{1}{2}}^{\frac{1}{2}} h(t)\Upsilon_z(t)dt,$$

where $\Upsilon_z(t) = \sum_{i=1}^{r} \Upsilon_{z,d_i}(t)$.

We need some properties of exponential Euler splines.

Lemma 2 *[8] For $d_i \in \mathbb{N}, n \in \mathbb{Z}$ and $z \in \mathbb{C} \setminus \{0\}$, we have*

(i) $\Upsilon_{z^{-1}}(-t) = \Upsilon_z(t)$,
(ii) $\Upsilon_z(t+n) = (z)^n \Upsilon_z(t)$,
(iii) $\frac{d}{dt}(\Upsilon_{z,d_i+1}(t)) = \left(1 - \frac{1}{z}\right)\Upsilon_{z,d_i}\left(t + \frac{1}{2}\right)$,
(iv) $\Upsilon_{-1,d_i}\left(\frac{1}{2}\right) = 0$ *and* $\Upsilon_{-1,d_i}(t) > 0$ *for* $t \in \left(-\frac{1}{2}, \frac{1}{2}\right)$.

In [8], it is shown that if the roots of $G_h(z)$ are simple and there is no root on the unit circle, then the local average sampling problem has a unique solution.

Theorem 2 *[8] Let $d_i \in \mathbb{N}$ and $h(t)$ be an integrable function satisfying conditions 1 and 2. If the roots of $G_h(z)$ are simple and there are no roots on the unit circle $|z| = 1$, then for a given sequence of numbers $\{y_n\}_{n\in\mathbb{Z}} \in \mathcal{D}_\gamma$, there exists a unique $f \in \mathcal{S}_{N,\gamma}$, such that*

$$f \star h(n) = y_n, n \in \mathbb{Z}. \tag{6}$$

Moreover, the solution can be written as

$$f(t) = \sum_{n\in\mathbb{Z}} y_n L(t-n),$$

where the reconstruction function L is given by $L(t) := \sum_{i=1}^r L_i(t) := \sum_{i=1}^r \sum_{n\in\mathbb{Z}} c_n \beta_{d_i}(t-n)$ and c_n are the coefficients of the Laurent series expansion of $G_h(z)^{-1}$. Further the reconstruction function L is of exponential decay.

3 Behaviour of $G_h(z)$

Proof (Main Theorem)

As a consequence of Theorem 2, it is sufficient to show that for $d_i \leq 4$ all the roots of $G_h(z)$ are simple and none of them are on the unit circle $|z| = 1$.

The Generalized Euler-Frobenius Laurent polynomial $G_h(z) = \sum_{i=1}^r G_{h,d_i}(z)$ can be written as

$$G_h(z) = \sum_{i=1}^r z^{\frac{-l_i}{2}} P_i(z)$$

where $l_i := \begin{cases} d_i + 1 & \text{if } d_i \text{ is odd} \\ d_i & \text{if } d_i \text{ is even} \end{cases}$ and $P_i(z)$ is a polynomial of degree l_i. Hence

$$G_h(z) = z^{\frac{-m}{2}} \sum_{i=1}^r z^{\frac{m-l_i}{2}} P_i(z) = z^{\frac{-m}{2}} P(z),$$

where $P(z)$ is a polynomial of degree $m = max(l_1, l_2, \ldots, l_r)$.

For $d_i \leq 4$, we get $m = 4$ and we can write

$$\begin{aligned} P(z) &= z^2 G_h(z) \\ &= z^4 \left\{h \star \beta_{d_4}(2) + h \star \beta_{d_3}(2)\right\} + z^3 \left\{h \star \beta_{d_4}(1) + h \star \beta_{d_3}(1) + h \star \beta_{d_2}(1) + h \star \beta_{d_1}(1)\right\} \\ &\quad + z^2 \left\{h \star \beta_{d_4}(0) + h \star \beta_{d_3}(0) + h \star \beta_{d_2}(0) + h \star \beta_{d_1}(0)\right\} \\ &\quad + z \left\{h \star \beta_{d_4}(-1) + h \star \beta_{d_3}(-1) + h \star \beta_{d_2}(-1) + h \star \beta_{d_1}(-1)\right\} \\ &\quad + \left\{h \star \beta_{d_4}(-2) + h \star \beta_{d_3}(-2)\right\}. \end{aligned}$$

We obtain $P(0) > 0$ and $P(1) > 0$.
We can write $P(z)$ as

$$P(z) = z^2 \sum_{i=1}^{4} \int_{-\frac{1}{2}}^{\frac{1}{2}} h(t) \Upsilon_{z,d_i}(t) dt. \tag{7}$$

By lemma 2 and Eq. (7), we obtain

$$P(-1) = \sum_{i=1}^{4} \int_{-\frac{1}{2}}^{\frac{1}{2}} h(t) \Upsilon_{-1,d_i}(t) dt > 0.$$

Since $\lim_{z \longrightarrow \infty} P(z) = \infty$, it suffices to find $z_0 \in (-1, 0)$ such that

$$\sum_{i=1}^{4} \Upsilon_{z_0,d_i}(t) < 0, \text{ for all} t \in \left(-\frac{1}{2}, \frac{1}{2}\right), \tag{8}$$

since for such a z_0, we have

$$P(z_0) = z_0^2 \sum_{i=1}^{4} \int_{-\frac{1}{2}}^{\frac{1}{2}} h(t) \Upsilon_{z_0,d_i}(t) dt < 0, z_0 \in (-1, 0)$$

and

$$P\left(\frac{1}{z_0}\right) = \frac{1}{z_0^2} \sum_{i=1}^{4} \int_{-\frac{1}{2}}^{\frac{1}{2}} h(t) \Upsilon_{z_0^{-1},d_i}(t) dt = \frac{1}{z_0^2} \sum_{i=1}^{4} \int_{-\frac{1}{2}}^{\frac{1}{2}} h(t) \Upsilon_{z_0,d_i}(-t) dt < 0, z_0^{-1} \in (-\infty, -1).$$

By solving $\sum_{i=1}^{4} \Upsilon_{z_0,d_i}\left(\frac{1}{2}\right) = 0$, we get a unique $z_0 \in (-1, 0)$.

In the next table, we obtain all possible cases of $d_i \leq 4$.

S. No.	d_1	d_2	d_3	d_4	$\sum_{i=1}^{4} \Upsilon_{z_0,d_i}\left(\frac{1}{2}\right) = 0$	Possible solutions
1	1	2	3	4	$\frac{3}{48} z_0^3 + \frac{93}{48} z_0^2 + \frac{93}{48} z_0 + \frac{3}{48}$	$z_0 = -1, -15 - 4\sqrt{14}, -15 + 4\sqrt{14}$.
2	1	–	3	4	$\frac{1}{16} z_0^3 + \frac{69}{48} z_0^2 + \frac{69}{48} z_0 + \frac{1}{16}$	$z_0 = -1, -11 + 2\sqrt{30}, -11 - 2\sqrt{30}$.
3	1	2	–	4	$\frac{1}{24} z_0^3 + \frac{35}{24} z_0^2 + \frac{35}{24} z_0 + \frac{1}{24}$	$z_0 = -1, -17 + 12\sqrt{2}, -17 - 12\sqrt{2}$.
4	–	2	3	4	$\frac{1}{16} z_0^3 + \frac{69}{48} z_0^2 + \frac{69}{48} z_0 + \frac{1}{16}$	$z_0 = -1, -11 + 2\sqrt{30}, -11 - 2\sqrt{30}$.
5	1	2	3	–	$\frac{1}{48} z_0^3 + \frac{71}{48} z_0^2 + \frac{71}{48} z_0 + \frac{1}{48}$	$z_0 = -1, -35 + 6\sqrt{34}, -35 - 6\sqrt{34}$.
6	–	2	–	4	$\frac{1}{24} z_0^3 + \frac{23}{24} z_0^2 + \frac{23}{24} z_0 + \frac{1}{24}$	$z_0 = -1, -11 + 2\sqrt{30}, -11 - 2\sqrt{30}$.
7	–	–	3	4	$\frac{1}{16} z_0^3 + \frac{45}{48} z_0^2 + \frac{45}{48} z_0 + \frac{1}{16}$	$z_0 = -1, -7 + 4\sqrt{3}, -7 - 4\sqrt{3}$.
8	–	2	3	–	$\frac{1}{48} z_0^3 + \frac{47}{48} z_0^2 + \frac{47}{48} z_0 + \frac{1}{48}$	$z_0 = -1, -23 + 4\sqrt{33}, -23 - 4\sqrt{33}$.
9	1	–	–	4	$\frac{1}{24} z_0^3 + \frac{23}{24} z_0^2 + \frac{23}{24} z_0 + \frac{1}{24}$	$z_0 = -1, -11 + 2\sqrt{30}, -11 - 2\sqrt{30}$.
10	1	–	3	–	$\frac{1}{48} z_0^3 + \frac{47}{48} z_0^2 + \frac{47}{48} z_0 + \frac{1}{48}$	$z_0 = -1, -23 + 4\sqrt{33}, -23 - 4\sqrt{33}$.
11	1	2	–	–	$1 + z_0$	$z_0 = -1$.

By this table, we get a unique solution $z_0 \in (-1, 0)$. For such a z_0 value, we obtain

$$\sum_{i=1}^{4} \Upsilon_{z_0,d_i}(t) < 0, \text{ for all} t \in \left(-\frac{1}{2}, \frac{1}{2}\right).$$

Thus all the roots of $G_h(z)$ are simple and there are no roots on the unit circle for $d_i \leq 4$. □

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