Chapter 3
A Concept of Multi Rough Sets Defined on Multi-contextual Information Systems

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Abstract Rough set theory, introduced by Pawlak in 1982 [1], is an important concept in constructing many applications of Data Mining and Knowledge Discovery. Rough set as a generalization of crisp set, deals with crisp granularity of objects by providing an alternative to formulate a given crisp set with imprecise boundaries. In rough set theory, a given crisp set of object is approximated into two different subsets derived from a crisp partition defined on the universal set of objects. The universal set of objects is characterized by a non-empty finite set of attributes, called data table or information system. The information system is formally represented by a pair (U, A) in which U is a universal set of objects and A is a finite set of attributes. In the real application, depending on the context, a given object may have different values of attributes. Thus, a given set of objects might be approximated based on multi-context of attributes, called multi-contextual information systems. Here, n context of attributes will provide n partitions. Clearly, a given set of object, $X \subseteq U$, may then be represented by n pairs of lower and upper approximations. The n pairs of lower and upper approximations are denoted as *multi rough sets* of X as already proposed in [2, 3]. This paper extends the concept of multi rough sets by providing more properties and examining more set operations.

Keywords Rough sets · Multi rough sets · Granular computing · Multi-context of attributes

3.1 Introduction

In the real application, depending on the context, a given object may be characterized by different values of attributes or different set of attributes. In this case, different set of attributes are considered as different contexts in which they may also

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provide different values of attributes for a given object based on different perceptions. Here, context may be viewed as background or situation by which somehow it is necessary to group some attributes as a subset of attributes and consider the subset as a certain context. For instance, considering human beings as a universal set of objects, every person (object) might be characterized by several sets of attributes corresponding to some contexts such as his or her status as employee, student, club member, family member, etc. In the context of student, student's set of attributes might be regarded as {*ID-Number*, *Name*, *Address*, *Supervisor*, *Major*, etc.}. We may consider different sets of attributes in the relation to the contexts of both employee and family member. Using the same example of human beings as universal set of objects, in the relation to perception-based data especially for fuzzy data, attributes such as *weight*, *age* and *height* might have different fuzzy values in describing a given certain object depending on contexts (perceptions) of Japanese, American and so on. For instance, Japanese may consider height of 175 cm as {*high*}, but American may consider it as {*medium*}. Therefore, it is necessary to consider multi-contextual information systems as an extension of information system (see Sect. 3.3). Here, every context as represented by a certain set of attributes may provide a certain partition of objects. Consequently, n contexts (n subsets of attributes) will provide n partitions. A given set of object, X, may then be approximated into n pairs of lower and upper approximations, called *multi rough sets* of X as already proposed in [4, 5]. In the relation to the concept of multi rough sets, more properties and more set operations are proposed and examined. Primary concern is also given to the generalization of contexts in the presence of multi-contextual information systems. Furthermore, three general contexts, namely AND-general context, OR-general context and OR$^+$ general context, are recalled. It can be proved that AND-general context and OR$^+$-general context provide (disjoint) partitions. On the other hand, OR$^+$-general context provides covering of the universe. Then, a summarized rough set of a given crisp set of objects is able to be derived from partitions as well as covering of the general contexts. Finally, relations among three general contexts are examined and summarized.

3.2 Concept of Rough Sets

Rough set theory, introduced by Pawlak in 1982 [6], plays essential roles in many applications of Data Mining and Knowledge Discovery. The theory offers mathematical tools to discover hidden patterns in data, recognize partial or total dependencies in data bases, remove redundant data, and others [7]. Rough set generalizes classical (crisp) set by providing an alternative to formulate sets with imprecise boundaries. A rough set is basically an approximate expression of a given crisp set in terms of two crisp subsets derived from a crisp partition defined on the universal set involved [3]. Two subsets are denoted as the lower and upper approximation. Every elemets in the crisp partition is related based on equivalence relation. Thus, an element belongs to the only one equivalence class and two distinct equivalence classes are disjoint. Formally, the concept of rough sets is defined precisely as follows. Let U denotes a non-empty universal set, and let R be an equivalence relation on U. The partition of the universe is referred to as the quotient set and is denoted by U/R, where $[x]_R$ represents the equivalence class in U/R that contains $x \in U$. A rough set of subset $A \subseteq U$ is represented by a pair of lower and upper approximation. The lower approximation [6],

$$\begin{aligned} Lo(A) &= \{x \in U | [x]_R \subseteq A\}, \\ &= \{[x]_R \in U/R | [x]_R \subseteq A\}, \end{aligned} \tag{3.1}$$

is the union of all equivalence classes in U/R that are contained in A. The upper approximation [6],

$$\begin{aligned} Up(A) &= \left\{x \in U | [x]_R \cap A = \emptyset\right\}, \\ &= \left\{[x]_R \in U/R | [x]_R \cap A = \emptyset\right\}, \end{aligned} \tag{3.2}$$

is the union of all equivalence classes in U/R that overlap with A.

3.3 Multi-contextual Information Systems

A *Multi Rough Sets* is considered as a generalized concept of rough set. A multi rough sets is constructed when a given set of objects is approximated into several partitions of objects in which every partition is constructed by a certain context of attributes. Every context of attributes is represented by a set of attributes. Clearly, multi-contexts of attributes may provide partitions of multi-contexts that are generated from multi-contextual information systems. Here, multi-contextual information systems [4, 5] is formally defined by a pair I = (U, $\mathbf{A}$), where U is a universal set of objects and $\mathbf{A}$ is a non-empty set of contexts such as $\mathbf{A} = \{A_1, ..., A_n\}$. $A_i \in \mathbf{A}$ is a certain set of attributes and denoted as a context. Every attribute, $a \in A_i$, is associated with a set of V_a as its attribute values called domain of a. It is not necessary for $i \neq j \Rightarrow A_i \cap A_j = \emptyset$. Attributes such as *weight* and *height* might be represented by different contexts (i.e. context of Japanese and context of American) in which they may provide different perceptions or values of a certain attribute concerning a given object. Therefore, for all $x \in U$, $a(x)^i \in V_a$ is denoted as the value of objects x in terms of attribute a based on the context $a \in A_i$. An equivalence relation (indiscernibility relation) is then defined in terms of context A_i as follows.

For $x, y \in U$ [4],

$$R_{A_i}(x, y) \Leftrightarrow a(x)^i = a(y)^i, \quad a(x)^i, a(y)^i \in V_a, \quad \forall a \in A_i. \tag{3.3}$$

Equivalence class of $x \in U$ in the context A_i is given by [4]

$$[x]_{A_i} = \{y \in U | R_{A_i}(x, y)\}. \tag{3.4}$$

It should be proved that for $i \neq j$, $\exists x \in U$, $[x]_{Ai} \neq [x]_{Aj}$, otherwise A_i and A_j are redundants in terms of providing similar crisp partitions. By eliminating all redundant contexts, the number of contexts in the relation to the number of objects satisfies the following equation [4].

$$\text{For}|U| = m, \quad |\mathbf{A}| \leq B(m), \quad B(m) = \sum_{i=0}^{m-1} C(m-1, i) \times B(i), \tag{3.5}$$

where $B(0) = 1$ and $C(n, k)$ is combination of size k from n elements given by:

$$C(n, k) = \frac{n!}{k!(n-k)!}.$$

$|U|$ and $|\mathrm{A}|$ are cardinalities of U and $\mathbf{A}$ representing the number of objects and the number of contexts, respectively. It can be clearly seen that a set of contexts $\mathbf{A}$ derives set of partitions of universal objects as given by $\{U/A_1, \ldots, U/A_n\}$, where U/A_i expresses a partition of the universe based on context A_i. Here, U/A_i contains all equivalence classes of $[x]_{Ai}$, $x \in U$.

3.4 Concept of Multi Rough Set

A multi rough set is introduced as an approximation of a given crisp set in the presence of a set of partitions derived from multi-contextual information systems providing set of rough sets corresponding to the set of partitions. Here, the multi rough sets may be provided regardless of redundant contexts in multi-contextual information systems. Clearly, every element of the multi rough set is a pair of lower and upper approximation in the relation to a given context of attributes. Formally, multi rough set is given by the following definition.

Definition 3.1 Let U be a non-empty universal set of objects. R_{A_i} and U/R_{A_i} represent equivalence relation and partition with respect to set of attributes in the context of A_i, respectively. For $X \subseteq U$, in the relation to a set of contexts, $\mathbf{A} = \{A_1, A_2, \ldots, A_n\}$, $\mathbf{X}$ is multi rough set of X as given by the following equation [4, 5].

$$X = \{(Lo(X_1), Up(X_1)), (Lo(X_2), Up(X_2)), \ldots, (Lo(X_n), Up(X_n))\}. \tag{3.6}$$

Thus, a pair of lower and upper approximations, $(Lo(X_i), Up(X_i))$ is an element of multi rough set in terms of context A_i. Similar to the definition of classical rough set, $Lo(X_i)$ and $Up(X_i)$ are then defined by following equations [4]

$$Lo(X_i) = \{u \in U | [u]_{A_i} \subseteq X\} = \bigcup \{[u]_{A_i} \in U/A_i | [u]_{A_i} \subseteq X\}, \tag{3.7}$$

and

$$Up(X_i) = \{u \in U | [u]_{A_i} \cap X \neq \emptyset\} = \bigcup \{[u]_{A_i} \in U/A_i | [u]_{A_i} \cap X \neq \emptyset\}, \tag{3.8}$$

respectively. Similar to crisp multi-set (*bags*) as discussed in [8], a multi rough set, **X** is characterized by a counting function $\Sigma_{\mathbf{X}}$ as given by:

$$\sum\nolimits_{\mathbf{X}} \mathbf{P}(U)^2 \rightarrow \mathrm{N}, \tag{3.9}$$

where $P\ (U)$ is power set of U and N is a set of non-negative integers. Here, $\Sigma_{\mathbf{X}}$ $((M, N))$ counts number of occurrences the pair (M,N) in the multi rough set **X** for any pair of lower and upper approximations $(M,N) \in P\ (U)^2$. It should be verified that

$$(M, N) \notin \mathbf{X} \Rightarrow \sum\nolimits_{\mathbf{X}} ((M, N)) = 0$$

Also, $\mathbf{X}^*$ denotes a support set of **X**. In its definition, $\mathbf{X}^*$ satisfies the following equation:

$$(M, N) \in \mathbf{X}* \Leftrightarrow \sum\nolimits_{\mathbf{X}} ((M, N)) > 0, \tag{3.10}$$

where $\forall (M, N) \in \mathbf{X}^*, \sum\nolimits_{\mathbf{X}*} ((M, N)) = 1$. It can be also verified that if $\mathbf{X} = \mathbf{X}^*$ then there is no redundancy in the set of contexts **A**, not vice versa. It is necessary to define some basic relations and operations concerning sets of pair lower and upper approximations as elements of multi rough set. For **X** and **Y** are two multi rough sets on U drawn from multi-contextual information systems **A**, where $|\mathbf{A}| = n$ [4, 5]:

i. Containment: $\mathbf{X} \subseteq \mathbf{Y} \Leftrightarrow (Lo(X_i) \subseteq Lo(Y_i), Up(X_i) \subseteq Up(Y_i)), \forall i \in \mathrm{N}_n$;
ii. Equality: $\mathbf{X} = \mathbf{Y} \Leftrightarrow (Lo(X_i) = Lo(Y_i), Up(X_i) = Up(Y_i)), \forall i \in \mathrm{N}_n$;
iii. Complement:
$\mathbf{Y} = \neg \mathbf{X} \Leftrightarrow (Lo(Y_i) = U - Up(X_i), Up(Y_i) = U - Lo(X_i)), \forall i \in \mathrm{N}_n$;
iv. Union: $\mathbf{X} \cup \mathbf{Y} \Leftrightarrow \{(Lo(X_i) \cup Lo(Y_i), Up(X_i) \cup Up(Y_i)) | \forall i \in \mathrm{N}_n\}$;
v. Intersection: $\mathbf{X} \cap \mathbf{Y} \Leftrightarrow \{(Lo(X_i) \cap Lo(Y_i), Up(X_i) \cap Up(Y_i)) | \forall i \in \mathrm{N}_n\}$;

where N_n is a non-negative set of integers which is less or equal to n. Obviously, the operations given in (i)–(v) are strongly related to the order of elements corresponding to set of contexts. In the relation to the occurrences of elements with regardless of the order of elements in the multi rough set, some more basic operations are defined as follows.

(a) Containment: $\mathbf{X} \prec \mathbf{Y} \Leftrightarrow \sum_{\mathrm{X}}((M,N)) \leq \sum_{\mathrm{Y}}((M,N)), \forall(M,N)$;
(b) Equality: $\mathbf{X} \equiv \mathbf{Y} \Leftrightarrow \sum_{\mathrm{X}}((M,N)) = \sum_{\mathrm{Y}}((M,N)), \forall(M,N)$;
(c) Union: $\sum_{\mathbf{X}\oplus\mathbf{Y}}((M,N)) = \mathbf{max}[\Sigma_{\mathbf{X}}((M,N)), \Sigma_{\mathbf{Y}}((M,N))], \forall(M,N)$;
(d) Intersection: $\sum_{\mathrm{X}\otimes\mathrm{Y}}((M,N)) = \mathbf{min}[\Sigma_{\mathrm{X}}((M,N)), \Sigma_{\mathrm{Y}}((M,N))], \forall(M,N)$;
(e) Insertion: $\sum_{\mathbf{X}+\mathbf{Y}}((M,N)) = \Sigma_{\mathbf{X}}((M,N)) + \Sigma_{\mathbf{Y}}((M,N)), \forall(M,N) \in \mathbf{X} \oplus \mathbf{Y}$;
(f) Minus: $\sum_{\mathbf{X}-\mathbf{Y}}((M,N)) = \mathbf{max}[\Sigma_{\mathbf{X}}((M,N)) - \Sigma_{\mathbf{Y}}((M,N)), 0], \forall(M,N) \in \mathbf{X}$;

It can be proved that the above basic operations satisfy the following properties:

1. Idempotent laws:

$$\mathbf{X}\cup\mathbf{X} = \mathbf{X}, \quad \mathbf{X}\cap\mathbf{X} = \mathbf{X}, \quad \mathbf{X} \oplus \mathbf{X} = \mathbf{X}, \quad \mathbf{X} \otimes \mathbf{X} = \mathbf{X};$$

2. Commutative laws:

$$\mathbf{X}\cup\mathbf{Y} = \mathbf{Y}\cup\mathbf{X}, \quad \mathbf{X}\cap\mathbf{Y} = \mathbf{X}\cap\mathbf{Y}, \quad \mathbf{X} \oplus \mathbf{Y} = \mathbf{Y} \oplus \mathbf{X},$$
$$\mathbf{X} \otimes \mathbf{Y} = \mathbf{Y} \otimes \mathbf{X}, \quad \mathbf{X} + \mathbf{Y} = \mathbf{X} + \mathbf{Y};$$

3. Associative laws:

$$\mathbf{W}\cup(\mathbf{X}\cup\mathbf{Y}) = (\mathbf{W}\cup\mathbf{Y})\cup\mathbf{X}, \quad \mathbf{W}\cap(\mathbf{X}\cap\mathbf{Y}) = (\mathbf{W}\cap\mathbf{X})\cap\mathbf{Y},$$
$$\mathbf{W} \oplus (\mathbf{X} \oplus \mathbf{Y}) = (\mathbf{W} \oplus \mathbf{Y}) \oplus \mathbf{X}, \quad \mathbf{W} \otimes (\mathbf{X} \otimes \mathbf{Y}) = (\mathbf{W} \otimes \mathbf{Y}) \otimes \mathbf{X},$$
$$\mathbf{W} + (\mathbf{X} + \mathbf{Y}) = (\mathbf{W} + \mathbf{Y}) + \mathbf{X};$$

4. Absorption laws:

$$\mathbf{X}\cap(\mathbf{X}\cup\mathbf{Y}) = \mathbf{X}, \quad \mathbf{X}\cup(\mathbf{X}\cap\mathbf{Y}) = \mathbf{X},$$
$$\mathbf{X}\otimes(\mathbf{X} \oplus \mathbf{Y}) = \mathbf{X}, \quad \mathbf{X} \oplus (\mathbf{X} \otimes \mathbf{Y}) = \mathbf{X},$$
$$\mathbf{X} \otimes (\mathbf{X} + \mathbf{Y}) = \mathbf{X}, \quad \mathbf{X} \oplus (\mathbf{X} + \mathbf{Y}) = \mathbf{X} + \mathbf{Y};$$

5. Distributive laws:

$$\mathbf{W}\cup(\mathbf{X}\cap Y) = (\mathbf{W}\cup\mathbf{X})\cap(\mathbf{W}\cup\mathbf{Y}),$$
$$\mathbf{W}\cap(\mathbf{X}\cup Y) = (\mathbf{W}\cap\mathbf{X})\cup(\mathbf{W}\cap\mathbf{Y}),$$
$$\mathbf{W} \oplus (\mathbf{X} \otimes Y) = (\mathbf{W} \oplus \mathbf{X}) \otimes (\mathbf{W} \oplus \mathbf{Y}),$$
$$\mathbf{W} \otimes (\mathbf{X} \oplus Y) = (\mathbf{W} \otimes \mathbf{X}) \oplus (\mathbf{W} \otimes \mathbf{Y}),$$
$$\mathbf{W} + (\mathbf{X} \otimes Y) = (\mathbf{W} + \mathbf{X}) \otimes (\mathbf{W} + \mathbf{Y}),$$
$$\mathbf{W} + (\mathbf{X} \oplus Y) = (\mathbf{W} + \mathbf{X}) \oplus (\mathbf{W} + \mathbf{Y});$$

6. Additive laws:

$$\mathbf{X} \oplus \mathbf{Y} = \mathbf{X} + \mathbf{Y} - (\mathbf{X} \otimes \mathbf{Y});$$

7. Double negation law:

$$\neg\neg\mathbf{X} = \mathbf{X};$$

8. De Morgan laws:

$$\neg(\mathbf{X}\cap\mathbf{Y}) = \neg\mathbf{X}\cup\neg\mathbf{Y}, \quad \neg(\mathbf{X}\cup\mathbf{Y}) = \neg\mathbf{X}\cap\neg\mathbf{Y};$$

9. Maximum multi rough sets ($\mathbf{U} = \{(U,U),\ldots\}$, $|\mathbf{U}| = |\mathbf{X}|$);

$$\mathbf{X}\cap\mathbf{U} = \mathbf{X}, \quad \mathbf{X}\cup\mathbf{U} = \mathbf{U};$$

10. Minimum multi rough sets ($\mathbf{E} = \{(\varnothing,\varnothing),\ldots\}$, $|\mathbf{E}| = |\mathbf{X}|$);

$$\mathbf{X}\cap\mathbf{E} = \mathbf{E}, \quad \mathbf{X}\cup\mathbf{E} = \mathbf{X};$$

11. Kleene's laws:

$$(\mathbf{X}\cap\neg X)\cap(\mathbf{Y}\cup\neg\mathbf{Y}) = (\mathbf{X}\cap\neg\mathbf{X}), \quad (\mathbf{X}\cap\neg\mathbf{X})\cup(\mathbf{Y}\cup\neg\mathbf{Y}) = (\mathbf{Y}\cup\neg\mathbf{Y});$$

Since basic operations defined in (iii)–(v) do not satisfy complementary laws ($(\mathbf{X}\cap\neg\mathbf{X}) \neq \mathbf{E}$ and $(\mathbf{X}\cup\neg\mathbf{X}) \neq \mathbf{U}$), they do not satisfy Boolean algebra but just Kleene algebra instead. We may apply operations of union and intersection to all pair elements of multi rough sets **X** in order to achieve a summary of the multi rough set as given by the following definition [4, 5]:

$$\Gamma(X) = \bigcup_i Up(X_i), \tag{3.11}$$

$$\Theta(X) = \bigcap_i Up(X_i), \tag{3.12}$$

$$\Phi(X) = \bigcup_i Lo(X_i), \tag{3.13}$$

$$\Psi(X) = \bigcap_i Lo(X_i), \tag{3.14}$$

where $Lo(\mathbf{X}) = \{(\Phi(X), \Psi(X))\}$ and $Up(\mathbf{X}) = \{(\Gamma(X), \Theta(X))\}$ are summarized multi rough set by which they have only one pair elements. It can be easily proved that their relationship satisfies the following equation:

$$\Psi(X)\subseteq\Phi(X)\subseteq X\subseteq\Theta(X)\subseteq\Gamma(X).$$

Here, pair of $(\Phi(X),\ \Theta(X))$ may be considered as a finer approximation of $X\subseteq U$. On the other hand, pair of $(\Psi(X),\ \Gamma(X))$ is a worse approximation of $X\subseteq U$.

Moreover, it can be followed that the definition of summary multi rough set satisfies some properties such as [4]:

(1) $X \subseteq Y \Leftrightarrow [\Psi(X) \subseteq \Psi(Y),\ \Phi(X) \subseteq \Phi(Y),\ \Theta(X) \subseteq \Theta(Y),\ \Gamma(X) \subseteq \Gamma(Y)]$,
(2) $\Psi(X) = \neg\Gamma(\neg X),\ \Phi(X) = \neg\Theta(\neg X),\ \Theta(X) = \neg\Phi(\neg X),\ \Gamma(X) = \neg\Psi(\neg X)$,
(3) $\Psi(U) = \Phi(U) = \Theta(U) = \Gamma(U) = U,\ \Psi(\emptyset) = \Phi(\emptyset) = \Theta(\emptyset) = \Gamma(\emptyset) = \emptyset$,
(4) $\Psi(X \cap Y) = \Psi(X) \cap \Psi(Y), \Phi(X \cap Y) = \Phi(X) \cap \Phi(Y), \Theta(X \cap Y) \leq \Theta(X) \cap \Theta(Y)$, $\Gamma(X \cap Y) \leq \Gamma(X) \leq \Gamma(Y)$,
(5) $\Psi(X \cup Y) \geq \Psi(X) + \Psi(Y) - \Psi(X \cap Y), \Phi(X \cup Y) \geq \Phi(X) + \Phi(Y) - \Phi(X \cap Y)$, $\Phi(X \cup Y) \leq \Phi(X) + \Phi(Y) - \Phi(X \cap Y), \Gamma(X \cup Y) \leq \Gamma(X) + \Gamma(Y) - \Gamma(X \cap Y)$.

Furthermore, we may consider two special characteristics of context, namely *total ignorance* and *identity*, as follows.

1. A_i is called *total ignorance* (τ) if $x \in U$, $[x]_\tau = U$.
   Therefore $\forall X \subseteq U$, $X \neq \varnothing \Rightarrow Lo(X_\tau) = \varnothing$, $Up(X_\tau) = \mathrm{U}$.
2. A_i is called *identity* (ι) if $\forall x \in U$, $[x]_\iota = \{x\}$.
   Therefore, $\forall X \subseteq U \Rightarrow Lo(X_\iota) = Up(X_\iota) = X$.

Clearly, in the relation to intersection and union operations, it is also satisfied some properties as follows. $\forall A_i \in \mathbf{A}$, $X \subseteq U$ [4],

– Union: $X \neq \varnothing \Rightarrow Up(X_i) \cup Up(X_\tau) = U$, $Lo(X_i) \cup Lo(X_\tau) = Lo(X_i)$,

$$Up(X_i) \cup Up(X_\iota) = Up(X_i), Lo(X_i) \cup Lo(X_\iota) = X.$$

– Intersection: $Up(X_i) \cap Up(X_\tau) = Up(X_i)$, $Lo(X_i) \cap Lo(X_\tau) = \varnothing$,

$$Up(X_i) \cap Up(X_\iota) = X, \quad Lo(X_i) \cap Lo(X_\iota) = Lo(X_i).$$

From the above relations dealing with union and intersection operations, ι is considered as identity context for intersection operation of lower approximation as well as for union operation of upper approximation. On the other hand, τ is considered as an identity context for intersection operation of upper approximation as well as for union operation of lower approximation.

Moreover, it is necessary to define two count functions in order to characterize multi rough sets based on the number of objects (elements of U) as follows:

Definition 3.2 $\eta_{\mathbf{X}}$: $U \rightarrow \mathrm{N}_n$ and $\sigma_{\mathbf{X}}$: $U \rightarrow \mathrm{N}_n$ are defined as two functions to characterize multi rough set by counting total number of copies of a given element of U in upper and lower sides of multi rough set $\mathbf{X}$, respectively, as given by:

$$\eta_X(x) = \sum_i^n \theta_{Up(X_i)}(x), \tag{3.15}$$

$$\sigma_X(x) = \sum_{i}^{n} \theta_{Lo(X_i)}(x), \tag{3.16}$$

where $|\mathbf{A}| = n$ and $\theta_M(x) = 1 \Leftrightarrow x \in M$, otherwise $\theta_M(x) = 0$.

These count functions are similar to one proposed in (Yager 1990) talking about *bags* (multi-set). Similar results will be found by firstly taking *insertion operation* to all lower side yielding a multi-set of lower side as well as all upper side yielding a multi-set of upper side. Then, the counting function is used to calculate number of copies each element in both multi-sets. In the relation to summary rough sets as discussed before, these two count functions, η and σ, satisfy some properties such as for $X,\ Y \in U$, $|\mathbf{A}| = n$ [4]:

1. $\eta_{\mathbf{X}}(y) \geq \sigma_{\mathbf{X}}(y), \forall y \in U,$
2. $\sigma_{\mathbf{X}}(y) > 0 \Rightarrow y \in X,$
3. $y \in X \Rightarrow \eta_{\mathbf{X}}(y) = n,$
4. $y \in \Theta(X) \Leftrightarrow \eta_{\mathbf{X}}(y) = n,$
5. $y \in \Psi(X) \Leftrightarrow \sigma_{\mathbf{X}}(y) = n,$
6. $\eta_{\mathbf{X}}(y) > 0 \Leftrightarrow \Gamma(X),$
7. $\sigma_{\mathbf{X}}(y) > 0 \Leftrightarrow \Phi(X),$
8. $\mathbf{X} \subseteq \mathbf{Y} \Rightarrow \eta_{\mathbf{X}}(y) \leq \eta_{\mathbf{Y}}(y),\ \sigma_{\mathbf{X}}(y) \leq \sigma_{\mathbf{Y}}(y), \forall y \in U,$
9. $\mathbf{X} = \mathbf{Y} \Rightarrow \eta_{\mathbf{X}}(y) = \eta_{\mathbf{Y}}(y),\ \sigma_{\mathbf{X}}(y) = \sigma_{\mathbf{Y}}(y), \forall y \in U,$
10. $\eta_{\mathbf{X}\cup\mathbf{Y}}(y) = \eta_{\mathbf{X}}(y) + \eta_{\mathbf{X}}(y) - \eta_{\mathbf{X}\cap\mathbf{Y}}(y),$
11. $\sigma_{\mathbf{X}\cup\mathbf{Y}}(y) = \sigma_{\mathbf{X}}(y) + \sigma_{\mathbf{X}}(y) - \sigma_{\mathbf{X}\cap\mathbf{Y}}(y),$
12. $\eta_{\mathbf{X}\oplus\mathbf{Y}}(y) = \eta_{\mathbf{X}}(y) + \eta_{\mathbf{X}}(y) - \eta_{\mathbf{X}\otimes\mathbf{Y}}(y),$
13. $\sigma_{\mathbf{X}\oplus\mathbf{Y}}(y) = \sigma_{\mathbf{X}}(y) + \sigma_{\mathbf{X}}(y) - \sigma_{\mathbf{X}\otimes\mathbf{Y}}(y),$

Simply, two membership functions denoted by $\mu_{\mathbf{X}}(y)$: $U \rightarrow [0,1]$ and $v_{\mathbf{X}}(y)$: $U \rightarrow [0,1]$ might be defined by dividing two previous count functions with total number of contexts ($|\mathbf{A}| = n$) as follows [4].

$$\mu_X(y) = \frac{\eta_X(y)}{n}, \tag{3.17}$$

$$v_X(y) = \frac{\sigma_X(y)}{n}, \tag{3.18}$$

where $\mu_X(y)$ and $v_X(y)$ represent membership value of y in upper and lower multi set $\mathbf{X}$, respectively. Actually, μ and ν are nothing but another representation of the count functions. However, $(v_X(y),\ \mu_X(y))$ might be considered as pair of an *interval membership function* of $y \in U$ in the presence of multi-contexts of attributes. Similarly, by changing n to 1 in Property (3)–(5), μ and ν have exactly the same properties as given by η and σ, respectively.

3.5 Generalization of Contexts

The objective behind generalization of contexts is to combine all contexts of attributes into one general context. In this case, three kinds of general context are introduced, namely AND-general context, OR-general context and OR^+-general context.

First, AND-*general context* is a general context which is provided by AND logic operator to all attributes of all contexts. It is simply built by unifying all elements of attributes of all contexts into one general context as given by the following definition.

Definition 3.3 Let A = $\{A_1, A_2, \ldots, A_n\}$ be set of contexts. $A_\wedge$ is defined as AND-general context by $A_\wedge = A_1 \wedge A_2 \wedge \cdots \wedge A_n$, where $A_\wedge$ is a result of summation of all conditions as given by all attributes of A_i, $\forall i \in \mathrm{N}_n$ or simply,

$$A_\wedge = A_1 + A_2 + \cdots + A_n \tag{3.19}$$

Similar to what has been discussed in Sect. 3.3, In Definition 3.3, it is also not necessary $i \neq j \Rightarrow A_i \cap A_j = \emptyset$. In this case, every attribute is assumed to provide unique and independent value of the attribute in the relation to a given object in terms of a certain context of attribute. It can be verified that $A_\wedge$ satisfies $|A_\wedge| = \sum_{i=1}^{n} |A_i|$. Also, $\forall [u]_{A_\wedge}, \forall i \in \mathrm{N}_n, \exists [u]_{A_i}$ such that $[u]_{A_\wedge} \subseteq [u]_{A_i}$.

For a given $X \subseteq U$, $Lo(X_\wedge)$ and $Up(X_\wedge)$ are denoted as lower and upper approximation of X based on set of attributes, $A_\wedge$. Approximation space constructed by AND-general context is considered as the finest disjoined partition by combining all partition of contexts and considering every possible area of intersection among equivalence classes as a equivalence class of AND-general context (see Fig. 3.1c). It can be clearly seen that AND-general context provides the finest approximation of rough set.

Second, the independency of every context persists in the process of generalization, if relationships among contexts are related by OR logic operator. Obviously, instead of a disjoint partition, it may provide a covering of the universal objects. Since OR-general context provides a covering [9, 1], it is also called *Cover-general context* (*C-general context*, for short) as given by the following definition.

Definition 3.4 Let A = $\{A_1, A_2, \ldots, A_n\}$ be set of contexts. $A_\vee$ is defined as C-general context as given by: $A_\vee = A_1 \vee A_2 \vee \cdots \vee A_n$, such that [4]

$$U/A_\vee = \bigcup_{i=1}^{n} U/A_i \tag{3.20}$$

where $U/A_\vee$ is a covering of the universe as union of all equivalence classes in U/A_i, $i \in \mathrm{N}_n$.

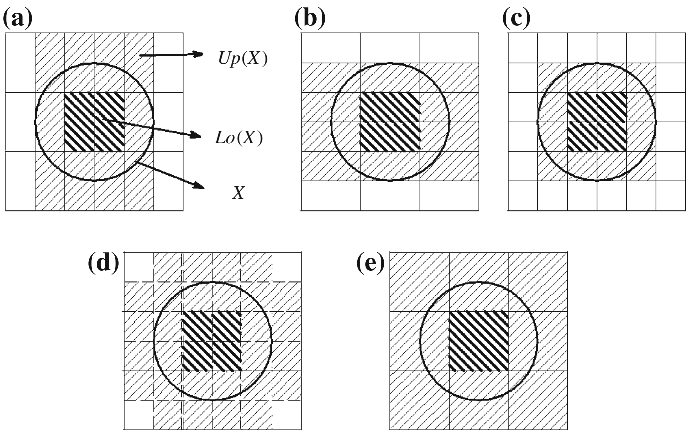

Fig. 3.1 Generalization contexts [4]. **a** U/A_1, **b** U/A_2, **c** $U/A_\wedge$, **d** $U/A_\vee$, **e** U/A_v^+

Consequently, $|U/A_\vee| \leq \sum_{i=1}^{n} |U/A_i|$ and $\forall C \in U/A_\vee, \forall i \in N_n, \exists [u]_{A_i}$ such that $C = [u]_{A_i}$, where C is a *similarity class* in covering and $[u]_{A_i}$ is an equivalence class in the partition of U/A_i. Here C is called similarity class in order to distinguish it from equivalence class provided by equivalence relation as usually used in partition. Here, every similarity class may take overlap one to each other in providing a covering. Therefore, a given object $u \in U$ is possibly a member or an element of more than one similarity classes. It can be proved that for $X \subseteq U$, $Lo(X_\vee)$ and $Up(X_\vee)$, as a pair of lower and upper approximations of X in terms of $A_\vee$, are calculated by the following equations [4],

$$Lo(X_\vee) = \bigcup_{i=1}^{n} Lo(A_i), \tag{3.21}$$

$$Up(X_\vee) = \bigcup_{i=1}^{n} Up(A_i), \tag{3.22}$$

where $Lo(X_i)$ and $Up(X_i)$ are lower and upper approximation of X dealing with the context A_i. It can be easily verified that iterative operation is applied in operation of the upper approximation as given by $Up(X_\vee) \subseteq Up(Up(X_\vee))$. In this case, $M(Up(X_\vee))$ is then considered as a maximum upper approximation as given by $Up(X_\vee) \subseteq Up(Up(X_\vee)) \subseteq \cdots \subseteq M(Up(X_\vee))$ in which the iterative operation is no longer applied in the maximum upper approximation as shown in the following relation: $Up(M(Up(X_\vee))) = M(Up(X_\vee))$. In the relation to the covering of the univeral objects, some properties have been already given and discussed in [2]. Furthermore, related to summary of multi rough set as discussed in the previous section, it can be verified that $Up(X_\vee) = \Gamma(X)$ and $Lo(X_\vee) = \Phi(X)$.

The third general context is called OR$^+$-*general context*. In OR$^+$-*general context*, transitive closure operation is applied to the covering as provided by OR-general context or C-general context. In other words, similarity classes of OR$^+$-general context are constructed by union of all equivalence classes of all partitions (of all contexts) that overlap one to each other. Similarity classes of OR$^+$-general context is then defined by the following definition.

Definition 3.5 Let A = $\{A_1, A_2, \ldots, A_n\}$ be set of contexts. $A_\vee^+$ is defined as OR$^+$-general context by: $A_\vee^+ = A_1 \circ A_2 \circ \cdots \circ A_n$, such that $y \in [x]_{A_\vee^+}$ iff [4]

$$(\exists C_i \in U/A_\vee,\, x, y \in C_i) \quad \text{OR} \quad (\exists C_{i1}, C_{i2}, \ldots, C_{im} \in U/A_\vee,\\ x \in C_{i1}, C_{ik} \cap C_{i(k+1)} \neq \emptyset, k = 1, \ldots, m-1, y \in C_{im}). \tag{3.23}$$

where $m \le n$ and $[x]_{A_\vee^+}$ is an equivalence class containing x in terms of $A_\vee^+$.

For $U/A_\vee$ be a set of equivalence classes provided by all contexts, equivalence classes generated by $A_\vee^+$ are able to be constructed by the following algorithm [4, 5]:

$S_i \in U/A_\vee^+, i \in$ N **:** Equivalence classes of OR$^+$-general context.
$p = 0$; SC = $U/A_\vee$ $U/A_\vee$;
while SC = Ø *do {*
$p = p + 1$; S_p = Ø;
SC = SC − $\{M\}$; M is an element (similarity class) of SC.
$S_p = M$;
SS = SC;
while SS ≠ Ø *do {*
    SS = SS − $\{M\}$; M is an element (similarity class) of SS.
    if $S_p \cap M \neq \emptyset$ *then {*
        $S_P = S_P \cup M$;
        SC = SC − $\{M\}$;
        }
    }
}

Finally, algorithm in Definition 3.5 shows that there will be p equivalence classes. Possibly, p might be equal to 1 in case all elements in $U/A_\vee$ transitively join each other. It can be verified that all equivalence classes in $U/A_\vee^+$ are disjoint. Also, $\forall S \in U/A_\vee^+$ such that $\forall i \in \mathrm{N}_n, \exists M \in U/A_i, M \subseteq S$. For a given $X \subseteq U$, $Lo(A_\vee^+)$ and $Up(A_\vee^+)$ are defined as lower and upper approximation of X, respectively provided by set of attributes, $A_\vee^+$. It can be easily seen that OR$^+$-general context will construct the worst disjoined partition. Hence, it will provide the worst approximation of rough set. In the relation to the maximum upper approximation based on OR-general context, it can be easily proved that apr

$Up\left(A_{\vee}^{+}\right) = M(Up(A_{\vee}))$. Compare to summary multi rough set in previous discussion and approximation based on AND-general context, we have the following relation [4].

$$Lo(A_{\vee}^{+}) \subseteq \Psi(X) \subseteq \Phi(X) \subseteq Lo(A_{\wedge}) \subseteq X \subseteq Up(A_{\wedge}) \subseteq \Theta(X) \subseteq \Gamma(X) \subseteq Up(A_{\vee}^{+})$$

To be more clearly understandable how generalization of contexts applied in the approximation of X, they are illustrated by Fig. 3.1. To simplify the problem, two contexts are given, A_1 and A_2, such as approximation of X is represented in Fig. 3.1a, b. Approximations of X based on AND, OR and OR^{+}-general context are given in Fig. 3.1c–e, respectively.

3.6 Conclusion

The concept of multi rough sets based on multi contextual information systems was proposed by Intan et al. (2003) [4, 5]. Here, context can be viewed as situation or background by which somehow it is necessary to group some attributes as a subset of attributes and consider the subset as a certain context. Multi rough sets were considered as a generalization of rough sets. This paper was an extended work by exploring more properties and set operations of the concept of multi rough sets based on multi-contextual information systems. Basic operations and some properties were examined. Two count functions as well as their properties were defined and examined to characterize multi rough sets. Finally, three types of general contexts, namely AND-general context, C-general context and OR^{+}-general context were proposed and discussed in order to aggregate contexts into one general context. This paper also discussed briefly relation among all approximations provided by the general contexts. In the future work, it is necessary to apply and implement the concept of multi rough sets in the real world application.

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