Romanovski Polynomials Method and Its Application for Non-central Potential System

A. Suparmi and C. Cari

Abstract The approximate analytical solution of Schrodinger equation for Eckart potential plus with trigonometric Poschl-Teller noncentral potential and trigonometric Rosen-Morse non-central potential systems are investigated using Romanovski polynomials. The approximate bound state energy eigenvalue of the first system is given in the close form and the corresponding approximate radial eigen functions is formulated in the form of Romanovski polynomials while the angular wave function is also expressed in Romanovski polynomials. The effect of the presence of trigonometric Poschl-Teller potential increases the angular wave function level. The presence of non-central potentials cause the orbital quantum numbers are mostly not integer.

1 Introduction

Schrodinger equations for a class of shape invariant potentials have been solved by using some methods such as SUSY WKB (SWKB) [1–4], SUSY operator and factorization method [5–9], NU method [10–14] and Romanovski polynomials [15–18]. Romanovski polynomials, which is a traditional method, consists of reducing Schrodinger equation by an appropriate change of the variable to the form of generalized hypergeometric equation [19]. The solution steps of Romaovski polynomials are rather similar to the steps applied in solution of Schrodinger equation by using NU method [20, 21], which is discussed in Chap. 5, and hypergeometric differential equation. The polynomial was discovered by Sir Routh [22] and rediscovered 45 years later by Romanovski [23]. Romanovski polynomial method is also called as finite Romanovski polynomial. The notion "finite" refers to the observation

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that, for any given set of parameters (i.e. in any potential) only a finite of polynomials appear orthogonal [24]. However the polynomial will not be finite if the certain condition is not satisfied and then leads to an infinite of polynomials that show orthogonal. It seems that NU and Romanovski polynomials methods are very similar in the way of variable substitution but they solve Schrodinger equation differently. NU method is applied wider than Romanovski polynomials since Romanovski polynomials could not be applied for potential solved using confluent hypergeometric differential equation.

In this chapter we discuss the Schrodinger equation solution for a particle which is in the field of Eckart potential with simultaneously presence of Poschl-Teller non-central potential and the polar Schrodinger equation for trigonometric Rosen-Morse potential using Romanovski polynomials. A non-central potential is potential as a function of radial and angular positions, it could be composed of radial function potentials and non-central potentials which are shown in Chap. 5. The simple choice of non-central potential is the separable potential [25–27]. The three dimensional Schrodinger equations of separable non-central potentials are exactly solvable as long the centrifugal term is approximated by hyperbolic function, trigonometric function or exponential function [28–32]. Due to the approximation of the centrifugal term, the energy spectra and the radial wave functions are approximately obtained for *l*-state solution and becomes exact solution for s-wave.

2 Romanovski Polynomials

One dimensional Schrodinger equation of potential of interest reduces to the differential equation of Romanovski polynomials by appropriate variable and wave function substitutions. The one dimensional Schrodinger equation is given as

$$-\frac{\hbar^2}{2M}\frac{\partial^2\,\Psi(x)}{\partial\,x^2}+V(x)\,\Psi(x)=E\,\Psi\,(x) \tag{1}$$

where V(x) is an effective potential which is mostly shape invariant potential. By suitable variable substitution $x = f(s)$ (1) changes into generalized hypergeometric type equation expressed as

$$\frac{\partial^2\,\Psi(s)}{\partial\,s^2}+\frac{\tilde{\tau}(s)}{\sigma\,(s)}\frac{\partial\,\Psi(s)}{\partial\,s}+\frac{\tilde{\sigma}\,(s)}{\sigma^2(s)}\Psi\,(s)=0 \tag{2}$$

with $\sigma(s)$ and $\tilde{\sigma}(s)$ are mostly polynomials of order two, $\tilde{\tau}(s)$ is polynomial of order one, of s, $\sigma(s)$, $\tilde{\sigma}(s)$, and $\tilde{\tau}(s)$ can have any real or complex values [21]. Equation (2) is solved by variable separation method. By introducing new wave function in (2) we obtain a hypergeometric type differential equation, which can be solved using finite Romanovski polynomials [16, 17, 21] which is expressed as

$$\sigma(s)y''(s) + \tau(s)y'(s) + \lambda y(s) = 0 \tag{3a}$$

with

$$\sigma(s) = as^2 + bs + c;\ \tau = fs + h \text{ and } -\{n(n-1) + 2n(1-p)\} = \lambda = \lambda_n \tag{3b}$$

Equation (3a) is described in the textbook by Nikiforov-Uvarov [21] where it is cast into self adjoint form and its weight function, $w(s)$, satisfies Pearson differential equation

$$\frac{d(\sigma(s)w(s))}{ds} = \tau(s)w(s) \tag{4}$$

The weight function, $w(s)$, is obtained by solving the Pearson differential equation expressed in (4) and by applying condition in (3b), that is

$$w(s) = \exp(\int \frac{(f-2a)s + (h-b)}{as^2 + bs + c} ds) \tag{5}$$

The corresponding polynomials are classified according to the weight function, and are built up from the Rodrigues representation which is presented as

$$y_n = \frac{B_n}{w(s)} \frac{d^n}{ds^n} \left\{ \left(as^2 + bs + c \right)^n w(s) \right\} \tag{6}$$

with B_n is a normalization constant, and for $\sigma(s) > 0$ and $w(s) > 0$, $y_n(s)$'s are normalized polynomials and are orthogonal with respect to the weight function $w(s)$ within a given interval (s_1, s_2), which is expressed as

$$\int_{-\infty}^{\infty} w(s) y_n(s) y_{n'}(s) ds = \delta_{nn'} \tag{7}$$

For Romanovski polynomials, the values of parameters in (3b) are:

$$a = 1,\ b = 0,\ c = 1,\ f = 2(1-p) \text{ and } h = q \text{ with } p > 0 \tag{8}$$

By inserting (8) into (5) we obtain the weight function as

$$\begin{aligned} w(s) &= \exp(\int \frac{(f-2a)s + (c-b)}{as^2 + bs + c}) = \exp(\int \frac{(2-2p-2)s + q}{s^2+1} ds) \\ w(s) &= (1+s^2)^{-p} e^{q \tan^{-1}(s)} \end{aligned} \tag{9}$$

This weight function first reported by Routh [19] and then by Romanovski [23]. The polynomial associated with (9) are named after Romanovski and will be denoted by $R_n^{(p,q)}(s)$. Due to the decrease of the weight function by s^{-2p}, integral of the type

$$\int_{-\infty}^{\infty} w^{(p,q)} R_n^{(p,q)}(s) R_{n'}^{(p,q)}(s) ds \tag{10}$$

will be convergent only if

$$n' + n < 2p - 1 \tag{11}$$

This means that only a finite number of Romanovski polynomials are orthogonal, and the orthogonality integral of the polynomial is expressed similar to the (7) where $y_n = R_n^{(p,q)}(s)$.

The differential equation satisfied by Romanovski Polynomial obtained by inserting (3b) and (8) into (3a) given as

$$\begin{aligned}(1+s^2)\frac{\partial^2 R_n^{(p,q)}}{\partial s^2} + \{2\,s(-p+1)+q\}\frac{\partial R_n^{(p,q)}(s)}{\partial s}\\ -\{n(n-1)+2n(1-p)\}R_n^{(p,q)}(s) = 0\end{aligned} \tag{12}$$

where $y_n = R_n^{(p,q)}(s)$. The heart of Romanovski polynomials method is in obtaining (12) from one dimensional Schrodinger equation. The Schrodinger equation of the potential of interest will be reduced into second order differential equation that is similar to (12) by an appropriate transformation of variable, for example, $r = f(s)$, and by introducing a new wave function which is given as

$$\Psi_n(r) = g_n(s) = (1+s^2)^{\frac{\beta}{2}} e^{\frac{-\alpha}{2}\tan^{-1}s} D_n^{(\beta,\alpha)}(s) \tag{13}$$

where $\Psi_n(s) = \Psi_n(x)$ is an eigen function of generalized hypergeometric equation in (2) which is the solution of Schrodinger equation for potential interest in (1), and

$$D_n^{(\beta,\alpha)}(s) = R_n^{(p,q)}(s) \tag{14}$$

From condition in (14) we get the relation between β with p, and α with q. The Romanovski polynomials obtained from Rodrigues formula expressed in (6) for the corresponding weight function in (9) is expressed as

$$R_n^{(p,q)}(s) = D_n^{(\beta,\alpha)}(s) = \frac{1}{(1+s^2)^{-p} e^{q\tan^{-1}(s)}} \frac{d^n}{ds^n}\left\{(1+s^2)^n (1+s^2)^{-p} e^{q\tan^{-1}(s)}\right\} \tag{15}$$

If the wave function of the nth level in (13) is rewritten as

$$\Psi_n(r) = \frac{1}{\sqrt{\frac{df(s)}{ds}}}(1+s^2)^{\frac{-p}{2}} e^{\frac{q}{2}\tan^{-1}(s)} R_n^{(p,q)}(s) \tag{16}$$

then the orthogonality integral of the wave functions expressed in (16) gives rise to orthogonality integral of the finite Romanovski polynomials, that is given as

$$\int_0^{\infty} \Psi_n(r)\Psi_{n'}(r)dr = \int_{-\infty}^{\infty} w^{(p,q)} R_n^{(p,q)}(s) R_{n'}^{(p,q)}(s)ds \tag{17}$$

In this case the values of p and q are not n-dependence where n is the degree of polynomials. However, if either (11) or (17) is not fulfilled then the Romanovski polynomials is infinity [16–18].

3 Application of Romanovski Polynomials for Energy Spectra and Wave Functions Analysis for Non-central Potential

Non-central potentials which consist are solvable by Romanovski polynomials. The non-central potentials that are solved using Romanovski polynomials are Eckart plus Poschl-Teller non-central potential system and polar Schrodinger equation of 3D oscillator harmonics plus trigonometric Rosen-Morse non-central potential system.

3.1 Eckart Plus Poschl-Teller Non-central Potential

The non-central potential is a potential of a function radial and angular simultaneously. The non-central potential which is constructed from Eckart potential and trigonometric Poschl-Teller non-cental potential given as

$$V(r,\theta) = \frac{\hbar^2}{2Ma^2}\left(V_0 \frac{e^{-r/a}}{(1-e^{-r/a})^2} - V_1 \frac{1+e^{-r/a}}{1-e^{-r/a}}\right) + \frac{\hbar^2}{2Mr^2}\left(\frac{\kappa(\kappa-1)}{\sin^2\theta} + \frac{\eta(\eta-1)}{\cos^2\theta}\right) \tag{18}$$

with V_0 and V_1 describe the depth of the potential well and are positives, $V_1 > V_0$, a is a positive parameter which to control the width of the potential well, M is the mass of the particle, and $0 < (r/a) < \infty,\ \kappa > 1,\ \eta > 1$. The non-central potential expressed in (18) is separable ones therefore the Schrodinger equation of this potential is solved using variable separation method.

The three dimensional time-independent Schrodinger equation for Eckart potential combined with trigonometric Poschl-Teller non-central potential is

$$\begin{aligned} & -\frac{\hbar^2}{2M}\left\{\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}\right\}\psi(r,\theta,\varphi) \\ & +\frac{\hbar^2}{2Ma^2}\left(V_0\frac{e^{-r/a}}{(1-e^{-r/a})^2}-V_1\frac{1+e^{-r/a}}{1-e^{-r/a}}\right)\psi(r,\theta,\varphi) \\ & +\frac{\hbar^2}{2Mr^2}\left(\frac{\kappa(\kappa-1)}{\sin^2\theta}+\frac{\eta(\eta-1)}{\cos^2\theta}\right)\psi(r,\theta,\varphi)=E\psi(r,\theta,\varphi) \end{aligned} \tag{19}$$

By using variable separation method we get radial, polar and azimuthal parts of Schrodinger equation as following:

$$\frac{1}{R}\frac{\partial}{\partial r}\left(r^2\frac{\partial R}{\partial r}\right)-\frac{r^2}{a^2}\left(V_0\frac{e^{-r/a}}{(1-e^{-r/a})^2}-V_1\frac{1+e^{-r/a}}{1-e^{-r/a}}\right)+\frac{2Mr^2}{\hbar^2}E=\lambda=l(l+1) \tag{20}$$

$$\left\{-\frac{1}{P\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial P}{\partial\theta}\right)-\frac{1}{\Phi\sin^2\theta}\frac{\partial^2\Phi}{\partial\varphi^2}\right\}+\left(\frac{\kappa(\kappa-1)}{\sin^2\theta}+\frac{\eta(\eta-1)}{\cos^2\theta}\right)=\lambda \tag{21}$$

$$\frac{1}{\Phi}\frac{\partial^2\Phi}{\partial\varphi^2}=-m^2 \tag{22}$$

The azimuthal part of wave function obtained from (22) is given, as usual, as

$$\Phi=A_m e^{im\varphi} \tag{23}$$

The radial and polar parts of the Schrodinger equations are solved using Romanovski Polynomials.

3.1.1 Solution of Radial Part of Schrodinger Equation

By substitution $\frac{2M}{\hbar^2}E=-\varepsilon^2$ and $R=\frac{\chi(r)}{r}$ in (20) we get

$$\frac{\partial^2\chi(r)}{\partial r^2}-\frac{1}{a^2}\left(V_0\frac{e^{-r/a}}{(1-e^{-r/a})^2}-V_1\frac{1+e^{-r/a}}{1-e^{-r/a}}\right)\chi(r)-\varepsilon^2\chi(r)-\frac{l(l+1)}{r^2}\chi(r)=0 \tag{24}$$

For $\frac{r}{a} <<< 1$ the approximation of the centrifugal term in (24) [18, 19] is given as $\frac{1}{r^2} \cong \frac{1}{4a^2}\left(d_0 + \frac{1}{\sinh^2(r/2a)}\right)$ with $d_0 = 1/12$. In term of hyperbolic functions, (24) is rewritten as

$$\begin{aligned} &\frac{d^2\chi(r)}{dr^2} - \frac{l(l+1)}{4a^2}\left(d_0 + \frac{1}{\sinh^2(r/2a)}\right)\chi(r) \\ &\quad -\frac{1}{a^2}\left(\frac{V_0}{4\sinh^2(r/2a)} - V_1\coth(r/2a)\right)\chi(r) - \varepsilon^2\chi(r) \\ &\quad = 0 \end{aligned} \tag{25}$$

and by making an appropriate change of variable, $r = f(x) = 2a\coth^{-1}(ix)$ in (25), we get

$$(1+x^2)\frac{d^2\chi}{dx^2} + 2x\frac{d\chi}{dx} - \left\{V_0 + l(l+1) + \frac{4V_1 ix}{(1+x^2)} - \frac{l(l+1)d_0 + 4\varepsilon^2a^2}{(1+x^2)}\right\}\chi = 0 \tag{26}$$

To solve (26) in terms of Romanovski polynomial, (13) suggests the substitution in (26) as [29]

$$\chi(f(x)) = g_n(x) = \left(1+x^2\right)^{\frac{\beta}{2}} e^{\frac{-\alpha}{2}\tan^{-1}x} D_n^{(\beta,\alpha)}(x) \tag{27}$$

where $1 < ix < \infty$.

By inserting (27) into (24) we obtain

$$\begin{aligned} &\left(1+x^2\right)\frac{\partial^2 D}{\partial x^2} + \{2x(\beta+1) - \alpha\}\frac{\partial D}{\partial x} \\ &\quad - \left\{\frac{\beta\alpha x - \frac{\alpha^2}{4} + \beta^2 + 4V_1 ix - (l(l+)d_0 + 4\varepsilon^2 a^2}{1+x^2} + V_0 + l(l+1) - \beta^2 - \beta\right\}D \\ &\quad = 0 \end{aligned} \tag{28}$$

Equation (28) reduces to differential equation that satisfied by Romanovski polynomials if the coefficient of $\frac{1}{1+x^2}$ term in (28) is set to be zero, that are

$$-\frac{\alpha^2}{4} + \beta^2 - (l(l+1)d_0 + 4\varepsilon^2a^2) = 0 \text{ and } \beta\alpha + 4V_1 i = 0 \tag{29}$$

and then (28) becomes

$$\left(1+x^2\right)\frac{\partial^2 D}{\partial x^2}+\{2x(\beta+1)-\alpha\}\frac{\partial D}{\partial x}-\left\{V_0+l(l+1)-\beta^2-\beta\right\}D=0 \tag{30}$$

It is seen that the structure of (30) is similar to (12), and thus we assume that $D_n^{(\alpha,\beta)}(x)\cong R_n^{(p,q)}(x)$. By comparing the parameters between (12) and (30) we obtain the following relation:

$$\begin{gathered}V_0+l(l+1)-\beta^2-\beta=n(n-1)+2n(1-p);\\ 2(\beta+1)=2(-p+1)\text{ and }\alpha=-q\end{gathered} \tag{31}$$

From (31) we have $p=-\beta$, since $p>0$ then the value of β obtained from (31) that has physical meaning is

$$\beta=\beta_n=-\sqrt{V_0+\left(l+\frac{1}{2}\right)^2}-n-\frac{1}{2} \tag{32}$$

From (29) and (32) we obtain

$$\alpha^2=-2\left(l(l+1)d_0+4\varepsilon^2a^2\right)\pm 2\sqrt{\{l(l+1)d_0+4\varepsilon^2a^2\}^2-16V_1^2} \tag{33}$$

$$\text{with }\alpha=\alpha_n=\frac{4V_1 i}{\sqrt{V_0+\left(l+\frac{1}{2}\right)^2}+n+\frac{1}{2}} \tag{34}$$

Finally, the energy spectrum of the system achieved from (33) and (34) is given as

$$E_n=-\frac{\hbar^2}{2M}\left\{\frac{\left(\sqrt{V_0+\left(l+\frac{1}{2}\right)^2}+n+\frac{1}{2}\right)^2}{4a^2}+\frac{V_1^2}{a^2\left(\sqrt{V_0+\left(l+\frac{1}{2}\right)^2}+n+\frac{1}{2}\right)^2}-\frac{l(l+1)d_0}{4a^2}\right\} \tag{35}$$

The energy spectra of Eckart potential obtained using Romanovski polynomials in (35) is in agreement with the energy spectra obtained using NU method which is presented in Chap. 5.

To determine the wave function, (31), (32) and (34) are inserted into (9) and (15) so that we obtain the weight function $w(x)$ and the Romanovski polynomials $R_n^{(p,q)}=R_n^{(-\beta,-\alpha)}(x)$ as

$$w^{(-\beta,-\alpha)}=\left(1+x^2\right)^{-\left(\sqrt{V_0+\left(l+\frac{1}{2}\right)^2}+n+\frac{1}{2}\right)}\exp\left(-\frac{4iV_1}{\sqrt{V_0+\left(l+\frac{1}{2}\right)^2}+n+\frac{1}{2}}\tan^{-1}(x)\right) \tag{36a}$$

and

$$D_n^{(\beta,\alpha)}(x) = R_n^{(p,q)}(x) = R_n^{(-\beta,-\alpha)}(x) \\ = \frac{1}{(1+x^2)^{\beta_n} e^{-\alpha_n \tan^{-1}(x)}} \frac{d^n}{dx^n}\left\{ \left(1+x^2\right)^{\beta_n+n} e^{-\alpha_n \tan^{-1}(x)} \right\} \tag{36b}$$

where β_n and α_n are expressed in (32) and (34). As a result, the wave function of the nth level is given by

$$\chi(f(x)) = g_n(x) = \left(1+x^2\right)^{\frac{\beta_n}{2}} e^{\frac{-\alpha_n}{2}\tan^{-1}x} R_n^{(-\beta,-\alpha)}(x) \tag{37}$$

By using trigonometric-hyperbolic functions relation

$$e^{-\frac{\alpha_n}{2}\tan^{-1}x} = e^{-\frac{\alpha_n}{2}\tan^{-1}(-i\coth(r/2a))} = \left(\frac{1+\coth(r/2a)}{1-\coth(r/2a)}\right)^{\frac{i\alpha_n}{4}} \tag{38}$$

then (37) becomes

$$\chi_n(r) = g_n(r) \\ = (1-\coth(r/2a))^{\frac{\beta_n}{2}-\frac{i\alpha_n}{4}} (1+\coth(r/2a))^{\frac{\beta_n}{2}+\frac{i\alpha_n}{4}} R_n^{(-\beta,-\alpha)}(-i\coth(r/2a)) \tag{39}$$

$$R_n^{(-\beta,-\alpha)}(r) = \frac{i^n}{(1-\coth(r/2a))^{\beta_n-\frac{i\alpha_n}{2}} (1+\coth(r/2a))^{\beta_n+\frac{i\alpha_n}{2}}} \\ \frac{d^n}{d(\coth(r/2a))^n}\left\{ (1-\coth(r/2a))^{\beta_n-\frac{i\alpha_n}{2}+n} (1+\coth(r/2a))^{\beta_n+\frac{i\alpha}{2}+n} \right\} \tag{40}$$

The radial wave functions for Eckart plus Poschl-Teller non-central potential in (39) and (40) are in agreement with the result achieved by using NU method.

Since the β_n and α_n parameters, expressed in (32) and (34), are n-dependence then the orthogonality of the wave functions may not produce to the orthogonality integral of the polynomials [29], as shown in (41),

$$\int_0^\infty \chi_n(r)\chi_{n'}(r)dr = \delta_{nn'} \neq \int_1^\infty w^{(-\beta,-\alpha)} R_n^{(-\beta,-\alpha)}(x) R_{n'}^{(-\beta,-\alpha)} dx \tag{41}$$

By carrying out the differentiations of (36b), we find the lowest four unnormalized Romanovski polynomials given as

$$R_0^{(-\beta_0,-\alpha_0)}(x) = 0 \tag{42}$$

$$R_1^{(-\beta_1,-\alpha_1)}(x) = (\beta_1 + 1)2x - \alpha_1 \tag{43}$$

$$R_2^{(-\beta_2,-\alpha_2)}(x) = 2(\beta_2 + 2)(2\beta_2 + 3)x^2 - 2\alpha_2(2\beta_2 + 3)x + \alpha_2^2 + 2\beta_2 + 4 \tag{44}$$

$$\begin{aligned} R_3^{(-\beta_3,-\alpha_3)}(x) &= 4x^3(\beta_3 + 3)(2\beta_3 + 5)(\beta_3 + 2) - 6a_3x^2(2\beta_3 + 5)(\beta_3 + 2) \\ &+ 2x(6\beta_3^2 + 3\alpha_3^2\beta_3 + 28\beta_3 + 6\alpha_3^2 + 34) - 2\alpha_3(2\beta_3 + 5) - \alpha(\alpha^2 + 2\beta + 6) \end{aligned} \tag{45}$$

where β_n and α_n are expressed in (32) and (34). The lowest four degrees of un-normalized radial wave functions for arbitrary values of l are calculated by using (41) and (42)–(45).

3.1.2 The Solution of Polar Schrodinger Equation for Eckart Potential Combined with Non Central Poschl-Teller Potential

To solve the polar Schrodinger equation expressed in (19), we set the polar wave function as

$$P = \frac{Q(\theta)}{\sqrt{\sin\,\theta}}, \frac{\partial P}{\partial \theta} = \frac{\frac{dQ}{d\theta}}{\sqrt{\sin\,\theta}} - \frac{1}{2}\frac{\cos\,\theta Q}{\sqrt{(\sin\,\theta)^3}} \tag{46}$$

where $Q(\theta)$ is the new polar wave function. By inserting (46) into (19) we obtain one dimensional polar Schrodinger equation as

$$\frac{d^2Q}{d\theta^2} - \left\{\frac{\kappa(\kappa-1) + m^2 - (1/4)}{\sin^2\,\theta} + \frac{\eta(\eta-1)}{\cos^2\,\theta}\right\}Q + (l(l+1) + (1/4))Q = 0 \tag{47}$$

Equation (47) to be solved using Romanovski polynomials, therefore we substitute the variable θ and introduce a new wave function such that (47) reduces to generalized hypergemetric type equation expressed in (2) or into second order differential equation of Romanovski polynomials expressed in (10). By making a change of variable in (47), $\cos 2\theta = is$ then (47) becomes

$$(1+s^2)\frac{\partial^2 Q}{\partial s^2} + s\frac{\partial Q}{\partial s} + \left[\frac{\kappa(\kappa-1) + m^2 - \frac{1}{4} + \eta(\eta-1)}{2(1+s^2)} + \frac{\kappa(\kappa-1) + m^2 - \frac{1}{4} - \eta(\eta-1)}{2(1+s^2)}is - \frac{(l+\frac{1}{2})^2}{4}\right]Q = 0 \tag{48}$$

Equation (48) will be reduced into differential equation of Romanovski polynomial by setting

$$Q(\theta) = g(s) = (1+s^2)^{\frac{\beta}{2}} e^{\frac{-\alpha}{2}\tan^{-1}s} D_n^{(\alpha,\beta)}(s) \tag{49}$$

By inserting (49) into (48) we get

$$\begin{gathered}(1+s^2)\frac{\partial^2 D}{\partial s^2} + \{s(2\beta+1)-\alpha\}\frac{\partial D}{\partial s}\\ -\left\{\frac{2\beta s\alpha - s\alpha - \frac{\alpha^2}{2} + 2\beta^2 - 2\beta - \left\{\kappa(\kappa-1)+m^2-\frac{1}{4}-\eta(\eta-1)\right\}is - \left\{\kappa(\kappa-1)+m^2-\frac{1}{4}+\eta(\eta-1)\right\}}{2(1+s^2)} + \frac{(l+\frac{1}{2})^2}{4} - \beta^2\right\}D = 0\end{gathered} \tag{50a}$$

Equation (50a) reduces to the differential equation satisfied by Romanovski polynomials given as

$$(1+s^2)\frac{\partial^2 D}{\partial s^2} + s(2\beta+1)-\alpha\}\frac{\partial D}{\partial s} - \left\{\frac{(l+\frac{1}{2})^2}{4} - \beta^2\right\}D = 0 \tag{50b}$$

when the coefficient of $\frac{1}{2(1+s^2)}$ is set to be zero, that are

$$-\left\{\kappa(\kappa-1)+m^2-\frac{1}{4}+\eta(\eta-1)\right\} - \frac{\alpha^2}{2} + 2\beta^2 - 2\beta = 0 \tag{51}$$

$$-\left\{\kappa(\kappa-1)+m^2-\frac{1}{4}-\eta(\eta-1)\right\}i + 2\beta\alpha - \alpha = 0 \tag{52}$$

By comparing (50b) and (10) we obtain

$$(2\beta+1) = 2(-p+1);\ a = -q \tag{53}$$

$$\text{and } \frac{\left(l+\frac{1}{2}\right)^2}{4} - \beta^2 = n(n-1) + 2n(1-p) \tag{54}$$

By using (51) and (52) and setting

$$\left(\kappa-\frac{1}{2}\right)^2 + m^2 - \frac{1}{4} = \kappa(\kappa-1)+m^2 = F, (\eta-\frac{1}{2})^2 = G \tag{55}$$

we obtain

$$\alpha = \pm i(\sqrt{F} \mp \sqrt{G}) \text{ and } \left(\beta-\frac{1}{2}\right) = \frac{(F-G)}{\pm 2i(\sqrt{F}\mp\sqrt{G})}i \tag{56}$$

To have physical meaning, the proper choice of the values of α and β in (56) are

$$\alpha = i\left(\sqrt{F} - \sqrt{G}\right) = i\left(\sqrt{\kappa(\kappa-1)+m^2} - \left(\eta - \frac{1}{2}\right)\right) \text{ and}$$
$$\beta = \frac{\left(\sqrt{F}+\sqrt{G}\right)+1}{2} = \frac{\sqrt{\kappa(\kappa-1)+m^2} + (\eta + \frac{1}{2})}{2} \tag{57}$$

From (54) and (57) we have

$$l = \left(\sqrt{(\kappa - \frac{1}{2})^2 + m^2 - \frac{1}{4}} + \eta + 2n\right) \tag{58}$$

Equation (58) shows that the values of l depend on the potential parameters, κ and η, and the degree of the Romanovski polynomial, $n = n_l$. The weight function obtained from (7), (54) and (57) is given as

$$w^{(p,q)} = w^{(-\beta+\frac{1}{2},-\alpha)} = (1+s^2)^{\frac{\sqrt{F}+\sqrt{G}}{2}} e^{-i(\sqrt{F}-\sqrt{G})\tan^{-1}s} \tag{59}$$

The Romanovski polynomials are obtained by using (13) and (61) as

$$R_n^{(-\beta+\frac{1}{2},-\alpha)}(s) = \frac{1}{(1+s^2)^{\frac{\sqrt{F}+\sqrt{G}}{2}} e^{-i(\sqrt{F}-\sqrt{G})\tan^{-1}s}} \frac{d^n}{ds^n}\left\{\left((1+s^2)^{\frac{\sqrt{F}+\sqrt{G}}{2}+n} e^{-i(\sqrt{F}-\sqrt{G})\tan^{-1}s}\right)\right\} \tag{60}$$

and the polar wave functions obtained from (11) and (60) is given as

$$Q(\theta) = g_n(s) = (1+s^2)^{\frac{(\sqrt{F}+\sqrt{G})+1}{4}} e^{-\frac{i}{2}(\sqrt{F}-\sqrt{G})\tan^{-1}s} R_n^{(-\beta+\frac{1}{2},-\alpha)}(s) \tag{61}$$

The polar eigen function obtained from (46) and (61) is given as

$$P_{nm\kappa\eta}(\theta) = \frac{Q_{nm\kappa\eta}}{\sqrt{\sin\,\theta}} = \frac{1}{\sqrt{\sin\,\theta}} \sqrt{(1+s^2)^{\frac{\sqrt{F}+\sqrt{G}+1}{2}} e^{-i(\sqrt{F}-\sqrt{G})\tan^{-1}s}} R_n^{(-\beta+\frac{1}{2},-\alpha)}(s) \tag{62a}$$

$$\text{or } P_{n_l m\kappa\eta}(\theta) = \frac{1}{\sqrt{\sin\,\theta}} (1-\cos\,2\theta)^{\frac{\sqrt{F}}{2}+\frac{1}{4}} (1+\cos\,2\theta)^{\frac{\sqrt{G}}{2}+\frac{1}{4}} R_{n_l}^{(-\beta+\frac{1}{2},-\alpha)}(-i\cos 2\theta) \tag{62b}$$

The orthogonality integral of the angular wave function obtained from (62a), (62b) is given as

$$\begin{aligned}\int_0^{\frac{\pi}{2}} Q_n^*(\theta)Q_{n'}(\theta)d\theta &= \int_{-1}^{1} (1+s^2)^{\frac{(\sqrt{F}+\sqrt{G})+1}{2}} e^{-\frac{i}{2}\left(\sqrt{F}-\sqrt{G}\right)\tan^{-1}s} R_n^{(-\beta+\frac{1}{2},-\alpha)}(s) R_{n'}^{(-\beta+\frac{1}{2},-\alpha)}(s)\frac{\partial\theta}{\partial(is)}d(is)\\ &= -\frac{1}{2}\int_{-1}^{1} (1+s^2)^{\frac{(\sqrt{F}+\sqrt{G})}{2}} e^{-\frac{i}{2}\left(\sqrt{F}-\sqrt{G}\right)\tan^{-1}s} R_n^{(-\beta+\frac{1}{2},-\alpha)}(s) R_{n'}^{(-\beta+\frac{1}{2},-\alpha)}(s)d(is)\end{aligned} \tag{63a}$$

where $\frac{\partial\theta}{\partial(is)} = -\frac{1}{2\sin 2\theta}$, $\left(1+s^2\right)^{1/2} = \sin 2\theta$ and from (63a) we have

$$\int_0^{\frac{\pi}{2}} Q_n^*(\theta)Q_{n'}(\theta)d\theta = -\frac{1}{2}\int_{-1}^{1} w^{\left(-\beta+\frac{1}{2},-\alpha\right)} R_n^{(-\beta+\frac{1}{2},-\alpha)}(s) R_{n'}^{(-\beta+\frac{1}{2},-\alpha)}(s)d(is) \tag{63b}$$

Equation (63b) shows that the orthogonality of Romanovski polynomials is produced from the orthogonality of wave function but (63a), (63b) is not convergent [29, 30] since

$$n + n' < 2p - 1 \text{ or } n < -\frac{\sqrt{\left(\kappa-\frac{1}{2}\right)^2 + m^2 - \frac{1}{4}} + \eta + \frac{1}{2}}{2} \tag{64}$$

and the interval of the variable is not in $-\infty < s < \infty$ interval.

The first four unnormalized Romanovski polynomials obtained from (60) are given as:

$$R_0^{(-\beta+\frac{1}{2},-\alpha)}(s) = 1 \tag{65}$$

$$R_1^{(-\beta+\frac{1}{2},-\alpha)}(s) = (\sqrt{F}+\sqrt{G}+2)s - i(\sqrt{F}-\sqrt{G}) \tag{66}$$

$$R_2^{(-\beta+\frac{1}{2},-\alpha)}(s) = \left[\left(\sqrt{F}+\sqrt{G}+4\right)\left(\sqrt{F}+\sqrt{G}+3\right)s^2 - 2is\left(\sqrt{F}-\sqrt{G}\right)\left(\sqrt{F}+\sqrt{G}+3\right) - \left(\sqrt{F}-\sqrt{G}\right)^2 + \left(\sqrt{F}+\sqrt{G}+4\right)\right] \tag{67}$$

$$R_3^{(-\beta+\frac{1}{2},-\alpha)}(s) = \begin{bmatrix} (\sqrt{F}+\sqrt{G}+6)(\sqrt{F}+\sqrt{G}+5)((\sqrt{F}+\sqrt{G})+4)s^3 - 3is^2(\sqrt{F}-\sqrt{G})(\sqrt{F}+\sqrt{G}+5)(\sqrt{F}+\sqrt{G}+4) \\ \left\{-3\,s(\sqrt{F}+\sqrt{G}+4)\left\{(\sqrt{F}-\sqrt{G})^2 - (\sqrt{F}+\sqrt{G}+6)\right\} + i\left\{(\sqrt{F}-\sqrt{G})^2 - (3\sqrt{F}+3\sqrt{G}+16)\right\}(\sqrt{F}-\sqrt{G})\right. \end{bmatrix} \tag{68}$$

The solution of the first four of un-normalized polar wave functions are obtained from (57), (62a), (62b) and (65)–(68). The polar wave function (62b) is in agreement to the wave function obtained using NU method [31], but only for even numbers of polynomial degrees (n_l).

If there is no the presence of trigonometric Poschl-Teller potential, where $\kappa = 0$ and $\eta = 0$ then the polar wave function reduces to associated Legendre polynomials and the orbital quantum number expressed in (36a), (36b) becomes $l = m + 2n_l$, with n_l is the degree of polynomial. However the associated Legendre polynomial obtained from this non-central potential are only those polynomials whose values of l and m are differed by even numbers since $l = m + 2n_l$. The effect of the presence of Poschl-Teller non-central potential to spherical harmonics is illustrated by using the three dimensional representation and the polar diagram of the absolute value of un-normalized angular wave functions obtained from (55), (62a), (62b), and (68) for $n = n_l = 3$. The 3D representations and polar diagram of $\left|Y_l^m\right|$ visualized using Mat Lab 7 are shown in Fig. 1 for $n_l = 3, \kappa = 0, \eta = 0, l = 6, m = 0$, Fig. 2 for $n_l = 3$, $\kappa = 2, \eta = 0, l = 7.4, m = 0$ and Fig. 3 for $n_l = 3, \kappa = 0, \eta = 2, l = 8, m = 0$. By comparing Figs. 2 and 3 with Fig. 1, it is concluded that there is a state change in angular wave function caused to the presence of Poschl-Teller potentials. Therefore it may be concluded that the number of the degeneracy of the system changes. By comparing Fig. 3 with Figs. 1 and 2 is concluded that the $\sec^2\,\theta$ causes the change of the angular wave function state, while the effect of $\mathrm{cosec}^2\,\theta$ term causes the absolute values of the angular wave function shifted to larger values of θ. Therefore the

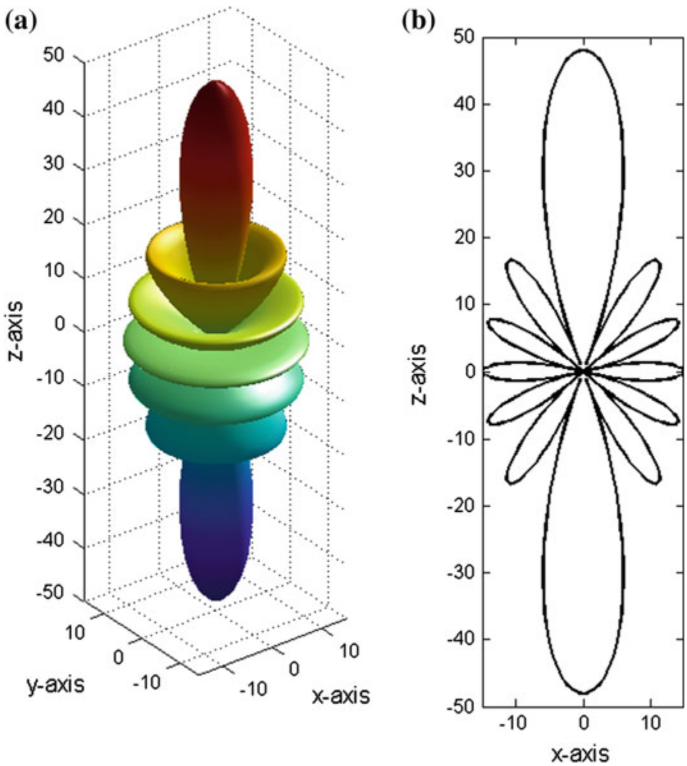

Fig. 1 **a** Three dimensional representation of $\left|Y_6^0\right|$ and **b** its polar diagram of $\left|Y_6^0\right| = \{86.63\cos^3 2\theta + 23.63\cos^2 2\theta - 55.13\cos 2\theta - 7.13\}$

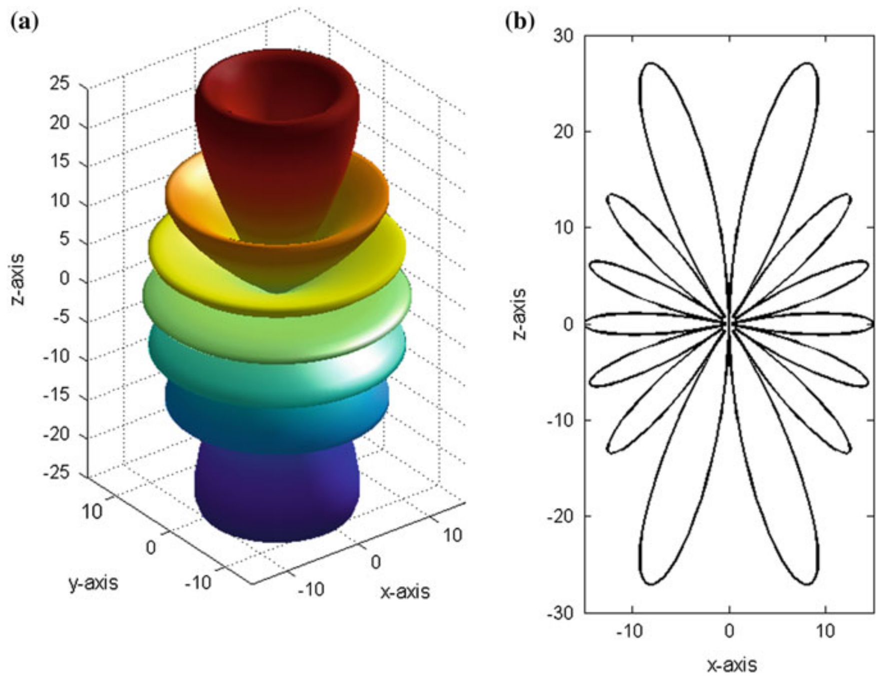

Fig. 2 **a** Three dimensional representation of $\left|Y_{7.4}^{0}\right|$ and **b** its polar diagram of $\left|Y_{7.4}^{0}\right| = \{199.48\cos^3 2\theta + 164.79\cos^2 2\theta - 48.36\cos 2\theta - 28.67\}\sin^{1.4}\theta$

dominant effect of the presence of Poschl-Teller potential is coming from the $\sec^2\theta$ term. The un-normalized angular wave functions illustrated in Figs. 1, 2 and 3 is in agreement with the result calculated using NU method [31]. By putting the new value of the orbital quantum number expressed in the energy of Eckart potential combined with trigonometric Poschl-Teller non-central potential is rewritten as

$$E_{n_r} = -\frac{\hbar^2}{2M}\left\{ \begin{array}{l} \dfrac{V_1^2}{a^2\left[\sqrt{V_o+\left(\sqrt{(\kappa(\kappa-1)+m^2)}+\eta+2n_l+\frac{1}{2}\right)^2}+n_r+\frac{1}{2}\right]^2} + \\ \dfrac{\left[\sqrt{V_o+\left(\sqrt{(\kappa(\kappa-1)+m^2)}+\eta+2n_l+\frac{1}{2}\right)^2}+n_r+\frac{1}{2}\right]^2}{4a^2} - \dfrac{l(l+1)d_0}{4a^2} \end{array} \right\} \quad (69)$$

where n_l is a new polar quantum number and its values are non-negative integer, while n_r is radial quantum number and is nonnegative integer. From (69) we can calculate the energy for special case, for Eckart potential, we set $\kappa = \eta = m = n_l = 0$, therefore the energy spectrum of Eckart potential is

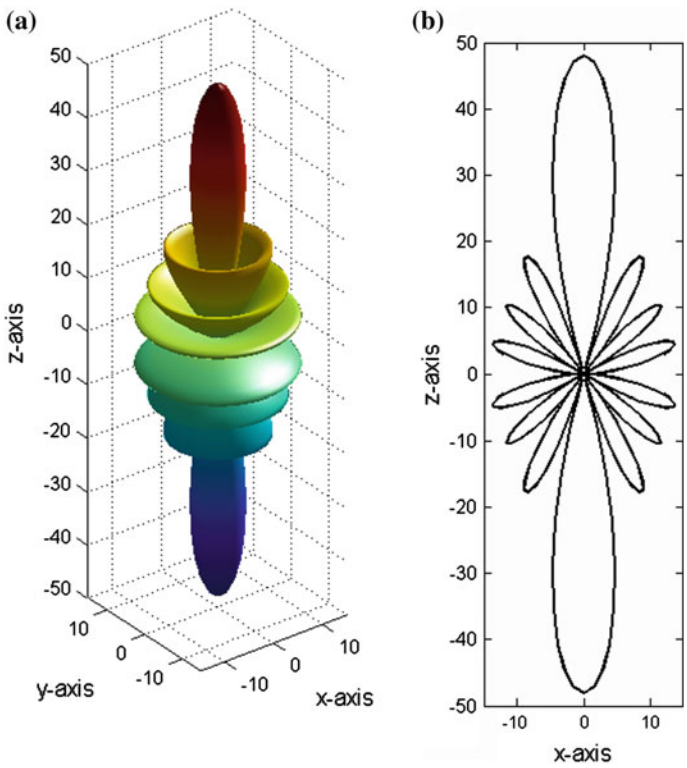

Fig. 3 **a** Three dimensional representation of $\left|Y_8^0\right|$ and **b** its polar diagram of $\left|Y_8^0\right| = \{268.13\cos^3 2\theta - 160.88\cos^2 2\theta - 86.63\cos 2\theta + 27.38\}\cos^2\theta$

$$E_{n_r} = -\frac{\hbar^2}{2M}\left\{\frac{V_1^2}{a^2\left[\sqrt{V_0 + \left(\frac{1}{2}\right)^2} + n_r + \frac{1}{2}\right]^2} + \frac{\left[\sqrt{V_0 + \left(\frac{1}{2}\right)^2} + n_r + \frac{1}{2}\right]^2}{4a^2} - \frac{l(l+1)d_0}{4a^2}\right\} \tag{70}$$

The total un-normalized wave function of the system obtained from (39) and (63a), (63b) is given as

$$\psi(r,\theta,\varphi) = (1-\coth(r/2a))^{\frac{\beta_{nr}}{2}-\frac{i\alpha_{nr}}{4}}(1+\coth(r/2a))^{\frac{\beta_{nr}}{2}+\frac{i\alpha_{nr}}{4}}R_{n_r}^{(-\beta_{nr},-\alpha_{nr})}(-i\coth(r/2a)) \times \frac{1}{\sqrt{\sin\theta}}(1-\cos 2\theta)^{\frac{\sqrt{F}}{2}+\frac{1}{4}}(1+\cos 2\theta)^{\frac{\sqrt{G}}{2}+\frac{1}{4}}R_{n_l}^{(-\beta+\frac{1}{2},-\alpha)}(-i\cos 2\theta)e^{im\varphi} \tag{71}$$

The wave function of the system in (68) reduces to the wave function of three dimensional Eckart potential by the absent of Poschl-Teller potential.

3.2 The Solution of Polar Schrodinger Equation for 3D Trigonometric Rosen Morse Non-central Potential

The polar part of the Schrodinger equation for trigonometric Rosen-Morse non-central potential is given as

$$\frac{\partial^2 P(\theta)}{\partial\theta^2}+\cot\theta\frac{\partial P(\theta)}{\partial\theta}-\left(\frac{v(v+1)+m^2}{\sin^2\theta}-2\mu\cot\theta\right)P(\theta)+l(l+1)P(\theta)=0 \tag{72}$$

By setting $P=\frac{Q}{\sqrt{\sin\theta}}$ in (72) then (72) becomes

$$\frac{d^2Q}{d\theta^2}-\left(\frac{v(v+1)+m^2-\frac{1}{4}}{\sin^2\theta}-2\mu\cot\theta\right)Q+\left(l(l+1)+\frac{1}{4}\right)Q=0 \tag{73}$$

To solve (73) we introduce a new variable $\cot\theta=s$ and (73) change into

$$\left(1+s^2\right)\frac{\partial^2 Q}{\partial s^2}+2s\frac{\partial Q}{\partial s}-\left\{\left(v(v+1)+m^2-\frac{1}{4}\right)-\frac{2\mu x}{(1+s^2)}-\frac{l(l+1)+\frac{1}{4}}{(1+s^2)}\right\}Q=0 \tag{74}$$

Equation (74) is solved in terms of Romanovski polynomial by setting

$$Q(\theta)=g_n(s)=\left(1+s^2\right)^{\frac{\beta}{2}}e^{-\frac{\alpha}{2}\tan^{-1}}D_n^{(\beta,\alpha)}(s) \tag{75}$$

for $0<s<\infty$

By inserting (75) into (74) we obtain

$$\begin{aligned}&\left(1+s^2\right)\frac{\partial^2 D}{\partial s^2}+\{2s(\beta+1)-\alpha\}\frac{\partial D}{\partial s}\\&\quad-\left\{\frac{\beta s\alpha-\frac{\alpha}{4}+\beta^2-2\mu s-\left(l(l+1)+\frac{1}{4}\right)}{1+s^2}+v(v+1)+m^2-\frac{1}{4}-\beta^2-\beta\right\}D\\&\quad=0\end{aligned} \tag{76}$$

Equation (76) reduces to differential equation satisfied by Romanovski polynomials

$$\left(1+s^2\right)\frac{\partial^2 D}{\partial s^2}+\{2s(\beta+1)-\alpha\}\frac{\partial D}{\partial s}-\left\{v(v+1)+m^2-\frac{1}{4}-\beta^2-\beta\right\}D=0 \quad (77)$$

for

$$\beta s\alpha-\frac{\alpha^2}{4}+\beta^2-2\mu s-\left\{l(l+1)+\frac{1}{4}\right\}=0 \quad (78)$$

By comparing (10) and (77) we obtain

$$\begin{aligned}(\beta+1)=(-p+1);\alpha=-q \text{ and } v(v+1)+m^2-\frac{1}{4}-\beta^2-\beta\\ =n(n-1)+2n(1-p)\end{aligned} \quad (79)$$

From (78) we have

$$-\frac{\alpha^2}{4}+\beta^2-\left\{l(l+1)+\frac{1}{4}\right\}=0; \quad \beta\alpha-2\mu=0 \quad (80)$$

that give

$$\beta^2=\frac{\left(l+\frac{1}{2}\right)^2\pm\sqrt{\left(l+\frac{1}{2}\right)^4+4\mu^2}}{2} \quad (81)$$

and

$$\alpha^2=\frac{8\mu^2}{\left(l+\frac{1}{2}\right)^2\pm\sqrt{\left(l+\frac{1}{2}\right)^4+4\mu^2}} \quad (82)$$

Using (79) we obtain

$$p=-\beta; q=-\alpha \quad (83)$$

and

$$v(v+1)+m^2=\left(\beta+n+\frac{1}{2}\right)^2 \quad (84)$$

Then from (84) we get

$$\beta=\sqrt{v(v+1)+m^2}-n-\frac{1}{2} \quad (85)$$

or

$$\beta = -\sqrt{v(v+1)+m^2} - n - \frac{1}{2} \tag{86}$$

By using (80) with (85) or with (86) we obtain

$$\alpha_n = \frac{2\mu}{\sqrt{v(v+1)+m^2} - n - \frac{1}{2}} \tag{87}$$

or

$$\alpha_n = -\frac{2\mu}{\sqrt{v(v+1)+m^2} + n + \frac{1}{2}} \tag{88}$$

From (85), (86), and (81) we obtain

$$\left(\sqrt{v(v+1)+m^2} - n - \frac{1}{2}\right)^2 = \frac{\left(l+\frac{1}{2}\right)^2 \pm \sqrt{\left(l+\frac{1}{2}\right)^4 + 4\mu^2}}{2} \tag{89}$$

$$\left(-\sqrt{v(v+1)+m^2} - n - \frac{1}{2}\right)^2 = \frac{\left(l+\frac{1}{2}\right)^2 \pm \sqrt{\left(l+\frac{1}{2}\right)^4 + 4\mu^2}}{2} \tag{90}$$

By imposing the condition that $p > 0$ then from (83) we have $-\left(\beta - \frac{1}{2}\right) > 0$ or $\left(\beta - \frac{1}{2}\right) < 0$, therefore the values of β and α that satisfy that condition are expressed in (86) and (88). The value of l that satisfies the system is obtained from (90), that is

$$l = l' = \sqrt{\left(\sqrt{v(v+1)+m^2} + n + \frac{1}{2}\right)^2 - \frac{\mu^2}{\left(\sqrt{v(v+1)+m^2} + n + \frac{1}{2}\right)^2}} - \frac{1}{2} \tag{91}$$

The weight function obtained from (7), (83), (86) and (88) is given as

$$w^{(-\beta,-\alpha)} = \left(1+s^2\right)^{\beta_{n_l}} e^{-\alpha_{n_l}\tan^{-1}s} \tag{92}$$

Using (13) and (92) we obtain the Romanovski polynomials given as

$$R_n^{(-\beta,-\alpha)}(s) = \frac{1}{(1+s^2)^{\beta_{n_l}} e^{-\alpha_{n_l}\tan^{-1}s}} \frac{d^{n_l}}{ds^{n_l}} \left\{\left(1+s^2\right)^{\beta_{n_l}+n_l} e^{-\alpha_{n_l}\tan^{-1}s}\right\} \tag{93}$$

The polar wave function obtained from (11) and (93) is given as

$$Q_{n_l}\left(\theta = \cot^{-1} s\right) = g_{n_l}(s) = \left(1+s^2\right)^{\frac{\beta_{n_l}}{2}} e^{\frac{-\alpha_{nl}}{2}\tan^{-1} s} R_{n_l}^{(-\beta,-\alpha)}(s) \tag{94}$$

The polar wave function obtained from (94) is given as

$$P_l^m(\theta) = \frac{Q_{n_l}(\theta)}{\sqrt{\sin\,\theta}} = \left(1+s^2\right)^{\frac{\beta_{n_l}}{2}+\frac{1}{4}} e^{\frac{-\alpha_{n_l}}{2}\tan^{-1} s} R_{n_l}^{(-\beta,-\alpha)}(s) \tag{95}$$

Due to the condition that β_n and α_n are n-dependence, thus the Romanovski polynomials is infinity [16, 17] and the orthogonality of polynomials is not produced from orthogonality integral of the wave function, that is

$$\int \frac{Q_{n_l}}{\sqrt{\sin\,\theta}}(\theta)\frac{Q_{n_{l'}}}{\sqrt{\sin\,\theta}}(\theta)\sin\,\theta d\theta = \delta_{n_l n_{l'}} \neq \int (1+s^2)^{\beta_{n_l}} e^{-\alpha_{n_l}\tan^{-1}(s)} R_{n_l}^{(-\beta,-\alpha)}(s) R_{n_l'}^{(-\beta,-\alpha)}(s)ds \tag{96}$$

Construction of Romanovski polynomial

The first four Romanovski polynomial are constructed using (93) are

$$R_0^{\left(-\beta+\frac{1}{2},-\alpha\right)}(s) = 1 \tag{97}$$

$$R_1^{\left(-\beta+\frac{1}{2},-\alpha\right)}(s) = 2(\beta_1+1)s - \alpha_1 \tag{98}$$

$$R_2^{\left(-\beta+\frac{1}{2},-\alpha\right)}(s) = \left\{4(\beta_2+2)(\beta_2+1.5)s^2 - 4\alpha_2(\beta_2+1.5)s + \alpha_2^2 + 2(\beta_2+2)\right\} \tag{99}$$

$$\begin{aligned} R_3^{\left(-\beta+\frac{1}{2},-\alpha_3\right)}(s) = {} & 8(\beta_3+3)(\beta_3+2.5)(\beta_3+2)s^3 - 12\alpha_3(\beta_3+2.5)s^2 \\ & + 6\left\{\alpha_3^2 + 2(\beta_3+3)\right\}(\beta_3+2)s - \alpha_3\left\{\alpha_3^2 + 6\beta_3 + 16\right\} \end{aligned} \tag{100}$$

The Romanovski polynomials expressed in (97–100) can be constructed manually or using computer programming with Mat Lab software. If Rosen-Morse noncentral potential is absent then

$$\beta_{n_l} = -m - n_l; \alpha_n = 0; l = n_l + m; \tag{101}$$

and the polar wave functions reduce to associated Legendre polynomials. The polar wave functions for $n_l = 3$ with different values of v and μ are shown in Table 1.

The third degree of Romanovski polynomials and the corresponding polar wave functions for $n_l = 3$, $m = 1$, $v = 0.2$ and $\mu = 0.2$ are calculated using (93), (95), (100) are listed in Table 1. The angular wave functions, $Y_{l'}^{m'}$, are obtained by multiplying the polar wave function listed in the last column at Table 1. with the

Table 1 The Romanovski polynomials and its corresponding un-normalized polar wave functions for Rosen-Morse potential for $n_l = 3$ and $m = 3$, with $v' + \frac{1}{2} = \sqrt{v(v+1) + m^2} = m'$

No	n_l	m	v	μ	$v' + \frac{1}{2}$	$l = l'$	$-\beta$	$R_{n_l}^{(-\beta,-\alpha)}(s)$	$P_{n_l m v\mu} = P_{l'}^{m}(\theta)$
1	3	0	0	0	0	3	3.5	$-6\cot^3\theta + 9\cot\theta$	$(\sin\theta)^3(-6\cot^3\theta + 9\cot\theta)$
2	3	1	0	0	1	4	4.5	$-60\cot^3\theta + 45\cot\theta$	$(\sin\theta)^4(-60\cot^3\theta + 45\cot\theta)$
3	3	1	2	0	2.65	5.65	6.15	$-381.17\cot^3\theta + 156.87\cot\theta$	$(\sin\theta)^{5.65}\begin{pmatrix}-381.17\cot^3\theta + \\ 156.87\cot\theta\end{pmatrix}$
4	3	1	0	2	1	3.98	4.5	$-60\cot^3\theta + 54\cot^2\theta + 32.85\cot\theta - 9.17$	$2(\sin\theta)^4 e^{0.45\tan^{-1}\cot\theta}\sin^4\theta\begin{pmatrix}-60\cos^3\theta\sin\theta + 54\cos^2\theta\sin^2\theta \\ +32.85\cos\theta\sin^3\theta - 9.17\end{pmatrix}$
5	3	1	2	2	2.65	5.64	6.15	$-381.71\cot^3\theta + 118.15\cot^2\theta + 146.35\cot\theta - 17.21$	$(\sin\theta)^{5.65}e^{0.33\tan^{-1}\cot\theta}\begin{pmatrix}-381.71\cot^3\theta + 118.15\cot^2\theta \\ +146.35\cot\theta - 17.21\end{pmatrix}$

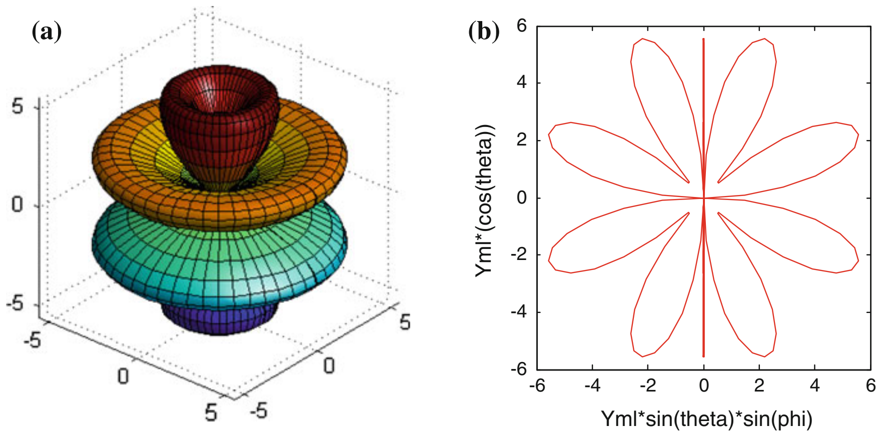

Fig. 4 **a** Three dimensional polar representation of absolute value of eigen function, $n_l = 3$, $m = 1$, $v = 0$ and $\mu = 0$ $Y_4^1 = 15\left(-4\cos^3\ \theta \sin\ \theta + 3\sin^3\ \theta \cos\ \theta\right)$. **b** Polar diagram of angular wave function $Y_4^1 = 15\left(-4\cos^3\ \theta \sin\theta + 3\sin^3\theta\cos\theta\right)$

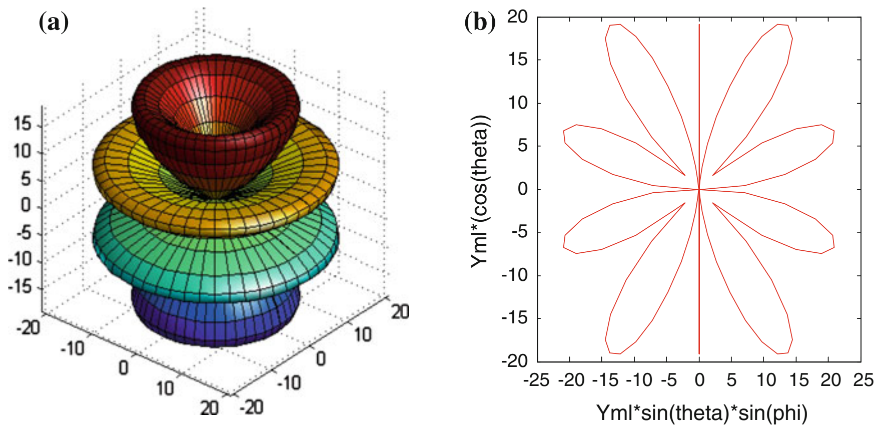

Fig. 5 **a** Three dimensional polar representation of absolute value of eigenfunction, $n_l = 3$, $m = 1$, $v = 2$ $\mu = 0$ $\left|Y_{5.65}^{2.65}\right| = (\sin\theta)^{5.65}(-381.17\cot^3\theta + 156.87\cot\theta)$. **b** Polar diagram of angular wavefunction $\left|Y_{5.65}^{2.65}\right| = (\sin\theta)^{5.65} \times (-381.17\cot^3\theta + 156.87\cot\theta)$

azimuthal wave function in (28). The polar diagram of orbital angular momentum eigen function, $Y_{l'}^{m'}$ and three dimensional polar representation of the absolute value of the angular wave functions listed in Table 1 for all *m = 1* are graphed using Math Lab software shown in Figs. 4, 5, 6 and 7. By comparing Fig. (6a) and (7a) it can be

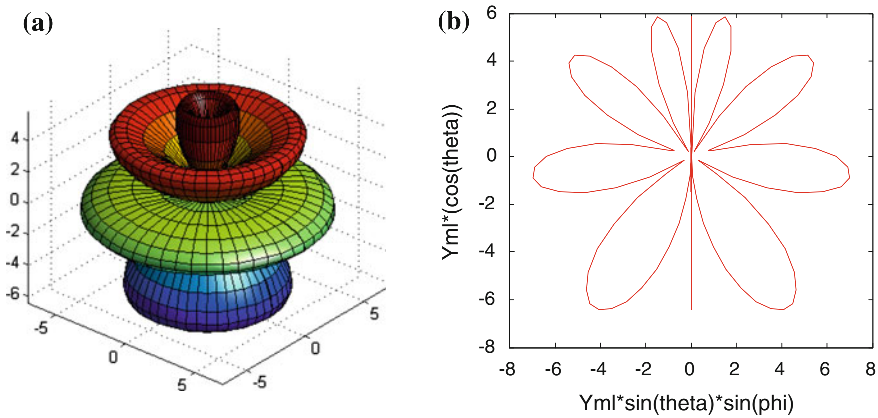

Fig. 6 **a** Three dimensional polar representation of absolute value of eigenfunction $n_l = 3$, $m = 1$, $v = 0$ and $\mu = 2$ $\left|Y^1_{3.98}\right| = 2(\sin\,\theta)^4 e^{0.45\tan^{-1}\cot\theta}\begin{pmatrix} -60\cos^3\theta\,\sin\,\theta \\ +54\cos^2\,\theta\,\sin^2\,\theta \\ +32.85\cos\,\theta\,\sin^3\,\theta \\ -9.17 \end{pmatrix}\sin^4\,\theta$. **b** Polar diagram of angular wavefunction $\left|Y^1_{3.98}\right| = 2(\sin\,\theta)^4 e^{0.45\tan^{-1}\cot\theta}\begin{pmatrix} -60\cos^3\,\theta\sin\,\theta \\ +54\cos^2\,\theta\sin^2\,\theta \\ +32.85\cos\,\theta\sin^3\,\theta \\ -9.17 \end{pmatrix}\sin^4\,\theta$

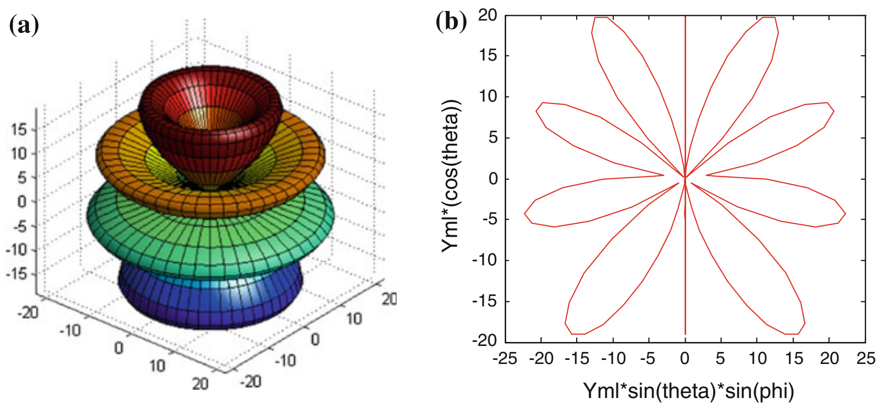

Fig. 7 **a** Three dimensional polar representation of absolute value of eigenfunction $n_l = 3$, m = 1,$v = 2$ and $\mu = 2$ $Y^{2.65}_{5.64} = (\sin\,\theta)^{5.65} e^{0.33\tan^{-1}\cot\theta}(-381.71\cot^3\,\theta + 118.15\cot^2\,\theta + 146.35$ $\cot\,\theta - 17.21)$. **b** Polar diagram of angular wavefunction $Y^{2.65}_{5.64} = (\sin\,\theta)^{5.65} e^{0.33\tan^{-1}\cot\theta}$ $(-381.71\cot^3\,\theta + 118.15\cot^2\,\theta + 146.35\cot\vartheta - 17.21)$

shown that the effect of cot θ is larger for lower level polar wave function. From Figs. (4a) and (5a) can be seen that $\csc^2\theta$ causes the increase of the absolute value of the angular wave function, while cot θ term causes the decrease of the absolute value of the angular wave function in the interval of $0<\theta<\frac{\pi}{2}$ but causes the increase in the $\frac{\pi}{2}<\theta<\pi$ as shown in Figs. (6a) and (7a).

The total u-normalized wave function for the n level is given as

$$\psi(r,\theta,\varphi) = e^{im\varphi}\left\{C_{n_r} r^{-(n_r+l'+\frac{1}{2})} e^{-\frac{\gamma r^2}{2}} \frac{d^{n_r}}{dr^{n_r}}\left(r^{2l'+1+2n_r} e^{-\gamma r^2}\right)\right\}$$
$$\left[(\csc^2\theta)^{-\frac{\beta_{n_l}}{2}+\frac{1}{4}} e^{\frac{\alpha_{n_l}}{2}\tan^{-1}(\cot\theta)} \frac{d^{n_l}}{d\cot\theta^{n_l}}\left\{(\csc^2\theta)^{\beta_{n_l}+n_l} e^{-\alpha_{n_l}\tan^{-1}(\cot\theta)}\right\}\right] \tag{102}$$

By the absent of Rosen-Morse potential the wave function in (102) reduces to the three dimensional spherical harmonics oscillator wave function.

The three dimensional Schrodinger equation for separable shape-invariant non-central potentials are solved using variable separation method. The 3D Schrodinger equation is separated into three one dimensional equations, radial and polar equations are solved using Romanovski polynomial while the azimuthal part is simple differential equation. The generalized hypergeometric type equation and so the Romanovski differential equations fall into two groups, first group, such as (26) and (74) have the same form, and so the (28) and (77). while the second group, such as (48) which is different to the form of (26) and (74) and so (50b) is different to the form of (28) and (77). Therefore by recognizing the form of the generalized hypergeometric type equation we can determine the form of differential equation of Romanovski polynomials. Even for complex variable, Romanovski polynomials method working very well in determining the energy spectra of the system but there is a limitation in producing the wave function, as in Poschl-Teller potential.

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