Application of Nikiforov-Uvarov Method for Non-central Potential System Solution

C. Cari and A. Suparmi

Abstract The energy eigenvalues and eigenfunctions of Schrodinger equation for a 3D harmonic oscillator potential plus Rosen-Morse non-central potential and Eckart plus trigonometric Poschl-Teller non-central potential are investigated using NU method. The bound state energy eigenvalues for both systems are given in a closed form and the corresponding radial wave functions are expressed in associated Laguerre polynomials for 3D harmonics oscillator while the radial and angular eigenfunctions are given in terms of Jacobi polynomials. The Rosen-Morse and Poschl-Teller potentials are considered to be the perturbation factors to the 3D harmonic oscillator and Eckart potentials that cause the decrease of angular momentum length but preserve the number of energy degeneracy.

1 Introduction

The Schrodinger equations of physical potentials have been studied intensively in recent years. Mostly methods used to obtain the exact solution of Schrodinger for physical potentials which consist of a class of shape invariant potentials are factorization method [1–3], super-symmetric quantum mechanics (SUSY QM) approach [4–8], Nikiforov-Uvarov (NU) method [9–13], and Romanovski polynomial [14–17]. Among these methods, there are some methods that interconnect to each other, SUSY QM with factorization method and WKB approach, NU method and Romanovski polynomial are developed based on hypergeometric differential equation. Shape invariant potentials is a class of one dimensional potentials (radial/central and angular functions potential) that obey to the properties proposed by Gendenshtein [8].

NU method, proposed by Nikiforov and Uvarov [18], has been widely used to solve second order linier differential equation without direct solution. The energy

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spectrum and the wave function of certain potential system are calculated using formulas derived from hypergeometric type equation which obtained by simple mathematical manipulation of the Schrodinger equation of the potential of interest. By suitable variable substitution, the Schrodinger equation for certain physical potential reduces to an generalized hypergeometric differential equation (GHDE), and by parameter and wave function substitutions to GHDE then GHDE reduces to hypergeometric/confluent hypergeometric differential equation that is called as hypergeometric type equation. As by product, a set of formulas, which is used to determine the energy spectrum and wave functions, are produced. Therefore the heart of NU method is determining the coefficients of the second, first, and zeroth derivatives from GHDE and plugging it into set of formulas.

In this chapter we analyze the energy spectra and wave functions of non-central potential: 3D oscillator harmonics plus trigonometric Rosen-Morse and Eckart plus Poschl-Teller non-central potentials using NU method. These two system potentials are separable potentials [19–21] therefore its Schrodinger equations solved using variable separation method. Three dimensional harmonics oscillator is one of exactly solvable potential that used to describe the nuclei, atomic or molecular vibration. Non-central potential composed of spherical harmonics oscillator with square of inverse potential together with ring-shaped non-central potential, or double ring shaped potential have been investigated intensively by some authors [22–25]. The Rosen-Morse potential is trigonometric potential which was proposed by Rosen-Morse [26] in 1935 and was used to describe the quark-gluon dynamics. The approximate bound state solution for trigonometric Rosen-Morse potential have also been studied for l-state solution [27–29], and Coulombic Rosen-Morse non-central potential, particularly for $cot\theta$ part, has been investigated intensively [27]. For $l \neq 0$ and r $\ll 0$, the centrifugal term is approximated by hyperbolic, trigonometric or exponential functions [ ] and leads to the exact analytical solution of the radial Schrodinger equation. The exact analytical solutions of Schrodinger equations for some physical potentials are very essential since the knowledge of wave functions and energy contains all possible important information of the physical properties of quantum system.

2 Non-central Potential

The Schrodinger equations for central potentials, which are shape invariant potentials such as three dimensional harmonics oscillator, Kepler problems, Wood Saxon potential, Kratzer molecular potential, have been solved exactly using SUSY QM, WKB with Langer correction, and hypergeometric type equations particularly only for $l = 0$. Non-central potentials, which are separable ones, are also exactly solvable for $l = 0$. The separable non-central potential is given as

$$V(r,\theta,\varphi) = V(r) + \frac{V(\theta)}{r^2} + \frac{V(\varphi)}{r^2 \sin^2\theta} \tag{1}$$

Schrodinger equation for non-central potential expressed in (1) is solved using variable separation method and is exactly solvable for $l = 0$ if $V(r)$, $V(\theta)$, and $V(\varphi)$ are shape invariants. The Schrodinger equation for non-central potential expressed in (1) are resolved into three second order linier differential equation: radial, polar and azimuthal Schrodinger equations. In the case of radial Schrodinger equation, for $l \neq 0$, however, is exactly solvable only if the contribution of centrifugal term, $\frac{l(l+1)}{r^2}$, for very small value of r, r ≪ 1, is approximated by

$$\frac{l(l+1)}{r^2} \cong \frac{l(l+1)}{\sinh^2 r} \cong \frac{l(l+1)}{\sin^2 r}, \tag{2a}$$

Approximation expressed in (2a, 2b) was initially proposed by Greene and Aldrich [L] and newly improved [M], with $d_0 = 1/12$, given as

$$\frac{l(l+1)}{r^2} \cong \frac{l(l+1)(1+d_0)}{\sinh^2 r} \cong \frac{l(l+1)(1+d_0)}{\sin^2 r} \tag{2b}$$

The radial parts of non-central potential, $V(r)$, that produce the exact solution within r approximation either expressed in (2a) or (2b) are including radial Eckart potential, radial hyperbolic Poschl-Teller potential, radial trigonometric Poschl-Teller potential, radial hyperbolic Rosen-Morse potential, radial trigonometric Rosen-Morse potential, radial hyperbolic Scarf potential and radial trigonometric Scarf potential, as shown in Table 1.

On the other hand, the polar parts of non-central potentials that have exact solutions are listing in Table 2.

By using (1) the non-central potentials are achieved by combining the radial function potentials listed in Table 1 with polar potential listed in Table 2.

Table 1 Lists of radial potential that are solvable using NU method

No	Potential's name	Potential's equation
1	Eckart	$V(r) = \frac{\hbar^2}{2Ma^2}\left(V_0 \frac{e^{-r/a}}{(1-e^{-r/a})^2} - V_1 \frac{1+e^{-r/a}}{1-e^{-r/a}}\right)$
2	Poschl-Teller (trigonometric)	$V(r) = \frac{\hbar^2\alpha^2}{2M}\left(\frac{\kappa(\kappa-1)}{\sin^2 \alpha r} + \frac{\eta(\eta-1)}{\cos^2 \alpha r}\right)$
3	Modified Poschl-Teller/(hyperbolic)	$V(r) = \frac{\hbar^2\alpha^2}{2M}\left(\frac{\kappa(\kappa-1)}{\sinh^2 \alpha r} - \frac{\eta(\eta+1)}{\cosh^2 \alpha r}\right)$
4	Hyperbolic Rosen-Morse	$V(r) = \frac{\hbar^2\alpha^2}{2M}\left(\frac{v(v-1)}{\sinh^2 \alpha r} - 2\mu \coth \alpha r\right)$
5	Trigonometric Rosen-Morse	$V(r) = \frac{\hbar^2\alpha^2}{2M}\left(\frac{v(v+1)}{\sin^2 \alpha r} - 2\mu \cot \alpha r\right)$
6	Trigonometric scarf II	$V(r) = \frac{\hbar^2\alpha^2}{2M}\left(\frac{b^2+a(a-1)}{\sin^2 \alpha r} - \frac{2b(a-\frac{1}{2})\cos \alpha r}{\sin^2 \alpha r}\right)$
7	Hyperbolic scarf II	$V(r) = \frac{\hbar^2\alpha^2}{2M}\left(\frac{b^2+a(a+1)}{\sinh^2 \alpha r} - \frac{2b(a+\frac{1}{2})\cosh \alpha r}{\sinh^2 \alpha r}\right)$

Table 2 Lists of polar function potentials that are solvable using NU method

No	Polar potential	Potential's mathematical equation
1	Poschl-Teller	$V(\theta)=\frac{\hbar^2}{2M}\left(\frac{\kappa(\kappa-1)}{\sin^2\theta}+\frac{\eta(\eta-1)}{\cos^2\theta}\right)$
2	Rosen-Morse	$V(\theta)=\frac{\hbar^2}{2M}\left(\frac{v(v+1)}{\sin^2\theta}-2\mu\cot\theta\right)$
3	Scarf	$V(\theta)=\frac{\hbar^2}{2M}\left(\frac{b^2+a(a-1)}{\sin^2\theta}-\frac{2b(a-\frac{1}{2})\cos\theta}{\sin^2\theta}\right)$

In this chapter we will solve Schrodinger equation for non-central potential with $V(\varphi)=1$,

$$V(r,\theta)=\frac{M\omega^2 r^2}{2}+\frac{\hbar^2}{2Mr^2}\left(\frac{v(v+1)}{\sin^2\theta}-2\mu\cot\theta\right) \tag{3}$$

and

$$V(r,\theta)=\frac{\hbar^2}{2Ma^2}\left(V_0\frac{e^{-r/a}}{(1-e^{-r/a})^2}-V_1\frac{1+e^{-r/a}}{1-e^{-r/a}}\right)+\frac{\hbar^2}{2Mr^2}\left(\frac{\kappa(\kappa-1)}{\sin^2\theta}+\frac{\eta(\eta-1)}{\cos^2\theta}\right), \tag{4}$$

and the one dimensional Schrodinger equations are solved using Nikiforov-Uvarov method. Special for three dimensional harmonics oscillator we do not need the approximation value of r.

3 Nikiforov-Uvarov Method

Nikiforov-Uvarov (NU) method was developed based on the hypergeometric diferential equation. In the following section the formulas used in NU method are derived from hypergeometric differential equation.

The hypergeometric differential equation expressed by Gau β [30] is given as

$$z(1-z)\frac{\partial^2\Phi}{\partial^2 z}+\{c-(a+b+1)z\}\frac{\partial\Phi}{\partial z}-ab\Phi=0 \tag{5}$$

Equation (1) has three regular singular points at $z=0$, $z=1$, $z=\infty$. By using Frobenius method [30], the general solution at around point $z=0$ is given as

$$\Phi(z)=A_2F_1(a,b;c;z)+Bz^{1-c}{}_2F_1 a+1-c,b+1-c:2-c;z) \tag{6}$$

where

$$
{}_2F_1(a,b;c;z) = \sum_{n=0} \frac{(a)_n (b)_n}{(c)_n} \frac{z^n}{n!} \\
= 1 + \frac{ab}{c}\frac{z}{1!} + \frac{a(a+1)\,b(b+1)}{c(c+1)}\frac{z^2}{2!} \\
+ \frac{a(a+1)(a+2)\,b(b+1)(b+2)}{c(c+1)(c+2)}\frac{z^3}{3!} + \ldots \tag{7}
$$

$$
(a)_n = a(a+1)(a+2)(a+3)\ldots(a+n-1) \tag{8}
$$

and c is not integer. When c is integer the second part of the solution becomes complicated and we have only the first part of the solution. For $a = -n$ or $b = -n$ the solution of (7) becomes finite. By substituting $z = \frac{x}{b}$ and for $b \to \infty$ then hypergeometric differential equation in (5) reduces to confluent hypergeometric equation given as

$$
x\frac{\partial^2\Phi}{\partial^2 z} + \{c - x\}\frac{\partial\Phi}{\partial z} - a\Phi = 0 \tag{9}
$$

The solution of confluent hypergeometric equation in (9) at around regular singular point $z = 0$ is given as

$$
\Phi(z) = A_1F_1(a;c;z) + (Bz^{1-c}{}_1F_1 a + 1 - c;\; 2 - c; z) \tag{10}
$$

with

$$
{}_1F_1(a;c;z) = \sum_{n=0} \frac{(a)_n}{(c)_n}\frac{z^n}{n!} = 1 + \frac{a}{c}\frac{z}{1!} + \frac{a(a+1)}{c(c+1)}\frac{z^2}{2!} + \frac{a(a+1)(a+2)}{c(c+1)(c+2)}\frac{z^3}{3!} + \ldots \tag{11}
$$

For $a = -n$ the polynomials in (11) becomes finite.

The one-dimensional Schrodinger equation of any shape invariant potential can be reduced into hypergeometric or confluent hypergeometric type differential equation, expressed in (5) or (9) by suitable variable transformation [12, 26]. The hypergeometric type differential equation, which is solved using Nikiforov-Uvarov method, is presented as

$$
\frac{\partial^2\Psi(s)}{\partial s^2} + \frac{\tilde{\tau}(s)}{\sigma(s)}\frac{\partial\Psi(s)}{\partial s} + \frac{\tilde{\sigma}(s)}{\sigma^2}\Psi(s) = 0 \tag{12}
$$

where $\sigma(s)$ and $\tilde{\sigma}(s)$ are polynomials at most in the second order, and $\tilde{\tau}(s)$ is first order polynomial. Equation (12) is obtained from the Schrodinger equation of the certain potential by suitable variable substitution. Equation (12) is solved by using separation of variable method which is expressed as

$$\Psi(s) = \phi(s) y(s) \tag{13}$$

for appropriate $\phi(s)$ function, and (12) reduces to

$$y''(s) + \left\{ 2\frac{\phi'(s)}{\phi(s)} + \frac{\tilde{\tau}(s)}{\sigma(s)} \right\} y'(s) + \left\{ \frac{\phi''(s)}{\phi(s)} + \frac{\phi'(s)\tilde{\tau}(s)}{\phi(s)\sigma(s)} + \frac{\tilde{\sigma}(s)}{\sigma^2(s)} \right\} y(s) = 0 \tag{14}$$

In order (14) is not more complex than (12), then the coefficient of $y'(s)$ in (14) has to be in the form of $\frac{\tau(s)}{\sigma(s)}$ that is

$$2\frac{\phi'(s)}{\phi(s)} + \frac{\tilde{\tau}(s)}{\sigma(s)} = \frac{\tau(s)}{\sigma(s)} \tag{15}$$

By rewriting

$$\frac{\phi'(s)}{\phi(s)} = \frac{\pi(s)}{\sigma(s)} \tag{16}$$

then we obtain

$$2\pi(s) = (\tau(s) - \tilde{\tau}(s)) \rightarrow \tau(s) = \tilde{\tau}(s) + 2\pi(s) \tag{17}$$

where the new parameter $\pi(s)$ is the first order polynomial. By expressing the $\frac{\phi''(s)}{\phi(s)}$ term in (14) as

$$\frac{\phi''(s)}{\phi(s)} = \left[\frac{\phi'(s)}{\phi(s)}\right]' + \left[\frac{\phi'(s)}{\phi(s)}\right]^2 = \left[\frac{\pi(s)}{\sigma(s)}\right]' + \left[\frac{\pi(s)}{\sigma(s)}\right]^2 \tag{18}$$

and by setting the coefficient of $y(s)$ in (14) to be equal to $\frac{\bar{\sigma}(s)}{\sigma^2(s)}$ then from (14) and (18) we have

$$\left[\frac{\pi(s)}{\sigma(s)}\right]' + \left[\frac{\pi(s)}{\sigma(s)}\right]^2 + \frac{\phi'(s)\tilde{\tau}(s)}{\phi(s)\sigma(s)} + \frac{\tilde{\sigma}(s)}{\sigma^2(s)} = \frac{\bar{\sigma}(s)}{\sigma^2(s)} \tag{19}$$

and (14) becomes

$$y''(s) + \frac{\tau(s)}{\sigma(s)} y'(s) + \frac{\bar{\sigma}(s)}{\sigma^2(s)} y(s) = 0 \tag{20a}$$

Equation (19) is rewritten as

$$\bar{\sigma}(s) = \tilde{\sigma}(s) + \pi^2(s) + \pi(s)(\tilde{\tau}(s) - \sigma'(s)) + \pi'(s)\sigma(s) \tag{20b}$$

In order the expression of (20a, 20b) as simple as possible then $\bar{\sigma}(s)$ in (20a, 20b) should be divisible by $\sigma(s)$ that is

$$\bar{\sigma}(s) = \lambda\sigma(s) \tag{21}$$

with λ is a constant, and thus (20a, 20b) reduces to

$$\sigma(s)y''(s) + \tau(s)y'(s) + \lambda y(s) = 0 \tag{22}$$

Equation (22) is called as hypergeometric type equation and its solutions as a functions of hypergeometric type, and (12) is called as generalized hypergeometric type equation [18]. The new parameter $\pi(s)$, which is the first order polynomial, is determined by using (20a, b) and (21) given as

$$\tilde{\sigma}(s) + \pi^2(s) + \pi(s)(\tilde{\tau}(s) - \sigma'(s)) - k\sigma(s) = 0 \tag{23}$$

with

$$\lambda - \pi'(s) = k \tag{24}$$

From (23) we have

$$\pi(s) = \frac{\sigma'(s) - \tilde{\tau}(s)}{2} \pm \sqrt{\left(\frac{\sigma'(s) - \tilde{\tau}(s)}{2}\right)^2 - \tilde{\sigma}(s) + k\sigma(s)} \tag{25}$$

Since the parameter $\pi(s)$ has the form of first order polynomial, then the expression under square root of (25) has to be a perfectly quadratic expression, that means the discriminate of the quadratic expression has to be zero, and so we obtain the value of k.

Before determining the solution of (22) it is necessary to show that the derivative of hypergeometric type differential equation is also a hypergeometric type differential equation. By setting $\upsilon_1(s) = y'(s) = \frac{\partial y(s)}{\partial s}$ in (22) that have been differentiated we have

$$\sigma(s)\upsilon_1''(s) + \tau_1(s)\upsilon_1'(s) + \mu_1\upsilon_1(s) = 0 \tag{26}$$

with

$$\tau_1(s) = \tau(s) + \sigma'(s) \text{ and } \mu_1 = \lambda + \tau'(s) \tag{27}$$

Since $\tau_1(s)$ is the first polynomial and μ_1 is a parameter that s independence therefore (26) is the hypergeometric type differential equation. By repeating the step in obtaining (26) by substituting $\upsilon_2(s) = y''(s) = \frac{\partial^2 y(s)}{\partial s^2}$ in (26) that has been differentiated we get

$$\sigma(s)\upsilon_2''(s) + \tau_2(s)\upsilon_2'(s) + \mu_2\upsilon_2(s) = 0 \tag{28}$$

with

$$\tau_2(s) = \tau_1(s) + \sigma'(s) = \tau(s) + 2\sigma'(s) \tag{29a}$$

$$\mu_2 = \mu_1 + \tau_1'(s) = \lambda + 2\tau'(s) + \sigma''(s) \tag{29b}$$

By repeating the differentiation of (22) n times with $\upsilon_n(s) = y^{(n)}(s)$ such that we have

$$\sigma(s)\upsilon_n''(s) + \tau_n(s)\upsilon_n'(s) + \mu_n\upsilon_n(s) = 0 \tag{30}$$

and $\tau_n(s)$ dan μ_n yaitu,

$$\tau_n(s) = \tau(s) + n\sigma'(s) \tag{31}$$

$$\mu_n = \lambda + n\tau'(s) + \frac{n(n-1)}{2}\sigma''(s) \tag{32}$$

If $\mu_n = 0$, then from we obtain,

$$\lambda = \lambda_n = -n\tau'(s) - \frac{n(n-1)}{2}\sigma''(s) \quad n = 0, 1, 2, 3, \ldots \tag{33}$$

The solution of (22) is obtained from condition that $y(s) = y_n(s)$, which is the nth order polynomial given as

$$y_n(s) = \frac{B_n}{\rho(s)}\frac{d^n}{ds^n}[\sigma^n(s)\rho(s)] \tag{34}$$

with B_n is normalization constant and $\rho(s)$ is weight function that satisfies the condition

$$\frac{d}{ds}[\sigma(s)\rho(s)] = \tau(s)\rho(s) \tag{35}$$

with $\sigma(s)$ and $\tilde{\sigma}(s)$ which are obtained from (12) are polynomials with mostly second order, and $\tilde{\tau}(s)$ is first order polynomial given as

$$\sigma(s) = as^2 + bs + c \text{ and } \tilde{\tau}(s) = fs + h \tag{36}$$

The useful formulas used to determine the energy spectra and the wave function of quantum system using NU method have been derived from hypergeometric differential equation.

4 Application of NU Method for Non-central Potential

By using (16), (17), (24), (25), (33), (34), and (35), the energy spectra and the corresponding wave functions of 3D oscillator harmonics plus trigonometric Rosen-Morse non-central potential and Eckart plus Poschl-Teller non-central potentials are calculated.

4.1 Energy Spectrum and Wave Function of 3D Oscillator Harmonics Plus Trigonometric Rosen-Morse Non-central Potential

The first part of this section discusses the solution of three dimensional Schrodinger equation for 3D HO potential with simultaneously the presence of trigonometric Rosen-Morse non-central potential whose potential is expressed as

$$\begin{aligned}&-\frac{\hbar^2}{2M}\left\{\frac{1}{r^2}\frac{\partial}{\partial r}\left((r^2\frac{\partial}{\partial r}) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\frac{\partial}{\partial\theta}) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}\right)\Psi(r,\theta,\varphi)\right.\\&\quad+\left\{\frac{M\omega^2r^2}{2} + \frac{\hbar^2}{2Mr^2}\left(\frac{v(v+1)}{\sin^2\theta} - 2\mu\cot\theta\right)\right\}\Psi(r,\theta,\varphi)\}\\&\quad= E\Psi(r,\theta,\varphi)\end{aligned} \tag{37}$$

The non-central potential is separable one then (37) is solved using variable separation method by setting the wave function in (37) as $\psi(r,\theta,\varphi) = R(r)P(\theta)\phi(\varphi)$ and we obtain

$$\begin{aligned}\frac{1}{R}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) - \frac{r^2}{\hbar^2}M^2\omega^2r^2 + \frac{2Mr^2}{\hbar^2}E &= -\frac{1}{P\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) - \frac{1}{\phi\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}\\&\quad+\left(\frac{v(v+1)}{\sin^2\theta} - 2\mu\cot\theta\right)\\&= \lambda = \ell(\ell+1)\end{aligned} \tag{38}$$

The azimuthal part obtained from the wave equation expressed in (38) is given as

$$\frac{1}{\phi}\frac{\partial^2}{\partial\varphi^2} = -m^2 \tag{39}$$

therefore the solution of azimuthal part of wave function as usual is

$$\phi = A_m e^{im\varphi} \tag{40}$$

From (38) we obtain the radial and angular parts of Schrodinger equation given as

$$\frac{1}{R}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) - \frac{r^2}{\hbar^2}M^2\omega^2 r^2 + \frac{2Mr^2}{\hbar^2}E = l(l+1) \tag{41}$$

$$-\frac{1}{P\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{m^2}{\sin^2\theta} + \left(\frac{v(v+1)}{\sin^2\theta} - 2\mu\cot\theta\right) = l(l+1) \tag{42}$$

From the solution of (41) will be obtained the energy spectrum of 3D HO potential and the radial part of wave function while from (42) will be obtained the angular wave function and the value of l, orbital quantum number.

4.1.1 Solution of Radial Schrodinger Equation for 3D Oscillator Harmonics Plus Trigonometric Rosen-Morse Non-central Potential

The radial Schrodinger equation in (41) is rewritten as

$$\frac{\partial^2 R}{\partial r^2} + \frac{2}{r}\frac{\partial R}{\partial r} + \left(-\frac{M^2}{\hbar^2}\omega^2 r^2 + \frac{2\mu}{\hbar^2}E\right)R - \frac{l(l+1)}{r^2}R = 0 \tag{43}$$

By setting

$$\frac{2M}{\hbar^2}E = \varepsilon^2 \quad \frac{M^2\omega^2}{\hbar^2} = \gamma^2 \quad \text{and} \quad R = \frac{\chi}{r} \tag{44}$$

and inserting it into (43) we get

$$\frac{\partial^2\chi}{\partial r^2} + \left(-\gamma^2 r^2 + \varepsilon^2 - \frac{l(l+1)}{r^2}\right)\chi = 0 \tag{45}$$

Making a change variable $r^2 = x$ in (42) and change it into equation which is given as

$$x\frac{\partial^2\chi}{\partial x^2} + \frac{1}{2}\frac{\partial\chi}{\partial x} - \left(\frac{\gamma^2x^2 - \varepsilon^2x + l(l+1)}{4x}\right)\chi = 0 \tag{46}$$

By comparing (12) and (46) we have

$$\sigma = x, \quad \tilde{\tau} = \frac{1}{2} \quad \tilde{\sigma} = -\left\{\frac{l(l+1) + \gamma^2x^2 - \varepsilon^2x}{4}\right\} \tag{47}$$

From (25) and (47) we get

$$\pi = \frac{1}{4} \pm \sqrt{\frac{1}{16} + \frac{l(l+1) + \gamma^2x^2 - \varepsilon^2x}{4} + kx} \tag{48}$$

The value of k is obtained from the condition that quadratic expression under the square root in (48) has to be completely square of first degree of polynomial therefore (48) is rewritten as

$$\pi = \frac{1}{4} \pm \frac{\gamma}{2}\left\{x + 2\frac{k - \frac{\varepsilon^2}{4}}{\gamma^2}\right\} \tag{49}$$

and the discriminate of the quadratic expression under the square root that has to be zero is given as

$$\left(k - \frac{\varepsilon^2}{4}\right)^2 - 4\left(\frac{\gamma^2}{4}\right)\frac{\left(l+\frac{1}{2}\right)^2}{4} = 0 \tag{50}$$

From (50) we obtain the value of k as

$$k_1 = \frac{\varepsilon^2}{4} + \frac{\gamma}{2}(l + \frac{1}{2}) \qquad \text{or} \qquad k_2 = \frac{\varepsilon^2}{4} - \frac{\gamma}{2}(l + \frac{1}{2}) \tag{51}$$

By imposing the condition that $\tau' < 0$ then from (49) and (51) we get

$$\text{for } k_1 \ \pi = \frac{1}{4} - \frac{\gamma}{2}\left\{x + 2\frac{\frac{\gamma}{2}(l+\frac{1}{2})}{\gamma^2}\right\} = -\frac{\gamma x}{2} - \frac{l}{2} \text{ or } \pi = \frac{1}{4} - \frac{\gamma}{2}\left\{x - 2\frac{\frac{\gamma}{2}(l+\frac{1}{2})}{\gamma^2}\right\} = -\frac{\gamma x}{2} + \frac{(l+1)}{2} \text{ for } k_2 \tag{52}$$

By using (17), (24), (47), (51), and (52) we obtain

$$\text{for } \mathrm{k}_1 : \lambda = \frac{\varepsilon^2}{4} + \frac{\gamma}{2}\left(l + \frac{1}{2}\right) - \frac{\gamma}{2} = \frac{\varepsilon^2}{4} + \frac{\gamma}{2}\left(l - \frac{1}{2}\right) \quad \text{or} \\ \lambda = \frac{\varepsilon^2}{4} - \frac{\gamma}{2}\left(l + \frac{3}{2}\right) \text{ for } \mathrm{k}_2 \tag{53}$$

$$\tau = \frac{1}{2} + 2\left(-\frac{\gamma x}{2} - \frac{1}{2}\right) = -\gamma x - \left(l - \frac{1}{2}\right) \text{ for } \mathrm{k}_1 \quad \text{or} \quad \tau = -\gamma x + \left(l + \frac{3}{2}\right) \text{ for } \mathrm{k}_2 \tag{54}$$

From (33), (47) and (54) we obtain the same values of λ_n either for k_1 or k_2, that is

$$\lambda = \lambda_n = -n_r(-\gamma) = \gamma n_r \tag{55}$$

The energy eigenvalue obtained by equating equations (53) and (55) is given as

$$\frac{\varepsilon^2}{4} = \gamma\left(n_r - \frac{\left(l - \frac{1}{2}\right)}{2}\right) \text{ for } \mathrm{k}_1 \quad \text{or} \quad \frac{\varepsilon^2}{4} = \gamma\left(n_r + \frac{\left(l + \frac{3}{2}\right)}{2}\right) \text{ for } \mathrm{k}_2 \tag{56}$$

To have physical meaning, the choice of the values of k, π, τ, λ and λ_n are all values for $k = k_2$ in (52)–(56) therefore the energy spectrum of 3D HO plus Rosen-Morse non-central potential is obtained from (44) which is given as

$$\varepsilon^2 = 2\gamma\left(2n_r + \left(l + \frac{3}{2}\right)\right) \rightarrow E = \hbar\omega\left(2n_r + l + \frac{3}{2}\right) \tag{57}$$

where n_r is radial quantum number, l is orbital quantum number and its values depend on the parameters of Rosen-Morse non-central potential. The orbital quantum number obtained from the solution of angular Schrodinger equation is expressed as

$$l = l' = \sqrt{\left(\sqrt{v(v+1) + m^2} + n_l + \frac{1}{2}\right)^2 - \frac{\mu^2}{\left(\sqrt{v(v+1) + m^2} + n_l + \frac{1}{2}\right)^2}} - \frac{1}{2} \tag{58a}$$

We can see from (58a) that for fixed values of principal quantum number, n_l, and radial quantum number n_r, the values of $l = l'$ is not fixed since it depends on the Rosen-Morse's parameter. Since the values of $l' \geq 0$ then from (58a) we obtain the condition that

$$\left(\sqrt{v(v+1)+m^2}+n_l+\frac{1}{2}\right)^4-\left(\frac{\sqrt{v(v+1)+m^2}+n_l+\frac{1}{2}}{2}\right)^2 \geq \mu^2 \tag{58b}$$

The radial wave functions are calculated using (13), (16), (34), (35), (47), (52), and (54). The first part of the wave function obtained using (16), (47) and (52) is given as

$$\phi(r) = (x)^{\frac{(l'+1)}{2}} e^{-\frac{\gamma x}{2}} \tag{59}$$

and by using (35), (47), and (54) we get the weight function which is given as

$$\rho(x) = x^{l^i+\frac{1}{2}} e^{-\gamma x} \tag{60}$$

The second part of radial wave function obtained using (34) and (60) is given as

$$y_{n_r}(x) = \frac{c_{n_r}}{\rho(x)} \frac{d^{n_r}}{dx^{n_r}} (\sigma^{n_r}(x)\rho(x)) = \frac{c_{n_r}}{x^{l'+\frac{1}{2}} e^{-\gamma x}} \frac{d^{n_r}}{dx^{n_r}} (x^{l'+\frac{1}{2}+n} e^{-\gamma x}) \tag{61}$$

From (61) we get the un-normalized first four of the second part of radial wave functions given as

$$y_0(x) = 1 \tag{62a}$$

$$y_1(x) = C_1(l' + 1.5 - \gamma x) \tag{62b}$$

$$y_2(x) = C_2((l'+2.5)(l'+1.5) - (2l'+5)\gamma x + \gamma^2 x^2) \tag{62c}$$

$$\begin{aligned} y_3(x) = C_3((l'+3.5)(l'+2.5)(l'+1.5) - (3l'+10.5)(l'+2.5)\gamma x \\ + (3l'+10.5)(\gamma x)^2) - (\gamma x)^3 \end{aligned} \tag{62d}$$

The second part of radial wave function change into associated Laguerre polynomials by setting $\gamma x = z$ in (61), that is

$$y_{n_r}(x) = \frac{c_{n_r}(\gamma)^{l'+\frac{1}{2}}}{z^{l'+\frac{1}{2}} e^{-z}} \frac{d^{n_r}}{dz^{n_r}} \left(z^{l'+\frac{1}{2}+n_r} e^{-z}\right) = C_{n_r}(\gamma)^{l'+\frac{1}{2}} n_r! L_{n_r}^{l'+\frac{1}{2}}(z) \tag{63}$$

The un-normalized first four radial wave function obtained using (44), (59) and (62a)–(62d) are given as

$$R_0(x) = C_{n_r} r^{l'} e^{-\frac{\mu\omega r^2}{2\hbar}} \tag{65a}$$

$$R_1(x) = C_1 r^{l'} e^{-\frac{\gamma r^2}{2\hbar}} (l' + 1.5 - \gamma r^2) \tag{65b}$$

$$R_2 = C_1 r^{l'} e^{-\frac{\gamma r^2}{2}}((l'+2.5)(l'+1.5)-(2l'+5)\gamma r^2+\gamma^2 r^4) \tag{65c}$$

$$R_3 = C_1 r^{l'} e^{-\frac{\gamma r^2}{2}}((l'+3.5)(l'+2.5)(l'+1.5)-((3l'+10.5)(l'+2.5)\gamma r^2 + (3l'+10.5)\gamma^2 r^4 - \gamma^3 r^6) \tag{65d}$$

The effect of the presence of Rosen-Morse non-central potential to radial wave function is represented by the value of orbital momentum numbers, l', that are not always be integer but it always be positive number. The normalization factor B_n in (51) can be obtained from the normalization condition of radial wave function which is expressed as

$$\int_0^\infty \chi_{n_{r'}}(r)\chi_{n_r}(r)dr = \delta_{n_{r'}n_r} \tag{66a}$$

By inserting (51) into (54) we have

$$\int_0^\infty B_{n_{r'}} z^{\frac{(l'+1)}{2}} e^{-\frac{z}{2}} L_{n_{r'}}^{l'+\frac{1}{2}}(z) B_{n_r} z^{\frac{(l'+1)}{2}} e^{-\frac{z}{2}} L_{n_r}^{l'+\frac{1}{2}}(z) \frac{dz}{2\sqrt{\beta z}} = \delta_{n_{r'}n_r} \tag{66b}$$

The normalization condition for associated Laguerre polynomials is given as

$$\int_0^\infty z^{l'+\frac{1}{2}} e^{-z} L_{n_{r'}}^{l'+\frac{1}{2}}(z) L_{n_r}^{l'+\frac{1}{2}}(z)dz = \frac{\left(n_r+l'+\frac{1}{2}\right)!}{n_r!}\delta_{n_{r'}n_r} \tag{66c}$$

From (66a) and (66b) we get the normalization factor of radial wave function given as

$$B_{n_r} = \sqrt{\frac{2\sqrt{\gamma}n_r!}{\left(n_r+l'+\frac{1}{2}\right)!}} \tag{66d}$$

The radial wave function of 3D HO plus Rosen-Morse non-central potential is expressed as associated Laguerre polynomials with the values of orbital quantum numbers are trigonometric Rosen-Morse's parameters dependence. By the absent of Rosen-Morse potential the radial wave function becomes the radial wave function of 3D HO. The effect of the presence of Rosen-Morse potential: the $\csc^2\theta$ term causes the increase in wave amplitude and the wavelength as shown in Fig. 1, while the $\cot\theta$ term causes the decrease of the amplitude and the wavelength as shown in Fig. 2.

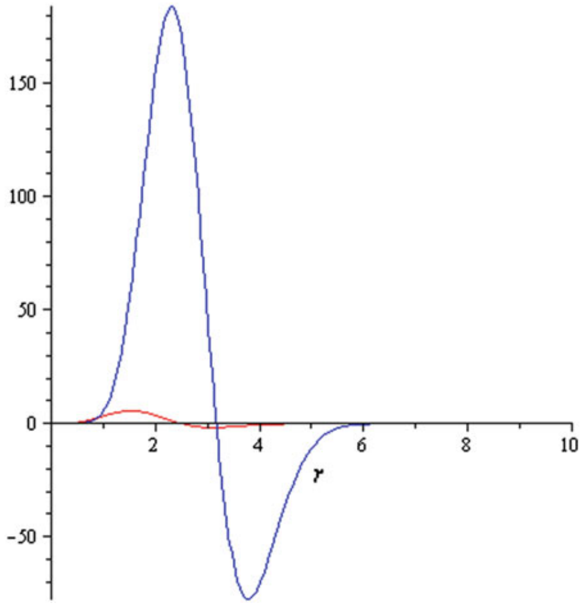

Fig. 1 The graph of 1st state of radial wave function for 3D HO-Rosen-Morse system for $\csc^2\theta$ term (red) $R_{n_l m v \mu n_r}(r) = R_{21201}(r) = C_1 r^{3.65} e^{-\frac{\gamma r^2}{2}} (6.15 - \gamma r^2)$ (blue) $R_{n_l m v \mu n_r}(r) = R_{21201}(r) = C_1 r^{3.65} e^{-\frac{\gamma r^2}{2}} (6.15 - \gamma r^2)$

4.1.2 The Solution of Angular Schrodinger Equation

The polar part of the Schrodinger equation expressed in (40) is rewritten as

$$\frac{\partial^2 P(\theta)}{\partial \theta^2} + \cot\theta \frac{\partial P(\theta)}{\partial \theta} - \left(\frac{v(v+1)+m^2}{\sin^2\theta} - 2\mu\cot\theta\right)P(\theta) + l(l+1)P(\theta) = 0 \tag{67}$$

By making a change of variable, $\cot\theta = is$, in (67) we have

$$\frac{\partial}{\partial\theta} = i(1-s^2)\frac{\partial}{\partial s} \text{ and } \frac{\partial^2}{\partial\theta^2} = i(1-s^2)\frac{\partial}{\partial s}\left\{i(1-s^2)\frac{\partial}{\partial s}\right\} \tag{68}$$

By inserting (68) into (67) we achieve

$$(1-s^2)\frac{\partial^2 P}{\partial s^2} - s\frac{\partial P}{\partial s} + \left\{m^2 + v(v+1) - \frac{2\mu is + l(l+1)}{1-s^2}\right\}P = 0 \tag{69}$$

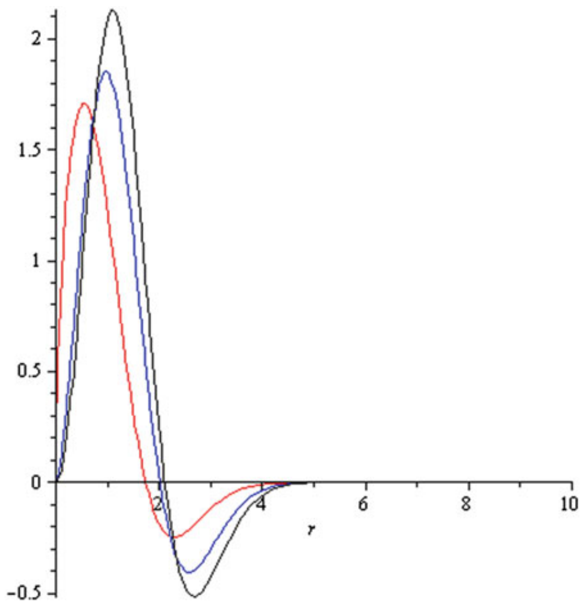

Fig. 2 The graph of 1st state of radial wave function for 3D HO-Rosen-Morse system for cot term: black line $R_{21061}(r) = C_1 r^{1.55} e^{-\frac{\gamma r^2}{2}} (4.05 - \gamma r^2)$, blue line $R_{n_l m v \mu n_r}(r) = R_{21021}(r) = C_1 r^{1.95} e^{-\frac{\gamma r^2}{2}} (4.45 - \gamma r^2)$, red line $R_{210101}(r) = C_1 r^{0.52} e^{-\frac{\gamma r^2}{2}} (3.02 - \gamma r^2)$

$$\sigma = 1 - s^2,\ \tilde{\tau} = -s,\ \tilde{\sigma} = \left\{m^2 + v(v+1)\right\}(1 - s^2) - \{2\mu i s + l(l+1)\} \tag{70}$$

By using (25) and (70) we get

$$\pi = -\frac{s}{2} \pm \sqrt{\frac{s^2}{4} - [m^2 + v(v+1) - l(l+1) - k] + 2\mu i s + [m^2 + v(v+1) - k] s^2} \tag{71a}$$

$$\pi = -\frac{s}{2} \pm \sqrt{\left[m^2 + (v + \frac{1}{2})^2 - k\right]} \left\{ s + \frac{2\mu i}{2\left\{m^2 + (v + \frac{1}{2})^2 - k\right\}} \right\} \tag{71b}$$

The value of k in (71a) is obtained from the condition that quadratic expression under the square root in (71a) has to be perfectly square of first degree of polynomial therefore (71a) reduces to (71b), and the discriminate of the quadratic expression under the square root in (71a) has to be zero given as

$$-\mu^2 + k^2 - k\left\{2m^2 + 2\left(v+\frac{1}{2}\right)^2 - \left(l+\frac{1}{2}\right)^2\right\} + \left\{m^2 + \left(v+\frac{1}{2}\right)^2 - \left(l+\frac{1}{2}\right)^2\right\}\left\{m^2 + \left(v+\frac{1}{2}\right)^2\right\} = 0 \tag{72}$$

The values of k obtained from (72)

$$k_1 = m^2 + \left(v+\frac{1}{2}\right)^2 - \frac{\left(l+\frac{1}{2}\right)^2 + \sqrt{\left(l+\frac{1}{2}\right)^4 + 4\mu^2}}{2} = m^2 + \left(v+\frac{1}{2}\right)^2 - p_1^2 \tag{73a}$$

$$k_2 = m^2 + \left(v+\frac{1}{2}\right)^2 - \frac{\left(l+\frac{1}{2}\right)^2 - \sqrt{\left(l+\frac{1}{2}\right)^4 + 4\mu^2}}{2} = m^2 + \left(v+\frac{1}{2}\right)^2 - p_2^2 \tag{73b}$$

with

$$p_1^2 = \frac{\left(l+\frac{1}{2}\right)^2 + \sqrt{\left(l+\frac{1}{2}\right)^4 + 4\mu^2}}{2} \text{ and } p_2^2 = \frac{\left(l+\frac{1}{2}\right)^2 - \sqrt{\left(l+\frac{1}{2}\right)^4 + 4\mu^2}}{2} \tag{74}$$

By inserting (73a, 73b) into (71b) we obtain the value of π that satisfies the condition $\tau' < 0$ given as

$$\pi = -\frac{s}{2} - p(s + \frac{\mu i}{p^2}) = -s\left(p+\frac{1}{2}\right) - \frac{\mu i}{p} \tag{75}$$

By inserting (70) and (75) into (17) we have

$$\tau = -s - s - 2p\left(s + \frac{\mu i}{p^2}\right) = -2s(1+p) - \frac{2\mu i}{p} \tag{76}$$

The eigen values of the system are obtained by manipulating (24), (33), (70), (73a, 73b), (74), and (76) as follows. By using (24), (73a, 73b) and (74) we have

$$\lambda = m^2 + \left(v+\frac{1}{2}\right)^2 - p^2 - (p + \frac{1}{2}) \tag{77}$$

and by using (33), (70), and (76) we obtain

$$\lambda_n = -n(-2p-2) + n(n-1) = 2np + n + n^2 \tag{78}$$

By equating equation (77) and (78) we have

$$m^2+\left(v+\frac{1}{2}\right)^2-p^2-(p+\frac{1}{2})=2np+n+n^2\rightarrow m^2+v(v+1)=(p+n+\frac{1}{2})^2 \tag{79}$$

By inserting (74) into (79) we get

$$\pm\sqrt{m^2+v(v+1)}-(n+\frac{1}{2})=p=\pm\sqrt{\frac{\left(l+\frac{1}{2}\right)^2\pm\sqrt{\left(l+\frac{1}{2}\right)^2+4\mu^2}}{2}} \tag{80}$$

To have physical meaning, from (80) we choose

$$-\sqrt{m^2+v(v+1)}-(n+\frac{1}{2})=p=-\sqrt{\frac{\left(l+\frac{1}{2}\right)^2+\sqrt{\left(l+\frac{1}{2}\right)^2+4\mu^2}}{2}}$$
$$\rightarrow\left\{\sqrt{m^2+v(v+1)}+(n+\frac{1}{2})\right\}^2=\frac{\left(l+\frac{1}{2}\right)^2+\sqrt{\left(l+\frac{1}{2}\right)^2+4\mu^2}}{2} \tag{81}$$

that gives

$$\left(l+\frac{1}{2}\right)=\sqrt{\left(\sqrt{m^2+v(v+1)}+n+\frac{1}{2}\right)^2-\frac{\mu^2}{\left(\sqrt{m^2+v(v+1)}+n+\frac{1}{2}\right)^2}} \tag{82}$$

Equation (82) shows that l, disturbed orbital momentum number, as a function of m, μ, v, and $n = n_l$ which is angular quantum number.

The first part of the polar wave function obtained from (16), (70) and (75) as follows:

$$\frac{\phi'}{\phi}=\frac{-s(p+\frac{1}{2})-\frac{\mu i}{p}}{1-s^2}=\frac{-s(p+\frac{1}{2})}{1-s^2}+\frac{-\frac{\mu i}{p}}{2(1-s)}+\frac{-\frac{\mu i}{p}}{2(1+s)} \tag{83}$$

$$\frac{d\phi}{\phi}=\frac{\frac{(p+\frac{1}{2})}{2}d(-s^2)}{1-s^2}+\frac{\frac{\mu i}{p}d(-s)}{2(1-s)}+\frac{-\frac{\mu i}{p}ds}{2(1+s)}\rightarrow\phi=(1-s)^{\frac{(p+\frac{1}{2})}{2}+\frac{\mu i}{2p}}(1+s)^{\frac{(p+\frac{1}{2})}{2}-\frac{\mu i}{2p}} \tag{84}$$

The weight function of the second part of wave function is obtained from (35), (70), and (76) given as

$$\frac{\partial(\sigma\rho)}{\partial r} = \tau(r)\rho(r) - 2s\rho + \left(1 - s^2\right)\rho' = \left(-2\boldsymbol{s}(\boldsymbol{p}+1) - \frac{2\boldsymbol{\mu i}}{\boldsymbol{p}}\right)\boldsymbol{\rho} \tag{85}$$

$$\frac{\rho'}{\rho} = \frac{-2sp - \frac{2\mu i}{p}}{1 - s^2} = \frac{-2sp}{1 - s^2} + \frac{-\frac{2\mu i}{p}}{2(1-s)} + \frac{-\frac{2\mu i}{p}}{2(1+s)} \rightarrow \rho = (1-s)^{p+\frac{\mu i}{p}}(1+s)^{p-\frac{\mu i}{p}} \tag{86}$$

The second angular part obtained using (34), (70), and (86) is given as

$$y_n(s) = \frac{B_n}{(1-s)^{p+\frac{\mu i}{p}}(1+s)^{p-\frac{\mu i}{p}}} \frac{d^n}{ds^n}\left((1-s)^{p+\frac{\mu i}{p}+n}(1+s)^{p-\frac{\mu i}{p}+n}\right) \tag{87}$$

The polar wave function obtained from (13), (84) and (87) is

$$P(s) = (1-s)^{\frac{(p+\frac{1}{2})}{2}+\frac{\mu i}{2p}}(1+s)^{\frac{(p+\frac{1}{2})}{2}-\frac{\mu i}{2p}} y_n(s) \tag{88}$$

The total wave function of the system obtained from (40), (44), and (88) is

$$\Psi(r,s,\varphi) = B_{n_r}\gamma^{\frac{l'}{2}} r^{l'} e^{-\frac{z}{2}} L_{n_r}^{l'+\frac{1}{2}}(\gamma r^2)(1-s)^{\frac{(p+\frac{1}{2})}{2}+\frac{\mu i}{2p}}(1+s)^{\frac{(p+\frac{1}{2})}{2}-\frac{\mu i}{2p}} y_n(s) e^{im\varphi} \tag{90}$$

with $s = -i\cot\theta$, and the energy spectrum is expressed in (57) and (58a).

4.2 Energy Spectrum and Wave Function of Eckart Plus Poschl-Teller Non-central Potential

The Schrodinger equation for for Eckart plus Poschl-Teller non-central potential is given as

$$\begin{aligned}
&-\frac{\hbar^2}{2M}\left\{\frac{1}{r^2}\frac{\partial}{\partial r}\left((r^2\frac{\partial}{\partial r}) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\frac{\partial}{\partial\theta}) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}\right)\Psi(r,\theta,\varphi)\right.\\
&+\frac{\hbar^2}{2Ma^2}\left(\frac{V_0 e^{\frac{-r}{a}}}{(1-e^{\frac{-r}{a}})^2} - \frac{V_1(1+e^{\frac{-r}{a}})}{(1-e^{\frac{-r}{a}})}\right)\Psi(r,\theta,\varphi)\}\\
&+\frac{\hbar^2}{2Mr^2}\left(\frac{\kappa(\kappa-1)}{\sin^2\theta} + \frac{\eta(\eta-1)}{\sin^2\theta}\right)\Psi(r,\theta,\varphi)\\
&= E\Psi(r,\theta,\varphi)
\end{aligned} \tag{91}$$

The three dimensional Schrodinger equation expressed in (12) is solved using variable separation method by setting $\psi(r,\theta,\varphi) = R(r)P(\theta)\phi(\varphi)$ so we get

$$\frac{1}{R}\frac{\partial}{\partial r}(r^2\frac{\partial R}{\partial r}) - \frac{r^2}{a^2}\left(\frac{V_0 e^{\frac{-r}{a}}}{(1-e^{\frac{-r}{a}})^2} - \frac{V_1(1+e^{\frac{-r}{a}})}{(1-e^{\frac{-r}{a}})}\right) + \frac{2MEr^2}{\hbar^2} = -\frac{1}{P\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\frac{\partial P}{\partial\theta}) - \frac{1}{\Phi\sin^2\theta}\frac{\partial^2\Phi}{\partial\varphi^2} + \frac{\kappa(\kappa-1)}{\sin^2\theta} + \frac{\eta(\eta-1)}{\sin^2\theta} = \lambda = l(l+1) \tag{92}$$

From (92) we get three differential equations with single variable as following:

$$\frac{1}{R}\frac{\partial}{\partial r}(r^2\frac{\partial R}{\partial r}) - \frac{r^2}{a^2}\left(\frac{V_0 e^{\frac{-r}{a}}}{(1-e^{\frac{-r}{a}})^2} - \frac{V_1(1+e^{\frac{-r}{a}})}{(1-e^{\frac{-r}{a}})}\right) + \frac{2MEr^2}{\hbar^2} = \lambda = l(l+1) \tag{93a}$$

$$-\frac{1}{P\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\frac{\partial P}{\partial\theta}) + \frac{m^2}{\sin^2\theta} + \frac{\kappa(\kappa-1)}{\sin^2\theta} + \frac{\eta(\eta-1)}{\sin^2\theta} = \lambda = l(l+1) \tag{93b}$$

and

$$\frac{1}{\Phi}\frac{\partial^2\Phi}{\partial\varphi^2} = -m^2 \rightarrow \Phi = A_m e^{im\varphi} \tag{94}$$

The radial part of Schrodinger equation is given as

By setting $\frac{2m}{\hbar^2}E = -\epsilon^2$, $R = \frac{\chi(r)}{r}$, applying an approximation for centrifugal term [14, 15], $\frac{1}{r^2} \cong \frac{1}{4a^2}\left(d_0 + \frac{1}{sinh^2\frac{r}{2a}}\right)$ for $\frac{r}{2a} <<< 1$ and $d_0 = \frac{1}{12}$ and changing the exponential term into hyperbolic function in (93a) we get

$$\frac{d^2\chi(r)}{dr^2} - \frac{1}{a^2}\left\{\frac{V_0 + l(l+1)}{4\sinh^2\frac{r}{2a}} - V_1\coth\frac{r}{2a} + \frac{l(l+1)d_0 + 4a^2\varepsilon^2}{4}\right\}\chi(r) = 0 \tag{95}$$

By making a coordinate transformation, $r = f(s) = 2acoth^{-1}(1-2s)$, in (95) we obtain

$$s(1-s)\frac{\partial^2\chi}{\partial s^2} + (1-2s)\frac{\partial\chi}{\partial s} + \left\{\frac{(V_0 + l(l+1))s(1-s)}{s(1-s)} + \frac{V_1(1-2s)}{s(1-s)} - \frac{\left(\frac{l(l+1)d_0}{4a^2} + \varepsilon^2\right)a^2}{s(1-s)}\right\}\chi = 0 \tag{96}$$

From (96) we get

$$\tilde{\tau} = 1 - 2s,\ \sigma = s(1 - s) \tag{97a}$$

$$\tilde{\sigma} = (V_0 + l(l+1))s(1 - s) + V_1(1 - 2s) - \left(\frac{l(l+1)d_0}{4a^2} + \varepsilon^2\right)a^2 \tag{97b}$$

Inserting (18) and (19) into (5) we have

$$\pi = \pm\sqrt{(V_0 + l(l+1) - k)s^2 - (V_0 + l(l+1) - 2V_1 - k)s - \left((V_1 - (\frac{l(l+1)d_0}{4a^2} + \varepsilon^2))a^2\right)} \tag{98}$$

Due to the condition that the expression under the square root of (98) must be square of first degree polynomial, then (98) is rewritten as

$$\pi = \pm\sqrt{(V_0 + l(l+1) - k)s^2}\left(s - \frac{V_0 + l(l+1) - 2V_1 - k}{2(V_0 + l(l+1) - k)}\right) \tag{99}$$

and the discriminate of the quadratic expression under the square root has to be zero, that is

$$(V_0 + l(l+1) - 2V_1 - k)^2 + 4(V_0 + l(l+1) - k)\left(V_1 - (\frac{l(l+1)d_0}{4a^2} + \varepsilon^2)a^2\right) = 0 \tag{100}$$

From (100) we get

$$k = (V_0 + l(l+1) - 2(\frac{l(l+1)d_0}{4a^2} + \varepsilon^2)a^2) \pm 2\sqrt{(\frac{l(l+1)d_0}{4a^2} + \varepsilon^2)^2 a^4 - V_1^2} = A - C \tag{101}$$

with

$$A = V_0 + l(l+1) \text{ and } C = 2(\frac{l(l+1)d_0}{4a^2} + \varepsilon^2)a^2) \mp 2\sqrt{(\frac{l(l+1)d_0}{4a^2} + \varepsilon^2)^2 a^4 - V_1^2} \tag{102}$$

By imposing $\tau' < 0$ we have

$$\pi = -\sqrt{C}\left(s - \frac{C - 2V_1}{2C}\right) \text{ and } \tau = 1 - 2s - 2\sqrt{C}\left(s - \frac{C - 2V_1}{2C}\right) \tag{103}$$

Using (6) and (7) together with the values of k, π, τ, and σ we get

$$\lambda = k + \pi' = A - C - \sqrt{C} \tag{104}$$

and

$$\lambda_n = -n\tau' - \frac{n(n-1)}{2}\sigma'' = -n(-2 - 2\sqrt{C}) + n(n-1) \tag{105}$$

By equating equations (104) and (105) we have

$$A - C - \sqrt{C} = n + n^2 + 2n\sqrt{C} \rightarrow V_0 + (l + \frac{1}{2})^2 = (n + \sqrt{C} + \frac{1}{2})^2 \tag{106a}$$

$$\begin{aligned}\sqrt{C} &= \pm\sqrt{V_0 + (l + \frac{1}{2})^2} - (n + \frac{1}{2}) \\ &= \pm\sqrt{2(\frac{l(l+1)d_0}{4a^2} + \varepsilon^2)a^2) \mp 2\sqrt{(\frac{l(l+1)d_0}{4a^2} + \varepsilon^2)^2 a^4 - V_1^2}}\end{aligned} \tag{106b}$$

The proper choice in (106b) that has physical meaning is

$$\begin{aligned}\sqrt{C} &= -\sqrt{V_0 + (l + \frac{1}{2})^2} - (n + \frac{1}{2}) \\ &= -\sqrt{2(\frac{l(l+1)d_0}{4a^2} + \varepsilon^2)a^2) + 2\sqrt{(\frac{l(l+1)d_0}{4a^2} + \varepsilon^2)^2 a^4 - V_1^2}}\end{aligned} \tag{107}$$

and the energy spectrum produced is

$$E = -\frac{\hbar^2}{2M}\left\{\frac{\left(\sqrt{V_0 + (l + \frac{1}{2})^2} + (n_r + \frac{1}{2})\right)^2}{4a^2} + \frac{V_1^2}{a^2\left(\sqrt{V_0 + (l + \frac{1}{2})^2} + (n_r + \frac{1}{2})\right)^2} - \frac{l(l+1)d_0}{4a^2}\right\} \tag{108}$$

The energy spectrum of Eckart potential with the absent of Poschl-Teller potential is produced from (108).

The first part of the wave function is obtained by using (97a) and (103)

$$\frac{\phi'}{\phi}=\frac{\pi}{\sigma}=\frac{-\sqrt{C}\left(s-\frac{C-2V_1}{2C}\right)}{s(1-s)}=\frac{-\sqrt{C}}{1-s}+\frac{C-2V_1}{s2\sqrt{C}}+\frac{C-2V_1}{(1-s)2\sqrt{C}}\rightarrow\phi(s)$$
$$=s^{\frac{C-2V_1}{2\sqrt{C}}}(1-s)^{\frac{C+2V_1}{2\sqrt{C}}} \tag{109}$$

The weight function of the radial part of the system is obtained using (35), (97a), and (103)

$$\frac{\partial(\sigma\rho)}{\partial s}=\tau(s)\rho(s)\rightarrow(1-2s)\rho+s(1-s)\rho'$$
$$=\left\{1-2s-2\sqrt{C}\left(s-\frac{C-2V_1}{2C}\right)\right\}\rho$$

that gives

$$\rho(s)=s^{\frac{C-2V_1}{\sqrt{C}}}(1-s)^{\frac{C+2V_1}{\sqrt{C}}} \tag{110}$$

The second part of the wave function is derived from Rodrigues relation expressed in (34). By inserting (97a) and (110) into (34) and by setting $\sqrt{C}=p$, C is (107) we get

$$y_n(s)=\frac{B_n}{s^{p-\frac{2V_1}{p}}(1-s)^{p+\frac{2V_1}{p}}}\frac{d^n}{ds^n}\left\{s^{p-\frac{2V_1}{p}+n}(1-s)^{p+\frac{2V_1}{p}+n}\right\} \tag{111}$$

We finally obtain the complete wave functions from (16), (109) and (111) and with $\coth(r/2a)=1-2s$ as

$$\chi(s)=s^{\frac{C-2V_1}{2\sqrt{C}}}(1-s)^{\frac{C+2V_1}{2\sqrt{C}}}y_n(s), \tag{112}$$

4.2.1 The Polar Schrodinger Equation Solution

By making a variable transformation $cos2\theta=s$ in (93b) we get

$$(1-s^2)\frac{\partial^2P}{\partial s^2}-\left(\frac{1}{2}+\frac{3}{2}s\right)\frac{\partial P}{\partial s}$$
$$-\left(\frac{2[\kappa(\kappa-1)+m^2](1+s)}{4(1-s^2)}+\frac{2\eta(\eta-1)(1+s)}{4(1-s^2)}-\frac{l(l+1)(1-s^2)}{4(1-s^2)}\right)P$$
$$=0 \tag{114}$$

The form of (114) is similar to the (12). The orbital momentum number and the polar wave function are achieved from (114) by applying (12), (13), (16), (17), (22), (24), (25), (33)–(35), with the solution steps similar to steps in Sect. 4.1.2 or 4.2.1. From (114) we have

$$\sigma = 1 - s^2 \; \tilde{\tau} = -\left(\frac{1}{2} + \frac{3}{2}s\right) \tag{115a}$$

$$\tilde{\sigma} = -\left\{\frac{[2(\kappa(\kappa-1)+m^2)+2\eta(\eta-1)-l(l+1)]}{4} + \frac{[2(\kappa(\kappa-1)+m^2)-2\eta(\eta-1)]s}{4} + \frac{l(l+1)s^2}{4}\right\} \tag{115b}$$

By imposing that $\tau' < 0$ and applying the condition of (25), then from (25) and (115a, 115b) we obtain

$$\pi = \frac{1-s}{4} - \sqrt{\left(\frac{l(l+1)}{4} - k + \frac{1}{16}\right)}\left(s + \frac{\frac{[2(\kappa(\kappa-1)+m^2)-2\eta(\eta-1)}{4} - \frac{1}{8}}{2\left(\frac{l(l+1)}{4} - k + \frac{1}{16}\right)}\right) \tag{116}$$

and

$$\begin{aligned}&\{\frac{2(\kappa(\kappa-1)+m^2)-2\eta(\eta-1)}{4} - \frac{1}{8}\}^2 \\ &\quad - 4\left\{\frac{l(l+1)}{4} - k + \frac{1}{16}\right\}\left\{\frac{[2(\kappa(\kappa-1)+m^2)+2\eta(\eta-1)-l(l+1)]}{4} + \frac{1}{16}\right\} \\ &\quad = 0\end{aligned} \tag{117}$$

The value of k obtained from (117) is

$$\begin{aligned}k &= \left(\frac{l+1/2}{2}\right)^2 - \left(\frac{\sqrt{\kappa(\kappa-1)+m^2} \pm (\eta-1/2)}{4}\right)^2 \\ &= \left(\frac{l+1/2}{2}\right)^2 - \frac{\left(\sqrt{o} \pm \sqrt{t}\right)^2}{2}\end{aligned} \tag{118}$$

with

$$o = \frac{2(\kappa(\kappa-1)+m^2)}{4} \text{ and } t = \frac{2\eta(\eta-1)}{4} + \frac{1}{8} = \frac{(\eta-1/2)^2}{2} \tag{119}$$

By inserting (118) into (116) we have

$$\pi = -s\left(\frac{\sqrt{o} \pm \sqrt{t}}{\sqrt{2}} + \frac{1}{4}\right) - \frac{\sqrt{o} \mp \sqrt{t}}{\sqrt{2}} + \frac{1}{4} \tag{120}$$

By using (17), (115a) and (120) we get

$$\tau = -2s\left(\frac{\sqrt{o} \pm \sqrt{t}}{\sqrt{2}} + 1\right) - 2\frac{\sqrt{o} \mp \sqrt{t}}{\sqrt{2}} \tag{121}$$

By using (24), (33), (115a), (118), (120), and (121) we obtain

$$\lambda = \frac{(l + 1/2)^2}{4} - \left(\frac{\sqrt{o} \pm \sqrt{t}}{\sqrt{2}}\right)^2 - \left(\frac{\sqrt{o} \pm \sqrt{t}}{\sqrt{2}} + \frac{1}{4}\right) \tag{122}$$

and

$$\lambda_n = 2n\left(\frac{\sqrt{o} \pm \sqrt{t}}{\sqrt{2}} + 1\right) + n(n+1) \tag{123}$$

By equating (122) and (123) and also together with (119) we get l and the proper choice of l given as

$$l = \sqrt{\kappa(\kappa - 1) + m^2} + (\eta - 1/2) + 2n + 1 - 1/2 = \sqrt{\kappa(\kappa - 1) + m^2} + \eta + 2n_l \tag{124}$$

The values of π and τ corresponding to the proper choice of l are

$$\pi = -s\left(\frac{\sqrt{o} + \sqrt{t}}{\sqrt{2}} + \frac{1}{4}\right) - \frac{\sqrt{o} - \sqrt{t}}{\sqrt{2}} + \frac{1}{4} \tag{120a}$$

$$\tau = -2s\left(\frac{\sqrt{o} + \sqrt{t}}{\sqrt{2}} + 1\right) - 2\frac{\sqrt{o} - \sqrt{t}}{\sqrt{2}} \tag{121a}$$

The first part of the wave function obtained by using (16), (115a) and (120) is

$$\phi(s) = (1 - s)^{\sqrt{\frac{o}{2}}}(1 + s)^{\sqrt{\frac{t}{2} + \frac{1}{4}}} \tag{125}$$

The weight function for the second part of the wave function obtained by using (35) (115a) and (121a) is

$$\rho(s) = (1 - s)^{2\sqrt{\frac{\varrho}{2}}}(1 + s)^{2\sqrt{\frac{t}{2}}} \tag{126}$$

By using (34), (115a) and (126) we obtain the second part of the polar wave function which is expressed in term of Jacobi polynomials given as

$$y_n(s) = \frac{B_n}{(1 - s)^{2\sqrt{\frac{\varrho}{2}}}(1 + s)^{2\sqrt{\frac{t}{2}}}}\frac{d^n}{ds^n}\left\{(1 - s)^{2\sqrt{\frac{\varrho}{2}}+n}(1 + s)^{2\sqrt{\frac{t}{2}}+n}\right\} \tag{127}$$

The total polar wave function achieved from (13), (125) and (127) is given as

$$\Psi(s) = B_n(1 - \cos 2\theta)^{-\sqrt{\frac{\varrho}{2}}} (1 + \cos 2\theta)^{-\sqrt{\frac{t}{2}+\frac{1}{4}}}\frac{d^n}{d(\cos 2\theta)^n}\left\{(1 - \cos 2\theta)^{2\sqrt{\frac{\varrho}{2}}+n}(1 + \cos 2\theta)^{2\sqrt{\frac{t}{2}}+n}\right\} \tag{128}$$

The total wave function of Eckart plus Poschl-Teller non-central potential found from (112) and (128) and the corresponding energy spectrum is expressed in (108).

The NU method is method developed based on hypergeometric differential equation but the application is wider since NU method is also applicable for problems that usually solved by confluent hypergeometric differential equation as for 3D harmonics oscillation.

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