A Multiscale Model of Stokes–Cahn–Hilliard Equations in a Porous Medium: Modeling, Analysis and Homogenization

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Abstract We consider a phase-field model for a mixture of two immiscible, incompressible porous media flow including surface tension effects. At micro-scale, the model comprises a strongly coupled system of Stokes–Cahn–Hilliard equations. An evolving diffuse interface having finite width independent of the scale parameter ε is separating the fluids in the considered model. In order to investigate the well-posedness of system at micro-scale, we first derived some a-priori estimates. With the help of two-scale convergence and unfolding operator technique we rigorously derived the homogenized equations for the microscopic model. For our purpose, we have used extensions theorems and well-known theories available in the literature beforehand.

Keywords Phase-field model · Porous media flow · Stokes equations · Cahn–Hilliard equations · Existence of solution · Homogenization · Asymptotic expansion method · Two-scale convergence · Periodic unfolding

1 Introduction

We study a binary-fluid model where the considered fluids are incompressible and immiscible. The domain $U \subset \mathbb{R}^n$, $n = 2, 3$ is occupied by the binary-fluid mixture. On the time interval $S = (0, T)$, the model comprises a system of steady Stokes–Cahn–Hilliard equations

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$$-\mu\Delta\mathbf{u} + \nabla p = \lambda w \nabla c \quad \text{in } (0,T)\times U, \tag{1.1a}$$
$$\nabla.\mathbf{u} = 0 \quad \text{in } (0,T)\times U, \tag{1.1b}$$
$$\partial_t c + \mathbf{u}.\nabla c = \Delta w \quad \text{in } (0,T)\times U, \tag{1.1c}$$
$$w = -\Delta c + f(c) \quad \text{in } (0,T)\times U, \tag{1.1d}$$

where $\mathbf{u}$ and w are the unknown velocity and chemical potential, respectively. μ is the viscosity and λ is the interfacial width parameter. Here c represents microscopic concentration of one of the fluids with values lying in the interval $[-1, 1]$ in the considered domain and $(-1, 1)$ within the thin diffused interface of uniform width proportional to λ. The term $f(c) = F'(c)$, where F is a homogeneous free energy functional that penalizes the deviation from the physical constraint $|c| \leq 1$. In our work, we consider F to be a quadratic double-well free energy functional, i.e., $F(s) = \frac{1}{4}(s^2 - 1)^2$. One can choose F as a logarithmic or a non-smooth (obstacle) free energy functional, cf. [3, 4]. The nonlinear term $c\nabla w$ in (1.1a) models the surface tension effects, and the advection effect is modeled by the term $\mathbf{u} \cdot \nabla c$ in (1.1c). The system (1.1a)-(1.1d) represent the steady Stokes equations for incompressible fluid and Cahn–Hilliard equations, respectively.

1.1 The Model

We consider U as a bounded domain with a sufficiently smooth boundary ∂U in $\mathbb{R}^n$, $n = 2, 3$, $S := (0, T)$ denotes the time interval for any $T > 0$, and the unit reference cell $Y := (0, 1)^n \subset \mathbb{R}^n$. Y_p and Y_s represent the pore and solid part of Y, respectively, which are mutually distinct, i.e., $Y_s \cap Y_p = \emptyset$, also $Y = Y_p \cup Y_s$. The solid boundary of Y is denoted as $\Gamma_s = \partial Y_s$, see Fig. 1. The domain U is assumed to be periodic and is covered by a finite union of the cells Y. In order to avoid technical difficulties, we postulate that: solid parts do not touch the boundary ∂U, solid parts do not touch each other and solid parts do not touch the boundary of Y. Let $\varepsilon > 0$

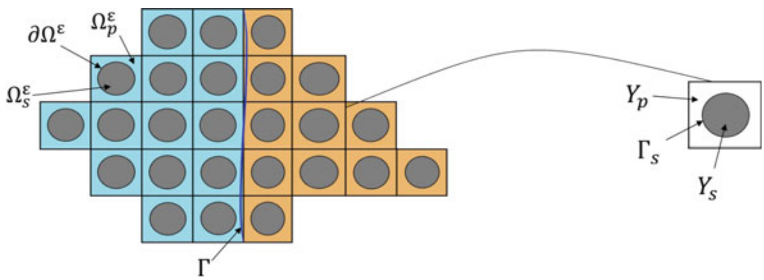

Fig. 1 (left) Porous medium $U = U_p^\varepsilon \cup U_s^\varepsilon$ as a periodic covering of the reference cell $Y = Y_p \cup Y_s$ (right). The blue interface Γ is the macroscopic interface between two fluids occupying the pore space U_p^ε

be the scale parameter. We define the pore space $U_p^\varepsilon := \bigcup_{\mathbf{k}\in\mathbb{Z}^n} Y_{p_k} \cap U$, the solid part as $U_s^\varepsilon := \bigcup_{\mathbf{k}\in\mathbb{Z}^n} Y_{s_k} \cap U = U \backslash U_p^\varepsilon$ and $\Gamma^\varepsilon := \bigcup_{\mathbf{k}\in\mathbb{Z}^n} \Gamma_{s_k}$, where $Y_{p_k} := \varepsilon Y_p + k$, $Y_{s_k} := \varepsilon Y_s + k$ and $\Gamma_{s_k} = \bar{Y}_{p_k} \cap \bar{Y}_{s_k}$.

Let $\chi(y)$ be the Y-periodic characteristic function of Y_p defined by

$$\chi(y) = \begin{cases} 1 & y \in Y^p, \\ 0 & y \in Y - Y^p. \end{cases} \tag{1.2}$$

We assume that U_p^ε is connected and has a smooth boundary. We consider the situation where the pore part U_p^ε is occupied by the mixture of two immiscible fluids separated by an evolving macroscopic interface $\Gamma : [0, T] \to U$ represented by the blue part in Fig. 1, and includes the effects of surface tension on the motion of the interface. We model the flow of the fluid mixture on the pore-scale using a phase-field approach motivated by the Stokes–Cahn–Hilliard system (1.1) in [2]. The velocity of the fluid mixture is assumed to be $\mathbf{u}^\varepsilon = \mathbf{u}^\varepsilon(t, x), (t, x) \in S \times U_p^\varepsilon$ which satisfies the stationary Stokes equation. The order parameter c^ε plays the role of microscopic concentration and the chemical potential w^ε satisfies the Cahn–Hilliard equation. p^ε is the fluid pressure. The term $\lambda c^\varepsilon \nabla w^\varepsilon$ models the surface tension forces which acts on the macroscopic interface between the fluids. Fluid density is taken to be 1. Then, the Stokes–Cahn–Hilliard system of equations is given by

$$-\mu\varepsilon^2 \Delta \mathbf{u}^\varepsilon + \nabla p^\varepsilon = -\lambda c^\varepsilon \nabla w^\varepsilon \qquad S \times U_p^\varepsilon, \tag{1.3a}$$

$$\nabla . \mathbf{u}^\varepsilon = 0 \qquad S \times U_p^\varepsilon, \tag{1.3b}$$

$$\mathbf{u}^\varepsilon = 0 \qquad S \times \partial U_p^\varepsilon, \tag{1.3c}$$

$$\partial_t c^\varepsilon + \varepsilon \mathbf{u}^\varepsilon . \nabla c^\varepsilon = \Delta w^\varepsilon \qquad S \times U_p^\epsilon, \tag{1.3d}$$

$$w^\varepsilon = -\varepsilon^2 \Delta c^\varepsilon + f(c^\varepsilon) \qquad S \times U_p^\epsilon, \tag{1.3e}$$

$$\partial_n c^\varepsilon = 0 \qquad S \times \partial U_p^\varepsilon, \tag{1.3f}$$

$$\partial_n w^\varepsilon = 0 \qquad S \times \partial U_p^\varepsilon, \tag{1.3g}$$

$$c^\varepsilon(0, x) = c_0(x) \qquad U_p^\varepsilon, \tag{1.3h}$$

where $\frac{\partial c^\varepsilon}{\partial \mathbf{n}} = \partial_n c^\varepsilon$ and $f(s) = s^3 - s = F'(s) = \frac{1}{4}(s^2 - 1)^2$ is the double-well free energy. The above scaling for the viscosity is such that the velocity $\mathbf{u}^\varepsilon$ has a nontrivial limit as ε goes to zero. Also, $0 \le \alpha, \beta, \gamma \le 2$ where $\alpha, \beta, \gamma \in \mathbb{R}$. We denote (1.3a)–(1.3h) by $(\mathcal{P}^\varepsilon)$.

2 Preliminaries and Notation

Let $\theta \in [0, 1]$ and $1 \le r, s \le \infty$ be such that $\frac{1}{r} + \frac{1}{s} = 1$. Assume that $\Xi \in \{U, U_p^\varepsilon, U_s^\varepsilon\}$ and $l \in \mathbb{N}_0$, then as usual $L^r(\Xi)$ and $H^{l,r}(\Xi)$ denote the Lebesgue and Sobolev spaces with their usual norms and they are denoted by $||.||_r$ and $||.||_{l,r}$, cf. [5]. The extension and restriction operators are denoted by E and R, respectively. The symbol $(.,.)_H$ represents the *inner product* on a *Hilbert space* H and $||.||_H$ denotes the corresponding norm. For a Banach space X, X^* denotes its dual and the duality pairing is denoted by $\langle . \,, \, . \rangle_{X^* \times X}$. By classical trace theorem on *Sobolev space* $H_0^{1,2}(\Xi)^* = H^{-1,2}(\Xi)$. The symbols $\hookrightarrow$, $\hookrightarrow\hookrightarrow$ and $\underset{\hookrightarrow}{d}$ denote the continuous, compact, and dense embeddings, respectively.

We define the function spaces:

$$\begin{gathered}\mathbf{H}^1(U) = H^1(U)^n, \quad \mathbf{H}_0^1(U) = H_0^1(U)^n, \\ \mathfrak{U}^\varepsilon := \mathbf{H}_{div}^1(U) = \{\eta : \eta \in \mathbf{H}_0^1(U), \nabla \cdot \eta = 0\}, \\ \mathfrak{C}^\varepsilon = \{c^\varepsilon : c^\varepsilon \in L^\infty(S; H^1(U_p^\varepsilon)), \partial_t c^\varepsilon \in L^2(S; H^1(U_p^\varepsilon)^*)\}, \\ \mathfrak{W}^\varepsilon = L^2(S; H^1(U_p^\varepsilon)) \text{ and } L_0^2(U) = \{\phi \in L^2(U) : \textstyle\int_U \phi \, dx = 0.\}.\end{gathered}$$

We choose $\mathbf{u}^\varepsilon \in \mathfrak{U}^\varepsilon$, $c^\varepsilon \in \mathfrak{C}^\varepsilon$, $w^\varepsilon \in \mathfrak{W}^\varepsilon$ and $p^\varepsilon \in L^2(S \times U_p^\varepsilon)$. We will now state few results and lemmas which are used in this paper and proofs of these can be found in literature.

Lemma 1 *Let E be a Banach space and E_0 and E_1 be reflexive spaces with $E_0 \subset E \subset E_1$. Suppose further that $E_0 \hookrightarrow\hookrightarrow E \hookrightarrow E_1$. For $1 < p, q < \infty$ and $0 < T < 1$ define $X := \{u \in L^p(S; E_0) : \partial_t u \in L^q(S; E_1)\}$. Then $X \hookrightarrow\hookrightarrow L^p(S; E)$.*

Lemma 2 (Restriction theorem) *There exists a linear restriction operator $R^\varepsilon : L^2(S; H_0^1(U))^d \longrightarrow L^2(S; H_0^1(U_p^\varepsilon))^d$ such that $R^\varepsilon u(x) = u(x)|_{U_p^\varepsilon}$ for $u \in L^2(S; H_0^1(U))^d$ and $\nabla \cdot R^\varepsilon u = 0$ if $\nabla \cdot R^\varepsilon u = 0$ if $\nabla \cdot u = 0$. Furthermore, the restriction satisfies the following bound*

$$||R^\varepsilon u||_{L^2(S \times U_p^\varepsilon)} + \varepsilon ||\nabla R^\varepsilon u||_{L^2(S \times U_p^\varepsilon)} \le C(||u||_{L^2(S \times U)} + \varepsilon ||\nabla u||_{L^2(S \times U)}),$$

where C is independent of ε.

Similarly, one can define the extension operator from $S \times U_p^\varepsilon$ to $S \times U$, cf. [1, 8].

Definition 1 (*Two-scale convergence*) A sequence of functions $(u^\varepsilon)_{\varepsilon>0}$ in $L^p(S \times U)$ is said to be two-scale convergent to a limit $u \in L^p(S \times U \times Y)$ if

$$\lim_{\epsilon \to 0} \int_{S \times U} u^\varepsilon(t, x) \phi\left(t, x, \frac{x}{\varepsilon}\right) dx\, dt = \int_{S \times U \times Y} u(t, x, y) \phi(t, x, y) \, dx\, dt\, dy$$

for all $\phi \in L^q(S \times U; C_\#(Y))$.

Lemma 3 *For $\varepsilon > 0$, let $(u^\varepsilon)_{\varepsilon>0}$ be a sequence of functions, then the following holds:*

(i) for every bounded sequence $(u^\varepsilon)_{\varepsilon>0}$ in $L^p(S \times U)$ there exists a subsequence $(u^\varepsilon)_{\varepsilon>0}$ (still denoted by same symbol) and an $u \in L^p(S \times U \times Y)$ such that $u^\varepsilon \overset{2}{\rightharpoonup} u$.

(ii) *let* $u^\varepsilon \to u$ *in* $L^p(S \times U)$, *then* $u^\varepsilon \stackrel{2}{\rightharpoonup} u$.

(iii) *let* $(u^\varepsilon)_{\varepsilon>0}$ *be a sequence in* $L^p(S; H^{1,p}(U))$ *such that* $u^\varepsilon \stackrel{w}{\rightharpoonup} u$ *in* $L^p(S; H^{1,p}(U))$. *Then* $u^\varepsilon \stackrel{2}{\rightharpoonup} u$ *and there exists a subsequence* $u^\varepsilon_{\varepsilon>0}$, *still denoted by same symbol, and an* $u_1 \in L^p(S \times U; H^{1,p}_\#(Y))$ *such that* $\nabla_x u^\varepsilon \stackrel{2}{\rightharpoonup} \nabla_x u + \nabla_y u_1$.

(iv) *let* $(u^\varepsilon)_{\varepsilon>0}$ *be a bounded sequence of functions in* $L^p(S \times U)$ *such that* $\varepsilon \nabla u^\varepsilon$ *is bounded in* $L^p(S \times U)^n$. *Then there exist a function* $u \in L^p(S \times U; H^{1,p}_\#(Y))$ *such that* $u^\varepsilon \stackrel{2}{\rightharpoonup} u$, $\varepsilon \nabla_x u^\varepsilon \stackrel{2}{\rightharpoonup} \nabla_y u$.

Definition 2 (*Periodic Unfolding*) Assume that $1 \le r \le \infty$. Let $u^\varepsilon \in L^r(S \times U)$ such that for every t, $u^\varepsilon(t)$ is extended by zero outside of U. We define the unfolding operator $T^\varepsilon : L^r(S \times U) \to L^r(S \times U \times Y)$ as

$$T^\varepsilon u^\varepsilon(t, x, y) = u^\varepsilon\left(t, \varepsilon\left[\frac{x}{\varepsilon}\right] + \varepsilon y\right) \quad \text{for a.e. } (t, x, y) \in S \times U \times Y, \tag{2.1a}$$

$$= 0 \quad \text{otherwise.} \tag{2.1b}$$

For the following definitions and results, interested reader can refer to [7] and references therein.

Definition 3 Assume that $1 \le r \le \infty$, $u^\varepsilon \in L^r(S \times U)$ and T^ε is defined as in Definition 3. Then we say that:
(i) u^ε is weakly two-scale convergent to a limit $u_0 \in L^r(S \times U \times Y)$ if $T^\varepsilon u^\varepsilon$ converges weakly to u_0 in $L^r(S \times U \times Y)$.
(ii) u^ε is strongly two-scale convergent to a limit $u_0 \in L^r(S \times U \times Y)$ if $T^\varepsilon u^\varepsilon$ converges strongly to u_0 in $L^r(S \times U \times Y)$.

Lemma 4 *Let* $(u^\varepsilon)_{\varepsilon>0}$ *be a bounded sequence in* $L^r(S \times U)$. *Then the following statements hold:*

(a) *if* $u^\varepsilon \stackrel{2}{\rightharpoonup} u$, *then* $T^\varepsilon u^\varepsilon \stackrel{w}{\rightharpoonup} u$, *i.e.,* u^ε *is weakly two-scale convergent to a* u.

(b) *if* $u^\varepsilon \to u$, *then* $T^\varepsilon u^\varepsilon \to u$, *i.e.,* u^ε *is strongly two-scale convergent to* u.

Lemma 5 *Let* $(u^\varepsilon)_{\varepsilon>0}$ *be strongly two-scale convergent to* u_0 *in* $L^r(S \times U \times \Gamma)$ *and* $(v^\varepsilon)_{\varepsilon>0}$ *be weakly two-scale convergent to* v_0 *in* $L^s(S \times U \times \Gamma)$. *If the exponents* $r, s, \nu \ge 1$ *satisfy* $\frac{1}{r} + \frac{1}{s} = \frac{1}{\nu}$, *then the product* $(u^\varepsilon v^\varepsilon)_{\varepsilon>0}$ *two-scale converges to the limit* $u_0 v_0$ *in* $L^\nu(S \times U \times Y)$. *In particular, for any* $\phi \in L^\mu(S \times U)$ *with* $\mu \in (1, \infty)$ *such that* $\frac{1}{\nu} + \frac{1}{\mu} = 1$ *we have*

$$\int_{S \times U} u^\varepsilon(t, x) v^\varepsilon(t, x) \phi(t, x)\, dx\, dt \stackrel{\varepsilon \to 0}{\longrightarrow} \int_{S \times U \times Y} u_0(t, x, y) v_0(t, x, y) \phi(t, x)\, dx\, dy\, dt.$$

Before we proceed with the weak formulation, we make the following assumptions for the sake of analysis of $(\mathcal{P}^\varepsilon)$.

A1. for all $x \in U$, $\mathbf{u_0}$, c_0 and $w_0 \geq 0$.
A2. $\mathbf{u_0} \in L^\infty(U) \cap H^1(U)$, $c_0 \in L^\infty(U) \cap H^1(U)$ and $w^0 \in L^\infty(U) \cap H^1(U)$ such that $\sup_{\varepsilon>0} ||\mathbf{u_0}||_{L^\infty(U)\cap H^1(U)} < \infty$, $\sup_{\varepsilon>0} ||c_0||_{L^\infty(U)\cap H^1(U)} < \infty$, $\sup_{\varepsilon>0} ||w_0||_{L^\infty(U)\cap H^1(U)} < \infty$.
A3. $p^\varepsilon \in L^2(S; H^1(U_p^\varepsilon))$ such that $\sup_{\varepsilon>0} ||p^\varepsilon||_{L^2(S;H^1(U_p^\varepsilon))} < \infty$.

2.1 Weak Formulation of ($\mathcal{P}^\varepsilon$)

Let the assumptions A1–A4 be satisfied. A triple $(\mathbf{u}^\varepsilon, c^\varepsilon, w^\varepsilon) \in \mathfrak{U}^\varepsilon \times \mathfrak{C}^\varepsilon \times \mathfrak{W}^\varepsilon$ is said to be the weak solution of the model $(\mathcal{P}^\varepsilon)$ such that $(\mathbf{u}^\varepsilon, c^\varepsilon, w^\varepsilon)(0, x) = (\mathbf{u}_0, c_0, w_0)(x)$ for all $x \in U$, and

$$\mu\varepsilon^2 \int_{S\times U_p^\varepsilon} \nabla\mathbf{u}^\varepsilon : \nabla\eta \, dx\, dt = -\lambda \int_{S\times U_p^\varepsilon} c^\varepsilon \nabla w^\varepsilon \cdot \eta \, dx\, dt, \tag{2.2a}$$

$$\int_S \langle \partial_t c^\varepsilon, \phi\rangle \, dt - \varepsilon \int_{S\times U_p^\varepsilon} c^\varepsilon \mathbf{u}^\varepsilon \cdot \nabla\phi \, dx\, dt + \int_{S\times U_p^\varepsilon} \nabla w^\varepsilon \cdot \nabla\phi \, dx\, dt = 0, \tag{2.2b}$$

$$\int_{S\times U_p^\varepsilon} w^\varepsilon \psi \, dx\, dt = \varepsilon^2 \int_{S\times U_p^\varepsilon} \nabla c^\varepsilon \cdot \nabla\psi \, dx\, dt + \int_S \langle f(c^\varepsilon), \psi\rangle \, dx\, dt, \tag{2.2c}$$

for all $\eta \in L^2(S; \mathbf{H}^1_{div}(U_p^\varepsilon))$ and $\phi, \psi \in L^2(S; H^1(U_p^\varepsilon))$.

We are now going to state the two main theorems of this paper which are given below.

Theorem 1 *Let the assumptions A1–A4 be satisfied, then there exists a unique positive weak solution* $(\mathbf{u}^\varepsilon, c^\varepsilon, w^\varepsilon) \in \mathfrak{U}^\varepsilon \times \mathfrak{C}^\varepsilon \times \mathfrak{W}^\varepsilon$ *of the problem* $(\mathcal{P}^\varepsilon)$ *which satisfies*

$$\begin{aligned} &||\mathbf{u}^\varepsilon||_{L^4(U_p^\varepsilon)} + \sqrt{\mu}\varepsilon||\nabla\mathbf{u}^\varepsilon||_{L^2(S\times U_p^\varepsilon)} + ||w^\varepsilon||_{L^2(S\times U_p^\varepsilon)} + \sqrt{\varepsilon\lambda}||\nabla w^\varepsilon||_{L^2(S\times U_p^\varepsilon)} \\ &\quad + ||c^\varepsilon||_{L^\infty(S;L^4(U_p^\varepsilon))} + \sqrt{\frac{\lambda}{2}}||\nabla c^\varepsilon||_{L^\infty(S);L^2(U_p^\varepsilon))} + ||\partial_t c^\varepsilon||_{L^2(S;H^1(U_p^\varepsilon)^*)} \\ &\quad \leq C < \infty \quad \forall\varepsilon, \end{aligned} \tag{2.3}$$

where the constant C is independent of ε.

Theorem 2 (Upscaled Problem $(\mathcal{P})$) *There exists* $(\mathbf{u}, c, w) \in \mathfrak{U} \times \mathfrak{C} \times \mathfrak{W}$ *which satisfies*

$$-\mu\Delta_y \mathbf{u} + \nabla_y p_1(x,y) + \nabla_x p(x) = -\lambda c\left(\nabla_x w(x) + \nabla_y w_1(x,y)\right), \quad S\times U\times Y_p, \tag{2.4a}$$

$$\nabla_y \cdot \mathbf{u}(x,y) = 0, \quad S\times U\times Y_p, \tag{2.4b}$$

$$\nabla_x \cdot \bar{\mathbf{u}}(x) = 0, \quad S\times U, \tag{2.4c}$$

$$\mathbf{u}(x,y) = 0, \quad S\times U\times \Gamma_s, \tag{2.4d}$$

$$\partial_t c(x,y) + \nabla_y \cdot c(x,y)\mathbf{u}(x,y) = \Delta_x w(x) + \nabla_x \cdot \nabla_y w_1(x,y), \quad S\times U\times Y_p, \tag{2.4e}$$

$$w(x,y) = -\Delta_y c(x,y) + f(c(x,y)), \quad S\times U\times Y_p, \tag{2.4f}$$

$$\nabla_y \cdot \{\nabla_x w(x) + \nabla_y w_1(x,y)\} = 0, \quad S\times U\times Y_p, \tag{2.4g}$$

$$\nabla_y \cdot \nabla_y w(x) = 0, \quad S\times U\times Y_p \tag{2.4h}$$

$$c(0,x) = c_0(x), \quad U. \tag{2.4i}$$

where $\bar{\kappa}(x) = \frac{1}{|Y_p|}\int_{\partial Y_p} \kappa(x,y)\,dy$, $x \in U$ *denotes the mean of the quantity* κ *over the pore space* Y_p.

The systems of equations (2.4a)–(2.4i) *is the required homogenized (upscaled) model of* (1.3a)–(1.3h).

3 Anticipated Upscaled Model via Asymptotic Expansion Method

We consider the following expansions

$$\mathbf{u}^\varepsilon = \sum_{i=0}^{\infty} \varepsilon^i \mathbf{u_i},\ c^\varepsilon = \sum_{i=0}^{\infty} \varepsilon^i c_i,\ w^\varepsilon = \sum_{i=0}^{\infty} \varepsilon^i w_i \text{ and } p^\varepsilon = \sum_{i=0}^{\infty} \varepsilon^i p_i, \tag{3.1}$$

where each term $\mathbf{u_i}$, p_i, c_i and w_i are Y-periodic functions in y-variable. We have $\nabla = \nabla_x + \frac{1}{\varepsilon}\nabla_y$. After the substitution of $\mathbf{u}^\varepsilon$, c^ε, w^ε, p^ε in the problem ($\mathcal{P}^\varepsilon$), we get from (1.3a)

$$
\begin{aligned}
&\varepsilon^{-1}(\nabla_y p_0) + \varepsilon^0(-\mu\Delta_y \mathbf{u_0} + \nabla_x p_0 + \nabla_y p_1) \\
&+\varepsilon[-\mu\{\Delta_y \mathbf{u_1} + (\nabla_x \cdot \nabla_y + \nabla_y \cdot \nabla_x)\mathbf{u_0}\} + \nabla_x p_1 + \nabla_y p_2] \\
&= \varepsilon^{-1}\{-\lambda(c_0 \nabla_y w_0)\} + \varepsilon^0[-\lambda\{c_1 \nabla_y w_0 + c_0(\nabla_x w_0 + \nabla_y w_1)\}] + \mathcal{O}(\varepsilon). \qquad (3.2)
\end{aligned}
$$

We use (3.1) in (1.3b) then

$$
\varepsilon^{-1}\nabla_y \cdot \mathbf{u_0} + \varepsilon^0(\nabla_x \cdot \mathbf{u_0} + \nabla_y \cdot \mathbf{u_1}) + \varepsilon(\nabla_x \cdot \mathbf{u_1} + \nabla_y \cdot \mathbf{u_2}) + \varepsilon^2(\ldots) = 0. \qquad (3.3)
$$

From (1.3d), after plugging the expansions, we obtain

$$
\begin{aligned}
&\partial_t(c_0 + \varepsilon c_1) + \varepsilon^0\{\nabla_y \cdot (c_0 \mathbf{u_0})\} + \varepsilon\{\nabla_y \cdot (c_0 \mathbf{u_1}) + \nabla_x \cdot (c_0 \mathbf{u_0}) + \nabla_y \cdot (c_1 \mathbf{u_0})\} \\
&= \varepsilon^{-2}\Delta_y w_0 + \varepsilon^{-1}\{\Delta_y w_1 + (\nabla_x \cdot \nabla_y + \nabla_y \cdot \nabla_x) w_0\} \\
&+\varepsilon^0\{\Delta_y w_2 + (\nabla_x \cdot \nabla_y + \nabla_y \cdot \nabla_x) w_1 + \Delta_x w_0\} + \mathcal{O}(\varepsilon). \quad (3.4)
\end{aligned}
$$

Next, we substitute the expansions for w_ε, c_ε in (1.3e) and use the Taylor series expansion of f around c_0 which leads to

$$
w_0 + \varepsilon w_1 = -\Delta_y c_0 + \varepsilon^1\{-\Delta_y c_1 - (\nabla_x \cdot \nabla_y + \nabla_y \cdot \nabla_x) c_0\} + f(c_0) + \mathcal{O}(\varepsilon). \qquad (3.5)
$$

Now we substitute the expansions in the boundary conditions. From (1.3c), we obtain

$$
\mathbf{u_0} + \varepsilon \mathbf{u_1} + \varepsilon^2 \mathbf{u_2} + \cdots = 0 \quad \text{on } (0, T) \times \partial U_p^\varepsilon. \qquad (3.6)
$$

From (1.3f) and (1.3g), we get

$$
\varepsilon^{-1}\nabla_y c_0 \cdot \mathbf{n} + \varepsilon^0(\nabla_x c_0 + \nabla_y c_1) \cdot \mathbf{n} + \varepsilon(\nabla_x c_1 + \nabla_y c_2) \cdot \mathbf{n} + \cdots = 0 \qquad (3.7)
$$

and

$$
\varepsilon^{-1}\nabla_y w_0 \cdot \mathbf{n} + \varepsilon^0(\nabla_x w_0 + \nabla_y w_1) \cdot \mathbf{n} + \varepsilon(\nabla_x w_1 + \nabla_y w_2) \cdot \mathbf{n} + \cdots = 0 \qquad (3.8)
$$

respectively.

We compare the coefficient of ε^0 from (3.5) and integrate it over Y_p, then using (3.7) we get

$$
w_0(t, x, y) = f(c_0(t, x, y)) \quad \text{in } S \times U \times Y_p \qquad (3.9)
$$

We equate the coefficient of ε^0 from (3.4) and integrate it over Y_p, then using (3.8) we obtain

$$
|Y_p|\{\partial_t c_0 + \mathbf{u}_0 \cdot \nabla_y c_0\} = \nabla_x \cdot \int_{Y_p} \{\nabla_y w_1 + \nabla_x w_0\}\, dy. \qquad (3.10)
$$

The coefficients of ε^{-2} and ε^{-1} from (3.4) give The coefficient of ε^{-1} from (3.4) gives

$$\Delta_y w_0 = 0 \qquad \text{and} \qquad \nabla_x \cdot \nabla_y w_0 + \nabla_y \cdot \{\nabla_x w_0 + \nabla_y w_1\} = 0 \tag{3.11}$$

From (3.8) and (3.11) we observe that

$$w_0 = w_0(t, x). \tag{3.12}$$

We equate the coefficients of ε^{-1} from (3.2), then using (3.12) we get

$$\nabla_y p_0 = 0 \qquad \text{for } y \in Y_p. \tag{3.13}$$

The coefficient of ε^0 from (3.2) along with (3.12) gives

$$-\mu\Delta_y \mathbf{u_0} + \nabla_x p_0 + \nabla_y p_1 = -\lambda\, c_0\, (\nabla_x w_0 + \nabla_y w_1). \tag{3.14}$$

Again, using (3.3) and (3.6) one can deduce

$$\nabla_x \cdot \int_{Y_p} \mathbf{u_0}(x, y)\, dy = 0 \quad \text{in } S \times U. \tag{3.15}$$

Equating ε coefficient from (3.5) we get using (3.7)

$$|Y_p| w_1 = -\nabla_x \cdot \int_{Y_p} \nabla_y c_0\, dy \tag{3.16}$$

4 Proof of Theorem 2.1

4.1 A Priori Estimates

We put $\eta = \varepsilon \mathbf{u}^\varepsilon$, $\phi = \lambda w^\varepsilon$, $\psi = \lambda \partial_t c^\varepsilon$ in (2.2), and using $\nabla(c^\varepsilon w^\varepsilon) = c^\varepsilon \nabla w^\varepsilon + w^\varepsilon \nabla c^\varepsilon$ it yields

$$\sqrt{\mu}\varepsilon||\nabla \mathbf{u}^\varepsilon||_{L^2(S\times U_p^\varepsilon)} + \sqrt{\lambda}||\nabla w^\varepsilon||_{L^2(S\times U_p^\varepsilon)} + \sqrt{\frac{\lambda}{2}}\varepsilon||\nabla c^\varepsilon||_{L^\infty(S;L^2(U_p^\varepsilon))} \leq C \tag{4.1}$$

as $\varepsilon^{\frac{3}{2}} < \varepsilon$ for $\varepsilon \in (0, 1)$.

Next, Young's inequality gives

$$\int_{U_p^\varepsilon} F(c^\varepsilon(t))\,dx = \frac{1}{4}\int_{U_p^\varepsilon} ((c^\varepsilon)^2-1)^2\,dx \le C \quad \Rightarrow \int_{U_p^\varepsilon} |c^\varepsilon|^4\,dx \le C \quad \forall t$$

$$i.e., \quad \sup_{\varepsilon>0} ||c^\varepsilon||_{L^\infty(S;L^4(U_p^\varepsilon))} \le C. \tag{4.2}$$

We set $\psi = 1$ as a test function in (1.3e) and then using Poincare's inequality, we get

$$||w^\varepsilon - \int_{U_p^\varepsilon} w^\varepsilon\,dx||_{L^2(U_p^\varepsilon)} \le C||\nabla w^\varepsilon||_{L^2(U_p^\varepsilon)} \quad \Rightarrow ||w^\varepsilon||_{L^2(S\times U_p^\varepsilon)} \le C. \tag{4.3}$$

By Gagliardo–Nirenberg–Sobolev inequality for Lipschitz domain, $||u^\varepsilon||_{L^4(Y)} \le C||\nabla u^\varepsilon||_{L^2(Y)}$, where C depend on n and Y. By imbedding theorem, $||u^\varepsilon||_{L^2(Y)} \le C||u^\varepsilon||_{L^4(Y)} \le C$. By a straightforward scaling argument, we obtain

$$||\mathbf{u}^\varepsilon||_{L^4(U_p^\varepsilon)} \le C. \tag{4.4}$$

From (2.2b) we get,

$$||\partial_t c^\varepsilon||_{L^2(S;H^1(U_p^\varepsilon)^*)} \le C \quad \forall \varepsilon > 0 \tag{4.5}$$

From proposition III.1.1 in [10] and (2.2a), there exist a pressure $p^\varepsilon := \partial_t P^\varepsilon \in W^{-1,\infty}(S,\, L_0^2(U_p^\varepsilon))$ such that

$$\langle \nabla P^\varepsilon(t), \eta\rangle \le \mu\varepsilon^2 \int_S ||\nabla \mathbf{u}^\varepsilon||_{L^2(U_p^\varepsilon)} ||\nabla \eta^\varepsilon||_{L^2(U_p^\varepsilon)}\,dt + \int_S ||c^\varepsilon||_{L^4(U_p^\varepsilon)} ||\nabla w^\varepsilon||_{L^2(U_p^\varepsilon)}\,dt.$$

Thus by (4.1) and (4.2) it immediately follows that

$$\langle \nabla P^\varepsilon(t), \eta\rangle \le C||\eta||_{H_0^1(U_p^\varepsilon)^n} \Rightarrow \sup_{t\in[0,T]} ||\nabla P^\varepsilon(t)||_{H^{-1}(U_p^\varepsilon)^n} \le C \quad \forall \varepsilon > 0. \tag{4.6}$$

Now, with the help of a-priori estimates from (2.3), the existence of solution of $(\mathcal{P}^\varepsilon)$ can be shown using Galerkin's method, cf. [6] and references therein.

5 Proof of Theorem 2 (Homogenization of Problem $(\mathcal{P}^\varepsilon)$)

We start with the construction of an extension of solution from U_p^ε to U in the lemma below.

Lemma 6 *There exists a positive constant C depending on c_0, $\mathbf{u_0}$, n, $|Y|$, λ and μ but independent of ε and extensions $(\tilde{c^\varepsilon},\, \tilde{w^\varepsilon},\, \tilde{\mathbf{u}}^\varepsilon,\, \tilde{P^\varepsilon})$ of the solution $(c^\varepsilon,\, w^\varepsilon,\, \mathbf{u}^\varepsilon,\, P^\varepsilon)$ to $S \times U$ such that*

$$||\tilde{\mathbf{u}}^\varepsilon||_{L^\infty(S;L^2(U)^n)} + ||\tilde{c}^\varepsilon||_{L^\infty(S;L^4(U))} + ||\tilde{w}^\varepsilon||_{L^2(S;H^1(U))} + \sqrt{\mu}\varepsilon||\nabla\tilde{\mathbf{u}}^\varepsilon||_{L^2(S\times U)^{n\times n}}$$
$$+\sqrt{\frac{\lambda}{2}}\varepsilon||\nabla\tilde{c}^\varepsilon||_{L^\infty(S;L^2(U)^n)} + \sqrt{\lambda}||\nabla\tilde{w}^\varepsilon||_{L^2(S\times U)^n} + ||\partial_t\tilde{c}^\varepsilon||_{L^2(S;H^1(U)^*)}$$
$$+ \sup_{t\in[0,T]} ||\tilde{P}^\varepsilon(t)||_{L_0^2(U)} \le C. \tag{5.1}$$

Lemma 7 *Let $(\mathbf{u}^\varepsilon, P^\varepsilon, c^\varepsilon, w^\varepsilon)_{\varepsilon>0}$ be the extension of the weak solution from Lemma 6 (denoted by the same symbol). Then there exists some functions $\mathbf{u} \in L^2(S \times U; H^1_\#(Y))^n$, $w \in L^2(S \times U)$, $P \in L^2(S \times U \times Y)$, c, $w_1 \in L^2(S \times U; H^1_\#(Y))$ and a subsequence of $(\mathbf{u}^\varepsilon, P^\varepsilon, c^\varepsilon, w^\varepsilon)_{\varepsilon>0}$, still denoted by the same symbol, such that the following convergences hold:*

(i) $(\mathbf{u}^\varepsilon)_{\varepsilon>0}$ *two-scale converges to* $\mathbf{u}$. *(ii)* $(c^\varepsilon)_{\varepsilon>0}$ *two-scale converges to* c.
(iii) $(w^\varepsilon)_{\varepsilon>0}$ *two-scale converges to* w. *(iv)* $(P^\varepsilon)_{\varepsilon>0}$ *two-scale converges to* P.
(v) $(\varepsilon\nabla_x c^\varepsilon)_{\varepsilon>0}$ *two-scale converges to* $\nabla_y c$. *(vi)* $(\varepsilon\nabla_x \mathbf{u}^\varepsilon)_{\varepsilon>0}$ *two-scale converges to* $\nabla_y \mathbf{u}$.
(vii) $(\nabla_x w^\varepsilon)_{\varepsilon>0}$ *two-scale converges to* $\nabla_x w + \nabla_y w_1$.

Proof The convergences follow from the estimates (5.1), Lemmas 3 and 4.

In the next lemma we will discuss the convergence of nonlinear terms for $\varepsilon \to 0$.

Lemma 8 *The following convergence results hold:*

(i) $(c^\varepsilon)_{\varepsilon>0}$ *is strongly convergent to c in $L^2(S \times U)$. Thus, $\mathcal{T}^\varepsilon(c^\varepsilon)$ converges to c strongly in $L^2(S \times U \times Y)$, i.e., $(c^\varepsilon)_{\varepsilon>0}$ is strongly two-scale convergent to c.*
(ii) $\mathcal{T}^\varepsilon\mathbf{u}^\varepsilon$ *is weakly convergent to $\mathbf{u}$ in $L^2(S \times U \times Y)^n$, i.e., $(\mathbf{u}^\varepsilon)_{\varepsilon>0}$ is weakly two-scale convergent to $\mathbf{u}$.*
(iii) $\mathcal{T}^\varepsilon[\varepsilon\nabla_x c^\varepsilon]$ *converges to $\nabla_y c$ weakly in $L^2(S \times U \times Y)^n$, i.e., $\varepsilon\nabla_x c^\varepsilon$ is weakly two-scale convergent to $\nabla_y c$.*
(iv) *The nonlinear terms $f(c^\varepsilon)$, $c^\varepsilon\nabla_x w^\varepsilon$ and $c^\varepsilon\mathbf{u}^\varepsilon$ two-scale converge to $f(c)$, $c(\nabla_x w + \nabla_y w_1)$ and $c\mathbf{u}$.*

Proof We will prove step by step. From estimate (5.1) for $(c^\varepsilon)_{\varepsilon>0}$ and Theorem 2.1 in [9], there exists a subsequence of $(c^\varepsilon)_{\varepsilon>0}$, still denoted by same symbol, such that $(c^\varepsilon)_{\varepsilon>0}$ is strongly convergent to a limit c. The rest of (i) and the proofs of (ii) and (iii) follow from Lemma 4. Following the similar arguments as in [2] we can prove (iv).

Proof *(Proof of Theorem 2)* (i) We choose a test function ϕ in (2.2b) defined as $\phi = \phi(t, x, \frac{x}{\varepsilon}) = \phi_0(t, x) + \varepsilon\phi_1(t, x, \frac{x}{\varepsilon})$, where the functions $\phi_0 \in C_0^\infty(S \times U)$ and $\phi_1 \in C_0^\infty(S \times U; C_\#^\infty(Y))$:

$$\int_S \langle\partial_t c^\varepsilon, \phi\rangle\, dt - \int_{S\times U_p^\varepsilon} c^\varepsilon\mathbf{u}^\varepsilon \cdot \varepsilon\nabla\phi\, dx\, dt + \int_{S\times U_p^\varepsilon} \nabla w^\varepsilon \cdot \nabla\phi\, dx\, dt = 0.$$

We extend the solution to U and pass $\varepsilon \to 0$ in the two-scale sense and get

$$
\begin{aligned}
&-\int_{S\times U} c(t,x,y)\partial_t\phi_0(t,x)\,dx\,dt - \int_{S\times U} c(t,x,y)\mathbf{u}(t,x)\cdot\nabla_y\phi_0(t,x)\,dx\,dt \\
&+\int_{S\times U}\{\nabla_x w(t,x)+\nabla_y w_1(t,x,y)\}\cdot\Big(\nabla_x\phi_0(t,x)+\nabla_y\phi_1(t,x,y)\Big)\,dx\,dt = 0.
\end{aligned} \tag{5.2}
$$

Setting $\phi_0 = 0$ and $\phi_1 = 0$ in (5.2) yield, respectively,

$$\nabla_y\cdot\{\nabla_x w(t,x)+\nabla_y w_1(t,x,y)\} = 0, \tag{5.3}$$

$$\partial_t c(t,x,y) + \nabla_y\cdot c(t,x,y)\mathbf{u}(t,x,y) = \Delta_x w(t,x) + \nabla_x\cdot\nabla_y w_1(t,x,y), \tag{5.4}$$

in $S\times U\times Y_p$. Similarly, choosing a function $\psi \in C_0^\infty(S\times U; C_\#^\infty(Y))$ in (2.2c) and passing the limit gives

$$w(t,x,y) = -\Delta_y c(t,x) + f(c(t,x,y)) \quad \text{in } S\times U\times Y_p. \tag{5.5}$$

(ii) We choose the test functions $\eta \in C_0^\infty(U; C_{\#)}^\infty(Y))^n$ and $\xi \in C_0^\infty(S)$ and proceed as in [2]. Then, using Lemmas 7 and 8, and passing to the two-scale limit

$$
\begin{aligned}
&\lim_{\varepsilon\to 0}\int_{S\times U_p^\varepsilon} P^\varepsilon(t,x)\Big\{\nabla_x\cdot\eta(x,\frac{x}{\varepsilon}) + \frac{1}{\varepsilon}\nabla_y\cdot\eta(x,\frac{x}{\varepsilon})\Big\}\partial_t\xi(t)\,dx\,dy\,dt \\
&= \int_{S\times U\times Y_p} P(t,x,y)\nabla_y\cdot\eta(x,y)\partial_t\xi(t)\,dx\,dy\,dt \\
&= 0
\end{aligned} \tag{5.6}
$$

We get the y-variable independency of the two-scale limit of the pressure P from (5.6). Further, we consider the function $\eta \in C_0^\infty(U; C_\#^\infty(Y))^n$ such that $\nabla_y\cdot\eta(x,y) = 0$, so that

$$
\begin{aligned}
\mu\varepsilon^2\int_{S\times U_p^\varepsilon}\nabla\mathbf{u}^\varepsilon(t,x):\nabla\eta(x,y)\xi(t)\,dx\,dt + \int_{S\times U_p^\varepsilon} P^\varepsilon(t,x)\nabla\cdot\eta(x,y)\partial_t\xi(t)\,dx\,dt \\
= -\lambda\int_{S\times U_p^\varepsilon} c^\varepsilon(t,x)\nabla w^\varepsilon(t,x)\cdot\eta(x,y)\xi(t)\,dx\,dt.
\end{aligned} \tag{5.7}
$$

We use the extensions of solution to U (using the same notations), and pass to the two-scale limit.

$$-\lambda \int_{S\times U\times Y_p} c(t,x,y)\{\nabla_x w(t,x) + \nabla_y w_1(t,x,y)\}\cdot \eta(x,y)\xi(t)\,dx\,dy\,dt$$
$$= \mu \int_{S\times U\times Y_p} \nabla_y \mathbf{u}(t,x,y) : \nabla_y \eta(x,y)\xi(t)\,dx\,dy\,dt$$
$$+ \int_{S\times U\times Y_p} P(t,x)\nabla_x \cdot \eta(x,y)\partial_t \xi(t)\,dx\,dy\,dt. \quad (5.8)$$

The existence of a pressure $P_1 \in L^\infty(S; L^2_0(U; L^2_\#(Y_p)))$ and two-scale convergence results are followed as in [2] for the final step of the upscaling of the model equations.

$$\int_{S\times U\times Y_p} P(t,x)\nabla_x \cdot \eta(x,y)\partial_t\xi(t)\,dx\,dy\,dt + \int_{S\times U\times Y_p} P_1(t,x,y)\nabla_y \cdot \eta(x,y)\partial_t\xi(t)\,dx\,dy\,dt$$
$$+\lambda \int_{S\times U\times Y_p} c(t,x,y)\{\nabla_x w(t,x) + \nabla_y w_1(t,x,y)\}\cdot\eta(x,y)\xi(t)\,dx\,dy\,dt$$
$$+\mu \int_{S\times U\times Y_p} \nabla_y \mathbf{u}(t,x,y) : \nabla_y \eta(x,y)\xi(t)\,dx\,dy\,dt$$
$$= 0. \quad (5.9)$$

for all $\eta \in C_0^\infty(U; C_\#^\infty(Y))^n$ and $\xi \in C_0^\infty(S)$.

From (5.9), we obtain

$$-\mu\Delta_y \mathbf{u}(x,y) + \nabla_x p(x) + \nabla_y p_1(x,y) = -\lambda c(x,y)\{\nabla_x w(t,x) + \nabla_y w_1(t,x,y)\} \quad (5.10)$$

in $S \times U \times Y_p$.

6 Conclusion

A two fluids' mixture in strongly perforated domain is considered in which the fluids are separated by an interface of thickness of λ in the pore part. From the modeling of such phenomena in the pore space, we got a strongly coupled system of Stokes–Cahn–Hilliard equations. The surface tension effects have been taken into account and the aforementioned interface is assumed to be independent of the scale parameter ε. Several a-priori estimates are derived and the well-posedness at the micro-scale is shown. Two-scale convergence, periodic unfolding, and the estimates after using extension theorems on them, yield the homogenized model.

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