Stability Analysis of Additive Time-Varying T–S Fuzzy System Using Augmented Lyapunov Functional

Bhuvaneshwari Ganesan and Manivannan Annamalai

Abstract This article discusses the stability analysis problem of Takagi–Sugeno (T–S) fuzzy system with additive time-varying delay components. To find a stability region and to stabilize the system, a state feedback control scheme is considered. A Lyapunov–Krasovskii functional is constructed to obtain less conservative results by utilizing the integral inequality based on non-orthogonal polynomials and the conditions are derived as linear matrix inequality form. The stability conditions are obtained for the system involving two delay components and the proposed result is validated through numerical examples.

Keywords Additive time-varying delays · T–S fuzzy system · Stability · Linear matrix inequality

1 Introduction

In real world, there exist delays in physical systems inherently. Avoiding these delays when modeling physical system into mathematical model gives only the approximated results. In order to get more accurate results, the time delays must be included in mathematical models. Time-delay systems are fundamental mathematical representations of real-world events such as chemical engineering system, power system, biological system, and so on. The presence of delay causes the system to be unstable and gives poor performance. As a result, substantial research has been focused on analysis and synthesis challenges of time-delayed systems. Researchers have been more focused on determining the stability of systems of various kinds, such as neutral system [4], stochastic system [10], fuzzy system [11], singular system [14], and hybrid system [15].

The majority of work focused on determining the maximum upper bound for delayed system and analyzing its stability. It has been accomplished through the appli-

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cation of Lyapunov stability theory by developing appropriate Lyapunov–Krasovskii functional (LKF). The construction of proper LKF ensures to get less conservative results in analyzing stability of the system. There are various types of LKF which have been used in the literature such as discretized LKF [5], polynomial-type LKF [6], augmented LKF [7], relaxed LKF [18], etc.

Takagi and Sugeno first introduced the concept of fuzzy IF-THEN rules for nonlinear systems to make it into linear subsystems by employing input–output data. Another primary role of T–S fuzzy system is that the control and stability conditions can be expressed as linear matrix inequality (LMI). This methodology is used in nonlinear systems, which has wide applications in many practical problems. Discrete-time [16] and continuous-time [13] systems are two types of time-varying T–S fuzzy systems. These systems addressed the problem with time delays such as constant delay, discrete delay, distributed delay, and additive time-varying delays. In order to handle system with such delays, various control methodologies have been employed to stabilize the system, such as state feedback control, sliding mode control, fuzzy logic control, and adaptive control.

Many researchers have investigated the stability of nonlinear system with additive time-varying delays. A new stability results have been studied for the nonlinear system with additive time-varying delays via new augmented LKFs in [2]. In [8], stability problem of a system involves two additive time-varying delays which have been investigated by using a quadratic function negative-determination lemma. Stabilization problem of switched T–S fuzzy system has been investigated with additive time-varying delays and robust stabilization is also investigated in [1]. In [20], a stability and stabilization problem via new LKFs has been studied for additive time-varying delayed T–S fuzzy system. In [21], a local stability and stabilization problem has been investigated for nonlinear systems with parameter uncertainty and two additive time-varying delays via T–S fuzzy model.

In this paper, a stability and stabilization problem for T–S fuzzy system with additive time-varying delays has been considered. A state feedback controller involves state with additive time-varying delays which is employed to stabilize the system. LKFs are considered in an augmented form and an integral inequality based on non-orthogonal polynomials has been applied to get less conservative results. Furthermore, the stability conditions have been obtained in the form of LMI. Finally, the advantages of proposed method have been validated through numerical example.

2 Problem Formulations

Consider the delayed T–S fuzzy model with additive time-varying delays as follows: Fuzzy Plant Rule $i (i = 1, 2, \ldots, p)$: IF s_1 is w_{i1}, and, ..., and s_q is w_{iq} THEN

$$\begin{cases} \dot{x}(t) = A_i x(t) + B_i x(t - \hbar_1(t) - \hbar_2(t)) + C_i u(t), \\ x(t) = \phi(t), \ t \in [-\bar{\hbar}, 0], \ t \geq 0, \end{cases} \tag{1}$$

where $x(t) \in \mathbb{R}^n$ represents the state vector and $u(t) \in \mathbb{R}^n$ is control input; s_m, w_{im} $(m = 1, \ldots, q)$ represents the premise variables and associated fuzzy sets, respectively; p denotes the number of IF-THEN rules; A_i, B_i and C_i are appropriate dimensional known matrices. $\hbar_1(t)$, $\hbar_2(t)$ are two additive positive time-varying bounded delays satisfying the following conditions:

$$0 \le \hbar_1(t) \le \hbar_1, \quad \dot{\hbar}_1(t) \le \mu_1 < 1, \quad 0 \le \hbar_2(t) \le \hbar_2, \quad \dot{\hbar}_2(t) \le \mu_2 < 1, \qquad (2)$$

and $\bar{\hbar} = \hbar_1 + \hbar_2$. $\phi(t)$ denotes initial condition and it is continuously differentiable function on $[-\bar{\hbar}, 0]$. $\hbar_1$ and $\hbar_2$ are constant and positive scalars which represent the upper bound of two additive time-varying delays.

By adopting standard fuzzy inference, the overall fuzziness of the design can be denoted as follows:

$$\begin{cases} \dot{x}(t) = \sum_{i=1}^{p} \zeta_i(s(t))\Big[A_i x(t) + B_i x(t - \hbar_1(t) - \hbar_2(t)) + C_i u(t)\Big], \\ x(t) = \phi(t), \ t \ \in \ [-\bar{\hbar}, 0], \ t \ge 0, \end{cases} \qquad (3)$$

where $s(t) = [s_1(t), \ldots, s_q(t)]$ and

$$\zeta_i(s(t)) = \frac{\psi_i(s(t))}{\sum_{i=1}^{p} \psi_i(s(t))} \ge 0, \text{ and } \psi_i(s(t)) = \prod_{m=1}^{q} w_{im}(s_m(t))$$

with $w_{im}(s_m(t))$ representing the grade membership of $s_m(t)$ in w_{im}. It is clear to see that

$$\psi_i(s(t)) > 0, \quad \forall i = 1, \ldots, p, \quad \sum_{i=1}^{p} \psi_i(s(t)) > 0, \quad \text{for any } s(t).$$

Hence $\zeta_i(s(t))$ satisfy, $\quad \zeta_i(s(t)) \ge 0, \quad \forall i = 1, \ldots, p, \quad \sum_{i=1}^{p} \zeta_i(s(t)) = 1$, for any $s(t)$.

Now, to stabilize the delayed T–S fuzzy system, consider the state feedback control design with additive time delay as follows:

Controller rule: IF s_1 is w_{i1} and , …,and s_q is w_{iq}, THEN

$$u(t) = K_{ai} x(t) + K_{bi} x(t - \hbar_1(t) - \hbar_2(t)),$$

where K_{ai} and K_{bi} are unknown control gain matrices. Therefore, the complete fuzzy control rule is inferred as

$$u(t) = \sum_{i=1}^{p} \zeta_i(s(t))[K_{ai} x(t) + K_{bi} x(t - \hbar_1(t) - \hbar_2(t))]. \qquad (4)$$

By adopting (4) in (3), the closed-loop system can be obtained as follows:

$$\begin{cases} \dot{x}(t) = \sum_{i=1}^{p}\sum_{l=1}^{p} \zeta_i(s(t))\zeta_l(s(t))\Big[A_i x(t) + B_i x(t - \hbar_1(t) - \hbar_2(t)) \\ \qquad\qquad + C_i\big(K_{al}x(t) + K_{bl}x(t - \hbar_1(t) - \hbar_2(t))\big)\Big], \\ x(t) = \phi(t),\ t\ \in\ [-\bar{\hbar}, 0],\ t \geq 0. \end{cases} \tag{5}$$

The major goal of this paper is to establish stability of additive time-varying delayed T–S fuzzy system (5). Besides that, the problem deals with finding the control gain matrices K_{al} and K_{bl} and to stabilize the system (5). Some important lemmas are introduced before deriving the main results as follows.

Most existing results for delayed systems have been used in memoryless controller design of the form $u(t) = Kx(t)$. The controller considered in this paper contains state vector, also a state with two additive time-varying delays of the form $u(t) = K_a x(t) + K_b x(t - \hbar_1(t) - \hbar_2(t))$.

2.1 Preliminaries

This section provides some lemmas that can be used in the main result to obtain stability criteria of the delay-dependent T–S fuzzy system.

Lemma 1 ([19]) *For two scalars a and b with $b > a$, a vector $z : [a, b] \to \mathbb{R}^n$, and $n \times n$ real matrices $R > 0$, $H_i (i = 1, 2)$ and $Y_j (j = 1, 2, 3)$ satisfying*

$$\Theta := \begin{bmatrix} Y_1 & Y_2 & H_1 \\ * & Y_3 & H_2 \\ * & * & R \end{bmatrix} \geq 0, \textit{ the following inequality holds:}$$

$$\int_a^b \dot{z}^T(s)R\dot{z}(s)ds \geq \frac{1}{b-a}\chi_1^T R\chi_1 + \chi_2^T\Big(H_1 + H_1^T - \frac{b-a}{3}Y_1\Big)\chi_2 + \chi_3^T\Big[15(H_2 + H_2^T) - 20(b-a)Y_3\Big]\chi_3 + 20\chi_3^T H_2^T L_2\chi_1.$$

$$\textit{Where}\quad \chi_1 := z(b) - z(a),\ \chi_2 := z(b) + z(a) - (2/(b-a))\int_a^b z(s)ds,$$

$$\chi_3 := \frac{4}{b-a}\int_a^b z(s)ds - \frac{8}{(b-a)^2}\int_a^b\int_\theta^b z(s)dsd\theta.$$

Lemma 2 ([17]) *For any constant positive symmetric matrix $L \in \mathbb{R}^{m\times m}$, scalar $\kappa > 0$, vector function $z : [0, \kappa] \to \mathbb{R}^m$ such that the integration concerned is well defined, then*

$$\kappa\int_0^\kappa z^T(s)Lz(s)ds \geq \Big(\int_0^\kappa z(s)ds\Big)^T L\Big(\int_0^\kappa z(s)ds\Big).$$

3 Main Results

In this section, the stability criteria conditions are derived by choosing suitable LKFs and using the above-mentioned lemmas. Now, the following notations are given to understand the main results:

$$
\begin{aligned}
e_i =& \left[0_{n\times(i-1)n}\ I_n\ 0_{n\times(15-i)n}\right]\ (i=1,\ldots,15),\\
\xi^T(t) =& \Big[x^T(t)\quad x^T(t-\bar{\hbar})\quad x^T(t-\hbar_1)\quad x^T(t-\hbar_2)\quad x^T(t-\hbar_1(t))\quad x^T(t-\hbar_2(t))\\
& x^T(t-\hbar(t))\quad x^T(t-\hbar_1(t)-\hbar_2(t))\quad \dot{x}^T(t)\quad \frac{1}{\hbar_2-\hbar_1}\int_{t-\hbar_2}^{t-\hbar_1} x^T(s)ds\quad \int_{t-\hbar_1}^{t} x^T(s)ds\\
& \int_{t-\hbar_2}^{t} x^T(s)ds\quad \frac{1}{(\hbar_2-\hbar_1)^2}\int_{t-\hbar_2}^{t-\hbar_1}\int_{\theta}^{t-\hbar_1} x^T(s)dsd\theta\quad \frac{1}{\hbar_2^2}\int_{t-\bar{\hbar}}^{t-\hbar_1}\int_{\theta}^{t-\hbar_1} x^T(s)dsd\theta\\
& \frac{1}{\hbar_1^2}\int_{t-\bar{\hbar}}^{t-\hbar_2}\int_{\theta}^{t-\hbar_2} x^T(s)dsd\theta\Big].
\end{aligned}
$$

Theorem 1 *For given scalars and control gain matrices $\hbar_1 > 0, \hbar_2 > 0, \mu_1, \mu_2$, K_{al}, K_{bl}, the system (5) with additive time-varying delays $\hbar_1(t), \hbar_2(t)$ satisfying condition (2) is asymptotically stable if there exist positive definite symmetric matrices P, Q_i, R_i, $S_i (i = 1, 2, 3)$, $T_i (i = 1, 2)$ and any matrices L_i, $Z_i (i = 1, 2, 3)$ such that the following LMI is satisfied:*

$$
\Omega_{i,l} = \begin{bmatrix}
\varphi_{1il}^1 & 0 & \varphi_1^3 & \varphi_1^4 & 0 & 0 & 0 & \varphi_{1il}^8 & \varphi_{1il}^9 & 0 & 0 & 0 & 0 & \varphi_1^{14} & \varphi_1^{15}\\
* & \varphi_2^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
* & * & \varphi_3^3 & \varphi_3^4 & 0 & 0 & 0 & 0 & 0 & \varphi_3^{10} & 0 & 0 & \varphi_3^{13} & \varphi_3^{14} & \varphi_3^{15}\\
* & * & * & \varphi_4^4 & 0 & 0 & 0 & 0 & 0 & \varphi_4^{10} & 0 & 0 & \varphi_4^{13} & \varphi_4^{14} & \varphi_4^{15}\\
* & * & * & * & \varphi_5^5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
* & * & * & * & * & \varphi_6^6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
* & * & * & * & * & * & \varphi_7^7 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
* & * & * & * & * & * & * & \varphi_8^8 & \varphi_{8il}^9 & 0 & 0 & 0 & 0 & 0 & 0\\
* & * & * & * & * & * & * & * & \varphi_9^9 & 0 & 0 & 0 & 0 & \varphi_9^{14} & \varphi_9^{15}\\
* & * & * & * & * & * & * & * & * & \varphi_{10}^{10} & 0 & 0 & \varphi_{10}^{13} & 0 & 0\\
* & * & * & * & * & * & * & * & * & * & \varphi_{11}^{11} & 0 & 0 & 0 & 0\\
* & * & * & * & * & * & * & * & * & * & * & \varphi_{12}^{12} & 0 & 0 & 0\\
* & * & * & * & * & * & * & * & * & * & * & * & \varphi_{13}^{13} & 0 & 0\\
* & * & * & * & * & * & * & * & * & * & * & * & * & \varphi_{14}^{14} & \varphi_{14}^{15}\\
* & * & * & * & * & * & * & * & * & * & * & * & * & * & \varphi_{15}^{15}
\end{bmatrix} < 0, \quad (6)
$$

where

$\varphi_{1il}^{1} = Q_2 + Q_3 + \hbar_1 T_1 + \hbar_2 T_2 - S_1 + 2\beta N A_i + 2\beta NCK_{al},\ \varphi_1^3 = \frac{\hbar_2^2}{2}P_{12} + S_1,\ \varphi_1^4 = \frac{\hbar_1^2}{2}P_{13},$
$\varphi_{1il}^{8} = \beta NB_i + \beta NCK_{bl},\ \varphi_{1il}^{9} = P_{11} + A_i^T N^T + K_{al}^T C^T N^T - \beta N,\ \varphi_1^{14} = -P_{12}\hbar_2^2,$
$\varphi_1^{15} = -P_{13}\hbar_1^2,\ \varphi_2^2 = -R_2 - R_3,\ \varphi_3^3 = (\hbar_2 - \hbar_1)R_1 + R_2 - S_1 - \frac{1}{\hbar_2 - \hbar_1}S_2$
$-(L_1 + L_1^T - \frac{\hbar_2 - \hbar_1}{3}Z_1),\ \varphi_3^4 = \frac{1}{\hbar_2 - \hbar_1}S_2 - (L_1 + L_1^T - \frac{\hbar_2 - \hbar_1}{3}Z_1),$
$\varphi_3^{10} = 2(L_1 + L_1^T - \frac{\hbar_2 - \hbar_1}{3}Z_1) - 80L_2^T,\ \varphi_3^{13} = 160L_2^T,\ \varphi_3^{14} = \frac{\hbar_2^2\hbar_1^2}{2}P_{14}^T,\ \varphi_3^{15} = \frac{\hbar_2^4}{2}P_{15},$
$\varphi_4^4 = -(\hbar_2 - \hbar_1)R_1 + R_3 - \frac{1}{\hbar_2 - \hbar_1}S_2 - (L_1 + L_1^T - \frac{\hbar_2 - \hbar_1}{3}Z_1),$
$\varphi_4^{10} = 2(L_1 + L_1^T - \frac{\hbar_2 - \hbar_1}{3}Z_1) + 80L_2^T,\ \varphi_4^{13} = -160L_2^T,\ \varphi_4^{14} = \frac{\hbar_1^4}{2}P_{15}^T,\ \varphi_4^{15} = \frac{\hbar_1^2\hbar_2^2}{2}P_{16}^T,$
$\varphi_5^5 = (1-\mu_1)Q_1 - (1-\mu_1)Q_2 + (1-\mu_1)S_3,\ \varphi_6^6 = -(1-\mu_2)Q_3,\ \varphi_7^7 = -(1-\mu_1-\mu_2)Q_1,$
$\varphi_8^8 = -(1-\mu_1-\mu_2)S_3,\ \varphi_{8il}^9 = B_i^T N^T + K_{bl}^T C^T N^T,\ \varphi_9^9 = \hbar_1^2 S_1 + (\hbar_2 - \hbar_1)S_2 - 2N,$
$\varphi_9^{14} = \hbar_1^2 P_{12},\ \varphi_9^{15} = \hbar_2^2 P_{13},\ \varphi_{10}^{10} = -4(L_1 + L_1^T - \frac{\hbar_2 - \hbar_1}{3}Z_1) - 16\big[15(L_2 + L_2^T) - 20(\hbar_2 - \hbar_1)Z_3\big],$
$\varphi_{10}^{13} = 32\big[15(L_2 + L_2^T) - 20(\hbar_2 - \hbar_1)Z_3\big],\ \varphi_{11}^{11} = \frac{-1}{\hbar_1}T_1,\ \varphi_{12}^{12} = \frac{-1}{\hbar_2}T_2,$
$\varphi_{13}^{13} = -64\big[15(L_2 + L_2^T) - 20(\hbar_2 - \hbar_1)Z_3\big],\ \varphi_{14}^{14} = -2\hbar_1^2\hbar_2^2 P_{14},\ \varphi_{14}^{15} = -\hbar_2^4 P_{15} - \hbar_1^4 P_{15},$
$\varphi_{15}^{15} = -2\hbar_1^2\hbar_2^2 P_{16}.$

Proof Construct the LKF in the following form:

$$V(x_t) = \sum_{\nu=1}^{5} V_\nu(x_t),$$

where

$$V_1(x_t) = \eta^T(t)P\eta(t),$$
$$V_2(x_t) = \int_{t-\hbar(t)}^{t-\hbar_1(t)} x^T(s)Q_1x(s)ds + \int_{t-\hbar_1(t)}^{t} x^T(s)Q_2x(s)ds + \int_{t-\hbar_2(t)}^{t} x^T(s)Q_3x(s)ds,$$
$$V_3(x_t) = (\hbar_2 - \hbar_1)\int_{t-\hbar_2}^{t-\hbar_1} x^T(s)R_1x(s)ds + \int_{t-\bar{\hbar}}^{t-\hbar_1} x^T(s)R_2x(s)ds + \int_{t-\bar{\hbar}}^{t-\hbar_2} x^T(s)R_3x(s)ds,$$
$$V_4(x_t) = \int_{-\hbar_1}^{0}\int_{t+\theta}^{t} x^T(s)T_1x(s)dsd\theta + \int_{-\hbar_2}^{0}\int_{t+\theta}^{t} x^T(s)T_2x(s)dsd\theta,$$
$$V_5(x_t) = \hbar_1\int_{-\hbar_1}^{0}\int_{t+\theta}^{t} \dot{x}^T(s)S_1\dot{x}(s)dsd\theta + \int_{-\hbar_2}^{-\hbar_1}\int_{t+\theta}^{t} \dot{x}^T(s)S_2\dot{x}(s)dsd\theta$$
$$+ \int_{t-\hbar_1(t)-\hbar_2(t)}^{t-\hbar_1(t)} x^T(s)S_3x(s)ds,$$

with $\eta = col\Big\{x(t),\ \int_{t-\bar{\hbar}}^{t-\hbar_1}\int_{\theta}^{t-\hbar_1} x(s)dsd\theta,\ \int_{t-\bar{\hbar}}^{t-\hbar_2}\int_{\theta}^{t-\hbar_2} x(s)dsd\theta\Big\}.$

The derivative of $V(x_t)$ is derived as follows:

$$\begin{aligned}\dot{V}_1(x_t)) =&2\eta^T(t)P\dot{\eta}(t),\\ =&2\xi^T(t)\left\{\begin{bmatrix} e_1 \\ \hbar_1^2 e_{14} \\ \hbar_2^2 e_{15}\end{bmatrix}^T \begin{bmatrix} P_{11} & P_{12} & P_{13} \\ * & P_{14} & P_{15} \\ * & * & P_{16}\end{bmatrix}\begin{bmatrix} e_9 \\ \frac{\hbar_2^2}{2}e_3 - \hbar_2^2 e_{14} \\ \frac{\hbar_1^2}{2}e_4 - \hbar_1^2 e_{15}\end{bmatrix}\right\}\xi(t) = \xi^T(t)\Upsilon_1\xi(t),\end{aligned} \tag{7}$$

$$\begin{aligned}\dot{V}_2(x_t) \leq&\xi^T(t)\Big\{e_1^T[Q_2+Q_3]e_1 + e_5^T[(1-\mu_1)Q_1 - (1-\mu_1)Q_2]e_5 - (1-\mu_1-\mu_2)e_7^T Q_1 e_7 \\ &- (1-\mu_2)e_6^T Q_3 e_6\Big\}\xi(t) = \xi^T(t)\Upsilon_2\xi(t),\end{aligned} \tag{8}$$

$$\begin{aligned}\dot{V}_3(x_t) =&\xi^T(t)\Big\{e_2^T\big[-R_2-R_3\big]e_2 + e_3^T\big[(\hbar_2-\hbar_1)R_1 + R_2\big]e_3 + e_4^T\big[-(\hbar_2-\hbar_1)R_1 \\ &+ R_3\big]e_4\Big\}\xi(t) = \xi^T(t)\Upsilon_3\xi(t),\end{aligned} \tag{9}$$

$$\dot{V}_4(x_t) \leq\xi^T(t)\Big\{e_1^T[\hbar_1 T_1 + \hbar_2 T_2]e_1 - \frac{1}{\hbar_1}e_{11}^T T_1 e_{11} - \frac{1}{\hbar_2}e_{12}^T T_2 e_{12}\Big\}\xi(t) = \xi^T(t)\Upsilon_4\xi(t), \tag{10}$$

$$\begin{aligned}\dot{V}_5(x_t) \leq&\xi^T(t)\Big\{\hbar_1^2 e_9^T S_1 e_9 - [e_1-e_3]^T S_1[e_1-e_3] + (\hbar_2-\hbar_1)e_9^T S_2 e_9 + (1-\mu_1)e_5^T S_3 e_5 \\ &- (1-\mu_1-\mu_2)e_8^T S_3 e_8\Big\}\xi(t) - \int_{t-\hbar_2}^{t-\hbar_1}\dot{x}^T(s)S_2\dot{x}(s)ds \\ =&\xi^T(t)\Upsilon_5\xi(t) - \int_{t-\hbar_2}^{t-\hbar_1}\dot{x}^T(s)S_2\dot{x}(s)ds.\end{aligned} \tag{11}$$

applying Lemma 1 in the integral $-\int_{t-\hbar_2}^{t-\hbar_1}\dot{x}^T(s)S_2\dot{x}(s)ds$ yields

$$\begin{aligned}-\int_{t-\hbar_2}^{t-\hbar_1}\dot{x}^T(s)S_2\dot{x}(s)ds \leq&\xi^T(t)\Big\{\frac{-1}{\hbar_2-\hbar_1}[e_3-e_4]^T S_2[e_3-e_4] - [e_3+e_4-2e_{10}]^T \\ &\times\big(L_1+L_1^T - \frac{\hbar_2-\hbar_1}{3}Z_1\big)[e_3+e_4-2e_{10}] \\ &- [4e_{10}-8e_{13}]^T\big(15(L_2+L_2^T) - 20(\hbar_2-\hbar_1)Z_3\big) \\ &\times[4e_{10}-8e_{13}]\Big\}\xi(t) = \xi^T(t)\Upsilon_6\xi(t).\end{aligned} \tag{12}$$

The following equation is obtained from the system (5) for any matrix N and any scalar β

$$\begin{aligned}0 =&[e_9+\beta e_1]2N\Big\{\sum_{i=1}^{p}\sum_{l=1}^{p}\zeta_i(s(t))\zeta_l(s(t))\Big[A_i e_1 + B_i e_8 + C_i(K_{al}e_1 + K_{bl}e_8\Big] - e_9\Big\} \\ =&\xi^T(t)\Upsilon_7\xi(t).\end{aligned} \tag{13}$$

From (7) to (13), the upper bound of $\dot{V}(x_t)$ is obtained as

$$\dot{V}(x_t) \leq \sum_{i=1}^{p}\sum_{l=1}^{p}\zeta_i(s(t))\zeta_l(s(t))\xi^T(t)\left\{\sum_{a=1}^{7}\Upsilon_a\right\}\xi(t) = \sum_{i=1}^{p}\sum_{l=1}^{p}\zeta_i(s(t))\zeta_l(s(t))\xi^T(t)\Omega_{i,l}\xi(t), \tag{14}$$

where $\xi(t)$ is given in the main results and $\Omega_{i,l}$ is given in (6). If the LMI (6) hold then the condition defined in (14) is satisfied. Thus the system (5) is asymptotically stable, this completes the proof.

Remark 1 In the derivative of $V_5(x(t))$ there exists single integral term $\int_{t-\hbar_2}^{t-\hbar_1}\dot{x}^T(s)S_2\dot{x}(s)ds$ in which integral inequality based on non-orthogonal polynomials has been applied. This integral inequality helps to derive a less conservative result.

Theorem 2 *For given scalars $\hbar_1 > 0, \hbar_2 > 0, \mu_1, \mu_2$ and unknown control gain matrices K_{al}, K_{bl}, the system (5) with additive time delays $\hbar_1(t)$, $\hbar_2(t)$ satisfying condition (2) is asymptotically stable if there exist positive definite symmetric matrices $\check{P}, \check{Q}_i, \check{R}_i, \check{S}_i (i = 1, 2, 3), \check{T}_i (i = 1, 2)$ and any matrices $\check{L}_i, \check{Z}_i (i = 1, 2, 3)$ such that the following LMI is satisfied:*

$$\check{\Omega}_{i,l} = \begin{bmatrix}
\check{\varphi}_{1il}^{1} & 0 & \check{\varphi}_{1}^{3} & \check{\varphi}_{1}^{4} & 0 & 0 & 0 & \check{\varphi}_{1il}^{8} & \check{\varphi}_{1il}^{9} & 0 & 0 & 0 & 0 & \check{\varphi}_{1}^{14} & \check{\varphi}_{1}^{15} \\
* & \check{\varphi}_{2}^{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
* & * & \check{\varphi}_{3}^{3} & \check{\varphi}_{3}^{4} & 0 & 0 & 0 & 0 & 0 & \check{\varphi}_{3}^{10} & 0 & 0 & \check{\varphi}_{3}^{13} & \check{\varphi}_{3}^{14} & \check{\varphi}_{3}^{15} \\
* & * & * & \check{\varphi}_{4}^{4} & 0 & 0 & 0 & 0 & 0 & \check{\varphi}_{4}^{10} & 0 & 0 & \check{\varphi}_{4}^{13} & \check{\varphi}_{4}^{14} & \check{\varphi}_{4}^{15} \\
* & * & * & * & \check{\varphi}_{5}^{5} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
* & * & * & * & * & \check{\varphi}_{6}^{6} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
* & * & * & * & * & * & \check{\varphi}_{7}^{7} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
* & * & * & * & * & * & * & \check{\varphi}_{8}^{8} & \check{\varphi}_{8il}^{9} & 0 & 0 & 0 & 0 & 0 & 0 \\
* & * & * & * & * & * & * & * & \check{\varphi}_{9}^{9} & 0 & 0 & 0 & 0 & \check{\varphi}_{9}^{14} & \check{\varphi}_{9}^{15} \\
* & * & * & * & * & * & * & * & * & \check{\varphi}_{10}^{10} & 0 & 0 & \check{\varphi}_{10}^{13} & 0 & 0 \\
* & * & * & * & * & * & * & * & * & * & \check{\varphi}_{11}^{11} & 0 & 0 & 0 & 0 \\
* & * & * & * & * & * & * & * & * & * & * & \check{\varphi}_{12}^{12} & 0 & 0 & 0 \\
* & * & * & * & * & * & * & * & * & * & * & * & \check{\varphi}_{13}^{13} & 0 & 0 \\
* & * & * & * & * & * & * & * & * & * & * & * & * & \check{\varphi}_{14}^{14} & \check{\varphi}_{14}^{15} \\
* & * & * & * & * & * & * & * & * & * & * & * & * & * & \check{\varphi}_{15}^{15}
\end{bmatrix} < 0, \tag{15}$$

where

$$\check{\varphi}_{1il}^{1} = \check{Q}_2 + \check{Q}_3 + \hbar_1\check{T}_1 + \hbar_2\check{T}_2 - \check{S}_1 + 2\beta A_i\check{N} + 2\beta CF_{al},\ \check{\varphi}_1^3 = \frac{\hbar_2^2}{2}\check{P}_{12} + \check{S}_1,\ \check{\varphi}_1^4 = \frac{\hbar_1^2}{2}\check{P}_{13},$$

$$\check{\varphi}_{1il}^{8} = \beta B_i\check{N} + \beta CF_{bl},\ \check{\varphi}_{1il}^{9} = \check{P}_{11} + \check{N}^T A_i^T + F_{al}^T C^T - \beta\check{N},\ \check{\varphi}_1^{14} = -\check{P}_{12}\hbar_2^2,\ \check{\varphi}_1^{15} = -\check{P}_{13}\hbar_1^2,$$

$$\check{\varphi}_2^2 = -\check{R}_2 - \check{R}_3,\ \check{\varphi}_3^3 = (\hbar_2 - \hbar_1)\check{R}_1 + \check{R}_2 - \check{S}_1 - \frac{1}{\hbar_2 - \hbar_1}\check{S}_2 - (\check{L}_1 + \check{L}_1^T - \frac{\hbar_2 - \hbar_1}{3}\check{Z}_1),$$

$$\check{\varphi}_3^4 = \frac{1}{\hbar_2 - \hbar_1}\check{S}_2 - (\check{L}_1 + \check{L}_1^T - \frac{\hbar_2 - \hbar_1}{3}\check{Z}_1),\ \check{\varphi}_3^{10} = 2(\check{L}_1 + \check{L}_1^T - \frac{\hbar_2 - \hbar_1}{3}\check{Z}_1) - 80\check{L}_2^T,$$

$$\check{\varphi}_3^{13} = 160\check{L}_2^T,\ \check{\varphi}_3^{14} = \frac{\hbar_2^2\hbar_1^2}{2}\check{P}_{14}^T,\ \check{\varphi}_3^{15} = \frac{\hbar_2^4}{2}\check{P}_{15},\ \check{\varphi}_4^4 = -(\hbar_2 - \hbar_1)\check{R}_1 + \check{R}_3 - \frac{1}{\hbar_2 - \hbar_1}\check{S}_2$$

$$-(\check{L}_1+\check{L}_1^T-\frac{\hbar_2-\hbar_1}{3}\check{Z}_1),\ \check{\varphi}_4^{10}=2(\check{L}_1+\check{L}_1^T-\frac{\hbar_2-\hbar_1}{3}\check{Z}_1)+80\check{L}_2^T,\ \check{\varphi}_4^{13}=-160\check{L}_2^T,$$
$$\check{\varphi}_4^{14}=\frac{\hbar_1^4}{2}\check{P}_{15}^T,\ \check{\varphi}_4^{15}=\frac{\hbar_1^2\hbar_2^2}{2}\check{P}_{16}^T,\ \check{\varphi}_5^5=(1-\mu_1)\check{Q}_1-(1-\mu_1)\check{Q}_2+(1-\mu_1)\check{S}_3,\ \check{\varphi}_6^6=-(1-\mu_2)\check{Q}_3,$$
$$\check{\varphi}_7^7=-(1-\mu_1-\mu_2)\check{Q}_1,\ \check{\varphi}_8^8=-(1-\mu_1-\mu_2)\check{S}_3,\ \check{\varphi}_{8il}^9=\check{N}^TB_i^T+F_{bl}^TC^T,$$
$$\check{\varphi}_9^9=\hbar_1^2\check{S}_1+(\hbar_2-\hbar_1)S_2-2\check{N},\ \check{\varphi}_9^{14}=\hbar_1^2\check{P}_{12},\ \check{\varphi}_9^{15}=\hbar_2^2\check{P}_{13},\ \check{\varphi}_{10}^{10}=-4(\check{L}_1+\check{L}_1^T-\frac{\hbar_2-\hbar_1}{3}\check{Z}_1)$$
$$-16\big[15(\check{L}_2+\check{L}_2^T)-20(\hbar_2-\hbar_1)\check{Z}_3\big],\ \check{\varphi}_{10}^{13}=32\big[15(\check{L}_2+\check{L}_2^T)-20(\hbar_2-\hbar_1)\check{Z}_3\big],\ \check{\varphi}_{11}^{11}=\frac{-1}{\hbar_1}\check{T}_1,$$
$$\check{\varphi}_{12}^{12}=\frac{-1}{\hbar_2}\check{T}_2,\ \check{\varphi}_{13}^{13}=-64\big[15(\check{L}_2+\check{L}_2^T)-20(\hbar_2-\hbar_1)\check{Z}_3\big],\ \check{\varphi}_{14}^{14}=-2\hbar_1^2\hbar_2^2\check{P}_{14},$$
$$\check{\varphi}_{14}^{15}=-\hbar_2^4\check{P}_{15}-\hbar_1^4\check{P}_{15},\ \check{\varphi}_{15}^{15}=-2\hbar_1^2\hbar_2^2\check{P}_{16}.$$

Then the control gain matrices can be constructed as $K_{al}=F_{al}\check{N}^{-1}$, $K_{bl}=F_{bl}\check{N}^{-1}$.

Proof Let us now consider $K_{al}\check{N}=F_{al}$, $K_{bl}\check{N}=F_{bl}$ and $\Gamma=col\{\check{N},\check{N},\check{N},\check{N},\check{N},\check{N},\check{N},\check{N},\check{N},\check{N},\check{N},\check{N},\check{N},\check{N},\check{N}\}$ where $\check{N}=N^{-1}$. Let us now consider the other matrices as $\check{P}=\check{N}P\check{N}$, $\check{Q}_i=\check{N}Q_i\check{N}$, $\check{R}_i=\check{N}R_i\check{N}$, $\check{S}_i=\check{N}S_i\check{N}$, $\check{T}_i=\check{N}T_i\check{N}$, $\check{L}_i=\check{N}L_i\check{N}$, $\check{Z}_i=\check{N}Z_i\check{N}$. Pre- and post-multiplication of Γ^T and Γ in LMI (6) leads to LMI (15). The proof is complete.

4 Numerical Examples

Example 1 Consider the delayed system (5) with parameters

$$A_1=\begin{bmatrix}-2.1 & 0.1\\ -0.2 & -0.9\end{bmatrix},\ B_1=\begin{bmatrix}-1.1 & 0.1\\ -0.8 & -0.9\end{bmatrix},\ C_1=\begin{bmatrix}0.14 & 0\\ 0.1 & 1.15\end{bmatrix},$$
$$A_2=\begin{bmatrix}-1.9 & 0\\ -0.2 & -1.1\end{bmatrix},\ B_2=\begin{bmatrix}-0.9 & 0\\ -1.1 & -1.2\end{bmatrix},\ C_2=\begin{bmatrix}0.13 & -0.1\\ 0 & 0.12\end{bmatrix}.$$

Membership function is chosen in the form that $\zeta_1(t)=\frac{1}{1+e^{-2x_1(t)}}$ and $\zeta_2(t)=1-\zeta_1(t)$. Moreover, let $\mu_1=0.1$, $\mu_2=0.1$, $\hbar_1=0.1$, $\beta=0.1$ and solving the LMIs in Theorem 2, the obtained maximum upper bound $\hbar_2$ is 3.2562. Also, the control gain matrices corresponding to Theorem 2 are obtained as

$$K_{a1}=\begin{bmatrix}-197.6648 & -197.9296\\ 19.2615 & 18.8824\end{bmatrix},\ K_{a2}=\begin{bmatrix}-197.6648 & -197.9296\\ 19.2615 & 18.8824\end{bmatrix},$$
$$K_{b1}=\begin{bmatrix}9.9122 & 2.6148\\ -0.1506 & 0.4711\end{bmatrix},\ K_{b2}=\begin{bmatrix}9.9122 & 2.6148\\ -0.1506 & 0.4711\end{bmatrix}.$$

The state response of the closed-loop system is obtained by assuming $\hbar_1(t)=0.4+0.1\sin t$, $\hbar_2(t)=0.8\sin t$ under initial condition $x(0)=[2\ \ -2]^T$. The state

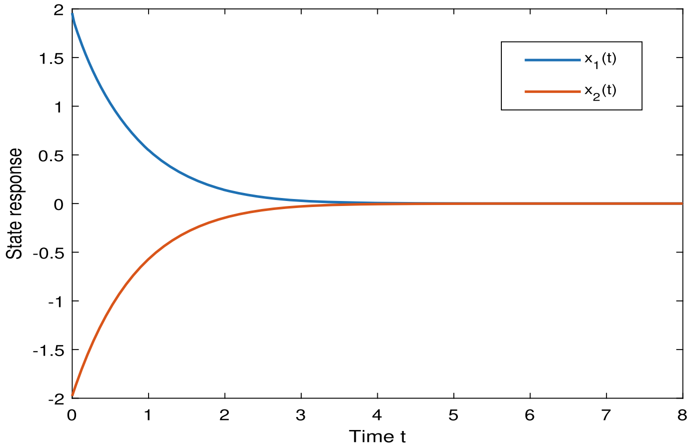

Fig. 1 State trajectories with $\hbar_1(t) = 0.4 + 0.1\sin t$, $\hbar_2(t) = 0.8\sin t$ (Example (1))

trajectory of the closed-loop system (5) under the obtained control gain matrices is expressed in Fig. 1. This implies that the additive time-varying delayed T–S fuzzy system converge to origin under the proposed controller.

Example 2 Consider the delayed system (5) with $C = 0$ gives

$$\dot{x}(t) = \sum_{i=1}^{p} \zeta_i(s(t))\Big[A_i x(t) + B_i x(t - \hbar_1(t) - \hbar_2(t))\Big], \tag{16}$$

and the parameters are as follows:

$$A_1 = \begin{bmatrix} -2 & 0 \\ 0 & -0.9 \end{bmatrix}, \; B_1 = \begin{bmatrix} -1 & 0 \\ -1 & -1 \end{bmatrix}.$$

Consider the LMIs in Theorem 2 with $C = 0$, for different values of $\hbar_1$ and $\mu_1 = 0.1$, $\mu_2 = 0.1$ the maximum allowable upper bound $\hbar_2$ is calculated and tabulated in Table 1, and for different values of $\hbar_2$ and $\mu_1 = 0.1$, $\mu_2 = 0.1$ the allowable upper bound $\hbar_1$ is calculated and tabulated in Table 2. When compared with the existing results, the acquired results, as shown in the table, are less conservative. Moreover, for the proposed T–S fuzzy system, the delay-dependent conditions obtained increase the delay bound.

Table 1 The obtained MAUBs $\hbar_2$ under $\mu_1 = 0.1,\ \mu_2 = 0.1$

Methods	$\hbar_1 = 1.0$	$\hbar_1 = 1.1$	$\hbar_1 = 1.2$	$\hbar_1 = 1.5$
[12]	1.198	1.027	0.980	0.610
[3]	0.9999	1.0770	0.9725	0.6807
[9]	1.2136	1.1136	1.0137	0.7137
Theorem 2	1.7231	1.6953	1.5135	1.2356

Table 2 The obtained MAUBs $\hbar_1$ under $\mu_1 = 0.1,\ \mu_2 = 0.1$

Methods	$\hbar_2 = 0.3$	$\hbar_2 = 0.4$	$\hbar_2 = 0.5$
[12]	1.708	1.645	1.574
[3]	1.8804	1.7798	1.6759
[9]	1.9137	1.8137	1.7136
Theorem 2	2.4135	2.3651	2.2355

5 Conclusion

The stability problem of T–S fuzzy system has been studied with two additive time-varying delays. A state feedback control design has been considered to stabilize the system. The control design takes the form of a state with additive time delays. In order to get less conservative results, augmented-type LKFs are constructed and an integral inequality based on non-orthogonal polynomials has been employed. The conservative results in the form of linear matrix inequalities have been obtained. Two numerical examples have been given to illustrate the improvement and efficacy of the proposed method.

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