The External Prestress Effect of Curved Tendons on the Natural Vibration Characteristics of Steel Beams

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Abstract In order to explore the external prestress effect of the curved tendons on the stiffness and natural vibration characteristics of the steel beam, this paper deduced the calculation equation of the natural frequency on the external prestressed simply supported steel beam of the curved arrangement, which was based on the Hamilton principle. The natural frequency is calculated by combining the example of I-shaped simply supported steel beam, which was analyzed and verified by establishing the finite element model. The results show that: the calculation of the equation is well demonstrated by the finite element results, and the validity of model equation was verified. When the applied prestress increases, the natural vibration frequency decreases and the change range is not large, which indicates that the magnitude of the prestress has little effect on the natural frequency of simply supported steel beams.

Keywords Simple supported beam · External prestress · Curved tendons · Natural frequency

1 Introduction

Prestress can affect the dynamic characteristics of the beam to a certain degree. When the anchoring methods of prestressed reinforcement and the beam are determined, the influencing factors mainly include the types of tendons, the magnitude of prestress and so on. There are three types of external prestressing tendons: straight-line tendons, broken line tendons and curve tendons. The convenient construction makes the first two more common, while the curve tendons help the components obtain more ideal stress distribution.

In summary, there is little research on the external prestress effect of the curved tendons on the natural vibration characteristics of the steel beam, and it is necessary to analyze and verify the characteristics by using different theories and research methods. Based on the Classical Dynamic theory, the authors apply the Hamilton

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principle to the calculation of natural vibration characteristics of the external prestressed simply supported steel beam of the curved arrangement for the first time. Then the vertical bending vibration differential equation of the beam is established by variational method, and the analytical expressions of its natural frequency and vibration shape are solved; Taking the curved externally prestressed I-shaped simply supported steel beam as an example and combining with the finite element calculation results, this paper analyzes the external prestress effect of the curved tendons on the natural vibration characteristics of the simply supported steel beam.

2 Establishment of the Vibration Equation Based on the Hamilton Principle

Hamilton principle:

$$\int_{t_1}^{t_2} \delta(T - V)dt + \int_{t_1}^{t_2} \delta W_{nc} dt = 0 \tag{1}$$

Kinetic energy generated by the beam during vibration:

$$T = \int_0^1 \frac{1}{2} m\dot{y}^2 dx \tag{2}$$

m—mass per unit length; $\dot{y}$— the first derivative of the dynamic deflection y of the beam with respect to time t.

The potential energy generated by the beam during vibration is divided into three parts:

One is the bending strain energy of the beam:

$$V_1 = \int_0^1 \frac{1}{2} E_c I_c (y'')^2 dx \tag{3}$$

where: $E_c I_c$ denotes the bending stiffness of simply supported beam itself; $(y'')^2$ means the second derivative of the dynamic deflection y of the beam to the spatial coordinate x.

The second one is the strain energy of prestressed reinforcement:

$$V_2 = \int_0^1 \frac{1}{2} E_p A_p (y'')^2 h dx \tag{4}$$

According to Reference [1], $h = \int_0^l \left((e(x))^2 (\cos\theta)^2 \cdot 10^{-3}\right) dx$ can be induced; Where, $\theta = de(x)\big/dx$; $e(x) = C_1 x^2 + C_2 x + C_3$; x represents the distance from the coordinate origin to the calculated section along the beam axis; E_p indicates elastic modulus of prestressed reinforcement; A_p refers to section area of prestressed reinforcement; $e(x)$ is the eccentricity of prestressed reinforcement; θ means included angle between prestressed reinforcement and beam axis.

The third is the tensile and compressive strain energy of the beam:

$$V_3 = \int_0^1 \frac{1}{2} E_c A_c r_q^2 h dx \tag{5}$$

where, $r_q = I_p y''$;$I_p = \left(E_p A_p\right)\big/(E_c A_c)$ represents the converted reinforcement ratio of externally prestressed tendons; E_c represents elastic modulus of beam; A_c means cross sectional area of beam body.

Therefore, the total potential energy is:

$$V = V_1 + V_2 + V_3 \tag{6}$$

When the beam vibrates, the work done by the non-conservative force is:

$$W_{nc} = \int_0^1 \frac{1}{2} N_{p\theta} \cos\theta_o (y')^2 dx \tag{7}$$

where, $N_{p\theta} \cos\theta_0$ stands for the horizontal component of initial prestress value; y' is the first derivative of dynamic deflection y to space coordinate t.

Variation of kinetic energy:

$$\int_{t_1}^{t_2} \delta T dt = -\int_{t_1}^{t_2} \int_0^l m\ddot{y} \delta y dx dt \tag{8}$$

Variation of bending strain energy:

$$\int_{t_1}^{t_2} \delta V_1 dxdt = \delta \int_{t_1}^{t_2} \int_0^l \frac{1}{2} E_c I_c (y'')^2 dxdt = \int_{t_1}^{t_2} \int_0^l \frac{1}{2} E_c I_c y^{(4)} \delta y dxdt + \int_{t_1}^{t_2} E_c I_c y_l'' \delta y_l' dt - \int_{t_1}^{t_2} E_c I_c y_0'' \delta y_0' dt - \int_{t_1}^{t_2} E_c I_c y_l^{(3)} \delta y_l dt + \int_{t_1}^{t_2} E_c I_c y_0^{(3)} \delta y_0 dt \quad (9)$$

Variation of strain energy of prestressed reinforcement:

$$\int_{t_1}^{t_2} \delta V_2 dxdt = \delta \int_{t_1}^{t_2} \int_0^l \frac{1}{2} E_p A_p (y'')^2 h dxdt = \int_{t_1}^{t_2} \int_0^l \frac{1}{2} E_p A_p h y^{(4)} \delta y dxdt + \int_{t_1}^{t_2} E_p A_p h y_l'' \delta y_l' dt - \int_{t_1}^{t_2} E_p A_p h y_0'' \delta y_0' dt - \int_{t_1}^{t_2} E_p A_p h y_l^{(3)} \delta y_l dt + \int_{t_1}^{t_2} E_p A_p h y_0^{(3)} \delta y_0 dt \quad (10)$$

Variation of beam tensile and compressive strain energy:

$$\int_{t_1}^{t_2} \delta V_3 dxdt = \delta \int_{t_1}^{t_2} \int_0^l \frac{1}{2} E_c A_c (I_P y'')^2 h dxdt = \int_{t_1}^{t_2} \int_0^l \frac{1}{2} E_c A_c I_p^2 h y^{(4)} \delta y dxdt + \int_{t_1}^{t_2} E_c A_c I_p^2 h y_l'' \delta y_l' dt - \int_{t_1}^{t_2} E_c A_c I_p^2 h y_0'' \delta y_0' dt - \int_{t_1}^{t_2} E_c A_c I_p^2 h y_l^{(3)} \delta y_l dt + \int_{t_1}^{t_2} E_c A_c I_p h y_0^{(3)} \delta y_0 dt \quad (11)$$

Variation of total potential energy:

$$\int_{t_1}^{t_2} \delta V dt = \int_{t_1}^{t_2} \int_0^l D y^{(4)} \delta d y dxdt + \int_{t_1}^{t_2} D y_l'' \delta y_l' dt$$

$$-\int_{t_1}^{t_2} Dy_0''\delta y_0' dt - \int_{t_1}^{t_2} Dy_l^{(3)}\delta y_l dt + \int_{t_1}^{t_2} Dy_0^{(3)}\delta y_0 dt \tag{12}$$

where:

$$D = E_c I_c + E_p A_p h + E_c A_c I_p^2 h = E_c I_c + E_p A_p h\left(1 + \frac{E_p A_p}{E_c A_c}\right) \tag{13}$$

Variation of non-conservative work:

$$\begin{gathered}\delta\int_{t_1}^{t_2} W_{nc} = \delta\int_{t_1}^{t_2}\int_0^l \frac{1}{2}N_{p\theta}\cos\theta_o(y')^2 dxdt = -\int_{t_1}^{t_2}\int_0^l N_{p\theta}\cos\theta_o y''\delta y dxdt \\ \int_{t_1}^{t_2} N_{p\theta}\cos\theta_0 y_l'\delta y_l dt - \int_{t_1}^{t_2} N_{p\theta}\cos\theta_0 y_0'\delta y_0 dt\end{gathered} \tag{14}$$

Through integrating Eqs. (12) and (14), the following can be obtained:

$$\begin{gathered}\int_{t_1}^{t_2}\int_0^l (-m\ddot{y} - D - N_{p\theta}\cos\theta_o y'')\delta y dxdt - \int_{t_1}^{t_2} Dy_l''\delta y_l' dt + \int_{t_1}^{t_2} Dy_0''\delta y_0' dt \\ + \int_{t_1}^{t_2} (Dy_l^{(3)} + N_{p\theta}\cos\theta_0 y_l')\delta y_l dt - \int_{t_1}^{t_2} (Dy_0^{(3)} + N_{p\theta}\cos\theta_0 y_0')\delta y_0 dt = 0\end{gathered} \tag{15}$$

According to the variational calculation, the variation is equal to zero on the boundary displacement condition, and the variation in the domain is arbitrary, then:

$$m\ddot{y} + D + N_{p\theta}\cos\theta_0 y'' = 0 \tag{16}$$

Therefore, Eq. (16) is the bending vibration equation of simply supported beam. From the solution of higher-order differential equation, the following can be obtained:

$$\begin{gathered}\omega_n = \left(\frac{n\pi}{l}\right)^2\sqrt{\frac{E_c I_c}{m}}\sqrt{1 - \frac{N_{p\theta}\cos\theta_0}{\left(\frac{n\pi}{l}\right)^2 E_c I_c}} \\ \sqrt{1 + \frac{E_p A_p h}{E_c I_c\left(1 - \frac{N_{p\theta}\cos\theta_0}{(\frac{n\pi}{l})^2 E_c I_c}\right)}(1 + I_p)}(n = 1, 2, \cdots, \infty)\end{gathered} \tag{17}$$

where: ω_n is the n-th natural vibration circle frequency; the fourth item is the correction item of natural frequency of simply supported beam after prestressing.

According to the relationship between natural circular frequency and natural frequency, the natural frequency of external prestressed simply supported beam is:

$$f_n = \frac{\omega_n}{2\pi} = \frac{n^2\pi}{2l^2}\sqrt{\frac{E_c I_c}{m}}\sqrt{1 - \frac{N_{p\theta}\cos\theta_0}{\left(\frac{n\pi}{l}\right)^2 E_c I_c}}$$
$$\sqrt{1 + \frac{E_p A_p h}{E_c I_c\left(1 - \frac{N_{p\theta}\cos\theta_0}{\left(\frac{n\pi}{l}\right)^2 E_c I_c}\right)}(1 + I_p)}(n = 1, 2, \cdots, \infty) \quad (18)$$

The vibration shape function: $\varphi(x) = A_3 \sin\frac{n\pi x}{l}$. Where, the magnitude of A_3 does not affect the variation law of vibration shape function.

3 Reliability Verification of Vibration Equation

Equation (17) can be simplified to $\omega_n = \left(\frac{n\pi}{l}\right)^2\sqrt{\frac{E_c I_c}{m}}(n = 1, 2, \cdots, \infty)$ without the effect of curve prestress, while in Reference [2], the natural vibration circle frequency is $\omega_n = n^2\pi^2\sqrt{\frac{EI}{mL^4}}(n = 1, 2, \cdots, \infty)$ without considering the moment of inertia and shear deformation, so the two equations are consistent. Equation (17) can be simplified as $\omega_n = \left(\frac{n\pi}{l}\right)^2\sqrt{\frac{E_c I_c}{m}}\sqrt{1 - \frac{N_{p\theta}\cos\theta_0}{\left(\frac{n\pi}{l}\right)^2 E_c I_c}}(n = 1, 2, \cdots, \infty)$ without the action of eccentricity of curved prestressing bar, while in Reference [2], the natural circular frequency of the beam under axial force is $\omega_n = n^2\pi^2\sqrt{1 - \frac{NL^2}{n^2\pi^2 EI}}\sqrt{\frac{EI}{mL^4}}(n = 1, 2, \cdots, \infty)$, so the two equations are consistent. After replacing the curve eccentricity influence coefficient h with linear eccentricity e_p^2, Eq. (17) can be simplified as $\omega_n = (\frac{n\pi}{l})^2\sqrt{\frac{E_c I_c}{m}}\sqrt{1 - \frac{N_{p\theta}\cos\theta_0}{(\frac{n\pi}{l})^2 E_c I_c}}\sqrt{1 + \frac{E_p A_p e_p^2}{E_c I_c\left(1 - \frac{N_{p\theta}\cos\theta_0}{(\frac{n\pi}{l})^2 E_c I_c}\right)}(1 + I_p)}(n = 1, 2, \cdots, \infty)$, while in reference [3], the natural circular frequency of external prestressed simply supported beams with eccentric linear reinforcement is

$$\omega_n = \left(\frac{n\pi}{l}\right)^2\sqrt{\frac{E_c I_c}{m}}\sqrt{1 + \frac{E_p A_p e_p^2}{E_c I_c\left(1 - \frac{N_{p0}\cos\theta_0}{\left(\frac{n\pi}{l}\right)^2 E_c I_c}\right)}(1 + I_p) - \frac{N_{p0} l^2}{n^2\pi^2 E_c I_c}}(n = 1, 2, \cdots, \infty),$$

so, the two equations consistent. By comparing the above equations with the equation in the special case of the existing reference, it can be seen that the equation established in this paper is more general.

4 Finite Element Verification

4.1 Influence of Prestress Value on Natural Frequency

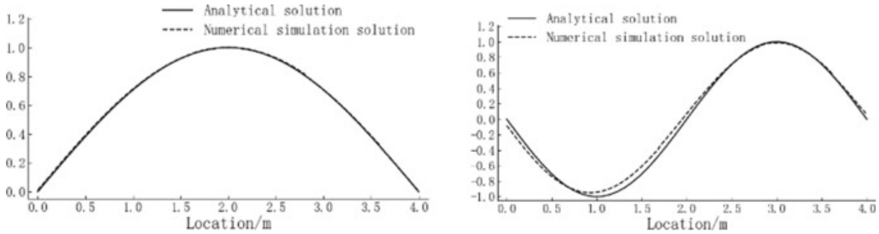
The cross-sectional area of prestressed cable is 139 mm^2, and the range of prestress value is 0–200 kn. The comparison between the calculated value of natural vibration frequency of simply supported beam and the finite element simulation value is shown in Table 1. It can be seen that under different prestress, the analytical solution of the first-order natural frequency of simply supported steel beam is very close to the finite element calculation results, the variation trend is consistent, and the maximum error is no more than 2%. It verifies the reasonableness of the calculation methods in this paper; The calculation error of the second-order natural frequency is relatively large, which is the result of the large degree of freedom of the finite element solution. The displacement element solution will lead to the small overall stiffness, and the larger solution order means the larger finite element calculation error, so it is reasonable to make the calculation result of the second-order frequency small [3].

In order to further verify the correctness of the equations established in this paper, the vibration shape calculation solution derived in this paper is compared with the vibration shape finite element solution calculated by ANSYS. The comparison results of the first-order vibration shape and the second one are shown in Fig. 1.

As can be seen from Fig. 1, the calculation results in this paper are in good agreement with the finite element simulation results, especially the first-order vibration shape, and the second-order vibration modes can basically coincide. The result further shows that the calculation method of the external prestressed simply supported steel beam of the curved arrangement in this paper is correct and reliable.

Table 1 The first two-order natural frequency and error with different preloads

Prestress (kN)	First-order frequency (Hz)			Prestress (kN)	Second-order frequency (Hz)		
	Calculated value	Finite element value	Error (%)		Calculated value	Finite element value	Error (%)
0	48.056	48.627	−1.20	0	192.225	174.486	9.23
50	47.477	46.682	1.67	50	191.859	173.904	9.36
100	46.819	46.140	1.45	100	191.235	173.423	9.31
150	46.152	45.591	1.22	150	190.561	172.839	9.30
200	45.474	45.034	0.97	200	189.908	172.353	9.24



（a）First-order vibration mode （b）Second-order vibration mode

Fig. 1 Vibration shape comparison chart

Table 2 The first-order natural frequency and error with different cross-sectional areas of tendons

Cross-sectional area (mm^2)	The first-order natural frequency (Hz)		
	Calculated value	Finite element value	Calculated value
54.8	44.568	54.8	44.568
98.7	45.130	98.7	45.130
139	45.474	139	45.474
191	45.716	191	45.716
285	45.996	285	45.996

4.2 *Influence of Prestressed Cable Section Area on Natural Frequency*

By using different specifications of steel strands, the cross-sectional areas of prestressed cables are changed and the variation law of natural frequency is analyzed. If the prestress value is 200 kN, the comparison between the calculated value of the first-order natural frequency of the simply supported beam and the finite element simulation value can be seen in Table 2. As the cross-sectional area of prestressed cable gradually increases, the natural frequency of simply supported steel beam gradually increases (as shown in Fig. 2).

5 Conclusions

(1) In this paper, based on Hamilton energy variational principle, this paper established the bending vibration differential equation of external prestressed simply supported steel beams with curved tendons for the first time, and then obtained the solution of the analytical expression of natural vibration frequency and the vibration shape function; The accuracy of the equation is verified by the comparative analysis of the equation and the finite element calculation results.

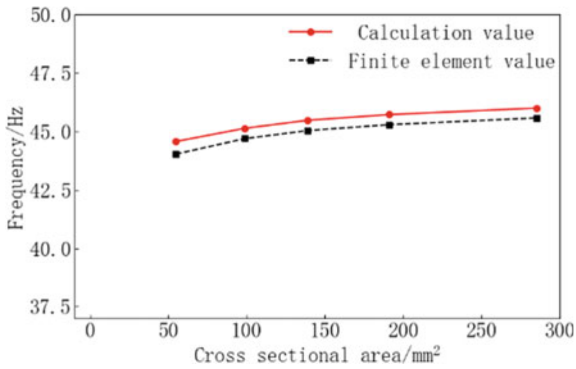


Fig. 2 Frequency comparison chart

(2) With the gradual increase of prestress, the natural frequency of simply supported steel beam decreases gradually, but the change range is small and less than 5%, indicating that the applied prestress has little effect on the natural frequency of simply supported steel beam; With the increasing cross-sectional area of prestressed cable, the natural frequency of simply supported steel beam increases gradually.

(3) The equations established in this paper are suitable for external prestressed simply supported steel beams with curved tendons. The adopted methods and the deduced equations carry a certain reference value for the calculation and analysis of prestressed concrete beams with unbonded [4] curved tendons.

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