Higher-Dimensional Geometry from Fano to Mori and Beyond

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Abstract Gino Fano's work has had a great impact on the development of modern projective geometry, in particular the studies of the varieties named after him.

Starting from Fano's results, a large number of mathematicians, often part of opposing schools, have constructed a bunch of theories in the last 50 years, which are among the most spectacular achievements of contemporary mathematics.

Keywords Fano varieties · Birational maps · Minimal model programme · Extremal rays · Rationally connected varieties

1 Introduction

The study of higher-dimensional varieties (higher than curves and surfaces) was started by B. Riemann in a remarkable lecture in 1854. Since then, the new concepts of *Mannigfaltigkeit* (variety or manifold) and of *Massverhältnisse* (metric relation) developed in various directions giving rise to different research areas in contemporary mathematics. All these theories are based on a very abstract way of thinking, similar to what happened in all arts in the same period, and they require a very strong mathematical capability and a great rigor.

The case of Algebraic Geometry was taken over soon by the Italian school at the end of 1800, for instance, by L. Cremona, G. Veronese, and C. Segre. They considered higher-dimensional projective space and properties of its linear subspaces and of its subvarieties. They studied the linear systems of divisors on these varieties, in particular the canonical system which contains information about the curvature. They understood that a classification of projective varieties should depend on the canonical divisor.

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G. Fano, a student of C. Segre, started a systematic study of projective varieties of dimension 3 in the early 1900. His pioneering work was remarkably original and deep, although at the time the necessary mathematical tools, especially in the field of Algebra, were not well developed. It is generally accepted that his proofs are not enough rigorous for the modern standard; on the other hand, they contain many intuitions on the geometry of projective threefolds, which turned out to be correct and fundamental.

Starting from Fano's results, a large number of mathematicians, often members of opposing schools, have constructed clever theories in the last 50 years, which are among the most spectacular achievements of contemporary mathematics. A starting point for the contemporary study of Fano's legacy is the work of V. Iskovskikh and V. Shokurov. The theory of minimal models developed by the Fields medalist S. Mori gave an enormous impulse; on the one hand, it changed the approach to classification of projective varieties and on the other hand gave to the objects studied by Fano a central place in the classification. In the last 15 years, many crucial conjectures were proved, among them the feasibility of the minimal model program in any dimension, under some assumptions, in the celebrated paper by C. Birkar et al. [10].

2 Fano Varieties and Fano-Mori Contractions

We consider normal projective varieties X defined over $\mathbb{C}$; if n is the dimension of X, we sometime call X and *n-fold*. We denote by K_X the *canonical sheaf*; we assume to have good singularities such that K_X, or a multiple of it, is a line bundle (a Cartier divisor).

Let $X \subset \mathbb{P}^N$ be a projective threefold such that for general hyperplanes $H_1 and H_2$, the curve $\Gamma := X \cap H_1 \cap H_2$ is canonically embedded into $H_1 \cap H_2$ (i.e., K_Γ embeds Γ). Fano called them *Varietá algebriche a tre dimensioni a curve sezioni canoniche* [20–23].

It is not difficult to prove that a smooth threefold X (one can allow mild singularities) whose general curve section Γ is canonically embedded has the anticanonical bundle, $-K_X$, very ample. Actually the anticanonical linear system, $|-K_X|$, embeds X as a threefold of degree $2g-2$ into a projective space of dimension $g+1$, $X := X^3_{2g-2} \subset \mathbb{P}^{g+1}$, where $g = g(\Gamma)$ is the genus of Γ.

An obvious example is given by the quartic threefold in $\mathbb{P}^4$, $X_4 \subset \mathbb{P}^4$.

Fano noticed that for such varieties, the following invariants are zero:

- $h^0(X, mK_X) = 0$ for all $m \geq 1$;
  $P_m(X) := h^0(X, mK_X)$ are called *m-th plurigenera*, and if they are all zero, we say that X has *Kodaira dimension* minus infinity, $k(X) = -\infty$.
- $h^i(\mathcal{O}_X) = 0$ for all positive i;
  in particular, the *irregularity* $q(X) = h^1(X, \mathcal{O}_X)$ is zero.

Varieties satisfying these two conditions were called by him *Varietá algebriche a tre dimensioni aventi tutti i generi nulli*.

Fano had the insight that this class of varieties contains varieties which are non-rational, in spite of the fact that they have all plurigenera and irregularity equal to zero; they would provide a counterexample to a Castelnuovo-type rationality criteria for threefolds. None of Fano's attempts to prove non-rationality has been considered acceptable.

The first proof of the non-rationality of (all) $X_4 \subset \mathbb{P}^4$ is the celebrated Iskovskih and Manin's [32]. B. Segre constructed some unirational $X_4 \subset \mathbb{P}^4$ [55]; therefore, these unirational but not rational $X_4 \subset \mathbb{P}^4$ represent counterexamples to Lüroth problem in dimension 3, as well as to a Castelnuovo-type rationality criteria.

In the same period, Clemens and Griffiths proved the non-rationality of the cubic threefold in $\mathbb{P}^4$ [18]. Both papers gave rise to subsequent deep results and theories aimed to determine the rationality or not of Fano varieties.

Nowadays, we define a Fano manifold as follows.

Definition 1 A smooth projective variety X is called a *Fano manifold* if $-K_X$ is ample.

If $Pic(X) = \mathbb{Z}$, then X is called a *Fano manifold of the first species* or a *prime Fano manifold*. In this case, if L is the positive generator of $Pic(X)$, we have $K_X = -rL$; the integer r is called the *index of* X. □

The following is a more general "relative" definition.

Let $f : X \to Y$ be a proper surjective map between normal varieties with connected fibers; we call such an f a *contraction*. If Y is affine, we say that f is a *local contraction*. The contraction can be birational with exceptional locus a divisor; in this case, it is called a *divisorial contraction*; it can be birational with exceptional locus of codimension ≥ 2; it is called a *small contraction*; if $dimX > dimY$, f is called of *fiber type*.

Definition 2 Let $f : X \to Y$ be a contraction and assume that X is smooth or with very mild singularities; f is called a *Fano-Mori contraction* (F-M for short) if $-K_X$ is f-ample.

If $Pic(X/Y) = \mathbb{Z}$, then X is called an *elementary Fano-Mori contraction*. In this case, if L is the positive generator of $Pic(X/Y)$, we have $K_X \sim_f -rL$; the rational number r is called the *nef value of* f.

A Fano manifold can be considered as a Fano-Mori contraction with $dimY = 0$. A general fiber of a Fano-Mori contraction is a Fano manifold. The property of being a Fano variety is not a birational property. Fano varieties and Fano-Mori contractions have been playing a crucial role for 50 years in the birational and biregular study and classification of projective varieties.

The definitions of Fano manifolds and of F-M contraction could be extended to the singular case. The definitions and the studies of the appropriate setting of singularities gave rise in the last 40 years to a fundamental theory intimately related to the properties of the canonical (and anticanonical) bundle. These singularities are ordered in a hierarchy which goes from the so-called terminal and canonical singularities up to log terminal and log canonical; we omit any further details, apart from the fact that on these singular varieties one can define the canonical sheaf K_X as well as concepts of positivity and ampleness. A detailed introduction can be found in the book of J. Kollár with S. Kovacs [38].

This is a beautiful example of a typical fact of mathematical theories in which a definition contains special properties, which are not explicitly mentioned at the beginning and remain obscure for a while. Subsequent researches bring out them, displaying the intrinsic power of the original definition. It is pretty clear, however, that Fano himself was conscious that his definition should include also the case with singularities.

3 Classifications of Fano Varieties and Fano-Mori Contractions

The minimal model program (MMP) aims to classify projective varieties. The Program was initiated by S. Mori (Fields medalist in 1990 for "the proof of Hartshorne's conjecture and his work on the classification of three-dimensional algebraic varieties"), thereafter it was developed by many mathematicians including C. Hacon and J. McKernan (Breakthrough Prize in Mathematics 2018 for "transformational contributions to birational algebraic geometry, especially to the minimal model program in all dimensions") and C. Birkar (Fields medalist in 2018 for "the proof of the boundedness of Fano varieties and for contributions to the minimal model program").

According to MMP, a projective variety, smooth or with at most Kawamata log terminal singularities, is birational equivalent either to a projective variety with positive (nef) canonical bundle or to a F-M contraction, $f : X \to Y$, of fiber typer $(dimX > dimY)$.

What is even more suggestive is the fact that the birational equivalence can be obtained via a finite number of either divisorial F-M contractions or flips of small F-M contractions. The existence of the MMP was proved in dimension 3 by S. Mori [46], while for higher dimension, it has been proved in many cases by C. Birkar et al. [10].

Because of the MMP, F-M contractions became the building blocks, or the atoms, of the classification of projective varieties; as a consequence, it is worth classifying them.

Fano started a biregular classification of Fano manifolds of dimension 3 [19–23]. His work contains serious gaps and many unsatisfactory technical tools.

V.A. Iskovskih, in a series of papers, [30] and [31], has taken up the classification, and using modern tools, he has been able to justify and amplify the work of Fano, obtaining a complete classification of prime Fano threefolds. If $g := \frac{1}{2}K_X^3 + 1$ (this is equal to the genus of the curve section), he proved that $3 \leq g \leq 12$ and $g \neq 11$. For every such g, he gave a satisfactory description of the associated Fano variety.

He used Fano's method of double projection from a line; in particular, he needs the existence of a line, a delicate result proved only later by Shokurov [57].

Among his results, a nice one is the construction of the Fano manifold $X_{22} \subset \mathbb{P}^{13}$; Fano in [23] discussed the existence of X_{22}, but this was omitted by Roth in [54]. Iskovskih proved that in this case, the double projection from a line, $\pi_{2Z} : X \cdots > W \subset \mathbb{P}^6$, goes into W, a Fano threefold of index 2, degree 5, $Pic(W) = \mathbb{Z}$, and at most one singular point. The inverse is given by the linear system $3H - 2C$, where H is the hyperplane and C is a normal rational curve of degree 5. X_{22} is rational.

Some years later, S. Mukai gave a new method to classify Fano-Iskovskikh threefolds based on vector bundle constructions [50]. He provided a third description of $X_{22} \subset \mathbb{P}^{13}$ (see also [52]).

In the same period, S. Mori and S. Mukai [49] gave a classification of all Fano threefold with Picard number greater or equal than 2, and this would have concluded the classification of Fano threefold. However, in 2002, at the Fano Conference in Torino, they announced that they have omitted one of them, namely, the blow-up of $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ along a curve of tridegree $(1, 1, 3)$ (erratum in [49]). It seems now clear that there are 88 types of non-prime Fano threefolds up to deformation. Their classification is based on Iskovskihk's and on the Mori theory of extremal rays, via the so-called two-ray game.

A classification of Fano manifolds of higher dimension is an Herculean task which, however, could be done in a *finite time*. Nadel and Kollár et al., [53] and [41], proved that Fano manifolds of a given dimension form a bounded family, meaning that they are classified by the points of finitely many algebraic varieties. The same results have been proved recently by C. Birkar in the singular case [9].

Fano manifolds of index $r \geq n = dimX$ are simply the projective spaces and the quadrics, and this was proved by Kobayashi and Ochiai [35]. Fano manifolds of index $(n - 1)$ are called del Pezzo manifolds; they were intensively studied by T. Fujita, who proved the existence of a smooth divisor in the linear system H generating $Pic(X)$ [26]. Mukai classified all Fano manifolds of index $= (n - 2)$ under the assumption that H has an effective smooth member [50]. M. Mella proved later that this assumption is always satisfied for Fano manifolds of index $= (n - 2)$, [42].

There are several projects aiming to classify singular Fano varieties in dimensions 3, 4, and 5. A very important one is carried out at Imperial College London under the guidance of A. Corti, and it is named *the periodic table of mathematical shapes*. It is estimated that 500 million shapes can be defined algebraically in 4 dimensions and a few thousand more in the fifth.

The following is a nice conjecture of Mukai [51], very useful for the classification.

Conjecture 1 Let X be a Fano manifold and ρ_X the Picard number of X, i.e., $\rho_X = dimN^1(X)$. Then

$$\rho_X(r_X - 1) \leq n.$$

More generally if $i_X = \min\{m \in \mathbb{N} \mid -K_X \cdot C = m, C \subset X$ rational curve $\}$ is the *pseudoindex* of X (note that $i_X = mr_X$), then

$$\rho_X(i_X - 1) \leq n \text{ with } = \text{ iff } \quad X \simeq (\mathbb{P}^{i_X-1})^{\rho_X}.$$

The conjecture holds for toric varieties [11] and in other special cases, for instance, for $n \leq 5$ [6].

In a fundamental paper, S. Mori [45], after developing his theory of extremal rays, classified all birational F-M contractions on a smooth threefold. This beautiful classification can be seen as the equivalent in dimension 3 of the Castelnuovo contraction criterion on smooth algebraic surfaces.

Later Kawamata described small local F-M contractions on a smooth fourfold [34].

Subsequently, Wisniewski and myself classified all the birational F-M contractions on a smooth fourfold [4]. All these classifications are based on a careful analysis of the deformations of rational curves contained in the fibers of the F-M contractions. The most difficult part is to construct explicit examples for all possible cases; some of them are quite peculiar and bizarre.

One can find several results on the classification of F-M contraction of fiber type on smooth threefolds and fourfolds. They range from the "classical" ones on conic bundles up to more recents which compared different birational models of F-M contractions via the so-called Sarkisov program. According to this program, every birational morphism between two fiber-type F-M contractions with the same target Y can be factorized via a finite number of few basic transformations.

In the 1980s, immediately after the introduction of the Mori theory, it appears with full evidence that the study of F-M contractions should be carried out in the singular setup. P. Francia constructed in 1981 [24] a brilliant example of commutative diagram of F-M contractions on threefolds which convinced everybody that a MMP can be performed only passing through singular cases. In particular, he showed that even on threefold with mild singularities, one can find small F-M contractions which need to be "flipped."

A careful classification of small F-M contractions on threefolds with terminal singularities, together with their flips, was given in a very deep paper by S. Mori [47] and then by S. Mori and J. Kollár [39].

Many authors, including Mori himself, are trying to obtain a complete classification of F-M contractions on threefolds with at most terminal singularities.

Based on the work of S. Mori, Y. Kawamata, Kawakita, and others on threefolds, I recently gave a characterization of birational divisorial contractions on n-fold with terminal singularities with nef value greater than $n - 3$: they are weighted blow-up of hyperquotient singularities [1].

4 Rational Curves on Fano Varieties: Rationally Connected

The name Fano variety is also used for some fundamental type of subvariety of the Grassmannian $\mathbb{G}(k, n)$ associated with a variety $X \subset \mathbb{P}^N$ (see, for instance, [28]). This is the variety of k-planes contained in X, that is,

$$F_k(X) := \{\Lambda : \Lambda \subset X\} \subset \mathbb{G}(k, n).$$

Fano studied $F_1(X)$ for some Fano manifolds X, for instance, for the cubic hypersurfaces $X_3 \subset \mathbb{P}^4$; in this case, $F_1(X_3) \subset \mathbb{G}(1, 4)$ is a surface of general type, called the Fano surface of X_3. It plays a crucial role in the proof of the irrationality of X_3 via the method of the intermediate Jacobian.

The idea of studying families of curves and not linear systems of divisors on a higher dimension variety (they coincide on surfaces), more precisely on Fano manifolds, was carried on in a spectacular way by S. Mori and developed by many other authors.

In [45], S. Mori proved the following results:

Theorem 3 *Let X be a Fano manifold. Then X contains a rational curve $f : \mathbb{P}^1 \to D \subset X$. In fact, through every point of X, there is a rational curve D such that*

$$0 < -(D \cdot K_X) \leq dimX + 1.$$

The proof is very nice, may be one of the nicest in the last years in algebraic geometry, and it can be quickly described, omitting some (difficult and deep) details.

Proof Take any curve C passing through the chosen point and consider its deformation space. By deformation theory and Riemann-Roch theorem, it has dimension greater or equal than

$$h^0(C, TX) - h^1(C, TX) - dimX = -C \cdot K_X - g(C) \cdot dimX.$$

Although by assumption $-C \cdot K_X$ is positive, the quantitative $-C \cdot K_X - g(C) \cdot dimX$ could not be positive, that is, the curve C may not deform. The idea of Mori at this point is to pass to a field of positive characteristic p and consider all the geometric objects over this new field, calling them X_p and C_p. There you have a new endomorphism, namely, the Frobenius endomorphism. One can change the curve C with another, which is the image of C_p via a number m of Frobenius endomorphism. Note that the genus of the curve remains $g(C)$. On the other hand, the above estimate changes by multiplying $-C_p \cdot K_{X_p}$ with p^m; in this way, one can make the quantity $-p^m \cdot C_p \cdot K_{X_p} - g(C_p) \cdot dimX_p$ positive.

Mori showed then that if a curve through a point on an algebraic variety moves, passing anyways from the point, it will "bend and break." More precisely, it will be algebraically equivalent to a reducible curve which has at least one rational component through the point. With a further step of "bend and break," he proves also that one can find a rational curve D_p with $-(D_p \cdot K_{X_p}) \leq dim X + 1$.

Having found in any characteristic a rational curve through the point, with bounded degree with respect to $-K_{X_p}$, one applies a *general principle*, based on number theory: if you have a rational curve (of bounded degree) through the point for almost all $p > 0$, then you have it also for $p = 0$.

An immediate consequence of the theorem is that a Fano variety is *uniruled*, i.e., it is covered by rational curve.

Campana [13] and Kollár et al. [40], [41] proved later that a Fano manifold is actually *rationally chain connected*, i.e., any two points can be connected by a chain of rational curves.

To be uniruled and rationally connected are birational properties.

It is straightforward to prove that if X is uniruled then $P_m(X) =$ for all $m > 0$, i.e., $k(X) = -\infty$. The converse is a long-lasting conjecture, stated by Mori in [47]:

Conjecture 2 Let X be a projective variety with canonical singularities; if $k(X) = -\infty$, then X is uniruled. □

The conjecture is false for more general singularities, for instance, for $\mathbb{Q}$-Gorenstein rational, as some examples of J. Kollár show [37]: they are rational varieties with ample canonical divisor.

Regarding rationally connectedness, we have the following conjecture of D. Mumford:

Conjecture 3 Let X be a smooth projective variety; if $H^0(X, (\Omega^1_X)^{\otimes m}) = 0$ for all $m > 0$, then X is rationally connected. □

Let me recall a curious remark of J. Harris during a school in Trento: "Mori's conjecture is well founded in birational geometry. Mumford's seems to be some strange guess, how did he come up with that?"

I think that J. Kollár was the first to notice that Mori's implies Mumford's; see [36], Chapter 4, Prop 5.7. His proof is based on the existence of the *MRC fibration* (see Theorem 9) and the *fibration theorem*, proved later by Graber-Harris-Mazur-Starr [27].

In [47], S. Mori introduced the definition of pseudo-effective divisor, i.e., a divisor contained in the closure of the cone of effective divisors in the vector space of divisors modulo numerical equivalence: $\overline{Eff(X)} \subset N^1(X)$.

He noticed that if K_X is not pseudo-effective, then $k(X) = -\infty$ and also that if X is uniruled, then K_X is not pseudo-effective. The non-pseudo-effectivity of K_X is therefore a condition in between uniruledness and negative Kodaira dimension.

The following result has been proved in [12] and in [10] using the bend and breaking theory of Mori.

Theorem 4 *Let X be a projective variety with canonical singularities; K_X is not pseudo-effective if and only if X is uniruled.* □

In a recent paper, together with C. Fontanari [2], we discuss other definitions in between uniruledness and negative Kodaira dimension which go under the title

"Termination of Adjuction." They have different levels of generality, and up to certain point, we prove the equivalence of these definitions with uniruledness. A more general definition, which has a classical flavor, was introduced by G. Castelnuovo and F. Enriques in the surface case.

Definition 5 (Termination of Adjunction in the Classical Sense) Let X be a normal projective variety and let H be an *effective Cartier divisor* on X. *Adjunction terminates in the classical sense* for H if there exists an integer $m_0 \geq 1$ such that

$$H^0(X, mK_X + H) = 0$$

for every integer $m \geq m_0$. □

It is easy to prove that uniruledness implies adjunction terminates for H and that this last condition implies that $k(X) = -\infty$.

We conjecture that if X has at most canonical singularities, then adjunction terminates for H is equivalent to uniruledness. This is true in dimension 2 by a theorem of Castelnuovo-Enriques. They proved it for *superficie adeguatamente preparate*; today, we would say for surfaces which are final objects of a MMP.

The following criteria for uniruledness were proved by Miyaoka [43]; their proof is based on a very general "bend and break technique."

Definition 6 T_X is *generically seminegative* if for every torsion-free subsheaf $E \subset T_X$, we have $c_1(E) \cdot C \leq 0$, where C is a curve obtained as intersection of high multiple of $(n-1)$ ample divisors. □

Theorem 7 *A normal complex projective variety X is uniruled if and only if T_X is not generically seminegative.* □

This criterion is a starting point to prove many nice result, including the following one of J. Wisniewski and myself [5], which is the generalization of the celebrated Frenkel-Hartshorne conjecture proved by S. Mori [44].

Theorem 8 *Let X be a projective manifold with an ample locally free subsheaf of $E \subset TX$.*
Then $X = \mathbb{P}^n$ and $E = \mathcal{O}(1)^{\oplus r}$ or $E = T_{\mathbb{P}^n}$. □

A nice conjecture in this setup has been formulated by F. Campana and T. Peternell [14].

Conjecture 4 A Fano manifold with nef tangent bundle is a rational homogeneous variety. □

Let's conclude this section with briefly mentioning two technical instruments developed in the last 30 years to study uniruled varieties. They are crucial in the proof of many deep theorems, including Theorem 8.

On a uniruled variety X, we can find a dominating family of rational curves (more precisely an irreducible component $V \subset Hom(\mathbb{P}^1, X)$ such that $Locus V = X$) having *minimal degree* with respect to some fixed ample line bundle. These families are extensively studied in the book of J. Kollár [36], and they are called *generically* *unsplit families*. This is a beautiful and useful extension of the concept of family of lines used by G. Fano in the study of his varieties.

For each $x \in X$, denote by V_x the family of curves from V passing through x. Let C_x be the subvariety of the projectivized tangent space at x consisting of tangent directions to curves of V_x, that is, C_x is the closure of the image of the *tangent map* $\Phi_x : V_x \to \mathbb{P}(T_xX)$. It has been considered first by S. Mori in [44] and then by many others. Hwang and Mok studied this variety in a series of papers (see, for instance, [29]) and called it *variety of minimal rational tangents* (in short, VMRT) of V.

The tangent map and the VMRT determine the structure of many Fano manifolds, for instance, of the projective space and of the rational homogeneous varieties.

Given a family of rational curves, $V \subset Hom(\mathbb{P}^1, X)$, one can define a *relation of rational connectedness with respect to* V, rcV relation for short, in the following way: $x_1, x_2 \in X$ are in the rcV relation if there exists a chain of rational curves parameterized by V which joins x_1 and x_2. The rcV relation is an equivalence relation, and its equivalence classes can be parameterized generically by an algebraic set. More precisely, we have the following result due to Campana [13] and to Kollár et al. [41].

Theorem 9 *There exist an open subset $X_0 \subset X$ and a proper surjective morphism with connected fibers $\phi_0 : X_0 \to Z_0$ onto a normal variety, such that the fibers of ϕ_0 are equivalence classes of the rcV relation.* □

We shall call the morphism ϕ_0 an *rcV fibration*. If Z_0 is just a point, then we will call X a rationally connected manifold with the respect to the family V.

More generally one can consider on a uniruled variety a rationally connectedness relation with respect to all rational curves $Hom(\mathbb{P}^1, X)$, denoted rc relation. Theorem 9 holds also in this case, and we obtain the so-called maximal rationally connected fibration (for short MRC), which we have quoted above.

The rcV and the MRC fibrations are very much connected to F-M contractions, and they are crucial tools for the study of uniruled varieties.

5 Elephants and Base Point Freeness

Let X be a Fano manifold, or more generally, let $f : X \to Y$ be a local F-M contraction. M. Reid created the neologism *general elephant* to indicate a *general element of the anticanonical system*, i.e., of the linear system $|-K_X|$.

The classification of Fano manifolds or of F-M contractions very often use and *inductive procedure* on the dimension of X, sometime called "Apollonius method", which (very) roughly speaking consists in the following steps:

1. Take a general elephant $D \in |-K_X|$, which is a variety of smaller dimension; by *adjunction formula*, it is in the special class of varieties with trivial canonical bundle.

2. *Lift up sections* of $(-K_X)_{|D}$ (or of other appropriate positive bundles) to sections of $-K_X$. This can be done via the long exact sequence associated with

$$0 \to \mathcal{O}_X \to -K_X \to (-K_X)_{|D} \to 0.$$

   This is possible thanks to the Kodaira *vanishing theorem*, which on a Fano manifolds gives $h^1(\mathcal{O}_X) = 0$.
3. Use the sections obtained in this way to study the variety X.

More generally, one can consider a line bundle L such that either $-K_X = rL$, where r is the index of X, or $-K_X \sim_f rL$, where r is the nef value of the F-M contraction $f : X \to Y$.

Take $D \in |L|$ and do an inductive procedure on D. By adjunction formula $-K_D = (r-1)L_D$, respectively, $-K_D \sim_f (r-1)L_D$, and by Kodaira vanishing theorem sections of L_D lifts to section of L.

The procedure has classical roots and can be traced back to the Italian school of projective geometry or, as the name used above, even to classical Greek geometry. Of course, it is not as smooth as in the above rough picture, and one runs soon in many delicate problems which were handled and solved by many distinguished mathematicians in the last 50 years. Besides S. Mori and others mentioned above, we must recall V. Shokurov, Y. Kawamata, and J. Kollár.

The first crucial problem is the *existence of a general elephant*, a question unexpectedly avoided by some authors. Moreover, it is needed that the singularities of the elephant are not worse than those of X; if X is smooth, we like that also the elephant is smooth.

For the second step, it is necessary to ensure the existence of enough sections of $(-K_X)_{|D}$, more generally of L_D. This is a very delicate problem, and it goes under the name *non-vanishing theorem*. In order to get non-vanishing sections in the linear systems $|L_D|$, sometime one changes slightly the line bundle L, introducing the so-called boundary or fractional divisors. If this is the choice, then the Kodaira vanishing theorem is not sufficient, and more powerful and suitable *vanishing theorems* are needed.

The contemporary theory of MMP and of the study of F-M contractions develops as a "game" between vanishing and non-vanishing. Two "teams" were competing and/or cooperating on this. On one side, there is the group of algebraic geometers, which uses boundary and fractional divisors and the so-called Kawamata-Viehweg vanishing theorem. They refer to Shokurov as the main master of the game, and his technique has been called "spaghetti-type proofs," an attribute to the Italian origins. On the other side, there is the group of analytic geometers or complex analysts, which uses the so-called Nadel ideals and Nadel vanishing theorem; besides Nadel, the two other main active figures are Y.T. Siu and J.P. Demailly.

Maybe the most important result proved with these methods is the existence of the MMP, in dimension 3 by S. Mori [48] and later in all dimension, under some assumptions, by Birkar et al.[10].

Regarding the existence and the regularity of the elephants among the many crucial technical steps in the last 50 years, I like to recall the following ones:

- The existence of a smooth general elephant on a smooth Fano threefold (more generally of an elephant with du Val singularities on a Fano threefold with Gorenstein canonical singularities), by V.V. Shokurov [56]. This assures completeness to the proof of the classification of smooth Fano threefolds started by Fano and concluded by Iskovskih.
- The existence of a general elephant with du Val singularities on a small F-M contraction on threefold with terminal singularities, by S. Mori [48] and by S. Mori and J. Kollár [39]. This is a fundamental step to prove the existence of the flip for every small contraction on a threefold with terminal singularities and, in turn, the existence of the MMP in dimension 3.
- The existence of a general elephant with du Val singularities on a divisorial F-M contraction on threefold with terminal singularities, by M. Kawakita in a series of paper from 2001 to 2005; see, for instance, [33].
- The existence of a smooth element in the linear system $|L|$ on a Fano manifold of index $r \geq (n-2)$, where $-K_X = rL$. This is "classical" for $r \geq n$; see, for instance, [35]. It has been proved for $r = (n-1)$ by T. Fujita in 1984 (see [26]) and for $r = (n-2)$ by M. Mella in 1999 (see [42]).
- The existence of an element in $|-mK_X|$ for a positive integer m depending only on d for any d-dimensional $\mathbb{Q}$-Fano variety X, by C. Birkar in 2019 [8]. This result is the starting step to prove the boundness of the number of families of $\mathbb{Q}$-Fano variety in any fixed dimension d (BAB conjecture) [9].
- On a local F-M contraction $f : X \to Y$ such that $-K_X \sim_f rL$, the line bundle L is base point-free at every point of a fiber F with $dim F < (r+1)$; if f is birational, then the same is true also for fibers F such that $dim \leq (r+1)$. This in turn, by Bertini's theorem, will give the existence of elements in $|L|$ with singularities not worse than those of X. This was proved for varieties X with klt singularities by Wisniewski and myself in 1993 [3] and extended to log canonical singularities by O. Fujino in 2021 [25].

6 Kähler-Einstein Metrics

On a Riemannian manifold (X, g), one can consider the Einstein field equations, a set of partial differential equations on the metric tensor g which describe how the manifold X should curve due to the existence of mass or energy. In a vacuum, where there is no mass or energy, the Einstein field equations simplify. In this case, the Ricci curvature of g, Ric_g, is a symmetric (2, 0) tensor, as is the metric g itself, and the equations reduce to

$$Ric_g = \lambda g$$

for a smooth function λ. A Riemannian manifold (X, g) solving the above equation is called an *Einstein manifold*. It can be proven that λ, if it exists, is a constant function.

If the Riemannian manifold has a complex structure J compatible with the metric structure (i.e., g preserves J and J is preserved by the parallel transport of the Levi-Civita connection), the triple (X, g, J) is called a *Kähler manifold*.

A *Kähler-Einstein manifold* combines the above properties of being Kähler and admitting an Einstein metric. A famous problem is to prove the existence of a Kähler-Einstein (K-E for short) metric on a compact Kähler manifold. It has been split up into three cases, depending on the sign of the first Chern class of the Kähler manifold.

If the first Chern class is negative, T. Aubin and S.T. Yau proved that there is always a K-E metric. If the first Chern class is zero, then S.T. Yau proved the Calabi conjecture, that there is always a K-E metric, which leads to the name Calabi-Yau manifolds. For this, he was awarded with the Fields medal.

The third case, which is the positive or Fano case, is the hardest. In this case, the manifold not always has a K-E metric; Y. Matsushima (1957) and A. Futaki (1983) gave necessary conditions for the existence of such metric. For instance, the blow-ups of $\mathbb{P}^2$ in one or two points do not have a K-E metric. G. Tian in [58] proposed a stability condition for a complex manifold M, called K*-stability*, connected with the existence of a K-E metric; in the same paper, he proved that there are Fano threefolds of type X_{22} which do not admit a K-E metric.

In 2012, Chen, Donaldson, and Sun proved that on a Fano manifold, the existence of a K-E metric is equivalent to K-stability. Their proof appeared in a series of articles in the *Journal of the American Mathematical Society* in 2014 [15–17].

Recently, many authors studied the existence of a K-E metric on the 105 irreducible families of smooth Fano threefolds, which have been classified by Fano, Iskovskikh, Mori, and Mukai. A very nice summary is contained in the forthcoming book by Carolina Araujo, Ana-Maria Castravet, Ivan Cheltsov, Kento Fujita, Anne-Sophie Kaloghiros, Jesus Martinez Garcia, Constantin Shramov, Hendrik Süß, and Nivedita Viswanathan; see [7]. For each family, they determine whether its general member admits a K-E metric or not; in many cases, this has been done also for the special members.

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