# The Genesis of the Italian School of Algebraic Geometry Through the Correspondence Between Luigi Cremona and Some of His Students

**Nicla Palladino and Maria Alessandra Vaccaro**

**Abstract** Luigi Cremona is considered the founder of the Italian school of algebraic geometry. He formed a group of students of great value, very active in scientific research. Examining the letters from Eugenio Bertini, Ettore Caporali, and Riccardo De Paolis to Cremona preserved in the archive of the Istituto Mazziniano in Genoa, we have reconstructed their biographies, careers, studies, and relationships with their teacher. They had the merit of cultivating the scientific innovations of the period and passing them on to the subsequent generations.

**Keywords** Italian school of algebraic geometry · Luigi Cremona correspondence · Bertini · Caporali · De Paolis

## 1 Introduction

In the first decades of the nineteenth century, the French and the German schools had built the foundations of modern geometry. The establishment of the Italian school of algebraic geometry was one of the aims of the broader post-Risorgimento plan of founding an Italian mathematical school related to the most advanced European studies.

Antonio Luigi Gaudenzio Giuseppe Cremona (Pavia 1830—Roma 1903) is considered one of the renovators of geometric studies in Italy. His fundamental scientific merit was probably the systematic study, since 1863, of "birational transformations" on the plane and the space named, after him, "Cremona transformations." See [1, 2]. In addition to the scientific ones, Cremona had the merit of having formed a group

N. Palladino (✉)
Dipartimento di Matematica e Informatica, University of Perugia, Perugia, Italy
e-mail: nicla.palladino@unipg.it

M. A. Vaccaro
Dipartimento di Matematica e Informatica, University of Palermo, Palermo, Italy
e-mail: marialessandra.vaccaro@unipa.it

 
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of students of great value, very active in scientific research and also as teachers or directors in various Italian schools.

From 1860 to 1866, Cremona was Professor of Higher Geometry at the University of Bologna. In 1866, he moved to the Polytechnic University of Milan, founded in 1863 by Francesco Brioschi, teaching higher geometry and graphical statics. In 1873, he was called to Rome, recently become the capital of Italy, to organize and direct the Scuola di applicazione degli ingegneri. He was also appointed Professor of Higher Mathematics at the University of Rome. He was a member of the senate of the Kingdom of Italy and briefly minister for education.

Among the students who studied with him and scientifically followed his footsteps, at least for some phase of their career, we can mention:

a) Eugenio Bertini (1846–1933, graduated in Pisa) among the Bolognese students
b) Angelo Armenante (1844–1878, graduated in Naples), Ferdinando Aschieri (1844–1907, graduated in Pisa), Giulio Ascoli (1843–1896, graduated in Pisa), Giuseppe Jung (1845–1926, graduated in Naples), Carlo Saviotti (1845–1928), and Emil Weyr (1848–1894) among the students of the Polytechnic of Milan (where they came to perfect their studies)
c) Riccardo De Paolis (1854–1892), Ettore Caporali (1855–1886), Giuseppe Veronese (1854–1917), and Giovanni Battista Guccia (1855–1914) among the Roman ones

The students who had direct contacts with Cremona form the “first nucleus” of young Italian algebraic geometers. Actually, Corrado Segre (1863–1924), Guido Castelnuovo (1865–1952), Federigo Enriques (1871–1946), and Francesco Severi (1879–1961), who brought Italian algebraic geometry to full maturity, constitute the second generation of Italian algebraic geometers.

> To create a school, the value of the teacher is not enough nor is it enough that he knows how to project a research plan so great that it surpasses his own workforce. It is also necessary that he be able to communicate his passion and faith to the students and know how to require and direct their collaboration. Luigi Cremona had these qualities in an eminent degree. The students who were lucky enough to listen to him in full zeal of research tell us that his enthusiasm for the problems he presented transpired during the lesson and was transmitted to the audience making them participate in the enjoyment of the discovery. With willpower, which was his main gift, he influenced young people and attracted them to the field of study he favored. This strength and faith were such that even his direct disciples were able to convey them to us, the disciples of the second generation.[1]

[1] “Per dar vita ad una scuola non basta il valore del maestro, né basta che egli sappia tracciare un piano di ricerche così vasto da superare la propria forza di lavoro. Occorre altresì che egli riesca a comunicare la sua passione e la sua fede ai discepoli e sappia esigerne e dirigerne la collaborazione. Queste doti possedeva in grado eminente Luigi Cremona. Raccontano gli allievi che ebbero la fortuna di ascoltarlo quando egli era nel pieno fervore della ricerca, che l’entusiasmo per le questioni da lui esposte traspariva durante la lezione e si trasmetteva all’uditorio rendendolo partecipe del godimento della scoperta. Con la forza di volontà, che era una sua dote precipua, suggestionava i giovani e li attirava verso l’indirizzo da lui prediletto. Tali furono questa forza e questa fede che noi stessi della seconda generazione ne subimmo la influenza, trasmessaci dai discepoli diretti.”

The above statement by Guido Castelnuovo in [3, pp. 615–616] well described Cremona's role in the creation of the Italian school of algebraic geometry.

In different ways and with different roles, and according to their possibilities, the aforementioned scholars contributed to raising the Italian mathematics by working with alacrity in scientific research. Following their career and their studies allows us to reconstruct the genesis of the Italian school of algebraic geometry; to do this, it was useful to make use of the unpublished correspondence they had with their professor, preserved in the archive of the Istituto Mazziniano in Genoa (Legato Itala Cozzolino Cremona). The archive contains a wide number of letters from almost all these scholars that give a complete idea of the role played by Cremona: Armenante (15 letters), Aschieri (5 letters), Ascoli (13 letters), Bertini (14 letters), Caporali (48 letters), De Paolis (1 letter), Guccia (44 letters), Jung (148 letters), and Weyr (1 letter). Furthermore, with regard to his direct students, 2 letters from Caporali, 6 letters from Saviotti, and 27 letters from Weyr to Cremona have been published in [4].

In this chapter, we directed our attention to Bertini, Caporali, and De Paolis not only because they are the first group of researchers who embraced Cremona's research, but above all because they can be credited with having represented with their studies a transition bridge to the next generation of algebraic geometers.

## 2 The First Student of Cremona: Eugenio Bertini

Bertini was in chronological order the first distinguished student of Cremona and, most likely, the one who had a role of major importance in the subsequent developments of algebraic geometry.

He was born in Forlì on 8 November 1846 from Vincenzo and Agata Bezzi and, after attending the technical school of the city, in 1863, at the age of 17, he entered the University of Bologna to study engineering. Following a mathematics course held by Cremona, he decided to devote himself to the studies of pure mathematics. In 1866, when the Third War of Independence broke out, Bertini interrupted the studies and he joined as a volunteer with Garibaldi. After starting again his studies, Cremona advised Bertini to move to the University of Pisa under the leadership of Enrico Betti and Ulisse Dini.

> With these few lines I recommend the young Eugenio Bertini from Forlì. He is a young man very dear to me for his excellent qualities: he is very clever and really wants to learn; if, as I do not doubt, he perseveres, he will be different from other scholars. Yesterday he passed the Higher Geometry exam which lasted more than an hour and made Professor Grassmann, who was present, amazed. But unfortunately Bertini knows little or nothing about analysis: and he comes to Pisa precisely to study algebra, calculus, etc. I kindly ask you to follow

> and advise him. Introduce him, also in my name, to Novi, to Dini, to those who can benefit him.[2]
>
> [Cremona to Betti. Bologna, 11 November 1866]

In 1867, Bertini graduated with honors in mathematics, and in 1868, as "aggregate student" of the Reale Scuola Normale Superiore, he obtained the teaching qualification by presenting a valuable thesis on Eulerian polyhedra, subsequently published in the first volume of Annali of the Reale Scuola Normale Superiore.

> As you already know, for Normale's thesis I continued to study those things on polyhedra I wrote to you about in a previous letter of mine, making various modifications. Pr. Betti and Dini were not dissatisfied and they told me that they would propose to publish them in the annali of mathematics.[3]
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> [Bertini to Cremona. Tredozio (on the Apennines), 7 September 1868]

Cremona's charisma attracted Bertini to Milan, the city where the geometer had moved since 1867. During the academic year 1868–1869, he had the opportunity to follow a three-part course held by Brioschi, Felice Casorati, and Cremona on abelian integrals, analyzed from three different points of view: Jacobi's analytical method, Riemann's topological one, and Clebsch's and Gordan's algebraic geometric one. The course had a profound and lasting influence on the young mathematician, to the point that in 1869 he was able to give one of the first and simplest geometric proofs on the genus invariance of an algebraic curve with birational transformations.[4] This proof, included in three classical texts [5, pp. 683–684; 6, vol. 2, pp. 131–135; 7, p. 314], helped to increase Bertini's estimation.

From December 1869 to 1872, Bertini taught at the Parini high school in Milan and in 1872–1873 at the Ennio Quirino Visconti high school in Rome. In that same year, by the will of Cremona who also settled in Rome, he taught courses in descriptive and projective geometry.

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[2] "Con queste poche righe ti raccomando il giovane Eugenio Bertini di Forlì. È un giovane a me assai caro per le egregie sue qualità: ha molto ingegno e molto desiderio di imparare; se, come non dubito, egli persevera, riuscirà qualche cosa di diverso dall'ordinario. Ha fatto l'altr'ieri un esame di geometria superiore che è durato più di un'ora e che ha fatto meravigliare il professore Grassmann ch'era presente. Ma disgraziatamente il Bertini sa poco o nulla di analisi: e viene a Pisa appunto per studiare algebra, calcolo, ecc. Ti prego caldissimamente di assisterlo e consigliarlo. Presentalo, anche a nome mio, al Novi, al Dini, a chi gli può giovare."

[3] "Come già le sarà noto, per la tesi di Normale seguitai quelle cose sui poliedri di cui le scrissi in una mia precedente, apportandovi varie modificazioni. Il Pr. Betti e il Dini non ne rimasero scontenti ed essi medesimi mi dissero che si sarebbero adoperati onde venissero pubblicate negli annali di matematica."

[4] E. Bertini, *Nuova dimostrazione del teorema: due curve punteggiate projettivamente sono dello stesso genere*, Giornale di Matematiche, 7, 1869, pp. 105–106.

> I was given the course of Descriptive Geometry and this too was approved by the superior council. I confess that this assignment gives me great pleasure and brings me a great deal of satisfaction. I know that for this too I am grateful to you for the words you wrote to Rome and that I doubt I deserve.[5]
>
> [Bertini to Cremona. Roma, 13 November 1872]

In 1875, Bertini won, with the support of Betti, the competition for the teaching position of Higher Geometry at the University of Pisa and after 3 years he became full professor. He remained in Pisa until 1880, when, by exchange with De Paolis, he moved to the University of Pavia, also attracted by the presence of two dear colleagues: Casorati and Eugenio Beltrami. In 1892, ceding to the friendly insistence of Luigi Bianchi and Ulisse Dini, Bertini returned to Pisa where the position of Higher Geometry had become vacant due to the untimely death of De Paolis and remained there for the rest of his life. In 1922, he retired at the age of 75 (succeeded by Carlo Rosati), but for 10 years more he continued to teach as professor emeritus a course of complements of projective geometry, whose lessons are published in a book he wrote at the age of 82 years. He died in Pisa on 24 February 1933.

Bertini is described by some of his direct students, such as Luigi Berzolari (1863–1949), Guido Fubini (1879–1943), and Gaetano Scorza (1876–1939), who affectionately named him "Papà Bertini," as a modest, selfless, good-natured person, unrelated to any form of exhibitionism, dedicated to his family and his work. "Passionate about teaching, he carefully prepared his lessons teaching with admirable clarity and precision" (Castelnuovo in [8, p. 748]). "For Bertini, research and teaching were simply two aspects of the same activity" (Berzolari in [9, p. 612]). "Often the subject of his lessons were the contributions he brought to science, so conversely many of his researches originated from didactic needs [ . . . ] many of Bertini's most beautiful works were precisely caused by the need to perfect this or that theory before presenting it into school, here filling a gap, there obviating possible objections" (Scorza in [10, pp. 110–111]).[6]

Educated at the Cremona school, Bertini soon departed from it, marking an interesting turning point in the history of Italian algebraic geometry. Between 1863 and 1865, Cremona had created the theory of birational transformations,[7] then named after him, first on the plane and then in space. He used them as a tool to reduce more complicated geometric entities into simpler ones and to convey the properties of the latter to them. Probably seduced by the many applications, it seems that Cremona had no interest in studying the group theoretical properties that remain unchanged under these transformations. On the contrary, in 1876, Bertini, through research on the plane involutions of pairs of points, decided to determine the

[5] "A me fu affidato l'incarico dell'insegnamento della Geometria descrittiva e anche questo fu approvato dal consiglio superiore. Le confesso che questo incarico mi fa molto piacere e mi reca moltissima soddisfazione. So che anche di ciò io debbo avere molta gratitudine a Lei che volle scrivere a Roma parole che io dubito di meritare."

[6] About Bertini's life, see also [11–13].

[7] With regard to the historical aspects of the Cremona transformations, see [14].

various typologies to which the plane involutions can be reduced through Cremona transformations.

> I also thank you for the inclusion of my work in the Annali [15]. Thinking in these days about the general question, I realized that some cases were missing. That is, when the fixed curve Γ does exist, it can be of order n with a (n-1)$^{\text{ple}}$ point and it can break in parts: furthermore it can even be the case where there is a unique fixed dotted line. I have now completed the study of these three cases where there are no serious difficulties, and it seems to me that all three are deduced from harmonic homology with repeated quadratic transformations. I will write these additions in the proofs, since, as far as I have sent it to you, there are no major changes. But please do me the courtesy of reviewing the second drafts. I will try to be sure of what I do: but, to tell you the truth, experience has taught me to be very cautious and I would be much calmer if you wanted to look at these insertions (in which there are some delicate considerations).[8]
>
> [Bertini to Cremona. Pisa, 23 July 1876]

A few months later, Caporali also devoted himself to studying Bertini's first work on involutions. In fact, he wrote to Cremona:

> I have read a note by Prof. Bertini on the Jonquières transformations which are also involutory. It seems to me that the consideration of other cases of the same problem may be interesting. When I was working on my thesis I encountered another class of involutory transformations, and it is the following. Fixed on the plane a net of curves of order n and genus one which have in common so many fixed points as to absorb n-2 intersections (which is generally possible in several ways), all the curves that pass through an arbitrary point in the plane also pass through another point that can be taken as corresponding to the first. The simplest case (for n = 3) served me in my thesis.[9]
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> [Caporali to Cremona. Roma, 11 September 1876]

Bertini's goal was clear, as he himself declared in the introduction to his Research [16, p. 244]: "Indicate all the possible involutory transformations of the plane, which

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[8] "La ringrazio altresì della inserzione del mio lavoretto negli Annali [15]. Pensando in questi giorni alla questione generale mi sono accorto che mancava la trattazione di alcuni casi. Cioè, quando esiste la curva unita Γ, questa può essere d'ordine $n$ con un punto $(n-1)$plo e può spezzarsi in parti: e può anche darsi che esista una sola retta punteggiata unita. Ho ormai condotto a termine lo studio di questi tre casi, ne' quali non s'incontrano gravi difficoltà, e mi pare che tutti tre si deducano dall'omologia armonica con ripetute trasformazioni quadratiche. Queste aggiunte le farò nelle bozze di stampa, giacché, per quello che le ho spedito, non vi sono modifiche di gran rilievo. Però voglia usarmi la cortesia di rivedere le seconde bozze. Cercherò di essere sicuro di ciò che faccio: ma, a dirle il vero, l'esperienza mi ha insegnato a diffidare assai e sarei assai più tranquillo se a queste aggiunte (nelle quali c'è qualche considerazione delicata) Ella volesse dare una occhiata."

[9] "Ho letto una memoria del Prof. Bertini sulle trasformazioni di Jonquières che sono anche involutorie. Mi pare che la considerazione di altri casi dello stesso problema, possa essere interessante. Quando facevo la mia tesi mi si era presentata un'altra classe di trasformazioni involutive, ed è la seguente. Fissata nel piano una rete di curve d'ordine n e di genere uno le quali abbiano in comune tanti punti fissi da assorbire n − 2 intersezioni (lo che è possibile in generale in più maniere), tutte le curve che passano per un punto preso ad arbitrio nel piano, passano anche per un altro punto che si può prendere come corrispondente al primo. Il caso più semplice (per n = 3) mi ha servito nella mia tesi."

are irreducible, that is, which cannot be inferred from each other by a series of quadratic transformations or, what is the same, by an one-to-one transformation."[10] He classified them into four irreducible types: the first two referable to involutory de Jonquières transformations; the third generated by the cubic plane curve passing through seven points, already considered by Geiser; and the fourth generated by sextics which have eight double points in common. To develop his studies, Bertini introduced some restrictive hypotheses that could have raised doubts about the incompleteness of this classification, a fear that was removed only later: "Other considerations would lead me to think that all the possible involutory transformations on the plane can be reduced to these cases. But I do not have a rigorous proof of this property. The difficulties arise above all from the consideration of those cases in which the fundamental curves are broken, and therefore several properties that exist in general are not valid."[11]

In [8, pp. 746–747], Castelnuovo affirmed that Bertini's important classification was received coldly by Cremona and that the latter, when Bertini personally communicated the result of his research to him, merely observed that he was already aware of the latest type of involution, thanks to an observation sent by the letter to Caporali. Thus, Bertini in [16, p. 273] inserted in a note the proof of Cremona introducing it in this way: "After having delivered this work to the editor of the Annali, I learned that prof. Cremona had already observed this property and communicated it by letter to Dr. Caporali."[12]

Castelnuovo in [8, p. 745] and Scorza in [10, p. 116] revealed that the good relations between Cremona and Bertini in a certain way deteriorated when the latter in 1875 left the chair of Descriptive Geometry in Rome to assume that of Higher Geometry in Pisa and that the two scholars reconciled only some years later. From the following letter, one infers how deeply disappointed Bertini was in 1878 by Cremona's behavior towards his last works:

> Professor Betti, returning from Rome, informed me that he had expressed to you my desire to ask for the chair of Higher Geometry and added that you thought it was more appropriate for me to ask for the chair of Projective Geometry. Now it is on this point that I would dare to write you some considerations.
>
> First of all, Betti assured me that you have definitively abandoned the intention of coming to Pisa. Consequently, together with the teaching of Projective Geometry, I should also take that of Higher Geometry. On the other hand, I do not hide to you that I care a lot about this last teaching, because from it I derive a lot of satisfaction and vigor for my

[10] "Indicare tutte le possibili trasformazioni involutorie del piano, che sono irriducibili, cioè non possono dedursi l'una dall'altra per una serie di trasformazioni quadratiche o, ciò che è lo stesso, per una trasformazione univoca."

[11] "Considerazioni di altra specie m'indurrebbero a pensare che a questi casi possano ridursi tutte le possibili trasformazioni involutorie del piano. Però una dimostrazione rigorosa di tale proprietà non mi è riuscita. Le difficoltà provengono soprattutto dalla considerazione di quei casi ne' quali le curve fondamentali si spezzano, e quindi cessano di essere vere parecchie proprietà che sussistono in generale."

[12] "Dopo aver consegnato il presente lavoro alla Direzione degli Annali, seppi che il prof. Cremona aveva già osservata questa proprietà e comunicatala per lettera al dott. Caporali."

studies. So I would be engaged in teaching at least as much as now, while it is one of my desires to be able to limit my occupations in order to make them as fruitful as possible.

But, leaving this consideration aside, I do not deny that, receiving the communication from Betti, I was truly disheartened. I thought that, after three years of teaching Higher Geometry and after having worked with the greatest possible effort, I would only be able to deserve the chair of Projective Geometry. I thought that a year ago I had already been judged worthy of this chair in the report of the commission for the Naples competition and that since then I have continued to work both for teaching and for science. I do not know if you have looked at my *Researches* but, if the love of a father does not deceive me, I believe that those researches (after the Naples competition) must be taken into account, regardless of the two little notes for the Lincei and the Giornale di Napoli[13] (even after the aforementioned competition) which are to be considered as a consequence of the same *Researches*. I am convinced that, if you want to examine this work carefully, you will find (I am sorry to enter into such particular facts, but I am not able to do without it) that the problem presented to me was not at all simple, that the appropriate way to solve it could not be predicted a priori, that the method I applied is (at least I think) new and applicable in similar research, and that finally, of the results I found, several are (or I am wrong) of interest. It is true that the problem was not entirely resolved, but you acknowledge that I have come very close to the solution; and I do not despair of completing it by resuming those researches in the future.[14]

[Bertini to Cremona. Pisa, 14 February 1878]

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[13] E. Bertini, *Una nuova proprietà delle curve di ordine n con un punto (n-2)-plo*, Trasunti della Reale Accademia Nazionale dei Lincei, (3), 1, 1876–77, pp. 92–96 and *Sulle curve razionali per le quali si possono assegnare arbitrariamente i punti multipli*, Giornale di Matematiche, 15, 1877, pp. 329–335.

[14] "Il Pr. Betti, ritornando da Roma, mi ha comunicato di averle manifestato il desiderio che io aveva di chiedere la titolarità di Geometria superiore e mi ha soggiunto che Ella credeva invece più opportuno che io domandassi la titolarità di Geometria projettiva. Ora è su ciò che io oserei scriverle alcune considerazioni. Anzitutto il Betti mi ha assicurato che Ella ha abbandonato in modo definitivo il proposito di venire a Pisa. Per conseguenza insieme all'insegnamento di Geometria projettiva, dovrei assumere anche quello di Geometria superiore. D'altra parte non Le nascondo che a quest'ultimo insegnamento io tengo assai, perché da esso traggo molta soddisfazione e vigoria ne' miei studï. Sicché mi troverei impegnato nell'insegnamento per lo meno quanto ora, mentre è uno de' miei desideri di riuscire a limitare l'estensione delle mie occupazioni onde rendere queste più fruttuose che sia possibile. Ma, lasciando a parte una tale considerazione, non Le nego che, ricevendo la comunicazione del Betti, sono rimasto veramente sconfortato. Ho pensato che, dopo tre anni di straordinariato di Geometria superiore e dopo avere lavorato colla maggiore intensità possibile, io sarei riuscito soltanto a meritarmi la titolarità della Geometria proiettiva. Ho pensato che di questa titolarità io era già giudicato degno un anno fà nel rapporto della Commissione pel concorso di Napoli e che da quell'epoca ho continuato a lavorare sia per l'insegnamento, sia per la scienza. Io non so se Ella abbia data un'occhiata alle mie Ricerche: ma, se non m'illude l'amore di padre, giudico che quelle ricerche (posteriori al concorso di Napoli) debbano essere tenute in qualche conto, prescindendo dalle due noticine pei Lincei e pel Giornale di Napoli (pure posteriori al suddetto concorso) che sono da riguardare come conseguenza delle medesime Ricerche. Io sono persuaso che, se Ella vorrà usarmi la gentilezza di esaminare attentamente questo lavoro troverà (mi duole di entrare in cosifatti particolari, ma per il mio assunto non ne posso a meno) che il problema propostomi non era affatto semplice, che non poteva prevedersi a priori la via opportuna a risolverlo, che il metodo seguito è (almeno credo) nuovo e applicabile in ricerche analoghe e che infine dei risultati trovati parecchi sono (o m'inganno) interessanti. Vero che il problema non fu interamente risoluto, ma Ella riconoscerà che mi sono avvicinato assai alla soluzione; e non dispero di compierla riprendendo in avvenire quelle ricerche."

In 1879, Caporali published the note [17] which, discrediting the real feasibility of the reduction in a finite number of irreducible involutions proposed by Bertini, approached the problem from another point of view. By introducing the concept of *class* of an involution, that is, the number of pairs of corresponding points lying on an arbitrary line, he determined all the first-class involutory transformations and applied the general results obtained to two particular cases.

In 1880 and thereafter in 1883, Bertini returned to this subject by publishing the two papers [18] and [19] where, accepting the notion of class defined by Caporali, he dealt with the construction of all first- and second-class involutions. Later in 1889, making use of simple geometric considerations, in [20] he demonstrated that each of the four types of involutions he determined generated a double transformation, that is, a rational involution. Actually, this result had already been determined in 1878 by Max Noether[15] and later treated by Jacob Lüroth.[16]

Bertini also devoted himself to the theory of linear systems through the publication of three works[17] of 1882, 1889, and 1901. In the first of them, he proved two classical theorems of algebraic geometry, generalized later by Enriques and Severi, which constitute the research subject of an article by Steven Kleiman [21].

Since 1885, Bertini turned his interest to studies on the projective geometry of hyperspaces, thanks to the contributions of Veronese and above all of Segre with whom he became friends. Subsequently, he collected his lectures on hyperspaces held in Pisa in the volume *Introduzione alla geometria proiettiva degli iperspazi* (1907), where he presented the results of last years with order and clarity.

A second group of works of the years 1889–1891 was aimed at simplifying the proof of a classical theorem of Noether developed through a purely algebraic treatment and enriched with new and important properties. Applications of these results can be found in the monograph of 1894,[18] where, following the approach of Alexander von Brill and Noether and making use of the correspondence with Segre and Noether, Bertini presented the geometry of linear series on a curve in a clear and systematic way. We owe both our author and Segre the credit for having contributed to the dissemination in Italy of the theory of the aforementioned German mathematicians, as Castelnuovo stated in [8, p. 748]: "This monograph and a contemporary one by Corrado Segre on the same topic discussed with a different method, have disseminated among us a theory that, originated in Germany, took

[15] M. Noether, *Über die ein-zweideutigen Ebenentransformationen*, Erlangener Berichte, 10, 1878, p. 81.

[16] J. Lüroth, *Rationale Flächen und involutorische Transformationen*, Prorektoratsrede, Freiburg, 1889, pp. 1–25.

[17] E. Bertini, *Sui sistemi lineari*, Rendiconti del Reale Istituto Lombardo di Scienze e Lettere, (2), 15, 1882, pp. 24–29; *Sulle curve fondamentali dei sistemi lineari di curve piane*, Rendiconti del Circolo Matematico di Palermo, 3, 1889, pp. 5–21 and *Sui sistemi lineari di grado zero,* Rendiconti della Reale Accademia Nazionale dei Lincei, (5), 10, 1901, pp. 73–76.

[18] E. Bertini, *La geometria delle serie lineari sopra una curva piana secondo il metodo algebrico*, Annali di Matematica pura ed applicata, (2), 22, 1894, pp. 1–40.

on thereafter new and extensive developments in Italy. Those two works gave the impetus to the subsequent research of our geometric school."[19]

## 3 Ettore Caporali in His Correspondence with Cremona

Ettore Caporali took his degree with Luigi Cremona in July 1875 in Rome. He was born in Perugia on 17 August 1855, from Vincenzo and Tecla Campi, had attended secondary schools in his hometown, and then studied mathematics at the University of Rome, under Giuseppe Battaglini and Eugenio Beltrami as well as with Cremona.

His thesis *Sulla superficie del quinto ordine dotata di una curva doppia di quinto ordine*[20] was about the rational surface of the fifth order of the ordinary three-dimensional space with a double line of the fifth order that he studied by means of its one-to-one representation on the plane. He derived this representation from a spatial construction of the surface through a particular birational transformation on the space. The research, which extends the results of Cremona, shows how Caporali faced problems that were of interest to the greatest experts of algebraic geometry at the time.

The archive of the Istituto Mazziniano in Genoa contains 48 letters from Caporali to Cremona, from 1876 to 1886. The unpublished correspondence with his teacher highlights a very close relation between the two scholars, both from a scientific and a human point of view. Immediately after his graduation, Caporali became teacher at the Spedalieri high school in Catania; however, he lived the year he spent in Sicily with boredom, and he often wrote to his teacher about this, complaining of not being in the right disposition to develop his studies and research and asking him for advice on changing living places and jobs. In 1876, he obtained the position of teaching assistant at the Scuola d'applicazione per gli ingegneri in Rome.

In this period, the scientific activity of Caporali became more intensive; he often discussed with his teacher by letter about the research they were developing: for example, in a letter, we read that Cremona had sent him the drafts of his note *Über die Polar-Hexaeder bei den Flächen dritter Ordnung*[21] and Caporali answered with observations and his original ideas:

> I was impressed by the simplicity of the analytical definition of your hexahedra: it is also remarkable that they relate the 27 straight lines to Sylvester's pentahedron with Reye's theorems: especially because it is perhaps possible, by means of the properties of third-class developable surfaces, to give this relation a simpler form than the one directly provided by Reye's note.

[19] "Questa monografia ed un'altra contemporanea di Corrado Segre sullo stesso argomento trattato con metodo diverso, hanno divulgato tra noi una teoria che, sorta in Germania, prese poi in Italia nuovi ed ampi sviluppi. Quei due scritti hanno dato l'impulso alle successive ricerche della nostra scuola geometrica."

[20] Printed by decision of the examining commission in the Annali di Matematica pura ed applicata, serie 2°, tomo 7°.

[21] L. Cremona, *Ueber die Polar-Hexaeder bei den Flächen dritter Ordnung*, Mathematischen Annalen, Band XIII, 1878, pp. 301–304.

> This topic made me think about Clebsch's Diagonalfläche: the 15 straight lines that determine it are arranged three by three in 15 planes; and since a pentahedron (the Sylvester pentahedron) is assumed as the starting point I believed your properties could provide a complete pentahedron theory. However, I realized shortly after that the hexahedron that should be deduced from the 15 lines is lost and with it almost all the properties. Since any edge of Sylvester's pentahedron absorbs three of the 60 Pascal lines, 30 lines remain that, perhaps, deserve to be studied.
>
> Moving from one idea to another, I thought about the problem of assuming six fundamental points in the plane in order to have the representation of the Diagonalfläche: I resolved it analytically.[22]
>
> [Caporali to Cremona. Massa Martana, 15 September 1877]

Caporali's ideas then converged into the work *Sull'esaedro completo*[23] which resumed Cremona's research of the aforementioned note, *Teoremi stereometrici dai quali si deducono le proprietà del l'esagrammo di Pascal* of Cremona[24] and *On Pascal's theorem* of Cayley.[25]

At that time, Caporali was trying to be hired at the University of Naples because some professorships had remained vacant after Battaglini's retirement. In 1878, being just 23 years old, he was then appointed extraordinary professor of Higher Geometry at the University of Naples. We can read the arguments of his course from a letter he wrote to Cremona:

> I began by rapidly re-introducing projectivity in the forms of the 1st kind: I will quickly repeat the main projective properties of conics and then I will present the real program of the course, which consists of the 2nd part of the Reye,[26] proceeding as far as possible.[27]
>
> [Caporali to Cremona. Napoli, 17 December 1878]

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22 "Mi ha colpito la semplicità della definizione analitica dei suoi esaedri: è poi notevole che essi mettono in relazione le 27 rette col pentaedro di Sylvester per mezzo dei teoremi del Reye: tanto più che è forse possibile, per mezzo delle proprietà delle sviluppabili di terza classe, di dare a questa relazione una forma più semplice di quella che è immediatamente fornita dalla memoria di Reye. Questo argomento mi aveva fatto pensare alla Diagonalfläche di Clebsch: le 15 rette che la determinano sono situate tre a tre in 15 piani; e siccome si assume come punto di partenza un pentaedro (di Sylvester) credevo che le di Lei proprietà potessero fornire una teoria del pentaedro completo. Però mi sono accorto poco dopo che l'esaedro che dovrebbe dedursi dalle 15 rette si perde e con esso quasi tutte le proprietà. Ma delle 60 rette di Pascal avviene che ogni spigolo del pentaedro di Sylvester ne assorbe tre, dimodoché ne rimangono 30: queste ultime meritano forse di essere studiate. Passando da un'idea all'altra ho pensato al problema di assumere sei punti fondamentali in un piano in modo da avere la rappresentazione della Diagonalfläche: l'ho risoluto analiticamente."

23 E. Caporali, *Sull'esaedro completo*, Rendiconti della Reale Accademia delle Scienze fisiche e matematiche di Napoli, fascicolo 3°, marzo 1881.

24 L. Cremona, *Teoremi stereometrici dai quali si deducono le proprietà dell'esagrammo di Pascal*, Memorie della Reale Accademia Nazionale dei Lincei, serie 3, vol. 1, 1876–1877, pp. 854–874.

25 A. Cayley, *On Pascal's theorem*, Quarterly Journal of Pure and Applied Mathematics, vol. IX, 1868, pp. 348–353.

26 T. Reye, *Die Geometrie der Lage*, Erste Abtheilung, Carl Rümpler, Hannover, 1866.

27 "Ho cominciato col ripresentare rapidamente la proiettività nelle forme di 1a specie: ripeterò di volo le proprietà principali proiettive delle coniche e poi passerò al vero programma del corso, che consiste nella 2a parte del Reye, arrivando fin dove si potrà."

In October 1878, the Società Italiana delle Scienze awarded him a prize for his two notes: *Sui complessi e sulle congruenze di secondo grado*[28] and *Sopra i piani ed i punti singolari della superficie di Kummer.*[29] The commission was formed by Cremona, Beltrami, and Battaglini.[30]

Throughout his life, Caporali was very attached to his family, plagued by various health problems which resulted in economic problems. For this reason, he often asked for loans from people closest to him, including Cremona. In 1884, he became full professor, but he went through a rough period for the health (his own and of his relatives), the household economy, and also the scientific production, as we can clearly read:

> At the beginning of the summer I was indisposed for a slight reappearance of nervous heartbeat, to which, if you remember, I was more severely subject in the past. [ . . . ] Anyway, the school year just ended was almost lost for my studies except for some reading. But it gets worse. And it is, that the few aptitudes that I previously seemed to have in studies, I now find considerably diminished, which makes me becoming greatly discouraged, as well as the mortification of finding myself unworthy of a position that I had accepted almost with a certain boldness.[31]
>
> [Caporali to Cremona. Genzano di Roma, 24 August 1882]

In these words, the despair that will lead Caporali to death is already beginning to be manifest. Despite the difficulties, he continued to devote himself to the elaboration of some works, as can be seen from a letter published in [4]:

> The first note refers to a $6^{\text{th}}$ degree complex which is the locus of the tangents of all cubics passing through 5 fixed points. It is a complex that can be represented on points in the space: and its representation is given by the surfaces of the $3^{\text{rd}}$ order that pass through the 5 points and the diagonal points of the nonplanar pentagon that they form.
>
> The second note contains a theorem on the tangents to a plane curve drawn from its multiple points. When the tangents emanate from an arbitrary point, in general the first polar of this point is the simplest curve that passes through the contact points of those tangents. But when the point is multiple, there are in general curves of minor orders that define the

[28] E. Caporali, *Sui complessi e sulle congruenze di secondo grado*, Memorie della Reale Accademia Nazionale dei Lincei, s. III, vol. II, 1877–1878, pp. 749–769.

[29] E. Caporali, *Sopra i piani ed i punti singolari della superficie di Kummer*, Memorie della Reale Accademia Nazionale dei Lincei, s. III, vol. II, 1877–1878, pp. 791–810.

[30] As it appears in the *Rapporti* of the *Memorie di Matematica e di Fisica della Società Italiana delle Scienze* (serie III, vol. IV, pp. XXI–XXIV). The *Società Italiana delle Scienze* was founded by Antonio Maria Lorgna (1735–1796) in 1782 in Verona as the Società Italiana, comprising 40 scientists from various parts of Italy. For this reason, it was also called *Accademia dei XL.*

[31] "Al principio dell'estate io sono stato indisposto per un leggero riapparire di cardiopalmo nervoso, incontro al quale, se si rammenta, andavo più gravemente soggetto per l'addietro. [ . . . ] L'anno scolastico testé finito, è stato però quasi perduto pei miei studi se se ne eccettui qualche lettura. Ma c'è di peggio. Ed è che le poche attitudini che prima mi pareva d'avere agli studi, le trovo ora notevolmente diminuite, cosa che mi fa provare grande scoraggiamento, oltre alla mortificazione di trovarmi inferiore ad una posizione che avevo accettato quasi con una certa baldanza."

> constraints on the contact points more simply. My theorem is very general and includes as a particular case that of Bertini, which you presented two or three years ago to the Lincei.[32]
>
> As soon as I have prepared these two notes for the press, I will resume certain studies on systems of lines that I interrupted for military service but which I have now been encouraged to resume by reading Stahl's work[33] on 2nd order and 3rd class systems published in latest issue of Crelle's Journal.[34]
>
> [Caporali to Cremona. Napoli, 29 May 1881]

In March 1881, Caporali was appointed honorary and resident member of the Reale Accademia delle Scienze fisiche e matematiche di Napoli. In 1882, the mathematician Seligmann Kantor, who had studied in Rome under Cremona in 1878, won the competition announced by the Mathematical Section of the Accademia of Naples on the subject: "Considering the birational transformation in two planes coinciding with each other, find the conditions so that by applying the same transformation several times, we return to the original figure."[35] Caporali wrote an elaborate report on Kantor's note, published on the Rendiconti dell'Accademia in December 1883.[36]

From the letters to Cremona, it comes out that in Naples, Caporali was also very active as a member of the faculty while trying to move to Rome taking advantage of Battaglini's wish to go back to Naples. He was also concerned with the preparation of the scientific cabinets of the Mathematics Institute with purchases from Germany and to ensure that Naples had a representative in the Consiglio Superiore della Istruzione Pubblica.

In his short life, Caporali received other awards for his scientific merits: on 31 December 1883, he became a corresponding member of the Reale Accademia

[32] E. Bertini, *Una nuova proprietà delle curve di ordine n con un punto (n-2)-plo*, Trasunti della Reale Accademia Nazionale dei Lincei, (3), 1, 1876–77, pp. 92–96.

[33] W. Stahl, *Das Strahlensystem dritter Ordnung zweiter Klasse,* Journal für die reine und angewandte Mathematik, 91, pp. 1–22.

[34] "La prima si riferisce ad un complesso di 6° grado che è il luogo delle tangenti di tutte le cubiche passanti per 5 punti fissi. È un complesso rappresentabile sui punti dello spazio: e la rappresentazione è data dalle superficie del 3° ordine che passano pei 5 punti e pei punti diagonali del pentagono gobbo che essi formano. La seconda nota contiene un teorema sulle tangenti condotte ad una curva piana da un suo punto multiplo. Quando le tangenti si conducono da un punto arbitrario, in generale la prima polare di questo punto è la curva più semplice che passa pei punti di contatto di quelle tangenti. Ma quando il punto è multiplo, vi sono in generale curve di ordine minore e che perciò definiscono più semplicemente i vincoli che legano i punti di contatto. Il mio teorema è molto generale e comprende come caso particolare quello del Bertini che Ella presentò due o tre anni fa ai Lincei. Appena avrò approntate queste due note per la stampa, riprenderò certi studi sui sistemi di rette che interruppi pel servizio militare ma che ora sono stato invogliato a riprendere per la lettura del lavoro di Stahl sui sistemi del 2° ordine e 3° classe pubblicato nell'ultimo fascicolo del giornale di Crelle."

[35] "Considerando la trasformazione birazionale in due piani tra loro coincidenti, trovare le condizioni affinché applicando più volte di seguito la stessa trasformazione, si ritorni alla figura da cui si parte."

[36] Competition report for the Premio accademico of 1882, published in the Rendiconti della Reale Accademia delle Scienze fisiche e matematiche di Napoli, issue 12°, December 1883.

Nazionale dei Lincei, and in May 1886, shortly before his death, ordinary and resident member of the Accademia Pontaniana of Naples.

On 2 July 1886, he committed suicide; for the circumstance, his colleague Dino Padelletti wrote these words in [22, pp. III–IV]: "He worried excessively about the diminished activity of his wits, which to him seemed to be irreparably decaying, turned his cruel hand against himself in a supreme moment of sadness and despair [...]. Important tasks entrusted to him by the unanimous vote of his colleagues showed the great trust that everyone had placed in that young man, in whom his judgment seemed to have preceded his age, and all whose actions were full of the very warm love of science and a feeling of irreproachable honesty."[37]

Even Guccia, in his letters to Cremona, wrote the reasons for his gesture: "He told me that intelligence abandoned him, he was no longer capable of creating anything, he did not find the usual mental energy to develop any research, he saw himself reached and overtaken by others, he read about various topics but did not feel the strength to deepen some of them; he saw the sacred fire of Science extinguish in him, he was discouraged"[38] (Guccia to Cremona. Palermo, 6 July 1886, in [23, p. 103]).

Guccia wrote again to Cremona: "Discouragement had invaded our friend in last years, due to a slow, but persistent, brain disease that increasingly lowered his mental faculties"[39] (Guccia to Cremona. Palermo, 8 August 1886, in [23, p. 109]).

Furthermore, Guccia expressed the estimation that they both had for Caporali: "Your beloved student, the best of your students [...]. I often collected evidences of estimation for the young Italian geometer. His works were more than a promise. The Cremona school has lost one of its best elements, as a man of science and as a professor"[40] (Guccia to Cremona. Palermo, 6 July 1886, in [23, p. 105]).

During his life, Caporali published 12 notes [22, 32], but among his manuscripts, other works were found which, although incomplete, were considered appropriate to publish. In 1889, 3 years after Caporali's death, Gino Loria published Caporali's entire scientific work in the Giornale di Matematiche di Battaglini [24], on

[37] "Egli accuorandosi oltre misura della scemata attività del suo ingegno, che a lui sembrava decadenza irrimediabile, rivolse in un momento supremo di tristezza e di sconforto la mano crudele contro sé stesso [...]. Importanti ufficii affidatigli dal voto unanime dei colleghi mostravano la grande fiducia da tutti riposta in quel giovane, in cui il senno sembrava avesse precorsa la età, e tutte le cui azioni erano informate dall'amore caldissimo della scienza e dal sentimento di una onestà intemerata."

[38] "Mi diceva che l'intelligenza lo abbandonava, non era più capace di crear nulla, non ritrovava la solita energia mentale per condurre a fine qualsiasi ricerca, si vedeva raggiunto e sorpassato da altri, curiosava intorno a diversi argomenti ma non si sentiva la forza di approfondirne alcuno; vedeva estinguersi in lui il fuoco sacro della Scienza, era scoraggiato."

[39] "Scoraggiamento di cui era invaso il nostro amico negli ultimi anni, per via di una lenta, ma persistente, malattia celebrale che abbassava vieppiù le sue facoltà mentali."

[40] "Suo amatissimo allievo, il migliore dei suoi allievi [...]. Raccolsi spesso all'estero delle autorevoli testimonianze di stima sul conto del giovane geometra italiano. I suoi lavori erano più che una promessa. La scuola del Cremona ha perduto uno dei migliori elementi, come uomo di scienza e come professore."

the initiative of a group of colleagues, friends, and admirers, to remember the distinguished extinct scholar in the most lasting way. In [24], Loria sorted Caporali's works into the following topics: third-order curves, fourth-order curves, nth-order curves, third-order surfaces, one-to-one transformations on the plane, representation of surfaces on the plane, geometry of the line, theory of configurations, four-dimensional geometry, manifolds, and geometric applications of algebraic forms.

Among the most important students of Caporali, we include Pasquale del Pezzo (1859–1936), who graduated in mathematics in 1882 and, among other things, was rector of the University of Naples and mayor of the city.

In 1892, Corrado Segre published the manuscript [25] to complete the fragments collected in *Sulla teoria delle curve piane del quarto ordine* that had been included in [22]. Segre reported two letters that Caporali had addressed to him in 1885 (11 August and 13 September). The letters seem important as evidence of the acknowledgement (by one of the best students of Cremona) of the emergence of a new referent for the Italian school of algebraic geometry. In [25, pp. 172–173], Caporali wrote to Segre:

> What you write to me about my studies on fourth order curves is interesting and shows that you immediately understood the essence of those researches. Although not very advanced, they have a complicated history that is related to various causes extraneous to science that have prevented me for three years from applying myself to study with that regularity and perseverance that would allow me to have good results. [...] When you published your Memoria sulla geometria delle coniche, I immediately saw the advantage that could be gained from the systematic use of that mode of representation and which is confirmed to me by your letter: I did not resume, however, nor will I be able to resume the research immediately, although precisely now, having acquired all their generality, they are in the most interesting stage.[41]
>
> [Caporali to Segre. Torre del Greco, 13 September 1885].

This letter is particularly significant because Caporali, who in 1885 had already been full professor in Naples for some time, addressed to Segre, graduated only 2 years before and assistant of his thesis supervisor Enrico D'Ovidio (1843–1933), as a new referent. Furthermore, the letter contains Caporali's work projects and shows how the hyperspace methods developed by Segre and Veronese fit perfectly into the framework of the research programs of the Cremona school. In this regard, see also [26].

[41] "Ciò che Ella mi scrive intorno ai miei studi sulle curve del quarto ordine è interessante e dimostra che Ella ha immediatamente penetrato lo spirito di quelle ricerche. Per quanto poco avanzate, esse hanno una storia complicata e in relazione con diverse cause estranee alla scienza che m'impediscono da tre anni di attendere allo studio con quella regolarità e quella perseveranza che sole permettono di cavarne buoni frutti. [...] Quando Ella pubblicò la sua Memoria sulla geometria delle coniche, vidi immediatamente il partito che si poteva trarre dall'uso sistematico di quel modo di rappresentazione e che mi è confermato dalla sua ... lettera: non ripresi però, né potrò subito riprendere le ricerche, benché precisamente ora, avendo acquistata tutta la loro generalità, siano nello stadio più interessante."

## 4 Riccardo De Paolis Through the Memory of Corrado Segre

The lives and careers of Riccardo De Paolis and Ettore Caporali followed parallel avenues, as suggested by De Paolis's commemoration by Corrado Segre in [27].

De Paolis was born in Rome on 9 January 1854 from Achille and Elena Chatelain; he studied in Rome immediately demonstrating a penchant for mathematics. In 1870, both Caporali and De Paolis enrolled at the University of Rome and became friends. During the first 2 years, they studied together solving many problems inspired by George Salmon and inventing new ones. Subsequently, they attended the Scuola di magistero per la matematica founded by Cremona in 1873, having as professors also Battaglini and Beltrami.

After graduating in July 1875,[42] De Paolis was appointed Professor of Mathematics in the high school of Caltanissetta while, as we have already said above, Caporali went to Catania. In 1876, both returned to Rome, De Paolis with the assignment of practical mathematics exercises at the University and then also as a substitute teacher for analytical geometry. In June 1878, he obtained his teaching qualification in analytical geometry and projective geometry. In these 2 years, he wrote and published some works on the double transformations on the plane that gave him notoriety in the scientific world. In the paper [28], he developed researches similar to those by Cremona for one-to-one transformations, providing a synthetic treatment. Segre drew attention to a substantial difference with the reduction of plane involutions through Cremona transformations made by Bertini in the same period. In fact, De Paolis did not deal with reducing plane (1, 2)-correspondences by applying Cremona transformations, but he analyzed their projective properties. This work was followed by two others on the study of two particular double transformations and their applications.[43]

In November 1878, winning a competition, he was appointed extraordinary professor of Algebra and Analytical Geometry at the University of Bologna and in January 1880, again by competition, he was called to Pavia as extraordinary professor of Higher Geometry. As already mentioned, the same year, he moved to Pisa in the same chair thanks to an exchange with Bertini and remained in Pisa for the rest of his life. In addition to higher geometry, De Paolis taught graphical statics from 1882–1883 to 1889, and projective and descriptive geometry from 1889–1890. In 1883, he was appointed, at the same time as Caporali, a corresponding member of the Accademia Nazionale dei Lincei and elected a member of the Consiglio direttivo of the Circolo Matematico di Palermo. De Paolis died in Rome on 24 June 1892 due to tuberculous peritonitis.

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[42] De Paolis' degree thesis *Sopra un sistema omaloidico formato da superficie d'ordine n con un punto (n−1)plo* was published in the Giornale di Matematiche, 13, 1875, pp. 226–248, 282–297.

[43] R. De Paolis, *La trasformazione piana doppia di secondo ordine e la sua applicazione alla geometria non euclidea*, Memorie della Reale Accademia Nazionale dei Lincei, (3), 2, 1878, pp. 31–50 and *La trasformazione piana doppia di terzo ordine, primo genere, e la sua applicazione alle curve del quarto ordine*, Memorie della Reale Accademia Nazionale dei Lincei, (3), 2, 1878, pp. 851–878.

In the note [29], De Paolis continued and completed what Karl von Staudt,[44] Felix Klein,[45] and Jean Gaston Darboux[46] had done to establish the fundamental theorem of projective geometry and to arrive, without measurement of quantities, at the projective coordinates of the forms of the first kind. The work consists of two parts: In the first one, following Klein, Lüroth, and Darboux, starting from the concept of Euclidean metrics on the projective line, he proved the fundamental theorem of projective geometry. In the second part, using projective concepts, De Paolis showed that it is possible to put the straight line in direct correspondence with rational numbers and therefore, postulating its completion, with real numbers. Segre stated: "This work gives us a first example of a particular tendency of De Paolis: that of always going back to the foundations, to the principles of theories. Perhaps he did not have this tendency from the beginning. On the occasion of his graduation, Cremona, doubting that De Paolis neglected special cases or the most elementary things in order to deal only with general, elevated issues, made him understand that this was a mistake. The student did not forget teacher's lesson!"[47]

In 1884, De Paolis published the book *Elementi di Geometria*,[48] and in the preface, he wrote: "I had a double purpose; to abandon the ancient separation of plane Geometry from solid, to try to rigorously establish the fundamental truths of Geometry and the theories of equivalence, limits, measure." [49] The originality of this work, which according to Segre had benefited well from the fusion between plane and solid geometry, was due to the fact that De Paolis, when he wrote it, knew no other elements than those of Euclid and during the writing he examined very few texts.

In 1885, De Paolis turned back to the topic of double transformations [30], and, analogously to what he had done in the case of the plane, he studied the ones in the space with the same methods, although here a larger number of cases occur. In [31], he analyzed those involutions in which the straight lines joining the pairs of homologous points are $\infty^2$ instead of $\infty^3$ many.

Even De Paolis, like Bertini and Caporali, can be considered the link between the first and second generations of Italian algebraic geometers. In fact, in addition

[44] K. G. C. von Staudt, *Geometrie der Lage*, Nürnberg Bauer & Raspe, 1847.

[45] F. Klein, *Über die sogenannte Nicht-Euklidische Geometrie*, Mathematische Annalen, 6, 1873, pp. 112–145.

[46] M. G. Darboux, *Sur le théorème fondamental de la géométrie projective* (*Extrait d'une lettre à M. Klein*), Mathematische Annalen, 17, 1880, pp. 55–61.

[47] "Questo lavoro ci dà un primo esempio di una particolare tendenza del De Paolis: quella di risalire sempre ai fondamenti, ai principi delle teorie. Forse questa tendenza egli non aveva da principio. Nell'occasione della laurea il Cremona, dubitando che il De Paolis per occuparsi solo di questioni generali, elevate, trascurasse i casi speciali o le cose più elementari, gli fece intendere che questo era un errore. Il discepolo non dimenticò la lezione del maestro!"

[48] R. De Paolis, *Elementi di geometria*, Torino, E. Loescher, 1884.

[49] "Si propose un doppio scopo; abbandonare l'antica separazione della Geometria piana dalla solida, tentare di stabilire rigorosamente le verità fondamentali della Geometria e le teorie dell'equivalenza, dei limiti, della misura."

to having had scientific relationships and an intense correspondence with Segre, he had Enriques among his students.

His latest research mainly concerned a question of method: Segre revealed that although the analytical method was widely used in previous writings, and more precisely the algebra of invariants, there was a strong preference for essentially geometric methods, a typical preference of the Cremona school. Following these methods, De Paolis proposed to go further.

In the archive of the Istituto Mazziniano in Genoa, a single letter from De Paolis to Cremona was found and, because it is very relevant, we choose to transcribe it almost entirely:

> Together with this letter of mine, I sent you two little notes, one on equivalent figures, which I published last year, the other one on projective involutions where I proved a theorem that I have developed about some research I am doing and I want to talk to you about it, surely I will not bore you, because I know that you remember me always fondly, by the time your old student, and you have always followed my publications with the satisfaction of a teacher. Here is what it is: I think I told you in some other occasion about a fixed idea that has haunted me for several years, that is, the idea of freeing Geometry from the aid of Algebra, founding true pure Geometry. After many efforts and many useless attempts, justified however by the difficulty of the problem, I can finally claim that I have completely resolved the question. It would be impossible for me to tell you in few words the path I followed, the method I used is completely new, and besides solving the problem I had posed, it also provides new results and new research to do. The plan I have proposed is vast and I still need a lot of time to fully implement it. **I start with the definitions of surface, line and point, I do not assume any knowledge of Geometry, and after some general considerations, I study the general theory of correspondences between the points of any two linear, surface or solid fields, assuming as a condition the continuity of the correspondences themselves. In this part my work can be considered as a geometric theory of continuous functions of 1, 2 or 3 real variables, and in fact I prove some fundamental geometric theorems, which correspond perfectly to the fundamental theorems of Analysis by Weierstrass, Cantor, etc. I then show that each continuous correspondence [m, n], between the points of two surfaces, can be replaced with another [1, 1] between the points of two other surfaces, which I call the Riemann surfaces of the given correspondence, in this way I acquire a powerful geometric research tool that will then be very useful to me later and especially in the study of the curves which up to now have been called algebraic and which I still do not know how to call; I introduce in Geometry those considerations that Riemann has used so much in Analysis. Then I am naturally led to study the connection of the surfaces, etc. etc. After these general considerations, I move on to establish the projective Geometry, with the same rigor held by Pasch, but much more simply. In a certain sense, after having elaborated that part of Geometry that corresponds to the general theory of functions, I move on to the part that corresponds to the study of algebraic functions. I analyze the part concerning projective Geometry up to the 2nd degree forms, also considering the imaginary elements, by Staudt's method. I show thereafter, based on some results due to Thieme, that systems $\infty^{n-1}$ of groups of n elements of a fundamental form of 1st kind can be built such that a group of the system is identified by any n-1 elements of its, I call these systems involutions of n degree and n-1 kind. After a complete study of these involutions and their linear systems, I prove, and herein lies the crux of the matter, that n of these involutions always have a group of n elements, real or imaginary, in common. I therefore study the intersections of these involutions and thus find the involutions of n degree and $r \leq$ n-1 kind, that are linear systems $\infty^{r}$ of groups of n elements. I finish this part of the work by showing that two involutions of 1st kind, projective and coincident,**

**always have m+n fixed elements, real or imaginary, in common. In order to arrive at this result, on which the whole theory of curves and surfaces can be based, I apply the properties of continuous correspondences, which I proved at the beginning. The theory of polar groups naturally arises from the theory of involutions. For now, with regard to curves and surfaces, I have only seen how they can be generated, with their linear systems, how it can be shown that two curves of order m, n always have mn common points, real or imaginary, etc. etc. Later I will go further and systematically evolve the whole theory with my method. But I will not stop there. I have also seen how the concept of multidimensional spaces can be introduced, and in a rigorous way, using a representation in our space. For multidimensional fields I also establish a theory of continuous correspondences, of connection, etc., and I introduce the concept of a multidimensional Riemann surface, a concept that I then use to determine the genus of the multidimensional surfaces. After this part, which corresponds to the study of continuous functions of n real variables, I proceed, with a method like that used for the two and three dimensions, to consider the elements that correspond to the algebraic surfaces.**

**You can see how the field I want to explore is large; but I have already found everything I need to easily deduce the remaining parts, in any case, the drafting of the work will still take me a long time, and I will not be able to finish it before the end of the year, although I work hardly on it.** I had already found the main results when I learned that part of the subject I was studying, the part, which refers to establishing the Geometry of algebraic curves without using Algebra, had twice been the topic of a competition by the Berlin Academy, and that the last time Kötter received the prize. Now, Kötter's work has not been published so far, I read the report they made at the Academy and it seems to me that he did not go as far as I did, that he applied another, less rigorous method, and rigor in some problems is everything. In any case, in order not to lose priority, I wrote the main results I obtained in 6 notebooks, closed them, sealed them and sent them to the Accademia dei Lincei to record the date. I then decided to submit all the work to the competition for the royal prize, which expires with the current year, but before deciding I would like to have your advice. Does it seem to you that the topic is sufficiently important, and that, if I had done it well, I would not make a bad impression even if I do not win the prize?[50]

[De Paolis to Cremona. Pisa, 3 March 1887]

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[50] "Insieme a questa mia le ho spedito due noticine, una, sulle figure equivalenti, che pubblicai l'anno scorso, l'altra sulle involuzioni proiettive, nella quale dimostro un teorema che ho trovato a proposito di alcune ricerche che sto facendo, e delle quali desidero parlarle, sicuro di non recarle noia, perché so che Ella si ricorda sempre affettuosamente di me, che sono ormai un suo antico allievo, e sempre ha tenuto dietro con soddisfazione di maestro alle pubblicazioni mie. Ecco di che cosa si tratta: Mi pare di averle altre volte parlato di una idea fissa che mi perseguita da più anni, dell'idea cioè di emancipare la Geometria dal sussidio dell'Algebra, di fondare la vera Geometria pura. Dopo molte fatiche e molti tentativi inutili, ma giustificati dalla difficoltà dell'argomento, posso finalmente asserire di avere completamente risoluto la quistione. Mi sarebbe impossibile dirle in poche parole la via che ho seguito, il metodo che ho adoperato è completamente nuovo, e non solo risolve il problema che mi ero proposto, ma fornisce anche nuovi risultati e nuove ricerche da fare. Il piano che mi sono proposto è vasto e ho bisogno ancora di molto tempo per svolgerlo completamente. Comincio dalle definizioni della superficie, della linea e del punto, non suppongo alcuna cognizione di Geometria, e dopo alcune considerazioni generali, studio la teoria generale delle corrispondenze tra i punti di due qualunque campi lineari, superficiali o solidi, ponendo per sola condizione la continuità delle corrispondenze stesse. In questa parte il mio lavoro si può considerare come una teoria geometrica delle funzioni continue di 1, 2 o 3 variabili reali, ed infatti dimostro alcuni teoremi geometrici fondamentali, che tengono perfettamente luogo di

On 1 July 1886, Ernst Kötter won the Steiner prize at the Berlin Academy of Sciences on the subject of founding a purely geometric theory of curves and surfaces

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teoremi fondamentali dell'Analisi, trovati da Weierstrass, Cantor, ecc. Faccio poi vedere che ogni corrispondenza continua [m,n], tra i punti di due superficie, si può sostituire con un'altra [1,1] tra i punti di altre due superficie, che chiamo le superficie di Riemann della corrispondenza data, così acquisto un potente mezzo di ricerca geometrico, mezzo che mi sarà poi utilissimo in seguito e specialmente nello studio delle curve che fin qui si sono chiamate algebriche e che io ancora non so come chiamare; introduco nella Geometria quelle considerazioni che Riemann ha tanto utilizzato nell'Analisi. Poi sono naturalmente condotto a studiare la connessione delle superficie, ecc. ecc. Dopo queste considerazioni così generali, passo a stabilire la Geometria proiettiva, con lo stesso rigore tenuto da Pasch, ma molto più semplicemente. In un certo senso, dopo avere svolto quella parte della Geometria che corrisponde alla teoria generale delle funzioni, passo alla parte che corrisponde allo studio delle funzioni algebriche. La parte che riguarda la Geometria proiettiva la spingo fino alle forme di 2° grado, comprese, considerando anche gli elementi immaginarî, col metodo di Staudt. Faccio poi vedere, fondandomi sopra alcuni risultati dovuti a Thieme, che si possono costruire sistemi $\infty^{n\text{-}1}$ di gruppi di n elementi di una forma fondamentale di $1^{a}$ specie, tali che un gruppo del sistema sia individuato da n-1 qualunque dei suoi elementi, questi sistemi li chiamo involuzioni di grado n e specie n-1. Dopo uno studio completo di queste involuzioni e dei loro sistemi lineari, dimostro, e qui sta il nodo della quistione, che n di queste involuzioni hanno sempre comune un gruppo di n elementi, reali o immaginari. Passo quindi a studiare le intersezioni di queste involuzioni e trovo così le involuzioni di grado n e specie $r \leq n-1$, che sono sistemi lineari $\infty^{r}$ di gruppi di n elementi. Termino questa parte del lavoro dimostrando che due involuzioni, di $1^{a}$ specie, proiettive e sovrapposte hanno sempre $m + n$ elementi uniti, reali o immaginarî, comuni. Per arrivare a questo risultato, sul quale si può basare tutta la teoria delle curve e delle superficie, applico le proprietà delle corrispondenze continue, che ho dimostrato sul principio. Dalla teoria delle suddette involuzioni sorge naturalmente quella dei gruppi polari. Per ora, riguardo alle curve e superficie, ho veduto solamente come si possono generare, insieme ai loro sistemi lineari, come si può dimostrare che due curve di ordine m, n hanno sempre mn punti comuni, reali o immaginarî, ecc. ecc. In seguito mi spingerò più innanzi e svolgerò col mio metodo sistematicamente tutta la teoria. Ma non mi fermerò qui. Ho pure veduto come si può introdurre il concetto di spazî a più dimensioni, e in modo rigoroso, ricorrendo ad una effettiva rappresentazione nello spazio nostro. Per i campi a più dimensioni stabilisco pure una teoria delle corrispondenze continue, della connessione, ecc., e introduco il concetto di superficie di Riemann a più dimensioni, concetto che poi utilizzo per determinare i generi delle superficie a più dimensioni. Dopo questa parte, che corrisponde allo studio delle funzioni continue di n variabili reali, passo, con metodo analogo a quello tenuto per le due e tre dimensioni, a considerare gli enti che corrispondono alle superficie algebriche. Ella vede quanto è vasto il campo che voglio esplorare; però ho già trovato tutto quello che mi serve per dedurre senza difficoltà le parti rimanenti, in ogni modo la redazione del lavoro mi porterà via ancora molto tempo, e non potrò terminarlo prima della fine dell'anno, per quanto ci lavori assiduamente. Già avevo trovato i risultati principali, quando seppi che una parte del tema che stavo trattando, quella che si riferisce allo stabilire la Geometria delle curve algebriche senza ricorrere all'Algebra, era stata per due volte consecutive messa a concorso dall'Accademia di Berlino, e che l'ultima volta un certo Kötter aveva ricevuto il premio. Ora il lavoro di Kötter non è stato fin qui pubblicato, ho letto la relazione che ne hanno fatta all'Accademia e mi pare che non si sia spinto innanzi come me, e che abbia tenuto un altro metodo, meno rigoroso, ed il rigore in certe quistioni è tutto. In ogni modo per non perdere la priorità ho appuntato, in 6 quinterni, i principali risultati che ho ottenuto, li ho chiusi, sigillati e spediti all'Accademia dei Lincei, perché se ne prenda data. Ho poi pensato di presentare tutto il lavoro al concorso per il premio reale, che scade coll'anno corrente, prima però di decidermi avrei piacere di sentire un suo consiglio. Le pare che l'argomento sia sufficientemente importante, e che, qualora lo avessi bene svolto, non farei una brutta figura anche se non vincessi il premio?"

of higher order. The day after writing to Cremona, De Paolis wrote a similar letter to Segre:[51]

> I read the report on Kötter's work in the publications of the Berlin Academy; I have been looking forward to its publication, but I do not know if it has been published yet. [ . . . ] For many years the idea of freeing Geometry from the use of Algebra, of founding true *pure Geometry* has haunted me. When I heard the news[52] I had completely solved the problem, I could not publish my research immediately, because I had not yet done the secondary parts of all the work I had proposed to do; so I thought of waiting for the publication of Kötter's essay, whatever could happen, all the more so since, from the report made on this essay, it seems to me that Kötter did not go as far as I did and that he used another method. [ . . . ] But now my work has gone further and I am tired of waiting; therefore, I wrote the main results I obtained in 6 notebooks, and I sent them, in a sealed envelope, to the Accademia dei Lincei, to record the date. So I hope to arrive, in any case, in time not to lose priority, or at least to be able to prove that I had solved the problem independently of Kötter. [ . . . ] It seems to me that Kötter's result, which I also obtained independently, in any case, is one of the possible applications of my method.[53]
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> [De Paolis to Segre. Pisa, 4 March 1887]

With these researches, in December 1887, De Paolis competed for the royal prize for mathematics at the Accademia Nazionale dei Lincei, presenting his essay *Fondamenti di una teoria, puramente geometrica, delle linee e delle superficie*. The paper was divided into three parts: general theory of correspondence between the points of several groups, general theory of projective correspondences in the basic forms with one dimension, and the one in two dimensions. The first part was published in 1890,[54] and the second and third parts[55] were published after his

[51] In order to avoid unnecessary repetition, the part selected in bold from the one sent to Cremona has been omitted from this letter.

[52] He is referring to the news of the result of the contest for the Steiner prize.

[53] "Lessi nei Rendiconti dell'Accademia di Berlino, la relazione del lavoro di Kötter; ho aspettato ansiosamente la sua pubblicazione, ma non so se ancora abbia veduto la luce. [ . . . ] Sono più anni che mi perseguita l'idea di emancipare la Geometria dal sussidio dell'Algebra, di fondare la vera *Geometria pura*. Quando seppi la notizia avevo completamente risolto la quistione, non potevo pubblicare subito le mie ricerche, perché ancora non avevo svolto le parti secondarie di tutto il lavoro che mi ero proposto di fare; perciò pensai di aspettare la pubblicazione della Memoria di Kötter, qualunque cosa potesse avvenire, tanto più che dalla relazione fatta su questa Memoria mi pare che Kötter non si sia spinto innanzi come me e che abbia tenuto un altro metodo. [ . . . ] Ora però il mio lavoro è spinto più innanzi e sono stanco di aspettare; perciò ho appuntato i principali risultati che ho ottenuto in 6 quinterni, e li ho spediti, in un plico sigillato, all'Accademia dei Lincei, perché se ne prenda data. Spero così di arrivare, in ogni caso, in tempo per non perdere la priorità, o almeno poter provare che avevo risoluto il problema indipendentemente dal Kötter. [ . . . ] Mi pare che il risultato di Kötter, che anche io del resto ho tenuto indipendentemente, in ogni modo sia una delle possibili applicazioni del mio metodo."

[54] R. De Paolis, *Teoria dei gruppi geometrici e delle corrispondenze che si possono stabilire tra i loro elementi*, Memorie della Società Italiana delle Scienze (detta dei XL), (3), 7, 1890.

[55] R. De Paolis, *Le corrispondenze proiettive nelle forme geometriche fondamentali*, Memorie della Accademia delle Scienze di Torino, (2), 42, 1892, pp. 495–584 and *Teoria generale delle corrispondenze proiettive e degli aggruppamenti proiettivi nelle forme fondamentali a due dimensioni*, Rendiconti della Reale Accademia Nazionale dei Lincei, (5), 3, 1894, pp. 225–227.

death in 1892 and in 1894, thanks to the contribution of Mario Pieri and Segre. The extracts from the second essay contain, according to the author's will, the dedication "If the work is worthy—of remembering a name—it must be the illustrious name—of Luigi Cremona—who in geometric science—I had as a teacher."[56]

## 5 Conclusions

The unpublished correspondence between Luigi Cremona and his students forms a rich source of evidences that highlight even better the dynamic that was developing in the period immediately after the unification of Italy, when the mathematicians of the Risorgimento succeeded in bringing Italian mathematics to the level of the European one. The numerous direct students of Cremona, including Bertini, Caporali, and De Paolis, formed a generation of scholars who had the merit of cultivating the scientific innovations of their teacher and passing them on to the subsequent generations, that is, those young mathematicians who were able to take up the baton by easily entering the framework of European research, having at their disposal cultural structures and tools (journals, libraries, scientific laboratories, etc.) by now largely sufficient for research developments.

From many points of view, Bertini, more than the others, served as a link between the Cremona work and that of the new school of algebraic geometry that was being formed. Caporali in Naples and De Paolis between Pavia and Pisa contributed to developing Cremona's ideas and researches and constantly considered him as a reference in their scientific life. Also, they both interacted with Segre, showing that he was already representing the new reference in the framework of the programs of research of the Cremona school.

The letters that the students of Cremona sent to their teacher form a new piece of that puzzle full of scientific and cultural ideas that has been reconstructing for years, making the influence that Cremona had in the scientific environment, the interactions, the collaborations that his students had with other European scientists, and the successes and results achieved nationally and internationally even more evident.

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[56] "Se l'opera è degna – di ricordare un nome – sia quello illustre – di Luigi Cremona – che nella scienza geometrica – ebbi maestro."

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