

Two Letters by Guido Castelnuovo



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Abstract In this expository paper, we transcribe two letters by Guido Castelnuovo, one to Francesco Severi and the other to Beniamino Segre, and explain the contents of both, which basically focus on the quest for an algebraic proof of the equality between the analytic and the arithmetic irregularity and of the closedness of regular 1-forms on a complex, projective, algebraic surface. Such an algebraic proof has been found only in the 1980s by Deligne and Illusie.

Keywords Guido Castelnuovo · Beniamino Segre · Francesco Severi · Holomorphic forms · Hodge theory · Frölicher spectral sequence

1 Introduction

As it is well-known, the treatise by Federigo Enriques epitomizing the celebrated classification of algebraic surfaces by the Italian school of algebraic geometry has been published posthumously in 1949, a few years after the sudden death of the author in 1946. As pointed out by Guido Castelnuovo in the preface (see [8]),

(...) dove il terreno è meno solido l'Autore mette sull'avviso lo studioso. Di questi punti ancora fluidi quello che presenta la difficoltà più ardua ed il maggiore interesse è *la teoria* dei sistemi continui di curve algebriche (...) che esistono sopra ogni superficie irregolare. (...) tutti i tentativi compiuti (...) per dimostrarla mediante considerazioni algebrico-geometriche si sono urtati contro difficoltà sinora insuperate. (...) l'Autore dà anche suggerimenti sopra una via da tentare per giungere alla meta. Debbo confessare che non vedo come quella via possa tradursi in un procedimento irreprensibile.

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(...) where the ground is less solid the Author warns the reader. Among these still unsteady points the most difficult and interesting one *is the theory* of the continuous systems of algebraic curves (...) existing on any irregular surface. (...) all attempts (...) towards an algebro-geometric approach have been frustrated by still insurmountable difficulties. (...) the Author provides some hints about a strategy to reach the goal. I should confess I cannot see how that strategy may be translated into a fully rigorous argument.

The two letters by Guido Castelnuovo that we transcribe and translate into English in Sect. 2, the first one addressed to Francesco Severi and dated 1947 and the second one addressed to Beniamino Segre and dated 1950, provide firsthand witness of Castelnuovo's attempts to a purely algebro-geometric understanding of irregular surfaces.

In Sect. 3 we explain the background of Castelnuovo's letters, using modern terminology. In particular we explain a classical method, very familiar to Castelnuovo and due to Picard and Severi, of constructing regular 1-forms on a surface. As explained in Sect. 3.3, one of the crucial points of Castelnuovo's approach is the attempt of proving the closedness of global regular 1-forms, a fact that today we are aware to strongly rely on the characteristic zero assumption. Indeed, Castelnuovo's remarks in his letters turn out to be quite inconclusive and sometimes even unprecise, as we discuss in Sect. 4 that is devoted to explaining most of the issues raised by Castelnuovo in his two letters. The algebro-geometric proof of the results that Castelnuovo sought (i.e., closedness of global regular 1-forms and equality of different definitions of irregularity, in characteristic zero) is now available; it is due to work by Deligne and Illusie in the 1980s and turns out to be completely out of reach of Castelnuovo's classical tools, since (somehow paradoxically) it involves a tricky reduction to the case of positive characteristic. Section 5 is devoted to give a brief account on how these algebraic proofs can be obtained using modern tools.

We stress that the present note does not contain any original result, but in our opinion the contents of Castelnuovo's letters are worthy of careful consideration from both a historical and a mathematical viewpoint. This paper is addressed to readers who are well aware of rather advanced concepts in algebraic geometry, so we do not dwell on explaining standard technical details when they occur.

2 The Letters

In this section we transcribe two letters of Castelnuovo, the first one of November 26, 1947, to Francesco Severi and the second one of January 15, 1950, to Beniamino Segre. The first letter belongs to the "Fondo Guido Castelnuovo" of the Accademia Nazionale dei Lincei that has been edited by Paola Gario, has been digitalized, and can be found on the web page

http://operedigitali.lincei.it/Castelnuovo/Lettere_E_Quaderni/menuL.htm

The second letter comes from the collection of documents of Beniamino Segre kept at the University of Caltech.

2.1 Guido Castelnuovo to Francesco Severi

Roma, 26 novembre 1947

Caro Severi,

aderendo al tuo desiderio ti comunico alcuni risultati sulle superficie irregolari; parecchi si ottengono senza difficoltà e possono servire come esercizio per i tuoi discepoli.

Lo scopo remoto ed ambizioso che mi proponevo era di costruire una teoria delle dette superficie indipendente dalla nozione di sistema continuo di curve, teoria in cui si ritrovassero il teorema sul numero $(p_g - p_a)$ dei differenziali totali indipendenti di prima specie, il teorema di Hodge, ecc.. Il programma è appena iniziato; ma si deve raggiungere la meta, a meno che la teoria delle superficie irregolari non riservi delle sorprese che non saprei nemmeno immaginare.

- I. Indico con $|C|$ un sistema regolare di grado n e genere π ; in molti casi occorre supporre che $|C|$ sia *abbastanza ampio*, contenga entro di sé il sistema canonico $|K|$ od anche un suo multiplo; ricercando caratteri invarianti, tutto ciò non ha importanza. Indico con χ il tuo invariante q' , cioè il numero delle curve indipendenti di $|2C + K|$ che passano per il gruppo jacobiano G_δ di un fascio $|C|$ e in conseguenza per il gruppo base G_n del fascio. Indico con G_k il gruppo dei k punti cuspidali di una superficie, d'ordine n , a singolarità ordinarie, le cui sezioni piane appartengano al sistema $|C|$.

Ecco un significato di χ che si raggiunge subito:

- (1) *E' χ la sovrabbondanza del sistema $|4C + 2K|$ rispetto al gruppo dei punti cuspidali G_k (cioè G_k impone $k - \chi$ condizioni al detto sistema).*

Invece G_k presenta condizioni indipendenti ai sistemi $|mC + K|$, $|mC + 2K|$, \dots , per $m \geq 5$.

Per $m = 4$ vi è un risultato di Enriques ottenuto indirettamente attraverso il computo dei moduli di una superficie, risultato che converrebbe dimostrare direttamente; lo ricordo perché interviene tra poco: "La sovrabbondanza del sistema $|4C + K|$ rispetto al gruppo G_k dei punti cuspidali è un invariante", che indicherò con Q' e di cui sotto darò l'espressione.

- (2) *La serie completa g_k determinata dal gruppo G_k sopra una curva di $|4C + K|$ passante per esso ha la dimensione χ .*
- (3) *La serie completa g_k determinata dal gruppo G_k sopra una curva di $|3C + K|$ passante per esso (ad es.: sulla $f = f'_x = 0$) ha la dimensione $2n - \pi + 2p_g + p_a - (I + 4) + \theta$ dove $0 \leq \theta \leq p_g - p_a$ (si suppone $|C|$ abbastanza ampio). E' θ un invariante?*

II. Il procedimento che ti ha condotto a stabilire l'invarianza di $q' = \chi$ fa vedere subito che:

(4) *E' invariante il numero delle curve linearmente indipendenti di $|2C + 2K|$ che passano per il gruppo jacobiano G_δ di un fascio $|C|$ ed anche per il gruppo base G_n ; indicherò questo invariante con Q .*

Si vede poi (se è esatto il risultato di Enriques sopra citato) che:

(5) *E' pure invariante il numero delle curve linearmente indipendenti di $|2C + 2K|$ che passano per il gruppo G_δ senza esser costrette a passare per G_n ; questo nuovo invariante uguaglia l'invariante di Enriques Q' .*

E' quindi invariante il numero delle condizioni che una curva di $|2C + 2K|$ passante per il gruppo jacobiano G_δ di un fascio $|C|$ deve soddisfare per contenere il gruppo base. Si dimostra che questo invariante soddisfa alla diseuguaglianza $Q' - Q \leq p_g$.

Quanto alle espressioni di Q e Q' posso dir questo.

Se le ∞^{Q-1} curve di $|2C + 2K|$ passanti per $G_\delta + G_n$ segano sopra una curva di $|2C + K|$ passante per lo stesso gruppo una serie *completa* (residua di $G_\delta + G_n$ rispetto alla serie canonica) allora:

$$Q - 1 = p_a + p_g + p^{(1)} - (I - 4) + \omega$$

dove $\omega (\leq p_g - p_a)$ è un nuovo invariante che ha un significato molto semplice: $I + 4 - \omega - 1$ è il numero delle condizioni che un gruppo G_{I+4} della tua serie d'equivalenza (in senso stretto) presenta alle curve *bicanoniche* costrette a contenerlo.

Se la serie lineare nominata non è completa, dall'espressione di $Q - 1$ va tolta la deficienza $\leq p_g - p_a$ della serie stessa.

Nello stesso ordine d'idee ti comunico ancora questo risultato:

(6) *La sovrabbondanza del sistema $|3C + K|$ rispetto al gruppo jacobiano G_δ di un fascio $|C|$ è un invariante e vale precisamente $2p_g$ (se il sistema completo $|C|$ cui il fascio appartiene è abbastanza ampio).*

III. Altre questioni. Come sai il teorema fondamentale da dimostrare è questo: una curva Γ di $|2C + K|$ passante per il gruppo jacobiano G_δ e il gruppo base G_n di un fascio $|C|$ sega sopra la curva generica C del fascio (fuori di G_n) un gruppo canonico *che non appartiene ad una curva aggiunta C'* . Ho cercato di trasformare la condizione in altre equivalenti. Tale è ad esempio questa: il gruppo G_n su quella curva deve presentare condizioni indipendenti alla serie caratteristica di Γ *resa completa*. Alla serie caratteristica in senso stretto, G_n presenta solo $n - 1$ condizioni.

Altra forma: Scriviamo la identità di Picard in coordinate omogenee:

$$Xf'_x + Yf'_y + Zf'_z + Tf'_t = 0$$

dove $X = 0, \dots$ sono superficie d'ordine $n - 3$. Occorre aggiungere la condizione (non detta esplicitamente) che le superficie $yX - xY = 0, \dots, tZ -$

$zT = 0$ siano aggiunte alla $f = 0$ d'ordine n . Segue che le $X = 0, \dots, T = 0$ passano semplicemente per i t punti tripli di f ed hanno inoltre in comune un gruppo di $(n - 4)d - 3t$ punti sulla curva doppia d'ordine d di f ; esse segano inoltre rispettivamente i piani $x = 0, \dots, t = 0$ in curve aggiunte d'ordine $n - 3$. Da ciò segue che quel gruppo di punti della curva doppia appartiene alla serie segata su questa dalle superficie d'ordine $n - 4$ passanti semplicemente per i t punti tripli, purch'è questa serie venga resa completa, mentre essa ha la deficienza $p_g - p_a$ per la definizione stessa di irregolarità. Orbene il teorema fondamentale equivale al seguente: *Per quel gruppo di $(n - 4)d - 3t$ punti della curva doppia non passa nessuna superficie d'ordine $n - 4$ che contenga i t punti tripli di f .* Questo enunciato si traduce in questo altro, molto elegante dal punto di vista analitico: *Non è possibile soddisfare una identità del tipo:*

$$\bar{X}f'_x + \bar{Y}f'_y + \bar{Z}f'_z + \bar{T}f'_t \equiv Qf$$

ove $\bar{X} = 0, \dots, \bar{T} = 0$ sono superficie aggiunte d'ordine $n - 3$ e $Q = 0$ una superficie d'ordine $n - 4$ se non nel caso banale $\bar{X} = \frac{1}{n}xQ, \dots, \bar{T} = \frac{1}{n}tQ$.

Ritornando all'identità di Picard scritta sopra, ti consiglio di far studiare da qualche discepolo la omografia tra il sistema di superficie non aggiunte di ordine $n - 3$ $\lambda X + \mu Y + \nu Z + \rho T = 0$ e il sistema di piani $\lambda x + \mu y + \nu z + \rho t = 0$, ognuno dei quali taglia la superficie corrispondente in una curva aggiunta. Nel caso delle rigate irrazionali dei primi ordini si trovano proprietà elegantissime.

IV. Finalmente alcune osservazioni che ti potranno servire se esporrai in lezione la tua Nota sugli integrali semiesatti.

Tu dimostri che ad ogni curva di $|2C + K|$ passante per il gruppo jacobiano e per il gruppo base di un fascio $|C|$ (curva covariante del fascio, come io la chiamo) si può associare una determinata curva covariante di ogni altro fascio $|D|$. Due curve associate segano sullo stesso gruppo di punti la curva di contatto di due fasci. Esse inoltre si segano in un gruppo G_{I+4} della tua serie. Si vede facilmente che questo gruppo è comune a tutta la famiglia di curve covarianti associate relative agli infiniti fasci esistenti sulla superficie. Ogni curva C di $f = 0$ è segata dalla curva covariante della famiglia in un gruppo canonico che dirò *gruppo traccia*.

Preso un punto P della superficie, esistono infinite curve per P per le quali P appartiene al gruppo traccia. *Tutte queste curve si toccano in P .* Vuol dire che ad ogni punto P di $f = 0$ è collegata una direzione tangente, o un elemento lineare uscente da P (indeterminato solo se P appartiene al gruppo G_{I+4}). Connettendo tutti questi elementi si viene a ricoprire la superficie con un fascio di curve (trascendenti) che risultano esser le curve integrali dell'equazione $Bdx - Ady = 0$, dove $A = 0$ e $B = 0$ sono due superficie aggiunte d'ordine $n - 2$ secanti su f le curve covarianti dei fasci $x = \text{cost.}$, $y = \text{cost.}$ Resterebbe naturalmente da far vedere che $1/f'_z$ è fattore integrante dell'espressione differenziale.

Al variare di P su f quella tangente in P descrive una congruenza algebrica di classe $2\pi - 2$ e di ordine $k - \nu = 6\pi - 6 + p^{(1)} - 1 - (I + 4)$ ove k è il numero di punti cuspidali e ν è l'ordine della curva $f = f'_x = 0$.

E qui termino questa lunghissima lettera che vorrei potesse spingere a colmare nella teoria delle superficie quella lacuna che tutti avvertiamo.

Cordiali saluti dal tuo aff.mo

GUIDO CASTELNUOVO

Rome, November 26, 1947

Dear Severi,

following your wishes I am going to tell you some results about irregular surfaces; many of them are easily obtained and may be useful as exercises for your students. My ambitious and ultimate purpose was to build a theory of such surfaces independent of the notion of continuous system of curves, a theory embracing the theorem on the number $(p_g - p_a)$ of independent global differentials of the first kind, the theorem of Hodge, etc.. This program has just started; but the goal should be achieved, unless the theory of irregular surfaces hide amazing things I could not even imagine.

- I. Let $|C|$ be a regular system of degree n and genus π ; in many cases we need to assume that $|C|$ is *sufficiently ample*, containing the canonical system $|K|$ or even one of its multiples; since we are looking for invariant characters, this is immaterial. I denote by χ your invariant q' , namely, the number of independent curves of $|2C + K|$ passing through the jacobian group G_δ of a pencil $|C|$, hence through the base locus G_n of the pencil. Let G_k be the group of the k cuspidal points¹ of a surface, of degree n , with only ordinary singularities, and whose plane sections belong to the system $|C|$.

Here is a meaning of χ which is immediate:

- (1) *The invariant χ is the superabundance of the system $|4C + 2K|$ with respect to the cuspidal points G_k (i.e. G_k imposes $k - \chi$ conditions to such system).*

On the other hand, G_k gives independent conditions to the systems $|mC + K|$, $|mC + 2K|$, \dots , per $m \geq 5$.

For $m = 4$ there is a result of Enriques, indirectly obtained by a moduli computation for a surface, but which should be directly proven; I recall it because it is coming into play shortly later: "The superabundance of the system $|4C + K|$ with respect to the group G_k of cuspidal points is an invariant", which I will denote Q' and whose expression I am going to give below.

- (2) *The complete series g_k determined by the group G_k on a curve of $|4C + K|$ passing through it has dimension χ .*

¹ The usual English term for *cuspidal points* is *pinch points*.

- (3) *The complete series g_k determined by the group G_k on a curve of $|3C + K|$ passing through it (for instance: on $f = f'_x = 0$) has dimension $2n - \pi + 2p_g + p_a - (I + 4) + \theta$ where $0 \leq \theta \leq p_g - p_a$ (assume $|C|$ sufficiently ample). Is θ an invariant?*

II. The same argument which led you to establish the invariance of $q' = \chi$ immediately shows:

- (4) *It is invariant the number of linearly independent curves of $|2C + 2K|$ passing through the jacobian group G_δ of a pencil $|C|$ and also through the base group of G_n ; I will denote this invariant by Q .*

Then one sees (if the aforementioned result of Enriques is correct) that:

- (5) *It is also invariant the number of linearly independent curves of $|2C + 2K|$ passing through the group G_δ without having to pass through G_n ; this new invariant is equal to Enriques invariant Q' .*

It is therefore invariant the number of conditions that a curve of $|2C + 2K|$ passing through the jacobian group G_δ of a pencil $|C|$ has to satisfy in order to contain the base group. One proves that this invariant satisfies the inequality $Q' - Q \leq p_g$.

Regarding the expressions of Q e Q' I can state the following.

If the ∞^{Q-1} curves of $|2C + 2K|$ passing through $G_\delta + G_n$ cut on a curve of $|2C + K|$ passing through the same group a *complete series* (residual of $G_\delta + G_n$ with respect to the canonical series) then:

$$Q - 1 = p_a + p_g + p^{(1)} - (I - 4) + \omega$$

where $\omega (\leq p_g - p_a)$ is a new invariant which has a very simple meaning: $I + 4 - \omega - 1$ is the number of conditions that a group G_{I+4} of your series of equivalence (in the strict sense) prescribes to the *bicanonical* curves forced to contain it.

If such a linear series is not complete, from the expression of $Q - 1$ one has to subtract the deficiency $\leq p_g - p_a$ of the series.

In the same circle of ideas I also tell you the following result:

- (6) *The superabundance of the system $|3C + K|$ with respect to the jacobian group G_δ of a pencil $|C|$ is an invariant and its value is precisely $2p_g$ (if the complete system $|C|$ to which the pencil belong is sufficiently ample).*

III. Other issues. As you know, the fundamental theorem to be proven is the following: a curve Γ of $|2C + K|$ passing through the jacobian group G_δ and the base group G_n of a pencil $|C|$ cuts on the generic curve C of the pencil (off G_n) a canonical group *which does not belong to an adjoint curve C'* . I tried to translate this condition into other equivalent formulations. Such is for instance the following one: the group G_n on that curve has to impose independent conditions to the characteristic series of Γ *made complete*. To the characteristic series in the strict sense, G_n imposes only $n - 1$ conditions.

Other formulation: Let us write Picard's identities in homogeneous coordinates:

$$Xf'_x + Yf'_y + Zf'_z + Tf'_t = 0$$

where $X = 0, \dots$ are surfaces of degree $n - 3$. We have to add the condition (not explicitly stated) that the surfaces $yX - xY = 0, \dots, tZ - zT = 0$ are adjoint to $f = 0$ of degree n .² It follows that $X = 0, \dots, T = 0$ pass simply through the t triple points of f and moreover share a group of $(n - 4)d - 3t$ points on the double curve of degree d of f ; furthermore, they cut the planes $x = 0, \dots, t = 0$, respectively, in adjoint curves of degree $n - 3$. Hence it follows that such group of points of the double curve belongs to the series cut on this curve by the surfaces of degree $n - 4$ passing simply through the t triple points, provided this series has been made complete, while it has deficiency $p_g - p_a$ by the very definition of irregularity. Now, the fundamental theorem is equivalent to the following: *Through such groups of $(n - 4)d - 3t$ points of the double curve it does not pass any surface of degree $n - 4$ containing the t triple points of f .*³ This statement translates into the following one, which is quite elegant from the analytic viewpoint: *It is impossible to verify an identity of the form:*

$$\bar{X}f'_x + \bar{Y}f'_y + \bar{Z}f'_z + \bar{T}f'_t \equiv Qf$$

where $\bar{X} = 0, \dots, \bar{T} = 0$ are adjoint surfaces of degree $n - 3$ and $Q = 0$ is a surface of degree $n - 4$ except in the trivial case $\bar{X} = \frac{1}{n}xQ, \dots, \bar{T} = \frac{1}{n}tQ$.

Going back to Picard's identity as written above, I suggest to you to propose to some student to investigate the homography between the system of non-adjoint degree $n - 3$ surfaces $\lambda X + \mu Y + \nu Z + \rho T = 0$ and the system of planes $\lambda x + \mu y + \nu z + \rho t = 0$, each cutting the corresponding surface in an adjoint curve. In the case of irrational ruled surfaces of low degree one finds very elegant properties.

IV. Finally, a few remarks you may find useful if you will present in a course your note about semiexact integrals.

You prove that to every curve of $|2C + K|$ passing through the jacobian group and the base group of a pencil $|C|$ (*covariant curve of the pencil*, as I call it) one can associate a unique covariant curve of every other pencil $|D|$. Two associated curves cut on the same group of points the contact curve of two pencils. They moreover cut each other in a group G_{I+4} of your series. One easily checks that this group is common to the whole family of associated covariant curves with

² This is clearly an error, Castelnuovo means adjoint of degree $n - 2$.

³ The right statement here would be: *Through such groups of $(n - 4)d - 3t$ points of the double curve it does not pass any surface of degree $n - 4$ containing the t triple points of f and not containing the double curve.*

respect to the infinitely many pencils on the surface. Every curve C of $f = 0$ is cut by the covariant curve of the family in a canonical group which I will call *trace group*.

Taken a point P of the surface, there exist infinitely many curves through P such that P belongs to the trace group. *All these curves intersect in P .* It means that to every point P of $f = 0$ is associated a tangential direction, or a linear element (not defined only if P belongs to the group G_{I+4}). By connecting all these elements the surface is covered by a pencil of (transcendental) curves which turn out to be the integral curves of the equation $Bdx - Ady = 0$, where $A = 0$ and $B = 0$ are two adjoint surfaces of degree $n - 2$ cutting on f the covariant curves of the pencils $x = \text{const.}$, $y = \text{const.}$ Of course one should show that $1/f'_z$ is an integral factor of the differential expression.

Varying P on f the tangent in P describes an algebraic congruence of class $2\pi - 2$ and degree $k - \nu = 6\pi - 6 + p^{(1)} - 1 - (I + 4)$ where k is the number of cuspidal points and ν is the degree of the curve $f = f'_x = 0$.

Here I stop this quite long letter I wish it could stimulate to fill in the theory of surfaces that gap we all perceive.

Best regards, yours friendly

GUIDO CASTELNUOVO

2.2 *Guido Castelnuovo to Beniamino Segre*

Roma, 15 genn. 50

Caro Professore,

In relazione alla nostra conversazione di venerdì scorso e al programma di ricerche di cui Le parlavo, penso di sottoporle una questione, risolta la quale si sarebbe compiuto un passo notevole verso la meta cui Le accennavo. Si tratta di una questione di geometria algebrica, la quale, ove si possano togliere alcune restrizioni forse non necessarie, si muta in una questione relativa alle equazioni alle derivate parziali con condizioni al contorno. Con i mezzi svariati e potenti di cui Ella dispone potrà affrontarla e pervenire alla risposta desiderata.

Sia $f(x, y, z, t)$ una superficie (in coord. omog.) d'ordine n , irriducibile, con singolarità ordinarie; e siano $X = 0$, $Y = 0$, $Z = 0$, $T = 0$ quattro superficie aggiunte d'ordine $n - 3$. Si tratta di dimostrare che un'identità del tipo

$$Xf'_x + Yf'_y + Zf'_z + Tf'_t = Qf, \quad (1)$$

con Q polinomio di grado $n - 4$, non può sussistere salvo nel caso banale (identità di Eulero) $X = \frac{1}{n}xQ, \dots, T = \frac{1}{n}tQ$. Per farle vedere l'interesse della questione Le dirò che se si toglie la condizione che le sup. X, \dots, T siano aggiunte, e si sostituisce con la condizione meno stretta che siano aggiunte le sei superficie

d'ordine $n - 2$ $yX - xY = 0, \dots$, allora la identità può sussistere con Q identicamente nulla; anzi di identità di quel tipo ve ne sono $p_g - p_a$ indipendenti per una superficie irregolare (Picard).

Ritornando alla (1), supposto che essa possa aver luogo, si vedrebbe che la superficie $Q = 0$ incontra la curva doppia di $f = 0$ nei punti tripli e nei punti ove $X'_x + Y'_y + Z'_z + T'_t = 0$, donde si concluderebbe che $Q \equiv X'_x + Y'_y + Z'_z + T'_t + \overline{Q}$ (salvo un fattore costante), essendo $\overline{Q} = 0$ una superficie aggiunta d'ordine $n - 4$ che darebbe luogo a un integrale doppio senza periodi; e di qua l'assurdo (Hodge). Ma io richiedo evidentemente una dimostrazione più diretta e più elementare di quella qui abbozzata.

Ci pensi quando ha tempo, perché mi pare ne valga la pena. Cordiali saluti; aff.mo

G. Castelnuovo

Rome, January 15, 1950

Dear Professor,

Concerning our conversation of last Friday and the research program I exposed to you, I am going to propose to you a question, whose solution would provide a remarkable step towards the goal I mentioned. It is a question in algebraic geometry, which, up to removing some maybe unnecessary restrictions, translates into a question in partial differential equations with boundary conditions. By applying the many and powerful tools you have at your disposal you could address it and obtain the desired answer.

Let $f(x, y, z, t)$ be a surfaces (in homogeneous coordinates) of degree n , irreducible, with ordinary singularities; let $X = 0, Y = 0, Z = 0, T = 0$ be four adjoint surfaces of degree $n - 3$. The point is to show that an identity of the form

$$Xf'_x + Yf'_y + Zf'_z + Tf'_t = Qf, \tag{2}$$

with Q polynomial of degree $n - 4$, is not satisfied unless in the trivial case (Euler identity) $X = \frac{1}{n}xQ, \dots, T = \frac{1}{n}tQ$. In order to show you the interest of the question I will tell you that if one drops the condition that the surfaces X, \dots, T are adjoint, and one replaces it by the less strict condition that the six degree $n - 2$ surfaces $yX - xY = 0, \dots$ are adjoint, then the identity may hold with Q identically zero; indeed, there are $p_g - p_a$ independent such identities for an irregular surface (Picard).

Coming back to (2), assuming it may hold, one would see that the surface $Q = 0$ meets the double curve of $f = 0$ in the triple points and in the points where $X'_x + Y'_y + Z'_z + T'_t = 0$, whence one would conclude that $Q \equiv X'_x + Y'_y + Z'_z + T'_t + \overline{Q}$ (up to a constant factor), where $\overline{Q} = 0$ would be an adjoint surface of degree $n - 4$ which would give rise to a double integral without periods, hence a contradiction (Hodge). But of course I am looking for a more direct and more elementary proof than the one sketched here.

Please think about that when you have time, because I believe it is worth the trouble. Best regards; yours friendly

G. Castelnuovo

3 Regular 1-Forms on a Surface

If X is a smooth, irreducible, projective surface over an algebraically closed field \mathbb{K} , the elements of $H^0(X, \Omega_X^1)$ are called *regular 1-forms* on X . We will denote the dimension of $H^0(X, \Omega_X^1)$ by $q_{\text{an}}(X)$ (or simply by q_{an} if there is no danger of confusion), and we will call it the *analytic irregularity* of X (see [6]).

In this section we want to explain the background of Castelnuovo’s letters, using modern terminology. In particular we want to explain a classical method, very familiar to Castelnuovo and due to Picard and Severi, of constructing regular 1-forms on a surface. In Sect. 3.3, we explain Castelnuovo’s viewpoint on the attempts of proving closedness of regular 1-forms on a surface.

3.1 The General Set Up

Let X be a smooth, irreducible, projective surface over an algebraically closed field \mathbb{K} . We may assume X to be linearly normally embedded as a surface of degree d in a projective space \mathbb{P}^r , with $r \geq 5$, in such a way that the following happens. If we consider a general projection π of S to \mathbb{P}^3 , whose image is a surface S of degree d , then S has *ordinary singularities* (see [20, Thm. 2]), i.e., it has:

- an irreducible *nodal* double curve Γ , i.e., S has normal crossings at the general point of Γ ,
- a finite number of triple points for both Γ and S ; the triple points for Γ are *ordinary*, i.e., the tangent cone there to Γ consists of the union of three non-coplanar lines, and the tangent cone there to S consists of the union of three distinct planes,
- finitely many *pinch points* on Γ ; we will denote by G_c the *pinch points scheme*, i.e., the reduced zero-dimensional scheme on X where the differential of π drops rank, so that G_c is mapped by π to the set of pinch points of S on Γ . We will set $\gamma = \text{length}(G_c)$.

The map $\pi : X \rightarrow S$ is the normalization map.

We will introduce homogeneous coordinates $[x_1, x_2, x_3, x_4]$ in \mathbb{P}^3 and related affine coordinates (x, y, z) , with

$$x = \frac{x_1}{x_4}, \quad y = \frac{x_2}{x_4}, \quad z = \frac{x_3}{x_4}$$

so that $x_4 = 0$ is the *plane at infinity*. We assume that the coordinates (i.e., the corresponding fundamental points) are general with respect to S . The homogeneous equation of S is of the form $F(x_1, x_2, x_3, x_4) = 0$, with F an irreducible homogeneous polynomial of degree d , and the affine equation of S is $f(x, y, z) = 0$, with $f(x, y, z) = F(x, y, z, 1)$. We will denote by f_x, f_y, f_z the partial derivatives of f with respect to x, y, z and by F_i the partial derivative of F with respect to x_i , for $1 \leq i \leq 4$ (we will use similar notations for other polynomials). Note that

$$f_x(x, y, z) = F_1(x, y, z, 1) \tag{3}$$

and similarly for the other derivatives. Therefore, by Euler’s identity, we have

$$d \cdot f(x, y, z) = xF_1(x, y, z, 1) + yF_2(x, y, z, 1) + zF_3(x, y, z, 1) + F_4(x, y, z, 1). \tag{4}$$

By the generality assumption of the coordinates with respect to S , we have that:

- the plane at infinity is not *tangent* to S , i.e., it cuts out on S a curve whose pullback on X via π is smooth;
- each of the pencils \mathcal{P}_i of planes with homogeneous equations $hx_i = kx_4$, with $(h, k) \in \mathbb{K} \setminus \{(0, 0)\}$, pulls back via π to a *Lefschetz pencil* \mathcal{X}_i on X , with $1 \leq i \leq 3$;
- the pullback Γ_i on X of the curve γ_i cut out on S off the double curve Γ by the *polar surfaces* $F_i = 0$ is smooth for $1 \leq i \leq 3$.

Remark 1 By the genericity of the position of S with respect to the coordinate system, one sees that the curves Γ_i , for $1 \leq i \leq 3$, contain the pinch points scheme G_c and intersect pairwise transversely there.

The singular points of the finitely many curves in the pencil \mathcal{X}_i are nodes and form a reduced zero-dimensional scheme \mathcal{J}_i on X , which is called the *Jacobian scheme* of \mathcal{X}_i , for $1 \leq i \leq 3$. We will assume that, for all $i \in \{1, 2, 3\}$, the image J_i of this scheme on S , called the *Jacobian scheme* of \mathcal{P}_i , has no intersection with the double curve Γ .

It is also easy to check that the curve Γ_i cuts out on Γ_j the divisor $G_c + \mathcal{J}_k$, where $\{i, j, k\} = \{1, 2, 3\}$. So, in particular, taking into account that $\Gamma_i \in |3C + K_X|$, for $1 \leq i \leq 3$ (we denote by C a hyperplane section of X), one has

$$\mathcal{O}_{\Gamma_3}(G_c + \mathcal{J}_1) = \mathcal{O}_{\Gamma_3}(G_c + \mathcal{J}_2) = \mathcal{O}_{\Gamma_3}(3C + K_X), \tag{5}$$

hence

$$\mathcal{O}_{\Gamma_3}(\mathcal{J}_1) = \mathcal{O}_{\Gamma_3}(\mathcal{J}_2). \tag{6}$$

Similar relations hold on Γ_2 and Γ_3 . Note that (5) implies that $|3C + K_X|$ has no fixed component and $(3C + K_X)^2 > 0$; hence, $3C + K_X$ is big and nef.

Let $e := e(X)$ be the *Euler–Poincaré characteristic* of X (i.e., the second *Chern class* of the tangent bundle of X) and g the arithmetic genus of the hyperplane sections of X . By the *Zeuthen–Segre formula* (see [10, p. 301]), the length δ of J_i is

$$\delta = e + 4(g - 1) + d.$$

3.2 The Expression of 1-Forms on a Surface

It is a result by Picard (see [19, p. 116]; Picard works over \mathbb{C} , but it is easy to check that his argument works on any algebraically closed field \mathbb{K}) that if ω is a regular 1-form on X , then it is the pullback on X of a rational 1-form of the type

$$\frac{A dy - B dx}{f_z} \tag{7}$$

where $A = 0, B = 0$ are affine equations of two *adjoint surfaces* of degree $d - 2$ to S . Recall that a surface is said to be adjoint to S if it contains the double curve Γ of S .

In the 1-form (7), we can make a change of variables passing from x, y to x, z . From the relation

$$f_x dx + f_y dy + f_z dz = 0$$

we deduce

$$dy = -\frac{f_x dx + f_z dz}{f_y}.$$

Substituting into (7) we find

$$\frac{-\frac{A f_x + B f_y}{f_z} dx - A dz}{f_y}$$

and this has to be of the same form as (7) with respect to the variables x, z . This implies that there must be a polynomial C of degree $d - 2$ such that $C = 0$ is the affine equation of an adjoint surface to S , such that

$$-\frac{A f_x + B f_y}{f_z} = C, \quad \text{modulo } f = 0.$$

This yields the *Picard’s relation*

$$A f_x + B f_y + C f_z = N f \tag{8}$$

where N is a suitable polynomial of degree $d - 3$. The Picard relation has some remarkable consequences, pointed out by Severi (see [22, §9]). Before stating Severi’s result, we recall the following:

Lemma 2 (Castelnuovo’s Lemma) *Let $g(x_1, x_2, x_3) = 0$ be the equation of an irreducible plane curve of degree n with no singular points except nodes. Then there is no non-trivial syzygy of degree $l \leq d - 2$ of the triple (g_1, g_2, g_3) of derivatives of g .*

For the proof see [22, §7] or [14, p. 34]. Next we can prove Severi’s result:

Proposition 3 *If A, B, C are non-zero polynomials verifying (8), then the (projective closure of the) surface with equation $A = 0$ [resp. $B = 0, C = 0$] contains the base line of the pencil of planes \mathcal{P}_1 [resp. of $\mathcal{P}_2, \text{ of } \mathcal{P}_3$] and also the Jacobian scheme J_1 [resp. J_2, J_3] of this pencil. Moreover the (projective closures of the) surfaces $A = 0, B = 0, C = 0$ cut out on the plane at infinity the same curve off the aforementioned lines.*

Proof First we prove that the surface with equation $A = 0$ contains the scheme J_1 . Let P be a point of J_1 . Then f, f_y, f_z vanish at P . Hence by (8), also Af_x vanishes at P . However f_x does not vanish at P because P does not belong to the double curve Γ of S . Hence A vanishes at P . Similarly for the surface with equation $B = 0$ [resp. $C = 0$] containing the scheme J_2 [resp. J_3].

Next, homogenize (8). By (3) (and the similar for the other derivatives), we get a relation of the form

$$\bar{A}F_1 + \bar{B}F_2 + \bar{C}F_3 = \bar{N}F$$

where we denote by the bars the homogenization of the corresponding polynomials. By taking into account the Euler identity, this relation takes the form

$$(d\bar{A} - x_1\bar{N})F_1 + (d\bar{B} - x_2\bar{N})F_2 + (d\bar{C} - x_3\bar{N})F_3 = x_4\bar{N}F_4.$$

Setting $x_4 = 0$ and taking into account Castelnuovo’s Lemma 2, we have identically

$$d\bar{A} - x_1\bar{N} \equiv 0, \quad d\bar{B} - x_2\bar{N} \equiv 0, \quad d\bar{C} - x_3\bar{N} \equiv 0$$

under the condition $x_4 = 0$. This implies that

$$dA_0 - x_1N_0 \equiv 0, \quad dB_0 - x_2N_0 \equiv 0, \quad dC_0 - x_3N_0 \equiv 0$$

where A_0, B_0, C_0 are the homogeneous components of A, B, C in degree $d - 2$ and N_0 is the homogeneous component of N of degree $d - 3$. The assertion follows right away. □

Remark 4 Note that the surfaces with equations $A = 0, B = 0, C = 0$ in Proposition 3 are not necessarily adjoint. Keeping the notation of the proof of

Proposition 3, set $N_0 = \vartheta$. Then we have identities of the form

$$A = x\vartheta + A_1, \quad B = y\vartheta + B_1, \quad C = z\vartheta + C_1, \quad N = d\vartheta + N_1 \tag{9}$$

where A_1, B_1, C_1 are (non-homogeneous) polynomials of degree at most $d - 3$ and N_1 has degree at most $d - 4$.

Severi next proved the following proposition (see [22, §9]):

Proposition 5 *Let $A = 0$ be the affine equation of an adjoint surface of degree $d - 2$ to S containing the scheme J_1 . Then there are uniquely determined adjoint surfaces of degree $d - 2$ to S with affine equations $B = 0$ and $C = 0$, containing the schemes J_2 and J_3 , respectively, such that (8) holds. Each of the polynomials A, B, C uniquely determines the other two.*

Proof Consider A as in the statement. The complete linear system $|2C + K_X|$ is the pullback to X of the curves cut out on S , off the double curve Γ , by the adjoint surfaces of degree $d - 2$. Looking at the exact sequence

$$0 \longrightarrow \mathcal{O}_X(-C) \longrightarrow \mathcal{O}_X(2C + K_X) \longrightarrow \mathcal{O}_{\Gamma_3}(2C + K_X) \longrightarrow 0$$

we see that $|2C + K_X|$ cuts out on Γ_3 a complete linear series ξ , because $h^1(X, \mathcal{O}_X(-C)) = 0$ (by the Kodaira vanishing theorem, see [11, p. 154]). Moreover, since $h^0(X, \mathcal{O}_X(-C)) = 0$, the restriction map

$$H^0(X, \mathcal{O}_X(2C + K_X)) \longrightarrow H^0(\Gamma_3, \mathcal{O}_{\Gamma_3}(2C + K_X))$$

is injective.

Let us abuse notation and denote by $A \in |2C + K_X|$ the pullback on X of the curve cut out on S by the (projective closure of the) surface $A = 0$ off Γ . Then A cuts out on Γ_3 a divisor of the form $\mathcal{J}_1 + Z \in \xi$. Since $\mathcal{J}_2 + Z \in \xi$ by (6), there is a unique curve $B \in |2C + K_X|$ that cuts out $\mathcal{J}_2 + Z$ on Γ_3 . By abusing notation, we denote by B a non-zero polynomial, uniquely defined up to a constant, such that $B = 0$ is the adjoint surface cutting out on S off Γ the curve whose pullback on X is B . The surfaces Af_x and Bf_y cut out on the curve γ_3 the same divisor; hence, there is a non-zero constant b such that $Af_x - bBf_y = 0$ on γ_3 . By substituting B with $-bB$, we may assume that $Af_x + Bf_y = 0$ on γ_3 .

Consider now the complete intersection scheme Y , whose ideal is generated by f and f_z , which consists of two components given by γ_3 and by Γ with a double structure. Since $Af_x + Bf_y$ vanishes on γ_3 and vanishes with multiplicity 2 on Γ , then $Af_x + Bf_y$ vanishes on Y , and therefore $Af_x + Bf_y$ is a combination of f and f_z , i.e., there are polynomials C and N , of degrees $d - 2$ and $d - 3$, respectively, such that (8) holds. Note that C cannot be identically zero. Otherwise we would have an identity of the sort

$$Af_x + Bf_y = Nf.$$

This is impossible, because then Bf_y would vanish along the curve γ_1 , but neither f_y nor B can vanish along this curve. Since Af_x , Bf_y , and f vanish doubly along Γ , then C vanishes along Γ so that $C = 0$ is adjoint to S . Moreover C is uniquely determined. In fact, from another identity of the form

$$Af_x + Bf_y + C'f_z = N'f,$$

subtracting memberwise from (8), we deduce

$$(C - C')f_z = (N - N')f$$

and f would divide the left-hand side, which is impossible because both factors there have degree smaller than f . The assertion follows. \square

By taking into account Proposition 3, one has the:

Corollary 6 *Every adjoint surface to S of degree $d - 2$ containing the scheme J_1 contains also the base line of the pencil \mathcal{P}_1 .*

We can state this corollary in an intrinsic form:

Corollary 7 *Let X be a smooth, irreducible, projective surface, C a very ample effective divisor on X , and \mathcal{P} a Lefschetz pencil in $|C|$. Then any curve in $|2C + K_X|$ containing the Jacobian scheme of the pencil \mathcal{P} (i.e., the scheme of double points of the singular curves in \mathcal{P}) also contains the base locus scheme of \mathcal{P} .*

Now, given an adjoint surface of degree $d - 2$ to S containing the scheme J_1 , with affine equation $A = 0$, consider the other two adjoint surfaces $B = 0$ and $C = 0$ existing by Proposition 5. We can consider the three regular 1-form pullbacks on X of the forms

$$\frac{A dy - B dx}{f_z}, \quad \frac{B dz - C dy}{f_x}, \quad \frac{C dx - A dz}{f_y}.$$

By the very proof of Proposition 3, we see that these forms are equal. In conclusion, if we consider the vector space $\text{Adj}_{d-2}(S)$ of (non-homogeneous) polynomials of degree (at most) $d - 2$ defining adjoint surfaces to S passing through J_1 , this determines an isomorphism

$$\varphi : \text{Adj}_{d-2}(S) \rightarrow H^0(X, \Omega_X^1). \tag{10}$$

The map φ sends a polynomial A to the 1-form pullback of the form (7) to X , where $B = 0$ is the adjoint surface of degree $d - 2$ described in Proposition 5. The same by exchanging J_1 with J_2 or J_3 .

3.3 Closedness of 1-Forms

The following result is well-known:

Proposition 8 *If X is a complex, smooth, compact surface, any regular 1-form on X is closed.*

Proof This proof is extracted from [2, p. 137–138].

Let $\omega \in H^0(X, \Omega_X^1)$ be a non-zero regular form. By Stokes' Theorem one has

$$\int_X d\omega \wedge d\bar{\omega} = \int_X d(\omega \wedge d\bar{\omega}) = 0. \tag{11}$$

Write down locally $d\omega = f dz_1 \wedge dz_2$. Then

$$d\omega \wedge d\bar{\omega} = -|f|^2 dz_1 \wedge d\bar{z}_1 \wedge dz_2 \wedge d\bar{z}_2 = 4|f|^2 dx_1 \wedge dy_1 \wedge dx_2 \wedge dy_2$$

where $z_j = x_j + iy_j$, for $1 \leq j \leq 2$, so that by (11) one gets $f = 0$, i.e., $d\omega = 0$. □

The proof of this proposition is analytic and does not hold in positive characteristic. In fact in positive characteristic there are counterexamples to Proposition 8 (see [16, Corollary]). There is then the problem, which was classically well-known (see [24, p. 185]) and considered also in the two letters by Castelnuovo, of finding a purely algebraic proof of Proposition 8. It is useful for us to review the classical viewpoint on this subject.

Let us keep the notation introduced above. Let ω be a regular 1-form on the surface X , which is the pullback on X of the rational 1-form (7). Then we have $d\omega = \phi dx \wedge dy$, with

$$\phi = \frac{\partial}{\partial x} \left(\frac{A}{f_z} \right) + \frac{\partial}{\partial y} \left(\frac{B}{f_z} \right)$$

where it is intended that the differentiations take place on the surface X , so that z is function of x, y implicitly defined by $f(x, y, z) = 0$. So, for instance,

$$\frac{\partial z}{\partial x} = -\frac{f_x}{f_z}$$

and

$$\begin{aligned} \frac{\partial}{\partial x} \left(\frac{A}{f_z} \right) &= \frac{\left(A_x + A_z \frac{\partial z}{\partial x} \right) f_z - A \left(f_{zx} + f_{zz} \frac{\partial z}{\partial x} \right)}{f_z^2} = \\ &= \frac{\left(A_x - A_z \frac{f_x}{f_z} \right) f_z - A \left(f_{zx} - f_{zz} \frac{f_x}{f_z} \right)}{f_z^2} = \\ &= \frac{f_z^2 A_x - f_z (A f_{zx} + A_z f_x) + f_{zz} A f_x}{f_z^3} \end{aligned}$$

and similarly

$$\frac{\partial}{\partial y} \left(\frac{B}{f_z} \right) = \frac{f_z^2 B_y - f_z (B f_{zy} + B_z f_y) + f_{zz} B f_y}{f_z^3}$$

so that

$$\phi = \frac{f_z^2 (A_x + B_y) - f_z (A f_{zx} + A_z f_x + B f_{zy} + B_z f_y) + f_{zz} (A f_x + B f_y)}{f_z^3}. \quad (12)$$

Taking into account (8) and the identity

$$\frac{\partial (A f_x + B f_y)}{\partial z} = A_z f_x + A f_{xz} + B_z f_y + B f_{yz},$$

Equation (12) becomes

$$\phi = \frac{1}{f_z^3} \left[f_z^2 (A_x + B_y + C_z - N) + f (N f_{zz} - f_z N_z) \right]$$

so that

$$\phi = \frac{A_x + B_y + C_z - N}{f_z}, \quad \text{modulo } f$$

and this is regular on X . Hence if we set

$$Q = A_x + B_y + C_z - N$$

the polynomial Q has to vanish on the double curve Γ of S , because it has to vanish where f_z vanishes.

A priori Q is a polynomial of degree $d - 3$, but one has actually:

Lemma 9 *In the above setting, Q has degree $d - 4$.*

Proof By taking into account the identities (9) in Remark 4, we have

$$A_x = \theta + x\theta_x + \frac{\partial A_1}{\partial x}, \quad B_y = \theta + y\theta_y + \frac{\partial B_1}{\partial y}, \quad C_z = \theta + z\theta_z + \frac{\partial C_1}{\partial z}$$

where θ is a homogeneous polynomial of degree $d - 3$. Hence, by Euler's identity, we get

$$\begin{aligned} A_x + B_y + C_z - N &= d\theta + \frac{\partial A_1}{\partial x} + \frac{\partial B_1}{\partial y} + \frac{\partial C_1}{\partial z} - (d\theta + N_1) = \\ &= \frac{\partial A_1}{\partial x} + \frac{\partial B_1}{\partial y} + \frac{\partial C_1}{\partial z} - N_1 \end{aligned} \quad (13)$$

which proves the assertion. \square

In conclusion, we have

$$\frac{\partial}{\partial x} \left(\frac{A}{f_z} \right) + \frac{\partial}{\partial y} \left(\frac{B}{f_z} \right) = \frac{Q}{f_z}$$

and with similar computations, one finds

$$\frac{\partial}{\partial y} \left(\frac{B}{f_x} \right) + \frac{\partial}{\partial z} \left(\frac{C}{f_x} \right) = \frac{Q}{f_x}, \quad \frac{\partial}{\partial z} \left(\frac{C}{f_y} \right) + \frac{\partial}{\partial x} \left(\frac{A}{f_y} \right) = \frac{Q}{f_y}.$$

In any event, the form ω as above is closed if and only if $Q = 0$ modulo f . But, since Q has degree smaller than d , this is the case if and only if Q is identically zero. So, taking into account (13), the problem of giving an algebraic proof of Proposition 8 translates in the following:

Problem 10 Find an algebraic proof that (8) implies either one of the two equivalent relations

$$N = A_x + B_y + C_z, \quad N_1 = \frac{\partial A_1}{\partial x} + \frac{\partial B_1}{\partial y} + \frac{\partial C_1}{\partial z} \quad (14)$$

each of which is called the *integrability condition*.

We want to stress that any solution of Problem 10 must use the fact that the base field \mathbb{K} has characteristic zero.

3.4 Homogeneous Form of Picard's Relation

It is useful to describe the homogeneous form of Picard's relation (8). This is contained in [19, p. 119], and we expose this here for the reader's convenience.

By (4), we can rewrite (8) as

$$\begin{aligned} & d \cdot A F_1(x, y, z, 1) + d \cdot B F_2(x, y, z, 1) + d \cdot C F_3(x, y, z, 1) = \\ & = N(x F_1(x, y, z, 1) + y F_2(x, y, z, 1) + z F_3(x, y, z, 1) + F_4(x, y, z, 1)) \end{aligned}$$

Set

$$X_1 = \overline{dA - xN}, \quad X_2 = \overline{dB - yN}, \quad X_3 = \overline{dC - zN}, \quad X_4 = -\bar{N}$$

where, as usual, the bars stay for homogenization. By (9), we have

$$X_1 = \overline{dA_1 - xN_1}, \quad X_2 = \overline{dB_1 - yN_1}, \quad X_3 = \overline{dC_1 - zN_1}, \quad X_4 = -\bar{N}$$

and the polynomials X_i , with $1 \leq i \leq 4$, are of degree $d - 3$. Then we have the relation

$$X_1 F_1 + X_2 F_2 + X_3 F_3 + X_4 F_4 = 0 \quad (15)$$

which is the *homogeneous Picard's relation*. If we consider the matrix

$$M = \begin{pmatrix} X_1 & X_2 & X_3 & X_4 \\ x_1 & x_2 & x_3 & x_4 \end{pmatrix}, \quad (16)$$

all minors of order 2 of M , after dehomogenization, are linear combinations of A , B , C and so are in $\text{Adj}_{d-2}(S)$.

Suppose the homogeneous Picard's relation (15) holds. Taking into account the expressions of the polynomials X_i , for $1 \leq i \leq 4$, and the relations (9), the integrability relation in the form of the right-hand side of (14) becomes

$$\frac{\partial X_1}{\partial x_1} + \frac{\partial X_2}{\partial x_2} + \frac{\partial X_3}{\partial x_3} + \frac{\partial X_4}{\partial x_4} = 0 \quad (17)$$

which is the *homogeneous integrability condition*. Problem 10 can now be expressed in homogeneous form as:

Problem 11 Find an algebraic proof that (15) (with all minors of order 2 of the matrix M in (16), after dehomogenization, in $\text{Adj}_{d-2}(S)$) implies the homogeneous integrability condition (17).

4 Comments on Castelnuovo's Letters

This section is devoted to explaining most of the issues raised by Castelnuovo in his two letters. Both letters focus on the understanding of the algebro-geometric meaning of the analytic irregularity q_{an} and on solving Problems 10 or 11.

In §§1 and 2 of the first letter, Castelnuovo suggests, with no proofs, various geometric interpretations of q_{an} . Analogous remarks have been partially included by Castelnuovo in the paper [4] published 2 years after this letter. Castelnuovo does not say it, but maybe he had in mind in the letter that the various geometric interpretations of q_{an} could have been useful to algebro-geometrically prove the equality between q_{an} and $q_a := h^1(X, \mathcal{O}_X)$ that we will call the *arithmetic irregularity*, an equality that Castelnuovo proved with analytic methods in the paper [3] of 40 years before (a different proof was given by Severi in [21]; see also [6]). Note that this equality does not hold in positive characteristic, as proved by Igusa in [12] (see also [17]).

Let us keep the notation introduced so far. The first result Castelnuovo states in his letter to Severi is the following:

Proposition 12 *Let $|C|$ be a very ample linear system on X . Then*

$$h^1(X, \mathcal{O}_X(4C + 2K_X) \otimes \mathcal{I}_{G_c|X}) = q_{\text{an}}.$$

Proof In Remark 1 we saw that $3C + K_X$ is big and nef. This implies that also $4C + K_X$ is big and nef.

Look at the exact sequence

$$0 \longrightarrow \mathcal{O}_X(C + K_X) \longrightarrow \mathcal{O}_X(4C + 2K_X) \longrightarrow \mathcal{O}_{\Gamma_3}(4C + 2K_X) \longrightarrow 0.$$

We have $h^i(X, \mathcal{O}_X(C + K_X)) = 0$ for $1 \leq i \leq 2$ (by the Kodaira vanishing theorem), and $h^1(X, \mathcal{O}_X(4C + 2K_X)) = 0$, because $4C + K_X$ is big and nef (by Mumford’s theorem, see [18, §II]). This implies that $|4C + 2K_X|$ cuts out on Γ_3 a complete, non-special linear series g''_n , where

$$r = n - p_a(\Gamma_3)$$

(recall the definition of the curves Γ_i , $i = 1, 2, 3$, from the beginning of Sect. 3.1).

Set now

$$h^1(X, \mathcal{O}_X(4C + 2K_X) \otimes \mathcal{I}_{G_c|X}) = h.$$

The linear system $|\mathcal{O}_X(4C + 2K_X) \otimes \mathcal{I}_{G_c|X}|$ cuts out on Γ_3 , off G_c , a complete linear series $\xi = g_{n-\gamma}^{r-\gamma+h}$ (recall that $\gamma = \text{length}(G_c)$), so that h is the index of speciality of ξ .

Let G be a general divisor of ξ , so that

$$\mathcal{O}_{\Gamma_3}(G + G_c) = \mathcal{O}_{\Gamma_3}(4C + 2K_X).$$

Let G' be a divisor on Γ_3 such that

$$\mathcal{O}_{\Gamma_3}(G' + \mathcal{J}_1) = \mathcal{O}_{\Gamma_3}(2C + K_X).$$

Adding up these two relations and subtracting (5), we get

$$\mathcal{O}_{\Gamma_3}(G + G') = \mathcal{O}_{\Gamma_3}(3C + 2K_X) = \omega_{\Gamma_3}.$$

So we get

$$h = h^0(\Gamma_3, \mathcal{O}_{\Gamma_3}(G')).$$

On the other hand, by looking at the exact sequence

$$0 \longrightarrow \mathcal{O}_X(-C) \longrightarrow \mathcal{O}_X(2C + K_X) \longrightarrow \mathcal{O}_{\Gamma_3}(2C + K_X) \longrightarrow 0,$$

since $h^1(X, \mathcal{O}_X(-C)) = 0$ (by the Kodaira vanishing theorem), we see that $|2C + K_X|$ cuts out on Γ_3 a complete linear series; hence

$$h = h^0(\Gamma_3, \mathcal{O}_{\Gamma_3}(G')) = h^0(X, \mathcal{O}_X(2C + K_X) \otimes \mathcal{I}_{\mathcal{J}_1|X})$$

and the assertion follows by the isomorphism φ in (10). □

After this Castelnuovo claims that

$$h^1(X, \mathcal{O}_X(nC + mK_X) \otimes \mathcal{I}_{G_c|X}) = 0$$

for $n \geq 5$ and $m \geq 1$. We have not been able to prove (or disprove) this assertion.

Another geometric interpretation of q_{an} that Castelnuovo suggests in the letter to Severi is the following: let D be a curve in $|4C + K_X|$ that contains G_c and it is smooth there; then

$$h^0(D, \mathcal{O}_D(G_c)) = q_{an} + 1. \tag{18}$$

Also for this statement, we could not come up with a proof (or a counterexample).

Remark 13 It looks rather difficult that (18) could hold. In fact, consider again the curve Γ_3 . Then D cuts out on Γ_3 a divisor $G_c + G$, where, by (5), one has

$$\mathcal{O}_{\Gamma_3}(G) = \mathcal{O}_{\Gamma_3}(\mathcal{J}_1 + H) \tag{19}$$

where H is a divisor cut out on Γ_3 by a hyperplane. By looking at the exact sequence

$$0 \longrightarrow \mathcal{O}_X(-C) \longrightarrow \mathcal{O}_X(3C + K_X) \longrightarrow \mathcal{O}_D(3C + K_X) \longrightarrow 0$$

and since $h^1(X, \mathcal{O}_X(-C)) = 0$, we see that $|3C + K_X|$ cuts out on D a complete linear series. Hence the linear series $|\mathcal{O}_D(G_c)|$ is cut out on D , off G , by the linear system $|\mathcal{O}_X(3C + K_X) \otimes \mathcal{I}_{G|X}|$, and therefore

$$h^0(D, \mathcal{O}_D(G_c)) = h^0(X, \mathcal{O}_X(3C + K_X) \otimes \mathcal{I}_{G|X}). \tag{20}$$

From the exact sequence

$$0 \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_X(3C + K_X) \otimes \mathcal{I}_{G|X} \longrightarrow \mathcal{O}_{\Gamma_3}(3C + K_X) \otimes \mathcal{I}_{G|X} \longrightarrow 0$$

we have

$$h^0(X, \mathcal{O}_X(3C + K_X) \otimes \mathcal{I}_{G|X}) \leq h^0(\Gamma_3, \mathcal{O}_{\Gamma_3}(3C + K_X) \otimes \mathcal{I}_{G|X}) + 1. \tag{21}$$

By (19), we have

$$\mathcal{O}_{\Gamma_3}(3C + K_X) \otimes \mathcal{I}_{G|X} = \mathcal{O}_{\Gamma_3}(2C + K_X) \otimes \mathcal{I}_{\mathcal{J}_1|X}.$$

Since, as we saw in the proof of Proposition 12, $|2C + K_X|$ cuts out on Γ_3 a complete linear series, we have

$$h^0(\Gamma_3, \mathcal{O}_{\Gamma_3}(2C + K_X) \otimes \mathcal{I}_{\mathcal{J}_1|X}) = h^0(X, \mathcal{O}_X(2C + K_X) \otimes \mathcal{I}_{\mathcal{J}_1|X}) = q_{an}.$$

Putting together this, (20) and (21), one gets

$$h^0(D, \mathcal{O}_D(G_c)) \leq q_{an} + 1.$$

Now the equality holds if and only if the restriction map

$$H^0(X, \mathcal{O}_X(3C + K_X) \otimes \mathcal{I}_{G|X}) \longrightarrow H^0(\Gamma_3, \mathcal{O}_{\Gamma_3}(3C + K_X) \otimes \mathcal{I}_{G|X})$$

is surjective. This looks difficult because the map

$$H^0(X, \mathcal{O}_X(3C + K_X)) \longrightarrow H^0(\Gamma_3, \mathcal{O}_{\Gamma_3}(3C + K_X))$$

is not surjective (it has corank q_a).

At the end of the first section of his letter to Severi, Castelnuovo claims that: if D is a curve in $|3C + K_X|$ containing G_c and smooth there, then

$$h^0(D, \mathcal{O}_D(G_c)) = 2d - g + 2p_g + \chi - e - 1 + \theta$$

with $0 \leq \theta \leq q_a$, and, as usual, $\chi = \chi(\mathcal{O}_X)$ and $p_g = h^0(X, \mathcal{O}_X(K_X))$. It is easy to check that this is equivalent to

$$2p_g + \chi - 1 \leq h^1(D, \mathcal{O}_D(G_c)) \leq 3p_g.$$

However we have not been able to prove this.

In the second section of the letter to Severi, Castelnuovo states the:

Proposition 14 *Let \mathcal{P} be a Lefschetz pencil in $|C|$, with Jacobian scheme \mathcal{J} . Then*

$$h^1(X, \mathcal{O}_X(4C + K_X) \otimes \mathcal{I}_{G_c|X}) = h^0(X, \mathcal{O}_X(2C + 2K_X) \otimes \mathcal{I}_{\mathcal{J}|X}).$$

The proof of this, not so different from the one of Proposition 12, is contained in [4, §3], and we do not reproduce it here. Let B be the base locus scheme of the Lefschetz pencil \mathcal{P} . Castelnuovo compares $h^0(X, \mathcal{O}_X(2C + 2K_X) \otimes \mathcal{I}_{\mathcal{J}|X})$ with $h^0(X, \mathcal{O}_X(2C + 2K_X) \otimes \mathcal{I}_{\mathcal{J}+B|X})$. This does not look particularly interesting, and we do not dwell on it here. Castelnuovo also claims that if \mathcal{P} is a Lefschetz pencil in $|C|$ with Jacobian scheme \mathcal{J} , then

$$h^1(X, \mathcal{O}_X(3C + K_X) \otimes \mathcal{I}_{\mathcal{J}|X}) = 2p_g$$

but we have not been able to prove it.

Let us jump for a moment to section 4 of the letter to Severi. In this part, as well as in the paper [4], Castelnuovo takes for granted the existence of the so-called Severi equivalence series. Severi claimed in [23] that, unless the surface has an irrational pencil, there exists the rational equivalence series, of dimension $q_{\text{an}} - 1$, of the zero-dimensional schemes of length e that are zeros of non-zero 1-forms in $H^0(X, \Omega_X^1)$. In addition Severi claimed that Ω_X^1 is generated by global sections. These claims are false in general, as shown by F. Catanese in [5, §6]. Hence the contents of section 4 of the letter and of the paper [4] have biases because of this.

Let us now go back to section 3 of the letter to Severi. The focus of this section is on Problems 10 or 11. Castelnuovo proposes a few equivalent formulations of these problems, the most interesting of which, in our opinion, is the following, which is also the topic of the letter to B. Segre.

Problem 15 Suppose there is a (homogeneous) relation of the form

$$Y_1 F_1 + Y_2 F_2 + Y_3 F_3 + Y_4 F_4 = QF \tag{22}$$

where $Y_i = 0$ are adjoint surfaces of degree $d - 3$ to S and $Q = 0$ is a surface of degree $d - 4$. Prove (algebraically) that Q is an adjoint surface and that

$$Y_i = \frac{1}{d} x_i Q.$$

The solution of this problem implies the solution of Problem 11. Indeed, we can rewrite (22) as

$$\sum_{i=1}^4 \left(Y_i - \frac{1}{d} Q x_i \right) F_i = 0$$

and this is a homogeneous Picard’s relation of the type (15), with

$$X_i = Y_i - \frac{1}{d} Q x_i, \quad 1 \leq i \leq 4.$$

Problem 11 asks to prove that (17) holds, whereas Problem 15 asks to prove much more, i.e., that $X_i = 0$, for $1 \leq i \leq 4$. So Problem 15 does not look equivalent to Problem 11, and it is not at all clear if it has a solution or not.

5 Algebraic Proofs via the Hodge–Frölicher Spectral Sequence

This section is devoted to giving a brief account on how algebraic proofs of the closedness of regular 1-forms and of the equality between algebraic and analytic irregularity (both in characteristic zero) can be obtained using modern tools.

5.1 Global Regular 1-Forms Are Closed in Characteristic Zero

Let X be a smooth, irreducible, and projective variety of arbitrary dimension over an algebraically closed field \mathbb{K} . Let Ω_X^i be the sheaf of algebraic differential i -forms on X . The exterior derivative $d : \Omega_X^i \rightarrow \Omega_X^{i+1}$ allows to define a complex (the so-called algebraic de Rham complex) and a spectral sequence (the so-called Hodge–Frölicher spectral sequence):

$$E_1 = \bigoplus_{i,j \geq 0} E_1^{i,j}$$

where

$$E_1^{i,j} := H^j(X, \Omega_X^i)$$

and

$$d_1 : E_1^{i,j} \rightarrow E_1^{i+1,j}$$

is given by

$$d : H^j(X, \Omega_X^i) \rightarrow H^j(X, \Omega_X^{i+1}).$$

If this spectral sequence degenerates at E_1 , then in particular we have $E_2^{1,0} = E_1^{1,0}$, i.e.,

$$\frac{\text{Ker}(H^0(X, \Omega_X^1) \rightarrow H^0(X, \Omega_X^2))}{\text{Im}(H^0(X, \mathcal{O}_X) \rightarrow H^0(X, \Omega_X^1))} = \text{Ker}(H^0(X, \Omega_X^1) \rightarrow H^0(X, \Omega_X^2)) = H^0(X, \Omega_X^1).$$

Hence we see that if the Hodge–Frölicher spectral sequence degenerates at E_1 , then all global regular 1-forms are closed.

An algebraic proof of the degeneration at E_1 of the Hodge–Frölicher spectral sequence in characteristic zero has been obtained by Deligne and Illusie in the paper [7] published in 1987 (see also [9] and [13] for more detailed and self-contained expositions). The strategy involves two steps: first, the result is proven under suitable

assumptions in positive characteristic; then, by applying standard “spreading out” techniques, it is extended to characteristic zero.

Theorem 16 ([13], Corollary 5.6) *Let k be a perfect field of characteristic p , and let X be a smooth and proper k -scheme of dimension $< p$. If X satisfies a technical assumption (namely, X can be lifted over the ring $W_2(k)$ of Witt vectors of length 2 over k), then the Hodge–Frölicher spectral sequence of X over k degenerates at E_1 .*

Corollary 17 ([13], Theorem 6.9) *Let \mathbb{K} be a field of characteristic zero, and let X be a smooth and proper \mathbb{K} -scheme of arbitrary dimension. Then the Hodge–Frölicher spectral sequence of X over \mathbb{K} degenerates at E_1 .*

For a friendly introduction to this circle of ideas, we refer the interested reader to the informal survey [15] (see also [17], which explains the role of Witt vectors in studying the irregularity in positive characteristic). Unluckily, it seems that in order to address the case of surfaces, one needs to apply the whole machinery developed for the general case.

5.2 Analytic Irregularity and Arithmetic Irregularity Coincide

Let X be a smooth and projective surface over the complex field \mathbb{C} . As already realized (at least implicitly) by Castelnuovo, the fact that the *analytic irregularity* $q_{\text{an}}(X) = h^0(X, \Omega_X^1)$ is equal to the *arithmetic irregularity* $q_a(X) = h^1(X, \mathcal{O}_X)$ (which holds in general only in characteristic zero) is strictly related to the closedness of global regular 1-forms.

A crucial additional ingredient for proving algebraically that $q_{\text{an}}(X) = q_a(X)$ is the following equality, which admits a purely algebraic proof (see, for instance, [24, Mumford’s remarks i) and iii) on p. 200], and [1, Theorem 5.1]):

$$h^1(X, \mathbb{C}) = 2h^1(X, \mathcal{O}_X) = 2q_a(X). \tag{23}$$

As in [2], proof of Lemma (2.6) on p. 139, there is a natural exact sequence

$$0 \rightarrow \mathbb{C} \rightarrow \mathcal{O}_X \rightarrow \mathcal{S} \rightarrow 0,$$

where \mathcal{S} denotes the sheaf of closed regular 1-forms on X . Since all global regular 1-forms are closed, we get an exact sequence

$$0 \rightarrow H^0(X, \Omega_X^1) \rightarrow H^1(X, \mathbb{C}) \rightarrow H^1(X, \mathcal{O}_X).$$

It follows that $h^1(X, \mathbb{C}) \leq h^0(X, \Omega_X^1) + h^1(X, \mathcal{O}_X)$ and together with (23) we may deduce

$$h^1(X, \mathcal{O}_X) \leq h^0(X, \Omega_X^1).$$

On the other hand, the opposite inequality turns out to be much subtler and seems to require the full strength of the Hodge–Frölicher spectral sequence. Indeed, if one defines the *algebraic de Rham cohomology* $H_{\text{dR}}^*(X/\mathbb{K})$ as the hypercohomology of the algebraic de Rham complex, then the equality

$$\dim(H_{\text{dR}}^1(X/\mathbb{K})) = q_{\text{an}}(X) + q_a(X)$$

is a formal consequence of the degeneration at E_1 of the Hodge–Frölicher spectral sequence (see for instance [15], Lemma 3.4). In particular, for $\mathbb{K} = \mathbb{C}$ we have

$$q_{\text{an}}(X) + q_a(X) = \dim(H_{\text{dR}}^1(X/\mathbb{K})) = h^1(X, \mathbb{C}) = 2q_a(X)$$

by (23); hence, we obtain $q_{\text{an}}(X) = q_a(X)$.

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