Nonlocal Fuzzy Solutions for Abstract Second Order Differential Equations

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Abstract This work considers the existence and uniqueness of fuzzy solutions for abstract second order differential systems. Since the nonlocal condition has more advantage than local condition, it is studied here. To establish the existence and uniqueness, we apply the concept of semi-group theory and suitable fixed point theorem. Finally, to explain the result, we give an example.

Keywords Mild solution · Fuzzy solution · Fixed point

1 Introduction

Byszewski [4] has studied the existence and uniqueness of strong, classical, and mild solutions of the nonlocal Cauchy problem. The nonlocal condition provides better results in physics when compared with the traditional local conditions. Inspired by this, many authors started to explore this type of equations. See for instance [6, 7], and its application in heat equation is seen in [8] and the references therein. Recently, [13] studied nonlocal conditions in fuzzy metric spaces.

On the other hand, when one is interested in modeling of real-world problems, it is also required sometimes to deal with uncertain phenomena. In this case, the concepts of a fuzzy set are one of the best approach, which leads us to inspect fuzzy differential equations. The implementation of the fuzzy differential equation in the model of population growth is discussed in [18]. Recently, the existence and uniqueness for the solutions to both local and nonlocal conditions for the fuzzy differential equations have been further examined and discussed by several researchers in various aspects, see the monograph of [12] and the papers [2, 3, 5, 9–11, 17]. The role of fuzzy

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numbers plays a significant role in dealing with vague numerical quantities. For more on fuzzy numbers and its properties, refer [1, 14, 16, 19].

We have tried in this work to describe the existence of fuzzy solutions for the following second order abstract differential system using α techniques of fuzzy numbers.

$$\frac{d^2}{dt^2}(p(t) - g(t, p(t))) = Ap(t) + F(t, p(t)), t \in [0, T] = J \tag{1}$$

$$p(0) = \mu(p_0, p), \tag{2}$$

$$\frac{d}{dt}(p(t) - g(t, p(t))|_{t=0} = \eta(q_0, p) \tag{3}$$

where $A : [0, T] \to E^n$ is a fuzzy coefficient. The continuous function $g, f : J \times E^n \to E^n$ are nonlinear, $p_0; q_0 \in E^n$ and $\mu, \eta : J \times E^n \to E^n$ are appropriate functions.

So far to the best of our understanding, the existence of solutions for the given differential system (1)–(3) with nonlocal conditions defined in the abstract form is not yet studied using fuzzy techniques, and this serves as a primary motivation for this present work. This work is structured as follows. In Sect. 2, some preliminary concepts of fuzzy sets and fuzzy numbers are provided, and in Sect. 3, the existence and uniqueness of fuzzy solutions are established for the system (1)–(3). Finally, an example is given in Sect. 4 to determine our result.

2 Preliminaries

Here, we review few basic concepts, remarks, and properties of fuzzy numbers which will be used through out this work are presented. For more on its properties refer [15, 19].

Let Q^n be the set of all non-empty compact, convex subsets of $\mathbb{R}^n$. For $C, D \in Q^n$ and for any $\beta \in \mathbb{R}$ the addition and multiplication operation are represented as

$$C + D = \{c + d/c \in C, d \in D\}, C = \{\beta c/c \in C\}$$

In the universe set X, a fuzzy set is defined as the mapping from $m \to [0, 1]$. Here, m is assigned as the degree of membership, and it is value lies between 0 and 1. For the fuzzy set m defined in n-dimensional space and for $\alpha \in (0, 1]$, we denote as,

$$[m]^\alpha = \{x \in R^n/m(x) \geq \alpha\}$$

If m be a fuzzy subset of X, the support of m, denoted as supp(m), is the crisp subset of X whose elements all have nonzero membership values in m, i.e., supp$(m) = \{x \in X|m(x) > 0\}$. For any $\alpha \in [0, 1]$, m is called compact if $[m]^\alpha \in Q^n$.

The collection of all fuzzy sets of R^n is called as E^n which satisfies the following conditions such as m is normal, fuzzy convex, upper semicontinuous, and $[m]^0$ is compact.
For any $m, n \in E^n$ the complete metric H_d is defined as

$$\mathrm{d}_\infty(m, n) = \sup_{0<\alpha\leq 1} H_d([m]^\alpha, [n]^\alpha)$$

Let $m, n \in C(J : E^n)$. Then supremum metric is defined as

$$H_1(m, n) = \sup_{0<\alpha\leq T} \mathrm{d}_\infty([m]^\alpha, [n]^\alpha)$$

3 Existence and Uniqueness Results

In this section, we define the existence and uniqueness of fuzzy solutions using Banach fixed point theorem for (1)–(3) .
We shall consider a space $\Gamma = p : J \to E^n$ to define the solution for (1)–(3)
And let us define $\Gamma^{'} = \Gamma \cap C([0, T] : E^n)$

Definition 3.1 A function $p : J \to E^n$ is an integral solution of (1)–(3), then

$$\begin{aligned} p(t) = {} & C(t)(\mu(p_0, p) - g(0, \mu(p_0, p))) + K(t)\eta(q_0, p) + g(t, p(t)) \\ & + \int_0^t AK(t-s)g(s, p(s))\mathrm{d}s + \int_0^t K(t-s)F(s, p(s))\mathrm{d}s, t \in J \end{aligned} \tag{4}$$

Assume:
(H1) Let $K(t)$ be a fuzzy number, where $[K(t)]^\alpha = [K_l^\alpha(t), K_r^\alpha(t)],\ K(0) = I$ and $K_j^\alpha(t)(j = l, r)$ is continuous with $|K_j^\alpha(t)| \leq m_1,\ m_1 > 0, |AK(t)| \leq m_0, m_0 > 0$ $\forall\, t \in J = [0, T]$.

(H2) Let $C(t)$ be a fuzzy number, where $[C(t)]^\alpha = [C_l^\alpha(t), C_r^\alpha(t)],\ C(0) = I$ and $C_j^\alpha(t)(j = l, r)$ is continuous with $|C_j^\alpha(t)| \leq m_2,\ m_2 > 0\ \forall\, t \in J = [0, T]$.

(H3) $\exists$ positive constants $d_g, d_f > 0$ for the functions g and f which are strongly measurable satisfying the Lipschitz conditions

$$H_d([g(t, p)]^\alpha, [g(t, q)]^\alpha) \leq d_g H_d([p(t)]^\alpha, [q(t)]^\alpha)$$

$$H_d([F(t, p)]^\alpha, [F(t, q)]^\alpha) \leq d_f H_d([p(t)]^\alpha, [q(t)]^\alpha)$$

are satisfied.

(H4) The continuous functions $\mu(p_0, .)$, $\eta(q_0, .)$ are locally bounded and $\exists$ positive constants $d_\mu, d_\eta > 0$ such that

$$H_d([\mu(p_0, p)]^\alpha, [\mu(p_0, q)]^\alpha) \leq d_\mu H_d([p(.)]^\alpha, [q(.)]^\alpha)$$

$$H_d([\eta(q_0, p)]^\alpha, [\eta(q_0, q)]^\alpha) \leq d_\eta H_d([p(.)]^\alpha, [q(.)]^\alpha)$$

(H5) $\left(m_1\left[d_\eta + \frac{d_g}{m_1} + (m_0 d_g + d_f)T\right] + m_2(d_\mu + d_g)\right) < 1$

Theorem 3.1 *Let $T > 0$. If the hypotheses (H1)–(H5) holds, then the system (1)–(3) has a unique fuzzy solution in $p \in \Gamma'$.*

Proof For each $p(t) \in \Gamma'$ and $t \in J$, define $(F_0 p)(t) \in \Gamma'$ by,

$$\begin{aligned}F_0 p(t) &= C(t)(\mu(p_0, p) - g(0, \mu(p_0, p))) + K(t)\eta(q_0, p) + g(t, p(t)) \\ &\quad + \int_0^t AK(t-s)g(s, p(s))\mathrm{d}s + \int_0^t K(t-s)F(s, p(s))\mathrm{d}s, \; t \in J.\end{aligned}$$

Now,

$$\begin{aligned}
&H_d([F_0 p(t)]^\alpha, [F_0 q(t)]^\alpha) \\
&\leq H_d\Bigg(\Bigg[C(t)(\mu(p_0, p) - g(0, \mu(p_0, p))) + K(t)\eta(q_0, p) + g(t, p(t)) \\
&\quad + \int_0^t AK(t-s)g(s, p(s))\mathrm{d}s + \int_0^t K(t-s)F(s, p(s))\mathrm{d}s\Bigg]^\alpha, \\
&\quad \Bigg[C(t)(\mu(p_0, q) - g(0, \mu(p_0, q))) + K(t)\eta(q_0, q) + g(t, q(t)) \\
&\quad + \int_0^t AK(t-s)g(s, q(s))\mathrm{d}s + \int_0^t K(t-s)F(s, q(s))\mathrm{d}s\Bigg]^\alpha\Bigg) \\
&\leq H_d\Bigg([C(t)\mu(p_0, p)]^\alpha + [C(t)g(0, \mu(p_0, p))]^\alpha + [K(t)\eta(q_0, p)]^\alpha + [g(t, p(t))]^\alpha \\
&\quad + \left[\int_0^t AK(t-s)g(s, p(s))\mathrm{d}s\right]^\alpha + \left[\int_0^t K(t-s)F(s, p(s))\mathrm{d}s\right]^\alpha, \\
&\quad [C(t)\mu(p_0, q)]^\alpha + [C(t)g(0, \mu(p_0, q))]^\alpha + [K(t)\eta(q_0, q)]^\alpha + [g(t, q(t))]^\alpha \\
&\quad + \left[\int_0^t AK(t-s)g(s, q(s))\mathrm{d}s\right]^\alpha + \left[\int_0^t K(t-s)F(s, q(s))\mathrm{d}s\right]^\alpha\Bigg) \\
&\leq H_d([C(t)\mu(p_0, p)]^\alpha, [C(t)\mu(p_0, q)]^\alpha) + H_d([C(t)g(0, \mu(p_0, p))]^\alpha, [C(t)g(0, \mu(p_0, q))]^\alpha) \\
&\quad + H_d([K(t)\eta(q_0, p)]^\alpha, [K(t)\eta(q_0, q)]^\alpha) + H_d([g(t, p(t))]^\alpha, [g(t, q(t))]^\alpha
\end{aligned}$$

$$+ H_d\left(\left[\int_0^t AK(t-s)g(s,p(s))\mathrm{d}s\right]^\alpha, \left[\int_0^t AK(t-s)g(s,q(s))\mathrm{d}s\right]^\alpha\right)$$
$$+ H_d\left(\left[\int_0^t K(t-s)F(s,p(s))\mathrm{d}s\right]^\alpha, \left[\int_0^t K(t-s)F(s,q(s))\mathrm{d}s\right]^\alpha\right)$$
$$\le m_2 d_\mu H_d([p(.)]^\alpha, [q(.)]^\alpha) + m_2 d_g H_d([p(t)]^\alpha, [q(t)]^\alpha)$$
$$+ m_1 d_\eta H_d([p(.)]^\alpha, [q(.)]^\alpha) + d_g H_d([p(t)]^\alpha, [q(t)]^\alpha)$$
$$+ m_0 m_1 \int_0^t d_g H_d([p(s)]^\alpha, [q(s)]^\alpha) ds + m_1 \int_0^t d_f H_d([p(s)]^\alpha, [q(s)]^\alpha)\mathrm{d}s$$

Therefore,

$$d_\infty(F_0 p(t), F_0 q(t))$$
$$= \sup_{0<\alpha\le 1} H_d([F_0 p(t)]^\alpha, [F_0 q(t)]^\alpha)$$
$$\le m_2 d_\mu \sup_{0<\alpha\le 1} H_d([p(.)]^\alpha, [q(.)]^\alpha) + m_2 d_g \sup_{0<\alpha\le 1} H_d([p(t)]^\alpha, [q(t)]^\alpha)$$
$$+ m_1 d_\eta \sup_{0<\alpha\le 1} H_d([p(.)]^\alpha, [q(.)]^\alpha) + d_g \sup_{0<\alpha\le 1} H_d([p(t)]^\alpha, [q(t)]^\alpha)$$
$$+ m_0 m_1 \int_0^t d_g \sup_{0<\alpha\le 1} H_d([p(s)]^\alpha, [q(s)]^\alpha)\mathrm{d}s$$
$$+ m_1 \int_0^t d_f \sup_{0<\alpha\le 1} H_d([p(s)]^\alpha, [q(s)]^\alpha)\mathrm{d}s$$

Hence,

$$H_1(F_0 p, F_0 q) = \sup_{0\le t\le T} d_\infty(F_0 p(t), F_0 q(t))$$
$$\le m_2 d_\mu \sup_{0\le t\le T} d_\infty([p(.)], [q(.)]) + m_2 d_g \sup_{0\le t\le T} d_\infty([p(t)], [q(t)])$$
$$+ m_1 d_\eta \sup_{0\le t\le T} d_\infty[p(.), [q(.)]) + d_g \sup_{0\le t\le T} d_\infty([p(t)], [q(t)])$$
$$+ m_0 m_1 d_g \sup_{0\le t\le T} \int_0^t d_\infty([p(s)], [q(s)])\mathrm{d}s$$
$$+ m_1 d_f \sup_{0\le t\le T} \int_0^t d_\infty([p(s)], [q(s)])\mathrm{d}s$$
$$\le \left(m_1\left[d_\eta + \frac{d_g}{m_1} + (m_0 d_g + d_f)T\right] + m_2(d_\mu + d_g)\right) H_1(p, q)$$

By hypothesis $(H5)$, F_0 is a contraction mapping. Therefore, by Banach fixed point theorem (1)–(3) has a unique solution in $p \in \Gamma^{'}$

4 Example

Here, we establish the results obtained in the previous section to investigate the existence of fuzzy solutions for the wave equation by taking the following second order nonlocal partial differential equation.

$$\frac{\partial}{\partial t}\left[\frac{\partial f(t,\xi)}{\partial t} + \int_0^{\pi} \kappa(\upsilon,\xi) f(t,\upsilon)\mathrm{d}\upsilon\right] = \frac{\partial^2}{\partial \xi^2} f(t,\xi) + G(t, f(t,\xi)), \quad (5)$$

$$f(t,0) = f(t,\pi) = 0, t \in [0,a] \quad (6)$$

$$f(0,\xi) = p(\xi) + \int_0^a m(f(s,\xi))\mathrm{d}s,\ \xi \in [0,\pi] \quad (7)$$

$$\frac{\partial}{\partial t} f(0,\xi) = q(\xi) + \int_0^a n(f(s,\xi))\mathrm{d}s \quad (8)$$

Assume:
(i) For the continuous function $G : [0,T] \times [0,\pi] \to \mathbb{R}$, $\exists$ a positive constant $d_G > 0$ such that

$$H_d([G(t,\xi_1)]^{\alpha}, [G(t,\xi_2)]^{\alpha}) \leq d_G H_d([p(\xi_1)]^{\alpha}, [q(\xi_2)^{\alpha}]),$$

(ii) The function $m, n : \mathbb{R} \to \mathbb{R}$ are continuous and $\exists$ constants d_m, d_n such that

$$H_d([m(f,\tau_1)]^{\alpha}, [m(f,\tau_2)]^{\alpha}) \leq d_m H_d([p(\tau_1)]^{\alpha}, [q(\tau_2)]^{\alpha})$$

$$H_d([n(f,\tau_1)]^{\alpha}, [n(f,\tau_2)]^{\alpha}) \leq d_n H_d([p(\tau_1)]^{\alpha}, [q(\tau_2)^{\alpha}])$$

It is evident that by applying the conditions (i) and (ii) in (5)–(8), Theorem (3.1) is satisfied. So, the system (5)–(8) has a unique fuzzy solution.

5 Conclusion

In this work, we have attempted to prove the existence and uniqueness of fuzzy solutions for second order nonlocal abstract differential equations. One can extend the same findings and study the impulsive nature of the systems.

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