Chapter 20
Vector Variational-Like Inequalities on the Space of Real Square Matrices

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Abstract In this paper, "Vector Variational-like Inequalities" and "Vector Optimization Problems" have been discussed over the space of real square matrices. Under the assumption of invexity, defined suitably in such space, the relationship between such problems has also been established.

Introduction

"Vector Variational Inequalities" (VVI) and "Vector Optimization Problems" (VOP) have been widely studied [1, 2, 6, 11–13] over the n-dimensional Euclidean space in the last few decades. Many valuable resources regarding vector optimization are available in [2]. It has been observed that many important optimization problems involve matrices as variables. Such problems arise from a wide range of diverse applications, e.g. image processing, graph optimization, nearest correlation matrix problem, etc. In this paper, "Vector Variational-like Inequalities" (VVLI) and VOP have been introduced over the space of real square matrices. The new class of VVLI problems defined on the space of real square matrices has been referred as "Matrix-Vector Variational-like Inequality Problems" (MVVLIP). Such class of problems

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is a generalized form of VVLI problems studied by Ruiz-Garzón et al. [13]. The new class of VOP defined on the space of real square matrices has been referred as "Matrix-Vector Optimization Problem" (MVOP).

Recently, researchers have found some applications of the theoretical developments of the concept of invexity and their generalizations. For example, Dinuzzo et al. [7] got a kernel function in machine learning which is neither convex nor quasiconvex but invex.

In this paper, the concept of invexity has also been suitably generalized on the space of real square matrices. It has been obtained that under the assumption of invexity, the solutions of the MVVLIP are the efficient points of the corresponding MVOP. Similar result has already been proved in the n-dimensional Euclidean space and hence validates the generalization proposed in this paper. Mishra et al. [12] considered weak VVI and established certain results related weak vector minimal points. In this paper, it has further been observed that under the assumption of pseudoinvexity, the weak MVVLIP and the corresponding weak MVOP admit the same set of solutions.

The overall contribution of the paper is the representation of VVLI and VOP in a completely newer space structure than what can be found in the existing literature. Interestingly, the widely discussed equality of solutions of such class of problems has been proved to be true in the proposed structure as well. In that sense, the work in this paper can be termed as an extension of many related existing works.

In Section "Matrix-Vector Variational-Like Inequality Problem", matrix derivative [10] and MVVLI have been discussed. In Section "Matrix-Vector Optimization Problem", MVOP has been defined with suitable example. The relationship between MVVLIP and MVOP has been established in Section "Solutions of MVVLIP and MVOP". Future direction of research in the proposed structure has been discussed briefly in the concluding section, i.e. Section "Conclusion".

Matrix-Vector Variational-Like Inequality Problem

Let us consider the following notations to be used in this paper:

$\mathbb{R}_t^{n \times n} :=$ The set of all $n \times n$ matrices with elements as real valued functions of a real variable t.

$$M = \left\{ F : F = \begin{pmatrix} f_{11}(t) \ f_{12}(t) \ ... \ f_{1n}(t) \\ f_{21}(t) \ f_{22}(t) \ ... \ f_{2n}(t) \\ ... \\ f_{n1}(t) \ f_{n2}(t) \ ... \ f_{nn}(t) \end{pmatrix} \in \mathbb{R}_t^{n \times n}, t \in \mathbb{R}, f_{ij} : \mathbb{R} \to \mathbb{R} \right\}.$$

Let $A, B \in \mathbb{R}_t^{n \times n}$ the Frobenius inner product is defined as $\langle A, B \rangle_F = Tr(A^T B)$ and Frobenius norm is defined as $\|AB\|_F = \sqrt{Tr(A^T B)}$.

Definition 20.2.1 Let $F = (f_{st})$ be an $n \times n$ matrix function of a scalar t, then

$$\frac{\delta F(t)}{\delta t} = \begin{pmatrix} \frac{\partial f_{11}(t)}{\partial t} & \frac{\partial f_{12}(t)}{\partial t} & \cdots & \frac{\partial f_{1n}(t)}{\partial t} \\ \frac{\partial f_{21}(t)}{\partial t} & \frac{\partial f_{22}(t)}{\partial t} & \cdots & \frac{\partial f_{2n}(t)}{\partial t} \\ \cdots & & & \\ \frac{\partial f_{n1}(t)}{\partial t} & \frac{\partial f_{n2}(t)}{\partial t} & \cdots & \frac{\partial f_{nn}(t)}{\partial t} \end{pmatrix}.$$

which has the same dimension as F.

Let M be a subset of $\mathbb{R}_t^{n \times n}$, $\mu : M \times M \to \mathbb{R}_t^{n \times n}$ be a matrix-valued function and $G : M \subset \mathbb{R}_t^{n \times n} \to \mathbb{R}_t^{n \times n}$ be differentiable. Let us consider the two definitions given below:

Definition 20.2.2 An MVVLIP is a problem that finds an $F(\bar{t}) \in M$, such that $\forall F(x) \in M$,

$$\left\langle \frac{\delta G(F(\bar{t}))}{\delta \bar{t}}, \mu(F(x), F(\bar{t})) \right\rangle_F > 0. \tag{MVVLIP}$$

Definition 20.2.3 A weak MVVLIP is a problem that finds an $F(\bar{t}) \in M$ such that, $\forall F(x) \in M$,

$$\left\langle \frac{\delta G(F(\bar{t}))}{\delta \bar{t}}, \mu(F(x), F(\bar{t})) \right\rangle_F \geq 0. \tag{WMVVLIP}$$

Definition 20.2.4 A "Matrix-Vector Variational Inequality Problem" (MVVIP) is defined as a problem that finds an $F(\bar{t}) \in M$ such that, $\forall F(x) \in M$,

$$\left\langle \frac{\delta G(F(\bar{t}))}{\delta \bar{t}}, F(x) - F(\bar{t}) \right\rangle_F > 0. \tag{MVVIP}$$

The weak MVVIP can also be defined in a similar way by weakening the inequality.

Matrix-Vector Optimization Problem

The following definitions are motivated from the related concepts discussed in many existing literatures which can collectively be found in [2, 3].

Definition 20.3.1 Let M be an open subset of $\mathbb{R}_t^{n \times n}$ and $G : M \subset \mathbb{R}_t^{n \times n} \to \mathbb{R}_t^{n \times n}$ be differentiable. A matrix $F(\bar{t})$ is called "efficient" if

$$\|G(F(x))\|_F - \|G(F(\bar{t}))\|_F > 0, \forall F(x) \in M.$$

$E(G, M)$ denotes the set of all such "efficient" matrices.

Given M and $G : M \subset \mathbb{R}_t^{n \times n} \to \mathbb{R}_t^{n \times n}$, think about finding $E(G, M)$ for the following MVOP:

$$\text{Minimize } G(F(t)),$$
$$\text{subject to } F(t) \in M.$$

Definition 20.3.2 Consider M to be an open subset of $\mathbb{R}_t^{n \times n}$ and $G : M \subset \mathbb{R}_t^{n \times n} \to \mathbb{R}_t^{n \times n}$ differentiable. A matrix $F(\bar{t})$ is defined as "weakly efficient" if

$$\|G(F(x))\|_F - \|G(F(\bar{t}))\|_F \geq 0, \forall F(x) \in M.$$

The set of "weakly efficient" points is written as $WE(G, M)$.

Assume M to be an open subset of $\mathbb{R}_t^{n \times n}$ and $G : M \subset \mathbb{R}_t^{n \times n} \to \mathbb{R}_t^{n \times n}$ be differentiable. Now, look into the problem of finding $WE(G, M)$ for the following WMVOP:

$$\text{Minimize } G(F(t)),$$
$$\text{subject to } F(t) \in M.$$

Example 20.3.1 Let us discuss the problem given below:

$$\text{Minimize } G(F(t)),$$
$$\text{subject to } F(t) \in M.$$

where $M = \left\{ F(t) : F(t) = \begin{pmatrix} t & -1 \\ 0 & t \end{pmatrix}, t \in (-\infty, -1] \cup [1, \infty) \right\}$. $G : M \to \mathbb{R}_t^{n \times n}$ is defined as $G(F(t)) = -F(t) = \begin{pmatrix} -t & 1 \\ 0 & -t \end{pmatrix}, t \in (-\infty, -1] \cup [1, \infty)$.

It is clear that for every $F(t) \in M$, $\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$ is a "weakly efficient solution" to the corresponding WMVOP.

According to the following example, an "efficient solution" of MVOP is not necessarily a solution of MVVIP.

Example 20.3.2

$$\text{Minimize } G(F(t)),$$
$$\text{subject to } F(t) \in M.$$

where $M = \left\{ F(t) : F(t) = \begin{pmatrix} 1 & t & t^3 \\ t & 2 & t^2 \\ t^2 & t & 1 \end{pmatrix}, t \in [-1, 1] \right\}$. $G : M \to \mathbb{R}_t^{n \times n}$ is defined

as $G(F(t)) = (F(t))^T = \begin{pmatrix} 1 & t & t^2 \\ t & 2 & t \\ t^3 & t^2 & 1 \end{pmatrix}, t \in [-1, 1]$.

Obviously, for every $F(t) \in M$, $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ is an efficient solution.

Now, $\frac{\delta G(F(t))}{\delta t} = \begin{pmatrix} 0 & 1 & 2t \\ 1 & 0 & 1 \\ 3t^2 & 2t & 0 \end{pmatrix}$ and at $t = 0$, $\langle \frac{\delta G(F(t))}{\delta t}, F(x) - F(t) \rangle_F = 2x +$

x^2.

When x is in $[-1, 0]$, $F(t) = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ is not a solution to the corresponding

MVVIP.

Definition 20.3.3 Let M be an open subset of $\mathbb{R}_t^{n \times n}$ and $G : M \subset \mathbb{R}_t^{n \times n} \to \mathbb{R}_t^{n \times n}$ be a differentiable function. Then, G is defined as *invex* if $\exists \mu : M \times M \to \mathbb{R}_t^{n \times n}$ such that,

$$\|G(F(x))\|_F - \|G(F(t))\|_F \geq \left\langle \frac{\delta G(F(t))}{\delta t}, \mu(F(x), F(t)) \right\rangle_F, \quad \forall F(t), F(x) \in M.$$

Furthermore, G is defined as *strictly invex* if $\exists \mu : M \times M \to \mathbb{R}_t^{n \times n}$ such that,

$$\|G(F(x))\|_F - \|G(F(t))\|_F > \left\langle \frac{\delta G(F(t))}{\delta t}, \mu(F(x), F(t)) \right\rangle_F,$$
$$\forall F(t), F(x) \in M, \ F(t) \neq F(x)$$

and G is defined as *pseudoinvex* if $\exists \mu : M \times M \to \mathbb{R}_t^{n \times n}$ such that,

$$\|G(F(x))\|_F - \|G(F(t))\|_F < 0 \Rightarrow \left\langle \frac{\delta G(F(t))}{\delta t}, \mu(F(x), F(t)) \right\rangle_F < 0,$$
$$\forall F(t), F(x) \in M.$$

Solutions of MVVLIP and MVOP

In this section, we shall extend the results of [6] to the case of functions on the space of real square matrices.

Theorem 20.4.1 Let M be an open set and $G : M \subset \mathbb{R}^{n \times n}_t \to \mathbb{R}^{n \times n}_t$ differentiable on M. If G is invex with respect to μ and $F(\bar{t})$ solves (MVVLIP) with respect to same μ, then $F(\bar{t})$ is an efficient solution to (MVOP).

Proof Let us assume the contrary. Let $F(\bar{t})$ is not an efficient point to (MVOP) which implies that $\exists F(x) \in M$, such that,

$$\|G(F(x))\|_F - \|G(F(\bar{t}))\|_F \leq 0. \tag{20.1}$$

Since G is invex with respect to the same μ, (20.1) implies that there exist $F(x) \in M$ such that,

$$\left\langle \frac{\delta G(F(t))}{\delta t}, \mu(F(x), F(t)) \right\rangle_F \leq 0. \tag{20.2}$$

The above inequality is a clear contradiction that proves the theorem. $\qquad \square$

In the following example, it has been shown that a solution of the (MVOP) will be the solution of (MVVLIP) with respect to μ though G is not invex with respect same to μ.

Example 20.4.1 Let us think about the MVOP given below:

$$\text{Minimize } G(F(x)),$$
$$\text{subject to } F(x) \in M.$$

where $M = \left\{ F(t) : F(t) = \begin{pmatrix} t & t^2 \\ 1 & t \end{pmatrix}, t \in \mathbb{N} \right\}$. $G : M \to \mathbb{R}^{n \times n}_t$ is defined as $G(F(t)) = (F(t))^T$.

It is clear that for every $F(t) \in M$, $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ is an "efficient solution".

Now, $\frac{\delta G(F(t))}{\delta t} = \begin{pmatrix} 1 & 0 \\ 2t & 1 \end{pmatrix}$ and at $t = 1$, $\langle \frac{\delta G(F(t))}{\delta t}, F(x) + F(t) \rangle_F \geq 0$.

Hence, $F(t) = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ is a solution to the corresponding MVVLIP but G is not invex with respect to same $\mu = F(x) + F(t)$.

The following theorem provides us with the conditions under which the solutions of WMVVLIP can be identified with the "weakly efficient points".

Theorem 20.4.2 If $F(\bar{t})$ is "weakly efficient" for (WMVOP) and G is invex then $F(\bar{t})$ solves (WMVVLIP). Moreover, if G is pseudoinvex then the converse is also true.

Proof Let $F(\bar{t})$ be the solution to (WMVOP). Then, $\nexists F(x) \in M$ such that,

$$\|G(F(x))\|_F - \|G(F(\bar{t}))\|_F < 0 \tag{20.3}$$

Now, since G is invex,

$$\left\langle \frac{\delta G(F(\bar{t}))}{\delta \bar{t}}, \mu(F(x), F(\bar{t})) \right\rangle_F \leq \|G(F(x))\|_F - \|G(F(\bar{t}))\|_F. \qquad (20.4)$$

From (20.3) and (20.4), we see that $\nexists F(x) \in M$ such that $\langle \frac{\delta G(F(\bar{t}))}{\delta \bar{t}}, \mu(F(x), F(\bar{t})) \rangle_F < 0$, i.e.

$$\left\langle \frac{\delta G(F(\bar{t}))}{\delta \bar{t}}, \mu(F(x), F(\bar{t})) \right\rangle_F \geq 0, \ \forall F(x) \in M \qquad (20.5)$$

Conversely, if $F(\bar{t})$ is not an "weakly efficient point", then $\exists F(x) \in M$ such that,

$$\|G(F(x))\|_F - \|G(F(\bar{t}))\|_F < 0 \qquad (20.6)$$

By the pseudoinvexity of F, (20.6) implies the existence $F(x) \in M$ such that,

$$\left\langle \frac{\delta G(F(\bar{t}))}{\delta \bar{t}}, \mu(F(x), F(\bar{t})) \right\rangle_F < 0 \qquad (20.7)$$

which contradicts that $F(\bar{t})$ is a solution of WMVVLIP.   □

Thus, if G is pseudoinvex, the (WMVVLIP) and the corresponding (WMVOP) admit the same set of solutions.

According to the following example, if G is pseudoinvex and if $F(t)$ is not a solution of (WMVOP), then it will not be a solution of (WMVVLIP).

Example 20.4.2   Let us consider the MVOP given below:

$$\text{Minimize } G(F(x)),$$
$$\text{subject to } F(x) \in M.$$

where $M = \left\{ F(t) : F(t) = \begin{pmatrix} t & -1 \\ 0 & t \end{pmatrix}, t \in [0, 1] \right\}.$

$G : M \to \mathbb{R}_t^{n \times n}$ is defined as $G(F(t)) = -F(t)$ and for $x < t; x, t \in [0, 1]$, G is pseudoinvex with respect to $\mu(F(t), F(x)) = F(t) + F(x)$.

Now, $\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$ neither a solution of WMVOP nor a solution of WMVVLIP with respect to same μ.

The following theorem tells us that under certain assumptions, a solution of the WMVOP will be a solution of the MVOP. It is to be noted that the converse is unconditionally true.

Theorem 20.4.3   Let $G : M \subset \mathbb{R}_t^{n \times n} \to \mathbb{R}_t^{n \times n}$ be differentiable and strictly invex with respect to μ. If $F(\bar{t})$ solves WMVOP, then $F(\bar{t})$ also solves MVOP.

Proof   Let us assume that $F(\bar{t})$ solves WMVOP but does not solve MVOP. Then, $\exists F(x) \in M$ such that,

$$\|G(F(x))\|_F - \|G(F(\bar{t}))\|_F \leq 0.$$

Now, since G is strictly invex with respect to μ, we have,

$$0 \geq \|G(F(x))\|_F - \|G(F(\bar{t}))\|_F > \left\langle \frac{\delta G(F(\bar{t}))}{\delta \bar{t}}, \mu(F(x), F(\bar{t})) \right\rangle_F. \quad (20.8)$$

which implies that $\exists F(x) \in M$ such that $\langle \frac{\delta G(F(\bar{t}))}{\delta \bar{t}}, \mu(F(x), F(\bar{t})) \rangle_F < 0$ and consequently $F(\bar{t})$ does not solve the (WMVVLIP). According to Theorem 20.4.2, this in turn implies that $F(\bar{t})$ does not solve the (WMVOP), which is a contradiction. $\quad \square$

Conclusion

In this brief article, a widely discussed equality of solutions of two specific class of problems related to optimization has been discussed in a completely new kind of space structure which makes the work a novel one. There are many aspects of the proposed class of optimization problems which can further be discussed, for example, the concept of duality of the proposed MVOP. Matrices and optimization problems are pivotal in several areas like Machine Learning, Epidemiological Modelling, etc. Once the proposed representation is further developed, there may be a scope of application of the same in those areas.

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