Lecture Notes in Operations Research

Angappa Gunasekaran
Jai Kishore Sharma
Samarjit Kar *Editors*

Applications of Operational Research in Business and Industries

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Preface

The International Conference on Applications of Operational Research in Business and Industries (AORBI 2021), 54th Annual Convention of ORSI, was organized at Indore during December 17–19, 2021. AORBI 2021 brought together leading international experts on production systems and business from academia, industry, and government to discuss the issues in intelligent manufacturing, operations management, financial management, supply chain management, and Industry 4.0 in the artificial intelligence era.

Operations research (OR) is an interesting and popularly used technique in activities involving grocery arrangement to military applications. Broadly, the OR allows problem-solving and decision-making by using systematic approaches. Engineering, management, and medical applications widely adopt OR techniques for reaching reasonable solutions. The conference provided a forum for scientists, researchers, software developers, and practitioners to exchange ideas and approaches, to present research findings and state-of-the-art solutions, to share their experience on potentials and limits, and to open new avenues of research and development, on all issues and topics related to operations research and applications in business and industry.

AORBI 2021 received overwhelming submissions covering different areas related to OR theory and its applications. With the help of our program committee and reviewers, these submissions went through an extensive peer-review process. This volume comprises thirty-two accepted papers, providing a comprehensive overview of the current research and future scope in OR models and applications.

The volume covers wide applications from the business and industry domains, including medical, engineering, and management. Broadly, the papers are based on two broad themes such as OR theories and models (for instance, optimization and control, combinatorial optimization, queuing theory, resource allocation models, linear and nonlinear programming models, dynamic optimization, evolutionary optimization, multi-objective decision models) and applications of OR models in real-life problems (domain-specific and interdisciplinary).

The important highlight of the current volume lies in the core theme of blending computing paradigms with OR. Specifically, industry-oriented computing paradigms such as big data, machine learning, and data science are some of the focus areas of the present volume, and an amalgamation of these concepts with OR forms an attractive book that adheres to the emerging hot zones of research and development. These core themes in the present volume will not only help academicians to get an insight into the research advancement in OR but will also support practitioners in organizations to try these techniques for effectively solving their problems.

Bakersfield, USA — Angappa Gunasekaran
Noida, India — Jai Kishore Sharma
Durgapur, India — Samarjit Kar

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Samarjit Kar
J. K. Sharma
Angappa Gunasekaran

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About the Editors

Prof. Angappa Gunasekaran is currently the Special Assistant to the Provost for Academic Affairs and Student Success Professor of Operations Management (Tenured) at California State University, Bakersfield (USA). He obtained Ph.D. in Industrial Engineering and Operations Research from IIT, Mumbai in 1988 and participated in several training programs/seminars/workshops organised by prominent institutions such as UMass Dartmouth, CSU Bakersfield, AACSB, NASPAA, NEASC, WASC, George Blumenthal Scholar, and CSUB Deans Academy to name a few. Apart from holding many key academic and administrative positions earlier, Professor Gunasekaran published over 400 research papers and authored several editorial notes on emerging areas of Operations Management and MIS. In addition to the recipient of many honours and awards, he is the Editor-in-Chief of several journals, including *OPSEARCH*.

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Samarjit Kar is currently a professor in the Department of Mathematics, National Institute of Technology Durgapur, India. He is also an active participant in Chinese academia having served as a visiting professor at Tsinghua University since 2009. His academic collaborations/co-authors include academics from China, Poland, Norway, Canada, Serbia, Lithuania and Turkey. His current research interests include operations research and optimization, soft computing, machine learning and uncertainty modelling. He has published over 160 referred articles in international journals and authored six edited book volumes in Springer. His articles have been cited more than 5200 times on Google Scholar and have appeared in prestigious journals. He is serving as an Associate Editor of *OPSEARCH* and Guest editors of *IEEE Transaction on Fuzzy Systems*, *Sustainability*, *Symmetry and Mathematics journals*. Presently he act as secretary to Operational Research Society of India (Kolkata Chapter).

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Chapter 1
Software Defect Prediction Through a Hybrid Approach Comprising of a Statistical Tool and a Machine Learning Model

Ashis Kumar Chakraborty and Barin Karmakar

Abstract Traditional statistical learning algorithms perform poorly in case of learning from an imbalanced dataset. Software defect prediction (SDP) is a useful way to identify defects in the primary phases of the software development life cycle. This SDP methodology will help to remove software defects and induce to build a cost-effective and good quality of software products. Several statistical and machine learning models have been employed to predict defects in software modules. But the imbalanced nature of this type of datasets is one of the key characteristics, which needs to be exploited, for the successful development of a defect prediction model. Imbalanced software datasets contain non-uniform class distributions with most of the instances belonging to a specific class compared to that of the other class. We propose a novel hybrid model based on Hellinger distance-based decision tree (HDDT) and artificial neural network (ANN), which we call as hybrid HDDT-ANN model, for analysis of software defect prediction (SDP) data. This is a newly developed model which is found to be quite effective in predicting software bugs. A comparative study of several supervised machine learning models with our proposed model using different performance measures is also produced. Hybrid HDDT-ANN also takes care of the strength of a skew-insensitive distance measure, known as Hellinger distance, in handling class imbalance problems. A detailed experiment was performed over ten NASA SDP datasets to prove the superiority of the proposed method.

Keywords Software defect prediction · Class imbalance · Hellinger distance · Artificial neural network · Hybrid model

Introduction

Software defect prediction (SDP) is an important topic in software reliability engineering literature. It helps software engineers to allocate the available resources to detect the defect-prone modules, if one can predict the number of defects in a module

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in the early stage of the software development process. Several researchers tried to estimate the number of defects present in a software system [20]. Different statistical techniques like capture-recapture sampling have been applied to estimate the number of defects. Some authors also have used the number of defects detected during code inspection as a co-variate and used regression type of analysis to find out the estimate of reliability of a software [20]. A variant of this problem has been considered to estimate the optimum time for software release [10, 14]. An important feature of SDP problem is that the real-life data are highly imbalanced in nature between defect and non-defect classes. This basically means that the incidence of one of the classes, say non-defect classes are much more compared to the occurrence of defect classes. Some studies [12] noted this important feature for these types of datasets and suggested a hybrid approach which uses statistical and machine learning methods together to improve prediction accuracy. The authors [12] noted an important property of sample defect prediction datasets, that is in majority cases the modules that are non-defective (majority) are much more higher in number in comparison with the defective modules (minority). Noticing this class imbalance problem, several researchers [29] started using imbalance learning techniques to improve the performance for defect prediction. Earlier several SDP models have been developed with a goal to predict the possibility of occurrence of a defect in the unseen (future) version of a software product [7, 22, 35]. A software module with defects often fails and prevents to produce preferable results; so, it is required to have an early warning (detection) system of software defects in the system. In the starting phases of software development life cycles, there is possibility to correct the set of identified defects [38] in order to achieve better performance of the system. Furthermore, if it is possible to restrict such defects to propagate to the latter stages, then this could be cost-saving and worthwhile for the producer. Thus, constructing an efficient SDP model will be helpful in order to develop better quality software products which will reduce maintenance costs and accounts for higher customer satisfaction.

In this article, we use a novel hybrid approach where decision tree, which is based on Hellinger distance, and artificial neural network with one hidden layer is used to predict the number of software bugs. To validate the power of the hybrid HDDT-ANN method, we performed this method on several standard imbalanced SDP datasets. This new approach helped to increase minority class prediction accuracy and appears to be more efficient than other traditional supervised classification models for predicting defects and improving the overall performance.

This article is constructed as follows. In Section "Literature Review", we provide the literature review, while in Section "Proposed Model" we develop the proposed hybrid model. Section "Experimental Analysis" gives experimental analysis of the proposed model, wherein we apply the model on several SDP datasets. We conclude this article in Section "Conclusion". Section "Code" refers to the computer code used for implementation of the model developed.

Literature Review

Earlier, several authors have developed SDP models which were applied for predicting the occurrence of a defect that may be present in unseen (future) version of a software product [22, 29]. So a valid SDP model is required to develop good quality software products which helps in reducing maintenance costs and also increases customer satisfaction. Recent studies demonstrated that around 80% of the defects occurred in a few modules which consists of 20% [51] of total number of modules. This fact suggests that the defective class can be considered as a minority class and non-defective class is the majority class. This kind of imbalanced class frequency distribution is an important feature of SDP datasets. It is noted further that the more penalty is associated when minority (defective) classes are misclassified though they are less in number. Hence, for defect prediction in SDP area, it is essential to address the issue of misclassification of the elements of minority class for the unseen data to reduce future defect percentages and successful development of a software. In the binary pattern classification problem, when the source dataset (labeled data) contains one class of data, which are very high in number, compared to the other class, the concept of imbalance in the dataset originates. By convention, the majority class is the over-represented class whereas minority class is the other class having a lower number of instances. In software defect prediction, the probability of finding defect instances, which is from the minority class, are very less compared to finding a non-defect case which is from the majority class.

The objective of imbalance learning is to develop classification model that can correctly classify the instances from minority and majority class simultaneously. This kind of problem is tackled through different approaches, but almost all the traditional classifiers used to put higher weightage to the instances from majority class and as a result the instances from minority classes are often misclassified compared to the majority class instances [25]. In order to tackle this data imbalance, problem several methods have been developed utilizing the properties of data and also at algorithm levels. Some methods based on utilizing the properties of the dataset use various sampling techniques, where by manipulating training data we try to balance the skewed class distributions. The methods based on algorithm have modified the training procedure and used cost-sensitive and ensemble learning techniques in order to achieve increased accuracy on the minority class instances. The most common technique which has been used for solving class imbalance problems, is the well-known sampling techniques where we oversample or undersample the original dataset to modify the data class distributions. Random oversampling (ROS) or undersampling is one of the common techniques used for imbalanced datasets. In random oversampling, one randomly selects examples from the minority class, with replacement, and add them to the training dataset; on the other hand, random undersampling technique randomly selects instances from the majority class and deletes them from the training set. Another way to balance the class distribution is to synthesize new examples from the minority class. SMOTE (Synthetic Minority Oversampling Technique) is the most widely used approach to synthesizing new examples. SMOTE selects instances that are adjacent within the feature space. After that it draws a line between the instances in the feature space and then it selects a new sample at a position along that line. Shatnawi [43] showed the efficacy of random oversampling (ROS) where training instances are added in the defective class of the datasets, and it is more effective compared to SMOTE technique. Lopez et al. [36] have presented that oversampling may effect in overfitting since there is possibility that the input training dataset may contain duplicate instances. These approaches have shortcomings of often overfitting or underfitting. To get rid of these problems, hybrid sampling, the technique where the approach is not only balancing the class distribution but also removing noisy instances which are lying on the wrong side of the decision boundaries, is used. Some examples are the combined techniques of oversampling and undersampling like synthetic minority oversampling technique (SMOTE) + Tomek link (TL) and ROS + TL [2, 23]. Oversampling like SMOTE alone is not enough to counter the problems with imbalanced structure, like overfitting. So, undersampling like Tomek link (TL) is further applied in the preprocessing stage to get rid of the problems like class clusters are overlapped between some majority and minority class space. Only majority class examples that participate in a Tomek link are removed on the basis of the following principles: If two instances create a Tomek link (TL), then at least one of the instances is noise or both are considered as borderline, since minority class examples are considered too rare to be discarded. These hybrid methods have been implemented on the SDP datasets, and it is found more effective than the individual methods. But ensemble methods [25] work better compared to these methods.

Some authors used ensemble learning techniques where they use combination of finite number of classifiers and allocate different values of weights to the component methods to tackle the imbalance classification problems. It is seen that ensemble learning utilizes the strength of individual learners and improves overall performance of the model by combining them. Wang and Yao [48] introduced a dynamic version of the AdaBoost.NC (ABNC) which is used in SDP for tackling the imbalanced classification problem. On the other hand AdaBoost-based kernel ensemble learning method was also experimentally presented to be quite effective for the projects which are made on NASA SDP datasets [29]. One can also think to formulate the SDP problems as a binary classification problem where software modules are classified as either defect-prone or non-defect-prone based on a set of features whose components are software metrics.

In software reliability engineering, a set of static code attributes like McCabe and Halstead are used to describe the complexity of a module, which is defined as the smallest unit of functionality and is extracted from previous software releases with the log files of defects. As the complexity of a module increases, the more likely it is to be fault-prone in nature. The values of the metrics which are mentioned above have been used as features to construct classifiers for defective module prediction for the upcoming phase of release. This leads to detect the segments of the software that have greater probability of containing defects. PROMISE repository [3] is an open-source of defect prediction datasets from real-world projects and several researchers used in their academic study. Several statistical and machine learning tools have been implemented to solve SDP problems, such as decision tree

[6, 30], Random Forest (RF) [27, 45], Deep Feed Forward Neural Network [28], support vector machines [26], and artificial immune systems [8]. Some authors also considered ensemble learning [46], kernel-based technique [26], learning based on cost-sensitive [49], improved subclass discriminant analysis (ISDA) technique [29], value-cognitive boosting along with support vector machine (VCB-SVM) method [40] transfer learning [47] in order to build software defect prediction models. It seems that no single method is good enough for developing models based on the SDP datasets. But all the methods like RF and cost-sensitive learning seem to be working better for the most of the datasets [48] compared to others.

However, many of the previous studies had not taken into account the class imbalance scenario of SDP datasets, though some researchers have implemented ensemble and other techniques for tackling the imbalanced structure of the dataset [25]. Some research demonstrated the effectiveness of resampling techniques in case of tree-based learners [48]. Ensemble and cost-sensitive learning approaches were also found quite effective in the cases where an appropriate cost ratio is fixed [33]. Recently, Gong et al. [25] proposed a new method known as stratification embedded nearest neighbor (STr-NN) approach. Also, Sun et al. [46] presented a ensemble learning approach which is coding-based. But these approaches have some deficiencies. For example, while using sampling techniques, the actual datasets are modified. Also, the choice of optimized weights in constructing ensembles is not straight forward. To get rid of these drawbacks, some authors introduced "imbalanced data-oriented" classifiers [17]. These classifiers, e.g., Hellinger distance-based decision tree (HDDT) and the ensemble technique like Hellinger distance-based random forest (HDRF) can tackle the problem of class imbalance without modifying the original dataset and hence it can be a possible solution to deal with SDP problems. Even though HDDT takes care of class imbalance, still it has some deficiencies. Since it is a greedy algorithm, i.e., at each stage, it finds the best feature for splitting and as a result it may stick to local minima and also may lead to overfitting [9] when the tree size is very large compared to the number of training data present. Also, HDDT does not use pruning techniques, where we add complexity parameter to obtain optimal subtree, so it may overfit the dataset since we are using the whole tree [4]. Some authors have tackled these deficiencies of HDDT problems using hybridization like Hellinger net [12], where the following steps are associated to generate the model: convert an HDDT into rules, construct two hidden layered DFFNN (Deep feed forward neural network) architecture from the generated rules, and finally train the DFFNN using stochastic gradient descent back propagation method.

Different hybrid models have been constructed combining the decision tree algorithm along with neural networks. Sirat and Nadal [44] introduced a method called Neural tree (NT), which is an example of this kind of model. Such techniques combine both CART which have the advantages of a hierarchical organization and ANN, because of the perceptron's ability to handle many input variables and they further proposed NT model which works for multiclass classification problems also. Sakar and Mammone [41] introduced a new method known as neural tree networks (NTN), where neural networks are associated in a tree structure. In neural tree network method, the neural networks are used recursively to segment the feature space into disjoint sub-spaces at each tree node, which results in a preferable efficient classification performance. Foresti and Dolso [24] introduced a new concept known as adaptive neural tree (ANT). Later on, flexible neural tree (FNT) is introduced by Chen et al. [15], which works as a feature selection technique and also deal with problem of intrusion detection. Chen et al. [16] presented an experimental result where it is shown that the efficiency of FNT model to forecast the small time scale traffic measurement. Bouaziz et al. proposed a model called Flexible Beta Basis Function NT model [5], and it was shown that the model performs better compared to other related methods in case of dealing with some standard problems drawn from the control system and time series prediction. Sethi introduced [42] Entropy nets, a mapping of decision trees into a multilayer neural network structure, which has superiority since it contains comparatively fewer number of neural connections. In this framework, the number of neurons present in the neural network's input layer equals to the number of decision tree's internal nodes. These neurons go through hidden layer, and the number of neurons in the output layer is same as number of distinct classes. A recent study [12] has shown how hybrid model famously known as Hellinger net helps to improve accuracy of the model in the context of dealing with imbalanced dataset. In this article, we propose a hybrid HDDT-ANN model which will increase the performance of the model in terms of different accuracy metrics.

Proposed Model

In software defect prediction, static code attributes are extracted from older releases of software containing the log files of defects and these are used to construct models with an objective to predict the defective modules for the future releases. This is an advantage to discover parts of the software which are more probable to contain defects. We have presented our proposed hybrid HDDT-ANN model to tackle the class imbalance problem in context of SDP in this section. We first discuss the theoretical frameworks for constructing Hellinger distance-based decision tree (HDDT) and later on the work flow of our proposed model where HDDT and ANN two distinct classifiers are used jointly to make a decision.

Hellinger Distance Decision Tree

Hellinger distance, which is a symmetric and non-negative measure of distributional divergence, is associated to the Bhattacharyya's distance and the Kullback–Leibler divergence. Chawla [17] proposed how to use Hellinger distance as a decision tree splitting criterion for modeling an imbalanced dataset for classification. This method considers two distributions say U and V, which is the normalized frequencies of feature values considering the problem as a binary classification. Cieslak and Chawla [17] have introduced the concept of affinity between two distributions U and V as a

criterion for tree splitting. The goal is to split tree nodes on the features which are having minimal affinity which implies maximal Hellinger distance. This approach uses the idea of splitting of features based on how well the examples seen so far in the training data can be discriminated by them. Whereas other tree-based models used to split on the feature, which represents the highest possible number of data points seen so far (for an example: information gain discussed in Breiman's CART [6]). Hellinger distance is skew insensitive in nature, since examples of one particular class are higher in number compared to other class which only leads to make its sample distribution more inclined to the actual distribution. Considering the situation that a feature is a good class discriminator, then whatever the class balance is, it will remain same. The formal definition of Hellinger distance is as follows.
Definition: Let (δ, γ) denote a measurable space and assume that U and V be two continuous distributions with respect to the parameter γ having the densities u and v in a continuous space Ω, respectively [1]. One can define Hellinger distance (HD) as follows:

$$d_{\mathrm{H}}(U, V) = \sqrt{\int_{\Omega} (\sqrt{u} - \sqrt{v})^2 \mathrm{d}\gamma}$$

Note that Hellinger distance does not depend on the choice of the parameter γ. Some important properties of Hellinger distance are given below.

1. $d_{\mathrm{H}}(U, V)$ is in $[0, \sqrt{2}]$.
2. Hellinger distance (HD) is non-negative $d_{\mathrm{H}}(U, V) \geq 0, \forall\, U, V$.
3. HD is symmetric, i.e., $d_{\mathrm{H}}(U, V) = d_{\mathrm{H}}(V, U), \forall\, U, V$.
4. As the Hellinger distance increases, the discrimination power of the feature also gets better.

Hellinger Distance Decision Tree (HDDT) utilizes Hellinger distance for tree splitting and constructs the tree on the basis of methods proposed by Breiman's CART [6] as discussed in [17]. In this methodology, once the root node is found, the criterion is applied recursively until next decision node is found. This criterion determines in which input direction split will take place and where the cut should be initiated. HDDT works on feature input spaces in order to make a hierarchical axes-parallel split of it, similar like CART, the only exception is the split criterion. Here, each of the tree node represents the partitioned subsets in the input feature space. In this case, we consider only binary tree consisting of two child nodes or zero child nodes are constructed [17]. The details of the algorithm of splitting criteria of HDDT are provided below:

Consider $\underline{X}$ to be the input feature space and let p be the number of attributes and let $\underline{Y}$ be the set of binary class labels which consists of elements from the response column. Suppose, there is a training sample (labeled dataset) with k as the number of instances, $D_k = \{(X_j, Y_j); j = 1, 2, \ldots k\}$ where $X_j = (X_{j1}, X_{j2}, \ldots, X_{jp}) \in \underline{X}$, and Y_j represent their respective labels. The source SDP dataset D_k contains identically distributed independent instances (X_j, Y_j), which are distributed as the pair denoted by ($\underline{X}$, $\underline{Y}$), where $X \in R_+^p$ and Y $\in \{0, 1\}$. One may note that for the source SDP dataset, the range of values of static code attributes of a software module are positive and numeric whereas response class value is discrete in nature (that is defective or non-defective). In our proposed hybrid approach [13], the second model is an artificial neural network model. Before modeling, we need to normalize the features of SDP dataset, i.e., we are mapping each element of each column to [0, 1] range and keeping the response column as it is, so that in latter stage, while we are training the neural network, the model becomes efficient. We are using min-max scaling for normalization. The problem is given with a total of n instances on p input features where $\underline{X} \in C^p$ and $\underline{Y} \in \{0, 1\}$. The goal is to construct a binary pattern classifier that predicts whether the software module is defect-prone or not, given the observed features of the software module. HDDT does hierarchical splitting of the input feature space. Each tree node in the feature space represents as one of the segregated subsets in C^p. Cieslak et al. [17] introduced a splitting criterion to create the HDDT is as follows:

$$\text{Hellinger distance (HD)} = d_{\text{H}}(X_+, X_-) = \sqrt{\sum_{r=1}^{p}\left(\frac{|X_{+r}|}{|X_+|} + \frac{|X_{-r}|}{|X_-|}\right)^2} \tag{1.1}$$

In this methodology, given there are two class distributions X_+ and X_-, the goal is to calculate the “distance” in terms of the normalized frequencies combined over all the partitions. Assuming feature space as a countable space, we are discretizing all continuous features into p partitions. $|X_+|$ denotes the frequency of instances which are from the majority class (non-defective class) in the training dataset and $|X_+r|$ is a subset of the training set for the feature X, comprising of majority class and with value r. A similar description follows for $|X_-|$ and $|X_-r|$ which are defined for the minority class (defect class). As Hellinger distance (HD) increases, the features become more discriminative in nature. HDDT methods also have a feature selection mechanism, and the first feature (first split node) selected has the property of carrying minimal affinity with respect to the classes. The Hellinger distance is not influenced by prior probability; hence, it is insensitive to class distribution. HDDT is also skew insensitive in nature since it is not taking account the prior probability while calculating distance. However, the split criterion mentioned in (1) only works in case of classification problem where only two distinct classes are present. But often HDDT may overfit the data. To get rid of this problem, we go for a hybrid method. Generally, we combine classifiers which use distinct pattern representations to make a decision. Pattern classifiers are designed in order to achieve the best possible classification performance for the unseen data or the task in hand [32]. Decision trees and artificial neural networks are both well-known and competitive methodologies for building classification problems. Classification trees are hierarchical classifiers in nature and it is comparatively superior to artificial neural network (ANN) in case of readability of knowledge [37]. ANN works better in case of implementation of comprehensive inference over the inputs [50].

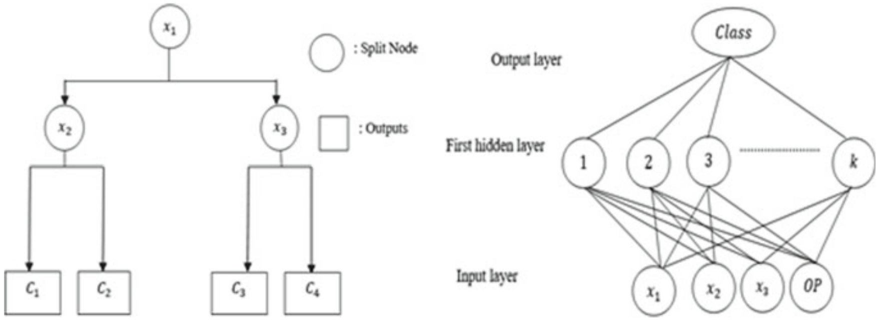


Fig. 1.1 Example of hybrid CT-ANN classification model where $x_i, i = 1, 2, 3$, represents important features extracted through CT and $c_i, i = 1, 2, 3$, are leaf nodes and OP as CT output [13]

Flowchart of the Proposed Model

In our proposed methodology, the feature space is split into disjoint segments by HDDT algorithm. The constructed tree chooses important features and removes redundant ones. Later, we build an ANN model with one hidden layer [39] where we have used the important features obtained through HDDT algorithm as input feature and additionally the prediction results produced by HDDT method is also used as input feature, i.e., if we obtain m number of important features then we provide $(m + 1)$ number of features in the input layer of neural networks. We first provide a picture in Fig. 1.1 of a simple hybrid model called CT-ANN Model [13]. This helps us to understand how a hybrid approach is used.

The informal work flow of the proposed hybrid model called HDDT-ANN model is given. Figure 1.2 represents the flowchart of the proposed HDDT-ANN model.

1. Normalize the source SDP dataset using min-max method.
2. Split the scaled dataset randomly by 80:20 ratio, where 80% data is used for training and 20% for testing.
3. Apply HDDT algorithm on training set to build a tree model which will extract important features.
4. HDDT shortlists the important features, which are contributing for building the tree and neglect the rest ones.
5. The prediction result produced by HDDT algorithm is considered as an additional feature along with the important features shortlisted, as described in step 5 are used in the input layer of Artificial neural network (ANN) model.
6. The important features extracted by HDDT method with an additional input feature are exported to the Artificial neural network (ANN) model with one hidden layer and with sigmoid activation function.
7. Then, we optimize the weights and number of neurons present in hidden layer of artificial neural network as mentioned in [13] and run ANN algorithm till acceptable accuracy is achieved.
8. Then, the classifier will be finally ready to produce results.

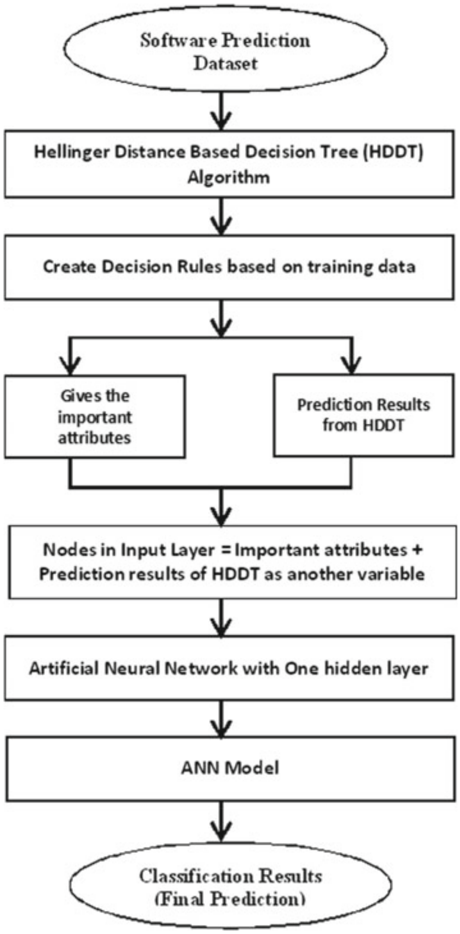


Fig. 1.2 Flowchart of HDDT-ANN model

Insights of the Proposed Model

The key point of our proposed HDDT-ANN model is the extraction of important features and utilization of the class levels produced through HDDT model, which is then followed by ANN model. Hornik and Stinchcombe [28] showed that if sufficient number of hidden units are present then standard multilayer feedforward networks consisting of one hidden layer is able to approximate any Borel measurable function from one finite dimensional space to another space to any desired level of accuracy. An additional input feature in the form of HDDT output with the extracted important features from HDDT method increases the dimension of feature space and will also increase the class separability, which will help ANN for drawing better decision boundary. Cover [18] proposed the idea that shows that if feature space is not densely populated, then in the higher dimensional space it becomes linearly separable, compared to a lower dimensional space in a complex pattern classification problem. Experimental results by Lee and Srihari [34] have presented that as more information is included, then the performance of combination of decision tree and ANN algorithm also increases. One of the key characteristics of our proposed hybrid model is choosing the optimum number of neurons to be used in ANN model in the hidden layer. It is shown that too few nodes in hidden layer may restrict network generalization capabilities, on the other side, too many hidden layers may lead into the situation of overtraining by the network [39]. To deal with this problem, we used the approach of optimizing weights and also varying the number of neurons to be used in the hidden layer until we achieve sufficient accuracy. This algorithm is a two-step problem-solving approach where it initially selects features based on HDDT and then use optimum ANN technique to improve the model. The theoretical properties of our proposed model have been proved in the article [11].

By implementing this model, future defects based on code attributes can be predicted accurately compared to other models and appropriate actions can be taken. We will experimentally illustrate that the proposed model is superior in comparison with the other supervised models, which is discussed in the literature for 10 NASA datasets available at promise repository. Our proposed methodology can be utilized for choosing features of items that will satisfy a specific goal and also can be engaged for modeling such complex scenarios.

Experimental Analysis

Description of Datasets

The SDP datasets consist of various features (numeric values) of a software module (e.g., various measures of lines of code, base Halstead measures, derived Halstead measures, McCabe metrics, and branch counts are some features of the SDP datasets), along with the response column consisting of class labels (true or false i.e., whether the module reported defects or not) [31]. Our objective is to classify between defect ("true" class) and non-defect ("false" class) distributions of software modules. The dataset is splitted into training set (80% of the dataset) for building the model and test set (20% of the dataset) to examine the performance of our model. Such a split is quite usual in literature [15]. The response variable indicates whether the software module is defective or non-defective. Sample datasets and their characteristics are shown in Table 1.1. Each of the datasets contains two classes and is highly imbalanced in nature. As an example, analysis of CM1 dataset is shown here. The number of attributes of CM1 dataset contains 22 features (lines of code measure of 5 different types, 3 different McCabe metrics, 4 different base Halstead measures, 8 different derived Halstead measures, a branch-count, and 1 goal field). There are no missing values in any of attributes. The class value (defects) is discrete in nature, and 90.87% are false instances compared to 9.13% true instances. We provide the summary of

Table 1.1 Description of 10 SDP datasets

Summary				
Dataset	Classes	Objects	Attributes	Defects %
CM1	2	327	22	9.13
KC3	2	194	39	18.6
MC1	2	1988	38	2.3
MC2	2	125	39	35.2
MW1	2	253	37	10.7
PC1	2	705	37	8.7
PC2	2	745	36	2.1
PC3	2	1077	37	12.4
PC4	2	1287	37	13.8
PC5	2	1711	38	27.5

ten SDP datasets in Table 1.1. It is clear from Table 1.1 that the true instances vary from 2.1 to 35.2%, which means that the false instances vary from 68.8 to 97.9% indicating a clear imbalance between the two class instances.

Performance Evaluation Metrics

For classification problems, the efficiency of a model is presented as a matrix representation of the classification results known as confusion matrix. The elements of the matrix are actual and predicted classification results produced by the classification model. After the confusion matrix for a model is generated, we calculate different metrics like precision, recall, F-measure and accuracy percentages to verify how good the model is performing. The ratio of number of positive instances which are predicted accurately to the total number of instances that are predicted positive is defined as precision and recall is defined as the ratio of number of positive instances which are predicted accurately to the total number of instances that belong to the actual class. F-measure is defined as the harmonic mean of precision and recall, and the ratio of number of correctly predicted data points to the total number of data points is defined as accuracy. In general, particularly dealing with balance data the model, which produces maximum accuracy may be considered as the best model but in case of imbalanced datasets, it may not be a good metric to evaluate model performance. For symmetric datasets where the number of false negatives and false positives are almost equal, the metric accuracy gives the best measure. However, in case of dealing with asymmetric class distribution in case of binary classification scenario, F-measure is usually more logical and capable than accuracy [19]. A F-measure value closer to 1 recommends a desired model. The formula of the different performance measures are given in the following:

$$\text{Precision} = \frac{\text{TP}}{\text{TP+FP}}$$

$$\text{Recall} = \frac{\text{TP}}{\text{TP+FN}}$$

$$\text{Specificity} = \frac{\text{TN}}{\text{TN+FP}}$$

$$\text{AUC} = \frac{\text{Recall+Specificity}}{2}$$

$$F\text{-measure} = \frac{2*\text{Precision}*\text{Recall}}{\text{Precision+Recall}}$$

$$\text{Accuracy} = \frac{\text{TP+TN}}{\text{TP+FP+TN+FN}}$$

where TP (True Positive) is number of correctly positive predictions and FP (False Positive) is number of positive prediction which are incorrect, whereas the number of correctly negative prediction is denoted as TN (True Negative) and FN (False Negative) is the number of negative prediction which are incorrect, done by the model.

Analysis of Results

The hybrid methodology presented in this article has been applied to ten NASA datasets [3] mentioned in Table 1.1. Since most of the datasets mentioned in Table 1.1 are mostly imbalanced in nature, the decision tree developed on the Hellinger distance [17] has been applied. We present here the analysis on CM1 (one of the 10 NASA SDP dataset) dataset. The HDDT has been grown on the randomly splitted training set and tested on remaining 20% instances. From there, we extract the important features. The important features are LO-Comment which is Halstead's count of lines of comments, LOC-COde-And-comment. We have predicted the class of each instances based on this HDDT model. Then, we took the prediction made by HDDT as an extra input along with the important features extracted by HDDT in the input layer of ANN. Several performance metrics have been considered in this context like F-measure, AUC, and Recall since data are highly imbalanced in nature. The F-measure of the HDDT model further improved to 90%, AUC is 66%, and recall is 88%. The optimal classification tree based on Hellinger distance is given in Fig. 1.3. ANN model is quite often used as a supervised learning methodology for predicting class of unknown instances in a classification problem. We have performed the normalization of the data prior to training a neural network, the major reason behind is that if we avoid normalization then the training process may become very difficult and time-consuming. We have used min-max normalization technique to scale the data in the interval [0, 1]. Hornik et al. [28] proved the universal consistency of the feedforward neural networks. We have chosen optimum number of neurons as mentioned in [13].

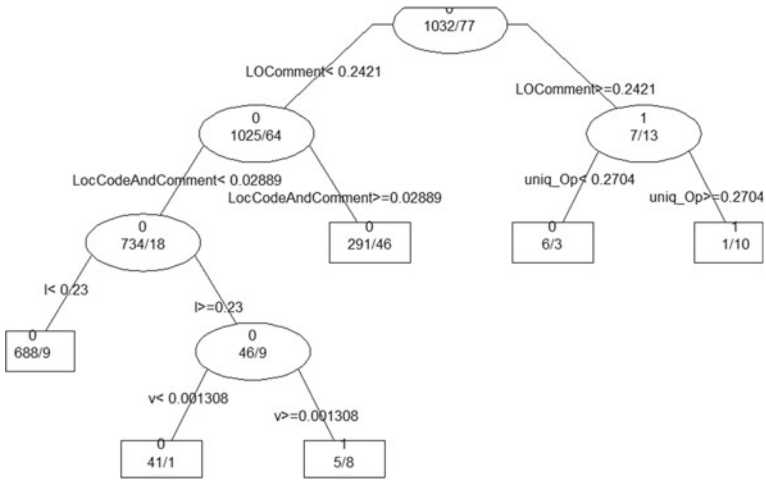
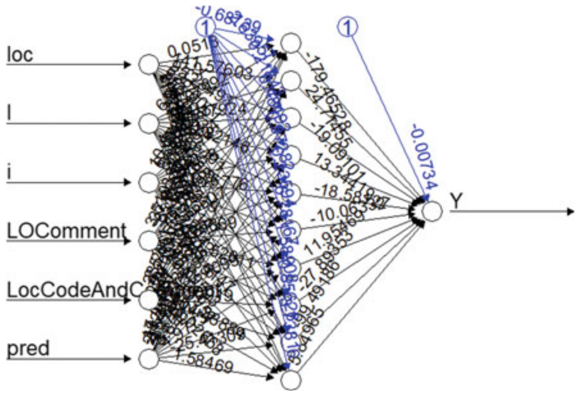


Fig. 1.3 Hellinger distance-based decision tree output on CM1 dataset

Fig. 1.4 Hybrid HDDT-ANN output on CM1 dataset



We have taken 10 neurons in the hidden layer in case of CM1 dataset. We used sigmoid activation function in the output layer.

The ANN model is presented in Fig. 1.4. After hybridization the F-measure improved to 96%, AUC improves to 90% and recall to 95%. Since the sample is drawn randomly, in each iteration we will get different values for each of metrics. We iterate 10,000 times and calculate mean and standard deviation for each of the metrics.

Table 1.2 gives the various performance metrics obtained for different models applied to these datasets, including the one of the recent methods [12]. The performance metrics of the proposed model are shown in the right most column of Table 1.2 and it clearly indicates that the proposed model is the best model so far used for imbalanced classification problems of this type, though for a few instances other models have done better occasionally.

Table 1.2 Mean (S.D.) of different metrics of various models applied on 10 SDP datasets

Dataset	Measure	ISDA	STr-NN	SMB+DT	VCB-SVM	ABNC+DT	HDDT	HDRF	Hellinger net	HDDT+ANN
CM1	*F*-measure	0.84 (0.02)	0.845 (0.012)	0.792 (0.01)	0.76 (0.03)	0.842 (0.02)	0.785 (0.015)	0.82 (0.054)	0.865 (0.052)	0.90 (0.031)
	AUC	0.55 (0.032)	0.566 (0.02)	0.653 (0.017)	0.595 (0.04)	0.66 (0.019)	0.573 (0.04)	0.671 (0.049)	0.706 (0.04)	0.66 (0.051)
	Recall	0.30 (0.01)	0.38 (0.02)	0.593 (0.001)	0.395 (0.04)	0.320 (0.005)	0.30 (0.01)	0.491 (0.049)	0.66 (0.034)	0.88 (0.053)
KC3	*F*-measure	0.73 (0.01)	0.77 (0.02)	0.78 (0.06)	0.80 (0.02)	0.732 (0.006)	0.892 (0.014)	0.902 (0.014)	0.832 (0.027)	0.94 (0.043)
	AUC	0.56 (0.05)	0.656 (0.04)	0.748 (0.03)	0.626 (0.017)	0.541 (0.07)	0.532 (0.12)	0.737 (0.01)	0.744 (0.02)	0.87 (0.045)
	Recall	0.35 (0.02)	0.5 (0.015)	0.348 (0.024)	0.465 (0.01)	0.33 (0.07)	0.584 (0.03)	0.536 (0.008)	0.60 (0.02)	0.91 (0.037)
MC1	*F*-measure	0.975 (0.001)	0.95 (0.003)	0.965 (0.002)	0.97 (0.001)	0.97 (0.005)	0.94 (0.005)	0.97 (0.004)	0.985 (0.003)	0.95 (0.002)
	AUC	0.64 (0.02)	0.76 (0.03)	0.69 (0.01)	0.66 (0.025)	0.59 (0.01)	0.532 (0.06)	0.70 (0.01)	0.80 (0.03)	0.80 (0.015)
	Recall	0.18 (0.00)	0.58 (0.001)	0.42 (0.001)	0.36 (0.008)	0.17 (0.00)	0.333 (0.003)	0.50 (0.01)	0.60 (0.001)	0.91 (0.005)
MC2	*F*-measure	0.67 (0.02)	0.71 (0.02)	0.66 (0.01)	0.61 (0.04)	0.75 (0.01)	0.57 (0.06)	0.68 (0.01)	0.73 (0.04)	0.94 (0.011)
	AUC	0.638 (0.01)	0.65 (0.01)	0.59 (0.01)	0.57 (0.02)	0.69 (0.01)	0.43 (0.06)	0.60 (0.01)	0.673 (0.008)	0.90 (0.025)
	Recall	0.60 (0.01)	0.5 (0.00)	0.33 (0.00)	0.26 (0.02)	0.4 (0.01)	0.5 (0.00)	0.55 (0.01)	0.54 (0.02)	0.92 (0.04)
MW1	*F*-measure	0.80 (0.04)	0.91 (0.014)	0.88 (0.144)	0.779 (0.132)	0.832 (0.090)	0.772 (0.120)	0.795 (0.134)	0.902 (0.017)	0.94 (0.04)
	AUC	0.612 (0.08)	0.816 (0.02)	0.70 (0.016)	0.643 (0.014)	0.58 (0.10)	0.535 (0.04)	0.627 (0.1)	0.790 (0.023)	0.56 (0.03)
	Recall	0.40 (0.01)	0.70 (0.02)	0.50 (0.06)	0.35 (0.02)	0.620 (0.05)	0.40 (0.01)	0.55 (0.04)	0.65 (0.03)	0.92 (0.015)
PC1	*F*-measure	0.915 (0.02)	0.890 (0.05)	0.89 (0.032)	0.88 (0.03)	0.92 (0.002)	0.882 (0.01)	0.902 (0.04)	0.945 (0.008)	0.96 (0.04)
	AUC	0.586 (0.08)	0.808 (0.027)	0.786 (0.008)	0.67 (0.010)	0.632 (0.07)	0.532 (0.05)	0.719 (0.08)	0.854 (0.022)	0.90 (0.04)
	Recall	0.842 (0.108)	0.858 (0.087)	0.796 (0.118)	0.838 (0.080)	0.815 (0.107)	0.832 (0.090)	0.839 (0.108)	0.87 (0.07)	0.95 (0.04)
PC2	*F*-measure	0.97 (0.00)	0.97 (0.00)	0.96 (0.01)	0.97 (0.01)	0.96 (0.00)	0.95 (0.00)	0.97 (0.00)	0.98 (0.005)	0.94 (0.001)
	AUC	0.5 (0.00)	0.5 (0.00)	0.56 (0.01)	0.45 (0.005)	0.620 (0.04)	0.432 (0.00)	0.570 (0.01)	0.66 (0.02)	0.74 (0.02)
	Recall	0 (0.00)	0.1 (0.00)	0 (0.00)	0 (0.00)	0.05 (0.00)	0.1 (0.00)	0.1 (0.00)	0.2 (0.00)	0.96 (0.04)
PC3	*F*-measure	0.888 (0.01)	0.840 (0.05)	0.82 (0.03)	0.81 (0.02)	0.880 (0.02)	0.796 (0.03)	0.883 (0.017)	0.907 (0.04)	0.94 (0.04)
	AUC	0.598 (0.018)	0.756 (0.012)	0.635 (0.03)	0.654 0.018	0.65 (0.02)	0.532 (0.04)	0.614 (0.02)	0.835 (0.009)	0.795 (0.001)
	Recall	0.48 (0.01)	0.67 (0.02)	0.45 (0.03)	0.48 (0.02)	0.38 (0.028)	0.49 (0.01)	0.64 (0.02)	0.70 (0.005)	0.93 (0.002)
PC4	*F*-measure	0.905 (0.015)	0.86 (0.01)	0.88 (0.01)	0.79 (0.02)	0.89 (0.04)	0.82 (0.006)	0.87 (0.01)	0.90 (0.02)	0.94 (0.022)
	AUC	0.805 (0.01)	0.836 (0.01)	0.80 (0.03)	0.67 (0.04)	0.744 (0.03)	0.679 (0.05)	0.783 (0.01)	0.842 (0.03)	0.72 (0.02)
	Recall	0.50 (0.06)	0.80 (0.03)	0.71 (0.01)	0.47 (0.07)	0.53 (0.04)	0.41 (0.09)	0.63 (0.03)	0.82 (0.03)	0.93 (0.01)
PC5	*F*-measure	0.74 (0.05)	0.72 (0.10)	0.75 (0.02)	0.78 (0.02)	0.73 (0.05)	0.78 (0.02)	0.80 (0.01)	0.825 (0.01)	0.96 (0.02)
	AUC	0.65 (0.003)	0.66 (0.04)	0.62 (0.06)	0.68 (0.01)	0.56 (0.05)	0.62 (0.005)	0.64 (0.06)	0.70 (0.05)	0.71 (0.04)
	Recall	0.45 (0.00)	0.360 (0.00)	0.40 (0.00)	0.26 (0.00)	0.35 (0.00)	0.25 (0.00)	0.40 (0.00)	0.50 (0.00)	0.90 (0.021)

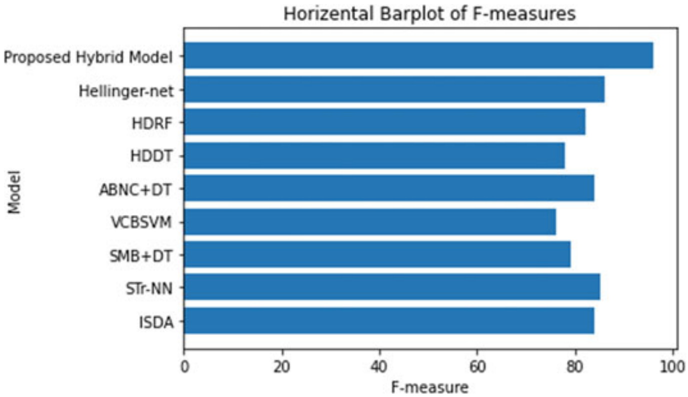


Fig. 1.5 Horizontal barplot of F-measure of different models applied on CM1 dataset

Conclusion

The objective of this article is to develop a model, which can predict whether a software module is defective or non-defective. In our study, we have produced a HDDT-ANN model which is a combination of artificial neural network and Hellinger distance-based classification tree, and it gives better results in terms of different metrics used for classification problems, specially for imbalanced datasets than all other traditional models as shown in Fig. 1.5. Splitting the data into training and testing set in the ratio 80:20 is quite popular in machine learning literature [13]. However, it is noted that when the data comes from an extremely imbalanced dataset, there are non-zero (although very low) chances of obtaining a sample which has only one class of data. But, since the samples generated randomly are in thousands, the effect of the positive probability of having a one-class sample will be much low, which can be neglected. In this study, we have used HDDT for extracting important features from imbalanced dataset and found that the hybrid HDDT-ANN model outperforms the other supervised models in the context of SDP problems. Significant accuracy compared to traditional machine learning algorithms has been achieved through the use of our experimentally optimized model for the 10 NASA SDP datasets. For a module which may be newly introduced, if we know the features of the new module in terms of Halstead's count of lines of comments, LOC-COde-And-comments, etc., then the proposed model will be able to say whether the module is defect-prone or not. The proposed hybrid HDDT-ANN model may be used for similar problems like in medical diagnosis contexts as well. Also, it is known that all defects are not of same importance. However, severity of the effects of a defect instance can be looked at as a future research topic. Some recent works [21] may be looked at for tackling such problems.

Code

We used R language to implement this hybrid HDDT-ANN methodology. The link for the code is given here: https://github.com/KARMAKAR03/Barin001.

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Chapter 2
Investigate the Reason for Students' Absenteeism in Engineering College in Fuzzy MCDM Environment

Sukarna Dey Mondal, Dipendra Nath Ghosh, and Pabitra Kumar Dey

Abstract For the progress of any nation, education system always plays a dynamic role. Best academic institutes are national assets, and students are major assets of any institute. However, today's students are less focused on their studies. As an effect, they are avoiding their important classes due to various reasons. A huge number of students are absent from class which may be destroyed their careers. These are the well-known images of the school and colleges. When a student is frequently absent from class, it has an unfavorable outcome on their academic performance and a few must repeat a grade level. Therefore, it is critical to conduct research that will point the way to understand the key that minimizes student absenteeism. So, an effort was made here to assess some criteria and sub-criteria using analytical hierarchy process (AHP) and techniques for order preference by similarity to ideal solution (TOPSIS) in Type 1 interval fuzzy (T1-IF) and Type 2 interval fuzzy (T2-IF) atmospheres for alternative ranking that depicts student absenteeism, as well as to make comparisons of T1-IF set and T2-IF set. In the end, the group decision-making (GDM) method is used. Questionnaire sessions are used to identify the major causes of student absenteeism. Finally, the outcome of the study builds a more practical way out to acknowledge the actual alternative which eradicates student absenteeism according to pre-assigned criteria and sub-criteria.

Keywords Student's absenteeism · T1-IF set · T2-IF set · AHP · TOPSIS · GDM method · Spearman's rank correlation coefficient (SRCC)

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Introduction

Higher education is one of the keys to growth performance, wealth, and competitiveness in national and universal economies. Higher education provides an individual an opportunity to get an accomplishment in today's world. Contemporary universities present their pupils with numerous programs which are arranged for them so that they can engage themselves in different economic sectors in the future. But in the contemporary situation, students are less interested to attend their classes regularly for various reasons. To reduce the crisis, a scientific model is proposed to achieve some ordinary and in general relevant solutions.

Engineering colleges are the guiding maps for the students who endeavor to become engineers. Students have to appear for WBJEE or AIEEE to get entrance into engineering colleges. Their view engineering education as a most simple path to building a bright career to make a future but they are unsuccessful to realize the weightage of awareness. Sometimes, the new college atmosphere, the new course make it complicated for the students to accustom to the engineering activities. Neither they can manage themselves for the engineering curriculum nor do their parents collect information from their wards. So as a result, absenteeism may be one solution to run away from the regular engineering schedule. Therefore, the idea behind the study was to investigate the fact which reduces student absenteeism.

In this study, a model is designed under the environment of T1-IF and T2-IF set to judge the appropriate alternative of "student absenteeism" according to some preassigned criteria and sub-criteria. The approach aims to present a conceptual method lucidly using MCDM problems to the student absenteeism problems.

Literature Review

A fuzzy set is a collection of substances with membership grades ranging from zero to one [1, 2]. This is called a membership function. In every field, fuzzy logic is a strong method to represent ambiguity. It is playing a vital role to solve the decision problems where the existing information is biased and imperfect [3–5], particularly at what time the objective is to discover a high-quality estimate explanation [6].

T2-IF set by Zadeh [1] is a continuum of the T1-IF set by Karnik and Mendel [29] and Mendel [30]. The common T2-IF set contains composite computational operations. So, normally T2-IF sets are not applicable [31]. On interval T2-IF sets, a ranking method was developed [7]. Pamucar developed a design of fuzzy decision in the field of waste management [8]. Biswas established an extended framework on picture fuzzy environment in social enterprise systems [9].

Saaty was developed AHP in 1980. This is the hierarchical model and pair-wise comparison method [10, 11]. AHP is a process to make a decision that combines numerous criteria into a hierarchy, considering the relative importance of criteria, evaluating alternatives for each criterion, and establishing an overall ranking of the alternatives [12]. Saaty's AHP was extended by integrating with fuzzy sets, i.e., invented by Buckley (1985). To attain the weightage of criteria, Buckley's idea is applied.

The principle of TOPSIS was first developed by Hwang and Yoon [13]. TOPSIS approach was extended by Abo-Sinna and Amer [14, 28] to solve multi-objective nonlinear programming problems. Jahanshahloo et al. [15] develop the idea of TOPSIS to build up a method to solve MCDM problems with interval data. Chen attempts in [16] to categorize suitable person inclinations and critically qualified assistances via information statistics. Kousalya et al. offered the usage of multi-requirements decision-making techniques for reputation alternatives to control student absenteeism in engineering colleges [17]. De et al. introduced a ranking of T2-IF numbers using GDM [18]. Biswas et al. developed a model for portfolio selection in an MCDM outline [19]. Biswas discussed a comparative analysis of MCDM in the field of management and engineering [20]. Das et al. established a framework on fuzzy soft set with respect to decision making [21, 25]. Das et al. developed an approach on intuitionistic trapezoidal fuzzy set using decision making [22]. SRCC enables to set on the degree of relation (including affirmative or negativity of a relation) among ranks attained through manner of means of extra everyday MCDM techniques [26, 27] and extraordinary decision-makers and extraordinary situations for a given set of alternatives. Additive ranking and multiplicative ranking are appreciably applied to compute the degree of the relation among strategies. Several critical values for the SRCC for numerous significance levels are furnished in [23].

Proposed Methodology

In the research, the proposed methodology is implemented to estimate the actual reason for student absenteeism at an engineering college. Six alternatives were observed and arbitrarily chosen. In beginning, a literature inspection was conducted. Then, an additional survey was arranged between experts (X, Y), for the addition/removal of criteria and sub-criteria. Experts' opinion is shown in Table 2.1. Linguistic terms are visible in (fuzzy sense) (Table 2.2). The shortlisting criteria and sub-criteria are shown below (Fig. 2.1):

Alternatives	***Criteria***
S1 Counseling	A Health
S2 Infrastructure	B Family problems
S3 Engagement of parents	C Impact of teachers
S4 Making lecture more interesting	D Class environment
S5 Amalgamation of students teachers	E Scarcity of inspiration
S6 Punishment	F Participation in seminars/conference/cultural activities

Sub-criteria

A1. Health
B1. Monetary problems
B2. Responsibilities being follow-up
B3. Uneducated parents
C1. Rude behavior
C2. No preparation/no commitment
D1. Irregular conduct of classes
D2. Language problem
D3. Disturbances of classes
E1. Indiscipline lifestyle
E2. Lack of interest in engineering education
E3. Effect of unsatisfactory campusing
F1. Participation in seminars/conferences/cultural activities

Proposed Flowchart

See Fig. 2.2.

Proposed Algorithm

See Fig. 2.3.

Analysis and Discussion

First, the expert's opinion has been taken on the students with respect to different literature survey reports upon several criteria. According to the view of experts', the relative comparison has been made to check the relative closeness. So, AHP has been used here to compute the weights of the criteria under the T1-IF and T2-IF environment which is given in Tables 2.3 and 2.4, respectively. Since TOPSIS can handle positive solutions as well as the negative solution and check the relative closeness between them, using that advantage TOPSIS has been applied. Here, the positive side, as well as the negative side, has been judged.

According to Karnik and Mendel [24], a T2-IF logic system can be a reflection of a compilation of several predetermined T1-IF logic systems. In principle fuzzy systems, T2-IF sets mitigate the impact of unpredictability. In computing each bound of the type reduced interval, the upper and lower membership functions of the same T2-IF set can be used concurrently. These characteristics do not exist in T1-IF sets. These are the merits of using T2-IF sets. In this study, AHP and TOPSIS together give the ranking of alternatives under the T1-IF and T2-IF environments. The concluding

Table 2.1 Experts' opinions against each alternative

Criteria	Sub-criteria	Experts'	Alternatives					
			S1	S2	S3	S4	S5	S6
A	A1	X	MH	M	M	H	VL	M
		Y	M	M	M	MH	H	M
B	B1	X	H	H	M	M	M	MH
		Y	M	VL	L	M	VL	MH
	B2	X	MH	L	M	M	MH	VL
		Y	M	M	M	L	H	VL
	B3	X	L	M	MH	M	VL	MH
		Y	M	H	M	L	L	M
C	C1	X	L	M	MH	L	MH	M
		Y	H	M	H	L	M	H
	C2	X	MH	VL	H	M	VL	VL
		Y	MH	H	H	H	MH	VL
D	D1	X	M	M	MH	L	VL	VL
		Y	L	MH	MH	M	M	L
	D2	X	VL	MH	M	VL	L	M
		Y	MH	M	M	L	L	L
	D3	X	MH	MH	H	M	MH	M
		Y	H	MH	H	M	H	MH
E	E1	X	MH	H	MH	M	VL	MH
		Y	M	VL	H	MH	M	VL
	E2	X	MH	MH	M	MH	M	VL
		Y	MH	M	M	VL	VL	H
	E3	X	M	MH	H	M	H	MH
		Y	MH	MH	H	MH	M	M
F	F1	X	MH	H	M	MH	M	MH
		Y	M	MH	M	L	MH	M

H—high; MH—medium high; M—medium; L—low; VL—very low

Table 2.2 Linguistic terms

Linguistic terms	Type 2 fuzzy set											
Very low (VL)	0.0	0.0	0.0	0.1	1.0	1.0	0.0	0.0	0.0	0.5	0.9	0.9
Low (L)	0.0	0.1	0.1	0.2	1.0	1.0	0.1	0.2	0.2	0.3	0.9	0.9
Medium (M)	0.3	0.5	0.5	0.7	1.0	1.0	0.4	0.5	0.5	0.6	0.9	0.9
Medium high (MH)	0.5	0.7	0.7	0.9	1.0	1.0	0.6	0.7	0.7	0.8	0.9	0.9
High (H)	0.9	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	0.9	0.9

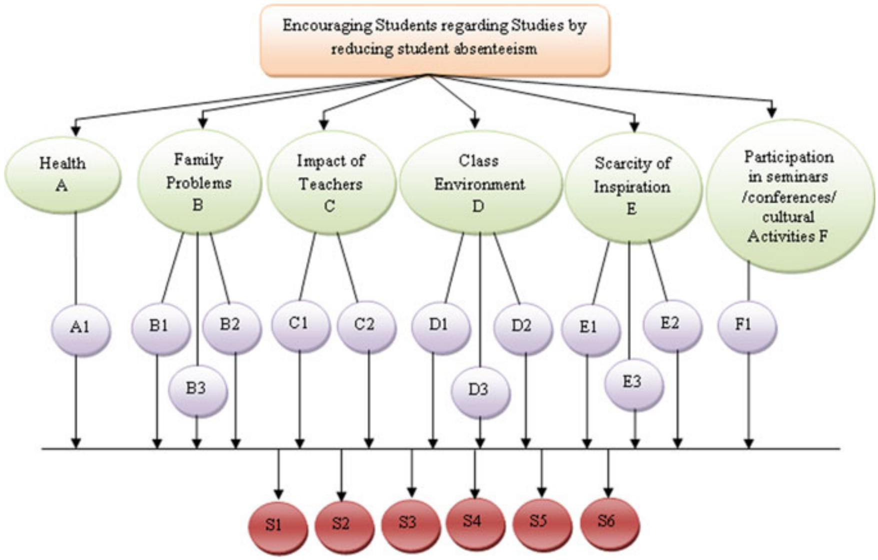

Fig. 2.1 Hierarchical decomposition of criteria and sub-criteria in student absenteeism

ranking of them in T1-IF (Table 2.5) and T2-IF (Table 2.6) are not the same. From Table 2.6, it is noticed that the alternative S4 "making lecture more interesting" which is ranked the first position using T1-IF, S1 "counseling" got the second position, S3 "engagement of parents" got the third position, and so on. Whereas from Table 2.6, S1 "counseling" got the first position, S3 "engagement of parents" got the second position and S2 "infrastructure" got the third position, and so on. These are very complicated rankings of alternatives from where to recognize the proper alternative of absenteeism.

To overcome these, the SRCC method is implemented using relative closeness under T1-IF and T2-IF in Table 2.7. According to SRCC characteristics, from Table 2.8, it has been possible to find an association between T1-IF and T2-IF. It is observed that from Table 2.9, a single ranking structure is originated which is co-related with T1-IF set and T2-IF set, where S4 "making lecture more interesting" got the first position, S1 "counseling" got the second position, S3 "engagement of parents" got the third position, and so on. That means apparently, it can be said that S4, S1, and S3 are the key factors than the others. It is important to focus on these. But logically after analyzing the ranking of Table 2.9, finally, it can be concluded that "S4" is the most vital alternative. Therefore, to eradicate absenteeism, it should give more stress to this particular factor. That's why, this is a very logical, realistic, and acceptable proposal for engineering colleges to reduce students' absenteeism.

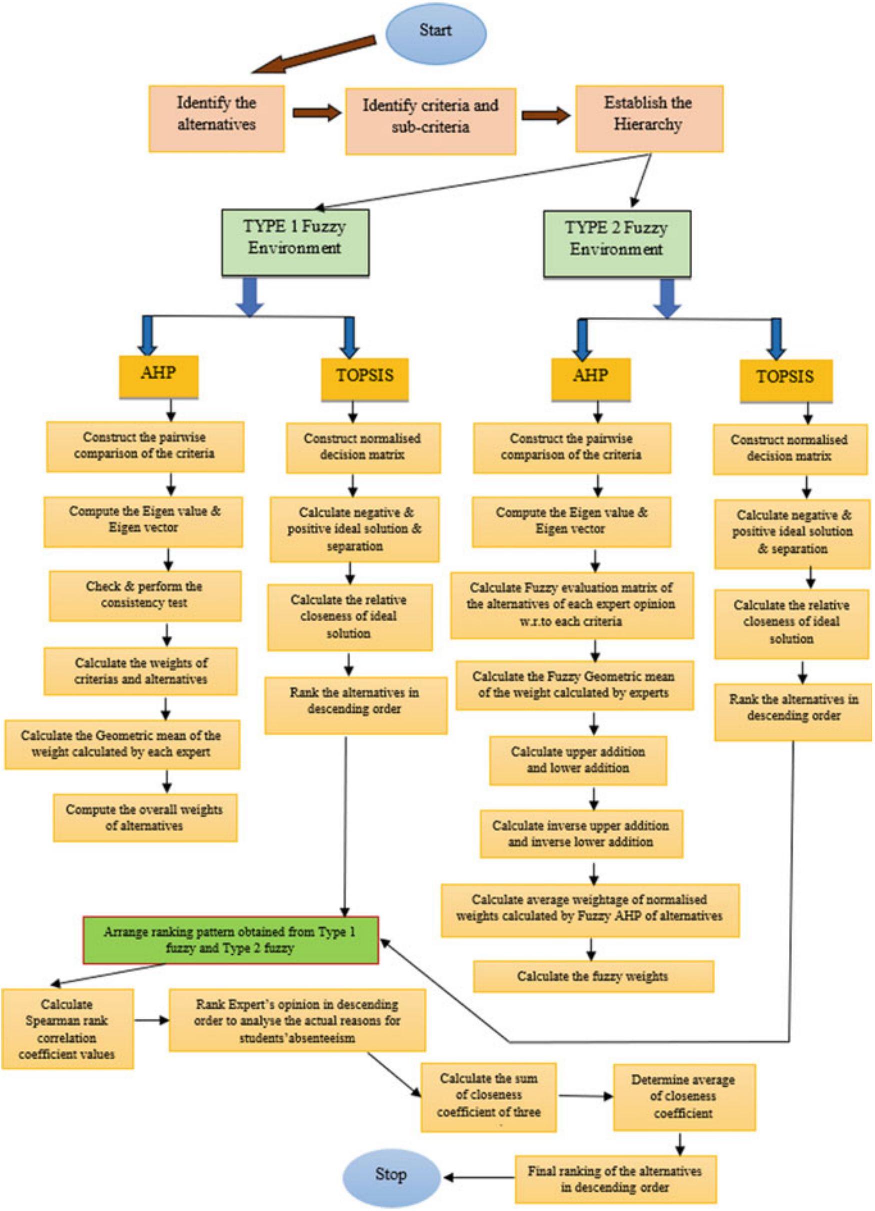

Fig. 2.2 Proposed flowchart for students' absenteeism in engineering college

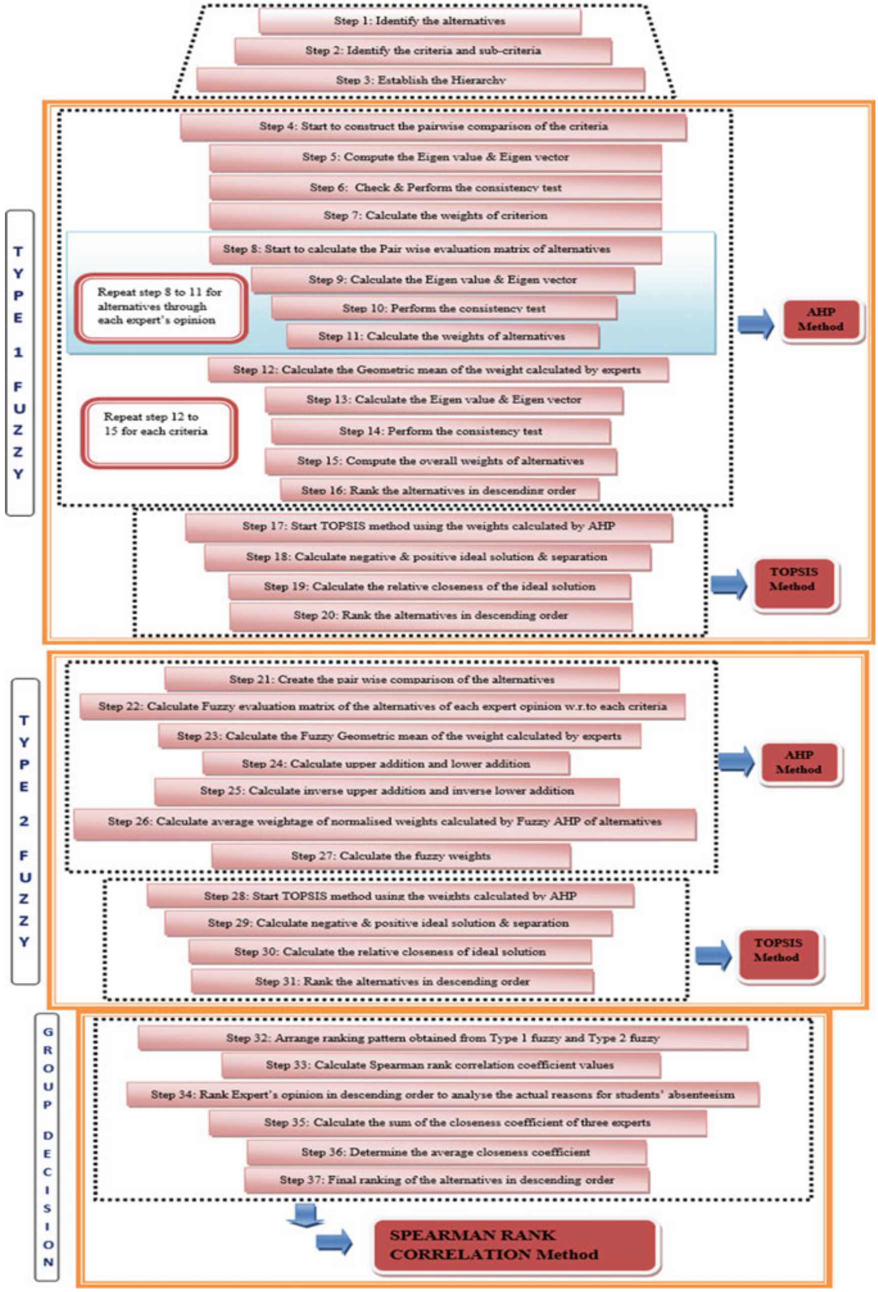

Fig. 2.3 Proposed algorithm for students' absenteeism in engineering college

Table 2.3 Weights of criteria under Type 1 fuzzy

Criteria	A1	B1	B2	B3	C1	C2	D1	D2	D3	E1	E2	E3	F1
Weight	0.2874	0.1064	0.0725	0.0391	0.0779	0.0720	0.0388	0.0746	0.0383	0.0537	0.0391	0.0202	0.0800

Table 2.4 Weights of criteria under Type 2 fuzzy

Criteria	Fuzzy weights					
A1	0.297	0.309	0.309	0.314	1.000	1.000
	0.288	0.297	0.296	0.299	0.900	0.900
B1	0.193	0.200	0.199	0.202	1.000	1.000
	0.182	0.182	0.187	0.191	0.900	0.900
B2	0.135	0.135	0.136	0.137	1.000	1.000
	0.133	0.131	0.130	0.132	0.900	0.900
B3	0.096	0.097	0.097	0.097	1.000	1.000
	0.101	0.102	0.101	0.102	0.900	0.900
C1	0.073	0.072	0.072	0.071	1.000	1.000
	0.076	0.076	0.076	0.075	0.900	0.900
C2	0.056	0.053	0.053	0.052	1.000	1.000
	0.059	0.058	0.058	0.056	0.900	0.900
D1	0.043	0.040	0.039	0.038	1.000	1.000
	0.045	0.044	0.044	0.042	0.900	0.900
D2	0.032	0.029	0.029	0.028	1.000	1.000
	0.034	0.033	0.033	0.031	0.900	0.900
D3	0.024	0.022	0.022	0.021	1.000	1.000
	0.026	0.025	0.025	0.024	0.900	0.900
E1	0.018	0.016	0.016	0.015	1.000	1.000
	0.020	0.019	0.018	0.017	0.900	0.900
E2	0.014	0.012	0.012	0.011	1.000	1.000
	0.015	0.014	0.014	0.013	0.900	0.900
E3	0.010	0.009	0.009	0.008	1.000	1.000
	0.012	0.011	0.011	0.010	0.900	0.900
F1	0.008	0.006	0.006	0.006	1.000	1.000
	0.009	0.008	0.008	0.007	0.900	0.900

Table 2.5 Ranking of alternatives under Type 1 fuzzy

Students	D_a^+	D_a^-	C_a	Rank
S1	0.0989	0.0630	0.3891	**2**
S2	0.1251	0.0619	0.3310	**5**
S3	0.1221	0.0691	0.3616	**3**
S4	0.0851	0.1163	0.5775	**1**
S5	0.1092	0.0554	0.3366	**4**
S6	0.1319	0.0473	0.2638	**6**

Table 2.6 Ranking of alternatives under Type 2 fuzzy

Students	*d*(Xi1, X1*) for A1		*d*(Xi2, X2*) for B1		*d*(Xi3, X3*) for B2		*d*(Xi4, X4*) for B3	
S1	0.0592	0.0370	0.1062	0.0955	0.0919	0.0750	0.0273	0.0245
S2	0.0305	0.0040	0.0530	0.0522	0.0448	0.0400	0.0742	0.0687
S3	0.0305	0.0040	0.0100	0.0159	0.0766	0.0530	0.0609	0.0544
S4	0.1334	0.1126	0.0563	0.0484	0.0448	0.0446	0.0273	0.0245
S5	0.0258	0.0137	0.0000	0.0110	0.1258	0.1061	0.0000	0.0000
S6	0.0305	0.0040	0.1015	0.0911	0.0000	0.0000	0.0609	0.0544
Students	*d*(Xi5, X5*) for C1		*d*(Xi6, X6*) for C2		*d*(Xi7, X7*) for D1		*d*(Xi8, X8*) for D2	
S1	0.0354	0.0334	0.0420	0.0409	0.0020	0.0038	0.0102	0.0110
S2	0.0337	0.0286	0.0286	0.0289	0.0156	0.0164	0.0184	0.0185
S3	0.0603	0.0562	0.0573	0.0579	0.0202	0.0214	0.0150	0.0147
S4	0.0000	0.0000	0.0433	0.0429	0.0020	0.0038	0.0000	0.0018
S5	0.0419	0.0372	0.0210	0.0205	0.0000	0.0024	0.0014	0.0029
S6	0.0520	0.0475	0.0000	0.0000	0.0000	0.0024	0.0082	0.0086
Students	*d*(Xi9, X9*) for D3		*d*(Xi10, X10*) for E1		*d*(Xi11, X11*) for E2		*d*(Xi12, X12*) for E3	
S1	0.0086	0.0094	0.0063	0.0066	0.0060	0.0067	0.0017	0.0021
S2	0.0051	0.0058	0.0045	0.0049	0.0047	0.0051	0.0025	0.0032
S3	0.0122	0.0131	0.0106	0.0115	0.0033	0.0052	0.0003	0.0000
S4	0.0000	0.0000	0.0063	0.0066	0.0014	0.0016	0.0017	0.0021
S5	0.0086	0.0094	0.0000	0.0000	0.0000	0.0000	0.0027	0.0033
S6	0.0025	0.0029	0.0019	0.0021	0.0034	0.0038	0.0017	0.0021
Students	*d*(Xi13, X13*) for F1		Dj		Ci		AVG	Rank
S1	0.0015	0.0016	0.3983	0.3461	0.6101	0.5381	0.5741	**1**
S2	0.0044	0.0062	0.3200	0.2764	0.4872	0.4481	0.4676	**3**
S3	0.0008	0.0006	0.3580	0.3074	0.5362	0.5327	0.5344	**2**
S4	0.0000	0.0000	0.3163	0.2890	0.4031	0.4180	0.4106	**4**
S5	0.0015	0.0016	0.2287	0.2066	0.3440	0.3394	0.3417	**6**
S6	0.0015	0.0016	0.2641	0.2189	0.4029	0.3719	0.3874	**5**

Table 2.7 Spearman rank correlation coefficient

	Type 1 fuzzy	Type 2 fuzzy
Type 1 fuzzy	1	0.4286
Type 2 fuzzy	0.4286	1

Table 2.8 SRCC characteristics

Correlation coefficient value	Nature of correlation	Remark
0.9–1.0	Very high	Very strong relationship
0.7–0.9	High	Marked relationship
0.4–0.7	*Moderate*	*Substantial relationship*
0.2–0.4	Low	Definite relationship
< 0.2	Slight	Small relationship

Table 2.9 Ultimate ranking formation obtained from group decision-making method

Students	Type 1 fuzzy	Type 2 fuzzy	Relative closeness (Type 1)	Relative closeness (Type 2)	Group decision making	Rank
S1	2	1	0.3891	0.5741	0.4816	**2**
S2	5	3	0.3310	0.4676	0.3993	**4**
S3	3	2	0.3616	0.5344	0.4480	**3**
S4	1	4	0.5775	0.4106	0.4940	**1**
S5	4	6	0.3366	0.3417	0.3391	**5**
S6	6	5	0.2638	0.3874	0.3256	**6**

Conclusion

Fuzzy logic can handle ambiguity and vagueness in a dataset of the same size. In the proposed study, different methods of fuzzy logic, specifically, T1-IF logic, T2-IF logic are used for decision making. T1-IF logic is a straightforward approach whereas T2-IF logic can present improved outcomes in numerous cases. In this study, the AHP and TOPSIS are used as they are always given a complete structure to solve MCDM problems. Since AHP and TOPSIS both are susceptible to rank reversal, so firstly, AHP and TOPSIS are applied under the T1-IF environment, secondly, AHP and TOPSIS are used under the T2-IF environment to check the ranking of alternatives are justified or not. But the ultimate ranking of alternatives is dissimilar. That implies the checking is not justified. If both rankings of alternatives will give the same ranking, that will be acceptable. But here the scenario is different. As it is known, whenever different MCDM methods are used in a particular problem, the ranking of alternatives may be a little bit different. In this condition, the GDM method plays a vital role to get a single ranking structure which is a very pragmatic and substantial proposal, will help engineering colleges to recognize the genuine explanations for students' absenteeism. This model is organized on students' absenteeism of engineering colleges established in West Bengal, which presents an enormous preference to the management of engineering colleges to enlarge their concern and maintain the tradition. However, not only would the educational organization benefit from this model, but it will also operate the same way where performance is assessed using judgment on the matter based on various criteria.

The consistency and precision of the data collected may not be sent percent. The number of expert opinions for ranking of the criteria/sub-criteria, alternatives are very limited in this case. Though there is no set minimum number in such a study, it is believed that a sample size of 50–70 would have been preferable. In the future, machine learning can be applied with increasing datasets to get an accurate result.

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Chapter 3
Solving a Real-World Non-convex Quadratic Assignment Problem

Badri Toppur

Abstract A manufacturing company has three plants, in India. They manufacture headlamps and tail-lights for the automotive industry. This paper looks at the facility location problem for one of the plants where 12 facilities have to be placed in a two-column, multi-row, cellular layout. The machining sequence for 20 parts conveyed amongst the 12 facilities is specified. The quadratic assignment problem (QAP) has been classified, as an NP-Hard problem for large instances. We have modelled the specific instance as a QAP and are reporting on the solution, obtained by an easily available generalized reduced gradient (GRG) nonlinear solver, and also the solution obtained from the fast *Gurobi* optimizer. The *Gurobi* optimizer gives evidence of global optimality, that the GRG solver does not.

Keywords Facility layout · Quadratic assignment problem · Non-convex objective · Case study · Nonlinear solvers

Mathematics Subject Classification 90B90, 90C90

Introduction

A company manufactures headlamps, rear combination lamps (RCL), and various other small lamps for automotive applications. Three plants produce about 2.8 million headlamps and 2.7 million RCL lamps per annum. The factory space is rectangular and cellular, with an aisle running down between two columns of cells. A number of facilities, wherein the parts are machined, need to be placed in the best cell location, so that the products do not require to be conveyed long distances frequently. Some products may have higher priority, or may be hazardous and have to be treated differently. This is known to be a very hard industrial problem to solve in the sense of optimizing the material flow, and many methods and algorithms have been developed

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to obtain good solutions. Benchmark problems are also available to test the efficiency of a new algorithm against the current best algorithms. The problem instance was solved by the company team, using evolutionary and metaheuristic algorithms. My goal was to obtain a solution using traditional nonlinear programming techniques; as to my knowledge, it had not been attempted for this problem instance. Observing and conveying the benefit of the optimized layout, to the businessman, is the primary goal of the study.

Section "Literature Review" contains the literature review relevant to this paper. Section "Problem Definition" describes the industrial problem faced by the automotive manufacturing company. Section "General Quadratic Assignment Model and Concrete Instance" explains the general formulation of the quadratic assignment problem (QAP) and the concrete instance for which the data was available. The last three sections are Section "Computational Results" that contain the computational results from the solver executions, Section "Managerial Insights" with some managerial insights about the scope and limitations of the numerical results, and finally Section "Conclusions" with general conclusions.

Literature Review

Rockefeller's treatise on convex analysis is most cited [20]; his classification of the objective function and constraints as convex, pseudoconvex, or quasiconvex or as strictly convex, strictly pseudoconvex, or strictly quasiconvex, has helped in structuring the space of nonlinear programmes, with a goal of developing a methodology for solving them. This codification means that one can determine which algorithm is suitable for a certain class of problems and which will not be. According to Bazaraa et al., "the reduced gradient method was developed by Wolfe, to solve a nonlinear programming problem having linear constraints. The method was later generalized by Abadie and Carpentier to handle nonlinear constraints" [1, 5].

The quadratic assignment problem becomes evident in a number of industrial settings. The Lawler form of the QAP which is a generalization of the Koopmans–Beckman form of the QAP is the traditional way to solve the problem [18]. Pioneering work was reported by Rosenblatt in 1986, and also by Gerd et al. in 1987 [11, 21]. A survey of dynamic layout problems was done in 1998, by Balakrishnan and Cheng [3]. A more recent literature survey was conducted in 2006, by Drira et al. [10]. Solutions approaches to the simplest instances of QAP are explained in the textbook by Burkard et al. [8]. Burkard et al. maintain QAPLib which is a library of benchmark problems about the quadratic assignment problem [7]. Another exposition, by example, about the partitioned matrix A, in the quadratic objective function $x^t Ax$, is formed, is given on page 7 of the NSF report by Bazaraa [4]. This is the methodology that I have used to formulate the problem. Advanced algorithms for the QAP are reported by Adams and Waddell [2]. An algorithm for the general QAP has been designed by Hahn et al. [15]. The QAP is NP-Hard, and only, problems of small size can be solved optimally in a reasonable amount of time [13]. Kothari and Ghosh have studied the single row facility layout problem, and even this problem is very complex [16]. This intractability has lead many researchers to design heuristics and metaheuristic approaches such as genetic algorithms, simulated annealing, ant colony optimization, particle swarm optimization, and other techniques to reach sub-optimal solutions in quick time. Ghassemi Tari et al. demonstrate a Tabu search approach for cellular layout design [14]. Prasad et al. present their CRAFT software package written in the Java programming language for solving facility layout problems [19]. Other recent papers on the quadratic assignment problem, applied to facility layout problems, are by Brosch and de Klerk, Cubukcuoglu et al., Fu et al., and Wu et al. [6, 9, 12, 22]. My interest in the study is in using current day, commercial and spreadsheet solvers, on the specific real-world instance of the problem. The first algorithm used was the generalized reduced gradient as designed by Lasdon et al. and implemented by Frontline Systems [17]. The GRG solver as implemented by Frontline Systems is easily available from the Office tools of the Windows operating system. It is an algorithm trusted for nonlinear optimization, over many decades. The *Gurobi* optimizer which has a number of methods incorporated in it was also used. The academic version of *Gurobi* optimizer is also available easily, and it is considered the world's fastest solver at present; it uses the multiple cores of the processor, for parallel processing of the subproblems.[1] This implies that the branch and bound method can be used to solve much larger problems, than it was possible to in the past. My contribution has been to compare the performance of the two solvers, in solving the given industrial problem.

Problem Definition

The 12 work centres occupy area that are given in Table 3.1. The area of these work centres, area either 100 ft^2, 200 ft^2 or 300 ft^2. The overall structure of the shop floor is two-column and multi-row, with each column having a width of 20 ft as shown in Fig. 3.1. There is an aisle between the two columns that is 10 ft wide. Thus the workspace in the two columns can be seen as a grid of 10 ft $\times$ 10 ft cells. One rule, that I have enforced to model the system, is that the 100 ft^2 facilities occupy one of these 10 ft $\times$ 10 ft cells, the 200 ft^2 facilities occupy two of the 10 ft $\times$ 10 ft cells and the 300 ft^2 facilities occupy three 10 ft $\times$ 10 ft cells. This rules out the placement of a 200 ft^2 facility in an area that has dimension 40 ft $\times$ 50 ft or 25 ft $\times$ 8 ft by straddling two or more cells partially. One can easily obtain a compact layout that accommodates these 12 work centres within 2200 ft^2. The minimum vertical length required for the two columns is 60 ft. Figures 3.1 and 3.2 show two feasible layouts for the twelve work centres. The work centres have been numbered and labelled according to the specification in Table 3.1. For example, the two work centres 1 and 2, that are 300 ft^2 and are labelled "2B" and "12L" on the diagram, are placed vertically; since if placed horizontally, they would obstruct the movement in the aisle. The six work centres,

[1] The preliminary results obtained with the GRG solver were presented at the virtual conference in 2020, of the Serbian OR Society, wherein a participant suggested the use of the *Gurobi* optimizer.

Table 3.1 Work centre area

Number	Work centre (W)	Area (ft^2)
1	A	200
2	B	300
3	C	100
4	D	200
5	E	100
6	F	200
7	G	200
8	H	100
9	I	200
10	J	200
11	K	100
12	L	300

that are 200 ft^2 in area, (A, D, F, G, I, J), have been placed horizontally, in the second column, and they are labelled, “1A”, “4D”, “6F”, “7G”, “9I” and “10J”, in the diagram. The four work centres that are 100 ft^2, (C, E, H, K), have been placed below the two work centres B and L, and are labelled, “3C”, “5E”, “8H” and “11K”. There are other feasible placements possible, with the twelve given work centres, but they are not more compact, in the sense of occupying less overall area, than this one. The rectilinear distance matrix obtained from them will vary marginally from these chosen layouts. Since my focus is more on facility location, than on facility layout, I have not shown all possible layouts. Indeed all feasible layouts for the existing plot, could be listed and searched. Rectilinear distances between the work centres can be calculated from the centroids of the work centres. This is the more appropriate metric, compared to the Euclidean norm, since there may be automatic vehicles and robots, that move along straight lines such as along the aisle, and turn into the work centres at right angles to the aisle. In the distance matrix, the distance to the centre of the aisle, from the centroid of the cell, and also the distance along the aisle have been added. I have also assumed that all material handling and movement is within the confines of the factory perimeter, and do not allow parts to be conveyed outside a work centre, and brought in again from the outside into another work centre. There are 20 parts that need to be machined at one or more of the 12 facilities. The sequence of jobs amongst the 12 facilities is given in Table 3.2. Also, provided in this table is the load that needs to be moved. These numbers can be consolidated to obtain the overall number of parts that need to be moved between the facilities, without differentiating between the parts. This data is displayed in Table 3.3, and the values in the first row indicate how many parts are moved from Facility 1, to the other eleven facilities. The other rows correspond to the flow from the other facilities. Note, that some times, the flow of parts is from a higher numbered facility to a lower numbered facility. This is because of the machining jobs required for some parts may be required much later in the sequence, then for other parts.

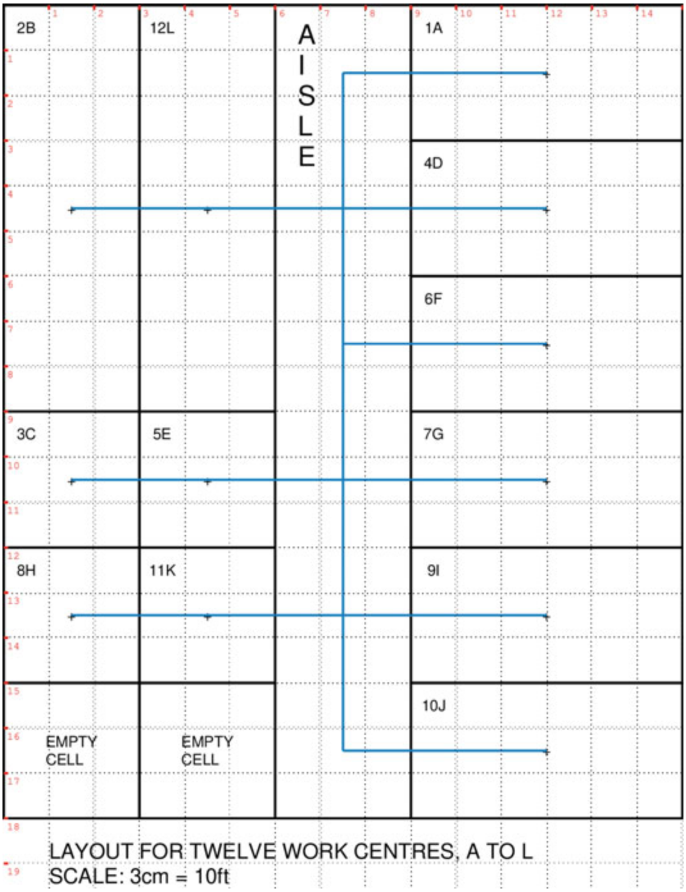

Fig. 3.1 Layout A

General Quadratic Assignment Model and Concrete Instance

Without explaining all the mathematical theory behind the model construction, I describe how the matrix in the quadratic objective function is formed. In the formulae below, f_{ir} is the fixed cost of locating facility i in work centre r. Since these fixed costs were not specified for the problem instance, it is assumed to be a constant for each r. The matrix element u_{ij} is the workflow from the work centre, i to work centre j. The matrix element d_{rs} is the rectilinear distance between work centre r and work centre s. This is a matrix of dimension $n^2 \times n^2$ for a problem involving n facilities. The elements of the matrix are obtained from the following formulae:

$$\Psi_{ij} = (\Psi_{rs}^{ij}) \qquad r = 1, 2, \ldots, n; s = 1, 2, \ldots, n \tag{3.1}$$

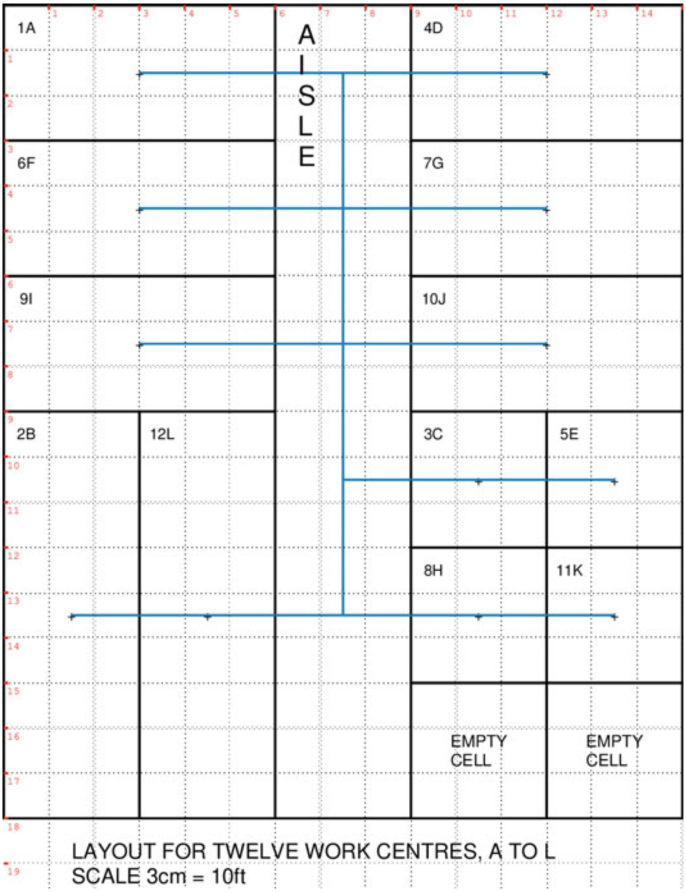

Fig. 3.2 Layout B

where if $i = j$, then

$$\Psi_{rs}^{ij} = \begin{cases} f_{ir} & \text{if } r = s \\ 0 & \text{if } r \neq s \end{cases} \tag{3.2}$$

and if $i \neq j$, then

$$\Psi_{rs}^{ij} = \begin{cases} 0 & \text{if } r = s \\ u_{ij} d_{rs} & \text{if } r \neq s \end{cases} \tag{3.3}$$

Table 3.2 Production sequence and load

Product	Sequence	Load	Product	Sequence	Load
α	1–2–6–7–8–10–12	40	λ	1–6–8–5–4–7–12	55
β	1–2–6–5–4–11–12	100	μ	1–4–5–10–11–12	35
γ	1–3–4–5–6–12	25	ν	1–2–5–6–4–7–12	65
δ	1–2–3–6–7–9–12	50	ξ	1–9–10–7–8–12	95
ϵ	1–3–5–4–8–11–12	90	o	1–8–6–7–11–5–2	150
ζ	1–3–5–4–7–10–12	30	π	1–4–6–10–8–12	20
η	1–5–3–6–7–8–12	45	ρ	1–8–11–10–7–9–12	35
θ	1–8–6–5–4–10–12	60	σ	1–9–6–5–4–10–12	45
ι	1–4–5–9–11–12	75	τ	1–2–3–4–5–6–11–12	50
κ	1–8–3–6–7–9–12	40	υ	1–2–6–8–10–11–12	40

Table 3.3 Flow matrix for parts

Fac.	1	2	3	4	5	6	7	8	9	10	11	12
1	0	345	145	130	45	55	0	285	140	0	0	0
2	0	0	100	0	65	180	0	0	0	0	0	0
3	0	0	0	75	120	135	0	0	0	0	0	0
4	0	0	0	0	185	20	150	90	0	105	100	0
5	0	0	0	380	0	140	0	0	75	35	0	0
6	0	0	0	65	205	0	325	95	0	20	50	25
7	0	0	0	0	0	0	0	180	125	30	150	120
8	0	0	40	0	55	210	0	0	0	80	90	160
9	0	0	0	0	0	45	0	0	0	95	75	125
10	0	0	0	0	0	0	0	0	0	0	40	175
11	0	0	0	0	150	0	0	0	0	35	0	390
12	0	0	0	0	0	0	0	0	0	0	0	0

The partitioned matrix Ψ has the structure as given below:

$$\begin{bmatrix} \Psi_{11} & \Psi_{12} & \dots & \Psi_{1n} \\ \Psi_{21} & \Psi_{22} & \dots & \Psi_{2n} \\ \dots & \dots & \dots & \dots \\ \Psi_{n1} & \Psi_{n2} & \dots & \Psi_{nn} \end{bmatrix}$$

Let e_n be a $n \times 1$ vector of ones and let I_n be the $n \times n$ identity matrix. Furthermore, let Φ be the $2n \times n^2$ assignment matrix:

$$\begin{bmatrix} e_n^t & & & & \\ & e_n^t & & & \\ & & e_n^t & & \\ & & & \ddots & e_n^t \\ I_n & I_n & I_n & \dots & I_n \end{bmatrix}$$

The location problem, or quadratic assignment problem that remains to be solved, is then:
Minimize $x^t \Psi x$
subject to $\Phi x = e_{2n}$
x binary.

Concrete Instance

The flow matrix given in Table 3.3 is obtained by summing all the flows between facilities, across all the twelve parts. For example, between facilities 1 and 2, note the flow of 40 units of part α, 100 units of part β, 50 units of part δ, 65 units of part ν, 50 units of part τ and 40 units of part υ, giving a total of 345 units. One can thus calculate the aggregated flow between any two facilities. For a few facility pairs, the index of the receiving facility is lower than the index of the sending facility, and these instances must also be initialized. Tables 3.4 and 3.5 give the rectilinear distance between the work centres, based upon the layout diagrams of Figs. 3.1 and 3.2.

Table 3.4 Rectilinear distance between work centres for first layout

D_{ij}	W1	W2	W3	W4	W5	W6	W7	W8	W9	W10	W11	W12
W1	–	45	65	40	55	50	60	75	70	80	65	35
W2		–	60	35	50	45	55	70	65	70	60	10
W3			–	55	10	45	35	50	45	55	40	50
W4				–	45	40	50	65	60	70	55	25
W5					–	35	25	40	35	45	30	40
W6						–	40	55	50	60	45	35
W7							–	45	40	50	35	45
W8								–	35	45	10	60
W9									–	40	25	55
W10										–	35	75
W11											–	50
W12												–

Table 3.5 Rectilinear distance between work centres for second layout

$D_i j$	W1	W2	W3	W4	W5	W6	W7	W8	W9	W10	W11	W12
W1	–	45	65	40	55	50	60	75	70	80	65	35
W2		–	60	35	50	45	55	70	65	70	60	10
W3			–	55	10	45	35	50	45	55	40	50
W4				–	45	40	50	65	60	70	55	25
W5					–	35	25	40	35	45	30	40
W6						–	40	55	50	60	45	35
W7							–	45	40	50	35	45
W8								–	35	45	10	60
W9									–	40	25	55
W10										–	35	75
W11											–	50
W12												–

Computational Results

I first ran Frontline Systems' GRG Solver, on a Lenovo desktop computer, equipped with an Intel core i3, 2.0 GHz, dual-core processor and 8 GB of RAM, to solve the mixed binary problem. The best location cost obtained in 2851.922 s (47.53 min) is 246,353. It took 3 iterations, and 364 subproblems had to be solved. The worst location cost obtained by maximizing the cost function was 285,678; for that solution, the solver took 0.515 s to obtain the solution. The algorithm took 1 iteration and solved zero subproblems. The difference between the best and the worst solution is 39,325, which is 15.96% as a percentage of the best solution. This implies, that there is significant gain from optimizing the location of the facilities, and it is not just an improvement of a few percentage points. For the alternative layout, the best solution obtained was 248,453, which is only 2100 more than the first layout. This solution took 3859.828 s (64.33 min) involving six iterations and 440 subproblems. The worst solution for the alternative layout was 293,178, and this took 3 iterations and zero subproblems. The difference between the bet and the worst is 44,725, which is 18% of the best solution. Thus it appears, that whichever layout, one chooses at the start, optimizing the assignment of machines to the work centres saves over 15%.

The other key conclusion is about the optimal assignment of facilities to work centres which is given in Table 3.6. Limited sensitivity analysis can be performed on the flow between the facilities, to determine the stability of this optimal assignment if the workflow between two facilities was to increase or decrease. This could reflect seasonal changes in the client's demand. To explore this, I took the workflow between Facility 1 and Facility 2, which is 345, and varied it in the range of 300–405. The optimal assignment of machines to work centres, was the same in four out of the eight scenarios, whereas the other four facility location assignments were all different. Thus

Table 3.6 Optimal assignment of facilities to work centres

Facility	Work centre
1	4
2	10
3	12
4	11
5	9
6	7
7	6
8	3
9	8
10	5
11	2
12	1

the workflow, between pairs of facilities, is not the sole determinant, of the optimal assignment, for then we would have got the same assignment in all eight scenarios. The interactions, between all the other pair-wise facilities, plays an important role in the stability of the solution.

Results from the Gurobi Optimizer

Convexity of the objective function in the quadratic assignment problem is an issue that should not be overlooked. This issue becomes important because one obtains different local minimums, when solving with multiple starting solutions to the algorithm. The eigenvalues of the matrix Ψ in the quadratic term $x^t \Psi x$ indicate whether the matrix is positive definite (PD) or positive semi-definite (PSD). Also, the constraint set may contain only equality constraints or may have nonlinear constraints too. These factors suggest which nonlinear solver is best suited for obtaining the solution. If some of the eigenvalues for the Q matrix are complex numbers, as we discover in this problem instance, it implies that the matrix is neither PD nor PSD. This means that the objective function is not strictly convex, and that the solution that is found by an algorithm may not be the global minimum. One way to determine if a solution to a nonlinear problem is the global minimum, is to solve the Lagrangian dual problem, and check if there is a duality gap. Since the matrix Ψ is not PSD, one cannot form Dorn's dual quadratic programme or similar Lagrangian dual and solve it. Moreover, since the variables are binary valued for this QAP, the dual integer programme is hard to interpret. A cutting plane method that is used successfully for QAPs with non-convex objective functions is the reformulation-linearization/convexification technique (RLT). The *Gurobi* solver uses RLT along

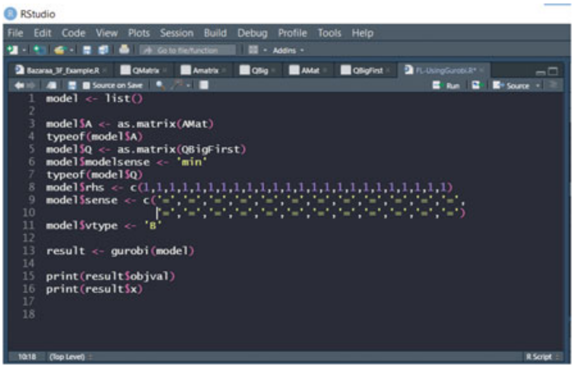

Fig. 3.3 R Script for calling Gurobi

with various other cutting plane techniques to solve the primal QAP, and also a dual formulation. Furthermore, one can observe the duality gap reduce to zero, confirming a global optimal solution.

The computer used for these experiments was a Lenovo, with an i5 processor running at 1 GHz speed, with four cores, and 16 GB RAM. The Gurobi optimizer used was version 9.1.1. The optimizer was called from the RStudio environment as in Fig. 3.3. For the first layout, the solver obtained a global optimum solution of 221,903. A total of 9,710,415 nodes were explored in 1358.68 s (22.65 min). The primal solution, in fact, converges to the optimal solution much earlier than the dual solution, in fact, when the duality gap is as high as 91.9%. This solution is much better than the minimum obtained from the GRG solver, with an origin start, which was 268,178. One also gets additional information from the Gurobi execution, about the cutting planes used—implied bounds: 8, MIR: 55, Flow Cover: 135, RLT: 2 and Relax-and-lift: 6. This indirectly reveals the structure of the class of problems. The mapping of machines to work centres is (10, 12, 4, 9, 5, 7, 6, 2, 8, 3, 1, 11). Screenshots of the programme running in RStudio are displayed in Figs. 3.4 and 3.5.

```
> result <- gurobi(model)
Gurobi Optimizer version 9.1.1 build v9.1.1rc0 (win64)
Thread count: 4 physical cores, 8 logical processors, using up to 8 threads
Optimize a model with 24 rows, 144 columns and 288 nonzeros
Model fingerprint: 0x772ae76a
Model has 6084 quadratic objective terms
Variable types: 0 continuous, 144 integer (144 binary)
Coefficient statistics:
  Matrix range     [1e+00, 1e+00]
  Objective range  [0e+00, 0e+00]
  QObjective range [2e+00, 9e+04]
  Bounds range     [0e+00, 0e+00]
  RHS range        [1e+00, 1e+00]
Found heuristic solution: objective 299228.00000
Presolve time: 0.03s
Presolved: 24 rows, 144 columns, 288 nonzeros
Presolved model has 6084 quadratic objective terms
Variable types: 0 continuous, 144 integer (144 binary)
```

Fig. 3.4 Presolve output

```
Cutting planes:
  Implied bound: 8
  MIR: 55
  Flow cover: 135
  RLT: 2
  Relax-and-lift: 6

Explored 9710415 nodes (125879767 simplex iterations) in 1370.21 seconds
Thread count was 8 (of 8 available processors)

Solution count 10: 221903 222753 224703 ... 239003

Optimal solution found (tolerance 1.00e-04)
Best objective 2.219030000000e+05, best bound 2.219030000000e+05, gap 0.0000%
> print(result$objval)
[1] 221903
> print(result$x)
  [1] 0 0 0 0 0 0 0 0 0 1 0 0 0 0 0 0 0 0 0 0 0 0 0 1 0 0 0 1 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 1 0 0
 [48] 0 0 0 0 0 1 0 0 0 0 0 0 0 0 0 0 0 0 0 1 0 0 0 0 0 0 0 0 0 0 1 0 0 0 0 0 0 0 1 0 0 0 0 0 0 0 0
 [95] 0 0 0 0 0 0 0 0 0 1 0 0 0 0 0 0 1 0 0 0 0 0 0 0 0 0 1 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0
[142] 0 1 0
>
```

Fig. 3.5 Objective value

For the second layout, the solver obtained a global optimum solution of 226,003 which is just a little higher than for the first layout. This solution explored 16,553,034 nodes in 1184.17 s (19.73 min). The primal converges to the optimal solution when the duality gap is 79.2%. This solution is also better than the corresponding minimum of 241,203 obtained from the GRG solver with an origin start, but only by about 6.7%. The additional information one obtains about the cutting planes used is—Gomory: 1, Implied Bound: 2, MIR: 1, StrongCG: 1, Flow Cover: 24 and RLT: 1. The mapping of the machines to work centres is (10, 3, 5, 9, 4, 11, 12, 1, 7, 8, 2, 6).

Managerial Insights

Solving industrial problems using systems modelling and quantitative techniques is very complex. The modelling of the system mathematically itself is a difficult task and requires ingenuity. In this case, mapping the facility location problem, to the quadratic assignment structure, is no small feat, even with a good sense for matrix dimensionality and products. The next part of solving it is equally difficult. An insight from all the numerical computation is that one can get a good solution with a generic solver, but if one uses the appropriate nonlinear solver for the mathematical model, the savings can be substantially better. Optimizing with a generic solver reduced the material handling cost by over 19% compared to the worst case location for both layouts; optimizing with the suitable solver reduces the cost obtained by the generic solver with an origin start by over 20% for the first layout and over 6% for the second layout. The main benefit of a suitable solver is that I know that the solution is the global optimum.

One limitation of the numerical results is that the sensitivity analysis output from the solvers is not comprehensive although in no sense is one suggesting that the solvers are simple. If the inputs to the problem are stochastic, then the improvement in the solution may vary too. There are probably functions available in the API, that are equivalent to the Solver Table for LP. The distance matrix is not going to change very much from layout to layout, but the volume of parts required or demand may change dynamically with the season and with business cycles, and the stability of the solution in response to such changes needs to be examined. Another limitation of the study is the issue about part priority, or hazardous nature of the part, or about the clock-wise or counter-clock-wise flow of the part in the facility. Customarily, clock-wise flow is preferred to a counter-clock-wise flow. These qualitative policy issues may be also incorporated in some way in a more elaborate quantitative model.

Conclusions

I have modelled a real instance of a facility layout/location problem, as a quadratic assignment problem. The optimal assignment can be found using an Excel add-in such as Frontline System's GRG solver or even the fastest solver in the world, from *Gurobi*. This model can be used by small and medium-sized industries whether they can afford the commercial solver or not. The distance matrix and the flow matrix can be modified easily, to do what-if analysis on the solution.

The solutions obtained from this method can be compared with the nature-inspired, metaheuristic methods, such as artificial neural networks (ANNs), genetic algorithms (GAs), particle swarm optimization, and bee colony algorithm, which are recommended for large problem instances, due to the exponential time complexity of the problem. Given the vector computing availability in modern day computers and languages, the old fashioned QAP formulation does not take as much time to be solved, as it used to. As a research question, the problem size, from where onwards, metaheuristics are better than solving quadratic assignment problems, can be investigated.

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Chapter 4
Production Inventory Model with Three Levels of Production and Demand for Deteriorating Item under Price, Stock and Advertisement Dependent Demand

Pankaj Narang, Mamta Kumari, and Pijus Kanti De

Abstract Demand rate is not constant in real-life situations. Many factors affect the demand rate like advertisement, selling price, and stock displayed in the market, and sometimes demand rate changes according to seasons and other factors, like demand for refrigerators increases in summer, but decreases in winter or demand of pharmaceutical products increases rapidly in pandemic situations. Also, some items deteriorate over time. By taking inspiration from these real-life situations, a production-inventory model with price, stock, and advertisement-dependent demand for instantaneously deteriorating items is developed here with three demand levels. To avoid shortages, three levels of production rate are considered here. We have assumed the instantaneous deterioration rate is a linear function of time. This model extends the basic EPQ model to three levels of production and demand with instantaneous deterioration items. The total cost function of the proposed model has been derived. The goal of the model is to determine the production time and production rate to the manufacturer to minimize the total average cost. To demonstrate the model's feasibility, numerical examples have been provided and solved by the graphical method. The total average cost's convexity is graphically illustrated. Sensitivity analysis is used to illustrate the findings of the suggested production inventory model and to demonstrate managerial insights. The significant finding of this work is that this model assists manufacturers in determining production rate and time according to different rates of demand. Sensitivity analysis helps them identify the critical cost parameter that significantly impacts the total average cost. This model can be used in many manufacturing processes where the demand for the product is highly volatile like seasonal products, etc.

Keywords EPQ · Price, stock and advertisement-dependent demand · Deterioration · Three-level production and demand

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Introduction

Many researchers developed production inventory models with constant demand in many earlier studies, but demand depends on various factors in actual life phenomena. For example, the demand for an item changes rapidly, like when a brand-new product is launched in the market, the demand for the previously existing product decreases. However, the product's demand is sometimes rapidly increased, like demand for the air conditioner in summer and the demand for the heater in winter.

In today's competitive market, one of the most critical factors that affect the product's demand is the selling price. It is well known that the increase in the selling price of an item causes a decrease in demand for that item and vice versa. Advertisement is the other factor that affects demand. It is observed that a product popularizes with the help of advertisements in well-known electronic or print media or by different modes to grow interest in customers' minds for their product. Demand is also affected by stock level. In today's market, it has been noticed that the demand rate is influenced by the inventory level displayed in supermarkets. i.e., the more items are shown in the supermarket increase the demand for that item. Mondal et al. [1] presented the inventory model for defective goods where selling price and advertisement cost affected demand rate with variable production cost. Palanivel et al. [2] presented a finite horizon EOQ model for non-instantaneously deteriorating goods, in which price and advertisement affected demand patterns. Bhunia et al. [3] presented a two-storage inventory model with a partial backlog for deteriorating products, where the demand rate is influenced by advertising and selling price. Geetha et al. [4] proposed a model to get an optimal replenishment policy for non-instantaneously deteriorating products when demand is affected by price and advertising. Singh et al. [5] proposed an EOQ model for deteriorating goods in which the demand of the product is stock-dependent. Jaggi [6] suggested an optimum replenishment strategy for deteriorating products with demand dependent on price and a time-varying storage cost. Panda et al. [7] used the credit policy approach to discuss an inventory model with two warehouses for deteriorating goods in which demand for the product is influenced by stock, advertisement and selling price under partial backlogging. Rathore [8] investigated a production reliability model where demand is affected by the selling price and advertising frequency.

The items that become damaged, decayed, or expired over time are known as deteriorating items. Things like food, radio-active substances, cosmetic goods, volatile items, etc., decrease because of their limited shelf-life. The inventory continuously decreases because of the market demand and little shelf life. Therefore, deterioration must be considered to find the actual inventory cost. An optimal inventory model for deteriorating goods with various market demands was explored by Yong He et al. [9]. Mukhopadhyay et al. [10] investigated joint pricing and ordering policies for deteriorating goods with a time proportional deterioration rate. Taleizadeh et al. [11] discussed a back-ordering inventory control problem for deteriorating goods with a constant deterioration rate. Manna et al. [12] created an EOQ model for deteriorating goods with time-dependent deterioration and demand rates. Maihami et al.

[13] established a combined pricing and inventory control system for deteriorating products with partial backlogs and demand reliant on price and time. An EOQ model is discussed by Ghosh et al. [14] with back-ordering for perishable goods where deterioration is dependent on both time and expiration date. Das et al. [15] explored a production lot size inventory model for a deteriorating item where the production rate constitutes regular, and overtime productions and demand depends on stock.

The primary goal of most manufacturing companies is to meet consumer demand and provide low-cost goods. The organization must have efficient inventory management and properly employ its resources to reach this goal. Sana et al. [16] created a production inventory model with time-dependent demand and a constant rate of deterioration. Rao et al. [17] established a production inventory model for deteriorating products in which the decay rate follows a Weibull distribution, and the demand rate is dependent on the production quantity. Sivashankari et al. [18] created a production inventory model that includes deteriorating products and shortages and two production levels. In 2014, Lakshmidevi et al. [19] created a three-level inventory model in which the demand rate is a quadratic function of time, and the deterioration rate is a linear function of time. Krishnamoorthi et al. [20] discussed a three-level production inventory model with deteriorating items and shortages when the demand rate is constant. Kumar et al. [21] proposed a two-level production inventory model where deterioration rate is linear and demand rate is an exponential function of time. Das et al. [22] developed an EPLS model for imperfect quality products by considering random machine failure in a fuzzy-stochastic environment where the demand rate is constant. Ghosh et al. [23] presented a supply chain coordination model for green items with selling price dependent on price, sales effort, and green-sensitive. An imperfect production model with two manufacturing plants is explored by Manna et al. [24] where demand is dependent on warranty period and CO_2 emission level.

This article presented a production inventory model where demand rate is affected by selling price, stock, and advertisement with three different rates in different time intervals for an instantaneous deteriorating item. Due to three different demand rates, three production levels are considered here to avoid shortages. It is feasible that manufacturing began at a fixed rate and then changed over time in response to the product's demand. This present model is different from existing models (Table 4.1) as we considered three different rates of demand, which are dependent on the advertisement, stock level, and selling price.

The objectives of this study are as follows:

1. The main objective of our study is to develop an EPQ model with varying demand rates that help the manufacturer determine the optimal production quantity and production time to minimize the system's total cost.
2. In this study, three different levels of production and demand are presented for different time intervals, with the deterioration rate being linearly dependent on time.
3. Nowadays, the item's demand is influenced by advertising and selling price. Considering this realistic situation, it is assumed that the product's demand rate depends on the frequency of advertisement, selling price, and inventory level.

The rest of the paper is structured as follows:

Assumptions and Notations, Mathematical formulation for the proposed model, Numerical example, Sensitivity Analysis, Conclusion.

Table 4.1 Brief literature review

Article	Inventory control system	Deterioration	Demand	Level of production
Mondal et al. [1]	EPQ	No	PD, AD(Advertisement dependent)	One
Palanivel et al. [2]	EOQ	Constant	PD, AD	–
Bhunia et al. [3]	EOQ	Constant	TD, SD, AD	–
Geetha et al. [4]	EOQ	Constant	PD, AD	–
Singh et al. [5]	EOQ	Constant	SD(Stock dependent)	–
Jaggi [6]	EOQ	Constant	PD	–
Panda et al. [7]	EOQ	Constant	SD, PD, AD	–
Rathore [8]	EOQ	Constant	SD, PD, AD	–
He et al. [9]	EPQ	Constant	Constant	–
Mukhopadhyay et al. [10]	EOQ	TD(Time dependent)	PD(Price dependent)	–
Taleizadeh et al. [11]	EOQ	Constant	Constant	–
Manna et al. [12]	EPQ	TD	TD	One
Maihami et al. [13]	EOQ	Constant	PD, TD	–
Ghosh et al. [14]	EOQ	TD	Constant	–
Das et al. [15]	EPQ	Constant	SD	One
Sana et al. [16]	EPQ	Constant	TD	–
Rao et al. [17]	EPQ	Weibull distribution	SD	One
Sivashankari et al. [18]	EPQ	Constant	Constant	Two
Lakshmidevi et al. [19]	EPQ	TD	TD	Three
krishnamoorthi et al. [20]	EPQ	Constant	Constant	Three
Narendra Kumar et al. [21]	EPQ	TD	Exponential function of time	Two
Das et al. [22]	EPQ	No	Constant	One
This paper	EPQ	Linear function of time	SD, PD, AD	Three

Motivation, Research Gap and Contribution

In real-life situations, the demand rate is not constant. Many factors influence demand, including advertising, selling price, and available stock in the market. Demand can fluctuate depending on seasons and other factors, such as demand for refrigerators increasing in the summer but decreasing in the winter, or demand for pharmaceuticals products rapidly increasing in pandemic situations. In addition, an item's value depreciates over time. Motivated by all these factors, this model is developed.

- The majority of existing production inventory models assume that the manufacturing unit's production rate is constant. However, the rate of production fluctuates depending on the product's demand.
- Analyzing the literature on the production inventory model, the demand rate is assumed to be constant. However, in reality, the demand rate varies depending on various factors such as advertisement, selling price, inventory level, etc.
- Most of the existing models assumed only one level of demand, but demand levels changes depending on seasons or when the same product is offered in the market by different companies, etc.
- The deterioration rate is not considered or assumed to be constant in most suggested work. However, this is not possible in reality.
- This suggested model can help manufacturers predict production rate and time-based on various demand rates. Sensitivity analysis can help them discover the crucial cost parameter that significantly impacts the total average cost. This model can be applied to various production processes where product demand is highly volatile.

Assumptions and Notations

The following assumptions and notations are used to develop the proposed model in this study.

Assumptions

1. The demand rate $D(A, p, I(t))$ is influenced by the frequency of advertisement (A), selling price (p) and inventory level $(I(t))$. , *i.e.*, $D(A, p, I(t)) = A^{\gamma}(a - \mathrm{bp} + \mathrm{cI}(t))$ where a, b, c, γ and A is an integer.
2. The instantaneous deterioration rate of the product is proportional to time, i.e., $\varphi(t) = \alpha + \beta t$
3. We are neglecting the second and highest powers of α and β.
4. Three different rates of demand are considered for different time intervals.

5. Three production levels are assumed to handle demand uncertainty, avoid shortages and prevent higher holding costs caused due to inventory overstock.
6. The starting and terminal inventory is nil.
7. The production rate is always higher than or equal to the sum of the demand rate and deterioration rate.
8. The production rate is constant.
9. Only one product is taken into account.
10. There have been no shortages.
11. There is no lead time.

Notations

Notations	Description
P	Rate of production per unit time
C_P	Cost of production per unit item
C_0	Setup cost
C_d	Cost of deterioration per unit item
C_h	Holding cost per unit item per unit time
C_A	Advertisement cost
$D(A, p, I(t))$	Demand rate
A	Frequency of advertisement
γ	The shape parameter of advertisement
$\varphi(t)$	Rate of deterioration
α	Deterioration parameter $(0 < \alpha \ll 1)$
β	Deterioration parameter $(0 < \beta \ll 1)$
a	Initial demand parameter
b	Price elasticity index
c	Parameter related to stock level
p	Selling price per unit item
T	Total time of inventory cycle
T_3	Production time

Mathematical Formulation of the Proposed Model

Initially, the production starts at a rate of P, the demand rate is D, and the product begins to deteriorate. During period $[T_1, T_2]$, due to some reasons, the demand rate changes to λD, so the production rate also changes to avoid shortages, and in this

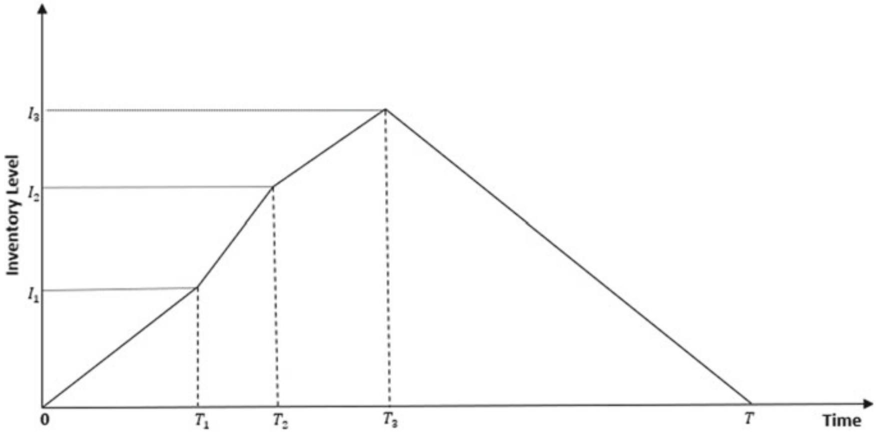


Fig. 4.1 Graphical representation of three-level production inventory model

interval, the stock level is $\lambda(P - D)$. The demand rate changes to ξD during time interval $[T_2, T_3]$, consequently production rate also changes, and the inventory level is $\xi(P - D)$. At time T_3, the production stops. During decline time $[T_3, T]$, the stock level gradually decreases due to deterioration and customers' demand. The stock level finally drops to zero at the completion of the cycle (Fig. 4.1).

The following differential equation illustrate the system in the interval [0, T]:

$$\frac{\mathrm{d}I(t)}{\mathrm{d}t} = P - D(A, p, I(t)) - \varphi(t)I(t); \qquad 0 \le t \le T_1 \tag{4.1}$$

$$\frac{\mathrm{d}I(t)}{\mathrm{d}t} = \lambda(P - D(A, p, I(t))) - \varphi(t)I(t); \quad T_1 \le t \le T_2 \tag{4.2}$$

$$\frac{\mathrm{d}I(t)}{\mathrm{d}t} = \xi(P - D(A, p, I(t))) - \varphi(t)I(t); \quad T_2 \le t \le T_3 \tag{4.3}$$

$$\frac{\mathrm{d}I(t)}{\mathrm{d}t} = -D(A, p, I(t)) - \varphi(t)I(t); \qquad T_3 \le t \le T \tag{4.4}$$

where $D(A, p, I(t)) = A^{\gamma}(a - \mathrm{bp} + cI(t)), \varphi(t) = \alpha + \beta t$

The boundary equations are $I(0) = 0$, $I(T) = 0$.

For the interval $[0, T_1]$, integrating the differential Eq. (4.1) and using boundary conditions, we get

$$\begin{aligned} I(t) =& \frac{(P - A^{\gamma}(a - \mathrm{bp}))}{A^{\gamma}c}\left[1 - e^{-A^{\gamma}\mathrm{ct}} + \alpha\left(\frac{-1}{A^{\gamma}c} + \frac{e^{-A^{\gamma}\mathrm{ct}}}{A^{\gamma}c} + te^{-A^{\gamma}\mathrm{ct}}\right)\right. \\ &\left. + \beta\left(\frac{-t}{A^{\gamma}c} + \frac{1}{(A^{\gamma}c)^2} - \frac{e^{-A^{\gamma}\mathrm{ct}}}{(A^{\gamma}c)^2} + \frac{t^2 e^{-A^{\gamma}\mathrm{ct}}}{2}\right)\right] \end{aligned} \tag{4.5}$$

Integrating the differential Eq. (4.2) for the interval $[T_1, T_2]$, we have

$$
\begin{aligned}
I(t) =&\frac{(P - A^{\gamma}(a - \mathrm{bp}))}{A^{\gamma}c}\Bigg[1 - e^{A^{\gamma}c((\lambda-1)T_1-\lambda t)} + \alpha\bigg(\frac{-1}{\lambda A^{\gamma}c} - \frac{e^{\lambda A^{\gamma}c(T_1-t)}}{A^{\gamma}c}\\
&+\frac{e^{A^{\gamma}c((\lambda-1)T_1-\lambda t)}}{A^{\gamma}c} + \frac{e^{\lambda A^{\gamma}c(T_1-t)}}{\lambda A^{\gamma}c} + te^{A^{\gamma}c((\lambda-1)T_1-\lambda t)}\bigg)\\
&+\beta\bigg(\frac{-t}{\lambda A^{\gamma}c} + \frac{1}{(\lambda A^{\gamma}c)^2} - \frac{T_1e^{\lambda A^{\gamma}c(T_1-t)}}{A^{\gamma}c} + \frac{e^{\lambda A^{\gamma}c(T_1-t)}}{(A^{\gamma}c)^2} - \frac{e^{A^{\gamma}c((\lambda-1)T_1-\lambda t)}}{(A^{\gamma}c)^2}\\
&+\frac{T_1e^{\lambda A^{\gamma}c(T_1-t))}}{\lambda A^{\gamma}c} - \frac{e^{\lambda A^{\gamma}c(T_1-t))}}{(\lambda A^{\gamma}c)^2} + \frac{t^2e^{A^{\gamma}c((\lambda-1)T_1-\lambda t)}}{2}\bigg)\Bigg]
\end{aligned}
\tag{4.6}
$$

On solving differential Eq. (4.3) for the interval $[T_2, T_3]$, we get

$$
\begin{aligned}
I(t) =&\frac{(P - A^{\gamma}(a - \mathrm{bp}))}{A^{\gamma}c}\Bigg[1 + e^{A^{\gamma}c((\lambda-1)T_1-\lambda T_2+\xi T_2-\xi t)}\bigg(-1 + \frac{\alpha}{A^{\gamma}c}\\
&-\frac{\beta}{(A^{\gamma}c)^2} + \alpha t + \frac{\beta t^2}{2}\bigg) + \frac{\beta}{\xi A^{\gamma}c}\bigg(-t + \frac{1}{\xi A^{\gamma}c}\bigg) - \frac{\alpha}{\xi A^{\gamma}c}\\
&+ e^{A^{\gamma}c(\lambda T_1-\lambda T_2+\xi T_2-\xi t)}\bigg(-\frac{\alpha}{A^{\gamma}c} + \frac{\alpha}{\lambda A^{\gamma}c} - \frac{\beta T_1}{A^{\gamma}c} + \frac{\beta}{(A^{\gamma}c)^2}\\
&+\frac{\beta T_1}{\lambda A^{\gamma}c} - \frac{\beta}{(\lambda A^{\gamma}c)^2}\bigg) + e^{\xi A^{\gamma}c(T_2-t)}\bigg(-\frac{\alpha}{\lambda A^{\gamma}c} - \frac{\beta T_2}{\lambda A^{\gamma}c}\\
&+\frac{\beta}{(\lambda A^{\gamma}c)^2} + \frac{\beta T_2}{\xi A^{\gamma}c} - \frac{\beta}{(\xi A^{\gamma}c)^2} + \frac{\alpha}{\xi A^{\gamma}c}\bigg)\Bigg]
\end{aligned}
\tag{4.7}
$$

Integrating the differential Eq. (4.4) for the interval $[T_3, T]$ and using boundary conditions, we have

$$
\begin{aligned}
I(t) =&\frac{A^{\gamma}(a - \mathrm{bp})}{A^{\gamma}c}\Bigg[-1 + \frac{1}{A^{\gamma}c}\bigg(\alpha + \beta t - \frac{\beta}{A^{\gamma}c}\bigg) + e^{A^{\gamma}c(T-t)}\,(1 + \alpha T\\
&-\frac{\alpha}{A^{\gamma}c} + \frac{\beta T^2}{2} - \frac{\beta T}{A^{\gamma}c} + \frac{\beta}{(A^{\gamma}c)^2} - \frac{\beta t^2}{2} - \alpha t\bigg)\Bigg]
\end{aligned}
\tag{4.8}
$$

The different costs associated with our proposed model are the cost of production, setup cost, advertisement cost, holding cost, and deterioration cost. These costs are determined as follows:

Production cost is calculated as

$$
\mathrm{PC} = C_P(\mathrm{PT}_1 + \lambda P(T_2 - T_1) + \xi P(T_3 - T_2)) \tag{4.9}
$$

Deterioration cost is determined as

$$
\begin{aligned}
\mathrm{DC} =& C_d\left[\int_0^{T_1}(\alpha+\beta t)I(t)\mathrm{d}t+\int_{T_1}^{T_2}(\alpha+\beta t)I(t)\mathrm{d}t+\int_{T_2}^{T_3}(\alpha+\beta t)I(t)\mathrm{d}t\right.\\
&\left.+\int_{T_3}^{T}(\alpha+\beta t)I(t)\mathrm{d}t\right]\\
=& C_d\frac{(P-A^\gamma(a-\mathrm{bp}))}{A^\gamma c}\left[-\frac{\alpha}{A^\gamma c}-\frac{\beta}{(A^\gamma c)^2}+\alpha T_3+\frac{\beta T_3^2}{2}\right.\\
&+e^{-A^\gamma cT_1}\left(\frac{\alpha}{A^\gamma c}+\frac{\beta}{(A^\gamma c)^2}+\frac{\beta T_1}{A^\gamma c}-\frac{\alpha}{\lambda A^\gamma c}-\frac{\beta}{(A^\gamma c\lambda)^2}-\frac{\beta T_1}{\lambda A^\gamma c}\right)\\
&+e^{-A^\gamma c(T_2\lambda-(\lambda-1)T_1)}\left(\frac{\alpha}{\lambda A^\gamma c}+\frac{\beta}{(A^\gamma c)^2}+\frac{\beta T_2}{\lambda A^\gamma c}-\frac{\alpha}{\xi A^\gamma c}-\frac{\beta}{(A^\gamma c\xi)^2}\right.\\
&\left.\left.-\frac{\beta T_2}{\xi A^\gamma c}\right)+e^{-A^\gamma c(\xi T_3-(\lambda-1)T_1+(\lambda-\xi)T_2)}\left(\frac{\alpha}{\xi A^\gamma c}+\frac{\beta}{(A^\gamma c\xi)^2}+\frac{\beta T_3}{\xi A^\gamma c}\right)\right]\\
&+\frac{C_d}{T}\left(\frac{A^\gamma(a-\mathrm{bp})}{A^\gamma c}\right)\left[-\alpha T+\alpha T_3-\frac{\alpha}{A^\gamma c}-\frac{\beta T^2}{2}+\frac{\beta T_3^2}{2}\right.\\
&\left.-\frac{\beta}{(A^\gamma c)^2}-\frac{\beta T}{A^\gamma c}+e^{-A^\gamma c(T_3-T)}\left(\frac{\alpha}{A^\gamma c}+\frac{\beta}{(A^\gamma c)^2}+\frac{\beta T_3}{A^\gamma c}\right)\right]
\end{aligned}
\tag{4.10}
$$

Setup cost is given by

$$
\mathrm{SC}=C_0 \tag{4.11}
$$

Advertisement Cost is given by

$$
\mathrm{AC}=C_A A \tag{4.12}
$$

Holding cost is calculated as

$$
\begin{aligned}
\mathrm{HC} =& \frac{C_h}{T}\left[\int_0^{T_1}I(t)\mathrm{d}t+\int_{T_1}^{T_2}I(t)\mathrm{d}t+\int_{T_2}^{T_3}I(t)\mathrm{d}t+\int_{T_3}^{T}I(t)\mathrm{d}t\right]\\
=& \frac{C_h}{T}\frac{(p-A^\gamma(a-\mathrm{bp}))}{A^\gamma c}\left[-\frac{1}{A^\gamma c}-\frac{\alpha T_1}{A^\gamma c}+\frac{2\alpha}{(A^\gamma c)^2}-\frac{\beta T_1^2}{2A^\gamma c}+\frac{\beta T_1}{(A^\gamma c)^2}\right.\\
&-\frac{\alpha}{\lambda(A^\gamma c)^2}+\frac{\alpha}{(\lambda A^\gamma c)^2}+\frac{\beta}{\lambda(A^\gamma c)^3}-\frac{\beta}{(A^\gamma c\lambda)^3}-\frac{\alpha T_2}{\lambda A^\gamma c}+\frac{\beta T_2}{(\lambda A^\gamma c)^2}\\
&+\frac{\alpha T_1}{\lambda A^\gamma c}-\frac{\beta T_1}{(\lambda A^\gamma c)^2}-\frac{\beta T_2^2}{2\lambda A^\gamma c}+\frac{\beta T_1^2}{2\lambda A^\gamma c}-\frac{\beta T_1}{\lambda(A^\gamma c)^2}+\frac{\beta T_1}{(\lambda A^\gamma c)^2}\\
&+T_3-\frac{\beta T_3^2}{2\xi A^\gamma c}+\frac{\beta T_2^2}{2\xi A^\gamma c}+\frac{\beta T_3}{(\xi A^\gamma c)^2}-\frac{\beta T_2}{(\xi A^\gamma c)^2}-\frac{\alpha T_3}{\xi A^\gamma c}+\frac{\alpha T_2}{\xi A^\gamma c}
\end{aligned}
$$

$$
\begin{aligned}
&+ \frac{1}{\xi A^{\gamma} c}\Bigg(-\frac{\alpha}{\lambda A^{\gamma} c} - \frac{\beta T_2}{\lambda A^{\gamma} c} + \frac{\beta}{(A^{\gamma} c\lambda)^2} + \frac{\beta T_2}{\xi A^{\gamma} c} - \frac{\beta}{\left(\xi A^{\lambda} c\right)^2} + \frac{\alpha}{\xi A^{\gamma} c}\Bigg)\\
&+ e^{-A^{\gamma} c T_1}\Bigg(\frac{1}{A^{\gamma} c} - \frac{2\alpha}{(A^{\gamma} c)^2} - \frac{\alpha T_1}{A^{\gamma} c} - \frac{\beta T_1}{\left(A^{\lambda} c\right)^2} - \frac{\beta T_1^2}{2A^{\gamma} c} - \frac{1}{\lambda A^{\gamma} c}\\
&+ \frac{\alpha}{\lambda(A^{\gamma} c)^2} + \frac{\alpha}{(\lambda A^{\gamma} c)^2} - \frac{\beta}{\lambda(A^{\gamma} c)^3} + \frac{\beta}{(A^{\gamma} c\lambda)^3} + \frac{\beta}{(\lambda A^{\gamma} c)^2}\\
&+ \frac{\alpha T_1}{\lambda A^{\gamma} c} + \frac{\beta T_1^2}{2A^{\gamma} c}\Bigg) + e^{-A^{\gamma} c(T_2\lambda-(\lambda-1)T_1)}\Bigg(\frac{1}{\lambda A^{\gamma} c} - \frac{\alpha}{\lambda(A^{\gamma} c)^2} - \frac{\alpha}{(\lambda A^{\gamma} c)^2}\\
&+ \frac{\beta}{\lambda(A^{\gamma} c)^3} - \frac{\beta}{(A^{\gamma} c\lambda)^3} - \frac{\beta}{(A^{\gamma} c\lambda)^2} - \frac{\alpha T_2}{\lambda A^{\gamma} c} - \frac{\beta T_2^2}{2\lambda A^{\gamma} c} - \frac{1}{\xi A^{\gamma} c}\\
&+ \frac{\alpha}{\xi(A^{\gamma} c)^2} - \frac{\beta}{\xi(A^{\gamma} c)^3} + \frac{\alpha}{(\xi A^{\gamma} c)^2} + \frac{\alpha T_2}{\xi A^{\gamma} c} + \frac{\beta}{(A^{\gamma} c\lambda)^3}\\
&+ \frac{\beta T_2}{(\xi A^{\gamma} c)^2} + \frac{\beta T_2^2}{2\xi A^{\gamma} c}\Bigg) + e^{-A^{\gamma} c\lambda(T_2-T_1)}\Bigg(\frac{\alpha}{\lambda(A^{\gamma} c)^2} - \frac{\alpha}{(\lambda A^{\gamma} c)^2}\\
&- \frac{\beta}{\lambda(A^{\gamma} c)^3} + \frac{\beta}{(A^{\gamma} c\lambda)^3} + \frac{\beta T_1}{\lambda(A^{\gamma} c)^2} - \frac{\beta T_1}{(\lambda A^{\gamma} c)^2} - \frac{\alpha}{\xi(A^{\gamma} c)^2}\\
&+ \frac{\alpha}{\lambda\xi(A^{\gamma} c)^2} - \frac{\beta T_1}{\xi(A^{\gamma} c)^2} + \frac{\beta}{\xi(A^{\gamma} c)^3} + \frac{\beta T_1}{\xi\lambda(A^{\gamma} c)^2} - \frac{\beta}{\xi\lambda^2(A^{\gamma} c)^3}\Bigg)\\
&+ e^{-A^{\gamma} c(\xi T_3-(\lambda-1)T_1+(\lambda-\xi)T_2)}\Bigg(1 - \frac{\alpha}{A^{\gamma} c} + \frac{\beta}{(A^{\gamma} c)^2} - \frac{\alpha}{(\xi A^{\gamma} c)^2} - \frac{\alpha T_3}{\xi A^{\gamma} c}\\
&- \frac{\beta}{(\xi A^{\gamma} c)^3} - \frac{\beta T_3}{(\xi A^{\gamma} c)^2} - \frac{\beta T_3^2}{2\xi A^{\gamma} c}\Bigg) - \frac{e^{-A^{\gamma} c(\xi T_3-\lambda T_1+(\lambda-\xi)T_2)}}{\xi A^{\gamma} c}\Bigg(-\frac{\alpha}{A^{\gamma} c} + \frac{\alpha}{\lambda A^{\gamma} c}\\
&- \frac{\beta T_1}{A^{\gamma} c} + \frac{\beta}{(A^{\gamma} c)^2} + \frac{\beta T_1}{\lambda A^{\gamma} c} - \frac{\beta}{(\lambda A^{\gamma} c)^2}\Bigg) - \frac{e^{-A^{\gamma} c\xi(T_3-T_2)}}{\xi A^{\gamma} c}\Bigg(-\frac{\alpha}{\lambda A^{\gamma} c} - \frac{\beta T_2}{\lambda A^{\gamma} c}\\
&+ \frac{\beta}{(A^{\gamma} c\lambda)^2} + \frac{\beta T_2}{\xi A^{\gamma} c} - \frac{\beta}{(\xi A^{\gamma} c)^2} + \frac{\alpha}{\xi A^{\gamma} c}\Bigg)\Bigg] + \frac{C_h}{T}\frac{(A^{\gamma}(a-bp))}{A^{\gamma} c}[-T + T_3\\
&+ \frac{\alpha T}{A^{\gamma} c} - \frac{\alpha T_3}{A^{\gamma} c} + \frac{\beta T^2}{2A^{\gamma} c} - \frac{\beta T_3^2}{2A^{\gamma} c} + \frac{\beta T_3}{(A^{\gamma} c)^2} - \frac{1}{A^{\gamma} c} + \frac{2\alpha}{(A^{\gamma} c)^2} + \frac{\beta T}{(A^{\gamma} c)^2}\\
&+ e^{-A^{\gamma} c(T_3-T)}\frac{1}{A^{\gamma} c} + \frac{\alpha T}{A^{\gamma} c} - \frac{2\alpha}{(A^{\gamma} c)^2} + \frac{\beta T^2}{2A^{\gamma} c} - \frac{\beta T}{(A^{\gamma} c)^2}\\
&- \frac{\beta T_3}{(A^{\gamma} c)^2} - \frac{\beta T_3^2}{2A^{\gamma} c} - \frac{\alpha T_3}{A^{\gamma} c}\Bigg)\Bigg)\Bigg]
\end{aligned}
\tag{4.13}
$$

Therefore, the total average cost in this proposed model is the sum of the cost of production, deterioration cost, setup cost, advertisement cost, and holding cost.

Total average cost

$$
\begin{aligned}
&= \frac{C_P}{T}(\mathrm{PT}_1 + \lambda P(T_2 - T_1) + \xi P(T_3 - T_2)) + \frac{C_d}{T}\frac{(P - A^{\gamma}(a - \mathrm{bp}))}{A^{\gamma}c}\Big[-\frac{\alpha}{A^{\gamma}c} \\
&- \frac{\beta}{(A^{\gamma}c)^2} + \alpha T_3 + \frac{\beta T_3^2}{2} + e^{-A^{\gamma}cT_1}\left(\frac{\alpha}{A^{\gamma}c} + \frac{\beta}{(A^{\gamma}c)^2} + \frac{\beta T_1}{A^{\gamma}c} - \frac{\alpha}{\lambda A^{\gamma}c}\right. \\
&\left.- \frac{\beta}{(A^{\gamma}c\lambda)^2} - \frac{\beta T_1}{\lambda A^{\gamma}c}\right) + e^{-A^{\gamma}c(T_2\lambda-(\lambda-1)T_1)}\left(\frac{\alpha}{\lambda A^{\gamma}c} + \frac{\beta}{(A^{\gamma}c)^2} + \frac{\beta T_2}{\lambda A^{\gamma}c}\right. \\
&\left.- \frac{\alpha}{\xi A^{\gamma}c} - \frac{\beta}{(A^{\gamma}c\xi)^2} - \frac{\beta T_2}{\xi A^{\gamma}c}\right) + e^{-A^{\gamma}c(\xi T_3-(\lambda-1)T_1+(\lambda-\xi)T_2)}\left(\frac{\alpha}{\xi A^{\gamma}c}\right. \\
&\left.\left. + \frac{\beta}{(A^{\gamma}c\xi)^2} + \frac{\beta_3}{\xi A^{\gamma}c}\right)\right] + \frac{C_d}{T}\left(\frac{A^{\gamma}(a - \mathrm{bp})}{A^{\gamma}c}\right)\Big[-\alpha T + \alpha T_3 - \frac{\alpha}{A^{\gamma}c} \\
&- \frac{\beta T^2}{2} + \frac{\beta T_3^2}{2} - \frac{\beta}{(A^{\gamma}c)^2} - \frac{\beta T}{A^{\gamma}c} + e^{-A^{\gamma}c(T_3-T)}\left(\frac{\alpha}{A^{\gamma}c} + \frac{\beta}{(A^{\gamma}c)^2}\right. \\
&\left.\left. + \frac{\beta T_3}{A^{\gamma}c}\right)\right] + \frac{C_o}{T} + \frac{C_A}{T}A + \frac{C_h}{T}\frac{(P - A^{\gamma}(a - \mathrm{bp}))}{A^{\gamma}c}\Big[-\frac{1}{A^{\gamma}c} - \frac{\alpha T_1}{A^{\gamma}c} \\
&+ \frac{2\alpha}{(A^{\gamma}c)^2} - \frac{\beta T_1^2}{2A^{\gamma}c} + \frac{\beta T_1}{(A^{\gamma}c)^2} - \frac{\alpha}{\lambda(A^{\gamma}c)^2} + \frac{\alpha}{(\lambda A^{\gamma}c)^2} + \frac{\beta}{\lambda(A^{\gamma}c)^3} \\
&- \frac{\beta}{(A^{\gamma}c\lambda)^3} - \frac{\alpha T_2}{\lambda A^{\gamma}c} + \frac{\beta T_2}{(\lambda A^{\gamma}c)^2} + \frac{\alpha T_1}{\lambda A^{\gamma}c} - \frac{\beta T_1}{(\lambda A^{\gamma}c)^2} - \frac{\beta T_2^2}{2\lambda A^{\gamma}c} \\
&+ \frac{\beta T_1^2}{2\lambda A^{\gamma}c} - \frac{\beta T_1}{\lambda(A^{\gamma}c)^2} + \frac{\beta T_1}{(\lambda A^{\gamma}c)^2} + T_3 - \frac{\beta T_3^2}{2\xi A^{\gamma}C} + \frac{\beta T_2^2}{2\xi A^{\gamma}c} + \frac{\beta T_3}{(\xi A^{\gamma}c)^2} \\
&- \frac{\beta T_2}{(\xi A^{\gamma}c)^2} - \frac{\alpha T_3}{\xi A^{\gamma}c} + \frac{\alpha T_2}{\xi A^{\gamma}c} + \frac{1}{\xi A^{\gamma}C}\left(-\frac{\alpha}{\lambda A^{\gamma}c} - \frac{\beta T_2}{\lambda A^{\gamma}c} + \frac{\beta}{(A^{\gamma}c\lambda)^2}\right. \\
&\left.+ \frac{\beta T_2}{\xi A^{\gamma}c} - \frac{\beta}{(\xi A^{\gamma}c)^2} + \frac{\alpha}{\xi A^{\gamma}c}\right) + e^{-A^{\gamma}cT_1}\left(\frac{1}{A^{\gamma}c} - \frac{2\alpha}{(A^{\gamma}c)^2} - \frac{\alpha T_1}{A^{\gamma}c}\right. \\
&- \frac{\beta T_1}{(A^{\gamma}c)^2} - \frac{\beta T_1^2}{2A^{\gamma}c} - \frac{1}{\lambda A^{\gamma}c} + \frac{\alpha}{\lambda(A^{\gamma}c)^2} + \frac{\alpha}{(\lambda A^{\gamma}c)^2} - \frac{\beta}{\lambda(A^{\gamma}c)^3} \\
&\left.+ \frac{\beta}{(A^{\gamma}c\lambda)^3} + \frac{\beta}{(\lambda A^{\gamma}c)^2} + \frac{\alpha T_1}{\lambda A^{\gamma}c} + \frac{\beta T_1^2}{2A^{\gamma}c}\right) + e^{-A^{\gamma}c(T_2\lambda-(\lambda-1)T_1)}\left(\frac{1}{\lambda A^{\gamma}c}\right. \\
&- \frac{\alpha}{\lambda(A^{\gamma}c)^2} - \frac{\alpha}{(\lambda A^{\gamma}c)^2} + \frac{\beta}{\lambda(A^{\gamma}c)^3} - \frac{\beta}{(A^{\gamma}c\lambda)^3} - \frac{\beta}{(A^{\gamma}c\lambda)^2} - \frac{\alpha T_2}{\lambda A^{\gamma}c} \\
&- \frac{\beta T_2^2}{2\lambda A^{\gamma}c} - \frac{1}{\xi A^{\gamma}c} + \frac{\alpha}{\xi(A^{\gamma}c)^2} - \frac{\beta}{\xi(A^{\gamma}c)^3} + \frac{\alpha}{(\xi A^{\gamma}c)^2} + \frac{\alpha T_2}{\xi A^{\gamma}c} \\
&\left.+ \frac{\beta}{(A^{\gamma}c\lambda)^3} + \frac{\beta T_2}{(\xi A^{\gamma}c)^2} + \frac{\beta T_2^2}{2\xi A^{\gamma}c}\right) + e^{-A^{\gamma}c\lambda(T_2-T_1)}\left(\frac{\alpha}{\lambda(A^{\gamma}c)^2} - \frac{\alpha}{(\lambda A^{\gamma}c)^2}\right. \\
&- \frac{\beta}{\lambda(A^{\gamma}c)^3} + \frac{\beta}{(A^{\gamma}c\lambda)^3} + \frac{\beta T_1}{\lambda(A^{\gamma}c)^2} - \frac{\beta T_1}{(\lambda A^{\gamma}c)^2} - \frac{\alpha}{\xi(A^{\gamma}c)^2} \\
&\left.+ \frac{\alpha}{\lambda\xi(A^{\gamma}c)^2} - \frac{\beta T_1}{\xi(A^{\gamma}c)^2} + \frac{\beta}{\xi(A^{\gamma}c)^3} + \frac{\beta T_1}{\xi\lambda(A^{\gamma}c)^2} - \frac{\beta}{\xi\lambda^2(A^{\gamma}c)^3}\right) \\
&+ e^{-A^{\gamma}c(\xi T_3-(\lambda-1)T_1+(\lambda-\xi)T_2)}\left(1 - \frac{\alpha}{A^{\gamma}c} + \frac{\beta}{(A^{\gamma}c)^2} - \frac{\alpha}{(\xi A^{\gamma}c)^2} - \frac{\alpha T_3}{\xi A^{\gamma}c}\right. \\
&\left.- \frac{\beta}{(\xi A^{\gamma}c)^3} - \frac{\beta T_3}{(\xi A^{\gamma}c)^2} - \frac{\beta T_3^2}{2\xi A^{\gamma}c}\right) - \frac{e^{-A^{\gamma}c(\xi T_3-\lambda T_1+(\lambda-\xi)T_2)}}{\xi A^{\gamma}c}\Big(-\frac{\alpha}{A^{\gamma}c}
\end{aligned}
$$

$$+\frac{\alpha}{\lambda A^{\gamma}c}-\frac{\beta T_1}{A^{\gamma}c}+\frac{\beta}{(A^{\gamma}c)^2}+\frac{\beta T_1}{\lambda A^{\gamma}c}-\frac{\beta}{(\lambda A^{\gamma}c)^2}\Bigg)$$
$$-\frac{e^{-A^{\gamma}c\xi(T_3-T_2)}}{\xi A^{\gamma}c}\Bigg(-\frac{\alpha}{\lambda A^{\gamma}c}-\frac{\beta T_2}{\lambda A^{\gamma}c}+\frac{\beta}{(A^{\gamma}c\lambda)^2}+\frac{\beta T_2}{\xi A^{\gamma}c}-\frac{\beta}{(\xi A^{\gamma}c)^2}$$
$$+\frac{\alpha}{\xi A^{\gamma}c}\Bigg)\Bigg]+\frac{C_h}{T}\frac{(A^{\gamma}(a-\mathrm{bp}))}{A^{\gamma}c}\Bigg[-T+T_3+\frac{\alpha T}{A^{\gamma}c}-\frac{\alpha T_3}{A^{\gamma}c}+\frac{\beta T^2}{2A^{\gamma}c}$$
$$-\frac{\beta T_3^2}{2A^{\gamma}c}+\frac{\beta T_3}{(A^{\gamma}c)^2}-\frac{1}{A^{\gamma}c}+\frac{2\alpha}{(A^{\gamma}c)^2}+\frac{\beta T}{(A^{\gamma}c)^2}+e^{-A^{\gamma}c(T_3-T)}\Bigg(\frac{1}{A^{\gamma}c}$$
$$+\frac{\alpha T}{A^{\gamma c}}-\frac{2\alpha}{(A^{\gamma}c)^2}+\frac{\beta T^2}{2A^{\gamma}c}-\frac{\beta T}{(A^{\gamma}c)^2}-\frac{\beta T_3}{(A^{\gamma}c)^2}-\frac{\beta T_3^2}{2A^{\gamma}c}-\frac{\alpha T_3}{A^{\gamma}c}\Bigg)\Bigg] \tag{4.14}$$

Equation (4.14) is our required total average cost. We aim to minimize this cost. This equation is highly non-linear and solved by Mathematica 11.2 software, and to show the convexity graphically, a 3D graph is directed.

Numerical Example

To demonstrate the feasibility of our proposed model a numerical example is created by considering the following values:

$$C_o=350,\ C_h=2,\ C_P=20,\ C_A=50,\ C_d=3,\ a=100,\ b=1.5,\ c=5,$$
$$\gamma=0.3,\ \alpha=0.2,\ \beta=0.8,\ p=40,\ A=9,\ T_1=0.6,\ T_2=1.1,\ T=2.$$

We take four different cases for λ and ξ.

1. The demand rate increases by 1.4 in the interval $[T_1, T_2]$, and the demand rate doubles in the interval $[T_2, T_3]$.
2. In this case, for the interval $[T_1, T_2]$, the demand rate becomes 0.7 to the initial demand rate, and in interval $[T_2, T_3]$, the demand rate increases by 1.4 to the initial demand rate.
3. In the interval $[T_1, T_2]$, the demand rate increases, and in the next interval, the demand rate decreases.
4. In this case, the demand rate decreases for both intervals, and the values are shown in Table 4.2.

Table 4.2 Different cases of λ and ξ

Cases	λ	ξ
1	1.4	2
2	0.7	1.4
3	1.4	0.7
4	0.6	0.8

Table 4.3 Optimum values of P and T_3

Cases	P	T_3	Total average cost
1	200	1.48	5006.36
2	200	1.56	3827.6
3	200	1.63	3881.78
4	200	1.56	3176.42

The optimal values of P and T_3 and the minimum total average cost are calculated by considering these initial values, and Table 4.3 shows the results.

Observations

When we compare the results of these four cases, we observed that

1. In all cases, the production rate is identical.
2. In case 1, the total average cost is maximum compared with other cases because the demand rate increases in both intervals, resulting in more production cost. However, the total average cost is minimum is case 4 as we assumed that the demand rate decreases in both intervals.
3. The production time is maximum in case 3 and minimum in case 1.
4. In case 2 and case 4, the rate and time of production are the same, but the total average cost is different.

The convexity of the total average cost with respect to production rate and production time is shown in Figs. 4.2, 4.3, 4.4, and 4.5. We noticed that the total average cost and the production time varies in all four situations, but the production rate is the same.

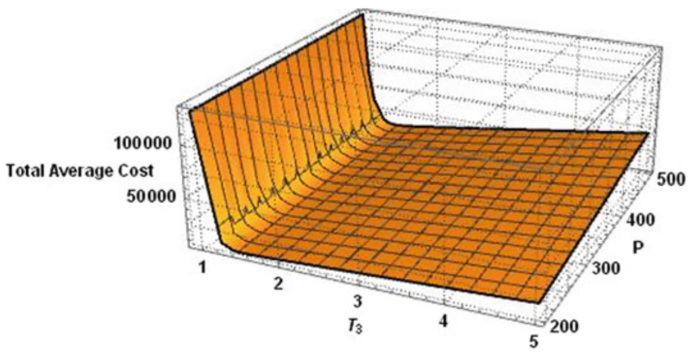


Fig. 4.2 Total average cost w.r.t. production rate (P) and production time (T_3) for case 1

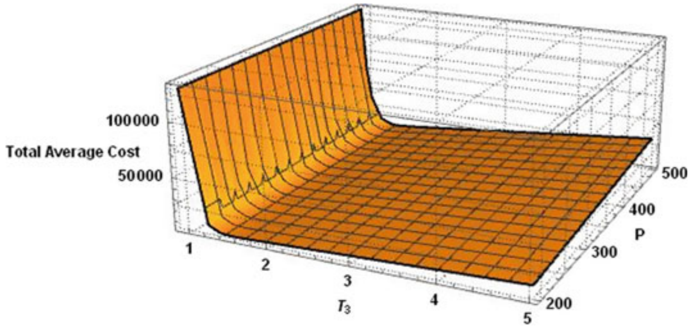


Fig. 4.3 Total average cost w.r.t. production rate (P) and production time (T_3) for case 2

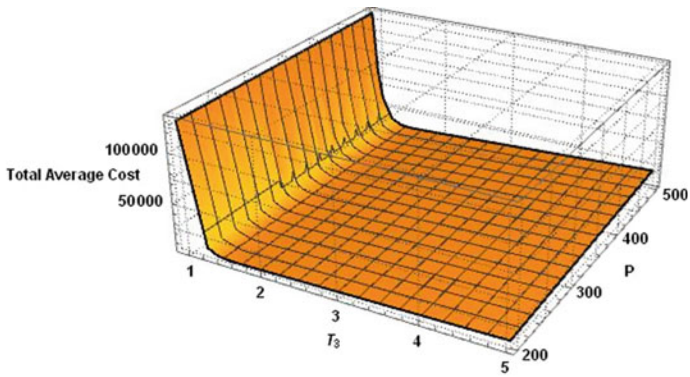


Fig. 4.4 Total average cost w.r.t. production rate (P) and production time (T_3) for case 3

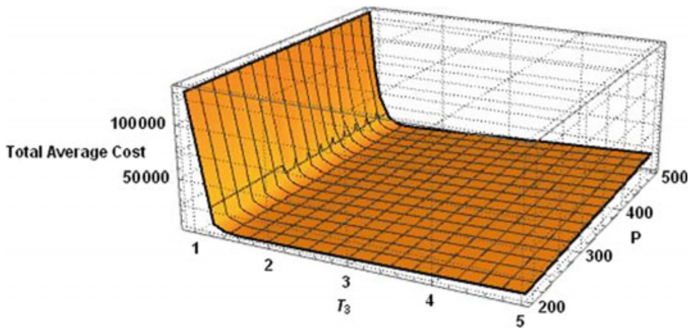


Fig. 4.5 Total average cost w.r.t. production rate (P) and production time (T_3) for case 4

Sensitivity Analysis

We conducted a detailed sensitivity analysis to examine the impact of changes in the numerical values used in the previous example. By changing the value of each parameter from − 20 to 20%, one at a time, this analysis is performed with the values of the other parameters being unchanged. The sensitivity analysis is carried out for case 1.

See Figs. 4.6, 4.7, 4.8, and 4.9 for graphical sensitivity analysis.

Observations

From the Table 4.4, we observed the following changes and compared them with the optimal solution we get in the above example: -

1. From the first four cases and Figs. 4.6, 4.7, 4.8, and 4.9, we conclude that when any cost related to the system decreases, then the total average cost also decreases, and when the cost increases, the total average cost increases, which satisfies the real-life phenomenon. However, the optimal values of P and T_3 do not change, indicating that the system's cost has no impact on the optimal solution.
2. In the case of advertisement frequency(A), when A's value decreases, the total cost also decreases. This indicates that by decreasing the advertisement frequency, the demand for the product decreases, which results in reducing production time. When A's value increases, the total cost and production time increase as the demand increases, but the production rate does not change.
3. When the value of 'a' decreases, it is observed that the total average cost and production time also decrease. It indicates that when demand decreases, the production of items decreases, resulting in decreased production time and total cost. When the value of 'a' rises, demand rises, resulting in more production time.
4. When the value of price elasticity index 'b' decreases, the product's demand increases, increasing production time and total cost, but the production rate is the same. As the value of 'b' rises, the demand for the product rises, reducing production time and total cost.
5. When the value of 'c' reduces, the demand for the product decreases. As a result, the production time decreases, and the total average cost decreases. When the value of 'c' rises, the demand increases, which means the production time increases, but the production rate for all values of c is the same. This indicates that we must fix the production rate to reduce the total cost.

Table 4.4 Sensitivity analysis of above example with respect to different parameters

Parameter	% change in parameter	P	T_3	TAC
C_h	− 30	200	1.48	4949.24
	− 20	200	1.48	4968.28
	− 10	200	1.48	4987.32
	+ 10	200	1.48	5025.4
	+ 20	200	1.48	5044.44
	+ 30	200	1.48	5063.49
C_o	− 30	200	1.48	4953.86
	− 20	200	1.48	4971.36
	− 10	200	1.48	4988.86
	+ 10	200	1.48	5023.86
	+ 20	200	1.48	5041.36
	+ 30	200	1.48	5058.86
C_A	− 30	200	1.48	4938.86
	− 20	200	1.48	4961.36
	− 10	200	1.48	4983.86
	+ 10	200	1.48	5028.86
	+ 20	200	1.48	5051.36
	+ 30	200	1.48	5073.86
C_d	− 30	200	1.48	4924.43
	− 20	200	1.48	4951.74
	− 10	200	1.48	4979.05
	+ 10	200	1.48	5033.67
	+ 20	200	1.48	5060.98
	+ 30	200	1.48	5088.29
A	− 30	200	1.41	4765.59
	− 20	200	1.48	4864.76
	− 10	200	1.48	4933.94
	+ 10	200	1.48	5082.06
	+ 20	200	1.48	5161.09
	+ 30	200	1.55	5209.84
a	− 30	200	1.34	4458.96
	− 20	200	1.41	4737.85
	− 10	200	1.48	4906.18
	+ 10	200	1.48	5106.54
	+ 20	200	1.56	5166.56
	+ 30	200	1.56	5210.47
b	− 30	200	1.56	5157.78
	− 20	200	1.48	5126.58
	− 10	200	1.48	5066.47
	+ 10	200	1.48	4946.25
	+ 20	200	1.48	4886.15
	+ 30	200	1.41	4780.8

(continued)

Table 4.4 (continued)

Parameter	% change in parameter	P	T_3	TAC
c	− 30	200	1.34	4511.84
	− 20	200	1.41	4716.25
	− 10	200	1.41	4894.9
	+ 10	200	1.56	5135.59
	+ 20	200	1.56	5214.29
	+ 30	200	1.56	5322.89

Fig. 4.6 Effect of C_h on total average cost

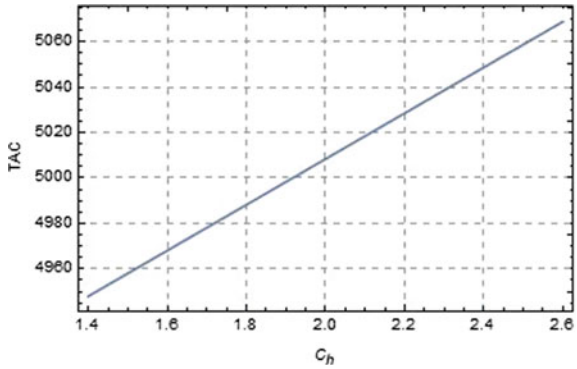
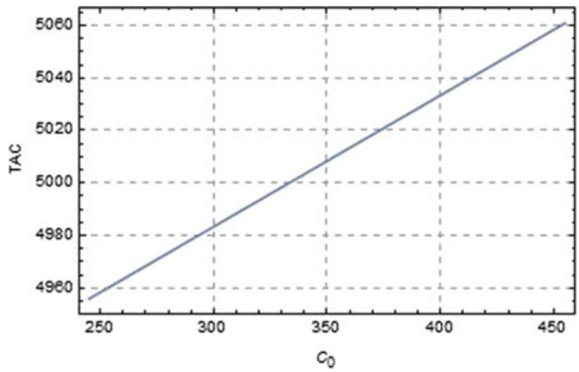


Fig. 4.7 Effect of C_0 on total average cost



Managerial Insights

This work provides valuable insights for an EPQ model, in which different actions can be taken at different phases. We evaluated various aspects of real-life situations like variable demand rate with variable production rate and deteriorating item. Managers can use this model to help them choose the optimal alternative for minimizing the total average cost. According to our study, from the above mention costs, deterioration cost is more sensitive than any other cost, i.e., a slight change in the deterioration

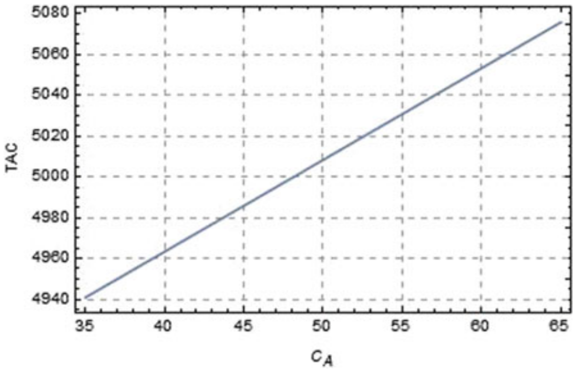


Fig. 4.8 Effect of C_A on total average cost

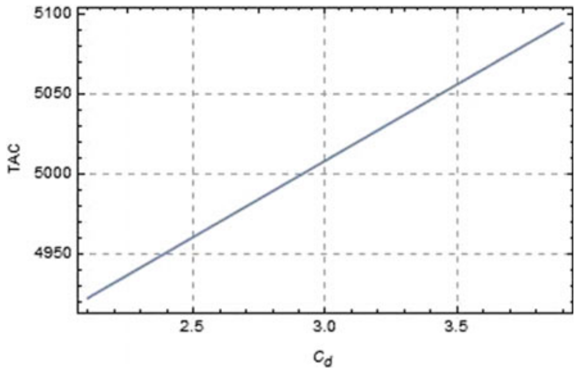


Fig. 4.9 Effect of C_d on total average cost

cost has a more significant impact on the total average cost of the system, and stock level displayed in the market has a crucial impact on the sales of the product.

Our results find the optimized production rate and time to minimize the system's total cost.

Our findings show that the frequency of advertisements has a considerable impact on the total average cost.

Conclusion

This article develops a production-inventory model for a deteriorating item with different levels of production and demand where demand rate is affected by selling price, advertisement, and stock. In general, the demand for a product is not constant, its changes according to seasons, time, etc. Therefore, to avoid shortages, the production rate changes over time according to the demand of the product. Nowadays, companies give advertisements on T.V, newspapers, etc., to increase the demand for their product. When supermarkets display products in large quantities in the showroom, customers express a strong urge to buy more. In this paper, the product's deterioration rate is considered as a function of time since, in real life, the product deteriorates with time. We considered the total duration of the cycle to be fixed. The total cost function is developed under these circumstances. The decision variables of our proposed model are production time and production rate that assist the manufacturer in reducing the total cost of the system. This function is highly nonlinear; it cannot be solved theoretically. So, we have utilized Mathematica 11.2 software to solve this problem. A numerical example has been provided with four different cases to illustrate the proposed model. Sensitivity analysis of some essential parameters has been performed. This proposed model helps the manufacturer determine production time, production rate, and total cost according to the demand of the product. In the future, this model extends by taking imperfect production processes with multiple items and by considering the reworking process for the defective items. It can be further extended by formulating an EPQ model with shortages. Also, the n-level of production can be included in this EPQ model.

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Chapter 5
Optimal Inventory Management Policies for Substitutable Products Considering Non-instantaneous Decay and Cost of Substitution

Ranu Singh and Vinod Kumar Mishra

Abstract Substitution is an important key of real retail business market. Nowadays, most of retailers offer similar product to the customer in out-of-stock period. This study presents an inventory management approach for substitutable non-instantaneous decaying products under joint ordering policy. This study deals all the possible cases: when a product is completely depleted due to requirement/demand and decay, its demand is partially met by other products, and remain unsatisfied demand is lost. In this article, demand and deterioration rate are taken as deterministic type. The purpose of this article is to find the optimal ordering quantities to optimize the total cost. This article presents solution methodology and provides a numerical simulation to illustrate the model's application. The numerical result shows a significant reduction in overall cost in the case of with substitution over without substitution case. Finally, the sensitivity of key parameters is examined and concludes this study with some future plans.

Keywords Substitutable product · Joint replenishment · Inventory management · Non-instantaneous decay

Introduction

In current retail business market, the incidence of partial stock-outs in the types of usually purchasing products is a very common phenomenon. In certain situations, it is regular for purchasers who are looking to purchase a particular product to buy a similar product in order to save time, rather of traveling to another supermarket/retail shop to find the first product. A product is replacement for other product only if it

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can be used in just the same way and follows the same requirements. In the case of alternatives, an increment in the cost of one of the products rises the substitute demand. Anupindi et al. [1] presented a study result in which he observed that 82–88% of the customer would be interested to purchase substitute goods if the requested goods are out of stock. In retail shops, substitution is a regular happening event for example between different brand of toothpaste, Coke and Pepsi, milk, etc. The effect of deterioration is very common in products, which play crucial role in the decision of ordering quantity. Hence, it would be beneficial for retailers to consider the effect of deterioration with substitution into account, when making inventory policies. To avoid extra transportation cost (ordering cost), and supplier cost, several organizations try to refill products jointly to take profit of joint ordering. In this research work, we present an inventory model having two non-instantaneous substitutable decaying products under joint ordering policy for a retailer perspective.

There is large number of literature available on product substitution, deterioration with joint replenishment separately. McGillivray and Silver [2] was the earliest researcher to establish an inventory model with similar cost of substitute product and shortage cost. Drezner et al. [3] presented an economic order quantity (EOQ) model with substitution and compared the output with no substitution. Salameh et al. [4] developed an inventory strategy by considering substitute product with joint ordering policy and show that substitution between items can be saved in a fixed order price, which is free from the quantity of items involved in the order. Krommyda et al. [5] expanded the previous model by incorporating a joint ordering strategy and a substitution rate ranging from 0 to 1. Many researchers categorized substitution into different parts. Tang and Yin [6] divided substitution into some categories: stock-out, price, and assortment-based substitution. They investigate how under variable and fixed pricing policies, the order size and selling price of two substitute products could be jointly determined by the retailer. A brief literature review on substitutable products in the inventory is given by [7–11].

Whitin [12] first investigated the inventory problem of deteriorating products, considering fashion products deteriorating at the end of storage interval. [13–18] presented extensive literature on decaying/deteriorating items in inventory system. Mishra [19] generalized Salameh et al. [4] model by taking deterioration of product into account. Tiwari et al. [20] developed an ordering policy inventory model for non-instantaneous deteriorating item with different realistic situation.

Based on the above literature review and our knowledge, no study has considered the effect of non-instantaneous deterioration for substitutable products. To fill this literature gap, authors have considered two mutually substitutable products considering non-instantaneous decay and the cost of substitution. The cost of substitution includes the promotional effects like (cost comes from advertisements and sales promotions). The objective of this study is to obtain the optimal ordering quantities to minimize the overall costs. A comparative study has been done in the with substitution and without substitution cases. A solution methodology is developed to determine the optimal values of decision parameters. The nature of the overall cost function is also discussed. The comparative study and gap analysis of the relevant paper are presented in Table 5.1.

Table 5.1 Comparative study and gap analysis

Researcher (s)	Model	Substitutable item	Cost of substitution	Non-instantaneous decaying product	Joint replenishment	Objective function
McGillivray and Silver [2]	EOQ	✓				Cost
Anupindi et al. [1]	–	✓				–
Gurler and Yilmaz [8]	–	✓				Profit
Maity and Maiti [10]	EOQ	✓				Profit
Krommyda et al. [5]	EOQ	✓			✓	Cost
Mishra [19]	EOQ	✓	✓		✓	Cost
Tiwari et al. [20]	EOQ			✓		Cost
Mishra and Mishra [9]	EOQ	✓	✓			Cost
Singh and Mishra [21]	EOQ	✓		✓	✓	Cost
This article	EOQ	✓	✓	✓	✓	Cost

The remaining part of this article is ordered as follows: The assumptions and notations of the study, model description and formulation, solution method, numerical application, and sensitivity of key variables are given in section-wise, respectively. The last section represents the conclusion of the research.

Assumptions and Notations

Assumptions

i. Lead time is assumed to be zero.
ii. Two substitutable products are taken, and each are ordered jointly in every cycle.
iii. Deterioration and demand rates are constant, and each product has distinct deterioration rates.
iv. Both products are partially substitutable, if one of them is out of stock, then the other can be a substitute for that product.
v. Substitution cost includes promotional effects like (cost comes from advertisements and sales promotions) are considered.

Notations

Symbol	Description
D_1, D_2	Demand rate of product 1 and product 2
C_1, C_2	Unit purchase cost of product 1 and product 2
θ_1, θ_1	Deterioration rate of product 1 and product 2
A_1, A_2	Setup cost for product 1 and product 2
$Q_i, i = 1, 2$	Stock level of product 1 and product 2
h	Holding cost of each product
$\alpha_i, i = 1, 2$	Substitution rate for product 1 and product 2
$\pi_i, i = 1, 2$	Unit lost sale cost of product 1 and product 2
$\mathrm{CS}_{12}, \mathrm{CS}_{21}$	Unit substitution cost for product 1 and product 2
l	Stock level of product when other product is completely depleted
s	Part of period when substitution occur
$I_i^1(t), i = 1, 2$	Stock level of product i when first product finishes before second
$I_3^1(t)$	Stock level of product 2 when first product finishes before second and substitution takes place
$\mathrm{TC}i(Q_1, Q_2), i = 1, 2$	Average total cost for case 1 and case 2
$\mathrm{TC}_{\mathrm{NS}}(Q_1, Q_2)$	Average total cost for case 3

Model Description and Formulation

Based on assumptions, we assume two substitutable non-instantaneous decaying products. At starting, the retailer's orders Q_1 and Q_2 quantity (unit) of product 1 and 2 jointly. The stock level of each product consumes due to demand in time period $[0, t_1]$ and deplete due to deterioration and demand in period $[t_1, t_2]$ and after that substitution occurs. Both products replenish jointly after consumption to acquire the benefits of joint ordering. At $t = t_2$, three possibilities arise as follows.

Case 1: Product 1 fully consumes before product 2, i.e., if product 1 is out of stock at time t_2, as seen in Fig. 5.1, then requirement of product 1 is partially satisfied by product 2 with substitution rate α_1 and rest demand for product 1 is lost with rate of $(1 - \alpha_1)$. The stock level of products is depicted by the given below differential equation, as seen in Fig. 5.1.

$$\frac{\mathrm{d}I_1^1(t)}{\mathrm{d}t} = -D_1; \quad 0 \le t \le t_1 \tag{5.1}$$

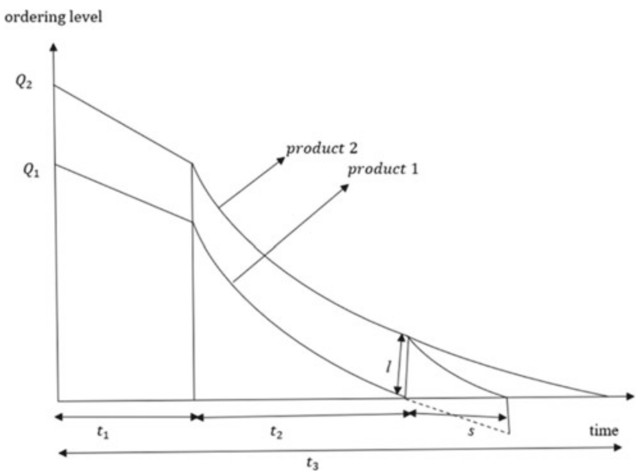


Fig. 5.1 Stock size for case 1 at time t

$$\frac{\mathrm{d}I_1^1(t)}{\mathrm{d}t} + \theta_1 I_1^1(t) = -D_1; \quad t_1 \le t \le t_2 \tag{5.2}$$

With $I_1^1(0) = Q_1$, $I_1^1(t_2) = 0$.

$$\frac{\mathrm{d}I_2^1(t)}{\mathrm{d}t} = -D_2; \quad 0 \le t \le t_1 \tag{5.3}$$

$$\frac{\mathrm{d}I_2^1(t)}{\mathrm{d}t} + \theta_2 I_2^1(t) = -D_2; \quad t_1 \le t \le t_2 \tag{5.4}$$

With $I_2^1(0) = Q_2$, $I_2^1(t_1) = Q_2 - D_2 t_1$, $I_2^1(t_2) = l$.

$$\frac{\mathrm{d}I_3^1(t)}{\mathrm{d}t} + \theta_2 I_3^1(t) = -(D_2 + \alpha_1 D_1); \quad t_2 \le t \le t_2 + s \tag{5.5}$$

With $I_3^1(t_2) = l$, $I_3^1(t_2 + s) = 0$.
The result of Eqs. (5.1–5.5) are

$$I_1^1(t) = Q_1 - D_1 t; \quad 0 \le t \le t_1 \tag{5.6}$$

$$I_1^1(t) = \frac{D_1}{\theta_1}\left(e^{\theta_1(t_2 - t)} - 1\right); \quad t_1 \le t \le t_2 \tag{5.7}$$

$$I_2^1(t) = Q_2 - D_2 t; \quad 0 \le t \le t_1 \tag{5.8}$$

$$I_2^1(t) = -\frac{D_2}{\theta_2} + e^{\theta_2(t_1-t)}\left(Q_2 - D_2t_1 + \frac{D_2}{\theta_2}\right); \quad t_1 \le t \le t_2 \tag{5.9}$$

$$I_3^1(t) = \frac{D_2 + \alpha_1 D_1}{\theta_2}\left(e^{\theta_2(t_2+s-t)} - 1\right); \quad t_2 \le t \le t_2 + s \tag{5.10}$$

Total cost of product 1 and product 2 during each cycle consists purchase, setup, and holding cost with their emissions cost and can be symbolized as

$$\mathrm{TC11}(Q_1, Q_2) = A_1 + C_1Q_1 + h\left(Q_1t_1 + \frac{D_1}{\theta_1}(t_1 - t_2) + \frac{D_1}{\theta_1^2}\left(e^{\theta_1(t_2-t_1)} - 1\right) - \frac{D_1t_1^2}{2}\right) \tag{5.11}$$

$$\begin{aligned} TC12(Q_1, Q_2) = A_2 + C_2Q_2 + (h)\Bigg(& Q_2t_1 + \frac{D_2}{\theta_2}(t_1 - t_2) \\ & + \frac{(Q_2\theta_2 - D_2t_1\theta_2 + D_2)}{\theta_2^2}\left(1 - e^{\theta_2(t_1-t_2)}\right) - \frac{D_2t_1^2}{2}\Bigg) \end{aligned} \tag{5.12}$$

The lost sale costs of product 1 per length cycle

$$= \pi_1(1-\alpha_1)D_1 \frac{\ln\left(\frac{((-D_2t_1+Q_2)\theta_2+D_2)e^{\theta_2(t_1-t_2)}+\alpha_1D_1}{D_2+\alpha_1D_1}\right)}{\theta_2} \tag{5.13}$$

Substitution cost due to product 1 partially replaced by product 2 per length cycle

$$= \mathrm{CS}_{12}\alpha_1D_1 \frac{\ln\left(\frac{((-D_2t_1+Q_2)\theta_2+D_2)e^{\theta_2(t_1-t_2)}+\alpha_1D_1}{D_2+\alpha_1D_1}\right)}{\theta_2} \tag{5.14}$$

Thus, the average total costs $\mathrm{TC}(Q_1, Q_2)$ for case 1 per unit time

$$\begin{aligned} \mathrm{TC1}(Q_1, Q_2) = \frac{1}{t_1+t_2+s}\Bigg[& A_1 + A_2 + C_1Q_1 + C_2Q_2 + h\Bigg\{Q_1t_1 + \frac{D_1}{\theta_1}(t_1 - t_2) \\ & + \frac{D_1}{\theta_1^2}\left(e^{\theta_1(t_2-t_1)} - 1\right) - \frac{D_1t_1^2}{2} + Q_2t_1 + \frac{D_2}{\theta_2}(t_1 - t_2) \\ & + \frac{(Q_2\theta_2 - D_2t_1\theta_2 + D_2)}{\theta_2^2}\left(1 - e^{\theta_2(t_1-t_2)}\right) - \frac{D_2t_1^2}{2} \\ & + \frac{(D_2 + \alpha_1D_1)}{\theta_2^2}\left(e^{\theta_2 s} - 1 - \theta_2 s\right)\Bigg\} \\ & + \pi_1(1-\alpha_1)D_1 \frac{\ln\left(\frac{((-D_2t_1+Q_2)\theta_2+D_2)e^{\theta_2(t_1-t_2)}+\alpha_1D_1}{D_2+\alpha_1D_1}\right)}{\theta_2} \\ & + \mathrm{CS}_{12}\alpha_1D_1 \frac{\ln\left(\frac{((-D_2t_1+Q_2)\theta_2+D_2)e^{\theta_2(t_1-t_2)}+\alpha_1D_1}{D_2+\alpha_1D_1}\right)}{\theta_2}\Bigg] \end{aligned} \tag{5.15}$$

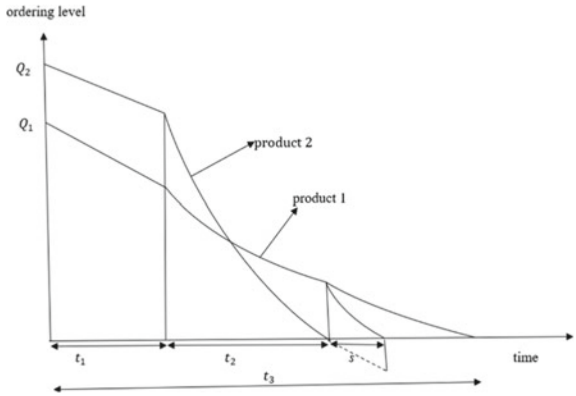


Fig. 5.2 Stock size for case 2 at time t

Case 2: Product 2 fully consumes before product 1, that means, if product 2 is out of stock at time t_2, as seen in Fig. 5.2, then requirement of product 2 is partially met by product 1 with substitution rate α_2 and rest demand for product 2 is lost with rate $(1-\alpha_2)$. The mathematical formation of case 2 is identical to case 1. The models are depicted in Fig. 5.2.

Therefore, the average total costs $\text{TC2}(Q_1, Q_2)$ of case 2 is

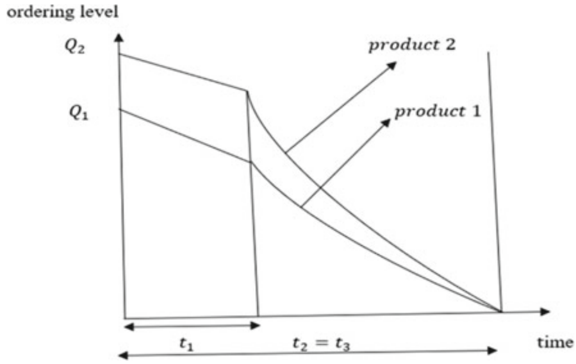
$$\begin{aligned}\text{TC2}(Q_1, Q_2) =& \frac{1}{t_1 + t_2 + s}\Bigg[A_1 + A_2 + C_1 Q_1 + C_2 Q_2 + h\Bigg\{Q_1 t_1 + \frac{D_1}{\theta_1}(t_1 - t_2) \\ &+ \frac{D_2}{\theta_2^2}\left(e^{\theta_2(t_2-t_1)} - 1\right) - \frac{D_1 t_1^2}{2} + Q_2 t_1 + \frac{D_2}{\theta_2}(t_1 - t_2) \\ &+ \frac{(Q_1\theta_1 - D_1 t_1 \theta_1 + D_1)}{\theta_1^2}\left(1 - e^{\theta_1(t_1-t_2)}\right) - \frac{D_2 t_1^2}{2} \\ &+ \frac{(D_1 + \alpha_2 D_2)}{\theta_1^2}\left(e^{\theta_1 s} - 1 - \theta_1 s\right)\Bigg\} + \pi_2(1 \\ &- \alpha_2) D_2 \frac{\ln\left(\frac{((-D_1 t_1 + Q_1)\theta_1 + D_1)e^{\theta_1(t_1-t_2)} + \alpha_2 D_2}{D_1 + \alpha_2 D_2}\right)}{\theta_1} \\ &+ \text{CS}_{21}\alpha_2 D_2 \frac{\ln\left(\frac{((-D_1 t_1 + Q_1)\theta_1 + D_1)e^{\theta_1(t_1-t_2)} + \alpha_2 D_2}{D_1 + \alpha_2 D_2}\right)}{\theta_1}\Bigg] \end{aligned} \tag{5.16}$$

Case 3 (without substitution): The stock level of each product is totally consumed at t_2, i.e., no substitution happens, as shown in Fig. 5.3. The stock level of each product is depleted at $t_2 = t_3$, i.e., $Q_1/D_1 = Q_2/D_2$ as displayed in Fig. 5.3.

Therefore, the average total costs $\text{TC}_{\text{NS}}(Q_1, Q_2)$ for case 3 is

$$\text{TC}_{\text{NS}}(Q_1, Q_2) = \frac{1}{t_1 + t_2}[A_1 + A_2 + (C_1)Q_1 + (C_2)Q_2 + (h)((Q_1 + Q_2)t_1$$

Fig. 5.3 Stock size for case 3 at time t



$$+\left(\frac{D_1}{\theta_1}+\frac{D_2}{\theta_2}\right)(t_1-t_2)-(D_1+D_2)\frac{t_1^2}{2}+\frac{D_1}{\theta_1^2}\left(e^{\theta_1(t_2-t_1)}-1\right)$$
$$+\frac{D_2}{\theta_2^2}\left(e^{\theta_2(t_2-t_1)}-1\right)\Bigg)\Bigg] \tag{5.17}$$

Solution Procedure

To obtain most desirable ordering quantity from overall cost function in the subsequent phase due to highly nonlinear function, we have graphically shown that the nature of overall cost expression is strictly convex nature. We develop an algorithm to achieve the most desirable ordering quantity.

Algorithm to achieve the optimal overall cost and ordering quantity.

Step 1: Solve the constraint-based problem

$$\min_{Q_1,Q_2} \mathrm{TC1}(Q_1, Q_2) \text{ Subject to, } \frac{Q_1}{D_1} \le \frac{Q_2}{D_2}$$

Step 2: Solve the constraint-based problem

$$\min_{Q_1,Q_2} \mathrm{TC2}(Q_1, Q_2) \text{ Subject to, } \frac{Q_1}{D_1} \ge \frac{Q_2}{D_2}$$

Step 3: To get optimum value, choose minimum between above two steps, i.e.,

$$\min[\mathrm{TC1}(Q_1, Q_2),\ \mathrm{TC2}(Q_1, Q_2)].$$

Numerical Example and Sensitivity Analysis

In this section, we first establish numerical experiment that demonstrates the implementation of proposed inventory model. The value of parameter of the numerical example is $C_1 = 3$, $C_2 = 2.5$, $A_1 = 300$, $A_2 = 250$, $h = 2$, $t_1 = 0.20$, $\theta_1 = 0.50$, $\theta_2 = 0.55$, $D_1 = 200$, $D_2 = 40$, $\alpha_1 = 0.2$, $\pi_1 = 0.4$, and $CS_{12} = 2$. We solve the above optimization problem with the above-developed algorithm and software maple. The outcomes of step 1 are $\text{TC1}(Q_1, Q_2) = 1288.56$, $Q_1 = 92.41$, and $Q_2 = 91.07$, and the outcomes of step 2 are $\text{TC2}(Q_1, Q_2) = 1350.79$, $Q_1 = 171.21$, and $Q_2 = 27.27$. On comparing both of outcomes in step 3 of the algorithm, step 1 outcomes dominance to the optimal values. Hence, the optimal total costs for with substitution is 1288.56 and optimal ordering quantities is $\left(Q_1^*, Q_2^*\right) = (92.41, 91.07)$. The optimal ordering quantities and overall cost for without substitution case is $(Q_1, Q_2) = (163.29, 32.65)$ and $\text{TC}_{\text{NS}}(Q_1, Q_2) = 1358.39$.

Comparing the outcomes of with substitution and without substitution, the beneficial difference in optimal total cost is 69.83 and the benefits of substitution over without substitution in percentage is 5.14%.

To investigate the essence of the total cost, we draw the overall cost function for product 1 and product 2 with different order quantities. Figure 5.4 shows the optimal total cost expression is strictly convex nature. Decision-making situations rarely remain fixed in the real world; thus, the parameters of the model change so, we perform sensitivity analysis numerically by varying the value of fixed parameter. Table 5.2 shows the numerical output of sensitivity analysis. The results of changes in parameters and percentage change are illustrate in Figs. 5.5, 5.6, 5.7 and 5.8.

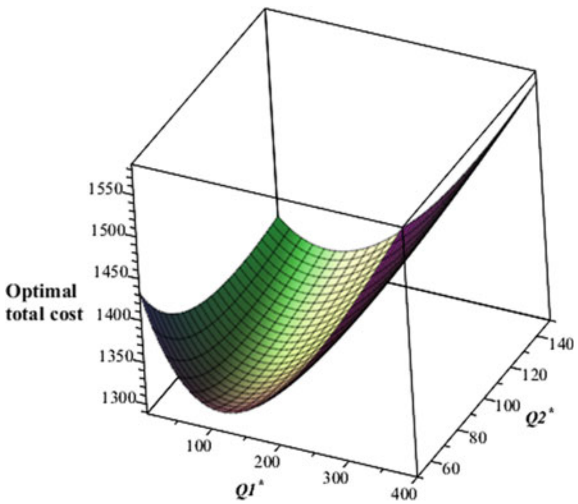


Fig. 5.4 Optimal overall cost versus optimal total quantities ($Q1^*$, $Q2^*$)

Table 5.2 Sensitivity analysis when key parameters change

Parameter	Data of parameter	With substitution			Without substitution			% Improvement in optimal total cost
		Q_1*	Q_2*	TC_d^2*	Q_1*	Q_2*	TC_d^{NS}*	
C_1	2.5	134.01	70.13	1243.71	173.91	34.78	1269.72	2.05
	3	92.41	91.07	1288.56	163.29	32.65	1358.39	5.14
	3.5	57.34	105.90	1316.47	153.50	30.70	1445.17	8.91
	4	27.36	116.14	1331.71	144.41	28.88	1530.13	12.97
C_2	2	84.84	101.56	1252.56	165.34	33.06	1340.81	6.58
	2.5	92.41	91.07	1288.56	163.29	32.65	1358.39	5.14
	3	100.26	81.24	1322.42	161.27	32.25	1375.89	3.89
	3.5	108.42	71.90	1354.07	159.28	31.85	1393.32	2.82
h	1	129.61	105.12	1222.37	201.83	40.36	1272.25	4.17
	2	92.41	91.07	1288.56	163.29	32.65	1358.39	5.14
	3	72.16	80.83	1341.08	139.24	27.84	1428.26	6.03
	4	59.43	73.09	1385.19	122.51	24.50	1487.33	6.85
A_1	250	92.41	80.04	1248.91	151.16	30.23	1303.32	3.92
	300	92.41	91.07	1288.56	163.29	32.65	1358.39	5.14
	350	92.41	101.43	1325.81	174.87	34.97	1410.89	6.10
	400	92.41	111.25	1361.12	185.97	37.19	1461.21	6.87
π_1	3	38.65	118.90	1189.72	163.29	32.65	1358.39	12.42
	4	92.41	91.07	1288.56	163.29	32.65	1358.39	5.14
	5	146.32	51.65	1346.32	163.29	32.65	1358.39	0.89
	6	166.01	33.20	1351.67	163.29	32.65	1358.39	0.49
CS_{12}	2	92.41	91.07	1288.56	163.29	32.65	1358.39	5.14
	3	105.87	82.47	1307.46	163.29	32.65	1358.39	3.75
	4	119.34	73.10	1323.65	163.29	32.65	1358.39	2.56
	5	132.82	62.88	1336.78	163.29	32.65	1358.39	1.59
α_1	0.2	92.41	91.07	1288.56	163.29	32.65	1358.39	5.14
	0.4	125.42	71.23	1339.41	163.29	32.65	1358.39	1.40
	0.6	165.86	33.17	1351.80	163.29	32.65	1358.39	0.49
	0.8	165.95	33.19	1351.97	163.29	32.65	1358.39	0.47
t_1	0.10	84.88	126.09	1407.07	201.05	40.21	1547.97	6.69
	0.20	92.41	91.07	1288.56	163.29	32.65	1358.39	4.54
	0.30	100.58	59.70	1174.95	135.36	27.07	1198.46	2.64
	0.40	109.35	31.22	1064.56	116.01	23.20	1066.48	1.08

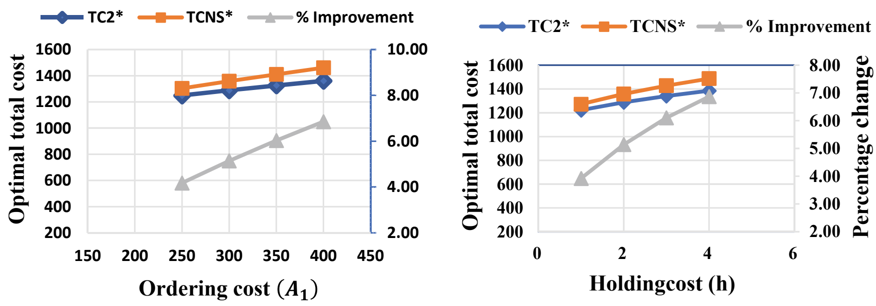


Fig. 5.5 Sensitivity analysis with variation in ordering cost (A_1) and holding cost (h)

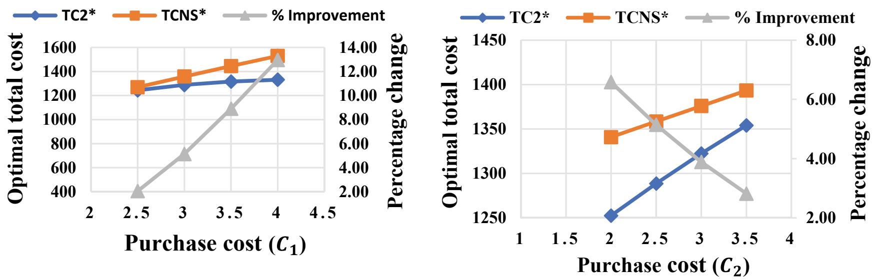


Fig. 5.6 Sensitivity analysis with variation in purchase cost (C_1, C_2)

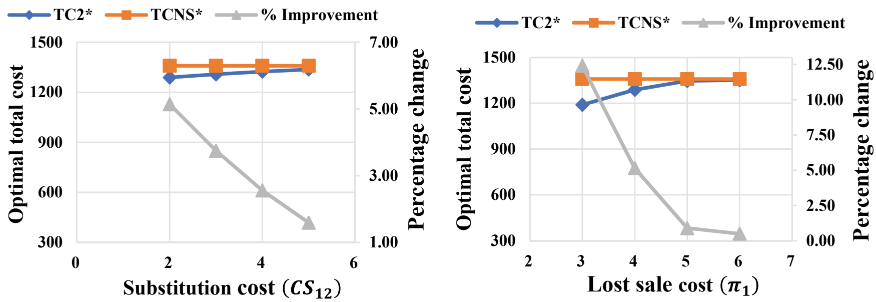


Fig. 5.7 Sensitivity analysis with variation in substitution cost (CS_{12}) and lost sale cost (π_1)

Conclusion

This study presents an inventory management model for two non-instantaneous decaying substitute products under joint ordering policy in each ordering cycle. At any time, if one of product is out of stock, a fraction of its demand may be met using the stock of the other product and other undesirable demand goes to loss. The suggested resulting model and solution method for determining the optimal

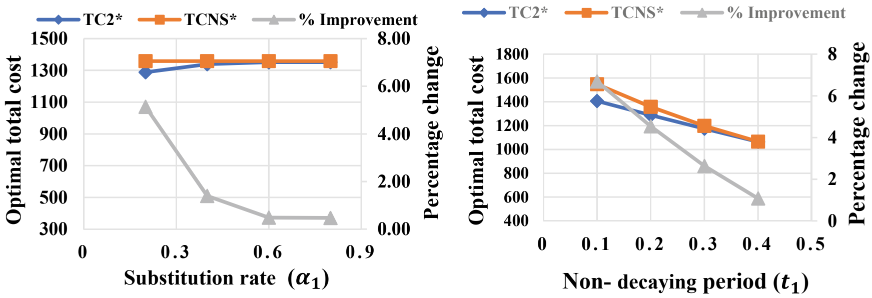


Fig. 5.8 Sensitivity analysis with variation in substitution rate (α_1) and non-decaying period (t_1)

ordering strategy is easy to understand and enforce. We first derive the overall cost expressions and illustrate graphically that they are convex. The numerical example demonstrates the application of the model, and the numerical output provides the percentage profitable improvement in overall cost, comparing with substitution and without substitution. This suggested that the impact of substitution with joint replenishment is a significant consideration that a retailer should consider into account, when picking inventory selections.

As the present model is restricted with two products which can be extended for multi-products. Another expansion of this study may be the consideration of trade credit systems and coordination between manufacturers and retailers. This study can also be extended for fuzzy stochastic environment, screening process, quality discounts, promotional effect, trade credit policies, and several other practical situations [22, 23].

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Chapter 6
A Decision Support System for Supplier Selection in Public Procurement: A Case of Banaras Locomotive Works, Varanasi

Gynaesh Tripathi and Ajinkya N. Tanksale

Abstract This study presents a mixed-method approach for the problem of supplier selection in the context of public procurement. A recent tender invited by BLW, a unit of Indian Railways producing 'diesel-electric' and electric locomotives for Indian Railways and export market for the propulsion system (the costliest and most important item in the assembly of the locomotive), has been considered in this work. Initially, the criteria for evaluation of vendors are identified through the survey of experts from Indian Railways, who are engaged in procurement for the organization. These criteria are used to calculate the inter-se ranking of the offers received against the procurement tender. For the purpose of evaluation, the weights of the attributes are calculated using the Analytic Hierarchy Process (AHP) by taking the inputs for relative comparison of the attributes from experts through a questionnaire-based survey. Then, the ranking of the offers, as well as the relative importance of offers, is calculated using the TOPSIS method. Further, a mixed-integer programming (MIP) problem has been formulated to ensure the optimal order allocation in the multi-sourcing environment with an objective of maximizing the total value of the purchase. The model is populated with constraints that limit the order quantity, like selection of suppliers, the order quantity, minimum order quantity, and minimum number of suppliers to be selected. Results of the case are presented considering various scenarios. As a result of the study, it has revealed that the multi-sourcing of the suppliers for a specific item in given situation has an additional penalty on the organization. The effective and optimal use of the resources can be ascertained with the help of the proposed study. The biggest advantage of the study is engagement of potential suppliers, evaluators, and the end user. Continual system improvement is additional and inherent advantage reaped through proposed model.

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Keywords Supplier selection · Public procurement · Indian railways · AHP · TOPSIS · MIP

Introduction

Supplier is the most important entity of any business and right selection of supplier is always a vital issue for the management. The selection of suppliers is a cumbersome process attempting to identify, and evaluate various parameters associated with supply chain and enter into agreement with suppliers for procurement of goods or service [1]. The appropriate supplier decision has long-term financial and strategic implications on the success of a business. The primary objective in the supply selection process is to make a purchase decision in a manner that all the challenges of supply chain management are effectively mitigated. Various risks associated with supplier's performance such as cost of failure in supply chain and cost associated with quality of the product/service being purchased need to be taken care, while making decisions for supplier's selection. In recent years, the business environment has undergone a fundamental change incorporating various factors (both internal and external) in the purchase process. Therefore, the criticality of the decision of supplier selection has become a point of concern in the business decision process.

Out of all the procurement practices employed in the business, public procurement is a special type of purchase exercise, where taxpayer's money is used to purchase goods and services. By definition the public procurement is the purchase done by the government and state-owned enterprises of goods, services, and works. Public procurement is a necessary evil for any public agency or department. For the limited sectors where the government produces final goods and services like railways and defense, procurement of raw material is essentially vital. In fact, procurement of goods and services is the primary channel through which money is passed on to private hands. The procurement in this case is accomplished through taxpayers' money and the high standards of conduct and impartial selection of the vendor become critical in ensuring high quality of services or goods being procured. That is why it is prone to misconduct and malpractices [2]. It is very frequently seen that the possibility of favoritism and manipulations thrives in the public procurements. The quality of the procurement is one of an inextricable area of leverage and possible misconducts and malpractices in public procurements. This makes the process of public procurement not only complex, but also marred with lack of transparency and technical inability for appropriate assessment of the quality of procurement. The complex environment and lack of responsibility have potential to promote the propensity for malpractices. Given all these factors in the play, the procurement decisions need to be taken with utmost care to prevent fraud, waste of money, corruption, or local protectionism. To prevent such possibilities, the government take numerous measures to ensure fairness in public procurement and the public procurement is regulated through various laws and regulations issued from time to time. These laws envisage procurement agencies to issue a public notice declaring the intention of the agencies to exercise the procurement process in an impartial manner and without any prejudice.

In public procurement the end user and the purchaser are usually different parties with different and even sometimes with conflicting priorities. The accountability of the procurement personnel is limited to the procurement exercise. Further, the parameters governing the procurement process and the ability of the suppliers are assessed by a different and unconnected agency. These factors influence the procurement process at basic level and develop tendencies for indulgence in compromises resulting in various complications in the entire process. Therefore, to avoid any biased to creep in, it demands a quantitative basis for the decision-making process which can be monitored, quantified, and improved as per the organizational goal. It calls for a systematic approach or a tailor-made decision support system to facilitate the supplier selection process. Present work is an attempt to develop such a support system for public procurement for one of the manufacturing units of Indian Railways to promote free and fair process with equal opportunity to every eligible supplier. This study proposes a mixed-method approach for the problem of supplier selection in the public procurement. In this study, we have solicited and incorporated the opinions of experts to identify the important criteria for supplier selection. To evaluate the weight of such identified criteria again a survey was conducted engaging experts from the field. AHP was applied for calculating the weights of these criteria identified in the survey. The weights so calculated are used with TOPSIS method to find the inter-se ranking of the offers received in response to a tender. Finally, a mix-integer linear program is used to decide the quantity to be placed on the firm, with compliance to government guidelines. Applicability of the proposed framework is demonstrated in the setting of procurement system of locomotive in Banaras Locomotive Works. Thus, we contribute to the practice of the supplier selection in public procurement. Although there are several studies that have adopted a mixed-method approach for supplier selection, to the best of our knowledge, this is the first attempt to do so for a very practical problem faced by the Indian Railways. The use of AHP, TOPSIS, and mixed Integer linear programming for solving complex problem with compliance to intricate Government's policy is the novelty of this study.

The rest of the article is organized as follows. A review of the extant literature is presented in Section "Literature Review". The problem description is given in Section "Problem Description". Section "Optimization Program for Multi-supplier Selection Through Open Tender" contains the mathematical model for optimization of total value of the procurement. The procedure to calculate inter-se ranking of the offer and relative weight is given in Section "Estimating Relative Vendor Ranking". A case study of the Banaras Locomotive Works is shown with experimental results in Section "The Case Study". Finally, Section "Conclusions" concludes the work.

Literature Review

Supplier selection has always been a topic of interest and a crucial decision parameter in supply chain decisions. The research on various aspects of supplier selection can be traced back to the early 1960s. This literature was summarized by Weber et al. [3]. Ghodsypour and O'Brien [4] have provided insightful research in supplier selection. Karpak et al. [5] discussed one of the 'user-friendly' alternative multiple criteria decision support systems—visual interactive goal programming (VIG). Nag [6] has examined issues, such as organizational structure, procurement organization, source selection methodology, procurement oversight and regulation and their impact on the economy, efficiency, transparency, and accountability aspects of procurement in Indian Railways. Various studies have been conducted to identify and include various parameters for supplier selection in different situations. Abdolshah [7] has presented a review of decision criteria reported in the literature for supporting the supplier selection process. Ellram [8] has studied the situation where the firm is considering partnership type of the relation. They have developed four categories of the factors for evaluating the supplier's potential. Goffin et al. [9] have stated that supplier management is one of the key issues of supply chain management because the cost of raw materials and component parts constitutes the main cost of a product and most of the firms must spend a considerable amount of their sales revenues on purchasing. Kilincci and Onal [10] have noticed that the selection is the one of the most significant decisions in the supply chain. For selection methodology, there have been numerous studies conducted on various areas. Ghobadian et al. [11] have proposed a computer-based vendor rating system. Segura and Maroto [12] have proposed a multiple criteria supplier segmentation using outranking and value function methods. An expert Systems with Applications was proposed by An et al. [13]. Önüt et al. [14] has proposed an analytical approach to supplier selection and used AHP as a selection tool. Deng et al. [15] have proposed supplier selection using AHP methodology extended by D numbers (D-AHP). Dweiri et al. [16] have designed a decision support system for supplier selection using AHP for the automotive industry in developing countries. Similarly, Önüt et al. [17] have used combined ANP and TOPSIS for long-term supplier selection problem in telecommunication company in GSM sector under the fuzzy environment where vagueness and subjectivity are handled with linguistic terms parameterized by triangular fuzzy numbers. Liao and Kao [18] proposed integrated fuzzy techniques for order similarity to ideal solution, TOPSIS and Multi-choice goal programming approach to solve supplier selection problem. A Fuzzy-Racsh-based COPRAS-G method was proposed by Chatterjee and Kar [19] for supplier selection in telecom industry. Multi-criteria analysis of supply chain risk management using interval valued fuzzy TOPSIS was proposed by [20]. Extension of TOPSIS and VIKOR methods was proposed by Si et al. [21]. Sharma et al. [22] have used multi-criteria decision-making techniques for prioritizing stations of Indian Railways. A Delphi-AHP TOPSIS-based framework for prioritization of intellectual capital indicators was proposed by Sekhar et al. [23]. Ramos et al. [24] propose to apply a multi-criteria decision model based on the

FAHP (Fuzzy Analytic Hierarchy Process) method for the selection of suppliers in a food company. Polat [25] has used AHP and PROMETHEE for selection of subcontractors in an international project. Mohanty and Mohanty [26] have proposed an efficient hybrid model for selection of best car models. The hybrid model constitutes Fuzzy AHP and PROMETHEE II techniques as both are best MCDM techniques for solving decision problems efficiently and economically. Kubt [27] has proposed a model for supplier selection using genetic algorithms and fuzzy-AHP. The proposed work is inspired from the various works cited above.

However, it is distinct from them in the following ways. (1) The criteria for evaluation have been identified after consultation from the experienced personnel having wide experience of procurement and technical evaluation of tender offers. (2) The calculation of the weights for these identified criteria is also done through surveys engaging the expert in the field of public procurement. (3) The study offers a mixed-integer program through which various regulations and guidelines form the government from time to time can be complied. Additionally, (4) It offers insight into the implications of various guidelines and their application cost on the procurement agencies. Therefore, the study may also be used as a tool for decision-making on application of various rules and ability to tweak the decision based on the mathematical model and helps in making informed decisions.

Problem Description

These challenges of free and fair procurement to ensure the best quality and genuine suppliers are always uphill task for the government agencies. These challenges become more pronounced for the simple reason of a non-incentivized procurement system for the persons engaged in procurement. The current system is marred with the lack of administrative capacity to handle these challenges. In practice while evaluating the offer, various methods are adopted such as Quality-Based Selection, Quality-Cost-Based Selection, and Least Cost Selection [28]. The least cost selection is the most popularly adopted method for general finalization of the tender called for procurement of goods and services. In recent years, the systems of procurement have been improved by incorporating concepts of vendor approval based on technical ability and capacity/capability assessments done by authorized agencies [6]. But this concept also inherits certain drawbacks and is crippled with its susceptibility for human judgments and manipulations to favor one or another supplier. Quantification of the subjective responses like numbers of failures, post-supply performance, and warranty obligation, while evaluating technical capability of the firm, have always been the bone of contention. The subjectivity has always manifested itself through compromise on the quality of the procurement or with the ability of the suppliers to dominate the process by finding a path to circumvent the rules and procedures. The parties involved in the process including supplier and the rating agencies evaluating the vendors, also have their limitations, on ability to distinguish the fine variations of the parameters in consideration. Other methods for ensuring free procurement of goods and services adopted in recent years are two packet systems and reverse auction systems, Indian [29]. These systems are effectively being used to ensure the targeted procurement with transparent methods to effect the best quality purchase. However, in both systems technical assessment and final decision for placing order to specific vendor depend on the human judgment. In many cases, decisions have been challenged through litigations and legal battles ensued. In certain cases, the concerned authorities have been subjected to legal scrutiny and derision. The subjectivity of technical evaluations and the inability to translate the technical parameters in quantifiable data have always been a question. However, it is noticed in recent years that the use of technology has proved to be an effective business tool which has improved performance, productivity, and effectiveness. But, the greatest challenge in adopting and implementing technological advancement includes preference of procurement personnel for legacy systems, issues in change management, and distrust on the ability of technological implementations. The intricacies of the latest systems and lack of opportunity to adopt the technological tools only aid in disregarding the utilization of the same. This necessitates a decision support system that will assist the supplier selection and order allocation in the public procurement.

In this work, we are proposing a mathematical model for optimization of intended procurement and its value with no human intervention. Wherever any human judgment is employed in the process of evaluation, it is collection of many individuals' judgments clubbed together with the idea to minimize individual influence on the final decision outcome to ensure free fair procurement. The proposed optimization program is presented in the next section.

Optimization Program for Multi-supplier Selection Through Open Tender

Sets and indices

I Set of Suppliers indexed by i.

Parameters

V_i The capacity of the ith supplier indexed by i.
R_i The rating of the supplier indexed by i.
D The demand for the item being procured.
n Minimum number of suppliers to be selected.
α Limit for minimum order quantity.
M A large positive number.

Decision Variable

$$y_i = \begin{cases} 1, \text{if Order is palced to } i\text{th supplier} \\ 0, \text{otherwise} \end{cases}$$

X_i The Quantity of the order placed to ith Supplier.

Model

Objective Functions

$$\max \sum_i R_i X_i \tag{6.1}$$

Constrains

$$\sum_i X_i = D \tag{6.2}$$

$$X_i \leq M y_i \quad \forall i \in I \tag{6.3}$$

$$X_i \geq (\alpha D) y_i \quad \forall i \in I \tag{6.4}$$

$$X_i \leq V_i \quad \forall i \in I \tag{6.5}$$

$$\sum_i y_i \geq n \tag{6.6}$$

$$X_i \in \mathbf{Z}^{\geq}, \quad \forall i \in I \tag{6.7}$$

$$\alpha_i \in \{0, 1\}, \quad \forall i \in I \tag{6.8}$$

The Objective function maximizes the total value of purchase. Constraint (6.2) ensures that total order quantity allotted to suppliers must meet the total demand of the item. Constraint (6.3) is to ascertain that order allocation to a supplier is possible only if the supplier is selected. Constrain (6.4) ensures the compliance of the regulation that in the case the order is placed on the firm, it should be above a certain percent (α) of the total tender quantity. In this study, we have assumed the minimum limit to be 10% and it varies from organization to organization. Constraint (6.5) restricts the order allocation with respect to supplier capacity. Constraint (6.6) is requirement imposed to promote the competition and avoid the eventual crises in case of the supplier's inability to supply owing to their internal or external factors. This number of minimum vendors for the item can be strategically decided by the procuring agency taking various parameters in consideration. Constraint (6.7) imposes the limit on the integrality of the distribution as in most of the cases a fraction of item is not desired and in certain cases it is not possible to purchase a part product. Constraint (6.8) indicates the nature of the decision variable.

Estimating Relative Vendor Ranking

Identification of the Criteria for Vendor Evaluation

The model presented in this work uses the criteria identified for selection of vendors through consultation and survey of the all-possible criteria for evaluation of performance of the vendor in respect of a single item already supplied to the procuring agency in the past. In fact, interrelation between the criteria identified should encompass all the performance parameters that the purchasing firm is expecting from the supplier. In practice they make a list of suppliers who are approved based on technical ability and continuous evaluation is done to keep the performance under check. However, there is always a subjective discretion available with decision-makers particularly in a situation where the difference in various evolutionary parameters is insignificant. The quantifiable attributes in such situations are ignored and the decision process becomes prone to favoritisms.

For identification of most relevant parameters governing the overall performance of a vendor, a survey was conducted among the officers of government of India who are engaged in purchase activity and in technical evaluation of the bid received through tender.

These evaluation criteria are:

A1. Cost of the item being offered by the vendor.
A2. In service failure of the item per unit item already in service.
A3. ‘On time Delivery’ performance of the vendor in the past orders.
A4. Compliance of warranty obligation and post-delivery performance of the vendor.

Calculation of Weightage of Criteria

In our study, the ranking of the vendor is calculated in terms of these parameters and their contribution on actual values of the attributes. These attributes are gathered from the literature and further refined to fit into the context by a survey.

The attributes listed above are used to calculate the inter-se ranking of the offers received against the procurement tender. We first calculate the weights of the attributes using the Analytic Hierarchy Process (AHP) proposed by [30]. The inputs (for relative comparison of the attributes) are taken from the experts through a questionnaires-based survey. The subjective judgments of the officers are quantified using a linear scale suggested by [31] to calculate weights of the attributes. The eigenvalue method of AHP is used for calculating the individual weights given by the experts. The individual weights are then aggregated to obtain final normalized weights for attributes. The weights obtained for criteria $A1$, $A2$, $A3$, and $A4$ using AHP are 0.41268955, 0.38512504, 0.11817148, and 0.084014, respectively.

Calculation of Relative Ranking and Absolute Values for Vendor Performance on Offer

In the next step, the data is the collection of data through the offers (quotations) received and normalization. Each element of the normalized decision matrix is multiplied by the weight of criteria obtained using AHP to obtain the weighted normalized decision matrix. Having calculated the weighted normalized decision matrix, the ranking of the vendor's offers, as well as the relative quantity of importance of offers, needs to be calculated. For this purpose, we use a multi-attribute decision-making technique—'Techniques for Order of Preference Similarity to the Ideal Solution' abbreviated as TOPSIS originally proposed by [32]. The steps of the TOPSIS method are described as calculation of the Zenith (an Ideal point) and Nadir (anti-ideal point) points for each criterion, calculating the distance for each criterion point to Zenith and Nadir points and calculation of the relative closeness coefficient, i.e., importance indexes are followed as reported in the literature [32].

The final ranking and the relative importance for case in consideration are depicted in Table 6.4. A summary of overall procedure is presented in Fig. 6.1.

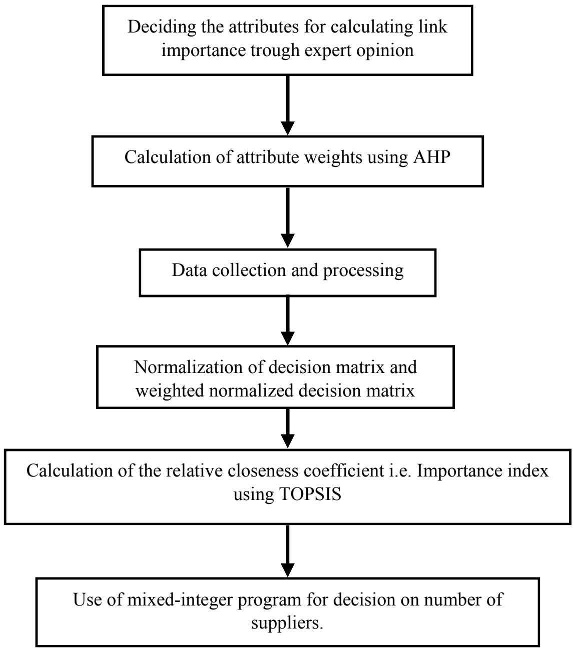

Fig. 6.1 Flow diagram depicting the steps involved in the algorithm

The Case Study

Case Description and Numeric Problem

The data for the case study has been collected form the recent tender invited by Banaras locomotive works, a unit of Indian Railways producing diesel and electric locomotives for Indian Railways and for export market. The item under study is an extremely critical item, i.e., the propulsion system which is the brain and heart of the locomotive. It is the costliest item in the complete assembly of the locomotive and costs around 20% of the total cost of the locomotive. The propulsion system is a technically complex and intricate item where firms with high caliber and proven track records are permitted to participate in the bid. The item is used with both goods and passenger locomotives. Therefore, the importance of the item is further multiplied because it impacts the safety of passengers. The propulsion system controls the functioning of complete operation and performance of the locomotive, including the braking system.

At present the supplier selection is based on the vendor directory methods, where suppliers are evaluated and enlisted as potential source for specific item, if they are found technically competent to produce the item [33]. The registration in the vendor directory is done only after successful completion of development, testing, and completion of mandatory trial without any major technical problem. Once registered, the supplier is groomed with hand-holding to bring the quality and capability to the level of expectation of Indian Railways by providing partial order for pre-defined developmental quantity. Many times, a separate tender is also called to promote more vendors for specific items where the number of approved suppliers is less than a certain number stipulated in the policy. However, there are constant disputes about the deliberations on various evaluation parameters and the procedure of evaluation of the vendors. For instance, the evaluation criterion of 'On Time Delivery' is seriously debatable. Though the vendors are penalized for delay in supplying the item, the fine is not commensurate with the consequential cost the organization has to bear for the delay. Further, the delay in supply is completely ignored while evaluating the performance of the vendor on other technical parameters. So, the criteria become a disincentive to the vendors, who are sincere and partnering with the organization at a cost to produce and deliver the items as per schedule of supply. The criteria like this are quantified in above study and evaluation is made considering the relative importance of the criteria as already explained in item no 3. Indian Railways is one of the world's largest public sector organizations, and it is the world's third largest railroad network under a single administration. It is the world's second largest railway passenger transport organization, carrying 7 billion passengers annually. It produces around 650 locomotives annually at its two manufacturing units at Chittaranjan and Varanasi. The total worth of this single item and its criticality is enough testimony for various possibilities of compromises in public procurement. The indicative data is selected for the purpose of study and to demonstrate the applicability of the concept proposed in this paper (Table 6.1).

Table 6.1 Summary of quotation received

Name of the firm	Offer cost (in Rs)	Failure/unit item in service*	On time delivery (percentage of the quantity supplied)*	Attention/failure of the item in a year*
AB	18,630,000	1.29444444	0.97	0.622318
BH	17,340,000	1.96296296	0.95	0.169811
Bomd	16,905,000	0.50420168	0.96	0.466667
CG	17,950,000	0.5	0.93	0.365854
Md	16,353,078	0.60550459	0.98	0.424242

* The data is assumed for calculation purpose

The offer received and the data collected

For calculation of weights Analytic Hierarchy Process (AHP) is used with eigenvalue method as explained in Section "Calculation of Weightage of Criteria". The weights obtained for criteria $A1$, $A2$, $A3$, and $A4$ using AHP are 0.41268955, 0.38512504, 0.11817148, and 0.084014.

Calculation of Rank Through TOPSIS

a. **Normalization of the metrics**

The performance, i.e., the actual data against each attribute criteria needs to be normalized in order to be able to compare the measure on the different units, e.g., cost in Cr. Rupees and failure are per unit items in service. The most popular method for normalization is the distributive normalization. In this method, the performances are divided by the square root of the sum of each squared element in a column (Table 6.2).

$$r_{ia} = \frac{x_{ia}}{\sqrt{\sum_{a=1}^{n} x_{ia}^2}}$$

b. Calculation of Nadir and zenith points from weightage normalized metrics. Weighted normalized matrix is obtained by multiplying the weights in each element of a column. These weighted scores will be used to compare each option to an ideal and anti-ideal option in the matrix (Table 6.3).

c. Collect the data of the best and worst performance on every criterion of the normalized decision matrix. For the ideal performance we have

$$X^+ = \left(v_1^+, v_2^2 \ldots\ldots\ldots\ldots\ldots v_m^+\right)$$

Table 6.2 Normalized matrix

Name of the firm	Offer cost (in Rs)	Failure/unit item in service	On time delivery (percentage of the quantity supplied)	Attention/failure of the item in a year
ABB	0.477354	0.511687	0.452742	0.639251
BHEL	0.4443	0.77595	0.443408	0.174432
Bombardier	0.433154	0.199308	0.448075	0.479365
CG	0.45993	0.197648	0.434073	0.375809
Medha	0.419013	0.239353	0.45741	0.435786

Table 6.3 Weighted normalized matrix

Weights	0.41269	0.385125	0.118171	0.084014
Firm	Cost	Failure/unit	On time delivery	Attention/failure
AB	0.196999	0.197064	0.053501	0.053706
BH	0.183358	0.298838	0.052398	0.014655
Bomd	0.178758	0.076759	0.05295	0.040273
CG	0.189808	0.076119	0.051295	0.031573
Md	0.172922	0.092181	0.054053	0.036612
Ideal best	0.172922	0.076119	0.054053	0.053706
Ideal worst	0.196999	0.298838	0.051295	0.014655

Table 6.4 Closeness coefficient and ranking of the offers

Firm	d_a^+	d_i^-	$d_a^+ + d_a^-$	$c_a = \frac{d_a^-}{d_a^+ + d_a^-}$	Relative ranking of the offers
AB	0.109031	0.123319	0.23235	0.469	4
BH	0.013685	0.226363	0.240048	0.057	5
Bomd	0.224301	0.014701	0.239002	0.938	1
CG	0.223476	0.027975	0.251451	0.889	3
Md	0.209228	0.023456	0.232684	0.899	2

where $v_i^+ = \max_a(v_{ai})$, if the criterion is to be maximized and $v_i^+ = \min_a(V_{ia})$, if the criterion is to be minimized.

d. The distance for each performance to the ideal action.

$$d_a^+ = \sqrt{\left(v_i^+ - v_{ia}\right)^2}, \quad a = 1\ldots, m$$

For the anti-ideal or zenith points

$$d_i^- = \sqrt{\left(v_i^- - v_{ia}\right)^2}, \quad a = 1\ldots m$$

e. Calculate the relative closeness of each performance:

$$c_a = \frac{d_a^-}{d_a^+ + d_a^-}$$

The closeness coefficient is always between 0 and 1, where 1 is the preferred performance. If a performance is closer to the ideal than the anti-ideal, then c_a approaches 1 and if the performance is closer to the anti-ideal then to the ideal, c_a approaches 0.

The relative values are taken as the indication of measures of performance of offer on various attributes (Table 6.4).

Result Discussion and Analysis

In the case of public procurement, the government issues instructions from time to time to promote multi-sourcing to discourage monopoly and encourage fair competition among the bidders. Therefore, the problem becomes complex while deciding the quantity of the orders to be placed on any firm to maximize the purchase value and promote the firms with equal opportunity to ensure compliance of the government's guidelines. The quantity of the procurement made in public procurement is also so large that it is normally not feasible for a single firm to supply the entire demand quantity. In our study, we have assumed both the situation and tried to demonstrate various possibilities in the case when the firm is competent to supply the complete demand or otherwise, with possible variations in the minimum numbers of the suppliers on which orders can be placed. 'n' is the number of suppliers on which the orders are finally placed in accordance with government guidelines or else to promote competition among the suppliers.

Scenario I

In case the capacity of all the suppliers is higher than the demand, the production capacity as quoted against the tender say 370 in this case.

Scenario II

In case the capacity of the suppliers is less than the demand. In above example, the capacity is assumed to be as mentioned in Table 6.5 (Fig. 6.2; Table 6.6).

Table 6.5 Capacity of suppliers

AB	260
BH	220
Bomd	285
CG	200
Md	225

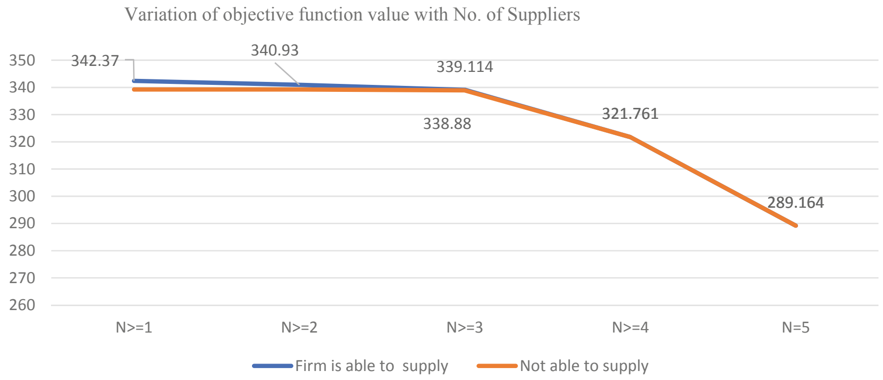

Fig. 6.2 Effect of variation in number of suppliers on value of purchase

Table 6.6 Iteration for variation of ‘n’ in both the scenarios

Condition	Capacity of supplier is higher than demand			Capacity of the supplier is less than the demand		
N	Variable	Variable Value	Objective function value	Variable	Variable Value	Objective function value
$N \geq 1$	x[Bomd]	365	342.37	x[Bomd] x[Md]	285 80	339.25
$N \geq 2$	x[Bomd] x[Md]	328 37	340.93	x[Bomd] x[Md]	285 80	339.25
$N \geq 3$	x[Bomd] x[CG] x[Md]	291 37 37	339.114	x[Bombd] x[CG] x[Med]	285 37 43	338.88
$N \geq 4$	x[AB] x[Bombdr] x[CG] x[Md]	37 254 37 37	321.761	x[AB] x[Bombd] x[CG] x[Md]	37 254 37 37	321.761
$N = 5$	x[AB] x[BH] x[Bombd] x[CG] x[Md]	37 37 217 37 37	289.164	x[AB] x[BH] x[Bombd] x[CG] x[Md]	37 37 217 37 37	289.164

Conclusions

From above discussion in both the scenarios where the supplier is either able to supply the desired quantity or not, the impact of variation of the number of suppliers is seen till the number is limited to 3. The best results are found for the cases where the suppliers are not able to fulfill the requirement and the number of orders is limited to 2 vendors. The objective value remains same for a single supplier and two supplier's scenario and deteriorates sharply afterward. To promote competition and fair practice in procurement, multi-sourcing is a proven tool in the supply chain. However, the study indicates that multi-sourcing has an associated cost and compromises need to be made to support it. The benefits of multi-sourcing are obviously proven to be overriding the marginal penalty which this program reflects to be making while selecting more than one vendor.

In real-life practice, multi-sourcing is not only a tool but also a compulsion to be accepted because of the size of the procurement by the public sector. The compromise being indicated in the study can be used as guiding tool for the policy-makers, while deciding the policy framework for multi-sourcing. The study may be quite insightful for the suppliers also, if their performance data is shared with them. The data-driven decision by supplies may also encourage improvement in performance on various parameters as indicated.

The study is an insightful exercise because it reveals the inability of procurement agencies to account for the associate performance of the firm like warranty obligation, on time delivery of the item, etc. These performance measures certainly have a cost implication for the suppliers, who tends to cut corners on the quality front to save money. Disregarding these indicators is proved to be a potential compromise. The results show that the application of a structured decision-making technique is vital, especially under the complex conditions that include both qualitative and quantitative criteria. The study is applicable in various fields of procurement and government agencies. The model can be used as the foundation for the development of complete support software for decision-making, eliminating or minimizing the human interventions in the procurement process. This way other related menaces in the system can be minimized ensuring saving on public money and best quality of procurement.

Future Research

This study further reveals that the weights associated with the parameters are critical in nature and detailed study on the impact of the variation of the weights on decision needs to be studied. In this study, the optimization program section offers the handy tool to accommodate and implement the government's rules and guidelines issued from time to time. The linear program has potential to be improved for more pragmatic situations, where running evaluation of multi-item through multi-vendor offers can be incorporated as future study.

As the potential of future extension of the study, sensitivity analysis of various factors such as weights associated may be conducted to find out the robustness of the support system being proposed through this study. Similarly, the sensitivity analysis of the various factors being considered may also be done to prove the impact of individual factors on overall performance of the vendor in specific tender offered, which can also act as advisory tool for the vendor to understand the focus performance area in the offer to remain competitive with respect to the other offers.

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Chapter 7
Single-Producer and Single-Retailer Integrated Inventory Model for Deteriorating Items Considering Three-Stage Deterioration

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Abstract This paper presents a single-producer single-retailer integrated policy for deteriorating items considering three-stage deterioration. This study considers three-stage deterioration of products during each phase of the supply system, such as production, retail, and transportation. The objective of this research is to obtain an optimal number of deliveries, lot size, and replenishment period to minimize the total integrated cost of the supply chain. In this paper, demand rate and deterioration rate are assumed deterministic and constant. The output of this study shows that the integrated approach is much more effective to fulfill the purpose as compared to the independent approach. To conclude the results, a numerical experiment and sensitivity analysis are performed.

Keywords Integrated inventory model · Deterioration · Optimal delivery · Production model

Introduction

In a traditional inventory system, researchers find the optimal policy separately for producers and retailers. This individual optimal strategy is no longer much beneficial for the whole supply chain system in today's global market. It has been confirmed that the integrated method results in a remarkable cost-saving and enhanced profit when it is compared with the individual decision made by the producer and the supplier [1]. Therefore, every organization adopted an integrated policy to achieve the mutual

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benefit of both producer and retailer. In the last few decades, many researchers developed numerous integrated inventory models based on several assumptions. Goyal [2] was the earliest researcher to establish an integrated vendor–buyer inventory model. He reveals that if we optimize joint economic cost for buyer and vendor instead of independent economic cost for buyer and vendor, then it will be profitable for both, i.e., for the whole supply chain. Benerjee [3] presented the integrated inventory model assuming JELS (joint economic lot size). Lu [4] extended single buyer into multiple buyers and developed single-vendor multi-retailer integrated model. So many researchers such as [5–10] have developed integrated inventory models considering deterioration for suppliers and retailers. Manna et al. [11] established a production method in a single production process, which is considered to minimize the production cost. Ghosh et al. [12] developed a single-producer multi-retailer supply chain system using the Stackelberg game method to maximize the profits of supply system participants.

Deterioration of items is the phenomenon of decay, dryness, vaporization, or spoilage of products in such a way that they lose their originality and sometimes become useless. Whitin [13] was first researcher who studied inventory of deteriorating items. Ghare and Schrader [14] also considered the impact of deterioration in inventory and developed a mathematical model for deteriorating products. Numerous researchers consider the effect of deterioration in their study and provide a fruitful optimal strategy for deteriorating items [15–17]. In most of the studies, it is assumed that deterioration of products starts from the instant of their arrival, i.e., at a retail stage but in reality, it can be seen easily that deterioration takes place during production as well as in transportation. Lin et al. [18] reveal that due to transport deterioration the amount of product acquired by the buyer is always lesser than the product ordered. Soysal et al. [19], Rong et al. [20] considered in-transit deterioration, and Wu and Sarker [9], Lin and Lin [21] have considered in-transit deterioration into an integrated model. Lin and Lin [21] stated that certain types of chemicals, foods, electronic components, etc. start deteriorating from the production stage. Mallick et al. [22] developed a two-level supply chain system for perishable items to optimize integrated overall profit by optimizing the retailer replenishment level and trade period. As soon as the production is completed these items start deteriorating instantly which affects the producer inventory level, thus the negative effect of deterioration during the production stage cannot be neglected. Since the model is deterministic therefore demand and deterioration rate are assumed to be known and constant. However, based on the above literature review, no researcher has considered the effect of deterioration during the production, transportation, and retail stage together in an integrated model. To fulfill this gap, we have considered a three-stage deterioration incorporated with integrated inventory model, which is more realistic for the inventory system.

Motivation and Research Contribution

As the authors have seen in the literature survey part, numerous researchers have worked on integrated model with the deteriorating nature of items. According to our examination of the literature, no researcher has examined the influence of deterioration during the production, transportation, and retail stages in an integrated model. This has motivated authors to consider three-stage deterioration in an integrated system.

This study examined a single-producer single-retailer integrated inventory model considering the three-stage deterioration nature of items. The goal of this research is to determine the ideal number of deliveries, lot size, and replenishment period to reduce the overall integrated cost of the supply chain. Here, deterioration rates in each stage are considered different. The convexity nature of the integrated cost function is represented graphically. A solution method is developed to determine optimal values of the integrated system. Finally, the developed model has been numerically analyzed by a numerical example.

Assumptions and Notations

To formulate the suggested model, we have assumed the below assumptions and notations:

Assumptions

1. One producer and one retailer are considered.
2. A single item having three-stage deterioration during each phase of the supply chain such as production, in-transit, and retail is considered.
3. The rate of production and demand is constant.
4. Shortages are not considered.
5. Deteriorated items are not repaired or replaced.
6. There is a single production cycle and N deliveries per order.

Notations

Symbol	Description
P	Production rate
D	Demand rate

(continued)

(continued)

Symbol	Description
Q	Quantity ordered by retailer's
q	Quantity received by retailer's
u	Cost of transportation per unit time
$I_{r1}(t)$, $I_{r2}(t)$	Retailer's stock level at t during in-transit period and retail period respectively
$I_{p1}(t_1)$, $I_{p2}(t_2)$	Stock level of producer in production period and non-production period respectively
F	Fixed ordering cost of retailer per unit time
T	Cycle length
T_{p1}, T_{p2}	Production period and non-production period for producer in each cycle
T/N	Stock cycle length per delivery for retailer
T_{r1}	Transportation time per shipment
θ_r	Rate of deterioration for item during in-transit and retail stage
θ_p	Rate of deterioration for item during production stage
h_r, h_p	Retailer's and producer's unit holding cost per item per unit time respectively
d_r	Retailer's unit deteriorated cost
d_p	Producer's unit deteriorated cost
ITC	Integrated total cost per unit time

Model Formulation

The status of inventory at any time for producer as well as for retailer is depicted in Fig. 7.2. At $t = 0$, production is started and when one shipment gets ready, it dispatches to the retailer. Therefore, producer's inventory falls due to demand and raises due to production. After some time, i.e., at $t = T_{p1}$ the producer stops production and produced items in stores, where inventory level falls due to demand of retailers and deterioration of items. Figure 7.1 shows the inventory status of both the retail and in-transit stages of retailers within one ordering cycle. At time $t = 0$, Q unit of the items are shipped to the retailer and T_{r1} is the transportation period per shipment. During in-transit stage $[0, T_{r1}]$, items deteriorate with the deterioration rate θ_r and therefore retailer get q units of items instead of Q units. During retail stage $\left(T_{r1}, \frac{T}{N} - T_{r1}\right)$, inventory level falls due to demand of customers and deterioration, and at $t = T/N$, retailer's inventory becomes zero and instantly the retailer's receive next lot dispatched by the producer, and the cycle continues.

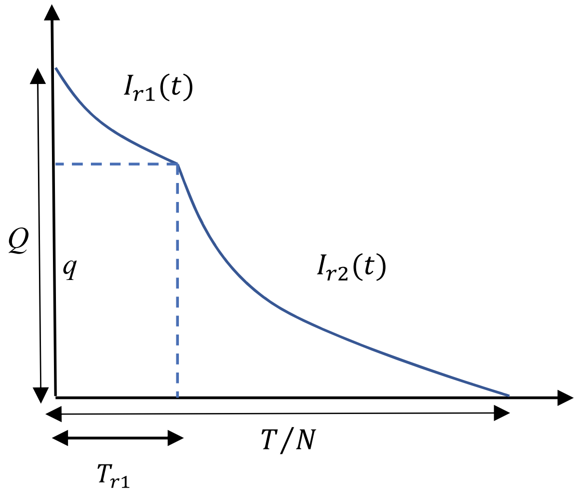

Fig. 7.1 Retailer's inventory level during in-transit and retail stage

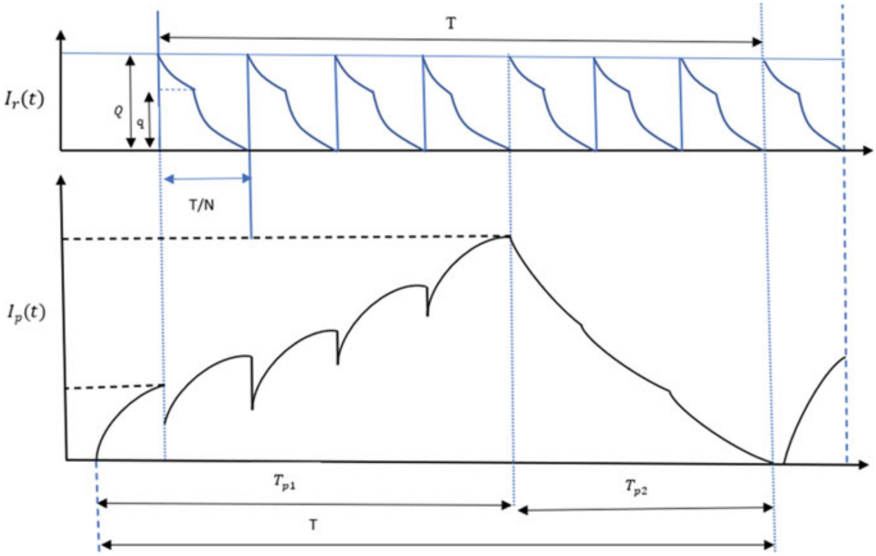

Fig. 7.2 Integrated inventory model for producer and retailer

Retailer's Perspective

The inventory differential equation of retailer during retail stage is as follows:

$$\frac{\mathrm{d}I_{r2}(t)}{\mathrm{d}t} = -\theta_r I_{r2}(t) - D, \qquad T_{r1} \le t \le \left(\frac{T}{N} - T_{r1}\right) \tag{7.1}$$

With the boundary condition $I_{r2}(T/N) = 0$, we get

$$I_{r2}(t) = \frac{D}{\theta_r}\left(e^{\theta_r\left(\frac{T}{N}-t\right)} - 1\right), \qquad T_{r1} \le t \le \left(\frac{T}{N} - T_{r1}\right) \tag{7.2}$$

$$\text{At}\, t = T_{r1}, q = \frac{D}{\theta_r}\left(e^{\theta_r\left(\frac{T}{N}-t\right)} - 1\right) \tag{7.3}$$

During in-transit stage, the retailer stock level $I_{r1}(t)$ can be represented by the given differential equation

$$\frac{\mathrm{d}I_{r1}(t)}{\mathrm{d}t} = -\theta_r I_{r1}(t), \qquad 0 \le t \le T_{r1} \tag{7.4}$$

With the boundary condition $I_{r1}(T_{r1}) = q$, $I_{r1}(0) = Q$, we can get

$$Q = \frac{De^{\theta_r T_{r1}}}{\theta_r}\left(e^{\theta_r\left(\frac{T}{N}-T_{r1}\right)} - 1\right) \tag{7.5}$$

This is the quantity ordered by the retailer.
The average cost for retailer per unit time:
Retailer's setup cost $= \frac{F+uL}{T}$.
Retailer's holding cost

$$= \frac{h_r N}{T}\int\limits_{T_{r1}}^{T/N} I_{r2}(t)\mathrm{d}t = \frac{h_r \mathrm{DN}}{T\theta_r^2}\left[e^{\theta_r\left(\frac{T}{N}-T_{r1}\right)} - \left(\frac{T}{N} - T_{r1}\right) - 1\right]$$

Deterioration cost for retailer's

$$= \frac{d_r N}{T}\left[Q - D\left(\frac{T}{N} - T_{r1}\right)\right] = \frac{d_r \mathrm{ND}}{2T}\theta_r\left[\left(\frac{T^2}{N^2} - T_{r1}^2\right)\right]$$

The retailer total average cost is given as:

$$\begin{aligned} &= \frac{(F + uT_{r1}N)}{T} + \frac{h_r \mathrm{DN}}{2T}\left[\left(\frac{T}{N} - T_{r1}\right)^2\left(1 + \frac{\theta_r}{3}\left(\frac{T}{N} - T_{r1}\right)\right)\right] \\ &+ \frac{d_r \mathrm{ND}}{2T}\theta_r\left[\left(\frac{T^2}{N^2} - T_{r1}^2\right)\right] \end{aligned} \tag{7.6}$$

Producer's Perspective

During production period the producer's inventory level is given by following differential equations

$$\frac{\mathrm{d}I_{p1}(t_1)}{\mathrm{d}t_1} = P - D \qquad 0 \le t_1 \le T_{p1} \tag{7.7}$$

With boundary condition $I_{p1}(0) = 0$ and solving Eq. (7.7),

$$I_{p1}(t_1) = (P - D)t_1 \tag{7.8}$$

During non-production period the producer's inventory level is given by

$$\frac{\mathrm{d}I_{p2}(t)}{\mathrm{d}t} = -\theta_p I_{p2}(t) - D, \qquad 0 \le t_2 \le T_{p2} \tag{7.9}$$

With boundary condition $I_{p2}\left(T_{p2}\right) = 0$ and solving Eq. (7.9),

$$I_{p2}(t_2) = \frac{D}{\theta_p}\left[e^{\theta_p(T_{p2}-t_2)} - 1\right] \tag{7.10}$$

At the point $I_{p1}(T_1) = I_{p2}(0)$, we get

$$T_{p1} = \frac{\mathrm{DT}_{p2}}{P - D}\left[1 + \frac{\theta_p T_{p2}}{2}\right] \tag{7.11}$$

From the relation,

$$T = T_{p1} + T_{p2} \tag{7.12}$$

$$T = \frac{T_2}{P - D}\left(p + \frac{1}{2}\mathrm{d}\theta T_2\right) \tag{7.13}$$

Producer's average cost per unit time.
Set-up cost for producer's $= \frac{S_p}{T}$.
Producer's holding cost

$$= \frac{h_p}{T}\left[\int_0^{T_{p1}} I_{p1}(t_1)\mathrm{d}t_1 + \int_0^{T_{p2}} I_{p2}(t_2)\mathrm{d}t_2 - N\int_0^{\frac{T}{N}} I_{r2}(t_2)\mathrm{d}t_2\right]$$

$$= \frac{h_p}{T}\left[(P-D)\frac{T_{p1}^2}{2} + \frac{\mathrm{DT}_{p2}^2}{2}\left(1+\frac{\theta_p T_p}{3}\right) - \frac{\mathrm{NDT}_{r1}^2}{2}\left(1+\frac{\theta_r T_{r1}}{3}\right)\right.$$
$$\left. - \frac{\mathrm{ND}}{2}\left(\frac{T}{N}-T_{r1}\right)^2\left(1+\frac{\theta_r}{3}\left(\frac{T}{N}-T_{r1}\right)\right)\right]$$

Deterioration cost for producer's

$$= \frac{d_{\mathrm{cp}}}{T}\left[\left(\mathrm{PT}_{p1}-\mathrm{DT}\right) - N\left(Q - D\left(\frac{T}{N}-T_{r1}\right)\right)\right]$$
$$= \frac{d_{\mathrm{cp}}}{T}\left[\left(\mathrm{PT}_{p1}-\mathrm{DT}\right) - \frac{\mathrm{ND}\theta_r}{2}\left(\frac{T^2}{N^2}-T_{r1}^2\right)\right]$$

Total cost for the producer is the resultant of setup, holding, and deterioration cost.

$$= \frac{S_p}{T} + \frac{h_p}{T}\left[(P-D)\frac{T_{p1}^2}{2} + \frac{DT_{p2}^2}{2}\left(1+\frac{\theta_p T_p}{3}\right) - \frac{\mathrm{NDT}_{r1}^2}{2}\left(1+\frac{\theta_r T_{r1}}{3}\right)\right.$$
$$\left. - \frac{\mathrm{ND}}{2}\left(\frac{T}{N}-T_{r1}\right)^2\left(1+\frac{\theta_r}{3}\left(\frac{T}{N}-T_{r1}\right)\right)\right] + \frac{d_{\mathrm{cp}}}{T}\left[\left(\mathrm{PT}_{p1}-\mathrm{DT}\right)\right. \tag{7.14}$$
$$\left. - \frac{\mathrm{ND}\theta_r}{2}\left(\frac{T^2}{N^2}-T_{r1}^2\right)\right]$$

Integrated total cost per unit time

$$= \frac{(F+uT_{r1}N)}{T} + \frac{h_r\mathrm{DN}}{2T}\left[\left(\frac{T}{N}-T_{r1}\right)^2\left(1+\frac{\theta_r}{3}\left(\frac{T}{N}-T_{r1}\right)\right)\right]$$
$$+ \frac{d_r\mathrm{ND}}{2T}\theta_r\left[\left(\frac{T^2}{N^2}-T_{r1}^2\right)\right] + \frac{S_p}{T} + \frac{h_p}{T}\left[(P-D)\frac{T_{p1}^2}{2} + \frac{\mathrm{DT}_{p2}^2}{2}\left[1+\frac{\theta_p T_p}{3}\right]\right.$$
$$\left. - \frac{\mathrm{NDT}_{r1}^2}{2}\left[1+\frac{\theta_r T_{r1}}{3}\right] - \frac{\mathrm{ND}}{2}\left(\frac{T}{N}-T_{r1}\right)^2\left(1+\frac{\theta_r}{3}\left(\frac{T}{N}-T_{r1}\right)\right)\right]$$
$$+ \frac{d_{\mathrm{cp}}}{T}\left[\left(\mathrm{PT}_{p1}-\mathrm{DT}\right) - \frac{\mathrm{ND}\theta_r}{2}\left(\frac{T^2}{N^2}-T_{r1}^2\right)\right] \tag{7.15}$$

Solution Method

The following procedure is followed to find out the optimal solution-

Step 1. Place the Eqs. (7.11) and (7.13) in Eq. (7.15).
Step 2. Substitute all the values of the parameter.
Step 3. Partially Derive ITC w.r.to T_{p2} and T_{r1} and put it equal to 0. Point out the resulting minimum value of T_{p2}, for each N.
Step 4. Evaluate the corresponding value of T_{p1} and T from Eq. (7.11) and (7.13), for each N.
Step 5. Evaluate the optimal number of deliveries N^* s.t. $\text{ITC}(N^* - 1) \geq \text{ITC}(N^*) \leq \text{ITC}(N^* + 1)$
Step 6. Evaluate the value of Q, from the Eq. (7.5).

Numerical Example and Sensitivity Analysis

To emphasize the applicability of this model, we assume a numerical example. The data of the numerical example is shown in Table 7.1. On implementing the above solution method, we get the optimal value of N and optimal integrated total cost as $N = 2$, $\text{ITC} = \$7667.74$. The corresponding value of production period, non-production period, and other time parameters are presented in Table 7.2. The optimal lot size $Q^* = 16$ units and optimal producer production quantity are 1200 units. The convexity of integrated total cost function (ITC) is graphically depicted in Fig. 7.3.

The output of sensitivity analysis of the major key parameters is shown in Table 7.3. The key parameters of the system are increased and decreased by 20% and the variational sensitivity outcome is depicted in Fig. 7.4. Based on the analysis, we can conclude that the deterioration cost parameters $\left(d_r, d_p\right)$ and deterioration rate $\left(\theta_r; \theta_p\right)$

Table 7.1 Values of parameter for numerical example

Parameters	Units	Values
Production rate (P)	Units/year	5000
Deterioration cost for producer d_p	\$/unit	80
Deterioration cost for retailer d_r	\$/unit	100
Demand rate (D)	Units/year	500
Deterioration rate for producer θ_p	–	0.30
Deterioration rate for retailer θ_r	–	0.35
Retailer holding cost h_r	\$/unit/year	60
Producer's set up cost S_p	\$/unit	1000
Transportation cost u	\$/delivery	50
Producer holding cost h_p	\$/unit/year	40
Retailer's fixed order cost F	\$/unit	150

Table 7.2 Optimal number of delivery (N) and optimal integrated total cost (ITC)

N	T_{r1}	T_{p2}	T_{p1}	T	ITC
1	0.1470	0.1875	0.0214	0.2089	7685.93
2*	**0.0884**	**0.2160**	**0.0247**	**0.2407**	**7667.74***
3	0.0642	0.2277	0.0261	0.2538	7721.32
4	0.0508	0.2340	0.0267	0.2607	7765.43
5	0.0423	0.2379	0.0265	0.2644	7798.79

* (Bold) stands for optimal value

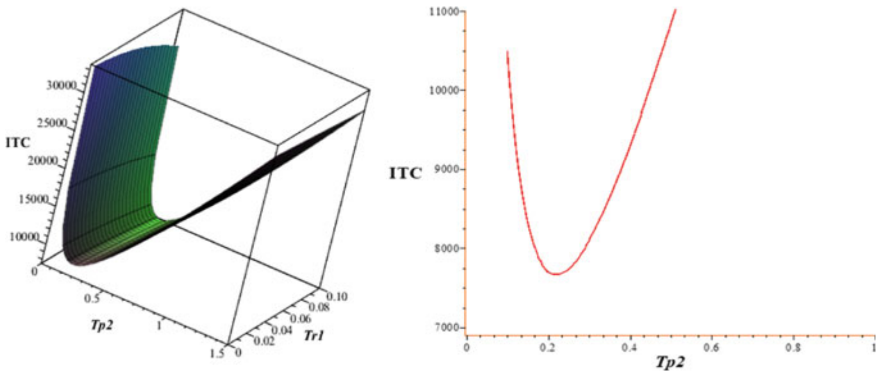

Fig. 7.3 Convexity of integrated total cost function versus time parameter

are highly sensitive because small increments in (d_r, d_p) and $(\theta_r; \theta_p)$ results high variation in N. Therefore, the manager should give priority to (d_r, d_p) and $(\theta_r; \theta_p)$ parameters.

Conclusion

In this study, we introduce an integrated single-producer single-retailer inventory model for three-stage deteriorating items. This paper considers deterioration during each stage like production, transportation, and retail. We have first derived the integrated cost expression and formulated a solution method to obtain optimal lot size, deliveries, replenishment period, and production quantity to minimize the integrated overall cost. Further, a numerical example has been carried out to elaborate the model and conducted the sensitivity analysis to find which key parameters affect the outcome with small change. This model is very realistic for highly volatile items. The convexity of cost expression is exhibited graphically with respect to variable parameters.

Table 7.3 Sensitivity analysis of key parameters

Parameter	Values of parameter	% change	Optimal number of deliveries (N*)	ITC*($)
P	4000	− 20	2	7515.18
	5000	0	2	7667.74
	6000	+ 20	2	7769.32
D	400	− 20	2	6979.24
	500	0	2	7667.74
	600	+ 20	2	8316.85
S_p	800	− 20	2	6954.37
	1000	0	2	7667.74
	1200	+ 20	2	8316.85
$h_r; h_p$	48; 32	− 20	2	7284.70
	60; 40	0	2	7667.74
	72; 48	+ 20	2	8118.51
$d_r; d_p$	80; 64	− 20	1	7231.56
	100; 80	0	2	7667.74
	120; 96	+ 20	2	8042.55
$\theta_r; \theta_p$	0.28; 0.24	− 20	1	7152.64
	0.35; 0.30	0	2	7667.74
	0.42; 0.36	+ 20	2	8178.05

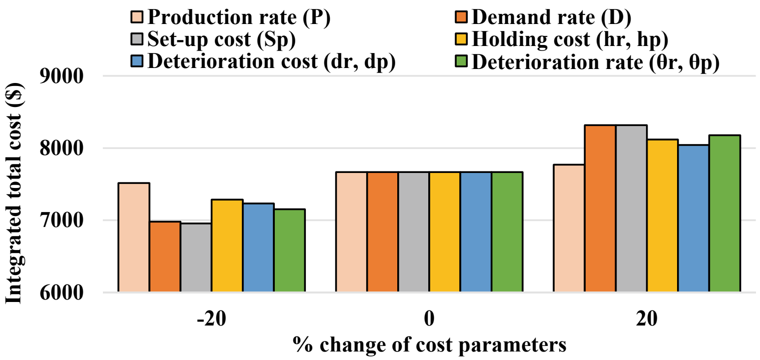

Fig. 7.4 Sensitivity results with respect to key parameters

This study is limited to single-producer single-retailer which can be extended by considering multi-producer multi-retailer. An interesting extension of this study one can include the carbon emission factor. This paper can also be generalized by assuming price-dependent demand, fuzzy stochastic environment, variable holding cost, and trade credit policies [23, 24].

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Chapter 8
Productivity Analysis of Structural Steel Fabrication in Construction Using Simulation

Prasanna Venkatesan Shanmugam and Justin Thomas

Abstract Analyzing the bottlenecks and productivity in construction using simulation has received increasing attention in recent years. This research proposes a Discrete-Event Simulation (DES) model of structural steel fabrication in construction to analyze bottlenecks and improve productivity. The data required for the simulation model are collected from a lead zinc beneficiation plant project executed by a large construction company in Rajasthan, India. Operational inefficiency and resource underutilization in the existing operations of the fabrication yard resulted in more than a 30% loss in productivity. Scenarios (by varying the cranes, manpower, number of weld beds, and job arrival) are proposed to improve the productivity of the fabrication yard. The model is developed using FlexSim software and validated by comparing the model result with the actual throughput obtained from the yard. It is proposed to revise the raw material interarrival time, the number of cranes, the number of weld beds, and the allocation of skilled workers. The productivity and cost associated with the proposed scenario are analyzed. Further, the results obtained are validated using paired *t*-test. The project managers have shown a reasonable agreement with the findings.

Keywords Construction · Labor productivity · Discrete-event simulation · FlexSim

Introduction

Steel is widely used in construction projects such as bridges, car parks, power plants, etc., because of the flexibility, ease of erection, and cost benefits it provides for the

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designers and the contractors. Structural steel is primarily fabricated in controlled shop environments off-site for better precision and accurate detailing and manufacturing [1]. The different parts are sub-assembled as modules at steel fabricators and finally assembled at the construction site according to fabrication drawings. Typically, fabrication yards are located near the project site. For a short-term project, yards are positioned away from the site in such a way that a single yard can feed for multiple projects. The major processes involved in steel fabrication are cutting, drilling, and welding usually carried out with automated machinery. The first operation in a steel fabrication yard is to cut the sections to the given length and profile the plates to the desired size or shape. After cutting processes, the joining of steel pieces to the specification in the drawings is done through welding. Steel fabrication is a project-based industry with low repetitiveness in production and a diverse range of products. The complexity of steel fabrication is originated in the uniqueness of projects, and this makes activities such as processing, routing, and resource planning different for each project. Traditionally, the project managers perform these activities based on their personal experience and with the use of simple tools, such as CPM/PERT, value stream mapping, which are often limited due to the complexities involved in structural steel fabrication [2]. Planning the above activities manually is cumbersome and may also result in improper routing and underutilization of resources. This further creates bottlenecks affect the fabrication productivity and the overall performance of any construction site. For any construction project, productivity, cost, quality, and time have been the main concern. Construction productivity is illustrated by an association between an output and an input [3]. Improving productivity in a structural steel fabrication construction project is a challenging task [4]. With advances in computer technologies, researchers create simulation models to help project managers plan and analyze the activities prior to commencing the actual project. Simulation models have been extensively developed and broadly used as a management tool within the manufacturing and business industries [5, 6]. Simulation models replicate the operations of real-life systems or processes considering various uncertainties. The ability of simulation to model various systems, complex interdependencies between operations, and use of resources, make them suitable for modeling structural steel fabrication [7]. This research proposes a Discrete-Event Simulation (DES) model of structural steel fabrication in construction to analyze and improve productivity. The data required for the simulation model are collected from a lead zinc beneficiation plant project executed by a large construction company in Rajasthan, India. Scenarios (varying the cranes, manpower, number of weld beds, and job arrival) are proposed to improve the productivity of the fabrication yard. The model is developed using FlexSim software, and the productivity and cost associated with each scenario are analyzed.

The remainder of this paper is organized as follows. Section "Literature Review" reports the literature review on simulation modeling in the construction industry. The problem description and simulation model development are presented in Section "Problem Description". Section "Results and Discussions" describes the results and discussion. Finally, the conclusions and future research directions are highlighted in Section "Conclusion".

Literature Review

García de Soto et al. [8] reported that the construction sector is facing several challenges to improve overall productivity. Qi et al. [9] analyzed the state-of-the-art research in the application of emerging technologies in industrialized construction. Dixit et al. [10] presented a systematic literature review of research in construction productivity. Tsehayae and Fayek [11] proposed a system model approach that integrates factor and activity models for better prediction of construction labor productivity. Hofacker and Gandhi [7] compared and evaluated fifty different simulation software to model and evaluate structural steel fabrication processes. Song [1] developed a virtual model capable of estimating, scheduling, and analyzing the production of a steel fabrication shop. Khalik et al. [12] proposed a two-stage integrated DES methodology to determine slipform operation's duration in silo construction projects. The performance of the proposed simulation model is validated by comparing results with a real system. Liu and An [13] proposed a DES methodology to study the interaction between cost, schedule, and emission performance for dam construction projects. Han et al. [5] proposed and validated a framework for productivity analysis in the construction process using the simulation model. Rustom and Yahia [14] demonstrated the use of simulation for real-time planning, scheduling, and control of the Gaza beach embankment protection project to find out the bottlenecks and improve the overall productivity. A hybrid system dynamics-discrete-event simulation approach was presented by Moradi et al. [15] to simulate the value of labor productivity considering the effects of both continuous (context) and discrete (operational) influencing variables. Shahin et al. [16] applied Agent-Based Modelling (ABM) as an effective tool for predicting the effects of congestion on labor productivity in construction projects. The above review highlights the increasing use of discrete-event simulation for productivity analysis in construction. However, it is observed that the use of simulation in structural steel fabrication has not received sufficient attention. To fulfill this research gap, this work proposes a DES model of structural steel fabrication in construction to analyze and improve productivity. The contribution and novelty of this work lie in addressing the following research objectives.

- To develop a FlexSim simulation model of structural steel fabrication yard to analyze bottlenecks and improve productivity.
- To validate the developed model by comparing the model result with the actual throughput obtained from the yard.
- To propose an improvement scenario and further validate the results.

Problem Description

Structural steel fabrication is one of the preliminary activities under mechanical work in any construction industry. On successful completion of structural fabrication, the remaining works, like piping, equipment erection, etc., are carried out. Hence, the fabrication of structural steel is considered a critical activity in construction projects. This research considers a lead zinc beneficiation plant project executed by a large construction company in Rajasthan, India. The layout of the steel fabrication yard is shown in Fig. 8.1, where most of the activities are labor intensive. Steel fabrication involves processes, such as detailing, fitting, welding, surface preparation, surface protection, and shipping. The detailing process consists of machining operations, such as cutting, drilling, and grinding, to shape steel components as specified in fabrication drawings and are carried out at fabrication beds 1–4. After all the steel piece components are detailed, they are stored in storage areas and are ready to be assembled either by a welded connection or a bolted connection, as specified in the fabrication drawing. For a welded connection, fitters fit and tack-weld detail components to the main component to temporarily assemble the steel piece until the final welding. Fitted parts are passed to welding beds 5–8 for final welding. Most welds made on structural steel/heavy plates are either groove welds, joining surfaces on the same plane, or filet welds, which join perpendicular edges. Surface preparation and protection are typically required for protecting steel pieces from oxidization and corrosion. Steel pieces are cleaned using an abrasive blasting machine before applying any protective coating. Abrasive blasting removes steel pieces' scale, rust, paint, and other surface contaminants. Once parts are cleaned, they can be painted or galvanized at paint beds 1 and 2 to protect the steel surfaces.

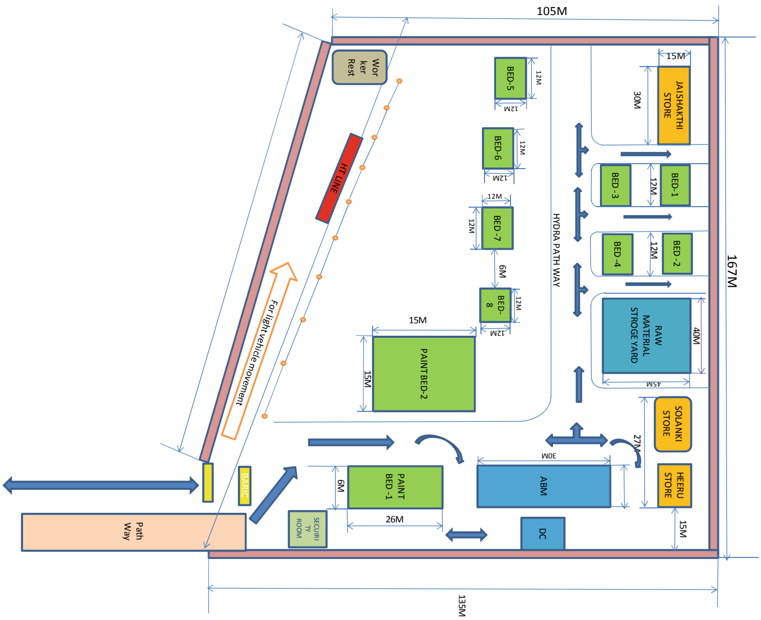

Fig. 8.1 Steel fabrication yard layout of a construction company

Finished pieces are shipped to the construction site for erection. At present, the project managers use simple tools, such as CPM/PERT, value stream mapping, and their personal experience for operations and resource planning. This resulted in the underutilization of some resources while others are over-utilized, which further creates a bottleneck and affects productivity. The actual throughput of the yard is 3.4 Mt/Mm (Metric Ton/Man-Month), while the obtained throughput is of the range of 2–2.5 Mt/Mm. The operational inefficiency in the various processes in the yard results in a loss of 1–1.5 Mm/MM. Also, it is reported that the non-availability of material at the right time in the yard keeps the resources idle for several days. Hence, the project managers are looking to minimize the bottlenecks and improve productivity using scientific tools instead of relying on their personal experience.

Simulation Model Development

Data Collection: Various activities (cutting, fit-up, welding, painting, and inspection) and sub-activities in the fabrication yard are observed based on a pilot study and the time required for each activity to produce one ton of finished product are noted. Further, a work sampling study is conducted at the steel fabrication yard to collect the required data. The observed data is additionally used as input for the simulation model. A typical illustration of the time needed for the sub-activities of cutting operations is shown in Table 8.1.

Model Assumptions

- The raw material input includes plates, angles, and beams of different dimensions. However, the developed simulation model considers only plates.

Table 8.1 Illustration of the time required for the sub-activities of cutting operations

Sub-activity		Time required (min)	No. of worker		
			Skilled	Semi-skilled	Unskilled
Preparation and cutting	Collect and study drawing	3.5	1		
	Prepare cutting plan	15	1		
	Marking 10 m length	15	1		
	Setting up the cutting machine	07	1	1	1
	Moving the cutting set	20			1
	Cut the plates	50	1		

Fig. 8.2 The column fabricated at the yard (before painting)

- The final product varies from project to project and depends on the design of columns. However, the present model developed is limited to the fabrication of columns with the design shown in Fig. 8.2.
- Worker absenteeism causes a delay in the fabrication processes. Hence, the probability of absenteeism is also included in the model.
- Delay due to safety hazards and natural calamities like heavy rainfall, flood, wind, etc., are also considered and included in the time.

The simulation model of the steel fabricating yard developed using FlexSim software is shown in Fig. 8.3. The developed model includes 60 workers with three skill levels, two processors each for cutting, fit-up, and painting operations, four processors for welding, and one for abrasive blasting. Queues are used to model the material storage yards. Four mobile cranes are modeled from the task executers to transport the materials to different processers. Three network nodes are used to make the task executers and workers move along a predefined path. The time required for the workers to perform a task varies with reference to skills. For each processor, the break downtime and time to repair were also included to make the model resemble the real fabrication yard.

Results and Discussions

The run length is set as one month, and 300 replications are made. The output obtained from the paint bed provides the measure of throughput as painting is the last operation carried out at the yard. From the results, an average of 141 Mt of products is fabricated at the end of one month. Considering 60 workers, the productivity obtained from the model is 2.35 Mt/Mm. It is also observed that the fabrication bed is underutilized due to the limited number of skilled workers assigned to this process as compared to welding and painting. A bottleneck at the fabrication bed further affects the subsequent processes. The developed model is validated by comparing the model

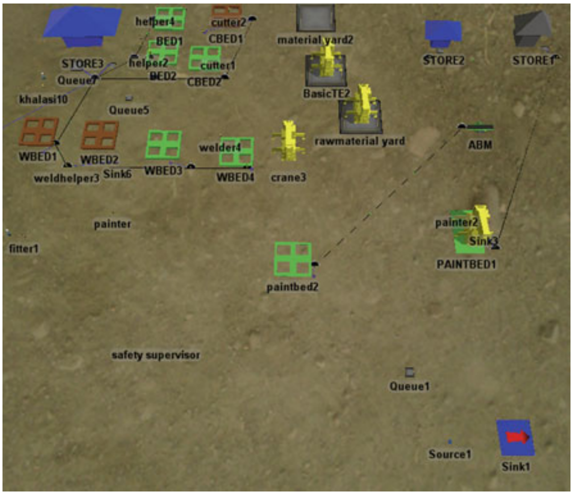

Fig. 8.3 Screenshot of FlexSim simulation model of the structural steel fabrication yard

result with the actual throughput obtained from the yard. In order to improve productivity, it is necessary to experiment with the model with user-defined scenarios. The proposed scenario includes a change in the resources (number of cranes, weld beds, and workers) and material arrival rate. The project managers are concerned about the cost overruns along with achieving productivity. The FlexSim software has an experimenter and an inbuilt financial analysis tool to carry out scenario and cost analysis, respectively. The running cost depends on the level of resource utilization. In the proposed scenario, the raw material interarrival time is set as 3 h, the number of cranes is reduced to 3, and the number of weld beds is set as 5. The total number of workers is reduced to 45 while the number of skilled workers allocated to fabrication beds is increased. These values are selected based on preliminary experiments conducted using FlexSim. The proposed scenario resulted in an average of 173 Mt of products fabricated at the end of one month, which is in line with the expected throughput of the yard. A paired *t*-test is performed, and the results are shown in Table 8.2, which indicates that there is no significant difference between the as-is model result and the actual system. From the *p*-value of the proposed scenario, it is found that there is a significant difference in productivity compared to the base case. The project managers have shown a reasonable agreement with the findings.

Table 8.2 Paired *t*-test result of the existing and proposed scenarios

Productivity (Mt/Mm)	Actual output	As-is simulation model	Proposed scenario
Mean	2.315	2.242	3.842
Variance	0.343	0.041	0.081
p-value		0.501	0.000

Conclusion

The construction industry is facing several challenges to improve overall productivity. This research aims to develop a FlexSim simulation model of structural steel fabrication to help the project managers to analyze bottlenecks and improve productivity. A case example of a lead zinc beneficiation plant project executed by a large construction company in Rajasthan, India, is presented. An improvement scenario is proposed, and the results are validated. This research contributes to the construction sector and validates the application of DES in evaluating the effect of strategies on productivity improvement. The future scope of this work is to extend the proposed approach as a hybrid model combining optimization and simulation.

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Chapter 9
Conservation of a Prey Species Through Optimal Taxation: A Model with Beddington–DeAngelis Functional Response

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Abstract The selective harvesting of prey population is discussed in the given article. Here, both prey and predator receive the functional response of Beddington–DeAngelis. Only a fraction of the prey species is allowed to be accessed by the predator. The study of the positivity of the solutions of the system of linear equations is made along with the existence of various equilibrium points and their stability. If the environmental carrying capacity of the prey species is used as bifurcation parameter, a Hopf bifurcation around the interior equilibrium may occur. Pontryagin's maximal principle is used to solve the problem of optimal taxation policy. The outputs of the proposed model are illustrated and verified with numerical examples using Maple.

Keywords Stability · Catchability coefficient · Harvesting · Hopf bifurcation · Carrying capacity · Pontryagin's maximum principle

Mathematics Subject Classification 92B05 · 92B10

Introduction

The dynamical relationship between predator and prey [1, 2] is the main motivation behind the study of ecology. Predator's functional response which is the predator's rate of feeding upon prey is the most notable factor of the prey-predator relationship.

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For various species, there are three basic forms of functional responses given by Holling in the year 1965 to model the phenomena of predation that eventually made the standard Lotka–Volterra [3, 4] system more realistic.

A functional response that was similar to Holling type-II was independently conceived by Beddington [5] and DeAngelis et al. [6] but that included an extra term that indicated correlative predator interference. As a result, a prey-predator model with Beddington–DeAngelis reaction looks like this

$$\begin{aligned}\frac{\mathrm{d}x}{\mathrm{d}t} &= rx\left(1-\frac{x}{k}\right)-\frac{mxy}{P+Qx+Ry}\\ \frac{\mathrm{d}y}{\mathrm{d}t} &= -ly+\frac{m\beta xy}{P+Qx+Ry}\end{aligned} \tag{9.1.1}$$

x and y represent the population density of the prey and predator species, respectively, whereas k represents the carrying capacity and r represents the intrinsic growth rate of the prey population. m is the predator's catching rate, β is the competence with which resources are converted to new consumers, P is the saturation constant, R is the impact of predator mutual interference on the feeding rate, Q (Units: 1/prey) is the effort of handling time on the feeding rate, and l is the predator's mortality rate.

Liu et al. [7] modeled with Beddington–DeAngelis functional response to study the asymptotical properties of a prey-predator system. Yu [8] studied a modified Leslie-Gower model using form (9.1.1).

In addition, Mehta et al. [9] and Abdulghafour and Naji [10] tweaked the response to investigate a prey-predator model.

Further, considering that the prey species is subjected to a harvesting effort, which has always fascinated the researchers [11–24], we take into account the widely accepted intra-specific rivalry among predator species in this paper, in addition to the above-mentioned criteria. This competition between the population of the same species is thought to result in an added prompt mortality rate which is proportional to the square of the population y, which alters the model proposed in (9.1.1) further.

Though other similar models have been published recently, the suggested model differs due to the reason that it includes the predator–prey model being exposed to the Beddington–DeAngelis functional response, which includes prey species that are harvested. When prey species are incorporated into harvesting, a portion of them becomes accessible to predators. The major goal of the article is to determine the appropriate taxation regime that will provide the most benefit from harvesting while also preventing the extinction of the species. The model becomes more ecologically accurate than previous models as a result of this added effect.

Section "Problem Formulation" goes through the construction and model assumptions. Section "Basic Property" discusses the basic property of the assumed model. In Section "The Existence of Equilibrium Points and Their Stability", the positivity and existence of the equilibrium point solutions, as well as their existence and stability, are addressed. Our work in the following part, i.e., Section "Hopf Bifurcation at

E_2 (x^*, y^*)", demonstrates the presence of Hopf bifurcation around the internal equilibrium. The maximum principle of Pontryagin is used to discuss the best taxation strategy in Section "Optimal Taxation". In Section "Numerical Results", we use Maple to mathematically verify all of our key discoveries. Finally, Section "Conclusion" offers a summary of the paper's main points as well as the ramifications of our findings.

Problem Formulation

The following system is regarded as a prey and predator population whose growth is in accordance with the dynamical system

$$\begin{aligned}\frac{\mathrm{d}x}{\mathrm{d}t} &= rx\left(1-\frac{x}{k}\right)-\frac{mxy}{P+Qx+Ry}-c_1Ex\\ \frac{\mathrm{d}y}{\mathrm{d}t} &= r_1y-r_{11}y^2+\frac{m\beta xy}{P+Qx+Ry}-ly\end{aligned} \tag{9.2.1}$$

having initial conditions

$$x(0)\geq 0,\ y(0)\geq 0 \tag{9.2.2}$$

The prey and predator species densities are denoted by $x(t)$ and $y\,(t)$, the catchability coefficient of the prey is c_1, E is the effort, r, k, l, m, β, P, Q and R are positive constants that have already been discussed in the Introduction part, the predator species' growth rate is defined by r_1, and the mutual rivalry rate among predators is denoted by r_{11}.

Basic Property

Theorem 3 *For every* $t \geq 0$ *and* $x \geq 0$, $y \geq 0$, *the solutions of the system (9.2.1) with initial conditions (9.2.2) exist in the intervals* $(0, \infty]$.

Proof The solution $(x(t), y(t))$ of (9.2.1) with initial conditions (9.2.2) exists and is unique on $(0, \xi]$ where $0 < \xi \leq +\infty$ [25], because the right-hand side of the system is totally continuous and locally Lipschitzian on C. We arrive at

$$x(t)=x(0)\exp\left[\int_0^{\infty}\left\{r\left(1-\frac{x(\theta)}{k}\right)-\frac{my(\theta)}{P+Qx(\theta)+Ry(\theta)}-c_1E\right\}\mathrm{d}\theta\right]\geq 0$$

$$y(t) = y(0)\exp\left[\int_0^\infty \left\{r_1 - r_{11}y(\theta) + \frac{m\beta x(\theta)}{P + Qx(\theta) + Ry(\theta)} - l\right\}\mathrm{d}\theta\right] \geq 0$$

which completes the proof.

The Existence of Equilibrium Points and Their Stability

The dynamical behavior of the system's probable equilibrium locations is discussed here, namely

1. Trivial equilibrium: $E_0(0, 0)$
2. Axial equilibrium: $E_1(\overline{x}, 0)$ where $\overline{x} = \frac{k(r - c_1 E)}{r}$
3. Interior equilibrium: $E_2(x^*, y^*)$.

Local Stability Analysis

The presence of the model system's non-trivial inner equilibrium is investigated and after evaluating $\frac{\mathrm{d}x}{\mathrm{d}t} = 0$, $\frac{\mathrm{d}y}{\mathrm{d}t} = 0$ are obtained as

$$Mx^3 + Nx^2 + Ox + P = 0 \tag{9.3.1}$$

where $M = \frac{R}{k}\left(r_{11}Q^2 r - \frac{R^2\beta r^2}{k}\right)$

$$N = r_1 R^2 Qr + \frac{r_{11}2PQRr}{k} + \frac{r_{11}Q^2 Rc_1}{E} - r_{11}Q^2 Rr + \frac{R^3\beta 2r^2}{k} - \frac{R^3\beta 2r}{k}\left(c_1 E + \frac{m}{R}\right) - \frac{lR^2 Qr}{k}$$

$$O = r_1 R^2 Q\left(c_1 E + \frac{m}{R}\right) + \frac{r_1 R^2 Pr}{k} - r_1 R^2 Qr + \frac{r_{11}P^2 Rr}{k} - r_{11}2PQm + \frac{r_{11}2PQRc_1}{E} + r_{11}2PQm - r_{11}2PQRr - R^3\beta\left(c_1 E + \frac{m}{R}\right)^2 - R^3\beta r^2 + R^3\beta 2r\left(c_1 E + \frac{m}{R}\right) - lR^2\left(\frac{Pr}{l} + Qc_1 E + \frac{Qm}{R} - Qr\right)$$

$$P = r_1 R^2 P\left(c_1 E + \frac{m}{R}\right) - r_1 R^2 Pr + \frac{r_{11}P^2 Rc_1}{E} - r_{11}P^2 Rr - lR^2 Pc_1 E - lRPm + lR^2 Pr$$

Computing the variational matrix corresponding to the system (9.2.1), we get

$$V = \begin{bmatrix} r - \frac{2rx}{k} - \frac{(P+Ry)my}{(P+Qx+Ry)^2} - c_1 E & -\frac{(P+Qx)mx}{(P+Qx+Ry)^2} \\ \frac{(P+Ry)m\beta y}{(P+Qx+Ry)^2} & r_1 - 2r_{11}y + \frac{(P+Qx)m\beta x}{(P+Qx+Ry)^2} - l \end{bmatrix}$$

Interpretation of E_0

The variational matrix of the system (9.2.1) at $E_0(0, 0)$ is given by

$$V(E_0) = \begin{bmatrix} r - c_1 E & 0 \\ 0 & r_1 - l \end{bmatrix}.$$

The eigenvalues of the characteristic matrix are $r - c_1 E, r_1 - l$. Now, $E > \frac{r}{c_1}$ and $l > r_1$, i.e., the eigenvalues are negative if mortality rate of the predator species is more than the growth rate. As a result, we have

Theorem 4.1.1 *The trivial equilibrium* $E_0(0, 0)$ *exists and is a stable node provided* $E > \frac{r}{c_1}$ *and* $l > r_1$.

Interpretation of E_1

The variational matrix of the system (9.2.1) at $E_1(\overline{x}, 0)$, $\overline{x} = \frac{k(r-c_1E)}{r}$ is given by

$$V(E_1) = \begin{bmatrix} r - \frac{2r\overline{x}}{k} - c_1 E & -\frac{\overline{x}m}{P+Q\overline{x}} \\ 0 & r_1 + \frac{\overline{x}m\beta}{P+Q\overline{x}} - l \end{bmatrix}$$

The eigenvalues of the characteristic matrix are $-r+c_1E$ and $r_1 + \frac{m\beta k(r-c_1E)}{Pr+Qkr-Qkc_1E} - l$. The first eigenvalue is negative provided $E < \frac{r}{c_1}$ (as a result, the existence of a stable node at the location of the trivial equilibrium point is breached) and the second eigenvalue is negative if

$$r_1 + \frac{m\beta k(r - c_1 E)}{Pr + Qkr - Qkc_1 E} < l$$

So, we arrive at the following theorem.

Theorem 4.1.2 *If* $E < \frac{r}{c_1}$ *and* $\frac{m\beta k(r-c_1E)}{Pr+Qkr-Qkc_1E} < l$ *are true, the system's axial equilibrium is a stable node.*

Which violates the fact that the trivial equilibrium at $E_0(0, 0)$ *is a stable node, and hence becomes a saddle point.*

Interpretation of E_2

The variational matrix of the system (9.2.1) at $E_2(x^*, y^*)$ is given by

$$V(E_2) = \begin{bmatrix} r - \frac{2rx^*}{k} - \frac{(P+Ry^*)my^*}{(P+Qx^*+Ry^*)^2} - c_1E & -\frac{(P+Qx^*)mx^*}{(P+Qx^*+Ry^*)^2} \\ \frac{(P+Ry^*)m\beta y^*}{(P+Qx^*+Ry^*)^2} & r_1 - 2r_{11}y^* + \frac{(P+Qx^*)m\beta x^*}{(P+Qx^*+Ry^*)^2} - l \end{bmatrix}$$

The characteristic equation of $V(E_2)$ is given by

$$\lambda^2 + a_1\lambda + a_2 = 0 \tag{9.3.2}$$

where $a_1 = -\left[r - \frac{2rx^*}{k} - \frac{(P+Qy^*)my^*}{(P+Qx^*+Ry^*)^2} - c_1E + r_1 - 2r_{11}y^* + \frac{(P+Qx^*)mx^*}{(P+Qx^*+Ry^*)^2} - l\right]$

$$a_2 = \left\{r - \frac{2rx^*}{k} - \frac{(P+Qy^*)my^*}{(P+Qx^*+Ry^*)^2} - c_1E\right\} \left\{r_1 - 2r_{11}y^* + \frac{(P+Qx^*)mx^*}{(P+Qx^*+Ry^*)^2}l\right\} + \frac{(P+y^*R)m\beta y^*(P+Qx^*)mx^*}{(P+Qx^*+Ry^*)^4}$$

According to the Routh Hurwitz criterion, eigenvalues have negative real parts if and only if

$$a_1 > 0 \quad \text{and} \quad a_2 > 0 \tag{9.3.3}$$

Theorem 4.1.3 *If and only if inequalities (9.3.3) are satisfied, the condition for the local asymptotical stability of the interior equilibrium holds.*

Global Stability Analysis

By developing a suitable Lyapunov function, we will investigate the global stability behavior of the systems (9.2.1) at the interior equilibrium point $E_2(x^*, y^*)$

$$S(x, y) = \left[x - x^* - x^* \ln \frac{x}{x^*}\right] + n_1\left[y - y^* - y^* \ln \frac{y}{y^*}\right]$$

where n_1 is a constant that will be determined in the next steps. It is straightforward to demonstrate that the function S is zero at the equilibrium point $E_2(x^*, y^*)$ and positive for all other values of x, y. When we differentiate S with respect to t, we get

$$\frac{dS}{dt} = \frac{x - x^*}{x}\frac{dx}{dt} + n_1\frac{y - y^*}{y}\frac{dy}{dt}$$

$$= \left(x - x^*\right)\left[r\left(1 - \frac{x}{k}\right) - \frac{my}{P + Qx + Ry} - c_1 E\right] + n_1\left(y - y^*\right)\left[r_1 - r_{11} y + \frac{m\beta x}{P + Qx + Ry} - l\right] \quad (9.3.4)$$

Moreover,

$$r\left(1 - \frac{x^*}{k}\right) - \frac{my^*}{P + Qx^* + Ry^*} - c_1 E = 0 \quad \text{and}$$
$$r_1 - r_{11} y^* + \frac{m_\beta \alpha x^*}{P + Qx^* + Ry^*} - l = 0 \quad (9.3.5)$$

in some neighborhood of (x^*, y^*), $\frac{\mathrm{d}S}{\mathrm{d}t}$ is negative semidefinite if

$$P + Qx^* + Ry^* > P + Qx + Ry \quad (9.3.6)$$

Theorem 4.2 *If inequality (9.3.6) is satisfied, the system's interior equilibrium point E_2 is globally asymptotically stable.*

Hopf Bifurcation at $E_2(x^*, y^*)$

The characteristic equation of the system (9.2.1) at E_2 is given by

$$\lambda^2 + a_1(k)\lambda + a_2(k) = 0 \quad (9.4.1)$$

where $a_1 = -\left[r - \frac{2rx^*}{k} - \frac{(P+Qy^*)my^*}{(P+Qx^*+Ry^*)^2} - c_1 E + r_1 - 2r_{11} y^* + \frac{(P+Qx^*)mx^*}{(P+Qx^*+Ry^*)^2} - l\right]$

$$a_2 = \left\{r - \frac{2rx^*}{k} - \frac{(P + Qy^*)my^*}{(P + Qx^* + Ry^*)^2} - c_1 E\right\} \left\{r_1 - 2r_{11} y^* + \frac{(P + Qx^*)mx^*}{(P + Qx^* + Ry^*)^2} l\right\} + \frac{(P + y^* R)m\beta y^*(P + Qx^*)mx^*}{(P + Qx^* + Ry^*)^4}$$

Using k as the bifurcation parameter to figure out the stability of the assumed model, let us state the following theorem by Murray [26].

Theorem 5.1 [26] *If $a_i(k)$, $i = 1, 2$ are smooth functions of k in an open interval about $k_c \in R$ such that the characteristic Eq. (9.4.1) has*

(i) *a pair of complex eigenvalues $\lambda = b_1(k) \pm ib_2(k)$ (with $b_1(k), b_2(k) \in R$) so that they become purely imaginary at $k = k_c$ and $\frac{\mathrm{d}b_1}{\mathrm{d}k}_{k=k_c} \neq 0$*
(ii) *Hopf bifurcation occurs around E_2 at $k = k_c$.*

Theorem 5.2 *If $a_2(k) > 0, a_1(k) = 0$, (9.2.1) has a Hopf bifurcation around the interior equilibrium when k goes through k_c.*

Proof The characteristic equation of (9.2.1) at the interior equilibrium point $\lambda^2 + a_2 = 0$, at $k = k_c$ results in roots $\lambda_1 = i\sqrt{a_2},\ \lambda_2 = -i\sqrt{a_2}$ which further gives rise to a pair of purely imaginary eigenvalues. Also $a_i(i = 1, 2)$ are smooth functions of k.

Considering k in a neighborhood of k_c, roots are $\lambda_1 = b_1(k) + ib_2(k), \lambda_2 = b_1(k) - ib_2(k)$ where $b_i(k), i = 1, 2$ are real.

Now, we are going to verify the condition $\frac{\mathrm{d}}{\mathrm{d}k}(\mathrm{Re}(\lambda_i(k)))|_{k=k_c} \neq 0,\ i = 1, 2.$

Substituting $\lambda(k) = b_1(k) + ib_2(k)$ in (9.4.1) we get

$$(b_1(k) + ib_2(k))^2 + a_1(b_1(k) + ib_2(k)) + a_2 = 0 \tag{9.4.2}$$

Differentiating both sides of (9.4.2) with respect to k, we obtain

$$\begin{aligned}2(b_1(k) + ib_2(k))\left(b_1^{'}(k) + ib_2^{'}(k)\right) + a_1^{'}(b_1(k) + ib_2(k))\\ + a_1\left(b_1^{'}(k) + ib_2^{'}(k)\right) + a_2^{'} = 0\end{aligned} \tag{9.4.3}$$

Analyzing like parts of (9.4.3), we have

$$\left(2b_1b_1' - 2b_2b_2' + a_1'b_1 + a_1b_1' + a_2'\right) = 0$$
$$\left(2b_2b_1' + 2b_1b_2' + a_1b_2' + a_1'b_2\right) = 0$$

i.e.,

$$D_1b_1' - D_2b_2' + D_3 = 0 \tag{9.4.4}$$

$$D_2b_1' + D_1b_2' + D_4 = 0 \tag{9.4.5}$$

where

$$\begin{aligned}D_1 &= 2b_1 + a_1\\ D_2 &= 2b_2\\ D_3 &= a_1'b_1 + a_2'\\ D_4 &= a_1'b_2\end{aligned}$$

Now, from (9.4.4) and (9.4.5), we have

$$b_1' = -\frac{D_1D_3 + D_2D_4}{D_1^2 + D_2^2} \tag{9.4.6}$$

At $k = k_c$

Case I $b_1 = 0,\ b_2 = \sqrt{a_2}$

$$D_1 = a_1$$

$$D_2 = 2\sqrt{a_2}$$

$$D_3 = a_2'$$

$$D_4 = a_1'\sqrt{a_2}$$

So, $D_1 D_3 + D_2 D_4 = a_1 a_2' + 2a_1' a_2 \neq 0$ at $k = k_c$.

Case II $b_1 = 0,\ b_2 = -\sqrt{a_2}$

$$D_1 = a_1$$

$$D_2 = -2\sqrt{a_2}$$

$$D_3 = a_2'$$

$$D_4 = -a_1'\sqrt{a_2}$$

So, $D_1 D_3 + D_2 D_4 = a_1 a_2' + 2a_1' a_2 \neq 0$ at $k = k_c$.
Therefore, $\frac{\mathrm{d}}{\mathrm{d}k}(\mathrm{Re}(\lambda_i(k)))|_{k=k_c} = -\frac{D_1 D_3 + D_2 D_4}{D_1^2 + D_2^2}|_{k=k_c} \neq 0.$

Optimal Taxation

In the harvesting literature, bionomic equilibrium is defined as when the total revenue earned from selling the captured biomass equals the whole cost of getting it. In this situation, the economic rent is completely vanquished. Let c represents the cost of harvesting per unit effort, and p_1 represents the price per unit of prey population. Then, at any time t, the economic net revenue is given by

$$\pi(x, y, E, t) = p_1 c_1 x E - cE$$

The bionomic equilibrium is $P_\infty(x_\infty, y_\infty)$ where the positive solutions of $\dot{x},\ \dot{y} = 0$ are x_∞, y_∞. The economic rent acquired from harvesting turns negative if $c > p_1 c_1 x$, i.e., the harvesting cost exceeds the revenue obtained from it. As a result, the process will be closed, and bionomic equilibrium will be lost and it is logical to suppose that $p_1 c_1 x > c$. If $E > E_\infty$, the overall cost of harvesting the population will be greater than the total revenue generated by the sector. As a result, some businessmen would lose money and, as a result, they would withdraw from the sector. If $E < E_\infty$, then the process is more profitable, it will attract more and more businessmen. As a result, some businessmen would lose money and, as a result, they would withdraw from the sector. Hence, the value of $E > E_\infty$ cannot be kept indefinitely.

The regulating agency's goal is to maximize the total discounted net profits generated by the business for the society. In terms of symbolism, this goal equates to maximizing the present value J of a continuous time-stream of revenue given by

$$J = \int_0^\infty \pi(x, y, E, t)\mathrm{e}^{-\delta t}\mathrm{d}t \tag{9.5.1}$$

where δ is the annual rate of discount at any given time [2]. The goal is to maximize J while keeping the state of equations (9.2.1) in mind. The constant price per unit biomass of the x species is represented by p_1, while the constant fishing cost per unit effort is represented by c.

The Hamiltonian of this control problem is given by

$$\begin{aligned} H &= (p_1 c_1 x E - cE)\mathrm{e}^{-\delta t} \\ &\quad + \lambda_1\left[rx\left(1 - \frac{x}{k}\right) - \frac{mxy}{P + Qx + Ry} - c_1 E x\right] \\ &\quad + \lambda_2\left[r_1 y - r_{11} y^2 + \frac{m\beta xy}{P + Qx + Ry} - ly\right] \\ &= \sigma(t)\mathrm{e}^{-\delta t} + \lambda_1\left[rx\left(1 - \frac{x}{k}\right) - \frac{mxy}{P + Qx + Ry} - c_1 E x\right] \\ &\quad + \lambda_2\left[r_1 y - r_{11} y^2 + \frac{m\beta xy}{P + Qx + Ry} - ly\right] \end{aligned} \tag{9.5.2a}$$

where $\lambda_1(t)$, $\lambda_2(t)$ are the adjoint variables and the switching function [2] is $\sigma(t) = (p_1 c_1 x - c)\mathrm{e}^{-\delta t} - c_1 x$.

Because H is linear in E, the ideal control approach is a combination of bang-bang and singular control. To maximize H, the best control $E(t)$ must meet the following constraints.

$$E = E_{\max},\ \text{as}\ \sigma(t) > 0,\ \text{which implies}\ \lambda_1(t)\mathrm{e}^{\delta t} < p - \frac{c}{c_1 x} \tag{9.5.2b}$$

$$E = 0,\ \text{as}\ \sigma(t) < 0,\ \text{which implies}\ \lambda_1(t)\mathrm{e}^{\delta t} > p - \frac{c}{c_1 x} \tag{9.5.2c}$$

The usual shadow price [6] is $\lambda_1(t)e^{\delta t}$ and $p - \frac{c}{c_1 x}$ is the net economic revenue on a unit harvest. This illustrates that the net economic revenue on a unit harvest is less than or larger than $E = E_{\max}$, or 0 according to the shadow price. In terms of economics, condition (9.5.2b) states that if the profit after all expenses is positive, harvesting up to the limit of possible effort is good. When the shadow price surpasses the fisherman's net economic revenue on a unit harvest, condition (9.5.2c) states that the fisherman will not exert any effort.

The Hamiltonian H becomes independent of the control variable $E(t)$ when $\sigma(t) = 0$, i.e., when the shadow price equals the net economic revenue on a unit harvest which implies $\frac{\partial H}{\partial E} = 0$. For the singular control $E^*(t)$ to be optimal over the control set $0 < E^* < E_{\max}$, this is a necessary condition.

Thus, the optimal harvesting policy is

$$E(t) = \begin{cases} E_{\max} & \sigma(t) > 0 \\ 0 & \sigma(t) < 0 \\ E^* & \sigma(t) = 0 \end{cases}$$

When $\sigma(t) = 0$ it follows that

$$\frac{\partial H}{\partial E} = (p_1 c_1 x - c)e^{-\delta t} - \lambda_1 c_1 x = 0 \tag{9.5.3}$$

$$\Rightarrow \lambda_1 = \frac{(p_1 c_1 x - c)e^{-\delta t}}{c_1 x} \tag{9.5.4}$$

The adjoint equations are

$$\begin{aligned} \frac{d\lambda_1}{dt} = -\frac{\partial H}{\partial x} = -\Bigg[p_1 c_1 E e^{-\delta t} + \lambda_1 \left\{ rx\left(1 - \frac{x}{k}\right) - \frac{mxy}{P + Qx + Ry} - c_1 E x \right\} \\ + \lambda_2 \frac{(P + Ry)m\beta y}{(P + Qx + Ry)^2} \Bigg] \end{aligned} \tag{9.5.5}$$

$$\begin{aligned} \frac{d\lambda_2}{dt} = -\frac{\partial H}{\partial y} = -\Bigg[-\lambda_1 \frac{(P + Qx)mx}{(P + Qx + Ry)^2} \\ + \lambda_2 \left\{ r_1 y - r_{11} y^2 + \frac{m\beta xy}{P + Qx + Ry} - ly \right\} \Bigg] \end{aligned} \tag{9.5.6}$$

We rewrite (9.5.6) by considering the inner equilibrium as to achieve an ideal equilibrium solution

$$\frac{d\lambda_2}{dt} - A_1 \lambda_2 = -A_2 e^{-\delta t}$$

where

$$A_1 = -r_{11}y^* - \frac{Rm\beta x^* y^*}{(P + Qx^* + Ry^*)^2} \text{ and } A_2 = \frac{(p_1 c_1 x^* - c)}{c_1 x^*} \cdot \frac{(P + Qx^*)mx^*}{(P + Qx^* + Ry^*)^2}$$

A solution of the above differential equation is

$$\lambda_2(t) = \frac{A_2}{A_1 + \delta} \cdot e^{-\delta t} \tag{9.5.7}$$

Similarly, from (9.5.5), we get

$$\lambda_1(t) = \frac{B_2}{B_1 + \delta} \cdot e^{-\delta t} \tag{9.5.8}$$

where $B_1 = -\frac{rx^*}{k} + \frac{Qmx^*y^*}{(P+Qx^*+Ry^*)^2}$ and $B_2 = p_1 c_1 E - \frac{A_2}{A_1+\delta} \cdot \frac{(P+Ry^*)m\beta y^*}{(P+Qx^*+Ry^*)^2}$.

Substituting the value of $\lambda_1(t)$ from Eq. (9.5.4) into Eq. (9.5.8) we have

$$\left(p_1 - \frac{c}{c_1 x^*}\right) = \frac{B_2}{B_1 + \delta} \tag{9.5.9}$$

The existence of an optimal equilibrium solution that satisfies the necessary conditions of the maximum principle has been established.

The following observations can be made:

(i) We can see from Eqs. (9.5.7) and (9.5.8) that $\lambda_i e^{\delta t}$ $(i = 1, 2)$ where λ_i's are adjoint variables, remain constant throughout time intervals in an optimum equilibrium, thereby satisfying the transversality condition, i.e., remaining bounded as $t \to \infty$.

(ii) Considering the interior equilibrium, Eq. (9.5.3) can be written as

$$\lambda_1 c_1 x^* = \left(p_1 c_1 x^* - c\right) \cdot e^{-\delta t} = e^{-\delta t} \cdot \frac{\partial \pi}{\partial E}$$

This means that the user's overall harvest cost per unit effort is equal to the future price's discounted value at the steady state effort level.

(iii) From Eqs. (9.5.7) and (9.5.8), we further get that

$$(p_1 c_1 x - c) = \frac{B_2 c_1 x}{B_1 + \delta} \to 0 \text{ as } \delta \to \infty$$

This shows that the net economic revenue to the society, $\pi(x_\infty, y_\infty) = 0$.

We can deduce that as tax rates rise, prey populations rise and harvesting effort falls, and an infinite discount rate leads to complete economic revenue dissipation.

Numerical Results

Validation of the derived results is required for analytical research to be complete. We will look at computer simulations of system (9.2.1) utilizing data from other relevant research articles to estimate parameter values in this section, e.g., Chattopadhyay et al. [27], Neverova et al. [14], Sharma and Samanta [11] and Zhang et al. [15]. We will look at two numerical examples here.

Example 1 We take the parameter values as $r = 14.0, r_1 = 13.0, k = 1000$, $m = 0.1, P = 12.0, Q = 12.0, R = 12.0, c_1 = 0.01, E = 1.0, r_{11} = 0.50$, $\alpha = 0.006$ in proper units. We discover that the equilibrium points for the above values (Fig. 9.1)

$$x = 999.2706342 \text{ and } y = 27.00009727.$$

Example 2 Getting the values for the parameters as $r = 3.0, r_1 = 0.4, k = 110, m = 2.5, P = 12.0, Q = 12.0, R = 1.0, c_1 = 0.1, E = 1.0, r_{11} = 0.01, \beta = 0.006$.

We find the equilibrium points in the appropriate units as

$$x = 103.4912777, y = 40.11996529$$

Furthermore, the phase plane trajectory is defined as in Fig. 9.2.

We find the given curve by plotting the prey and predator populations against time t (Fig. 9.3).

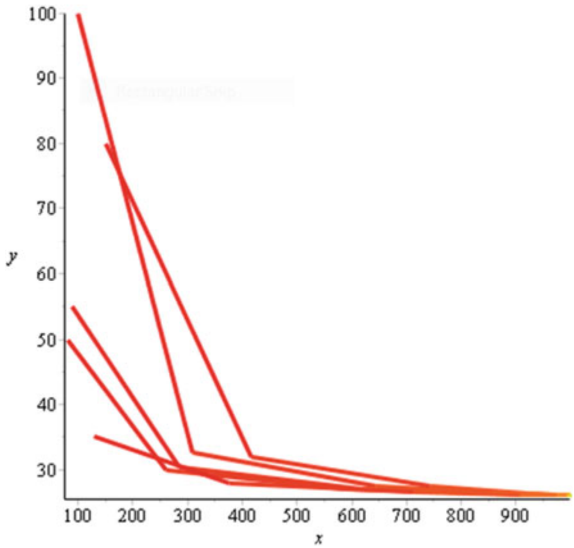

Fig. 9.1 Figure shows phase plane trajectories of the prey-predator system with various initial values matching to the above data set Sharma and Samanta [11]

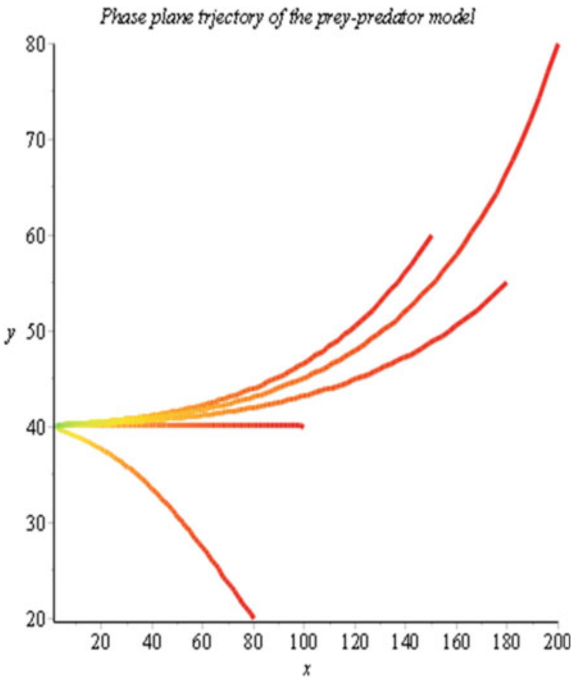

Fig. 9.2 Figure shows the prey-predator system with phase plane trajectories with varying initial values corresponding to above-mentioned data Neverova et al. [14]

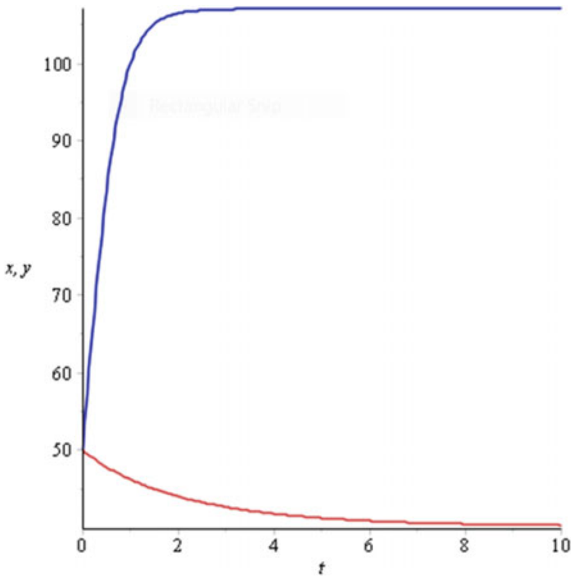

Fig. 9.3 For a period of up to 10 units, the figure depicts the solution curve of the prey-predator population Zhang et al. [15] and Chattopadhyay et al. [27]

Conclusion

We provide a system in which the former population is harvested and the later is subjected to intra-specific competition, both of which are influenced by the Beddington–DeAngelis functional response in this work. The system's dynamical behaviors and stability at various equilibrium points are then described. Here, we deal with three equilibrium points: trivial, axial and interior. If $E > \frac{r}{c_1}$ and $l > r_1$, the trivial equilibrium point produces a stable node in this case. Although there is an axial equilibrium, depending on the parameters, it is either a saddle point or an unstable node. A suitable Lyapunov function is constructed for the global stability study. Pontryagin's maximal principle is used to discuss the best harvesting policy. The study proposed in the text varies from prior recent studies in that as it includes the Beddington–DeAngelis functional response on a harvested prey species. Under the influence of intra-specific competition, a predator species emerges, enriching the system's dynamics. We looked into the conditions that cause a limit cycle to form due to Hopf bifurcation. The carrying capacity of the prey species k is critical for population stability, and if this value is preserved as a bifurcation parameter, a Hopf bifurcation may occur at the interior equilibrium point. If the prey species' carrying capacity, k, continues below a threshold value, the prey species' stability will be jeopardized. When we used biological approaches to find an optimal tax policy and an internal equilibrium matching to this tax policy, we discovered that an infinite discount rate resulted in the complete dissipation of economic income. As a result, the model can be enhanced further by factoring in the infected population's logistic growth. In the infected population model, refuge can also be included. It is also possible to contemplate predator species harvesting. However, the predator's consumption of both susceptible and infected prey species does not occur immediately; there must be some gestation time. As a result, there is plenty of room in the model to include this delay to make it more realistic. However, the predator's consumption of both susceptible and infected prey species does not occur immediately; there must be some gestation time. As a result, there is plenty of room in the model to include this delay to make it more realistic. However, the predator's eating of both susceptible and infected prey species is not an instantaneous process; there must be some gestation delay. As a result, there is plenty of room to include this delay in the model to make it even more realistic.

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Chapter 10
Optimization of an Inventory Model with Demand Dependent on Selling Price and Stock, Nonlinear Holding Cost Along with Trade Credit Policy

Mamta Kumari, Pankaj Narang, Pijus Kanti De, and Ashis Kumar Chakraborty

Abstract The demand for a product is influenced by a number of factors, including the selling price and the displayed stock level, among others. Considering this scenario, an EOQ inventory model is developed where demand is a function of both selling price and the inventory level which is one of the main contributions of this research work. Holding cost is assumed to be nonlinearly dependent on stock. Besides that, supplier grants a full trade credit policy to the retailer. This policy is very advantageous for both the counterpart—the supplier as well as the retailer. The supplier can attract more customers by offering a delay period whereas the latter enjoys the benefit of getting goods without instant payment. The proposed mathematical model aims to find out the optimal selling price and optimal length of the replenishment cycle so as to maximize the total profit of the retailer per unit time. Several theorems are well-established in order to reach to the optimal solution. A numerical example is also presented to demonstrate the suggested inventory model, and a sensitivity analysis is executed to highlight the findings of the inventory model and put forward valuable managerial insights. This research work can be helpful to the business communities facing nonlinear demand patterns. Businesses that want to offer trade credit policies but are dealing with nonlinear holding costs may also find it helpful. Sensitivity analysis can be useful in determining the impact of various cost parameters on the total generated profit.

Keywords Inventory · Selling price · Stock-dependent · Trade credit

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Introduction

Majority of the inventory models presume demand rate to be constant throughout the inventory cycle. In real life, demand is observed to be dependent on many factors such as selling price and displayed stock level. Customers show a strong desire to buy more products when supermarkets showcase them in huge quantities in the showroom according to Levin et al. [15]; this frequently increases product demand. Hence, displayed stock plays a significant role in the analysis of inventory. The inventory model presented by Datta and Pal [7] assumes that demand is a function of inventory level. Wu and his co-researchers [19] presented an inventory model for non-instantaneous deteriorating products with partial backlogging as well as stock level dependent demand.

Pricing is another key aspect that influences the demand for a product. A vital question that arises in the inventory model is what should be an item selling price so that the seller gains maximum benefits while satisfying customer needs. Various researchers and academicians have formulated different inventory models with price-dependent demand. Alfares and Ghaithan [1] formulated an inventory model with price-dependent demand as well as quantity discounts where holding cost was allowed to vary with time. Chang and his collaborators [4] portrayed an integrated model of inventory with the policy of trade credit tied with order size and price-dependent demand. In that same year, an integrated inventory system was framed with trade credit to be dependent on the size of the order [18].

One another assumption that is prevalent in economic order quantity (EOQ) inventory models is that the buyer should pay the seller upon receipt of goods. Small businesses that do not have enough money to pay for the ordered goods instantly suffer a lot due to this policy. Trade credit policy provides a chance to the buyer to buy goods without paying instantly. The buyer is granted a trade credit period in order to settle accounts. The trade credit policy is often utilized to stimulate demand. Haley and Higgins [10] were the first to frame an EOQ inventory model considering allowable delay in payments. Following that, Goyal [9] investigated an EOQ model of inventory with trade credit where the buyer is exempted from clearing the payment and earns interest throughout the credit period. Various trade credit policies have been developed as a result of diversification of trade and changes in the business environment. Musa and Sani [17] constructed a model of inventory for decaying items allowing delay in payments. After that, Khanra and his collaborators [13] formulated an inventory model considering shortages and permissible delay in payments for a single item where demand of customers is a quadratic function of time. An inventory model was developed with demand dependent on stock, partial backlogging, and nonlinear holding cost by Cárdenas-Barrón et al. [3]. The retailer was allowed a full trade credit period to clear the debt by the supplier. Ghosh et al. [8] constructed an inventory model with multiple advanced and delayed payment policies along with complete backordering for perishable items. Similary, other authors have made valuable contributions to the existing literature [5, 6, 11, 12, 14].

Some of the researchers have formulated inventory models where demand is dependent on both selling price and displayed stock level. By taking into account the exhibited level of stock and selling price-dependent demand, Hsieh and Dye [11] formulated an inventory model for deteriorating items. To find out the optimal solution, particle swarm optimization is applied. An EOQ inventory model with demand dependent on stock and selling price for deteriorating items was analyzed by Mishra et al. [16] where an estimation of the optimal order quantity, selling price, and preservation technology investment was made from the retailer's perspective.

The objectives of this study are as follows:

- Majority of the existing works are based on the assumption that demand always remains constant which is far from reality. Many factors influence demand, notably selling price, displayed goods, and so on. The aim of this study is to build an inventory model that considers demand as a function of selling price and displayed stock.
- Recent review of the literature suggests that holding cost is considered to be linearly dependent on stock which need not to be always. It totally depends on the nature of the item. In this paper, we have considered holding cost to be nonlinearly dependent on stock. Besides that, the supplier-retailer relationship is analyzed where the supplier offers a full trade credit period to the retailer.
- The main goal of the proposed study is to find out the optimal selling price at which goods are to be sold and the optimal replenishment length of the cycle so as to maximize the retailer's total generated profit considering demand to be nonlinear, nonlinear holding cost along with the trade credit policy. A numerical example is also presented to demonstrate the findings, and a sensitivity analysis is done to highlight the results.

The paper is further organized as follows: Section "Assumptions and Notations" proposes the assumptions and notations required to establish the inventory model mathematically. Section "Mathematical Model" constructs the mathematical model considering demand to be nonlinear, nonlinear holding cost, and trade credit. Section "Theoretical Results and Optimization Procedure" discusses some important theoretical results and optimization procedures necessary to reach to the optimal solution. Section "Numerical Example" illustrates a numerical example. In Section "Sensitivity Analysis", sensitivity analysis is done by changing one parameter at a time and keeping rest of the parameter's constant. Section "Managerial Insights" highlights some important results. Finally, Section "Conclusion" concludes the findings as well as suggests some important future research directions.

Assumptions and Notations

Assumptions

1. The inventory system's planning horizon is infinite.
2. Demand is assumed to be a function of stock level and selling price given by

$$D(t) = \alpha(a - bp)[q(t)]^{\beta} \quad \text{when } q(t) \geq 0$$

3. Holding cost is nonlinearly dependent on stock. It is represented as follows:

$$H(t) = h[q(t)]^{\gamma} \quad \text{when } \gamma > 0$$

   If $\gamma = 1$, then holding cost is considered to be linearly dependent on stock.
4. Rate of replenishment is instantaneous with negligible lead-time.
5. Retailer is granted a full trade credit period by the supplier.

Notations

See Table 10.1.

Table 10.1 Notations used to establish the inventory model

Parameter	Unit	Description	Parameter	Unit	Description
c	\$/unit	Purchasing price per unit	M	unit time	Trade credit period offered by the supplier to the retailer
o	\$/order	Ordering cost per order	I_p	%/unit time	Interest paid by the retailer
h	\$/unit/unit time	Holding cost per unit item per unit time	γ		Holding cost elasticity; $\gamma > 0$
β		Demand elasticity rate; $0 \leq \beta < 1$	α		Demand rate scale parameter
			Decision variables		
I_e	%/unit time	Interest earned by the retailer	p		Selling price per unit item
$TP(p, T)$	\$/unit time	Total profit per unit time	T		Replenishment cycle length

Mathematical Model

An inventory model is formulated where the cost of holding goods varies nonlinearly with stock. Initially, Q units of product exist in the inventory of retailer. Demand is dependent upon the level of stock carried in the inventory and selling price. The level of inventory decreases due to demand during the interval $[0, T]$. At time $t = T$, it drops down to zero. After that, again a replenishment order of Q units is placed. The beginning of the next inventory cycle is marked by the arrival of products. The supplier also offers a full trade credit period M to the retailer additionally. The inventory situation is best described by the following differential equations:

$$\frac{\mathrm{d}q(t)}{\mathrm{d}t} = -\alpha(a - bp)[q(t)]^{\beta}, \qquad 0 \le t \le T \tag{10.1}$$

With the following boundary conditions: $q(0) = Q$ and $q(T) = 0$. Solving the differential equation (10.1) with the above-mentioned boundary conditions, the following results are obtained:

$$q(t) = \left[Q^{1-\beta} - \alpha(a - bp)(1 - \beta)t\right]^{\frac{1}{1-\beta}}, \qquad 0 \le t \le T \tag{10.2}$$

$$Q = [\alpha(a - bp)(1 - \beta)T]^{\frac{1}{1-\beta}} \tag{10.3}$$

Various costs related with the proposed inventory model are as follows:

(1)

$$\text{Ordering cost} = o \tag{10.4}$$

(2)

$$\text{Purchasing cost} = c[\alpha(a - bp)(1 - \beta)T]^{\frac{1}{1-\beta}} \tag{10.5}$$

(3

$$\text{Sales revenue collected} = p[\alpha(a - bp)(1 - \beta)T]^{\frac{1}{1-\beta}} \tag{10.6}$$

(4)

$$\begin{aligned}\text{Inventory holding cost} &= h\int_0^T [q(t)]^{\gamma}\,\mathrm{d}t \\ &= h\int_0^T \left[Q^{1-\beta} - \alpha(a - bp)(1 - \beta)t\right]^{\frac{\gamma}{1-\beta}}\mathrm{d}t\end{aligned}$$

$$= h\frac{[\alpha(a-bp)(1-\beta)T]^{\frac{\gamma+1-\beta}{1-\beta}}}{\alpha(a-bp)(\gamma+1-\beta)} \tag{10.7}$$

Since the supplier offers a full trade credit period to the retailer, following two sub-cases arise:

Case-1: $M \leq T$
Case-2: $M \geq T$.

Case-1: Trade Credit Period Is Less Than or Equal to the Cycle Length ($M \leq T$)

The trade credit period granted by the supplier to the retailer is less than or equal to the cycle length in this scenario. After the end of credit period, the retailer has to bear the interest charges and needs to pay interest during the interval $[M, T]$. Consequently, the amount of interest paid is computed as follows:

$$\mathrm{IP} = \frac{cI_\mathrm{p}\left[[\alpha(a-bp)(1-\beta)(T-M)]^{\frac{2-\beta}{1-\beta}}\right]}{\alpha(a-bp)(2-\beta)} \tag{10.8}$$

The retailer earns interest during the credit period up to $t = M$.

$$\mathrm{IE} = pI_\mathrm{e}\Bigg[M[\alpha(a-bp)(1-\beta)T]^{\frac{1}{1-\beta}} + \frac{1}{\alpha(a-bp)(2-\beta)} \left\{\{\alpha(a-bp)(1-\beta)(T-M)\}^{\frac{2-\beta}{1-\beta}} - \{\alpha(a-bp)(1-\beta)T\}^{\frac{2-\beta}{1-\beta}}\right\}\Bigg] \tag{10.9}$$

The total profit per unit time is calculated as follows:

$$\mathrm{TP}_1(p, T) = \frac{\mathrm{SR} + \mathrm{IE} - o - \mathrm{PC} - \mathrm{HC} - \mathrm{IP}}{T}$$

Therefore,

$$\begin{aligned}\mathrm{TP}_1(p, T) = \left[\frac{1}{T}\right]\Bigg[& p[\alpha(a-bp)(1-\beta)T]^{\frac{1}{1-\beta}} \\ &+ pI_\mathrm{e}\Bigg[M[\alpha(a-bp)(1-\beta)T]^{\frac{1}{1-\beta}} \\ &+ \frac{1}{\alpha(a-bp)(2-\beta)}\Big\{\{\alpha(a-bp)(1-\beta)(T-M)\}^{\frac{2-\beta}{1-\beta}} \\ &- \{\alpha(a-bp)(1-\beta)T\}^{\frac{2-\beta}{1-\beta}}\Big\}\Bigg] - o\end{aligned}$$

$$- c[\alpha(a - bp)(1 - \beta)T]^{\frac{1}{1-\beta}} - h\frac{[\alpha(a - bp)(1 - \beta)T]^{\frac{\gamma+1-\beta}{1-\beta}}}{\alpha(a - bp)(\gamma + 1 - \beta)}$$
$$\left. - \frac{cI_{\mathrm{p}}\Big[[\alpha(a - bp)(1 - \beta)(T - M)]^{\frac{2-\beta}{1-\beta}}\Big]}{\alpha(a - bp)(2 - \beta)}\right] \tag{10.10}$$

Problem 1

Maximize $\mathrm{TP}_1(p, T) = \frac{S_1}{T}$

where $S_1 = \mathrm{SR} + \mathrm{IE} - o - \mathrm{PC} - \mathrm{HC} - \mathrm{IP}$

subject to $M \leq T$.

Case-2: Trade Credit Period Is Greater Than or Equal to the Cycle Length ($M \geq T$)

In this case, the trade credit period granted to the retailer by the supplier is greater than or equal to the cycle length. In this scenario, the retailer does not need to pay interest. Therefore,

$$\mathrm{IP} = 0 \tag{10.11}$$

The retailer earns interest during the credit period up to $t = M$.

$$\mathrm{IE} = pI_{\mathrm{e}}\left[M[\alpha(a - bp)(1 - \beta)T]^{\frac{1}{1-\beta}} - \frac{[\alpha(a - bp)(1 - \beta)T]^{\frac{2-\beta}{1-\beta}}}{\alpha(a - bp)(2 - \beta)}\right] \tag{10.12}$$

The total profit per unit time is computed as follows:

$$\mathrm{TP}_2(p, T) = \frac{\mathrm{SR} + \mathrm{IE} - o - \mathrm{PC} - \mathrm{HC} - \mathrm{IP}}{T}$$

Therefore,

$$\mathrm{TP}_2(p, T) = \left[\frac{1}{T}\right]\Big[p[\alpha(a - bp)(1 - \beta)T]^{\frac{1}{1-\beta}} + pI_{\mathrm{e}}\Big[M[\alpha(a - bp)(1 - \beta)T]^{\frac{1}{1-\beta}}$$
$$\left. - \frac{[\alpha(a - bp)(1 - \beta)T]^{\frac{2-\beta}{1-\beta}}}{\alpha(a - bp)(2 - \beta)}\right] - o - c[\alpha(a - bp)(1 - \beta)T]^{\frac{1}{1-\beta}}$$
$$\left. - h\frac{[\alpha(a - bp)(1 - \beta)T]^{\frac{\gamma+1-\beta}{1-\beta}}}{\alpha(a - bp)(\gamma + 1 - \beta)}\right] \tag{10.13}$$

Problem 2
Maximize $\mathrm{TP}_1(p, T) = \frac{S_2}{T}$

where $S_2 = \mathrm{SR} + \mathrm{IE} - o - \mathrm{PC} - \mathrm{HC} - \mathrm{IP}$

subject to $M \geq T$.

Theoretical Results and Optimization Procedure

The problem is solved using the following theorem of generalized concave functions. If $f(x)$ is non-negative, differentiable, and strictly concave, and $g(x)$ is positive, differentiable and convex, then the real-valued function

$$z(x) = \frac{f(x)}{g(x)} \tag{10.14}$$

is strictly pseudo-concave in nature. The detailed proof can be seen in Cambini and Martein [2].

$M \leq T$

Theorem 1 *For any given p,*

(a) $\mathrm{TP}_1(p, T)$ *is a strictly pseudo-concave function in T, hence there exists a unique maximum solution* T_1^*.
(b) *If* $M \leq T_1^*$, *then* $\mathrm{TP}_1(p, T)$ *subject to* $M \leq T$ *is maximized at* T_1^*.
(c) *If* $M \geq T_1^*$, *then* $\mathrm{TP}_1(p, T)$ *subject to* $M \leq T$ *is maximized at M.*

Proof See Appendix 1.

To find T_1^*, for any given p, taking the first-order derivative $\mathrm{TP}_1(p, T)$ with respect to T, setting the result to zero and rearranging terms we get

$$\begin{aligned}\frac{\partial \mathrm{TP}_1(p, T)}{\partial T} = &\left[\frac{1}{T}\right]\Big[(p-c)\alpha(a-bp)[\alpha(a-bp)(1-\beta)T]^{\frac{\beta}{1-\beta}} \\ &- h[\alpha(a-bp)(1-\beta)T]^{\frac{\gamma}{1-\beta}} - cI_\mathrm{p}[\alpha(a-bp)(1-\beta)(T-M)]^{\frac{1}{1-\beta}} \\ &+ pI_\mathrm{e}\Big[\alpha(a-bp)M[\alpha(a-bp)(1-\beta)T]^{\frac{\beta}{1-\beta}} \\ &+ [\alpha(a-bp)(1-\beta)(T-M)]^{\frac{1}{1-\beta}} - [\alpha(a-bp)(1-\beta)T]^{\frac{1}{1-\beta}}\Big]\Big] \\ &- \left[\frac{1}{T^2}\right]\Big[p[\alpha(a-bp)(1-\beta)T]^{\frac{1}{1-\beta}} + pI_\mathrm{e}\end{aligned}$$

$$\left[M[\alpha(a-bp)(1-\beta)T]^{\frac{1}{1-\beta}} + \frac{1}{\alpha(a-bp)(2-\beta)}\right.$$
$$\left.\left\{\{\alpha(a-bp)(1-\beta)(T-M)\}^{\frac{2-\beta}{1-\beta}} - \{\alpha(a-bp)(1-\beta)T\}^{\frac{2-\beta}{1-\beta}}\right\}\right]$$
$$- o - c[\alpha(a-bp)(1-\beta)T]^{\frac{1}{1-\beta}} - h\frac{[\alpha(a-bp)(1-\beta)T]^{\frac{\gamma+1-\beta}{1-\beta}}}{\alpha(a-bp)(\gamma+1-\beta)}$$
$$\left. - \frac{cI_{\mathrm{p}}\left[[\alpha(a-bp)(1-\beta)(T-M)]^{\frac{2-\beta}{1-\beta}}\right]}{\alpha(a-bp)(2-\beta)}\right] = 0 \tag{10.15}$$

Similarly, for any given T, taking the first-order derivative of $\mathrm{TP}_1(p, T)$ with respect to p, setting the result equal to zero and rearranging terms we get

$$\frac{\partial \mathrm{TP}_1(p, T)}{\partial p} = \left[\frac{1}{T}\right]\left[[\alpha(a-bp)(1-\beta)T]^{\frac{1}{1-\beta}}\right.$$
$$- b\alpha(p-c)T[\alpha(a-bp)(1-\beta)T]^{\frac{\beta}{1-\beta}}$$
$$- h\left[-\frac{bT[\alpha(a-bp)(1-\beta)T]^{\frac{\gamma}{1-\beta}}}{(a-bp)}\right.$$
$$\left.+\frac{b[\alpha(a-bp)(1-\beta)T]^{\frac{\gamma+1-\beta}{1-\beta}}}{\alpha(\gamma+1-\beta)(a-bp)^2}\right]$$
$$- cI_{\mathrm{p}}\left[-\frac{b(T-M)[\alpha(a-bp)(1-\beta)(T-M)]^{\frac{1}{1-\beta}}}{(a-bp)}\right.$$
$$\left.+\frac{b[\alpha(a-bp)(1-\beta)(T-M)]^{\frac{2-\beta}{1-\beta}}}{\alpha(2-\beta)(a-bp)^2}\right]$$
$$+ I_{\mathrm{e}}\left[M[\alpha(a-bp)(1-\beta)T]^{\frac{1}{1-\beta}} + \frac{1}{\alpha(a-bp)(2-\beta)}\right.$$
$$\left.\left\{\{\alpha(a-bp)(1-\beta)(T-M)\}^{\frac{2-\beta}{1-\beta}} - \{\alpha(a-bp)(1-\beta)T\}^{\frac{2-\beta}{1-\beta}}\right\}\right]$$
$$+ pI_{\mathrm{e}}\left[-b\alpha MT[\alpha(a-bp)(1-\beta)T]^{\frac{\beta}{1-\beta}}\right.$$
$$- \frac{b(T-M)[\alpha(a-bp)(1-\beta)(T-M)]^{\frac{1}{1-\beta}}}{(a-bp)}$$
$$+ \frac{b[\alpha(a-bp)(1-\beta)(T-M)]^{\frac{2-\beta}{1-\beta}}}{\alpha(2-\beta)(a-bp)^2}$$
$$+ \frac{bT[\alpha(a-bp)(1-\beta)T]^{\frac{1}{1-\beta}}}{(a-bp)}$$

$$-\frac{b[\alpha(a-bp)(1-\beta)T]^{\frac{2-\beta}{1-\beta}}}{\alpha(2-\beta)(a-bp)^2}\Bigg]\Bigg]=0 \quad (10.16)$$

Theorem 2 *For any given* $T > 0$, *if* $Z_1 = \frac{\partial^2 f}{\partial p^2} < 0$ *where* $f = \Big[p[\alpha(a-bp)(1-\beta)T]^{\frac{1}{1-\beta}} + pI_e\Big[M[\alpha(a-bp)(1-\beta)T]^{\frac{1}{1-\beta}} + \frac{1}{\alpha(a-bp)(2-\beta)}\Big\{\{(a-bp)(1-\beta)(T-M)\}^{\frac{2-\beta}{1-\beta}} - [\alpha(a-bp)(1-\beta)T]^{\frac{2-\beta}{1-\beta}}\Big\}\Big] - o - c[\alpha(a-bp)(1-\beta)T]^{\frac{1}{1-\beta}} - h\frac{[\alpha(a-bp)(1-\beta)T]^{\frac{\gamma+1-\beta}{1-\beta}}}{\alpha(a-bp)(\gamma+1-\beta)} - \frac{cI_p\Big[[\alpha(a-bp)(1-\beta)(T-M)]^{\frac{2-\beta}{1-\beta}}\Big]}{\alpha(a-bp)(2-\beta)}\Big]$, *then* $\mathrm{TP}_1(p,T)$ *is a strictly pseudo-concave function in* p, *hence there exists a unique maximum solution* p^*.

$M \geq T$

Theorem 3 *For any given* p,

(a) $\mathrm{TP}_2(p,T)$ *is a strictly pseudo-concave function in* T, *hence there exists a unique maximum solution* T_2^*.
(b) *If* $M \geq T_2^*$, *then* $\mathrm{TP}_2(p,T)$ *subject to* $M \geq T$ *is maximized at* T_2^*.
(c) *If* $M \leq T_2^*$, *then* $\mathrm{TP}_2(p,T)$ *subject to* $M \geq T$ *is maximized at* M.

Proof See Appendix 2.

To find T_2^*, for any given p, taking the first-order derivative $\mathrm{TP}_2(p,T)$ with respect to T, setting the result to zero and rearranging terms we get

$$\begin{aligned}\frac{\partial \mathrm{TP}_2(p,T)}{\partial T} &= \left[\frac{1}{T}\right]\Big[(p-c)\alpha(a-bp)[\alpha(a-bp)(1-\beta)T]^{\frac{\beta}{1-\beta}} \\ &- h[\alpha(a-bp)(1-\beta)T]^{\frac{\gamma}{1-\beta}} + pI_e[\alpha(a-bp) \\ &M[\alpha(a-bp)(1-\beta)T]^{\frac{\beta}{1-\beta}} - [\alpha(a-bp)(1-\beta)T]^{\frac{1}{1-\beta}}\Big]\Big] \\ &- \left[\frac{1}{T^2}\right]\Big[p[\alpha(a-bp)(1-\beta)T]^{\frac{1}{1-\beta}} \\ &+ pI_e\Big[M[\alpha(a-bp)(1-\beta)T]^{\frac{1}{1-\beta}} \\ &- \frac{[\alpha(a-bp)(1-\beta)T]^{\frac{2-\beta}{1-\beta}}}{\alpha(a-bp)(2-\beta)}\Big] - o - c[\alpha(a-bp)(1-\beta)T]^{\frac{1}{1-\beta}} \\ &- h\frac{[\alpha(a-bp)(1-\beta)T]^{\frac{\gamma+1-\beta}{1-\beta}}}{\alpha(a-bp)(\gamma+1-\beta)}\Big] = 0 \end{aligned} \quad (10.17)$$

Similarly, for any given T, taking the first-order derivative of $\mathrm{TP}_2(p,T)$ with respect to p, setting the result equal to zero and rearranging terms we get

$$\frac{\partial \mathrm{TP}_2(p,T)}{\partial p} = \left[\frac{1}{T}\right]\Bigg[[\alpha(a-bp)(1-\beta)T]^{\frac{1}{1-\beta}} - b\alpha(p-c)T[\alpha(a-bp)(1-\beta)T]^{\frac{\beta}{1-\beta}} - h\left[-\frac{bT[\alpha(a-bp)(1-\beta)T]^{\frac{\gamma}{1-\beta}}}{(a-bp)} + \frac{b[\alpha(a-bp)(1-\beta)T]^{\frac{\gamma+1-\beta}{1-\beta}}}{\alpha(\gamma+1-\beta)(a-bp)^2}\right] + I_e\left[M[\alpha(a-bp)(1-\beta)T]^{\frac{1}{1-\beta}} - \frac{[\alpha(a-bp)(1-\beta)T]^{\frac{2-\beta}{1-\beta}}}{\alpha(a-bp)(2-\beta)}\right] + pI_e\left[-b\alpha MT[\alpha(a-bp)(1-\beta)T]^{\frac{\beta}{1-\beta}} + \frac{bT[\alpha(a-bp)(1-\beta)T]^{\frac{1}{1-\beta}}}{(a-bp)} - \frac{b[\alpha(a-bp)(1-\beta)T]^{\frac{2-\beta}{1-\beta}}}{\alpha(2-\beta)(a-bp)^2}\right]\Bigg] = 0 \tag{10.18}$$

Theorem 4 *For any given* $T > 0$, *if* $Z_1 = \frac{\partial^2 f}{\partial p^2} < 0$ *where* $f = \Big[p[\alpha(a-bp)(1-\beta)T]^{\frac{1}{1-\beta}} + pI_e\Big[M[\alpha(a-bp)(1-\beta)T]^{\frac{1}{1-\beta}} - \frac{[\alpha(a-bp)(1-\beta)T]^{\frac{2-\beta}{1-\beta}}}{\alpha(a-bp)(2-\beta)}\Big] - o - c[\alpha(a-bp)(1-\beta)T]^{\frac{1}{1-\beta}} - h\frac{[\alpha(a-bp)(1-\beta)T]^{\frac{\gamma+1-\beta}{1-\beta}}}{\alpha(a-bp)(\gamma+1-\beta)}\Big]$, *then* $\mathrm{TP}_2(p,T)$ *is a strictly pseudo-concave function in p, hence there exists a unique maximum solution p*.*

For any given p, let us define the discriminant term

$$\Delta = \left[\frac{1}{M}\right]\Bigg[(p-c)\alpha(a-bp)[\alpha(a-bp)(1-\beta)M]^{\frac{\beta}{1-\beta}} - h[\alpha(a-bp)(1-\beta)M]^{\frac{\gamma}{1-\beta}} + pI_e\Big[\alpha(a-bp)M[\alpha(a-bp)(1-\beta)M]^{\frac{\beta}{1-\beta}} - [\alpha(a-bp)(1-\beta)M]^{\frac{1}{1-\beta}}\Big]\Bigg] - \left[\frac{1}{M^2}\right] \Big[p[\alpha(a-bp)(1-\beta)M]^{\frac{1}{1-\beta}} + pI_e$$

$$\left[M[\alpha(a-bp)(1-\beta)M]^{\frac{1}{1-\beta}} - \frac{[\alpha(a-bp)(1-\beta)M]^{\frac{2-\beta}{1-\beta}}}{\alpha(a-bp)(2-\beta)}\right]$$
$$-o-c[\alpha(a-bp)(1-\beta)M]^{\frac{1}{1-\beta}}$$
$$\left.-h\frac{[\alpha(a-bp)(1-\beta)M]^{\frac{\gamma+1-\beta}{1-\beta}}}{\alpha(a-bp)(\gamma+1-\beta)}\right] \tag{10.19}$$

Theorem 5 *For any given p,*

(1) *If* $\Delta > 0$, *then the total profit is maximized at* T_1^*.
(2) *If* $\Delta = 0$, *then the total profit is maximized at* M.
(3) *If* $\Delta < 0$, *then the total profit is maximized at* T_2^*.

Proof See Appendix 3.

Numerical Example

The proposed inventory model is illustrated with a numerical example. The aim is to find out the optimal selling price p^* and optimal replenishment length of the cycle T^* so as to maximize the overall profit earned by the retailer per unit time. The input values of various parameters are as follows:

$a = 90$; $b = 0.5$; $I_p = 12\%$/year; $I_e = 7\%$/year; $M = 180/365$ year; $\alpha = 0.4$; $o = \$200$/order; $h = \$10$/unit/year; $c = \$50$/unit; $\gamma = 1.1$; $\beta = 0.4$.

Since $\Delta > 0$, it falls into the category 4.1. Hence, the optimal solution is

$p^* = 113.696$; $T^* = 3.07143$; $Q^* = 205.796$; $\text{TP}^*(p, T) = 2708.96$. The profit function is concave in nature as shown in Fig. 10.1.

Sensitivity Analysis

The illustrated numerical example is used to study the impact of under or overestimation of input parameters on the optimal values of selling price (p^*), replenishment cycle length (T^*), order quantity (Q^*), and the total profit per unit time ($\text{TP}^*(p, T)$) of the inventory system. The sensitivity analysis is carried out by changing the input parameters from − 20 to 20%. It is done by changing the input parameters one at a time and keeping the other parameters constant. The results are presented in Table 10.2. From Table 10.2, following results are obtained:

(1) With the increase in purchasing price c, the optimal selling price (p) increases. It is also observed that the optimal replenishment cycle length (T), optimal order quantity (Q), and the total profit per unit time ($\text{TP}(p, T)$) decreases. It

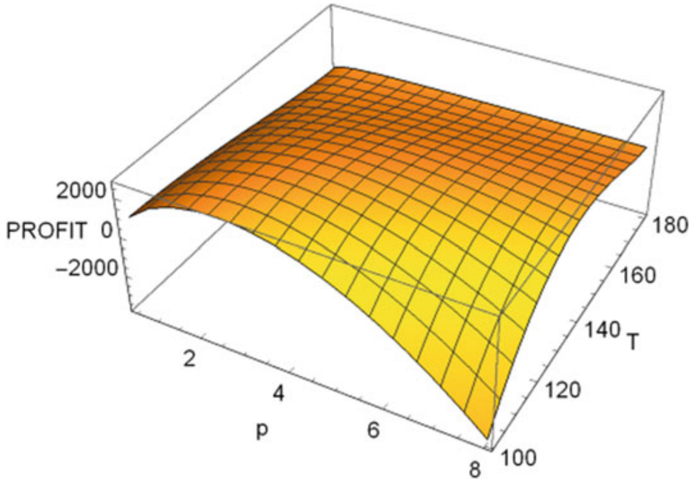

Fig. 10.1 Change in the profit function with respect to p and T

is obvious that if the purchasing price of goods increases, the selling price will also increase.

(2) As holding cost h increases, optimal replenishment cycle length (T), optimal order quantity (Q), and the total profit per unit time (TP(p, T)) decreases. Selling price (p) remains constant up to some time and then increases. With the increase in holding cost, inventory carrying cost increases, and hence profit decreases.

(3) With the increase in the value of γ, the optimal replenishment cycle length (T), optimal order quantity (Q), and the total profit per unit time (TP(p, T)) decreases whereas selling price (p) increases. With the increase in the value of γ, holding cost increases, hence retailer's order quantity decreases, and total profit decreases.

(4) As a increases, it is observed that optimal selling price (p), optimal replenishment cycle length (T), optimal order quantity (Q), and the total profit per unit time (TP(p, T)) increases. With the rise in the value of b, optimal selling price (p), optimal replenishment cycle length (T), optimal order quantity (Q), and the total profit per unit time (TP(p, T)) decreases.

(5) As the value of β increases, optimal replenishment cycle length (T), optimal order quantity (Q), and the total profit per unit time (TP(p, T)) increases. As α increases, order quantity (Q) and the total profit per unit time (TP(p, T)) increases. It is observed that as ordering cost increases, total profit per unit time decreases.

(6) As I_p increases, total profit per unit time decreases. It is observed that as the value of I_e increases, total profit per unit time increases. As M increases, the retailer has the chance to sell more goods and collect sales revenue. The retailer

Table 10.2 Sensitivity analysis with respect to input parameters

Parameter	% Change in parameter	Change in optimal values			
		p^*	T^*	TP*	Q^*
c	− 20	109.464	3.39286	3507.8	269.317
	− 10	110.875	3.23214	3088.72	240.166
	10	116.518	2.91071	2363.57	175.005
	20	117.929	2.75	2050.39	153.346
h	− 20	113.696	3.55357	2990.11	262.408
	− 10	113.696	3.23214	2840.04	224.054
	10	113.696	2.91071	2591.97	188.163
	20	115.107	2.75	2486.34	165.14
γ	− 20	113.696	7.08929	4410.35	829.608
	− 10	113.696	4.51786	3466.79	391.522
	10	115.107	2.10714	2163.59	105.955
	20	116.518	1.625	1778.25	66.2433
a	− 20	101	2.75	917.614	83.1708
	− 10	105.232	2.91071	1669.58	145.258
	10	123.571	3.39286	4099.84	294.543
	20	132.036	3.71429	5902.13	418.713
b	− 20	136.268	3.875	4873.87	339.656
	− 10	123.571	3.39286	3608.74	258.273
	10	106.643	2.75	2049.78	155.914
	20	101	2.58929	1556.68	126.736
β	− 20	115.107	2.58929	1702.19	99.6341
	− 10	115.107	2.75	2112.68	132.749
	10	115.107	3.39286	3602.23	305.972
	20	113.696	3.71429	5024.96	511.158
α	− 20	115.107	3.23214	1879.38	149.027
	− 10	113.696	3.07143	2280.27	172.653
	10	113.696	3.07143	3163.33	241.226
	20	113.696	3.07143	3642.28	278.872
o	− 20	113.696	3.07143	2721.98	205.796
	− 10	113.696	3.07143	2715.47	205.796
	10	113.696	3.07143	2702.45	205.796
	20	113.696	3.07143	2695.94	205.796
I_p	− 20	113.696	3.23214	2769.02	224.054
	− 10	113.696	3.07143	2738	205.796
	10	113.696	3.07143	2679.93	205.796

(continued)

Table 10.2 (continued)

Parameter	% Change in parameter	Change in optimal values			
		p^*	T^*	TP*	Q^*
	20	113.696	2.91071	2652.47	188.163
I_e	− 20	113.696	3.07143	2702.18	205.796
	− 10	113.696	3.07143	2705.57	205.796
	10	113.696	3.07143	2712.35	205.796
	20	113.696	3.07143	2715.74	205.796
M	− 20	113.696	3.07143	2666.35	205.796
	− 10	113.696	3.07143	2687.58	205.796
	10	113.696	3.07143	2730.5	205.796
	20	113.696	3.07143	2752.19	205.796

has to pay interest charges for a lesser number of goods, hence total profit per unit time increases.

Managerial Insights

In order to compete in the business era, it is very important to decide the selling price of the item since it directly impacts customer choice. The retailer must appropriately decide the selling price in order to generate profit rather than suffer loss. Another aspect is to decide efficiently the replenishment cycle length so as to avoid shortages and run the business smoothly. To provide a better insight, demand is supposed to be a function of selling price as well as the inventory level. It is also important to efficiently manage the total holding cost. Trade credit policy allows the buyer to purchase goods without paying instantly. While this policy looks lucrative, a better insight can only be provided if the effect of this policy on the buyer's profit is analyzed from all aspects which is one of the intentions of this research work. This research work can also help the managers in analyzing the impact of the important parameters like purchasing price, ordering cost, demand parameters, etc., and improve the efficacy of the supply chain.

Conclusion

With the increase in globalization, the demand for an efficient and effective supply chain management has increased. Selling price plays a significant role in deciding the demand of a product. Along with selling price customer's demand is also determined by exhibited stock. In this research work, an inventory model is presented considering demand to be a function of selling price as well as displayed stock along with trade credit policy. In general, the common perception is that cost of holding goods in the inventory is always linearly dependent on stock which need not to be. In this paper, holding cost is considered to be nonlinearly dependent on stock. Trade credit policy allows the buyer to purchase goods without paying instantly. In this research work, the supplier grants a full trade credit period to the retailer. The main aim of the proposed model of inventory is to determine the optimal selling price, optimal replenishment cycle length so as to maximize the total profit earned by the retailer per unit time.

This research work can be extended along many directions such as: partial trade credit policy, inflation, fuzzy-valued inventory costs, credit-dependent demand function among others.

Appendix 1

$$
\begin{aligned}
f(T) = \Big[& p[\alpha(a-bp)(1-\beta)T]^{\frac{1}{1-\beta}} \\
& + pI_e\Big[M[\alpha(a-bp)(1-\beta)T]^{\frac{1}{1-\beta}} \\
& + \frac{1}{\alpha(a-bp)(2-\beta)}\Big\{[(a-bp)(1-\beta)(T-M)]^{\frac{2-\beta}{1-\beta}} \\
& - [\alpha(a-bp)(1-\beta)T]^{\frac{2-\beta}{1-\beta}}\Big\}\Big] \\
& - o - c[\alpha(a-bp)(1-\beta)T]^{\frac{1}{1-\beta}} \\
& - h\frac{[\alpha(a-bp)(1-\beta)T]^{\frac{\gamma+1-\beta}{1-\beta}}}{\alpha(a-bp)(\gamma+1-\beta)} \\
& - \frac{cI_p\Big[[\alpha(a-bp)(1-\beta)(T-M)]^{\frac{2-\beta}{1-\beta}}\Big]}{\alpha(a-bp)(2-\beta)}\Big]
\end{aligned}
$$

$$
g(T) = T
$$

If $f''(T) < 0$, then by using the theoretical result in (10.14) it can be proved that $TP_1(p, T)$ is a strictly pseudo-concave function in T, which completes the proof of Part (1). The proof of Part (2) and Part (3) follows immediately from the proof of Part (1) of Theorem 1.

Appendix 2

Similar to Appendix 1.

Appendix 3

Since $\mathrm{TP}_1(p, T)$ is a strictly pseudo-concave function in T, we know that $\frac{\partial \mathrm{TP}_1(p,T)}{\partial T}$ is a decreasing function in T. If $\Delta > 0$, then $\lim\limits_{T\to\infty} \frac{\partial \mathrm{TP}_1(p,T)}{\partial T} < 0$.

By applying mean value theorem, we know that there exists a unique $T_1^* \in (M, \infty)$ such that $\frac{\partial \mathrm{TP}_1(p,T)}{\partial T} = 0$. By this, we complete the proof of $\Delta > 0$. Similarly, other theorems of Theorem 5 can be proved.

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Chapter 11
Buying Guide for Best Car in India: An Application of Data Envelopment Analysis

Neelanghsu Ghosh

Abstract Buying the right car is tricky proposition for middle-class people in India. With the availability of number of mid-range cars, it is difficult to choose the suitable automobile, which is pocket friendly, having good looks and value for money. Buying that dream car gives a bargain hunter immense pleasure and satisfaction and a sense of heavenly accomplishment. In this paper, data envelopment analysis (DEA) is applied to find the best vehicle in the offering within a budget of four to six lakhs of Indian rupees. The methodology involved and the input–output combination chosen is justified thoroughly. Interestingly, it is found that most of the Indian car manufacturers know the mindset of Indians and make their four-wheelers appropriate for an Indian buyer. Nearly, all of the available brands in India have at least one car that falls in the category of most desirable car for the middle class.

Keywords Pocket friendly · DEA · Input–output combination · Quantitative and qualitative factors · Most desirable car

Introduction

There are a number of cars available for middle-class people in India to buy and fulfill their dreams of buying a four-wheeler, but not all of them come with an affordable price or nicer looks or best mileage. People of India have choices—they can select from a wide range of cars, be it a hatchback or a sedan or convertible. But which one is the top four-wheeler to buy? People, for whom buying a car is a luxury wants to put their best foot forward while purchasing a vehicle. Middle income group people always have that vision of buying the best automobile at an affordable price. They have a perception that their best car will be reasonably priced with moderate looks and comfortable to use. Generally, most people in India would like to buy a hatchback as their first automobile. With so many cars available in the market, it is important for

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them to know what features they are getting in that price. For the people in middle-income group, they want to buy a vehicle that gives them satisfaction in terms of mileage, comfort and status. It is also very important to them that they are getting their money's worth by buying their "dream" vehicle. People want to make the best use of their money they are paying.

Data envelopment analysis (DEA) is an efficiency measurement tool in mathematics. Efficiency is measured as the total output obtained through total input, it is worthwhile using DEA to find the best four-wheeler. Although there are several applications of DEA in automobile industry, not many could be found in terms of efficiency measurement of cars. Performance measurement of automobile industry in India [1] and Taiwan [2] was done earlier using DEA. Evaluation of performance for Turkey's car industry was done in the year 2018 [3]. Papagapiou et al. used DEA to obtain purchasing decisions to buy used cars [4]. In this paper, DEA is applied to find the car, which gives best output using similar or less inputs. The best car is defined as the one that gives the maximum output in terms of satisfaction, status or comfort using minimum input in terms of price, mileage, etc., among others. The inputs and outputs related to a four-wheeler will be discussed extensively.

Determining Inputs and Output for a Car

As it is said earlier, in India, there are a number of cars available. In this section, the discussion is restricted within the assortment of cars, which are reasonably priced for the middle-class people. Since only the affordable vehicles in India are considered, the discussion is confined to cars, which run on petrol, having manual transmission and price is between four lakhs to six lakhs of rupees. There are some cars available below that range, but they lack the styling, safety and comfort components. Also there is enough number of automobiles available above that range but those seemed too costly and thus not affordable for middle-class people of India. As in DEA methodology, these characteristics are termed as "homogeneous" and thus, this paper is limited within that domain. It is a pre-condition for DEA that all DMU's must be homogeneous in nature. In that price choice, several cars from the different automobile giants are available in India. The discussion is restricted to the popular models within the variety. Middle-class people can choose several cars from the brand name of "Maruti". They are Alto 800 VXI+, Wagon R LXI, Baleno Sigma, Swift LXI, Dezire LXI, Ignis Sigma, Celerio LXI, Eeco AC 5 Seater and S-Presso LXI. There are Elite i20 Era, Grand i10 Magna, Aura E, Santro Magna, Accent and Grand i10 Nios ERA to pick from brand "Hyundai". From brand "Tata", middle-class people can choose Tiago Petrol version, Altroz XE or Tigor. From the "Mahindra" group, there is 100KUVNxt. Also, there are Figo Ambiente, Freestyle and Aspire from the oldest car manufacturer of the world "Ford". There are Polo 1.0 MPI from "Volkswagen", KWID Climber 1.0 MT or Triber RXL from "Renault", GO Plus (A) Petrol or GO from "Datsun" and finally Punto EVO from brand "Fiat". Other models of different brands have not been considered as either they do not belong to that price category or they do not satisfy the homogeneity criteria. Several models of different brands in that price bracket are not considered because either they run on diesel or they may not be having a manual transmission. Only one variant of a particular brand is considered (e.g., for Alto only VXI model is considered and not LXI model and similarly for other brands). Hence, all the above-mentioned cars are taken as DMUs. In the next section, the different inputs and outputs are considered in case of buying a car.

Quantitative Factors

The next step is to determine the inputs that decide buyer's choice. Firstly, the most prominent and important factor that influences a middle-class person's buying decision is the "price" of a four-wheeler. They always look for the car, which is more pocket friendly for them. They also look for what they get in that affordable price. Then, the other factors like appearance, comfort, safety come into picture. Buyers then try to judge what they are getting in that price. There are some measurable quantities, which buyers get by purchasing a car: the engine, the boot space, power or torque generated by the engine, interior and exterior décor, capacity of the fuel tank, ground clearance and wheelbase. All these factors add value to the car. All of them contribute to the comfort, safety, styling and satisfaction of a buyer. There are different definitions of a vehicle's price. There is "ex-showroom price", and there is "on-road price". Generally, speaking the "ex-showroom price" includes the price of the car and some basic accessories, whereas, the "on-road price" includes the ex-showroom price + insurance + registration charges + lifetime road tax. The "ex-showroom price" varies from place to place as well as registration and road tax. To maintain the homogeneity, the "ex-showroom price" at Kolkata for each vehicle is considered in this paper. The insurance of a four-wheeler is also a very important component while purchasing a vehicle. It saves the driver from the legal repercussions in case of an accident. Also, it bears the damages to another person's property or injury to the driver as well as pedestrian because of accident, so insurance value of the car is chosen as the second input.

Thirdly, the next deciding factor is the "mileage". As a famous automobile manufacturer in India puts it in its advertisement: a customer goes to a showroom and the first question is asked about the car is the mileage. Incidentally, the fuel required per kilometer to run the car is also the supreme important factor in car buying decision. Because running cost and maintenance cost of a vehicle will always be considered by a rational buyer as those add to the cost of the car after it is purchased. Therefore, less fuel consumption is better for an automobile. Hence, petrol requirement is also one of the powerful factors that decide which car to buy. Therefore, in this paper, the third and the last input is the amount of fuel required per kilometer.

Qualitative Factors

But there are some qualitative factors too. What about good looks of a car? The looks of an automobile indicate the status of its owner. People view different cars in diverse ways. It is their perception about a car that establishes their status. As the saying goes- “if looks could kill”; hence design and appearance of the vehicle is one of the most important output to decide the choice of a car. Driving pleasure is also an intangible factor that helps in making up a buyer’s mind. However, for a middle-class car consumer, that kind of first hand driving experience may not be present. Not every purchaser can have an experience of test-drive unless he/she knows how to drive. Also a single test-drive may not be able to provide with the driving pleasure or vehicle handling experience.

Thirdly, the next important qualitative criterion is the comfort level in riding the car. People may not have the riding experience of each and every four-wheeler within the given price range, but they can have their own observations about the comfort levels provided by different cars. Also, they can check out the different models in the websites, gather information from different magazines or can talk to people who are actually using them. They can form their own opinion using all this information. After that, they can pick the vehicle that suits their requirement. Hence, comfort criterion is taken into consideration.

Fourthly, the performance level of a car is another essential decisive factor in making a verdict. The level of performance depends upon its engine: the power it generates, the torque it delivers, how quickly it picks up are some measures of engine benchmarks. Along with these, the riding comfort and driving pleasure adds to the level of performance. Fortunately, some of these quantities are measurable. The engine displacement, the power generated by the engine or the torque created is measurable for each automobile.

Finally, another important aspect for car buying judgment is the safety provided by it. Nowadays, cars come with several important safety features. Whether it is the anti-lock braking system (ABS) or child safety system or airbags, they provide value added safety features for a vehicle. These safety equipments have become essential parts of each and every car. Obviously, these features have increased the cost of the car, but these have become necessity for a buyer.

Hence, the qualitative factors like riding experience, styling, seating and riding comfort and degree of safety are very difficult to measure, and they also vary according to the perception of the buyer. But, it is certain that all of them add to the value of the car and hence, the cost of the four-wheeler increases consequently. Without the loss of generality, it can be said that a car will be more costly with greater styling quotient, better riding comfort and more safety equipment.

Instead of taking these qualitative factors, some measurable quantities can be chosen as their alternative. One of the proxies that can be selected against the safety is the ground clearance of an automobile. Ground clearance or in other words ride height of a car is one of the most fundamental but very important common dimensions of a vehicle. It is defined as the minimum distance between the lower end of the car’s body (i.e., the chassis) and the road. In other words, it indicates the elevation of the lowermost part of the automobile with respect to ground [online] (www.carbiketech.com). Generally, most of the manufacturers specify this dimension in unladen[1] condition. Actually, when a vehicle is loaded with passengers and cargo, the available ride height is always lower than specified in the actual running condition. On rough and uneven roads, higher the ground clearance better it is for the car. This is because it avoids the roughing up of the underbelly of the motor vehicle.

Ground clearance is one of the complicated dimensions to determine for a vehicle as maneuvering the vehicle directly depends on it. The higher the clearance, the higher is the position of center of gravity (CG) of the vehicle. And the higher position of CG means that the vehicle is prone to toppling over. That means vehicles with higher clearance have more possibility of roll over than the vehicles with lower ground clearance. Thus, it influences the handling of the car. On the other hand, vehicles with lower ride height offer excellent handling features due to lower height of vehicle CG. Hence, measure of ground clearance is a quantifiable substitute of safety and riding comfort. As ground clearance is a risky proposition, this study avoids taking it as input.

Safety features cannot be neglected as a contributory option to finalize a vehicle. New Car Assessment Program (NCAP) highlights the safety features in automobiles including how well the vehicle performs in various crash test (frontal, side). The idea of this test is to diagnose the safety of the ocupants in case of collision and how well the safety gears and features provide protection on the whole to car occupants including adults and children. Various NCAP Agencies—Global NCAP [online] (www.globalncap.org/about),[2] Euro NCAP, ASEAN NCAP and Australian NCAP are well-known in providing ratings to various models sold in India. Hence, alternately the safety rating offered by NCAP could be taken as input for safety performance of a car.

The second substitute that can be used against the riding comfort and safety is the measurement of wheelbase. It is the center distance between the front and the rear axle of the vehicle. It is one of the important dimensions of the vehicle that is specified by the vehicle manufacturers. The main significance of wheelbase lies in the fact that it gives an indication of the dynamic characteristics of vehicle, i.e., ride and handling. Along with the wheelbase, ground clearance also affects the handling of the vehicle. Vehicles with the longer wheelbase are believed to have better ride comfort than the vehicles with shorter wheelbase. This is mainly because such vehicles have a reduced tendency to pitch and roll compared to other vehicles. Also, a long wheelbase allows the vehicle manufacturers to allocate a larger area for passenger compartment. It makes the cabin more spacious and thus the ride more comfortable. On the other hand, shorter wheelbase vehicles are better in maneuvering turns at high speed. Therefore, the wheelbase helps in riding comfort for the passengers on long journeys.

[1] Without any load of cargo or passengers.

[2] Global NCAP's serves as a platform for co-operation among New Car Assessment Programs worldwide and promotes the universal adoption of the United Nation's most important motor vehicle safety standards worldwide (http://www.globalncap.org/about/).

The manufacturers prefer to design the vehicles with longer wheelbase to maximize the comfort for the passengers. Hence, length of the wheelbase in millimeter (mm) of a car could be chosen as input.

One crucial deciding factor for purchasing a four-wheeler is its features and styling quotient. The color, design, aerodynamics and looks of an automobile have immense impact on buyers' decisions. A buyer will stop to spend his hard-earned money on a car if it does not look appealing to others. It gives the buyer a sign of accomplishment, a status symbol with his choice of vehicle. Despite having good mechanical properties, if a car does not follow good interior or exterior design that will affect its sales. It is a key feature to attract people's attention, which can hide other flaws in a vehicle. By the design and appearance, some people get fooled, with low efficient engine and safety. Some concentrate more on design and ruin the looks. For some purchaser, the one with best look wins, not the one with best quality. Hence, features and styling could be chosen as one output.

There are quite a few more input-outputs possibilities, but for this particular study only the above-discussed input–output combinations are considered.

Methodology: Selection of DEA Model

There exist different models of DEA in the literature. Production efficiency concept was first introduced by Farrell [5] which led to the development of series of work in the economics literature beginning with Aigner and Chu [6]. DEA was introduced by Charnes et al. (CCR Model) [7, 8] and further extended by Banker et al. [9] (BCC Model) and other alternative DEA models [10, 12]. It provides a non-parametric and extremal method for estimating production frontiers and evaluating relative efficiency of decision-making units (DMU). The first DEA model (CCR) assumed constant returns to scale (CRS), while BCC extended the model to the situation of variable returns to scale (VRS).

For this present DEA study, it is decided that the input-oriented DEA model with constant returns to scale or the CCR-I model will be used. The reason behind choosing an input oriented model is that keeping the internal features, external attributes, engine, safety and other features intact, how the inputs could be improved to obtain the optimum score. In other words, which cars are providing better output features (power, boot, safety, etc.) using smaller inputs (price, petrol consumption, etc.). That is, the vehicle which uses lesser input contributions and provides higher output utilization is always a superior choice to buy. A constant return to scale (CRS) model is selected because from a buyer's point of view all cars input–output gives similar returns, i.e., he understands that higher priced automobiles will give him better proportionate safety and features than lower priced ones. Hence, choosing the CCR-I model fits the bill.

Input–Output Combination and Data

It is very important to consider the number of inputs and outputs associated with the problem while selecting a particular model. If the number of DMUs (n) is less than the combined number of inputs (m) and outputs (s), that is, if $n < (m + s)$, then a lot of DMUs will become efficient. Hence, efficiency of the DMUs would be questionable due to inadequate number of degrees of freedom. Hence, it is suggested that "n" exceeds "$m + s$" by several times. As a rule, it can be chosen as "n" equal to or greater than [max $\{m * s, 3 * (m + s)\}$] [11]. Thus, the selection of input and output items along with the number of DMU is crucial for successful application of DEA.

Hence, the following input–output combinations are considered for this study.

The inputs are:

(i) ex-showroom price (Kolkata) in rupees
(ii) insurance of the car in rupees
(iii) fuel required in cubic centimeters (cc) per kilometer.

The corresponding outputs are:

(i) power generated by the engine in bhp/rpm (brake-horse-power per rotation per minute)
(ii) torque produced by the engine in Nm/rpm (Newton-meter per rotation per minute)
(iii) boot space (space for luggage) in liter
(iv) wheelbase in mm
(v) safety rating based on NCAP[3]
(vi) standardized rating for featured and styling out of 100.[4]

The dataset used for this paper for 28 different four-wheeler models of several brands is collected from the website [online] (www.zigwheels.com) on a particular date, so that standardization of the data could be maintained. All the information about the cars, i.e., all inputs and five of the six output details are collected from the same website. The output-safety rating based on NCAP is collected from different NCAP websites[5] and another website [online] (www.mycarhelpline.com). Also, standardized feature and styling data is collected from www.zigwheel.com on the same date.

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[3] The rating is calculated based on the mathematical formula safety rating = (1.26 + 2.79 * star rating for adult) + (− 0.79 + 9.28 * star rating for child). It is obtained using linear regression. The maximum possible rating is 66.03.

[4] Standardized rating for featured and styling = (average rating of each car *100/ $\bar{X}$) / maximum possible rating, where. $\bar{X} = \sum$ (average rating of each car * number of purchaser of each car)/total number of purchasers.

[5] http://www.globalncap.org/, https://aseancap.org/, https://www.euroncap.com/ and https://www.ancap.com.au/.

Results of DEA

Now, the best car is found using the CCR-I[6] model of DEA on the above input–output combination. The dream car will have a score 1. Those cars will be both affordable, i.e., input efficient as well as provides best output.

The following Table 11.1 shows the cars with scores 1 are suitable to buy.

The fifth column of Table 11.1 concludes each affordable (with score 1) automobile is peer to a number of cars. For Alto 800, VXI + the number "4" in the fifth column suggests that it is peer to four vehicles other than itself. For KUV100Nxt, the number "0" in the fifth column imply that it is peer to only itself. Now for a vehicle with score < 1, the fifth column proposes the weighted combination of peer set to bring that vehicle to the buyable frontier. For example, the car at serial number 11 in the above table, Celerio LXI (score—0.996369, rank—14) has [1(0.02) 2(0.02) 5(0.76) 6(0.01) 12(0.19)] in the fifth column. It implies that Celerio LXI can bring itself on the efficient frontier using the combination of 2% of Alto 800 VXI + (the car at Sl no. 1) + 2% of Baleno Sigma (the car at Sl no 2) + 76% of Wagon R LXI (the car at Sl no 5) + 1% of Tiago Petrol (the car at Sl no. 6) + 19% of S-Presso LXI (the car at Sl no. 12). Table 11.1 also advocates that Tata Tiago Petrol is peer to 15 of the 28 cars except itself.

Table 11.2 represents results of DEA of input excess and target percent of improvement required in different inputs for inefficient DMU's.

The following Table 11.3 demonstrates the amount of shortage in each output for every non-buyable vehicle.

The following Table 11.4 also shows the target improvement required for three different outputs.

Discussion

The DEA results show that there are 13 out of 28 cars available within 4 lakhs to 6 lakhs of rupees in the Indian market, which are appropriate to buy and thus a middle-class person can be satisfied. Brand "Maruti" has Alto 800 VXI+, Wagon R LXI, Baleno Sigma, Dezire LXI, Ignis Sigma, Eeco AC 5 Seater and S-Presso LXI, which could fall into the dream car category. A buyer can buy any of them as his pocket permits. If he is not willing to buy brand "Maruti", he can go for "Tata" Altroz XE or Tiago or Tigor. Still, he has other alternatives from other brands. He can settle for KWID Climber 1.0 MT from "Renaults" or Punto EVO from "Fiat" or KUV100NXT from "Mahindra". The DEA result shows these cars belong to the efficient frontier. The amount of money buyer is paying to purchase these cars or the insurance money he is paying to insure or fuel required to travel per kilometer

[6] DEA-solver is used to solve.

Table 11.1 Results of DEA for cars showing score, rank and peer group

Sl. No.	Car	Score	Rank	Peer to number of cars	Peer group
1	Alto 800 VXI+	1	1	4	Alto 800 VXI+
2	Baleno Sigma	1	1	6	Baleno Sigma
3	Swift LXI	0.94155	19	5 (0.09) 6 (0.63) 7 (0.02) 8 (0.28)	Wagon R LXI, Tata Tiago Petrol, Dzire LXI, KWID Climber 1.0 MT
4	Elite i20 Era	0.88228	26	1 (0.01) 2 (0.11) 5 (0.15) 6 (0.70) 12 (0.11)	Alto 800 VXI+, Baleno Sigma, Wagon R LXI, Tata Tiago Petrol, S-Presso LXI
5	Wagon R LXI	1	1	10	Wagon R LXI
6	Tiago Petrol	1	1	15	Tata Tiago Petrol
7	Dzire LXI	1	1	5	Dzire LXI
8	KWID Climber 1.0 MT	1	1	5	KWID Climber 1.0 MT
9	Triber RXL	0.94743	18	6 (0.52) 7 (0.15) 8 (0.43)	Tata Tiago Petrol, Dzire LXI, KWID Climber 1.0 MT
10	Grand i10 Magna	0.8334	28	5 (0.13) 6 (0.72) 8 (0.15)	Wagon R LXI, Tata Tiago Petrol, KWID Climber 1.0 MT
11	Celerio LXI	0.99637	14	1 (0.02) 2 (0.02) 5 (0.76) 6 (0.01) 12 (0.19)	Alto 800 VXI+, Baleno Sigma, Wagon R LXI, Tata Tiago Petrol, S-Presso LXI
12	S-Presso LXI	1	1	4	S-Presso LXI
13	Ignis Sigma	1	1	3	Ignis Sigma
14	Altroz XE	1	1	4	Tata Altroz XE
15	Polo 1.0 MPI Trendline	0.89884	24	2 (0.45) 6 (0.07) 12 (0.04) 13 (0.33) 14 (0.11)	Baleno Sigma, Tata Tiago Petrol, S-Presso LXI, Ignis Sigma, Tata Altroz XE
16	Huyndai Aura E	0.97747	16	5 (0.07) 6 (0.02) 7 (0.20) 22 (0.71)	Wagon R LXI, Tata Tiago Petrol, Dzire LXI, Tata Tigor
17	Grand i10 Nios ERA	0.92639	20	2 (0.01) 5 (0.19) 6 (0.80) 8 (0.01)	Baleno Sigma, Wagon R LXI, Tata Tiago Petrol, KWID Climber 1.0 MT

(continued)

Table 11.1 (continued)

Sl. No.	Car	Score	Rank	Peer to number of cars	Peer group
18	Eeco 5 Seater AC	1	1	1	Eeco 5 Seater AC
19	GO Plus (A) Petrol	0.94886	17	5 (0.35) 6 (0.38) 7 (0.12) 18 (0.17)	Wagon R LXI, Tata Tiago Petrol, Dzire LXI, Eeco 5 Seater AC
20	Figo Ambiente	0.91703	21	6 (0.70) 13 (0.27) 14 (0.08)	Tata Tiago Petrol, Ignis Sigma, Tata Altroz XE
21	Santro Magna	0.89357	25	1 (0.18) 5 (0.10) 6 (0.50) 8 (0.23)	Alto 800 VXI+, Wagon R LXI, Tata Tiago Petrol, KWID Climber 1.0 MT
22	Tata Tigor	1	1	3	Tata Tigor
23	Hyundai Xcent	0.98285	15	6 (0.02) 14 (0.04) 22 (0.92)	Tata Tiago Petrol, Tata Altroz XE, Tata Tigor
24	Ford Freestyle	0.86793	27	2 (0.11) 6 (0.75) 13 (0.13) 14 (0.06)	Baleno Sigma, Tata Tiago Petrol, Ignis Sigma, Tata Altroz XE
25	Ford Aspire	0.90681	22	2 (0.05) 5 (0.32) 6 (0.32) 7 (0.14) 22 (0.25)	Baleno Sigma, Wagon R LXI, Tata Tiago Petrol, Dzire LXI, Tata Tigor
26	Fiat Punto EVO	1	1	0	Fiat Punto EVO
27	KUV100NXT	1	1	0	KUV100NXT
28	Datsun GO	0.90568	23	1 (0.04) 5 (0.13) 6 (0.59) 12 (0.26)	Alto 800 VXI+, Wagon R LXI, Tata Tiago Petrol, S-Presso LXI

Source Calculated by the author

is entirely acceptable to the levels of output provided by these cars. Unfortunately, the same could not be concluded about the other 15 cars. In fact, according to the outcome there is no car available from the brands “Hyundai”, “Volkswagen”, “Ford” or “Datsun” which gives desirable result.

Table 11.2 explains the reason, why the other 15 cars are unsuitable to buy for a middle-class individual. Either they are using excess amount of inputs for the similar amount of outputs or their current input should be reduced to achieve the optimum output. Maruti Swift LXI, Renault Triber RXL, Hyundai Grand i10 Magna, Hyundai Xcent and Ford Freestyle are costlier (excess ex-showroom prices) in contrast to the equivalent cars. The insurance costs of Hyundai Aura E, GO Plus (A) Petrol, Hyundai

Table 11.2 Results of DEA of input excess and target percent of improvement required in different inputs for inefficient DMU's

Car	Excess in ex-showroom price (rupees)	Excess in insurance (rupees)	Excess in the amount of fuel in liter required per km travel	Target percentage of improvement required for ex-showroom price[7]	Target percentage of improvement required in insurance cost[8]	Target percentage of improvement required in the amount of fuel needed per km travel[9] (%)
Swift LXI	15,340.27	0.00	0.0000	− 8.80	− 5.84	− 5.84
Elite i20 Era	0.00	0.00	0.0000	− 11.77	− 11.77	− 11.77
Renault Triber RXL	25,945.08	0.00	0.0000	− 9.71	− 5.26	− 5.26
Grand i10 Magna	39,241.59	0.00	0.0000	− 23.16	− 16.66	− 16.66
Celerio LXI	0.00	0.00	0.0000	− 0.36	− 0.36	− 0.36
VW Polo 1.0 MPI	0.00	0.00	0.0000	− 10.12	− 10.12	− 10.12
Huyndai Aura E	0.00	88.16	0.0000	− 2.25	− 2.52	− 2.25
Grand i10 Nios ERA	0.00	0.00	0.0000	− 7.36	− 7.36	− 7.36
GO Plus (A) Petrol	0.00	897.68	0.0000	− 5.11	− 8.10	− 5.11
Ford Figo Ambiente	0.00	0.00	0.0021	− 8.30	− 8.30	− 12.24
Santro Magna	0.00	0.00	0.0000	− 10.64	− 10.64	− 10.64
Hyundai Xcent	10,878.75	318.28	0.0000	− 3.57	− 2.68	− 1.72
Ford Freestyle	12,372.48	0.00	0.0000	− 15.31	− 13.21	− 13.21
Ford Aspire	0.00	0.00	0.0000	− 9.32	− 9.32	− 9.32
Datsun GO	0.00	734.76	0.0000	− 9.43	− 11.88	− 9.43

Source Calculated by the author using OSDEA-GUI

[7] (−) sign indicates ex-showroom price should be reduced.
[8] (−) sign indicates insurance amount should be reduced.
[9] (−) sign indicates less fuel should be consumed for travel.

Table 11.3 Results of DEA of output shortage in different inefficient DMU's

Car	Shortage of safety rating	Shortage of torque in Nm/rpm	Shortage of wheelbase in mm	Shortage of boot space in liters	Shortage of power in bhp/rpm	Shortage of standardized features and styling rating out of 100
Swift LXI	15.30	0.0034	0	0	0.0000	7.99
Elite i20 Era	7.58	0.0042	0	0	0.0008	0.00
Renault Triber RXL	12.93	0.0032	0	216	0.0033	0.00
Grand i10 Magna	32.51	0.0030	0	6	0.0000	1.38
Celerio LXI	15.23	0.0000	0	88	0.0010	0.00
VW Polo 1.0 MPI	0.00	0.0020	0	23	0.0015	0.00
Huyndai Aura E	0.00	0.0035	0	0	0.0005	6.13
Grand i10 Nios ERA	7.85	0.0045	0	7	0.0003	0.00
GO Plus (A) Petrol	19.27	0.0046	0	0	0.0000	6.30
Ford Figo Ambiente	0.00	0.0058	37	9	0.0000	11.75
Santro Magna	11.00	0.0059	0	14	0.0000	6.57
Hyundai Xcent	0.00	0.0054	0	0	0.0003	5.34
Ford Freestyle	0.00	0.0060	33	14	0.0000	0.00
Ford Aspire	11.69	0.0045	123	0	0.0000	0.00
Datsun GO	17.32	0.0051	0	0	0.0000	16.01

Source Calculated by the author using OSDEA-GUI

Xcent and Datsun GO are in excess, whereas Ford Figo Ambiente is using excess petrol per kilometer. Table 11.2 also shows the reduction of input combinations required for each of the inefficient vehicle. For example, Maruti Swift LXI has to reduce 8.8% of ex-showroom price, 5.84% of insurance cost and 5.84% fuel consumption to achieve target as good as its peer set's (Wagon R LXI, Tata Tiago Petrol, Dzire LXI and KWID Climber 1.0 MT) input combination. Then, only it can

Table 11.4 Results of DEA of output target percentage improvement in different inefficient DMU's

Car	Target percentage improvement of safety rating (%)	Target percentage improvement of torque (%)	Target percentage improvement of wheelbase (%)	Target percentage improvement of boot space in (%)	Target percentage improvement of power (%)	Target percentage improvement of standardized features and styling rating out of 100 (%)
Swift LXI	62.16	0.00	0.00	0.00	0.00	10.24
Elite i20 Era	20.66	0.00	0.00	0.00	0.00	0.00
Renault Triber RXL	52.53	0.00	0.00	256.95	0.00	0.00
Grand i10 Magna	333.40	0.00	0.00	2.41	0.00	1.65
Celerio LXI	156.23	0.00	0.00	37.57	0.00	0.00
VW Polo 1.0 MPI	0.00	0.00	0.00	8.36	0.00	0.00
Huyndai Aura E	0.00	0.00	0.00	0.00	0.00	7.66
Grand i10 Nios ERA	21.41	0.00	0.00	2.71	0.00	0.00
GO Plus (A) Petrol	153.66	0.00	0.00	0.00	0.00	8.76
Ford Figo Ambiente	0.00	20.87	1.47	3.61	0.00	14.69
Santro Magna	44.68	26.84	0.00	5.86	0.00	8.88
Hyundai Xcent	0.00	19.14	0.00	0.00	0.00	6.36
Ford Freestyle	0.00	21.33	1.34	5.47	0.00	0.00
Ford Aspire	42.65	0.00	4.93	0.00	0.00	0.00
Datsun GO	79.40	19.48	0.00	0.00	0.00	23.55

Source Calculated by the author using OSDEA-GUI become appropriate to buy for a middle-class buyer. Interestingly, Table 11.2 does not show any excess input for Elite i20 Era, Celerio LXI, VW Polo 1.0 MPI, Grand i10 Nios, Santro Magna and Ford Aspire but still they are not apt for purchase. Their peer set of cars use lesser combination of inputs. Hence, their target is to diminish the percentage of input use to bring them on the efficient frontier.

Table 11.3 shows most of these vehicles are unreasonable to buy as their safety rating and styling ratings are low. Renault Triber RXL has a very low boot space. The result recommends all the unproductive cars need to improve at least three of the six outputs individually. It also suggests that out of these six key features—safety, torque and styling are the three vital components, which are in deficit for most of the cars. Realistically, these three features are related to safety of the user, performance of the car and status of the owner. And for overall choice of a car today's learned buyers would not like to compromise on them. Upgrading of these features will help the non-desirable vehicles to achieve desired sales target in the future. According to Table 11.1, the last three ranks are taken by two cars from Hyundai and one car from Ford. Table 11.3 confirms that these cars have shortages in four of the six outputs, so they need to improve upon these components for better desirable result.

Table 11.4 emphasizes the observation from Table 11.3. It depicts the target percentage improvement necessary for each of the less significant vehicle to reach the projected frontier as their peer set of cars. Here, also it shows the three most important components for advancement are safety, styling and torque. For Swift LXI and GO Plus (A) Petrol, their hindrances are safety rating and standardized features and styling value. This implies people are not very comfortable with their safety and looks. Interestingly, all the cars have the potential power output same as their benchmark set, but most cars need to enlarge their boot. For example, Santro Magna must lift up four of its six outputs. Its upgrading targets are 44.68% in safety rating, 26.84% in torque, 5.86% in boot space and 8.88% in features and styling.

Conclusion

The study concludes that 13 affordable cars from different brands are available within the budget of four to six lakhs of rupees for the middle-class buyers. They can have their choice of brands between Maruti, Tata, Renault, Mahindra or Fiat. Indian brands like Maruti, Tata and Mahindra are doing really well. They are providing the bargain hunters with best affordable cars. They understand the mindset of Indian people and try to manufacture affordable cars with necessary comfort, safety and styling level. In fact, for brand Maruti, out of nine cars appearing as the DMU, seven of them are suitable to buy. The other two, Swift LXI and Celerio LXI are not far behind. Their scores are 94.3% and 99.6%, respectively. As a result, all models of brand Maruti are performing well in the market. If car sales statistics is checked in different websites,[10] it can be seen that the results obtained through DEA show absolute resemblance. The same could not be said about brand Hyundai. None of its six cars could bring itself to the frontier fitting to pay for. Only Xcent has a score of 98.28%, and it is performing quite well in sales volume compared to other models of brand Hyundai. Similarly, brand Ford has a dismal sales report, and the reason is reflected in the DEA result of this study.

However, this is not an exhaustive study. For different input–output combination, the result may be different. However, there must be a way to define the qualitative characteristics of a car suitably to find a better result. If more cars of diverse brands and different models could be incorporated, it is expected that the added diversity will display an improved result. There is a lot of scope of more exhaustive research. The results can have a better way for the buyers, and they may have more options to choose. Similar research can be done in case of cars running on diesel or the budget can be extended but that will be another research.

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Chapter 12
Supplier Selection and Order Allocation in Highway Construction Projects Using a Hybrid MCDM Approach

Prasanna Venkatesan Shanmugam and T. P. Vivek

Abstract Supplier selection and order allocation (SSOA) is an important phase in any construction project that affects the project cost, quality, and time. Supplier selection is a multi-criteria decision-making (MCDM) problem as it involves a number of qualitative and quantitative criteria based on which the project manager selects the fittest suppliers. This work proposes a hybrid analytic network process (ANP) and preference ranking organization method for enrichment evaluations (PROMETHEE) approach to rank the suppliers for a major construction company in India. At first, the materials are grouped into clusters based on their consumption value. The criteria weights are computed using ANP. The suppliers in each product cluster are then ranked using PROMETHEE. Following the company's policy, the initial orders are allocated to the top two suppliers in each product cluster. Upon the receipt of the materials, the supplier's performance is evaluated using a linear averaging method. The order allocation for the subsequent periods is revised considering the deviations in the supplier's performance. The proposed approach is developed as an add-in in Excel to support the company's supplier selection process.

Keywords Supplier selection · Order allocation · Construction supply chain · ANP · PROMTHEE

Introduction

The material cost in any construction project typically falls in the range from 50 to 60% of the total project cost [1]. Minimizing the procurement costs could result in

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reducing the overall project costs. The selection and maintenance of a competent pool of suppliers are one of the major decisions in construction projects. Supplier selection involves finding the appropriate suppliers who can provide the buyer with the right quality products at the right price, in the right quantities and at the right time [2]. Supplier selection is a multi-criteria decision-making (MCDM) problem as it involves a number of qualitative and quantitative criteria based on which the project manager selects the fittest suppliers. MCDM models for SSOA in construction projects deserve attention. The research objectives of this work are as follows.

- At first, we use a hybrid MCDM approach based on analytic network process (ANP), [3] and preference ranking organization method for enrichment evaluations (PROMETHEE) [4], to rank the suppliers for a major construction company in India. The materials are grouped into clusters based on their consumption value. The criteria weights are computed using ANP. The suppliers in each product cluster are then ranked using PROMETHEE.
- Next, the initial orders are allocated to the top two suppliers in each product cluster following the company's policy. Upon the receipt of the materials, the supplier's performance is evaluated using a linear averaging method. The order allocation for the subsequent periods is revised considering the deviations in the supplier's performance.
- The proposed approach is developed as an add-in in Excel to support the company's supplier selection process.

This paper is organized as follows. Section "Literature Review" reports the literature on supplier selection in construction projects. The proposed hybrid MCDM methodology is elaborated in Section "Proposed Methodology". Section "Case Example" presents the case example, and the results are discussed in Section "Results and Discussions". Finally, Section "Conclusion" concludes the paper with future research directions.

Literature Review

Construction supply chain management (CSCM) has become a new challenge for construction managers in order to procure the right quantities of materials to the construction site on time and within the pre-defined budget [5]. Supplier selection is a multi-criteria decision-making (MCDM) problem. MCDM approaches are generally grouped as a single synthesizing criterion approach, outranking synthesizing approach, and a hybrid approach [6]. Chai et al. [7] reviewed the decision-making techniques in supplier selection. A recent review by Tan et al. [8] analyzed the MCDM techniques utilized with building information modeling (BIM) in architecture, engineering, and construction (AEC) industry and reported that individual MCDM approaches such as analytic hierarchy process (AHP) and technique for order of preference by similarity to ideal solution (TOPSIS) are widely applied in different application domains. Khoso et al. [9] presented a review of MCDM applied in contractor selection in construction and reported that the hybrid approaches and AHP technique are widely used by researchers.

The reviewed articles on supplier selection in the construction supply chain are presented in Table 12.1. The literature is classified based on the criteria, and the MCDM approaches used. It is observed cost, quality, delivery, and past performance criteria have received significant attention, while sustainability, flexibility, and visibility deserve further attention. It is also observed that hybrid approaches combining the merits of two approaches have been widely used by researchers. Among the approaches, PROMETHEE is found to be more stable for ranking the finite number of alternatives according to several conflicting criteria [10]. In this research, a hybrid approach combing ANP and PROMETHEE is developed to solve the supplier selection problem for a large construction company in India. To the best of our knowledge, the combined ANP and PROMETHEE methods have not been attempted for SSOA in construction projects. The ANP method is used to analyze the structure of the supplier selection problem and determine the criteria weights, and the PROMETHEE method is used for final ranking. The proposed hybrid approaches enhance PROMETHEE with the interdependency handling features of ANP.

Overview of ANP and PROMETHEE

Analytic Network Process (ANP): ANP was developed by Saaty [3] to solve decision-making problems involving dependence on criteria and alternatives [18]. The steps of ANP are as follows.

Step1: The first step involves determining the relationships between/among the criteria, sub-criteria, and alternatives.

Step 2: The pairwise comparisons based on Saaty's scale are performed, and the local priority values are calculated as shown in Eq. 12.1. The pairwise comparison matrix (A), the eigenvector (W), and eigenvalue ($\lambda_{\max}$) are shown in the equation.

$$\lambda_{\max} \times W_1 = A \times W_1 \tag{12.1}$$

The obtained vectors are further normalized to get the local priority vector W_2. The normalized pairwise comparison matrix (B) and the eigenvector (W_2) are obtained as shown in Eqs. 12.2 and 12.3, respectively.

$$b_{ij} = \frac{a_{ij}}{\sum_{i=1}^{n} a_{ij}} \tag{12.2}$$

Table 12.1 Some literature on supplier selection in the construction supply chain

Authors and year	Criteria									MCDM method	
	Cost	Quality	Delivery	Technical support	Location	Past performance	Others			Single	Hybrid
							Sustainability	Visibility	Flexibility		
Aretoulis et al. [11]	•	•	•	–	–	–	–	–	–	–	Simos and mathematical model
Safa et al. [12]	•	•	•	–	–	•	–	–	–	TOPSIS	–
Polat et al. [13]	•	–	•	•	–	•	–	–	–	–	AHP, ER
Luzon et al. [14]	•	•	•	–	•	•	–	–	–	–	AHP, Delphi
Cengiz et al. [5]	•	•	•	•	•	•	•	–	–	ANP	–
Tamošaitienė et al. [15]	•	–	-	–	–	•	–	–	•	–	AHP, ARAS
Marzouk and Sabbah [16]	–	–	-	–	–	–	•	–	–	–	AHP, TOPSIS
Marovic et al. [17]	•	•	•	–	–	•	•	–	–	–	AHP, PROMETHEE
Present research	•	•	•	•	•	•	–	•	–	–	ANP, PROMETHEE

ER evidential reasoning, *ARAS* additive ratio assessment, *AHP* analytic hierarchy process, *TOPSIS* technique for order of preference by similarity to ideal solution, *ANP* analytic network process, *PROMETHEE* preference ranking organization method for enrichment evaluations

$$w_2 = \begin{bmatrix} w_1 \\ w_2 \\ . \\ . \\ . \\ w_n \end{bmatrix} \text{where } w_i = \frac{\sum_{i=1}^{n} b_{ij}}{n} \forall, i = 1, 2, \ldots n \tag{12.3}$$

Step 3: The interdependence among the criteria is considered in step 3. By locating the local priority vectors generated by the pairwise comparison matrix on convenient columns, the super-matrix is obtained.

Step 4: The final step is to get the interdependence priorities of the criteria by synthesizing the results. To obtain the limit, we must raise the matrix to powers until all the row values converge to a single value.

Preference Ranking Organization Method for Enrichment Evaluations (PROMETHEE): PROMETHEE is an MCDM method based on the outranking relation between pairs of alternatives [4]. The steps involved in PROMETHEE are adapted from the first author's published work [6] and are described below.

Step 1: The first step involves the construction of the evaluation matrix by comparing the alternatives based on each criterion. The difference between the alternatives is computed within each criterion and transformed into a preference degree ranging from 0 to 1.

Step 2: The aggregated preference indices are calculated using Eq. (12.4)

$$\prod(a, b) = \sum_{i=1}^{n} p_n(a, b)\omega_n \tag{12.4}$$

where ω_n denotes the weights associated with criteria 'n' and $0 \leq \prod(a, b) \leq 1$; The degree to which alternative 'a' is preferred over alternative 'b', considering all the criteria simultaneously, is represented as $\prod(a, b)$.

Step 3: The positive and negative outranking flows from the resulting preference indices are calculated as shown in Eqs. (12.5) and (12.6), respectively.

$$\emptyset^{+}(a) = \frac{1}{n-1} \sum_{x \in A} \prod(a, b) \tag{12.5}$$

$$\emptyset^{-}(a) = \frac{1}{n-1} \sum_{x \in A} \prod(a, b) \tag{12.6}$$

The net flow $\emptyset(a)$ which refers to the balance between positive and negative outranking flows is computed as shown in Eq. (12.7). The higher the net flow, the better the alternative.

$$\emptyset(a) = \emptyset^{+}(a) - \emptyset^{-}(a) \tag{12.7}$$

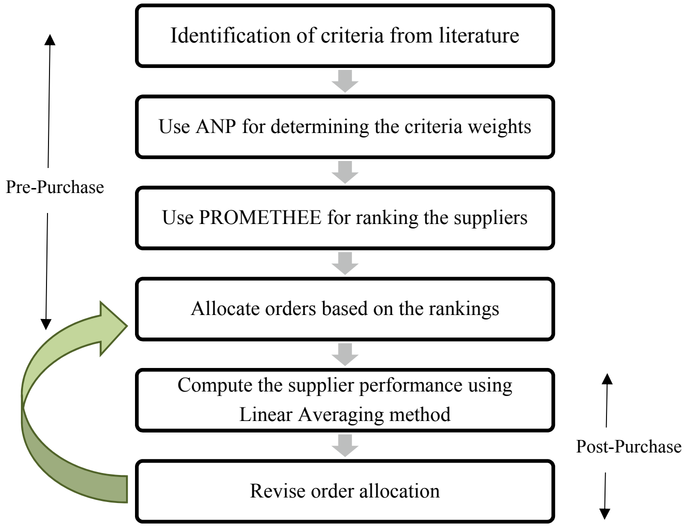

Fig. 12.1 Proposed methodology for supplier selection and order allocation

Proposed Methodology

The flow chart of the proposed methodology for supplier selection and order allocation is shown in Fig. 12.1. The pre-purchase includes supplier selection and evaluation. The post-purchase consists of order allocation and revision based on the actual performance. The ANP is used to determine the criteria weights, and the PROMETHEE is used for the final ranking of the suppliers. The initial orders are allocated top two suppliers in each product cluster. Upon the receipt of the materials, the supplier's performance is evaluated using a linear averaging method.

The order allocation for the subsequent periods is revised considering the deviations in the supplier's performance. The proposed approach is developed as an add-in in Microsoft Excel to support the company's supplier selection process.

Case Example

Managing relationships with all stakeholders in a construction supply chain is a challenging task. The selection of the appropriate supplier for the materials is a significant issue. A case example based on the real-time data from a construction company located in Telangana, India, undertaking a highway construction project is presented. There are around 9000 materials which are grouped into 20 product clusters like reinforcement steel, cement, hardware items, etc. After analyzing the company's supplier selection practices, the proposed hybrid MCDM approach is presented to the experts. Based on our interaction with the company's top management, a team consisting of five experts from the planning and purchasing departments is formed considering their qualifications and supply chain expertise. The members of the team have an average experience of more than ten years within the company. These professionals were interviewed personally to collect the data needed for the study. To filter out the suppliers in each product cluster, the ABC analysis is performed at first. From the literature, various criteria which are related to the construction industry are identified. These criteria are then shortlisted based on their applicability in the construction supply chain. Based on the expert's opinion, the cost, quality, delivery, past performance, technical support, location, and visibility criteria have been finalized. These criteria are defined as follows.

Cost (C_1): The cost refers to the total product price, including logistics, tax, etc. The discount offered by the suppliers is also considered.

Service (C_2): Service criteria include quality and fraction of materials rejected during delivery.

Delivery (C_3): Delivery considers the lead time, responsiveness, and delivery reliability.

Past performance (C_4): The relationship between supplier and company, reputation, and position in the market, etc., are considered to evaluate the past performance.

Technical support (C_5): This refers to the level of after-sales service and technical support offered by the supplier.

Location (C_6): Geography considers the location of the supplier and

Visibility (C_7): Visibility refers to the availability of product flow information along the supply chain.

Results and Discussions

The proposed approach is tested for a single product cluster, namely reinforcement steel and the results obtained are discussed below.

Calculating weights using ANP: The weights of the criteria are calculated based on the expert's opinion using Saaty's nine-point scale. The decision-makers are asked to evaluate the proposed criteria pairwise. They responded to questions such as among all the criteria, which do you consider as important and how strongly with reference to the goal of selecting the best supplier? The individual responses are aggregated, and the results obtained are presented in Table 12.2. To obtain the relative importance, the weights in each column are normalized to sum up to one. The super-matrix is further obtained by locating the local priority vectors generated by the pairwise comparison matrix on convenient columns. To obtain the converged set of weights, we raise the super-matrix (Table 12.3) to a large power. The matrix with the converged set of weights is called the limit matrix.

Table 12.2 Pairwise comparison matrix without interdependence

Criteria	C_1	C_2	C_3	C_4	C_5	C_6	C_7
C_1	1.00	2.20	3.50	2.69	4.21	3.02	2.63
C_2	0.45	1.00	1.56	4.56	3.67	3.22	4.06
C_3	0.29	0.64	1.00	3.88	4.59	2.33	4.08
C_4	0.37	0.22	0.26	1.00	2.25	2.55	3.60
C_5	0.24	0.27	0.22	0.44	1.00	0.76	3.23
C_6	0.33	0.31	0.43	0.39	1.32	1.00	0.83
C_7	0.38	0.25	0.25	0.28	0.31	1.20	1.00

Table 12.3 Weighted super-matrix obtained for the case example

Criteria	C_1	C_2	C_3	C_4	C_5	C_6	C_7
C_1	–	0.284	0.357	0.264	0.241	0.297	0.310
C_2	0.314	–	0.223	0.258	0.255	0.217	0.202
C_3	0.228	0.225	–	0.184	0.182	0.197	0.173
C_4	0.176	0.208	0.181	–	0.159	0.087	0.187
C_5	0.105	0.136	0.119	0.135	–	0.083	0.063
C_6	0.123	0.089	0.069	0.099	0.095	–	0.065
C_7	0.055	0.057	0.051	0.060	0.068	0.118	–

The criteria weights obtained from the ANP method are shown in Table 12.4. The consistency ratio value obtained was less than 0.1 for each of the pairwise comparison matrices; hence, the obtained criteria weights are accepted. From the result, it is observed that the cost [C_1], service [C_2], and delivery [C_3] are the most important criteria.

Ranking of suppliers using PROMETHEE: Experts were asked to evaluate the performance of the alternate suppliers under each criterion based on a five-point Likert scale (1 = very poor, 5 = very high). The evaluation matrix is formed as shown in Table 12.5. All the criteria have to be maximized, and a linear preference function is preferred with indifference and preference threshold set as 0 and 2, respectively, for all the criteria. The screenshot of the proposed approach developed as an add-in in Microsoft Excel is shown in Fig. 12.2.

The aggregated preference indices are then obtained using the criteria weights. The positive, negative, and net outranking flows for different alternatives (suppliers)

Table 12.4 Criteria weights obtained using ANP

Criteria	C_1	C_2	C_3	C_4	C_5	C_6	C_7
Weights	0.23	0.20	0.17	0.15	0.10	0.09	0.06

Table 12.5 PROMETHEE evaluation matrix for the case example

Suppliers	Criteria						
	C_1	C_2	C_3	C_4	C_5	C_6	C_7
A_1	4.67	3.83	4.00	3.33	2.83	3.00	3.00
A_2	4.00	4.50	2.00	2.33	3.33	4.00	3.00
A_3	4.33	3.33	2.17	3.67	2.67	3.00	4.00
A_4	2.83	2.50	3.33	3.17	2.00	3.00	3.00
A_5	3.83	3.33	3.50	4.00	3.67	4.00	2.00

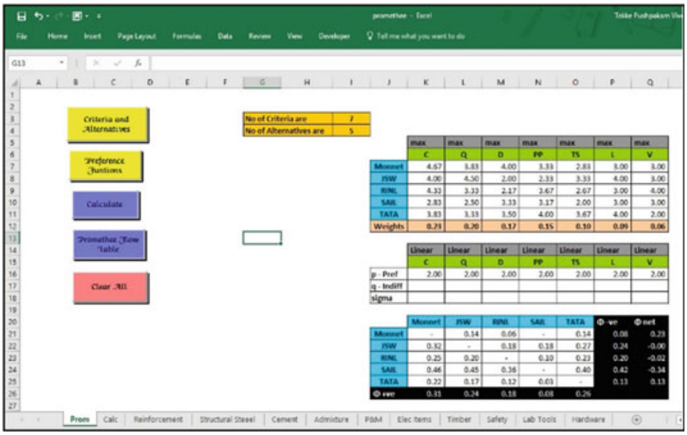

Fig. 12.2 Screenshot of the proposed approach developed as an add-in in Microsoft Excel

are calculated, and the results are reported in Table 12.6. Based on the net flow, the complete ranking of the suppliers is obtained and is given in Fig. 12.3.

Order allocation: Initial order is allocated to the top-ranked suppliers. These suppliers are then evaluated once the order is received by the customer. The linear averaging method is used for evaluating the suppliers based on cost, service, and delivery. The suppliers are evaluated for every order, and deviations in their performances are

Table 12.6 PROMETHEE flows obtained for the case example

Alternatives	Φ^+	Φ^-	Φ
A_1	0.31	0.08	0.23
A_2	0.24	0.24	0.00
A_3	0.18	0.20	− 0.02
A_4	0.08	0.42	− 0.34
A_{13}	0.26	0.13	0.13

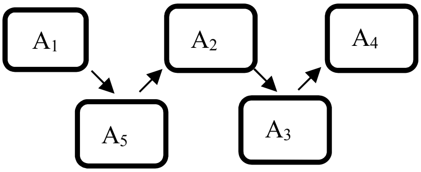

Fig. 12.3 Complete ranking of suppliers using the proposed ANP-PROMETHEE

observed. Orders are then reallocated to the suppliers if the observed deviation is 15–30%.

Conclusion

This research aims to facilitate the supplier selection and order allocation process of a major construction company in India. A hybrid ANP-PROMETHEE approach is proposed to evaluate and rank the suppliers based on cost, quality, delivery, past performance, technical support, location, and visibility. It is found that the ANP-PROMETHEE methodology is a practical and efficient tool for ranking the candidate suppliers. The proposed hybrid approach significantly reduces the ambiguities and vagueness that are inherent in the field of supplier selection in the construction supply chain. The future scope of this research is to incorporate fuzzy numbers [19, 20] and group decision-making [21] to handle imprecise judgments by experts.

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Chapter 13
Bibliometric Analysis of Supply Chain Contracts Under Disruption Risk

Imnatila Pongen and Pritee Ray

Abstract Supply chain management is continuously evolving from local and regional activities to the current global supply network. Owing to the ever-growing supply chain, many risks and uncertainties exist in the environment. Among the strategies adopted to manage risks and uncertainties of channel members, supply chain contract is widely studied by many researchers for channel optimization and coordination. Given the growing research volume in this area, it is highly desirable to study the bibliometric structure and discover the intellectual core, i.e., themes and sub-themes, in this rich body of the literature. In this paper, we study different types of contract adoption under disruption risk from data obtained from SCOPUS database. Citation, co-citation, and co-occurrence analyses are carried out to identify and validate the themes in this research area. The main study areas can be summarized into five broad categories as risk assessment and management, planning and controlling production cost under disruption, optimization of the supply chain under disruption, contract management and coordination under disruption, and profit maximization under information asymmetric.

Keywords Supply chain contracts · Bibliometric analysis · Network analysis · VOSviewer · Disruption

Introduction

The product life-cycle is becoming shorter by the day, due to the rapid advancement of technology and trends, and changing environment. Consumer electronics, fashion goods, seasonal products, and other products that fall under short life-cycle (SLC)

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play an important role in contributing to the economy. For products having a short life-cycle with different ordering and selling times, the demand uncertainty is much higher than for products having longer life-cycle. Flexible inventory and order management practices are adopted by manufacturers of SLC products to mediate the gap between demand and supply. Companies such as ITC, Nestle, Starbucks, Walmart, Apple, and HP, for example, have adopted varied flexible policies in this aspect.

The ever-growing market and development in the environment have led to major changes and uncertainties in demand–supply situations. To cope with such uncertainties, organizations design their supply chain and adopt various strategies to meet the growing needs of end consumers. Supply chain contract is one of the strategies studied by many researchers for channel optimization and coordination, as it can take into consideration different risks and uncertainties in the environment. Supply chain contract is a widely used method to balance the demand–supply in the market. Many studies have highlighted the use of different supply chain contracts in the literature. Some of the most used contracts include wholesale prices contract, two-part tariff contract, revenue-sharing contract, incentive contract, buy-back contact, cost-sharing contract, quantity discount contract, quantity flexibility contract, and option contract. When there is uncertainty in demand, risk hedging and flexibility in supply becomes important to manage market demand during the selling season, and therefore, the selection of appropriate contracts can impact all the channel members and the supply chain as a whole. Uncertainty in demand can occur due to channel members' attitudes toward risk, such as risk-seeking, risk-averse, and risk-neutral, which in turn leads to uncertainty in the market demand and supply situations. Researchers have studied risk aversion using several methodologies such as value-at-risk, mean–variance, and conditional value-at-risk. Studies have incorporated these methodologies under various contract types such as wholesale pricing contract, option contract, and commitment contract, to model real-life scenarios. Given the growing research volume, it is highly desirable to study the bibliometric structure and identify the intellectual core, i.e., themes and sub-themes, in this rich body of literature.

The remainder of the paper is organized as follows. We extract the bibliographic data for scientometric analysis from SCOPUS database and described the methodology in Section "Methods and Material". We use different methods of scientometrics including citation, co-citation, and co-occurrence analysis for identifying and validating the research themes. We then present the discussion and inferences from different analyses in Section "Results and Analysis". We conclude the paper with limitations and future scope in Section "Conclusion".

Methods and Material

Search Criteria

All the different combinations of keywords are considered and examined the results that each respective search has provided. It is found that using the phrase “supply chain AND contract AND disruption” yields relevant studies that help in forming a good corpus. The inclusion of conference proceedings in the corpus is also allowed. Many recent studies and research are usually presented in the conferences, and it is appropriate to include the conference proceedings as they provide useful and emerging research ideas. Scopus is used as the source of citation database. The data from the Scopus database for the keywords “supply chain AND contract AND disruption”, “supply chain AND contract AND uncertainty”, and “supply chain AND contract AND risk” search up to 2020 is retrieved in December 2020. The search generated 2088 articles. Different threshold levels are set in the study to ensure the selection of papers that are relevant for different bibliometric analyses.

Analytical Procedure

Bibliometric analysis is a quantitative literature review technique that uses mathematical and statistical methods for analyzing the development in research areas based on the existing literature [47]. Specific bibliometric techniques are used to analyze the literature on contracts under supply chain disruption: (i) citation, (ii) co-citation, and (iii) co-occurrence analysis. Citation analysis ranks and groups cited authors or journals based on how frequently they are cited and how important they are [12]. The term “co-citation analysis” refers to the process of determining the relationship between existing research in terms of authors, journals, or references that form an undirected network [30], through mentioning of two or more other studies by a third study. Co-occurrence analysis identifies the themes of research area by grouping common keywords across papers [8].

Various software applications are available for bibliometric analysis, such as VOSviewer [37], Pajek [4], NodeXL [31], BibExcel [27], Gephi [3], R’s Biblioshiny [1], and others. VOSviewer and Biblioshiny are used to perform the analyses in this study. The used software generates clusters in the form of visual maps like density or network maps which are easy to understand.

The corpus retrieved from SCOPUS database is first analyzed systematically to determine the relevance of the articles for the study. The downloaded files are reviewed, and duplicate files are removed. All the remaining articles are then read to determine the relevance for further analysis. A total of 228 articles are shortlisted for analysis using the title, abstract, and keyword as the inclusion criteria to determine the relevance of the articles. The process is summarized below.

Step 1: Search articles in SCOPUS database using different keywords
Step 2: Remove duplicate files from the corpus
Step 3: Based on the inclusion criteria, retain, or delete articles for bibliometric analysis.

Results and Analysis

Descriptive Analysis of the Corpus

Table 13.1 shows the summary of descriptive analysis. The extracted corpus consisted of documents published between the years 2005 and 2020. The resulting corpus has articles as the highest number of documents. There are 503 authors for the 228 documents extracted from the Scopus database. Various bibliometric analyses, namely citation, co-citation, and co-occurrence, are carried out on the extracted corpus. These analyses are performed to identify high-value papers and prominent themes in the literature.0020.

Analysis of Trend of Publication Year-Wise

From Table 13.1, it is observed that there are 228 articles found on closed-loop supply chain with contract in SCOPUS. An analysis of these articles on the year-wise trend of publication is provided in Fig. 13.1. It is observed that 169 out of 228 articles are published in the years 2011–2020. The figure shows an increasing trend from the early 2000–2020 which can indicate an increase in the number of researches on the topic each year by researchers showing its interest among the research community. The highest number of publications is found to be from the year 2020.

Analysis of Top Authors

Table 13.2 displays the top 10 authors who contributed to studies on supply chain contract under disruption. The 228 papers considered in this study received a total of 9124 citations based on authorship and are authored by 286 authors. We observe from the table that Yang Z. received the highest citations followed by Aydin G. The top 10 authors received 22% of the citations, while the top 20 authors contributed to 49% of the citations. It shows that studies are not widely distributed among many researchers. Zhao I. published the highest articles (seven papers) followed by Chen J. (seven papers).

Table 13.1 Summary of descriptive analysis

Description	Results
Main information about data	
Timespan	2005:2020
Sources (journals, books, etc.)	140
Documents	228
Average citations per document	14.81
References	6183
Document types	
Article	152
Article in press	1
Book	1
Book chapter	4
Conference paper	62
Conference review	1
Note	2
Review	5
Short survey	1
Authors	
Authors	503
Author appearances	660
Authors of single-authored documents	24
Authors of multi-authored documents	479

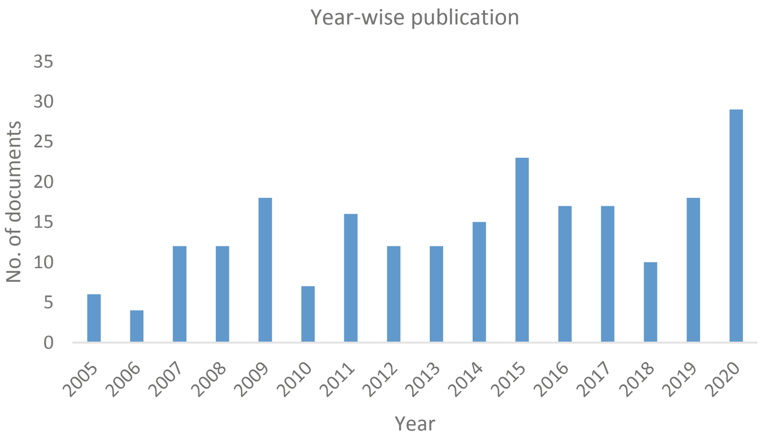

Fig. 13.1 Number of publications per year on supply chain contract under disruption

Table 13.2 10 Top authors for supply chain contract under disruption based on number of citations

Author	Number of citations	Number of articles	Average citations per article
Yang Z.	309	4	77.25
Aydin G.	304	2	152
Babich V.	304	2	152
Beil D. R.	304	2	152
Yu G.	269	5	53.8
Xia Y.	181	2	90.5
Cao E.	158	4	39.5
Chen J.	154	7	22
Yu H.	150	6	25
Li H.	126	2	63

Analysis of Top Countries

Figure 13.2 depicts a TreeMap of the countries with the highest number of citations on supply chain contracts under disruption. The figure shows that papers from the United States has the highest citation of 2472 followed by China with 1886 citations. A simple calculation shows that China, the United States, and Denmark received 55.44% of the citations from a total of 5826 citations. The top 5 countries received a total of 75% of the citations. This observation denotes that the focus on supply chain contract under disruption is concentrated in only a few countries.

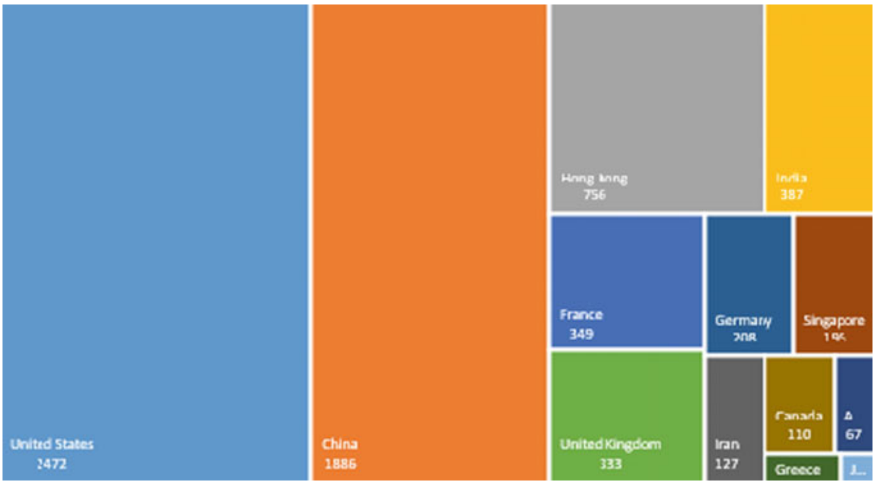

Fig. 13.2 TreeMap of 10 countries with the highest number of citations on supply chain contract under disruption

Citation Analysis by Document

First, we performed various iterations to identify the network where distinct clusters could be ascertained (the same process is carried out for all the subsequent analyses in this study). As the citation count is done by the application on the Scopus data that is imported into it, the count is based on the total corpus of references that is present in the data. The different documents and the total citation count of the 10 most cited documents are presented in table (Table 13.3). We set the threshold criteria of documents to five for citation analysis by documents and found that 61 documents meet the threshold.

The network map (Fig. 13.3) identified 11 clusters among the selected documents, with a total of 14, 7, 6, 6, 5, 5, 5, 4, 4, 3, and 2 documents under clusters 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, and 11, respectively. Cluster 1 focuses on stochastic demand. The topics like supply chain coordination with buy-back contract [49] impact on the optimal pricing and production decisions [48], establishing equilibrium with quantity discount contract [35], and supply chain coordination with an improved revenue-sharing contract [26] under disruption fall under this cluster. Cluster 2 focuses on demand disruption and the use of different contracts to mitigate the disruption in dual-channel supply chains. Some of the contributing documents include studies like supply chain coordination with multiple retailers using revenue-sharing to investigate the optimal strategies of players [7], determination of the pricing and production decisions using an improved revenue-sharing contract to coordinate a dual-channel supply chain [6], and coordination of a dual-channel supply chain with wholesale price, direct channels price, and lump sum fee approaches [50].

Cluster 3 focuses on risk-sharing among supply chain members. Some of the contributing documents include studies like the impact of information acquisition and sharing using risk-sharing contracts [38], adoption of option and firm order contracts with demand and supply risk-sharing in a decentralized supply chain [39], and application of subsidy contract with risk-sharing for addressing double marginalization and information problems [11]. Cluster 4 focuses mainly on the supplier side

Table 13.3 Citation by document

Document	Number of citations
Snyder [32]	283
Yang et al. [45]	220
Xiao et al. [40]	138
Zhang et al. [51]	106
Wakolbinger and Cruz [38]	103
Lei et al. [20]	90
Yang et al. [44]	84
Cao et al. [7]	82
Gümüş and Gurnani [15]	76
Asian and Nie [2]	73

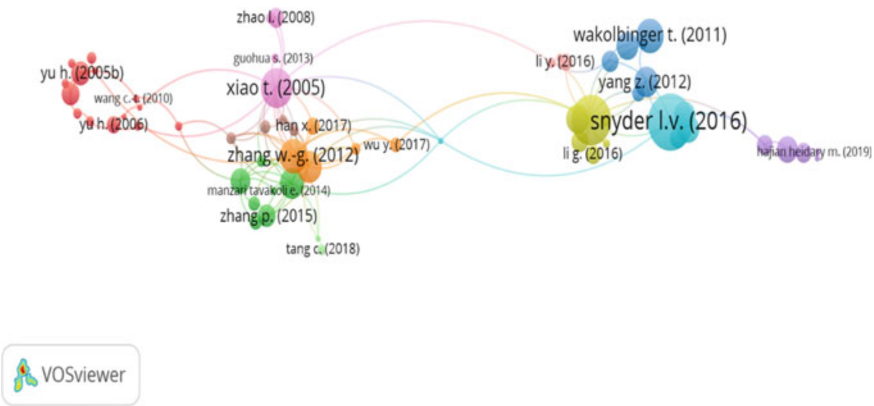

Fig. 13.3 Citation by document

disruption and reliability of suppliers with asymmetric information. Some of the contributing documents consist of studies like optimal contract menu development by the manufacturer for different levels of supplier reliability when there is information asymmetry related to supply-side disruption [46], and offering price and quantity guarantee to suppliers for gaining true risk information from unreliable suppliers [15]. Cluster 5 focuses on achieving supply chain resilience through sourcing strategies. Some of the contributing documents consist of studies like risk modeling, options contracts, and demand and spot price correlation for supply portfolio procurement decisions [25], and formation of a contract with a backup supplier while monitoring to obtain disruption risk information of the primary suppliers [29].

Cluster 6 focuses on risk hedging related to production downtime and capacity. Some of the contributing documents consist of studies like optimizing supplier selection for hedging against capacity disruption [43] and the adoption of performance-based contracting to mitigate downtime loss [18]. Cluster 7 focuses on cost disruption. Some of the contributing documents consist of studies like the adoption of linear contract menus as risk management strategies when cost disruption is private information [20], and the use of revenue-sharing contract for coordinating a close-loop supply chain under remanufacturing cost disruption [16]. Cluster 8 focuses on coordinating supply chains when the retailer is dominant under demand disruption. Some of the contributing documents include studies like coordinating supply chain when there is government price regulation policy using a revenue-sharing contract [24] and coordinating the supply chain using revenue-sharing contract when the dominant retailer influences the market retail price and sales promotion opportunity [21].

Cluster 9 focuses on the closed-loop supply chain with multiple demand markets under disruption. Some of the contributing documents include studies like coordinating a closed-loop supply chain using a buy-back contract and quantity discount contract when both retailers and third-party logistics can help mitigate the market demand deviation [23, 52]. Cluster 10 focuses on the financial aspect of the supply chain. Some of the contributing documents consist of studies like the integration of financial assistance and the non-delivery penalty for the supplier in contract formation under supply disruption [22] and coordination of supply chain by augmenting wholesale price contact with inventory-holding cost subsidy to the retailer [10]. Cluster 11 focuses on the impacts of simultaneous disruption on supply chain coordination. Contributing documents include studies by [33] for coordinating dual-channel supply chain under simultaneous disruption of demand and cost using a revenue-sharing contract and [53] for coordinating supply chains under simultaneous disruption of demand by retail price and non-price marketing effort using effort and revenue-sharing contract.

Co-citation Analysis by Document

The cluster of different documents and the total citation count of the 10 most cited documents are presented in (Table 13.4). We set the threshold criteria of documents to five for co-citation analysis by documents and found that 40 documents meet the threshold 4 clusters are identified among the selected documents, with a total of 14, 10, 9, and 7 documents under clusters 1, 2, 3, and 4, respectively (Fig. 13.4).

Cluster 1 focuses on the classification of risks and mitigation strategies. Studies in this cluster include mitigating disruption risk of primary supplier with a backup supplier with infinite capacity [36], building a framework for risk assessment and management arising out of normal activities like natural disasters, economic disruptions, etc. [19] and developing a unified framework to classify various supply chain risk management articles, and providing guidance to develop models for supply chain disruption mitigation [34]. Cluster 2 focuses on the coordination of supply chain with competing retailers under cost disruption. Studies in this cluster include adoption of all-unit quantity discount and incremental quantity discount contracts

Table 13.4 Co-citation by document

Document	Number of citations
Qi et al. [28]	37
Tomlin [36]	20
Chen and Xiao [9]	12
Giannoccaro and Pontrandolfo [14]	12
Xiao and Qi [41]	12
Cachon and Lariviere [5]	11
Kleindorfer and Saad [19]	10
Tang [34]	10
Hendricks and Singhal [17]	8
Lei et al. [20]	8

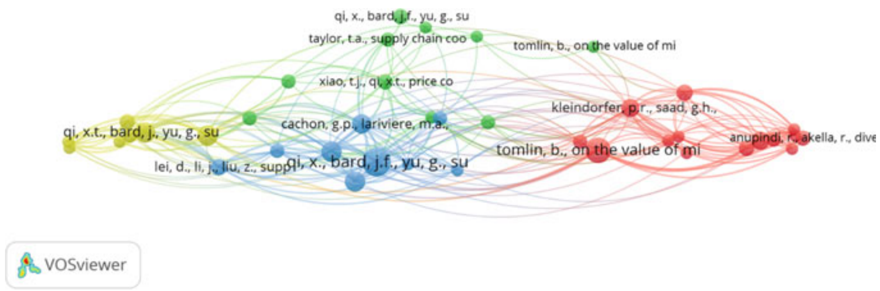

Fig. 13.4 Co-citation by document

for coordinating competing retailers under manufacturer's production cost disruption [41] and application of linear quantity discount schedule or an all-unit quantity discount schedule under production cost disruption [42].

Cluster 3 focuses on demand disruption. Some of the contributing documents consist of studies like the adoption of linear contract menus as risk management strategies when cost disruption is private information [20], adoption of linear quantity discount schedule and wholesale quantity discount policies for coordinating the supply chain with profit maximization [28], and adoption of wholesale price schedule for coordinating a supply chain with one dominant retailer and multiple fringe retailers [9]. Cluster 4 focuses primarily on the application of revenue-sharing contract under supply chain disruption. Some of the contributing documents consist of studies like revenue-sharing contract adoption to coordinate a three-level supply chain [14], a general supply chain model where the retailer's purchase quantity and price determines the total revenue [5] and an assembly system vendor-managed inventory [13].

Co-occurrence Analysis

The cluster of different keywords and the total count of the 10 most used keywords are presented in table (Table 13.5). We set the threshold criteria of keywords to six for co-occurrence analysis and found that 70 keywords meet the threshold.

The network map (Fig. 13.5) classifies the selected keywords into five clusters. Cluster 1 mainly consists of keywords like risk analysis, risk assessment, risk management, and demand uncertainty. Cluster 2 mainly consists of keywords like production plan, production cost, closed-loop supply chain, and production control. Cluster 3 mainly consists of keywords like optimization, optimal strategy, coordination mechanism, and disruption management. Cluster 4 mainly consists of keywords like the revenue-sharing contract, wholesale price contract, and demand disruptions. Cluster 5 mainly consists of keywords like profitability, asymmetric information, and supply chain contracts. The clusters can be summarized as risk assessment and

Table 13.5 Co-occurrence analysis

Keywords	Occurrences
Supply chains	132
Supply chain management	67
Costs	50
Supply chain coordination	42
Sales	41
Disruption management	40
Profitability	34
Supply chain	34
Commerce	31
Revenue-sharing contracts	29

management, planning and controlling production cost under disruption, optimization of the supply chain under disruption, contract management and coordination under disruption, and profit maximization when information is asymmetric for the clusters 1, 2, 3, 4, and 5, respectively. These are important areas of inquiry, but the papers (units) in the clusters are found to be interconnected thus illustrating the trans-disciplinary mode of research.

The word cloud (Fig. 13.6) can be used to infer the types of contracts commonly used by the authors in the corpus. Four main types of contracts can be found, namely

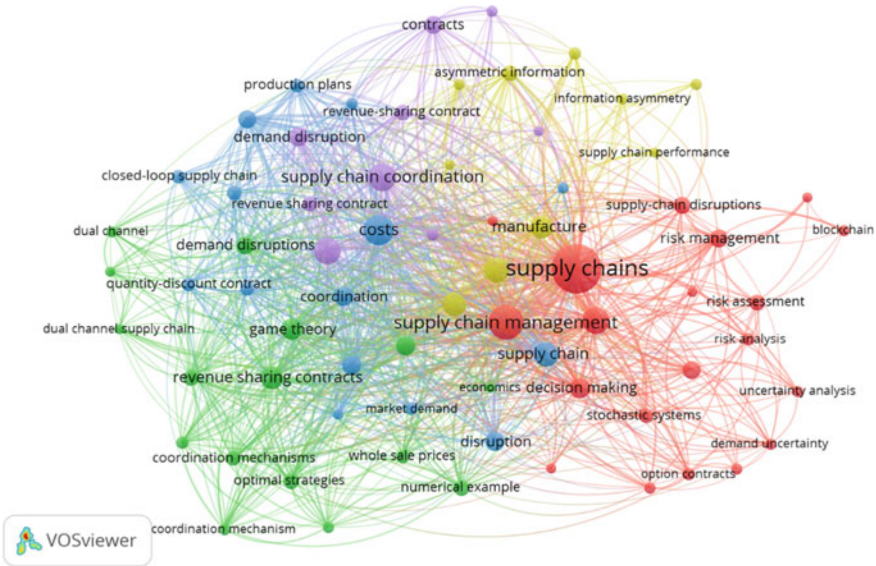

Fig. 13.5 Co-occurrence analysis

Fig. 13.6 Word cloud of keywords using Biblioshiny

revenue-sharing contract, wholesale contract, quantity discount contract, and option contract.

Conclusion

Supply chain management is continuously evolving from local and regional activities to the current global supply network. Due to the rapid advancement of technology and trends, and changing environment, many products such as consumer electronics, fashion goods, seasonal products, and other products that fall under short life-cycle (SLC) face supply–demand uncertainties. Given the growing concern, studies are carried out in large volume applying supply chain contracts to mitigate this problem. Thus, it is highly desirable to study the bibliometric structure and discover the intellectual core in this rich body of literature.

In this study, we examined studies on different types of contract adoption under disruption risk from data obtained from SCOPUS database using scientometric analysis. The extracted corpus consisted of documents published between the years 2005–2020, which are used to identify research trends across journals and countries. Various bibliometric analyses, namely citation, co-citation, and co-occurrence, are carried out on the extracted corpus. These analyses are performed to identify high-value papers and prominent themes in the literature. Through the results from various analyses, we could identify some of the most influential studies based on authors, papers, and country. The citation and co-citation analyses helped in assessing studies related to supply chain contracts under disruption.

The findings reveal that many aspects of supply chain disruption have been studied by researchers. This study emphasized the adoption of various contracts for mitigating disruptions in the supply chain. Thus, this study provides an overall bibliometric analysis of supply chain contract adoption under disruption, and the themes identified through the scientometric analysis form the intellectual core. This study contributes to the literature on supply chain contracts under disruption in many ways. Firstly, by applying bibliometric and network analysis (citations, co-citations, and co-occurrence), our study contributes to the literature by identifying and comparing studies by researchers. Secondly, through citations, co-citations, and co-occurrence analyses, we identified five themes constituting the intellectual core: risk assessment and management, planning and controlling production cost under disruption, optimization of the supply chain under disruption, contract management, and coordination under disruption, and profit maximization under information asymmetric. These are important areas of inquiry, but the papers (units) in the clusters are found to be interconnected thus illustrating the trans-disciplinary mode of research.

The citation analysis by document identified eleven themes among the selected documents: stochastic demand, demand disruption and the use of different contracts to mitigate the disruption in dual-channel supply chains, risk-sharing among supply chain members, supplier side disruption and reliability of suppliers with asymmetric information, achieving supply chain resilience through sourcing strategies, risk hedging related to production downtime and capacity, cost disruption, coordinating supply chains when the retailer is dominant under demand disruption, the closed-loop supply chain with multiple demand markets under disruption, the financial aspect of the supply chain, and impacts of simultaneous disruption on supply chain coordination. The co-citation analysis by document identified four themes: classification of risks and mitigation strategies, coordination of supply chain with competing retailers under cost disruption, demand disruption, and application of revenue-sharing contract under supply chain disruption.

Implications from this study can be incorporated into the future based on the core themes identified. With regards to risk assessment and management, studies can be done to identify practices that are followed in different industries and implement the ones that are suitable for the industry under study. For optimization, contract management, and profit maximization of the supply chain, studies can be focused on contracting with manufacturing units and/or suppliers who can operate under disruption. The area of study must be on identifying the key features that make them reliable and resilient under disruption. Further, investigations on coordination of supply chain could address the research questions on how each member in the chain manages resource flows while cooperating with other channel members. There are some limitations of this study, which can be addressed in the future studies. This study is done using data from SCOPUS database. In the future studies, we can consider adding more articles from other sources like Web of Science. In this study, we applied only document citation, document co-citation, and keyword co-occurrence analyses. In the future studies, we can employ methods such as closeness measure, betweenness measure, factor analysis, and cluster analysis to identify and complement the findings. Here, we used only one combination of keywords for extracting and analyzing the data. As future research, we can add keywords in our keyword set and extract data from the same repository or other similar repositories for identifying research trends.

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Chapter 14
Pricing Decisions in a Heterogeneous Dual-Channel Supply Chain Under Lead Time-Sensitive Customer Demand

Sarin Raju, T. M. Rofin, and S. Pavan Kumar

Abstract Internet facilities helped retailers to sell through online channels, and as a result, e-tailers rose into prominence and started competing with retailers. But, the e-commerce industry always confronted the issue of lead delivery time, hindering the growth of many e-tailers. We observed scant literature that studies the impact of delivery lead time on a dual-channel supply chain consisting of retailer and e-tailer. This research paper uses game theory to verify the impact of delivery lead time on pricing decisions of a heterogeneous dual-channel supply chain consisting of the manufacturer, retailer, and e-tailer. We used the Stackelberg game to study the manufacturer's and downstream partners' interaction: retailers and e-tailers. A horizontal Nash game was used to model the interaction between the downstream partners. We had analytically modeled how the lead delivery time significantly affects the channel partner's optimal pricing, sales volume, and profitability. We also did sensitivity analysis to check the influence of the customers' channel preference coefficient toward a particular channel and its cross-effects on the pricing policies when the customer is also lead time-sensitive. The study revealed that irrespective of large delivery time or next day delivery time, customers' preference toward a particular channel didn't affect the manufacturer's profit, whereas it affected the profit of the retailer and e-tailer. On the other hand, the increase in lead time-sensitivity coefficient severely affected the profit of all the supply chain partners. By analyzing the pricing decisions, we found that both the customer preference and lead time-sensitivity coefficients affected the pricing decisions, but customers' channel preference coefficient failed to mitigate the effect of lead delivery time. The inputs from this study can be used by practicing managers to develop decision support systems and as an input in multi-agent systems for converting lead time-sensitive supply chains to robust and resilient ones.

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Keywords Dual-Channel supply chain · Delivery time · Game theory · Multi-agent systems · Resilient supply chain · e-tailer · Disruption

Introduction

With the advent of the Internet and rapid technological growth, people have become increasingly accustomed to purchasing goods online. Consequently, this period witnessed the rise of e-commerce firms [2, 3, 12, 35], which facilitates the e-tailer to successfully provide a large volume and variety of products, allowing the customers to order goods in home comfort. Many e-tailers like WS retail, Cloudtail, etc., have their logistics owned by their parent company, whereas many other e-tailers depend on third-party logistic firms like Delhivery, Blue Dart, FedEx, etc., for the end customer delivery support [26]. Though many choices are available, delivery lead time remains a core concern for the e-tailer when competing with the traditional retailer. Empirical researches prove that, like pricing, delivery service is also an equally important factor for delivery lead time conscious customers [6, 11, 30]. Many researchers claim that the customers are ready to pay more price for the quick delivery of their goods [15], and aligning with these researches, companies like Amazon have badged many customers 'Prime' and promised them fast delivery with extra prime membership. This paper, for the first time, analytically models the pricing decisions of a heterogeneous dual-channel supply chain (HDCSC) consisting of retailers and e-tailers for a delivery lead time conscious customer. This paper also checks the variation of sales volume and profitability of the customer with the change in delivery lead time.

For analysis, we assume an HDCSC [14, 33, 36] consisting of a manufacturer and two downstream channel partners, retailer, and e-tailer [26–28]. We employ the Stackelberg game (S Game) [8, 9, 13, 37] to study the interaction between the upstream channel and downstream partners with the manufacturer as the Stackelberg leader. We used the horizontal Nash (HN) game to check the game between the downstream channel partners [28]. With these assumptions, we are addressing the following overarching research questions:

1. What is the influence of the delivery lead time on pricing decisions of the product for the retailer and e-tailer?
2. How the delivery lead time significantly impacts the sales volume of the channel partners?
3. What is the effect of delivery lead time on the profit of the channel partners?
4. What is the impact of customer channel preference coefficient and lead time-sensitivity coefficient on pricing decisions?
5. Can the customers' preference toward the e-tailers mitigate the effects of the lead delivery time?

Based on the analysis of the above questions, we deduce the following academic and managerial contributions. The optimal decisions of all the channel partners, namely retailer, e-tailer, and manufacturer, were derived, and the impact of lead delivery time was shown. Using the optimal decisions and S and HN game analytics, the sales volume of the channel partners was derived and analyzed. Later, optimal profit of both the upstream and downstream channel partners was derived, and the impact of delivery lead time was analyzed. The study also examined the effect of customers' channel preference coefficient and lead time-sensitivity coefficients on the optimal decisions. We find that the pricing decisions are affected by the lead time-sensitivity coefficient irrespective of whether the e-tailer delivered the product in the next day or had a considerable delivery time, whereas the channel preference coefficient is indifferent toward the optimal profit of the manufacturer. We also find that the profit of both the upstream and downstream channel partners decreased with an increase in lead time-sensitivity coefficient, and the decrease is severe for the e-tailer. To our surprise, the optimal profit of the retailer also decreased with an increase in lead time-sensitivity coefficient. The study also revealed that the impact of lead time sensitivity for all the supply chain partners could be reduced if the e-tailer can ably deliver the product in the next day and customers' channel preference toward e-tailers cannot mitigate the effects of delivery lead time. The optimal decisions of the study can act as an input for decision support systems and multi-agent systems in making the HDCSC robust and resilient.

In the next section, we report a brief account of existing literature in the field.

Literature Review

The Internet reached every nook and corner by the twenty-first century, and as a result, the e-commerce industry came into prominence. Observing the benefits of the e-commerce industry, many manufacturers started opening their own online channel along with the traditional brick and mortar retailers, thereby maintaining two channels simultaneously. The presence of two channels, i.e., one company-owned online channel and the traditional brick and mortar retailer, gives rise to the dual-channel supply chain (DCSC) concept. After that, there were many studies in the field of DCSC which concentrated on pricing [16, 20], inventory policies [34, 38], channel coordination [5, 32], disruption [17, 24, 25], etc. Later, e-tailers were introduced to the DCSC studies by Rofin and Mahanty [28], thereby bringing heterogeneous dual-channel supply chains (HDCSC) to the game-theoretic studies. Later, the same researchers introduced channel power structures in HDCSC [29].

During COVID-19 disruptions, the e-commerce industry faced severe lead delivery time-related issues, and many researchers started studying the impact of delivery lead time and its mitigating strategies [1, 10, 15, 19, 23]. Delivery lead time-depended stochastic customer demand was analyzed by Modak and Kelle [23] in a DCSC consisting of the traditional retailer and retailer-owned online channel and derived analytical models with the objective of profit maximization. The impact of delivery lead time on channel selection and pricing was studied by Hu et al. [15] using a mixed DCSC of manufacturer and retailer. The study suggested using consumer delivery lead time preference in retailers' decision-making. Delivery time was used in a game-theoretical approach in green DCSC by Alizadeh-Basban and

Taleizadeh [1], and equilibrium conditions were derived using the manufacturer-S game, distributor-S game, and Nash game. Delivery phase failures in e-retailing were analyzed, and recovery measures were suggested by Jafarzadeh et al. [19]. The study provides insights into mitigating the effects on the criticality of situation and brand equity due to delivery phase failures. The delivery lead time competition between e-tailers was analyzed by Raju et al. [26], and pricing decisions were modeled using the Stackelberg game and horizontal Nash game when one e-tailer is delivering the product the next day and the second e-tailer takes taking long duration for the delivery.

Critical observation and analysis of the abovementioned studies, the researchers found that most of the studies are concentrating on DCSC consisting of the retailer and online channel owned by the manufacturer, and there is very scant literature in the field of HDCSC consisting of retailer and e-tailer as downstream partners (which is a different story and analytics), which concentrates on lead delivery time and its impact on the decision variables of channel partners. This study mainly focuses on this research gap.

In the next section, we elaborate on the planned research method.

Research Method and Propositions

The basic linear demand function, $D = a - \lambda p$ [25, 26], is employed to establish the relationship between the demand and price. Here, a denotes base market potential. We also assume that the demand of the product is sensitive to the price [18, 25] and is denoted by own-price elasticity, λ. It can be defined as the change in demand due to a unit change in price. We use θ as the customer's preference toward the e-tailer channel where $0 \leq \theta \leq 1$ and is known as the customers' channel preference coefficient. Empirical researches prove that, like pricing, delivery service is also an equally important factor for delivery lead time conscious customers [6, 11, 30]. So, we fixed delivery lead time and price as decision variables. With these assumptions, the following equations were derived.

$$\text{Demand for the retailer, } D_r = (1 - \theta)a - \lambda P_r + \gamma P_e + \beta L \tag{14.1}$$

$$\text{Demand for the e-tailer, } D_e = \theta a - \lambda P_e + \gamma P_r - \Omega L \tag{14.2}$$

Here, suffix r and e denote retailer channel and e-tailer channel, and γ represents cross-price elasticity. We also assume that $\lambda, \gamma > 0$ and $\lambda > \gamma$. In the study, we assume delivery lead time-sensitive customers, and L indicates the delivery lead time, and β and Ω represent the lead time-sensitivity coefficients of the demands of the retailer and e-tailer, respectively. It means that if L increases by one unit, Ω units of the customer will be lost by the e-tailer, and from that β units will be gained by the retailer. For mathematical and practical correctness, we have also assumed $\Omega > \beta$. The actual decision-making sequence starts with the manufacturer announcing the

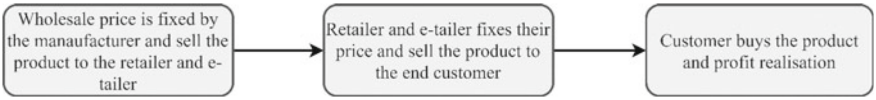

Fig. 14.1 Actual decision-making process

wholesale price, followed by the retailer and e-tailer fixing their price. The decision-making process ends with the customer buying the product and profit realization. The entire flow is shown in Fig. 14.1.

Here, we use two game theory analytics to study the interaction. An S game analytics is used to investigate the interaction between the upstream and downstream channel partners. Here, the manufacturer will have channel power over the other partners, and consequently, we assigned Stackelberg leadership to the manufacturer. We used second sub-game analytics to examine the downstream partners' interaction. Since the downstream channel partners have comparable channel powers, we used the HN game to derive the equilibrium conditions. The Stackelberg leader fixes the wholesale price, w, and the followers will use this price to derive their profit.

$$\begin{aligned}\text{Profit of the Retailer} = \pi_r &= (P_r - w)D_r \\ &= (P_r - w)\{(1-\theta)a - \lambda P_r + \gamma P_e + \beta L\}\end{aligned} \tag{14.3}$$

$$\begin{aligned}\text{Profit of the e-tailer} = \pi_e &= (P_e - w)D_e \\ &= (P_e - w)(\theta a - \lambda P_e + \gamma P_r - \Omega L)\end{aligned} \tag{14.4}$$

$$\text{Profit of the Manufacturer} = \pi_m = (w - s)(Q_r + Q_e) \tag{14.5}$$

Here, s denotes the unit production cost, and Q_r, Q_e are, respectively, the sales volumes of retailers and e-tailers. Wholesale price, w, can be derived using the principle of backward induction. The analytics of the backward induction is shown in Fig. 14.2.

After modeling the scenario, to get better managerial insights, we compare the pricing decisions and profitability of the channel partners when the e-tailer delivers the product the next day, and he took a higher delivery time.

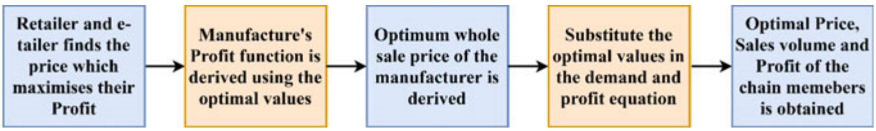

Fig. 14.2 Backward induction analytics

Propositions

For $\frac{\partial^2 \pi_r}{\partial P_r^2}$ and $\frac{\partial^2 \pi_e}{\partial P_e^2} < 0$, P_r and P_e can be obtained by taking first-order conditions (FOC) of π_r and π_e.

Propositions 1 *The optimal price of the retailer and e-tailer when the downstream channel partners are engaged in HN game is given by*

$$P_r = \frac{\gamma\theta a + 2a\lambda + 2L\beta\lambda + w\gamma\lambda - 2a\theta\lambda + 2w\lambda^2 - L\gamma\Omega}{4\lambda^2 - \gamma^2} \tag{14.6}$$

$$P_e = \frac{a\gamma\theta - a\gamma - L\beta\gamma - w\gamma\lambda - 2\theta a\lambda - 2w\lambda^2 + 2L\lambda\Omega}{4\lambda^2 - \gamma^2} \tag{14.7}$$

Substituting the values of P_r and P_e *in* (14.1) *and* (14.2)*, we will get the following corollary.*

Corollary 1 *The sales volume of the retailer and e-tailer when the downstream channel partners are engaged in HN game is given by*

$$Q_r = \frac{\lambda\left(2a(\theta - 1)\lambda - w\left(\gamma^2 + \gamma\lambda - 2\lambda^2\right) - \gamma\theta a\right) + L\left(2\beta\lambda^2 + \gamma\lambda\Omega - \beta\gamma^2\right)}{\gamma^2 - 4\lambda^2} \tag{14.8}$$

$$Q_e = \frac{a\gamma(\theta - 1)\lambda - 2\theta a\lambda^2 - w\lambda\left(\gamma^2 + \gamma\lambda - 2\lambda^2\right) + L\left(\gamma^2 - 2\lambda^2\right)\Omega - L\beta\gamma\lambda}{\gamma^2 - 4\lambda^2} \tag{14.9}$$

For $\frac{\partial^2 \pi_m}{\partial w^2} < 0$, *the optimal wholesale price of the manufacturer can be obtained by taking the FOC of* (14.5).

Proposition 2 *The upstream and downstream channel partners are assumed to be engaged in HN game. The optimal wholesale price obtained at the equilibrium point of the game is given by*

$$w = \frac{a - 2s\gamma - a\theta + \theta a + 2s\lambda}{4\lambda - 4\gamma} + \frac{L(-\beta + \Omega)}{4\lambda} \tag{14.10}$$

Substituting the respective optimal values of price, sales volume in (14.3), (14.4), *and* (14.5)*, we will get the optimal profit of both the downstream and upstream partners.*

Proposition 3 *The profit of the retailer, e-tailer, and manufacturer is given by*

$$\pi_r = \frac{\begin{array}{c}\left(2(a+L\beta-a\theta)\lambda + w\left(\gamma^2+\gamma\lambda-2\lambda^2\right)+\gamma(\theta a - L\Omega)\right)\\ \times\left(\lambda\left(\gamma\theta a - 2a(\theta-1)\lambda + w\left(\gamma^2+\gamma\lambda-2\lambda^2\right)\right)+L\left(\beta\left(\gamma^2-2\lambda^2\right)-\gamma\lambda\Omega\right)\right)\end{array}}{\left(\gamma^2-4\lambda^2\right)^2} \tag{14.11}$$

$$\pi_e = \frac{\begin{array}{c}\left(L\beta\gamma + w\gamma^2 + a(\gamma-\gamma\theta) + w\gamma\lambda + 2\theta a\lambda - 2w\lambda^2 - 2L\lambda\Omega\right)\\ \times\left(-a\gamma(\theta-1)\lambda + \lambda\left(2\theta a\lambda + w\left(\gamma^2+\gamma\lambda-2\lambda^2\right)\right)+L\left(\beta\gamma\lambda-\gamma^2\Omega+2\lambda^2\Omega\right)\right)\end{array}}{\left(\gamma^2-4\lambda^2\right)^2} \tag{14.12}$$

$$\pi_m = \frac{(s-w)(\lambda(a+2w\gamma-a\theta+\theta a-2w\lambda)+L(\gamma-\lambda)(\beta-\Omega))}{\gamma-2\lambda} \tag{14.13}$$

Numerical Analysis

In this section, we analyze the impact of delivery lead time using numerical analysis [7–9, 22, 31]. For numerical analysis, values were assigned based on previous research and underlying assumptions between various parameters [4, 21]. The base market potential, a is fixed as 150, and the unit product cost is assumed to be 5. The own-price elasticity (λ) and cross-price elasticity (γ) are assumed to be 1.5 and 1.3, respectively. The customer preference toward the online channel, θ, is assumed to be varied from 0.1 to 0.9. Ω is assumed to take the value 1.9.

Case 1: Same β and large delivery time

For case 1, we assumed that the lead time coefficient, β, remains the same, and delivery time is very large. For varying customers' channel preference coefficient, the change in optimal decisions and profits are shown in Table 14.1. We found that initially, the optimal price, sales volume, and profit of the e-tailer was very small owing to the large delivery time and very small customers channel preference coefficient. But, with an increase in customers' preference toward the online channel, the e-tailer could take the leverage irrespective of very large delivery time. As a result, the e-tailer's price successfully surpassed the retailer's price. A similar trend was observed for both the sales volume and profit. The customer preference toward the channel helped the e-tailer to overcome the demerit created by the very large delivery time. While analyzing the profit of the manufacturer, it is found that the profit is independent of the customer preference toward the channel.

Case 2: Same β and next day delivery time

Though we observed a similar trend during case 2 (see Table 14.2), much to our surprise, we found that if the e-tailer ably delivers the product in next day, it will help both the retailer and e-tailer to increase the price. The ability of the e-tailer to

Table 14.1 Performance of the HDCSC under the same β and large delivery time

θ	P_r	P_e	Q_r	Q_e	π_r	π_e	π_m
0.1	220	184	57	5	2200	15	10,964
0.3	213	191	47	15	1471	155	10,964
0.5	206	198	37	26	889	440	10,964
0.7	199	205	26	36	452	872	10,964
0.9	192	212	16	47	162	1449	10,964

Table 14.2 Performance of the HDCSC under the same β and next day delivery time

θ	P_r	P_e	Q_r	Q_e	π_r	π_e	π_m
0.1	225	196	54	11	1941	81	11,965
0.3	218	203	43	21	1261	308	11,965
0.5	211	210	33	32	727	681	11,965
0.7	204	217	23	42	340	1199	11,965
0.9	197	224	12	53	98	1864	11,965

provide the product in the next day benefitted both the downstream chain partners, and their profit significantly increased (See Tables 14.1 and 14.2). We also observed that the manufactures profit also improved when the delivery lead time was less.

Case 3: Varying β and large delivery time

For analyzing the impact of the lead time-sensitivity coefficient, we varied the β from 1.5 to 5 and reported the performance of the channel partners. We observed that the pricing decisions and sales volume of the e-tailers were severely affected by the lead time-sensitivity coefficient. Though the change in sales volume for the retailer was minimal, he couldn't take the leverage of the condition, and his optimal price also decreased with increase in β. Consequently, these pricing decisions affected the profit, and all the chain partners experienced smaller profit during high β.

Case 4: Varying β and next day delivery time

When the e-tailer ably delivers the product in next day, the lead time-sensitivity coefficient couldn't influence the decision variables much, as shown in Table 14.4.

Discussions and Results

The optimal price of the retailer is maximum when the e-tailer ably delivers the product in the next day. But, it is inferred that customers' channel preference coefficient failed to control the impact of delivery lead time significantly. Even though the optimal price decreased for the retailer and increased for the e-tailer with an increase in the customers' channel preference coefficient, the difference among the optimal price of the retailer and e-tailer remains constant when the e-tailer provided large delivery and next day delivery (Difference between the optimal price values of the retailer in Tables 14.1 and 14.2 remains constant with increase in customers' channel preference coefficient. A similar trend was observed for e-tailer also). Thus, it can be inferred that while considering lead delivery time, the optimal price of both the downstream partners increases with the decrease in the delivery lead time of the e-tailer but was independent of the customers' channel preference as it failed to overpower the effect of lead time significantly.

While analyzing the pricing decisions with the lead time-sensitivity coefficient (See Tables 14.3 and 14.4), we find that the optimal price of downstream partners decreased with the increase in the lead time-sensitivity coefficient, and thus the lead time-sensitivity coefficient can significantly disturb the pricing decisions of all the downstream channel partners in an HDCSC. This impact of lead time-sensitivity coefficient can be decreased if the e-tailer successfully delivers the product in the next day. Though the optimal price decreased with an increase in lead time-sensitivity coefficient during next day delivery, that decrease was trivial when compared with the large delivery time.

For better interpretation, we have compared the profit of the supply chain partners under different conditions and scenarios, and we find that if the e-tailer can deliver the product the next day, the optimal profit will increase for the e-tailer and manufacturer and will decrease for the retailer irrespective of the channel preference of the customer and lead time-sensitivity coefficients (See Tables 14.1, 14.2, 14.3 and 14.4). By analyzing the impact of customers' channel preference, we find that the optimal profit of the manufacturer is not impacted by the customers' channel preference.

Table 14.3 Performance of the HDCSC under varying β and large delivery time

β	P_r	P_e	Q_r	Q_e	π_r	π_e	π_m
1.5	211	204	37	27	898	496	11,595
2	204	197	36	25	886	427	10,809
3	192	182	36	21	864	304	9320
4	179	167	36	17	842	202	7941
5	166	151	35	13	820	121	6673

Table 14.4 Performance of the HDCSC under varying β and next day delivery time

β	P_r	P_e	Q_r	Q_e	π_r	π_e	π_m
1.5	212	211	33	32	728	687	12,030
2	211	210	33	32	727	679	11,949
3	210	209	33	32	725	662	11,787
4	208	207	33	31	723	646	11,627
5	207	206	33	31	721	629	11,467

As expected, the optimal profit of the retailer decreased and increased for the e-tailer, with the increase in customers' channel preference coefficient. But, the retailer could reduce this effect of customers' channel preference if the e-tailer could deliver the product the next day. While analyzing the impact of the lead time-sensitivity coefficient, we find that, with the increase in the lead time-sensitivity coefficient, the optimal profit of all the channel partners considerably decrease. This can only be reduced by controlling the delivery time, as this substantial diminution in the profit decreased when the e-tailer delivered the product in the next day.

Conclusion

A heterogeneous dual-channel supply chain comprising retailer and e-tailer as downstream partners is explored for studying the impact of delivery lead time. We assumed channel leadership for the manufacturer and employed the Stackelberg game to study the interaction between the manufacturer and downstream channel partners. We adopted second game-theoretic analytics in the form of horizontal Nash game to check the game within the downstream channel partners. We have modeled the interactions and later used numerical analysis to derive the influence of customers' channel preferences and delivery lead time coefficient. We have also numerically analyzed the impact of next day delivery and large delivery time and done sensitivity analysis to check the influence of the customers' channel preference coefficient toward a particular channel and its cross-effects on the pricing policies when the customer is also lead time sensitive.

The study revealed that irrespective of large delivery time or next day delivery time, customers' preference toward a particular channel didn't affect the upstream channel partner's profit, whereas it affected the profit of both the downstream partners. On the other hand, the increase in lead time-sensitivity coefficient severely affected the profit of all the supply chain partners. By analyzing the pricing decisions, we found that both the customer preference and lead time-sensitivity coefficients affected the pricing decisions, but customers' channel preference coefficient failed to mitigate the effect of lead delivery time.

Supply chain practitioners can apply the findings from the study in developing decision support systems and as an input to multi-agent systems. This will help the supply chains to predict the impact of lead time during the disruption period and thereby can help to build robust and resilient supply chains. Academicians can use this model as a base for future studies in heterogeneous dual-channel supply chain, which analyzes the impact of delivery time. The study is limited to the analytical modeling of the pricing decisions, sales volume, and optimal profit when the manufacturer is not discriminating the wholesale price. The study can be further expanded by analyzing the market condition of discriminatory wholesale prices for both the channel partners. The model developed in this study can be explored further in a more practical way if Python game theory software is used to create intelligent systems that can act as a multi-agent system. The model can be further validated by empirically checking the performance of the decision variables of both downstream and upstream channel partners. Future researchers can start from here.

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Chapter 15
Bayesian Estimation of the Parameters in a Bulk Service Queuing Model with Poisson Arrival and Exponential Service Time Distribution

Kuntal Bakuli

Abstract In this paper, a Bayesian approach has been used to estimate the system parameters of the queuing system $M/M^{(a,b)}/1$, when the system is in steady state. Data has been collected after each service completion. Two different joint prior distributions have been suggested. A method based on MCMC algorithm has been used to generate sample from posterior distributions. Further, 95% credible region in the restricted parameter space has been derived using genetic algorithm.

Keywords Bayesian estimation · Bulk service queue · MCMC algorithm · Credible region

Introduction

In this paper, a bulk service queuing model with fixed server capacity has been considered. Several bulk service queuing models have been studied in literature, but the problem estimating the system parameters for such models has not been explored. The existing literature considers only estimation of parameters only for models where a server serves one customer at a time. Basawa and Prabhu [1] and Basawa and Prabhu [2] discussed the estimation of the parameters for GI/G/1 queuing system. Baswa and Bhat [3] discussed the sequential inference for GI/G/1 queuing system. Thiruvaiyaru and Baswa [4] introduced the Bayesian approach. Chowdhury and Mukherjee [5] used Beta-Stacy distribution as joint prior distribution of arrival rate and service rate to estimate the system parameters of a M/M/1 queuing model. They also used independent Gamma priors for the same queuing model. Srinivas and Kale [6] discussed the MLE and UMVUE for M/D/1 queuing system. Here, we will confine ourselves in bulk service queuing system with single arrival.

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Armero and Consea [7] used Bayesian statistical tools to analyze the congestion of a single-stage system with various independent bulk service queues with the same characteristics and running in equilibrium. They considered Poisson arrival and exponential service time distribution with general bulk service rule. They used a Bayesian computation technique based on MCMC algorithm to estimate posterior predictive distribution of the usual performance measures any bulk service queuing system. Inter-arrival times of several groups of customers and several service times of the server separately have been considered in their data set.

In this paper, we have developed estimate of customers' mean arrival rate and service rate for a bulk service queuing system. Here, server serves minimum "a" number of customers and maximum "b" number of customers at a time, with general bulk service rule. Customers arrive in Poisson fashion with mean arrival rate λ, and the service time is exponentially distributed with mean $\frac{1}{\mu}$. Data has been collected after completion of consecutive services, because it will make data collection easier, convenient and the number of data points will be lesser. A Bayesian approach has been used to estimate the values of λ and μ. Two different prior distributions have been studied. Further, 95% credible region for λ and μ has been obtained using genetic algorithm.

This paper is organized as follows. In Sect. "Description of the Queuing Model", we have described the queuing system and its assumptions. In Section "Data Collection and the Likelihood Function", we have described data and derived the likelihood function. In Section "Bayesian Estimation of the System Parameters", we have explained a Bayesian approach to make inference about the system parameters. This section is divided into two subsections for different choices of joint prior distributions of the system parameters. In Section "Discussion", we have discussed how our prior belief can influence the resulting inference in this approach and a possible way to avoid this influence in further study. Here, we have also mentioned about the implementation of the suggested approach to other queuing systems.

Description of the Queuing Model

In this paper, we have considered a queuing system, which is governed by the following assumptions:

(i) Customers arrive to the system one by one as a Poisson process with mean arrival rate λ.
(ii) The service times are independently and identically distributed as exponential with mean $1/\mu$.
(iii) The waiting time of the server since last service until the number of waiting customers is at least equal to its serving capacity b does not affect the next service time.
(iv) The service station has an infinite waiting capacity.
(v) $\lambda < b\,\mu$.

Data Collection and the Likelihood Function

Here, we consider that the data has been collected from any arbitrary time point, and we record the number of customers present in the system thereafter at the end of each of the *n*-1 consecutive services. We gather more information by recording the times of the consecutive service completions and attempt to use this information to estimate the parameters.

Let, X_1 = number of customer in the system at the arbitrary time point from where we start data collection.

X_i = number of customers in the system after the following $i - 1$th service completion $\forall i = 2(1)n$.

τ_j be the length of time interval between *j*th and $(j + 1)$th service completion. Hence, the likelihood function is as follows:

$$L(\lambda, \mu) = \left\{ \prod_{m=2}^{n} P\left(X_m = x_m, \tau_{m-1} = t'_{m-1} | X_{m-1} = x_{m-1}\right) \right\} \pi(x_1). \quad (15.2.1)$$

$\pi(.)$ is the stationary distribution of the system size (number of customer present in the system at any point of time). To obtain the stationary distribution, we have used the north-west corner truncation method suggested by Bakuli and Pal [8]. There are several approaches to obtain the stationary distribution on the basis of the real root lying in (0, 1) of the polynomial, $\rho = r + r^2 + \ldots + r^b$. "*r*" exists when the system is in steady state. However, there is no generic form of analytic expression of *r* based on λ, μ and *b*.

Stationary distribution of the system size exist when the system is in steady state, i.e., $\lambda < b\mu$. Here, the likelihood function is defined over the parameter space, $\Omega = (\lambda, \mu) | \lambda \rangle 0, \mu > 0, \lambda < b\mu\}$.

Now,

$$P(X_m = k, \tau_{m-1} = t | X_{m-1} = j) = \frac{\partial Q_{jk}(t)}{\partial t} \quad (15.2.2)$$

where

$$Q_{jk}(t) = P(X_m = k, \tau_{m-1} \leq t | X_{m-1} = j) \quad (15.2.3)$$

Thus,

$$\begin{aligned} & P(X_m = k, \tau_{m-1} = t | X_{m-1} = j) \\ & = \int_0^t \frac{\mu \lambda^{k+a-j} x^k (t-x)^{a-j-1} \mathrm{e}^{-(\mu x + \lambda t)}}{k!(a-j-1)!} \mathrm{d}x \quad \text{when } j < a \\ & = \frac{\mu (\lambda t)^k \mathrm{e}^{-(\lambda+\mu)t}}{k!} \quad \text{when } b \geq j \geq a \end{aligned}$$

$$= \frac{\mu(\lambda t)^{k+b-j} e^{-(\lambda+\mu)t}}{(k+b-j)!} \quad \text{when } j > b, k \geq j - b \tag{15.2.4}$$

Bayesian Estimation of the System Parameters

In this section, we discuss Bayesian estimation of the arrival rate (λ) and the service rate (μ). We use Gibbs sampling with metropolis hasting algorithm for the purpose.

We start with the simulated data, generated using $\lambda = 1.5$ and $\mu = 0.4$. We consider an appropriate prior for λ and μ. Since direct sampling from the posterior distribution of λ and μ is difficult, we apply Gibbs sampling technique to generate posterior samples. The convergence of the Gibbs sampler is checked using the convergence diagnosis suggested by Raftery and Lewis [9]. We then determine the burn-in period and the thinning length and make the histogram plot and the density plot. We find the quantiles of the system parameters from their marginal posterior distributions. Finally, we find a 95% credible region in the parameter space. As we are using specific values of the system parameters to generate the data, we can numerically check the accuracy of our analysis.

The Gibbs sampling method is briefly stated below.

Gibbs sampler:

Let $P(\lambda, \mu|Y)$ be the posterior joint distribution of λ and μ, for given Y, and $f_1(\lambda)$ and $f_2(\mu|\lambda)$ be, respectively, the prior distribution of λ and the conditional prior distribution of μ given λ. Let, $L(\lambda, \mu|Y)$ be the likelihood function of λ and μ. Our objective is to generate a sample from $P(\lambda, \mu|Y)$ and then find the marginal distributions of each of the system parameters. Now,

$$P(\lambda, \mu|Y)L(\lambda, \mu|Y) f_1(\lambda)\, f_2(\mu|\lambda) = g(\lambda, \mu|Y)(\text{say}) \tag{15.2.5}$$

Knowing the likelihood function, and by fixing the prior distributions, we can compute the function $g(.,.|Y)$ for any values of λ and μ.

Let $P_1(\lambda|\mu, Y)$ and $P_2(\mu|\lambda, Y)$ be the full conditional distribution of λ and μ, respectively.

It can be shown that $P_1(\lambda|\mu, Y) \propto P(\lambda, \mu|Y)$, and hence, $P_1(\lambda|\mu, Y) \propto g(\lambda, \mu|Y)$.

Similarly, $P_2(\mu|\lambda, Y) \propto g(\lambda, \mu|Y)$.

The idea of Gibbs sampling is that we can update the two parameters by sampling just one parameter at a time, cycling through all parameters and repeating. To perform the update for one particular parameter, we substitute the current values of all other parameters.

Here, we have a joint posterior distribution for the two parameters λ and μ. According to the idea of Gibbs sampling, we have the following steps:

Step 1. First, we make an initial choice of λ and μ, say $\lambda = \lambda_0$ and $\mu = \mu_0$.

To generate the ith sample, $i = 1, 2, \ldots$, we follow Steps 2 and 3.

Step 2. We generate a sample observation λ_i from the distribution $P_1(\lambda|\mu_{i-1}, Y)$.

Step 3. Replace the previous value λ_{i-1} of λ with λ_i. Then, we generate a sample observation μ_i from $P_2(\mu|\lambda_i, Y)$.

Repeat Steps 2 and 3 "n" times to generate n samples from the posterior distribution of λ and μ.

One Gibbs cycle would include an update for each of the parameters.

In our case, we do not have an explicit expression of our likelihood function, $L(\lambda, \mu)$. We cannot, also, obtain the expressions of the full conditional distributions $P_1(\lambda|\mu, Y)$ and $P_2(\mu|\lambda, Y)$ of λ and μ, respectively. However, for given λ and μ, we can find the likelihood, function and hence compute the function $g(\lambda, \mu|Y)$ for any combination (λ, μ). So, since $P_1(\lambda|\mu, Y) \propto g(\lambda, \mu|Y)$ and $P_2(\mu|\lambda, Y) \propto g(\lambda, \mu)$, we use metropolis hastings algorithm in each of the Steps 2 and 3 considering the function $g(\lambda, \mu|Y)$.

Determination of 95% credible region:

To obtain credible region for (λ, μ), we have to decide upon the prior distribution of (λ, μ). To do so, we assume that we have reasons to believe that $\lambda < b\mu$, where "b" is the maximum number of customers served at a time. Further, we consider two situations which give specific information about the system.

Case 1

Suppose from past experience we believe that on an average α number of customers arrive in β units of time and most of the services take more than θ units of time, but very few take more than γ units of time, $\theta < \gamma$, such that $\alpha > b\beta/\gamma$. For $\alpha > b\beta/\gamma$, if the server takes more than γ unit of time to serve a batch most of the time, then the system could explode due to rapid increase in the number of waiting customers.

Let us assume some specific values of a, b, α, β, θ, γ, namely $a = 2$, $b = 5$, $\alpha = 80$, $\beta = 60$ min, $\theta = 1$ min, $\gamma = 5$ min.

Let,

$\lambda \sim$ gamma (α, β), where α is the shape parameter and β is the rate parameter.

$\mu|\lambda \sim$ truncated normal (μ_0, σ^2), truncated to the left at the point λ/b,

where "b" denotes the maximum number of customers served at a time.

Here, we choose $\mu_0 = ((\alpha + 10)/b)/\beta$ so that $1/\mu_0 < \gamma$.

We make such choices of μ_0 and the truncation in order to ensure that the system is in equilibrium.

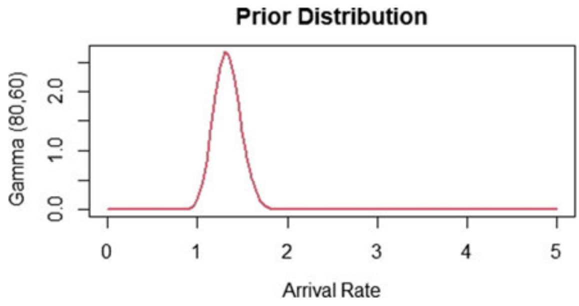

Fig. 15.1 Density plot of the distribution gamma (80, 60)

The value of σ can be chosen in accordance with the confidence of the designer in his belief about the service time. Here, we choose $\sigma = 0.25$.

To check whether the prior distribution of λ is going with our belief, we plot the density curve of Gamma (80, 60), and then, we find the quantiles of the prior predictive distribution of the number of the customers arriving per hour, which is a negative binomial distribution (Fig. 15.1).

The quantiles of the prior predictive distribution of the total number of arrivals in 60 min:

Q_1 = 1st quantile	Q_2 = 2nd quantile	Q_3 = 3rd quantile
72	79	87

From the graph of the prior distribution and the values of the quantiles, we can say that our prior distribution reflects the belief of the designer.

Now, we check the marginal distribution of the service rate and the prior predictive distribution of the service time from the density plot. Further, we check the quantiles of the prior predictive distribution of the service time (Figs. 15.2 and 15.3).

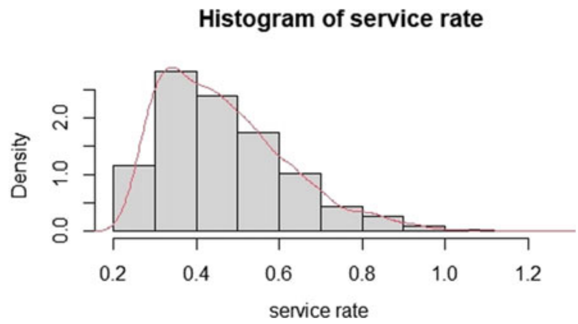

Fig. 15.2 Marginal prior distribution of service rate

From the prior predictive distribution of the service time, we find that Pr(service time > 1) > 0.60 and P(service time < 5) > 0.80. Thus, we can say that the choice of the prior reflects our perception about the service time.

The quantiles of the prior predictive distribution of service time:

$Q_1 = $ 1st quantile	$Q_2 = $ 2nd quantile	$Q_3 = $ 3rd quantile
0.62	1.58	3.26

Generating sample from the posterior distribution of λ and μ:

Once we set the informative prior, we run the Gibbs sampler to simulate 2×10^5 sample observations from the joint posterior distribution of λ and μ. As we are using a method based on MCMC algorithm, it is necessary to know how long the simulation needs to be run. It is also preferable to discard a first few iterations as burn-in period to avoid the influence of the initial point. Consecutive simulations are sometimes highly dependent. It is a good idea to store only every 10th or 20th or 100th observations simulated from the Markov chain to nullify the dependence in sample observations. It is called thinning of the chain. If we store every 50th observation, then 50 is the thinning rate.

To select a burn-in period and thinning rate so that we can explore the posterior distribution up to a desired level, we have used Raftery diagnosis using the package "CODA" in software R. We discard first 2000 simulation as burn-in period and use 500 as thinning rate to store more than 4000 observations from the posterior distribution. According to the Raftery diagnosis, we need around 4000 simulated observations. Further to check the convergence of the Markov chain, we make the trace plot. We also make a plot of autocorrelation function with respect to different lag values (Figs. 15.4 and 15.5).

Fig. 15.3 Prior predictive distribution of service time

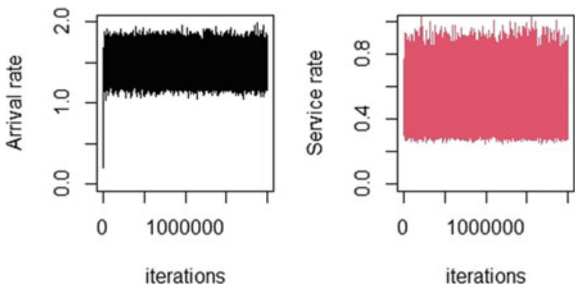

Fig. 15.4 Trace plot of the simulation

From the trace plot, it is clear that the chain has little influence of initial values in the first few iterations, while later it has converged. The autocorrelation plot suggests that 500 is sufficient as thinning rate for this chain as we are interested in the marginal posterior distributions of λ and μ.

From the thinned chain, we obtain the histogram of the marginal posterior distributions of λ and μ as follows (Fig. 15.6).

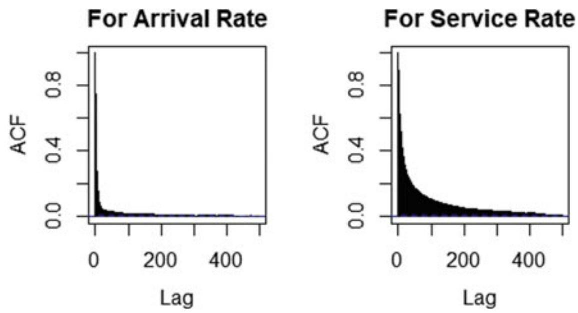

Fig. 15.5 Autocorrelation plot

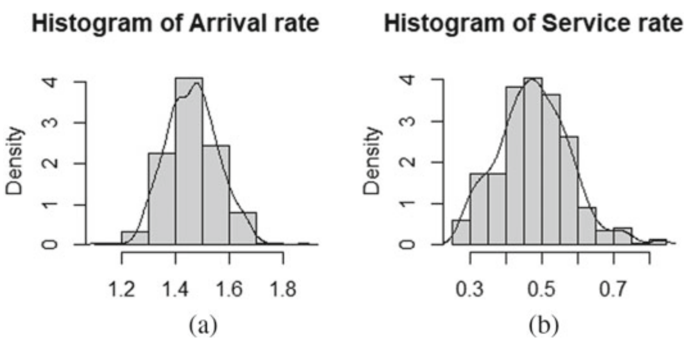

Fig. 15.6 Marginal posterior distribution of—**a** arrival rate and **b** service rate

We draw the approximate density curves over the corresponding histograms. From this figure, it is very clear that the density around the true values of λ (true value of λ is 1.5) and μ (true value of μ is 0.4) is very high. Now, we simulate sample observations of inter-arrival time and service time from the sample observations generated from posterior distributions. The histograms of the posterior predictive distributions of inter-arrival time of the customers and service time are as follows (Fig. 15.7).

We have drawn the density curve of the original inter-arrival time and the service time with dotted lines (which we have used to generate the data) on the respective histograms so that we can justify our model visually. In this graph, we also have drawn the approximate density curve of the marginal density functions with solid lines. It is observed that the dotted lines and the solid ones almost coincide with each other, thus indicating that the marginal posterior distributions will not mislead us.

Quantiles of the marginal posterior distributions of λ and μ:

	Q_1 = 1st quartile	Q_2 = 2nd quartile	Q_3 = 3rd quartile
Λ	1.393	1.461	1.529
μ	0.403	0.475	0.546

Using the package "bayestestR" in *R*, we obtain the high density 95% credible intervals of λ and μ separately as (1.28, 1.66) and (0.28, 0.66), respectively, 95% credible region for (λ, μ) from the joint posterior distribution:

Due to pre-assumed dependence structure of λ and μ, we cannot make our choice for λ and μ from their credible intervals independently. Moreover, we cannot say that the original value of the tuple (λ,μ) belongs to the interval [1.28, 1.66] × [0.28, 0.66] with probability 95%. Some points of the interval [1.28, 1.66] × [0.28, 0.66] do not belong to the parameter space, $\Omega = \{(\lambda, \mu)|\lambda\rangle 0, \mu > 0, \lambda < b\mu\}$.

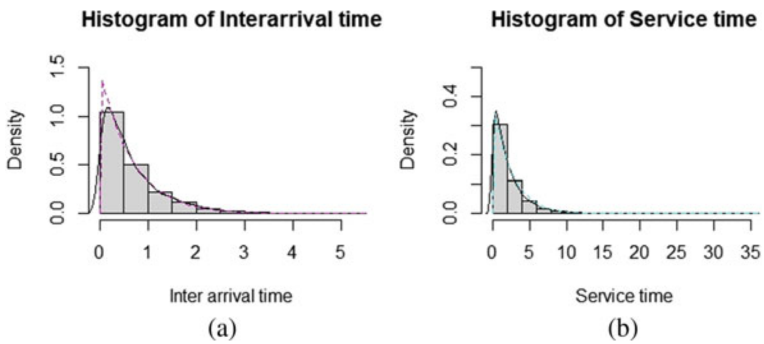

Fig. 15.7 Posterior predictive distribution of **a** inter-arrival time and **b** service time

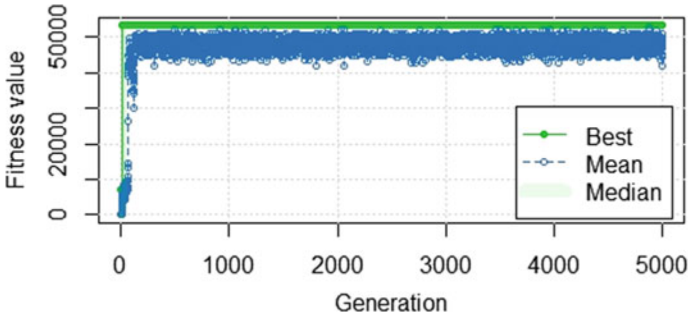

Fig. 15.8 Graph of the fitness function of genetic algorithm

We need a region $S = \{(\lambda, \mu)| \, 0 < a_1 < \lambda < b_1, 0 < a_2 < \mu < b_2\} \cap \Omega$ such that the posterior probability is S is 95%.

We use genetic algorithm to find such subset of parameter space.

The fitness function of the algorithm is follows:

$$F_1(a_1, b_1, a_2, b_2) = w_1|p_S - .95|^{-1} + w_2 A_s^{-1} \text{ when } 0 < a_1 < b_1 \text{ and } 0 < a_2 < b_2$$
$$= 0 \text{ otherwise}, \quad (15.3.1)$$

where p_S is the proportion of the sample observations from the posterior distribution that belongs to S, A_S is the area of S, and w_1 and w_2 are two chosen positive quantities to help the genetic algorithm converge quickly. Here, $w_1 > w_2$ because we give more importance to the fact that the set S must contain 95% of the sample from the joint posterior distribution. We have used reciprocal of the area of S because we do not want a very large 95% credible region.

To check the convergence of the genetic algorithm, we plot the fitness value with respect to the iterations as follows (Fig. 15.8):

Finally, we obtain a credible region as follows:

$$S_{obt} = \{(\lambda, \mu)|1.234 < \lambda < 1.655, 0.279 < \mu < 0.766\} \cap \Omega. \quad (15.3.2)$$

We find that 94.95% of the observations generated from the posterior distribution belong to S_{obt}.

Case 2

In this case, we first gather some experience of the running system. Suppose we observe the system for 30 min and during this time we note that there are 36 arrivals and 11 service completions. We use this experience to choose our prior distribution of (λ, μ). We take.

$$\lambda \sim \text{Gamma}(36, 30)$$
$$\mu|\lambda \sim \text{Truncated Gamma}(11, 30), \text{ truncated to the left at the point } \lambda/b.$$

where "b" denotes the maximum number of customers served at a time.

As we have considered the gamma prior for λ, the prior predictive distribution of the number of arrivals per unit time (hour) will follow negative binomial distribution.

We simulate sample observations from the joint prior distribution. From the following histogram of the marginal distribution of μ, we can have an idea of the marginal prior distribution of the mean service rate. We draw an approximate density function of that marginal distribution on the histogram. Further, we simulate sample from the prior predictive distribution of the service time (Fig. 15.9).

The quantiles of the prior predictive distribution of the total number of arrivals in 60 min are as follows:

Q_1 = 1st quartile	Q_2 = 2nd quartile	Q_3 = 3rd quartile
62	71	81

Here, the prior distribution of λ is a little bit flatter than the prior considered in the previous case, while the interquartile range of the distribution is slightly higher.

The histogram of the marginal prior distribution of μ and the prior predictive distribution of service time are (Fig. 15.10):

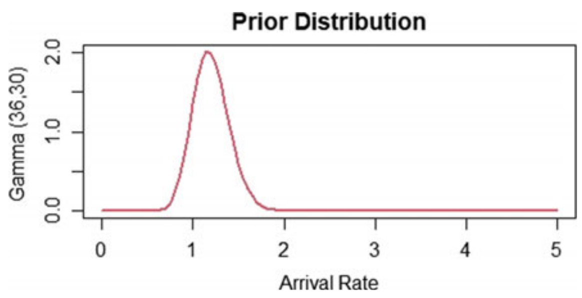

Fig. 15.9 Density plot of the distribution gamma (36, 30)

Here, we run the Gibbs sampler 2×10^5 times. Though Raftery and Lewis [9] diagnoses suggest that 10^4 iteration with burn-in period 1000 and thinning rate 200 would be sufficient, we discard first 1000 iteration and then check the trace plot and the autocorrelation plot which are given (Fig. 15.11):

From the trace plot, it is clear that the chain has converged. Autocorrelation plot suggests 250 as thinning rate. From the thinned chain, we obtain histogram of the marginal posterior distributions of λ and μ as follows (Fig. 15.12).

We draw the approximate density curves over the corresponding histograms. It is clear that the density around the true value of λ (true value of λ is 1.5) is very high. Density around true value of μ (true value of μ is 0.4) is high but not higher than the previous case. Now, we simulate sample observations of inter-arrival time and service time from the sample observations generated from the joint posterior distribution. The histograms of the posterior predictive distributions of inter-arrival time of the customers and service time of the server are as follows (Fig. 15.13).

We have drawn the density curve of the original inter-arrival time and the service time with dotted line (which we have used to generate the data) on the respective histograms so that we can justify our model visually. In this graph, we also have drawn the approximate density curve of the marginal density functions with solid lines. The dotted lines and the solid ones almost coincide with each other.

The 95% high-density credible intervals of λ and μ are then obtained as (1.28, 1.66) and (0.27, 0.66), respectively.

We also find the 95% credible region as in the previous case, and it comes out to be

$$S_{\text{obt}} = \{(\lambda, \mu) | 1.1177 < \lambda < 1.6513, 0.2793 < \mu < 0.7581\} \cap \Omega. \tag{15.3.3}$$

The credible region contains 94.998% of the simulated observations.

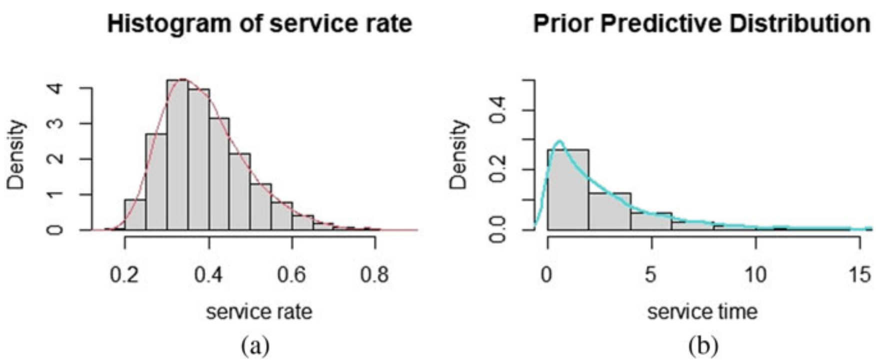

Fig. 15.10 Histogram of **a** marginal prior distribution of service rate and **b** prior predictive distribution of service time

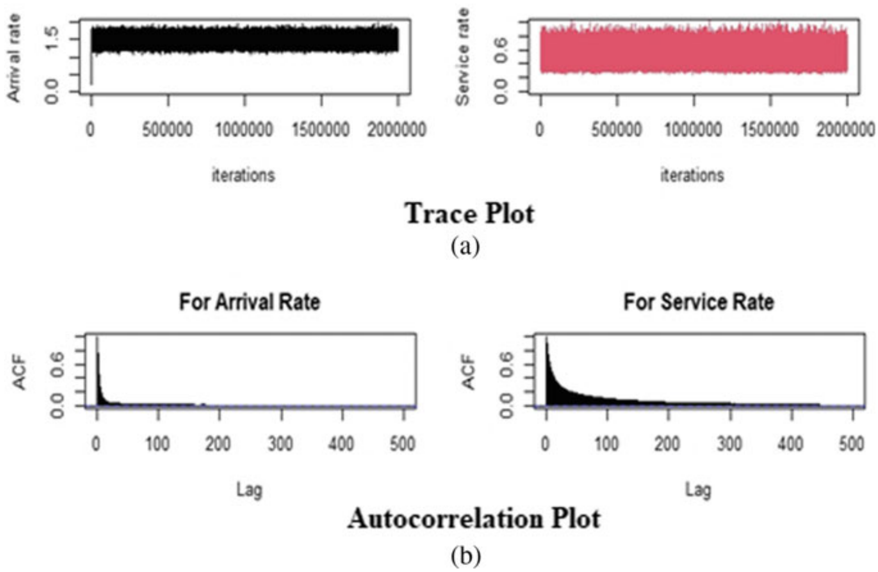

Fig. 15.11 **a** Trace plot and **b** autocorrelation plot

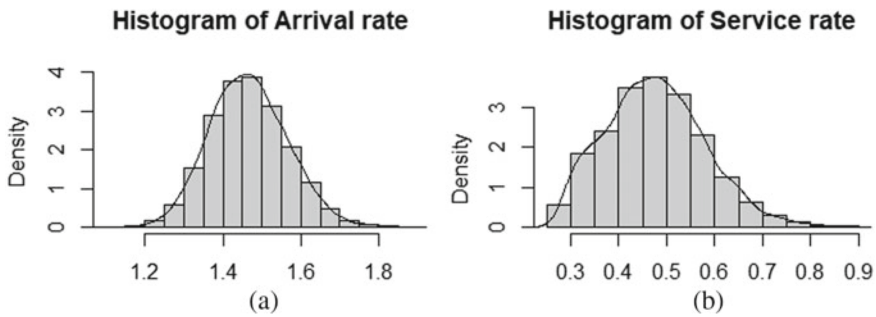

Fig. 15.12 Histogram of marginal posterior distribution of **a** arrival rate and **b** service rate

Discussion

In this study, we have used two different joint prior distributions. In our first case, we have used the information based on the experience of the designer. According to designer's experience, we have assumed that more or less 80 customers arrive in an hour and we have chosen our joint prior distribution accordingly. What if the information provided by the designer does not match with the real situation? Here in this paper, we have generated the data by using $\lambda = 1.5$ and $\mu = 0.4$. Suppose the designer inform us that he has experienced that more or less 120 customers arrive per hour; then, according to the provided information, we consider gamma (120,

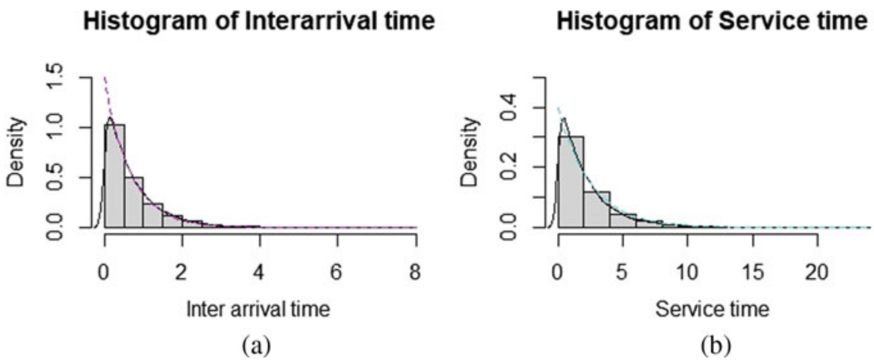

Fig. 15.13 Histogram of prior predictive distribution of **a** inter-arrival time and **b** service time

60) as prior distribution of λ. In this case, the conditional prior distribution of μ is truncated normal distribution with mean 13/30 and standard deviation 0.25, truncated to the left at the point λ/b. We have found that the different choices of the joint prior distribution influence the resulting inference, but with the choice of higher data size, we can overcome this influence to some extent. Here, we have considered the data of size 90 and made histogram plots of the posterior distributions of the system parameters and the posterior predictive distributions of inter-arrival time and service time as follows (Figs. 15.14 and 15.15).

The histograms are very similar to the respective histograms of Sect. 15.4.1, except the marginal posterior distribution of λ. Now if we look at figures of the first case study made in 3.1 and second case study made in 3.2, then we can notice that graphs of the respective prior distributions, the posterior distributions and the posterior predictive distributions are very similar. The credible regions obtained in both case studies are very similar. Thus, we can conclude that in these two case studies where two completely different joint prior distributions reflect similar kind of prior belief result almost similar posterior inference about the values of the system

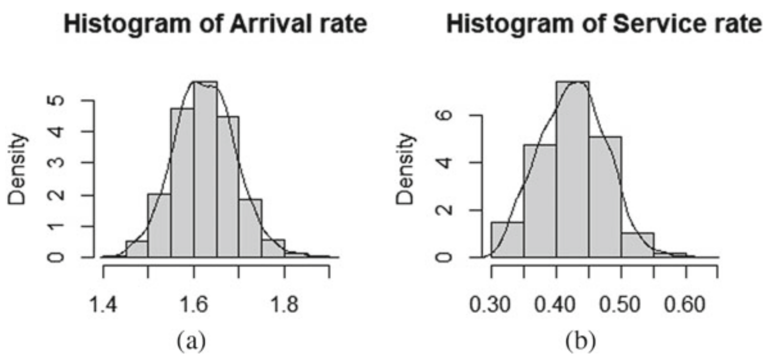

Fig. 15.14 Histogram of marginal posterior distribution of **a** arrival rate and **b** service rate

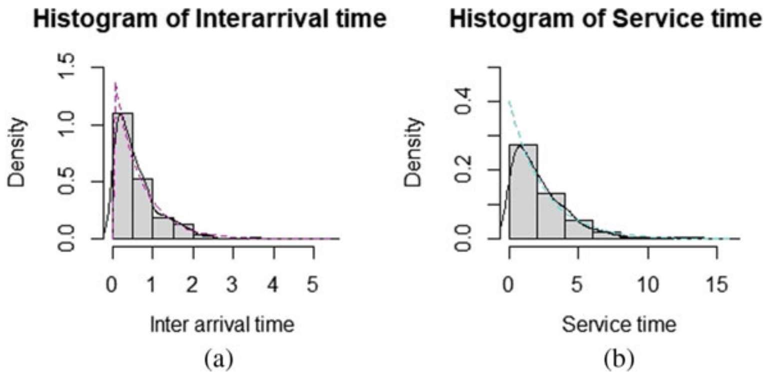

Fig. 15.15 Histogram of posterior predictive distribution of **a** inter-arrival time and **b** service time

parameters. We have found that the different choices of the hyper-parameters of the same joint prior distribution can influence the posterior distribution, which can be overcome to some extent, by increasing the data size. To avoid such influence of the prior belief, Bayesian hierarchical model with the suitable choice of joint prior distribution of the system parameters can be used to study further. We can use this Bayesian approach to infer about the system parameters for other bulk service queuing systems like $M/G^{(a,b)}/1$, $M/D^{(a,b)}/1$, etc., where we do not have explicit expression of the stationary distribution of the system size but we can obtain the likelihood function as we have obtained in Sect. 15.4.

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Chapter 16
Performance Analysis of an M/M/2 Queue with Partially Active Server Subject to Catastrophe

B. Thilaka, B. Poorani, and S. Udayabaskaran

Abstract An M/M/2 queue with the server operating in three modes—maintenance mode, sleep mode (no server is available), and active mode (partially active state where only a single server is available and fully active state where both servers are available for service) subject to catastrophe is considered. Catastrophes occur in both the sleep and active mode. Customer impatience is allowed only when the server is in the sleep mode. The objective of the paper is to determine the performance measures of the system. The transient and steady-state probabilities of the system, the steady-state performance measures are derived.

Keywords Catastrophe · Sleep mode · Maintenance mode · Transient and steady-state probabilities

Introduction

Multi-server queueing systems play a significant role in addressing various issues in production management, traffic control, computer and communication networks. In communication networks, multiple servers are used to reduce traffic congestion and improve server performance. Multi-server queueing models may deal with homogeneous servers as well as heterogeneous servers. A transient analysis of an M/M/2 queueing system was studied by Krishna Kumar and Madheswari [1]. Krishna Kumar

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and Madheswari [2] analyzed the transient behavior of an M/M/2 queue with heterogeneous servers subject to catastrophe.

Customer impatience in multi-server queues has been analyzed by various researchers. Kyoo and Henk [3] studied the M/M/c queue in which a customer is permitted to leave the system before commencement of service. Al-Seedy et al. [4] obtained a transient solution of M/M/c queue with balking and reneging. Krishnamoorthy and Sreenivasan [5] considered an M/M/2 queueing model with heterogeneous servers one of which is always available for service.

Ammar Sherif [6] obtained the transient analysis of an M/M/2 queueing system with heterogeneous servers with impatient customers. A two-heterogeneous server queue (M/M/2) with system disaster, server failure, and repair was studied by Sudhesh et al. [7]. Dharmaraja and Rakesh [8] derived the time-dependent solutions of the heterogeneous multiple server M/M/c queueing system with catastrophes.

Yonatan and Yri [9] discussed multi-server queue with exponentially distributed vacations. Nobuko [10] studied M/M/2 queue where only one server is permitted to take vacation when the system becomes empty. Tian Naishuo and Jinhua [11] analyzed the conditional stochastic decomposition of an M/M/c queue with server vacations. George and Naishuo [12] studied an M/M/c queue in which d servers ($0 \leq d \leq c$) take a vacation of random period together when they are finished with serving customers and find no customers in the waiting line. Naishuo and George [13] considered a two threshold vacation policy in multi-server queueing systems. Yue Dequan and Guoxi [14] studied an M/M/c queueing system with impatient customers and a synchronous vacation policy, where customer impatience is due to the server vacation.

In communication networks, multiple servers are used to reduce the traffic congestion and improve server performance. Multiple servers are required for high-efficiency transmissions. Bell Colin [15]studied an M/M/c queueing system in which the number of working servers can be adjusted to 0, 1 ... or c for minimizing the average cost. Zhang Jie and Yujie [16] considered queuing theory-based (M/M/c system) co-channel interference analysis approach for high-density wireless networks.

This has motivated the analysis of the behavior of an M/M/2 queueing system with the server in maintenance mode, sleep mode, and active mode (comprises partially active state and fully active state) subject to catastrophe. Customer impatience occurs only when the server is in sleep mode. Transitions can occur from sleep mode to the partially active state of the active mode and also from partially active state to fully active state of the active mode with the rate η (Fig. 16.1).

Model Description and Analysis

This section describes the mathematical model, and closed-form expressions for the transient probabilities of the system are derived. An M/M/2 queueing system with the server under maintenance subject to catastrophe is considered. The server is said to be in sleep mode (both the servers are not available for service), partially active

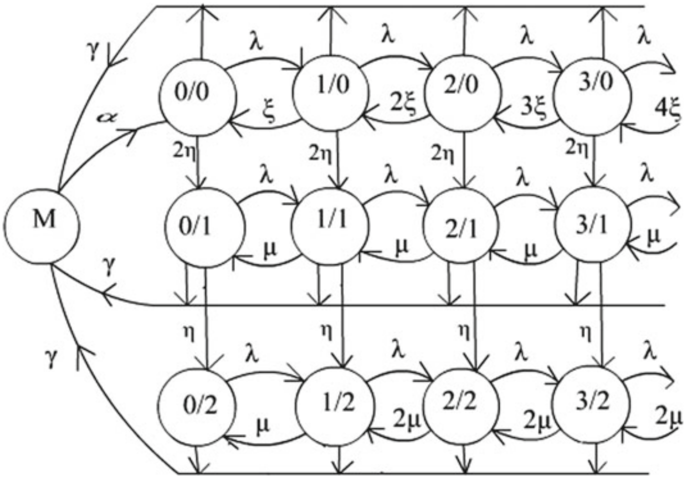

Fig. 16.1 Transition diagram

state of active mode (one server is available for service), and fully active state of active mode (both the servers are available for service). The server is said to be in state 0(D) if it is in sleep mode, state 1 if it is in partially active state of active mode , in state 2 if it is in fully active state of active mode, and state M if it is in maintenance mode. Customers arrive according to a Poisson process to any mode with arrival rate λ. The service rate for each server is assumed to be exponential at rate μ. The sleep mode and two states of the active modes may also be interpreted as follows: Both the servers maybe considered to be on vacation in sleep mode, while one server is on vacation in partially active state and neither server is on vacation in fully active state of the active mode. Customer impatience is assumed to occur only in sleep mode in the rate of ξ.

Catastrophes are assumed to occur independently when the system in both sleep mode and active mode and follow exponential distribution with parameter γ. Once a catastrophe occurs, all the customers are wiped out from the system, and the system enters the maintenance state. During the maintenance period, no customer is allowed to enter the system. At the end of the maintenance period, the server automatically moves to sleep mode with rate α. Sleep mode D comprises all the states $(n, 0)$ where $n = 0, 1, 2, \ldots$

Let $X(t)$ denote the number of customers in the system at time t and $J(t)$ denote the state of the server at time t. Then $X(t) \in \{0, 1, 2 \ldots\}$ and

$$J(t) = \begin{cases} M & \text{if the server is in maintenance mode} \\ 0(D) & \text{if the server is in sleep mode} \\ 1 & \text{if the server is in partially active state of the active mode} \\ 2 & \text{if the server is in fully active state of the active mode} \end{cases}$$

The joint process $\{X(t), J(t), t \geq 0\}$ is Markov.
The state space of the system is given by

$$\Omega = \{(0, M), (0, 0), (1, 0), (2, 0), \ldots\} \bigcup \{(0, 1), (1, 1), (2, 1), \ldots\} \bigcup \{(0, 2), (1, 2), (2, 2), \ldots\}$$

The system is initially assumed to be at state 0, leading to the initial condition $p(0, 0, 0) = 1$, $p(M, 0) = 0$, $p(n, 0, 0) = 0, n \geq 1$,
$p(n, 1, 0) = 0, n \geq 0$, $p(n, 2, 0) = 0, n \geq 0$.

By using probability laws

$$p(i, j, t) = P[X(t) = i, J(t) = j; t/i = 0, 1, 2, \ldots \infty, j = 0, 1, 2]$$

(Since the system may be in state (0, 0) at time t in any of the following ways:

(i) on arrival from state M at a rate α,

(ii) on arrival from state (1, 0) at a rate ξ)

$$p(0, 0, t) = \mathrm{e}^{-(\lambda+2\eta+\gamma)t} + [p(1, 0, t)\,\xi + p(M, t)\,\alpha] * \mathrm{e}^{-(\lambda+2\eta+\gamma)t} \tag{16.1}$$

$$p(i, 0, t) = [p(i-1, 0, t)\,\lambda + p(i+1, 0, t)\,(i+1)\xi] * \mathrm{e}^{-(\lambda+2\eta+\gamma+i\xi)t}, i \geq 1 \tag{16.2}$$

$$p(0, 1, t) = [p(0, 0, t)\,2\eta + p(1, 1, t)\,\mu] * \mathrm{e}^{-(\lambda+\eta+\gamma)t} \tag{16.3}$$

$$p(i, 1, t) = [p(i, 0, t)\,2\eta + p(i-1, 1, t)\,\lambda + p(i+1, 1, t)\,\mu] * \mathrm{e}^{-(\lambda+\eta+\gamma+\mu)t}, i \geq 1 \tag{16.4}$$

$$p(0, 2, t) = [p(0, 1, t)\,\eta + p(1, 2, t)\,\mu] * \mathrm{e}^{-(\lambda+\gamma)t} \tag{16.5}$$

$$p(1, 2, t) = [p(1, 1, t)\,\eta + p(0, 2, t)\,\lambda + p(2, 2, t)\,2\mu] * e^{-(\lambda+\mu+\gamma)t} \tag{16.6}$$

$$p(i, 2, t) = [p(i, 1, t)\,\eta + p(i-1, 2, t)\,\lambda + p(i+1, 2, t)\,2\mu] * \mathrm{e}^{-(\lambda+2\mu+\gamma)t}, i \geq 2 \tag{16.7}$$

$$p(M, t) = \sum_{i=0}^{\infty}[p(i, 0, t)\,\gamma + p(i, 1, t)\,\gamma + p(i, 2, t)\,\gamma] * e^{-\alpha t} \tag{16.8}$$

Taking Laplace Transform of (16.1) to (16.8) with respect to time and denoting the transform variable by s

$$(s+\lambda+2\eta+\gamma)\,p^*(0,0,s)=1+p^*(1,0,s)\,\xi+p^*(M,s)\,\alpha \tag{16.9}$$

$$(s+\lambda+2\eta+\gamma+i\xi)\,p^*(i,0,s)=p^*(i-1,0,s)\,\lambda+p^*(i+1,0,s)\,(i+1)\xi,\, i\geq 1 \tag{16.10}$$

$$(s+\lambda+\eta+\gamma)\,p^*(0,1,s)=p^*(0,0,s)\,2\eta+p^*(1,1,s)\,\mu \tag{16.11}$$

$$\begin{aligned}(s+\lambda+\eta+\gamma+\mu)\,p^*(i,1,s)&=p^*(i,0,s)\,2\eta+p^*(i-1,1,s)\,\lambda\\&+p^*(i+1,1,s)\,\mu,\, i\geq 1\end{aligned} \tag{16.12}$$

$$(s+\lambda+\gamma)\,p^*(0,2,s)=p^*(0,1,s)\,\eta+p^*(1,2,s)\,\mu \tag{16.13}$$

$$(s+\lambda+\mu+\gamma)\,p^*(1,2,s)=p^*(1,1,s)\,\eta+p^*(0,2,s)\,\lambda+p^*(2,2,s)\,2\mu \tag{16.14}$$

$$\begin{aligned}(s+\lambda+2\mu+\gamma)\,p^*(i,2,s)&=p^*(i,1,s)\,\eta+p^*(i-1,2,s)\,\lambda\\&+p^*(i+1,2,s)\,2\mu,\, i\geq 2\end{aligned} \tag{16.15}$$

$$(s+\alpha)\,p^*(M,s)=\sum_{i=0}^{\infty}[p^*(i,0,s)\,\gamma+p^*(i,1,s)\,\gamma+p^*(i,2,s)\,\gamma] \tag{16.16}$$

Defining the Probability Generating Functions as

$$G_j^*(u,s)=\sum_{i=0}^{\infty}p^*(i,j,s)\,u^i;\ j=0,1,2 \tag{16.17}$$

$$G^*(u,s)=p^*(M,s)+G_0^*(u,s)+G_1^*(u,s)+G_2^*(u,s) \tag{16.18}$$

Hence from (16.10)

$$\frac{\partial G_0^*(u,s)}{\partial u}-\left[\frac{\lambda}{\xi}-\frac{s+2\eta+\gamma}{\xi(1-u)}\right]G_0^*(u,s)=\frac{K(s)}{\xi(1-u)}$$

where $K(s)=\xi p^*(1,0,s)-(s+\lambda+2\eta+\gamma)p^*(0,0,s)$
The integrating factor is $\mathrm{e}^{-\frac{\lambda}{\xi}u}(1-u)^{\frac{s+2\eta+\gamma}{\xi}}$

$$\frac{\partial}{\partial u}\left[G_0^*(u,s)\,\mathrm{e}^{-\frac{\lambda}{\xi}u}(1-u)^{\frac{s+2\eta+\gamma}{\xi}}\right]=\frac{K(s)}{\xi}\mathrm{e}^{-\frac{\lambda}{\xi}u}(1-u)^{\frac{s+2\eta+\gamma}{\xi}-1}$$

The integrating factor is $e^{-\frac{\lambda}{\xi}u}(1-u)^{\frac{s+2\eta+\gamma}{\xi}}$

$$\frac{\partial}{\partial u}\left[G_0^*(u,s)\,e^{-\frac{\lambda}{\xi}u}(1-u)^{\frac{s+2\eta+\gamma}{\xi}}\right]=\frac{K(s)}{\xi}e^{-\frac{\lambda}{\xi}u}(1-u)^{\frac{s+2\eta+\gamma}{\xi}-1}$$

$$G_0^*(u,s)=\left[p^*(0,0,s)+\frac{K(s)}{\xi}\int_0^u e^{-\frac{\lambda}{\xi}v}(1-v)^{\frac{s+2\eta+\gamma}{\xi}-1}dv\right]e^{\frac{\lambda}{\xi}u}(1-u)^{-\left(\frac{s+2\eta+\gamma}{\xi}\right)}$$

$$G_0^*(1,s)=\left[p^*(0,0,s)+\frac{K(s)}{\xi}\int_0^1 e^{-\frac{\lambda}{\xi}v}(1-v)^{\frac{s+2\eta+\gamma}{\xi}-1}dv\right]e^{\frac{\lambda}{\xi}}\lim_{u\to 1}(1-u)^{-\left(\frac{s+2\eta+\gamma}{\xi}\right)}$$

Since $G_0^*(1,s)=\sum_{i=0}^{\infty}p^*(i,0,s)$ is well defined and $\lim_{u\to 1}(1-u)^{-\left(\frac{s+2\eta+\gamma}{\xi}\right)}=\infty$

$$\therefore K(s)=-\frac{\xi p^*(0,0,s)}{\int_0^1 e^{-\frac{\lambda}{\xi}v}(1-v)^{\frac{s+2\eta+\gamma}{\xi}-1}dv}$$

Substituting $K(s)$ in $G_0^*(u,s)$ yields

$$G_0^*(u,s)=\frac{e^{\frac{\lambda}{\xi}u}}{(1-u)^{\frac{s+2\eta+\gamma}{\xi}}}\left[1-\frac{\int_0^u e^{-\frac{\lambda}{\xi}v}(1-v)^{\frac{s+2\eta+\gamma}{\xi}-1}dv}{\int_0^1 e^{-\frac{\lambda}{\xi}v}(1-v)^{\frac{s+2\eta+\gamma}{\xi}-1}dv}\right]p^*(0,0,s)$$

In the limiting case as $u\to 1$ and applying L'Hospital rule,

$$G_0^*(1,s)=\frac{\xi}{(s+2\eta+\gamma)\int_0^1 e^{-\frac{\lambda}{\xi}v}(1-v)^{\frac{s+2\eta+\gamma}{\xi}-1}dv}p^*(0,0,s)$$

Let $I_\theta=\int_0^1 e^{-\frac{\lambda}{\xi}v}(1-v)^{\theta-1}dv=\frac{1}{\theta}\left[1-\frac{\lambda}{\xi}I_{\theta+1}\right]$
By iterating,

$$I_\theta=\frac{1}{\theta}\left[1+\sum_{n=1}^{m}\left(\frac{\lambda}{\xi}\right)^n\frac{(-1)^n}{\prod_{j=1}^{n}\frac{(\theta+j\xi)}{\xi}}+\left(\frac{\lambda}{\xi}\right)^{m+1}\frac{(-1)^m}{\prod_{j=1}^{n}\frac{(\theta+j\xi)}{\xi}}I_{\theta+m+1}\right]$$

Taking $m\to\infty$, since $\lim_{m\to\infty}I_{\theta+m+1}=0$

$$I_\theta = \frac{\xi}{s+2\eta+\gamma}\left[1+\sum_{n=1}^{m}\left(\frac{\lambda}{\xi}\right)^n \frac{(-1)^n}{\prod_{j=1}^{n}\frac{(s+2\eta+\gamma+j\xi)}{\xi}}\right]$$

$$G_0^*(1,s) = \frac{1}{1+\sum_{n=1}^{m}\left(\frac{\lambda}{\xi}\right)^n \frac{(-1)^n\xi^n}{\prod_{j=1}^{n}(s+2\eta+\gamma+j\xi)}} p^*(0,0,s) \tag{16.19}$$

From (16.12) and (16.15)

$$G_1^*(u,s) = \frac{2\eta u G_0^*(u,s) - \mu(1-u)p^*(0,1,s)}{-\lambda u^2 + (s+\lambda+\eta+\gamma+\mu)u - \mu} \tag{16.20}$$

$$G_2^*(u,s) = \frac{\eta u G_1^*(u,s) - 2\mu(1-u)p^*(0,2,s) - \mu u(1-u)p^*(1,2,s)}{-\lambda u^2 + (s+\lambda+\gamma+2\mu)u - 2\mu} \tag{16.21}$$

Substituting $u = 1$ in (16.20), (16.21) and using (16.19) in total probability law yield

$$p^*(0,0,s) = \frac{(s+\gamma)(s+\alpha)}{s(s+\alpha+\gamma)(s+2\eta+\gamma)}\left[1+\sum_{n=1}^{m}\left(\frac{\lambda}{\xi}\right)^n \frac{(-1)^n\xi^n}{\prod_{j=1}^{n}(s+2\eta+\gamma+j\xi)}\right]$$

$$= \left\{\frac{\alpha\gamma}{(\alpha+\gamma)(2\eta+\gamma)}\frac{1}{s} + \frac{\alpha\gamma}{(\alpha+\gamma)(\alpha-2\eta)}\frac{1}{s+\alpha+\gamma} + \frac{2\eta(\alpha-2\eta-\gamma)}{(2\eta+\gamma)(\alpha-2\eta)}\frac{1}{s+2\eta+\gamma}\right\}$$
$$\left[1+\sum_{n=1}^{m}(-1)^n\lambda^n\left(\frac{1}{\xi^{n-1}}\sum_{j=1}^{n}\frac{(-1)^{j-1}}{(j-1)!(n-j)!(s+2\eta+\gamma+j\xi)}\right)\right]$$

$$\begin{aligned} p(0,0,t) = {} & \alpha\gamma e^{-\gamma t}\left[\frac{1}{2\eta+\gamma} + \frac{1}{\alpha-2\eta}e^{-(\alpha+\gamma)t}\right] + \frac{2\eta(\alpha-2\eta-\gamma)}{(2\eta+\gamma)(\alpha-2\eta)}e^{-(2\eta+\gamma)t} \\ & -\lambda\left(\alpha\gamma e^{-\gamma t}\left[\frac{1}{2\eta+\gamma} + \frac{1}{\alpha-2\eta}e^{-(\alpha+\gamma)t}\right] + \frac{2\eta(\alpha-2\eta-\gamma)}{(2\eta+\gamma)(\alpha-2\eta)}e^{-(2\eta+\gamma)t}\right) \\ & * e^{-(2\eta+\gamma+\xi)t} e^{-\frac{\lambda}{\mu}(1-e^{-\xi t})} \end{aligned} \tag{16.22}$$

To find $p^*(k,0,s)$, using (16.19)
Let

$$\Psi(u,s) = \frac{G_0^*(1,s)}{p^*(0,0,s)} = \frac{e^{\frac{\lambda}{\xi}u}}{(1-u)^{\frac{s+2\eta+\gamma}{\xi}}}\left[1 - \frac{\int_0^u e^{-\frac{\lambda}{\xi}v}(1-v)^{\frac{s+2\eta+\gamma}{\xi}-1}dv}{\int_0^1 e^{-\frac{\lambda}{\xi}v}(1-v)^{\frac{s+2\eta+\gamma}{\xi}-1}dv}\right]$$

Differentiating n times with respect to u, and applying Leibtnitz's rule of successive differentiation

$$(1-u)\Psi_{n+2}(u,s)-\left[\frac{\lambda}{\xi}(1-u)+\frac{s+2\eta+\gamma}{\xi}+n+1\right]\Psi_{n+1}(u,s)+\frac{\lambda}{\xi}(n+1)\Psi_n(u,s)=0, n=1,2,\ldots$$

where $\Psi_n(u,s)$ denote the n^{th} derivative of $\Psi(u,s)$

$$\Psi_{n+1}(1,s)=\frac{\lambda(n+1)}{s+2\eta+\gamma+(n+1)\xi}\Psi_n(1,s)$$

$$\therefore \Psi_n(1,s)=\frac{n!\lambda^n}{\prod_{j=1}^{n}(s+2\eta+\gamma+j\xi)}\left[\frac{\xi}{(s+2\eta+\gamma)\int_0^1 \mathrm{e}^{-\frac{\lambda}{\xi}v}(1-v)^{\frac{s+2\eta+\gamma}{\xi}-1}dv}\right]$$

$$=\frac{n!\lambda^n}{\prod_{j=1}^{n}(s+2\eta+\gamma+j\xi)}\left[\frac{1}{1+\sum_{n=1}^{m}\left(\frac{\lambda}{\xi}\right)^n\frac{(-1)^n\xi}{\prod_{j=1}^{n}(s+2\eta+\gamma+j\xi)}}\right]$$

Expanding $\Psi(u,s)$ as a Taylor's series about $u=1$,

$$\frac{G_0^*(1,s)}{p^*(0,0,s)}=\Psi(u,s)=\Psi(1,s)+\sum_{n=1}^{\infty}\frac{\Psi_n(1,s)}{n!}(u-1)^n$$

$$G_0^*(u,s)=\frac{(s+\gamma)(s+\alpha)}{s(s+\alpha+\gamma)(s+2\eta+\gamma)}\left[1+\sum_{n=1}^{\infty}\frac{\lambda^n(u-1)^n}{\prod_{j=1}^{n}(s+2\eta+\gamma+j\xi)}\right]$$

$$=\frac{(s+\gamma)(s+\alpha)}{s(s+\alpha+\gamma)(s+2\eta+\gamma)}\left[1+\sum_{n=1}^{\infty}\frac{(-1)^n\lambda^n}{\prod_{j=1}^{n}(s+2\eta+\gamma+j\xi)}+\sum_{k=1}^{\infty}\left\{\sum_{r=0}^{\infty}(-1)^r\binom{k+r}{r}\frac{\lambda^{k+r}}{\prod_{j=1}^{k+r}(s+2\eta+\gamma+j\xi)}\right\}u^n\right]$$

Equating the coefficients of u^k

$$p^*(k,0,s) = \frac{(s+\gamma)(s+\alpha)}{s(s+\alpha+\gamma)(s+2\eta+\gamma)} \sum_{r=0}^{\infty}(-1)^r \binom{k+r}{r} \frac{\lambda^{k+r}}{\prod_{j=1}^{k+r}(s+2\eta+\gamma+j\xi)}, k=1,2,.. \tag{16.23}$$

$$p(k,0,t) = \sum_{r=0}^{\infty}(-1)^r \binom{k+r}{r} f(t) * g_{k+r}(t), k=1,2,\ldots \tag{16.24}$$

where

$$f(t) = \frac{\alpha\gamma}{s+\gamma}\left[\frac{1}{2\eta+\gamma} + \frac{1}{\alpha-2\eta}\mathrm{e}^{-(\alpha+\gamma)t}\right] + \frac{2\eta(\alpha-2\eta-\gamma)}{(2\eta+\gamma)(\alpha-2\eta)}\mathrm{e}^{-(2\eta+\gamma)t}$$

$$g_n(t) = \frac{\mathrm{e}^{-(2\eta+\gamma)t}}{\xi^{n-1}} \sum_{j=1}^{n} \frac{(-1)^{j-1}\mathrm{e}^{-j\xi t}}{(j-1)!(n-j)!} = \frac{\mathrm{e}^{-(2\eta+\gamma+\xi)t}(1-\mathrm{e}^{-\xi t})^{n-1}}{\xi^{n-1}(n-1)!}$$

To obtain the probabilities of $p(n,1,t), n=0,1,2,\ldots$:
From (16.20),the zeros of $-\lambda u^2 + u(s+\lambda+\mu+\gamma+\eta) - \mu = 0$ are given by

$$u_1, u_2 = \frac{(s+\lambda+\mu+\gamma+\eta) \mp \sqrt{(s+\lambda+\mu+\gamma+\eta)^2 - 4\lambda\mu}}{2\lambda};$$
$$|u_1| < 1, |u_2| > 1$$

Due to analyticity of $G_1^*(u,s)$,

$$G_1^*(u,s) = \frac{2\eta u G_0^*(u,s) - \mu(1-u)\frac{2\eta u_1}{\mu(1-u_1)}G_0^*(u_1,s)}{-\lambda(u-u_1)(u-u_2)}$$

$$= \frac{2\eta}{\mu}\left[\sum_{j=1}^{\infty}\sum_{m=0}^{\infty} u_1^{m+j} p^*(0,0,s) + \sum_{j=1}^{\infty}\sum_{l=1}^{\infty}\left(\frac{\lambda}{\mu}\right)^l u^l u_1^{l+j} p^*(0,0,s) + \sum_{l=1}^{\infty}\sum_{m=0}^{\infty}\sum_{j=1}^{\infty}\sum_{k=1}^{l} \left(\frac{\lambda}{\mu}\right)^{l-k} u^l u_1^{l-k+j} p^*(m+k,0,s) - \sum_{l=1}^{\infty}\sum_{m=0}^{\infty}\sum_{j=1}^{\infty}\sum_{k=1}^{l}\left(\frac{\lambda}{\mu}\right)^{l-k} u^l u_1^{l-k+j+1} p^*(m+k,0,s)\right]$$

Equating the coefficients of u^l

$$p^{*}\left(0,1,s\right)=\frac{2\eta}{\mu}\sum_{j=1}^{\infty}\sum_{m=0}^{\infty}u_{1}^{m+j}\,p^{*}\left(m,0,s\right) \tag{16.25}$$

$$p^{*}\left(l,1,s\right)=\frac{2\eta}{\mu}\left[\sum_{j=1}^{\infty}\left(\frac{\lambda}{\mu}\right)^{l}u_{1}^{l+j}\,p^{*}\left(0,0,s\right)+\sum_{m=0}^{\infty}\sum_{j=1}^{\infty}\sum_{k=0}^{l}\left(\frac{\lambda}{\mu}\right)^{l-k}\right.$$
$$\left.u_{1}^{l-k+j}\,p^{*}\left(m+k,0,s\right)-\sum_{m=0}^{\infty}\sum_{j=1}^{\infty}\sum_{k=1}^{l}\left(\frac{\lambda}{\mu}\right)^{l-k}u_{1}^{l-k+j+1}\,p^{*}\left(m+k,0,s\right)\right];l\geq 1 \tag{16.26}$$

Taking the inverse Laplace transform,

$$L^{-1}\left[\left(s-\sqrt{s^{2}-a^{2}}\right)^{k}\right]=\frac{ka^{k}}{t}I_{k}\left(at\right),$$

$$\Phi_{1;r}\left(t\right)=L^{-1}\left(u_{1}^{r}\right)=\mathrm{e}^{-(\lambda+\mu+\gamma+\eta)t}\left(\frac{\mu}{\lambda}\right)^{r/2}\frac{r\,I_{r}\left(2\sqrt{\lambda\mu}t\right)}{t}$$

$$p\left(0,1,t\right)=\frac{2\eta}{\mu}\sum_{m=0}^{\infty}\sum_{j=1}^{\infty}\Phi_{1;m+j}*p\left(m,0,t\right) \tag{16.27}$$

$$\begin{aligned}p\left(l,1,t\right)=\frac{2\eta}{\mu}&\left[\sum_{j=0}^{\infty}\left(\frac{\lambda}{\mu}\right)^{l}\Phi_{1;l+j}*p\left(0,0,t\right)\right.\\&+\sum_{m=0}^{\infty}\sum_{j=1}^{\infty}\sum_{k=0}^{l}\left(\frac{\lambda}{\mu}\right)^{l-k}\Phi_{1;l-k+j}*p\left(m+k,0,t\right)\\&\left.-\sum_{m=0}^{\infty}\sum_{j=1}^{\infty}\sum_{k=1}^{l}\left(\frac{\lambda}{\mu}\right)^{l-k}\Phi_{1;l-k+j+1}*p\left(m+k,0,t\right)\right];l\geq 1\end{aligned} \tag{16.28}$$

To obtain the probabilities of $p\left(n,2,t\right), n=0,1,2,\ldots$:
From (16.21), the zeros of $-\lambda u^{2}+u\left(s+\lambda+2\mu+\gamma\right)-2\mu=0$ are given by

$$u_{3},u_{4}=\frac{\left(s+\lambda+2\mu+\gamma\right)\mp\sqrt{\left(s+\lambda+2\mu+\gamma\right)^{2}-8\lambda\mu}}{2\lambda};\ |u_{3}|<1,\ |u_{4}|>1$$

Due to analyticity of $G_{2}^{*}\left(u,s\right)$,

$$p^{*}(0,2,s)=\frac{\eta u_3 p^{*}(0,1,s)}{4\mu\left\{1-\frac{1}{2}\left(1-\frac{\lambda u_3}{2\mu}\right)u_3\right\}}+\frac{\eta u_3 G_1^{*}(u_3,s)}{4\mu\left\{1-\frac{1}{2}\left(1-\frac{\lambda u_3}{2\mu}\right)u_3\right\}(1-u_3)}$$

$$=\eta\left\{\sum_{k=0}^{\infty}\sum_{r=0}^{k}\binom{k}{r}\frac{(-1)^r\lambda^r}{2^{k+r+2}\mu^{r+1}}u_3^{k+r+1}p^{*}(0,1,s)+\sum_{l=0}^{\infty}\sum_{m=0}^{\infty}\sum_{k=0}^{\infty}\sum_{r=0}^{k}\binom{k}{r}\right.$$
$$\left.\frac{(-1)^r\lambda^r}{2^{k+r+2}\mu^{r+1}}u_3^{m+k+r+l+1}p^{*}(l,1,s)\right\} \quad (16.29)$$

$$\Phi_{3;r}(t)=L^{-1}\left(u_3^r\right)==\mathrm{e}^{-(\lambda+2\mu+\gamma)t}\left(\frac{\mu}{\lambda}\right)^{r/2}\frac{rI_r\left(2\sqrt{\lambda\mu}t\right)}{t}$$

taking inverse Laplace transform of (16.29)

$$p(0,2,t)=\beta\left\{\sum_{k=0}^{\infty}\sum_{r=0}^{k}\binom{k}{r}\frac{(-1)^r\lambda^r}{2^{k+r+2}\mu^{r+1}}\Phi_{3;k+r+1}*p(0,1,t)\right.$$
$$\left.+\sum_{l=0}^{\infty}\sum_{m=0}^{\infty}\sum_{k=0}^{\infty}\sum_{r=0}^{k}\binom{k}{r}\frac{(-1)^r\lambda^r}{2^{k+r+2}\mu^{r+1}}\Phi_{3;m+k+r+l+1}*p(l,1,t)\right\} \quad (16.30)$$

$$p^{*}(1,2,s)=\frac{\eta G_1^{*}(u_3,s)}{\mu(1-u_3)}-\frac{\eta G_1^{*}(u_3,s)}{2\mu\left\{1-\frac{1}{2}\left(1-\frac{\lambda u_3}{2\mu}\right)u_3\right\}(1-u_3)}-\frac{\eta p^{*}(0,1,s)}{2\mu\left\{1-\frac{1}{2}\left(1-\frac{\lambda u_3}{2\mu}\right)u_3\right\}}$$

$$=\frac{\eta}{\mu}\sum_{i=0}^{\infty}\sum_{l=0}^{\infty}u_3^{i+l}p^{*}(i,1,s)-\eta\sum_{i=0}^{\infty}\sum_{l=0}^{\infty}\sum_{k=0}^{\infty}\sum_{r=0}^{k}\binom{k}{r}\frac{(-1)^r\lambda^r}{2^{k+r+1}\mu^{r+1}}$$
$$u_3^{k+r+i+l}p^{*}(i,1,s)-\eta\sum_{k=0}^{\infty}\sum_{r=0}^{k}\binom{k}{r}\frac{(-1)^r\lambda^r}{2^{k+r+1}\mu^{r+1}}u_3^{k+r}p^{*}(0,1,s) \quad (16.31)$$

$$p\left(1,2,t\right)=\frac{\eta}{\mu}\sum_{i=0}^{\infty}\sum_{l=0}^{\infty}\Phi_{3;i+l}*p\left(0,1,t\right)$$
$$-\eta\sum_{i=0}^{\infty}\sum_{l=0}^{\infty}\sum_{k=0}^{\infty}\sum_{r=0}^{k}\binom{k}{r}\frac{(-1)^{r}\lambda^{r}}{2^{k+r+1}\mu^{r+1}}$$
$$-\Phi_{3;k+r+i+l}*p\left(i,1,t\right)$$
$$-\eta\sum_{k=0}^{\infty}\sum_{r=0}^{k}\binom{n}{r}\frac{(-1)^{r}\lambda^{r}}{2^{k+r+1}\mu^{r+1}}\Phi_{3;k+r}*p\left(0,1,t\right) \tag{16.32}$$

To find $p^{*}\left(n,2,s\right),n=2,3,\ldots$:

$$G_{2}^{*}\left(u,s\right)=\frac{\eta uu_{3}\left[(1-u_{3})G_{1}^{*}\left(u,s\right)-(1-u)G_{1}^{*}\left(u_{3},s\right)\right]}{(u-u_{3})(1-u_{3})(2\mu-\lambda uu_{3}}+\frac{2\mu(1-u)p^{*}\left(0,2,s\right)}{2\mu-\lambda uu_{3}}$$

Equating the coefficients of u^{n}, $n=2,3,\ldots$,

$$p^{*}\left(n,2,s\right)=\eta\sum_{l=0}^{\infty}\frac{\lambda^{n-1}u_{3}^{n+l}}{2^{n}\mu^{n}}p^{*}\left(0,1,s\right)+\left[\frac{\lambda^{n}u_{3}^{n}}{2^{n}\mu^{n}}-\frac{\lambda^{n-1}u_{3}^{n-1}}{2^{n-1}\mu^{n-1}}\right]p^{*}\left(0,2,s\right)$$
$$+\eta\sum_{l=0}^{\infty}\sum_{m=0}^{\infty}\sum_{k=1}^{n}\frac{\lambda^{n-k}u_{3}^{n-k+m+l+1}}{2^{n-k+1}\mu^{n-k+1}}p^{*}\left(m+k,1,s\right)-\eta\sum_{l=0}^{\infty}\sum_{m=0}^{\infty}\sum_{k=2}^{n}\frac{\lambda^{n-k}u_{3}^{n-k+m+l+2}}{2^{n-k+1}\mu^{n-k+1}}p^{*}\left(m+k,1,s\right)$$

Inverting this

$$p\left(n,2,t\right)=\eta\sum_{l=0}^{\infty}\frac{\lambda^{n-1}}{2^{n}\mu^{n}}\Phi_{3;n+l}*p\left(0,1,t\right)+\frac{\lambda^{n}}{2^{n}\mu^{n}}\Phi_{3;n}*p\left(0,2,t\right)-\frac{\lambda^{n-1}}{2^{n-1}\mu^{n-1}}$$
$$\Phi_{3;n-1}*p\left(0,2,t\right)+\eta\sum_{l=0}^{\infty}\sum_{m=0}^{\infty}\sum_{k=1}^{n}\frac{\lambda^{n-k}}{2^{n-k+1}\mu^{n-k+1}}\Phi_{3;n-k+m+l+1}*p\left(m+k,1,t\right)$$
$$-\eta\sum_{l=0}^{\infty}\sum_{m=0}^{\infty}\sum_{k=2}^{n}\frac{\lambda^{n-k}}{2^{n-k+1}\mu^{n-k+1}}\Phi_{3;n-k+m+l+2}*p\left(m+k,1,t\right) \tag{16.33}$$

Steady-State Distribution

Steady-state probabilities of the system are derived in this section. The steady-state probabilities are given by

$$\pi\,(i,j)=\lim_{t\to\infty}p\,(i,j,t)\,,i=0,1,2,\ldots,j=0,M,1,2$$

$$\pi\,(0,0)=\frac{\gamma\alpha}{(\alpha+\gamma)(2\eta+\gamma)}\left[1+\sum_{n=1}^{\infty}\frac{(-1)^n\lambda^n}{\prod_{j=1}^{n}(2\eta+\gamma+j\xi)}\right]$$

$$\pi\,(k,0)=\frac{\gamma\alpha}{(\alpha+\gamma)(2\eta+\gamma)}\sum_{r=0}^{\infty}(-1)^r\binom{k+r}{r}\frac{\lambda^{k+r}}{\prod_{j=1}^{k+r}(2\eta+\gamma+j\xi)},k=1,2,..$$

From $p^*\,(0,1,s)$ and $p^*\,(l,1,s)$

$$\pi\,(0,1)=\frac{2\eta}{\mu}\sum_{j=1}^{\infty}\sum_{m=0}^{\infty}z_1^{m+j}\pi\,(m,0)$$

$$\pi\,(l,1)=\frac{2\eta}{\mu}\left[\sum_{j=1}^{\infty}\left(\frac{\lambda}{\mu}\right)^l z_1^{l+j}\pi\,(0,0)+\sum_{m=0}^{\infty}\sum_{j=1}^{\infty}\sum_{k=0}^{l}\left(\frac{\lambda}{\mu}\right)^{l-k}\right.$$
$$\left.z_1^{l-k+j}\pi\,(m+k,0)-\sum_{m=0}^{\infty}\sum_{j=1}^{\infty}\sum_{k=1}^{l}\left(\frac{\lambda}{\mu}\right)^{l-k}z_1^{l-k+j+1}\pi\,(m+k,0)\right],l=1,2..$$

From $p^*\,(0,2,s)$, $p^*\,(1,2,s)$ and $p^*\,(n,2,s)$

$$\pi\,(0,2)=\eta\left\{\sum_{k=0}^{\infty}\sum_{r=0}^{k}\binom{k}{r}\frac{(-1)^r\lambda^r}{2^{k+r+2}\mu^{r+1}}z_3^{k+r+1}\pi\,(0,1)\right.$$
$$\left.+\sum_{l=0}^{\infty}\sum_{m=0}^{\infty}\sum_{k=0}^{\infty}\sum_{r=0}^{k}\binom{k}{r}\frac{(-1)^r\lambda^r}{2^{k+r+2}\mu^{r+1}}z_3^{m+k+r+l+1}\pi\,(l,1)\right\}$$

$$\pi\,(1,2)=\frac{\eta}{\mu}\sum_{i=0}^{\infty}\sum_{l=0}^{\infty}z_3^{i+l}\pi\,(i,1)-\eta\sum_{i=0}^{\infty}\sum_{l=0}^{\infty}\sum_{k=0}^{\infty}\sum_{r=0}^{k}\binom{k}{r}\frac{(-1)^r\lambda^r}{2^{k+r+1}\mu^{r+1}}$$
$$z_3^{k+r+i+l}\pi\,(i,1)-\eta\sum_{k=0}^{\infty}\sum_{r=0}^{k}\binom{k}{r}\frac{(-1)^r\lambda^r}{2^{k+r+1}\mu^{r+1}}z_3^{k+r}\pi\,(0,1)$$

$$\pi\ (n,2)=\eta\sum_{l=0}^{\infty}\frac{\lambda^{n-1}z_3^{n+l}}{2^n\mu^n}\pi\ (0,1)+\left[\frac{\lambda^n z_3^n}{2^n\mu^n}-\frac{\lambda^{n-1}z_3^{n-1}}{2^{n-1}\mu^{n-1}}\right]\pi\ (0,2)+\eta\sum_{l=0}^{\infty}\sum_{m=0}^{\infty}\sum_{k=1}^{n}\frac{\lambda^{n-k}z_3^{n-k+m+l+1}}{2^{n-k+1}\mu^{n-k+1}}\pi\ (m+k,1)-\eta\sum_{l=0}^{\infty}\sum_{m=0}^{\infty}\sum_{k=2}^{n}\frac{\lambda^{n-k}z_3^{n-k+m+l+2}}{2^{n-k+1}\mu^{n-k+1}}\pi\ (m+k,1)\,,n\geq 2$$

where

$$z_1=\lim_{s\to 0}u_1=\frac{(\lambda+\eta+\gamma+\mu)-\sqrt{(\lambda+\eta+\gamma+\mu)^2-4\lambda\mu}}{2\lambda}$$

$$z_3=\lim_{s\to 0}u_3=\frac{(\lambda+2\mu+\gamma)-\sqrt{(\lambda+2\mu+\gamma)^2-8\lambda\mu}}{2\lambda}$$

Steady-State Performance Measures

In this section, mean number of customers in various mode under steady-state conditions is deduced.

Mean Number of Customers in Sleep Mode

Mean number of customers in sleep mode is obtained as

$$E(D)=\sum_{k=1}^{\infty}k\pi\ (k,0)=\lim_{s\to 0}s\left[\frac{\partial G_0^*(u,s)}{\partial u}\right]_{u=1}$$

$$=\frac{\alpha\lambda\gamma}{(\alpha+\gamma)(2\eta+\gamma)(2\eta+\gamma+\xi)}$$

Mean Number of Customers in Partially Active State

Mean number of customers in partially active state is given as

$$E(PA) = \sum_{k=1}^{\infty} k\pi\ (k, 1) = \lim_{s \to 0} s \left[\frac{\partial G_1^*(u, s)}{\partial u}\right]_{u=1}$$

$$= \frac{2\alpha\eta\gamma\left[\lambda(\eta+\gamma) - (\mu-\lambda)(2\eta+\gamma+\xi)\right]}{(\alpha+\gamma)(2\eta+\gamma)(2\eta+\gamma+\xi)(\eta+\gamma)^2} + \frac{\mu}{\eta+\gamma}\pi(0, 1)$$

Mean Number of Customers in Fully Active State

Mean number of customers in fully active state is obtained as

$$E(FA) = \sum_{k=1}^{\infty} k\pi\ (k, 2) = \lim_{s \to 0} s \left[\frac{\partial G_2^*(u, s)}{\partial u}\right]_{u=1}$$

$$= \frac{2\alpha\eta^2}{(\alpha+\gamma)(2\eta+\gamma)(\eta+\gamma)}\left[\frac{\lambda(\eta+\gamma) - (\mu-\lambda)(2\eta+\gamma+\xi)}{(2\eta+\gamma+\xi)(\eta+\gamma)} + \frac{\lambda-2\mu}{\gamma}\right] + \frac{\eta\mu}{\gamma(\eta+\gamma)}\pi(0, 1) + \frac{2\mu}{\gamma}\pi(0, 2) + \frac{\mu}{\gamma}\pi(1, 2)$$

Mean Number of Customers in the System

Let X be the number of customers in the steady-state condition and mean number of customers in the system is denoted by $E(X)$

$$E(X) = E(D) + E(PA) + E(FA)$$

Numerical Illustration

The effects of the various parameters on the performance measures are studied by means of numerical illustration in this section. The values assumed are $\lambda = 0.3$, $\mu = 0.7$, $\gamma = 0.5$, $\alpha = 2$, and η, ξ, $\rho_1 (= z_1)$ are chosen so as to satisfy the stability condition. The steady-state distribution is shown in Table 16.1. Using steady-state distribution, the values of some performance measures are given in Table 16.2 (Fig. 16.2).

Table 16.1 Steady-state distribution

$\pi(M) = 0.1544$	$\pi(0, 0)$	$\pi(i, 1)$	$\pi(i, 2)$
	$\pi(0, 0) = 0.2310$	$\pi(0, 1) = 0.2051$	$\pi(0, 2) = 0.0655$
	$\pi(1, 0) = 0.0317$	$\pi(1, 1) = 0.0892$	$\pi(1, 2) = 0.0501$
	$\pi(2, 0) = 0.0083$	$\pi(2, 1) = 0.0193$	$\pi(2, 2) = 0.0354$
	$\pi(3, 0) = 0.0013$	$\pi(3, 1) = 0.0038$	$\pi(3, 2) = 0.0050$
	$\pi(4, 0) = 0.0001504$	$\pi(4, 1) = 0.0006887$	$\pi(4, 2) = 0.00008$
	$\pi(5, 0) = 0.0000143$	$\pi(5, 1) = 0.0001094$	$\pi(5, 2) = 0.000012$
	$\pi(i, 0) = 0, i \geq 6$	$\pi(i, 1) = 0, i \geq 6$	$\pi(i, 2) = 0, i \geq 6$

Table 16.2 Performance measures

Probability that the server is in maintenance mode	0.1544
Probability that the server is in sleep mode	0.3706
Probability that the server is in partially active mode	0.3182
Probability that the server is in fully active mode	0.1568

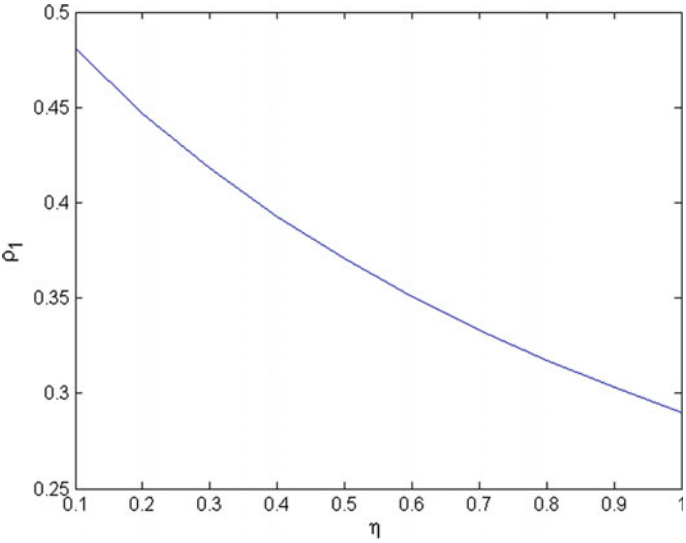

Fig. 16.2 Variation of ρ_1 versus η

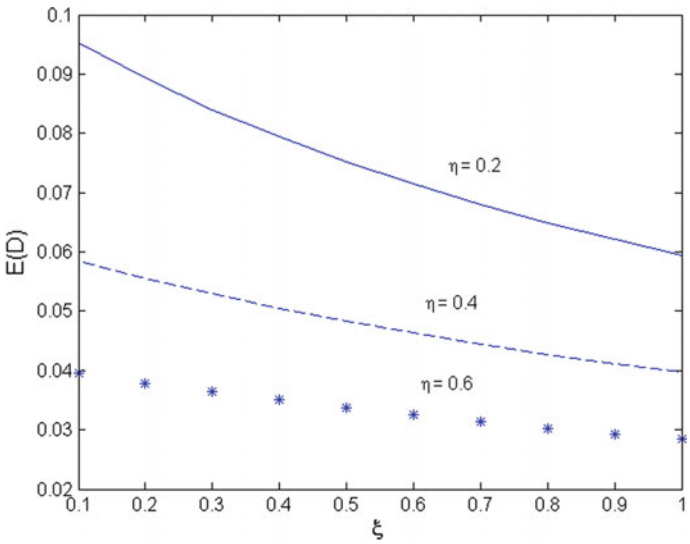

Fig. 16.3 Mean number of customers in sleep mode versus ξ

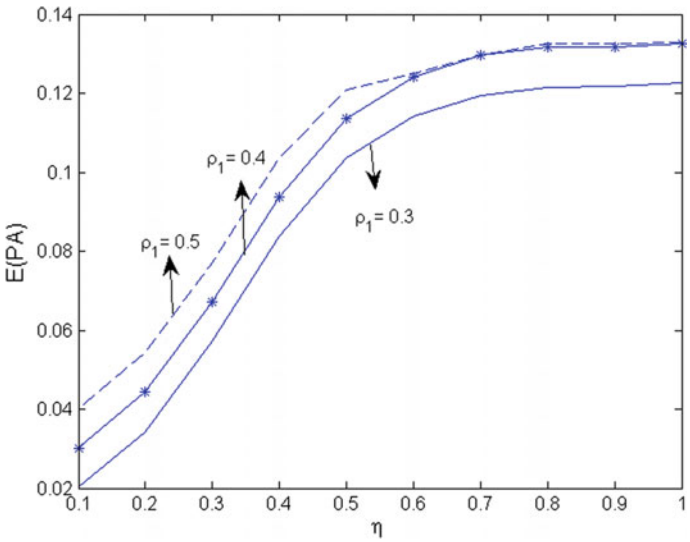

Fig. 16.4 Mean number of customers in partially active state versus η

From Fig. 16.3, it is observed that the average number of customers in the sleep mode decreases when the customer impatience rate ξ increases, and also, the average number of customers in sleep mode increases as η decreases.

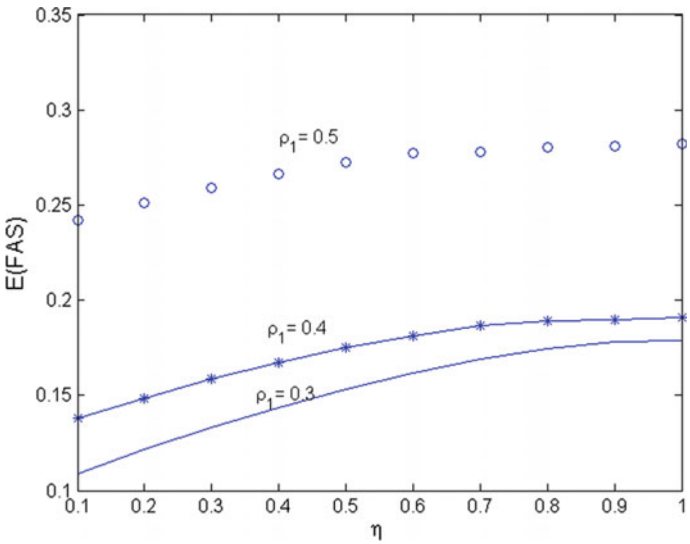

Fig. 16.5 Mean number of customers in fully active state versus η

Figure 16.4 shows that, the average number of customers in the partially active state increases when η increases for various values of ρ_1; also, the average number of customers increases when ρ_1 increases.

Figure 16.5 shows that the average number of customers in the fully active state increases when η increases; also, the average number of customers increases as ρ_1 increases.

Conclusion

In this paper, an M/M/1 queueing system with the server operating in three modes—maintenance mode, sleep mode, and active mode (busy state and working vacation state where working vacation state incorporates customer impatience) subject to catastrophes is considered. Explicit expressions are obtained for the transient probabilities of the system in the three different modes. The steady-state probabilities and some steady-state performance measures are obtained. Finally, graphical illustrations are presented, and the effects of various parameters on the system performance measures are discussed.

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Chapter 17
Is Bio-Supply Chain a Feasibility in India? An Uncertainty-Based Study

Kapil Gumte and Kishalay Mitra

Abstract Aligning with environment-friendly move by Indian government to blend gasoline and diesel with 20% bio-products, a countrywide multi-product supply chain network has been designed. The sustainable, techno-economic, and environment-friendly model has been formulated considering conflicting objectives involving net present value and greenhouse gas emissions while determining the network design and flow decisions using a multi-period mixed integer linear programming approach. To face the special challenge of continuous feed supply across the year in India, model allows handling multiple feed and their respective technologies considering spatial–temporal distribution of biomass. Imports are blended with the indigenous products to maintain product quantity and production shortfall. The model generates revenue by selling the bio-products and greenhouse gas carbon credits. The uncertainties involved in product demand, externally imported price and biomass raw material feed, are handled via robust optimization methodology considering realizations of possible uncertain scenarios. Based on real demand data of products during 2018–2026, the deterministic and stochastic nature of results have been studied thoroughly. The sensitivity analysis implicates atleast 45% of biomass raw material supply, to run the bioenergy sector successfully under worst possible scenario.

Keywords Bio-supply chain network design · Bio-fuel blending · Robust optimization · Mixed integer linear programming · Uncertainty handling

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Introduction

Due to depleting fossil fuel reserve, environmental issues, and international political tensions [1], there has been imbalance in demand and supply for the conventional fuels, which has led to uncertainty in India's energy domain. Subsequently, there is solid need to move toward elective climate amicable, financially practical, and environmentally friendly power sources like hydro, solar, geothermal, and bioenergy [2]. Amidst the renewable capacity, the bioenergy related to biomass sector is contributing around 13% as shown in Fig. 17.1 and this trend is about to rise in future [3].

Among different environmental-friendly power sources, biomass has acquired prevalence in enterprises as variety of bio-items can be produced from biomass prompting the approach to have abundance from squander. Subsequently, environmental-friendly power delivered from biomass has good future perspective from point of view of sustainability. India, being an agrarian country with broadened scope of harvests, has an exceptional possibility using the gigantic measure of biomass it produces consistently, which can be used effectively to meet the order of 20% biofuel mixing. To extract benefit out this study, a countrywide supply chain network (SCN) is required covering all the stages from beginning of raw material to final biofuel product.

In the literature survey, researchers have developed SCN model [4, 5]. These models are deterministic in nature which lacks the dynamicity of the real-time conditions involving uncertainty such as future demand, price, and feed supply for the sake of ease in analysis and due lack of data availability. However, a strategic supply chain model that can handle optimization of its network under uncertainty would be the most appropriate to deal with such situations. Hence, while handling uncertainty in bio-supply chain network design (SCND), probabilistic two-stage stochastic programming (TSSP) has been used most often [6]. Assuming the data and nature of uncertain information can be made available, TSSP solves the statistical average of performance metric, e.g., cost, over several uncertain scenarios, which often tags the results obtained by it as conservative [7]. Another drawback of TSSP is that

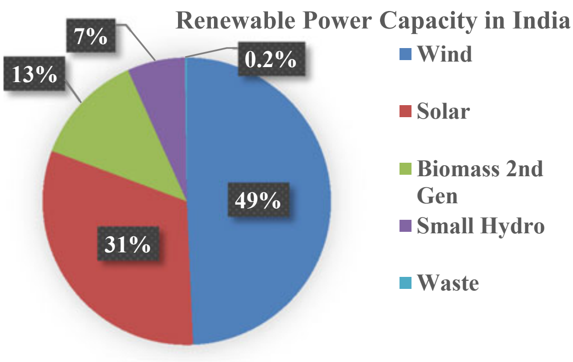


Fig. 17.1 Distribution of unconventional energy sources in India

the issue turns out to be computationally difficult to give sensibly precise solutions inside a given time period as the stochastic parameters, and its scenarios are increased in numbers. Indeed, even disintegration of the given problem into two phases isn't straight forward [7, 8].

Apart from TSSP, there are several other uncertainty handling methods such as multi-stage stochastic programming (MSSP) [9], chance constrained programming (CCP) [10], and fuzzy [11], which can be utilized in planning under different sources of uncertainty [7]. Methods like fuzzy optimization is utilized when the distribution information of dispersion data is missing and whatever data is present, is in ambiguous and possibilistic way [12]. For example, Babazadeh et al. [13] used possibilistic fuzzy programming to mitigate the uncertainty in biodiesel SCND in Iran to deal with variable cultivation cost [13], whereas other methods work under the assumption of availability of the underlying distribution information of the uncertain parameters, specifically in CCP. Also, in CCP, the hard constraints can be relaxed up to certain extent. Here, the methodology of robust optimization (RO) is an exception, which can work under both the conditions [14]. RO can work without knowing the underlying distribution and can handle the uncertainty by providing both best and worst-case solution providing the user to choose the most extreme solution and all the solution range lying in between worst and best cases [15].

From the literature study, many points can be noted down as research gap or knowledge gap. First, the most of the supply chain (SC) model developed covers the economic part which misses out the environmental greenhouse emission (GHGe) calculation part and its carbon credits. Second, the economic part either maximizes the profit or reduces the cost, but they miss to incorporate the money's time value and depreciation. Third, the biofuel demand is satisfied with indigenous production, and in case of stock out, penalty is applied. The method for satisfying the demand and avoid penalty simultaneously is missing in many bio-SC models. Fourth and most vital, math model related to bio-SCND specifically for Indian subcontinent is missing involving uncertainty or stochasticity handling approach. Hence, to overcome all the missing aspects of previous work, current work provides following *contribution and novelties*:

(i) The proposed mixed integer linear programming (MILP) supply chain (SC) model not only considered the entire four-layered supply chain, but also included the greenhouse gas emissions (GHGe) calculations of biomass cultivation before the supplier layers and GHGe factors of SC life cycle assessment (LCA) including manufacturing, storage, and transport layers (well to tank approach). This enables the carbon emissions saved due to biofuel with respect to the fossil fuels to be used as economic resource to enhance the profit.

(ii) The proposed techno-economic and environmental-friendly model provides dynamic SC solution in response to the changing futuristic demand of fuel grade biodiesel and bioethanol over 9 years (2018 to 2026).

(iii) Idea of blending externally imported industry grade fuel has been implemented at distributor storage unit to deal with the insufficient indigenous production because of feed shortage as well as maintenance of consistent product quality.

(iv) The proposed model assumes uncertainty in the demand and proposes solution via eight (multi-feed) biomass feed and its respective technologies based on feed composition to have consistent manufacturing across the year.
(v) The RO not only handles the stochasticity but gives the solution of best and worst case along with range of solutions lying in between them for instances of uncertain parameter providing feasible SC.
(vi) The profit is estimated via net present value (NPV) approach considering the money's time value.

The manuscript is sectioned as mentioned in this passage. Section "Materials and Methods" mentions the problem statement and model formulation followed by the implementation of RO technique. Section "Computation Analysis" gives insight of model's computational aspect. Section "Results and Discussion" expounds on results with detailed discussion. Section "Conclusion and Future work" concludes with major findings.

Materials and Methods

Problem Statement and Formulation

SC consisting of 4 different echelons, namely supplier layer, manufacturer layer, distributor layer (with inventory), and retailer layer is capable of multi-sites/nodes at each SC layer which is presenting its potential geographic location across India (see Fig. 17.2). These locations as per their climate and soil provide eight types of inedible biomass raw material to maintain consistent yearly supply to generate biodiesel and bioethanol. The Indian terrain is split into 12 zones with each zone involving (see Fig. 17.3) states and union territories (UT).

In each zone, three sites are assigned for manufacturer, supplier, and inventory distributor in conjugation with one demand retailer as potential candidate site. Hence, overall, it leads to thirty-six suppliers g_1–g_{36}, thirty-six manufacturers g_{37}–g_{72}, thirty-six distributors g_{73}–g_{108}, and twelve retailers g_{109}–g_{120}. Additionally, 2 external import sites/nodes g_{121} and g_{122} are considered.

Model assumption for creating equations:

(a) The binary (0 and 1) decision variables provide the site location and its connectivity.
(b) Mass flows in series, i.e., sequence of suppler layer, manufacturer layer, distributor layer, retailer layer without intermittent and backflow of feed and product.
(c) Transport can happen among and across the zones to maintain consistent material supply of feed and product.
(d) Material flow is in opposite direction of monitory flow.
(e) Pre-treatment of raw biomass is done at the site of manufacturing.

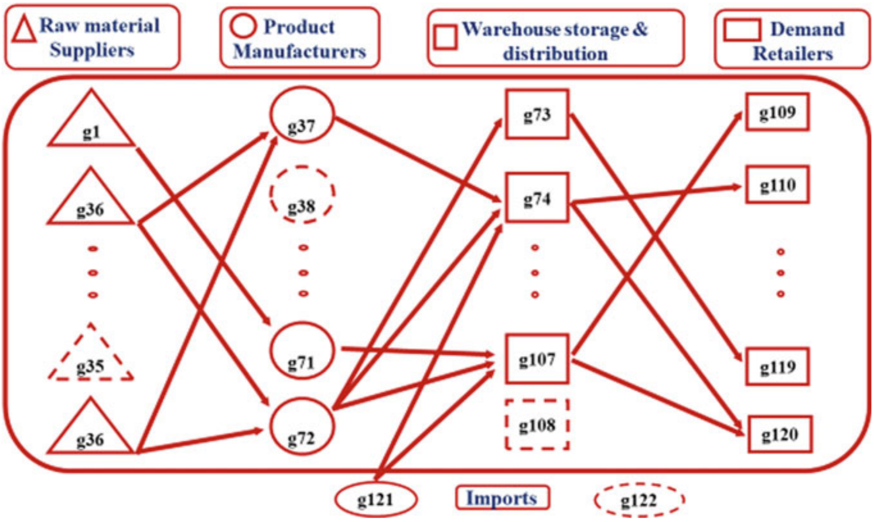


Fig. 17.2 Bio-(SCND) schematic view. The mass flow is from left to right with binary (1) showing site selection with solid outline

(f) Based on types of biomass feed (four for biodiesel, four for bioethanol), the product conversion technology is implemented.
(g) Imports are used to satisfy the product demand in case of indigenous stock outs. Imports are used to not only maintain quantity but also the quality (research octane number RON) as per demand market.
(h) All SC sites are connected via road and rail.

Model Equations

Please go through the nomenclature section in the last before reference section to understanding the equations.

Max NPV

$$\mathrm{NPV} = \sum_{t,p} \frac{1}{(\mathrm{alpha}+1)^{nn-1}}((\mathrm{EeR}_t - \mathrm{OPX}_t - \mathrm{DEP}) \times (1-\Phi_p) + \mathrm{DEP}) - \sum_t \frac{1}{(\mathrm{alpha}+1)^{nn-1}} \times \mathrm{CPX}_t \tag{17.1}$$

$$\mathrm{EeR}_t = \sum_{p,r} (\mathrm{dem}_{\mathrm{p,r},t} \times \mathrm{sp}_{\mathrm{p},t}) + \mathrm{GHR}_t \tag{17.2}$$

$$\mathrm{OPX}_t = \mathrm{TRC}_t + \mathrm{IVC}_t + \mathrm{PRC}_t + \mathrm{IMC}_t \tag{17.3}$$

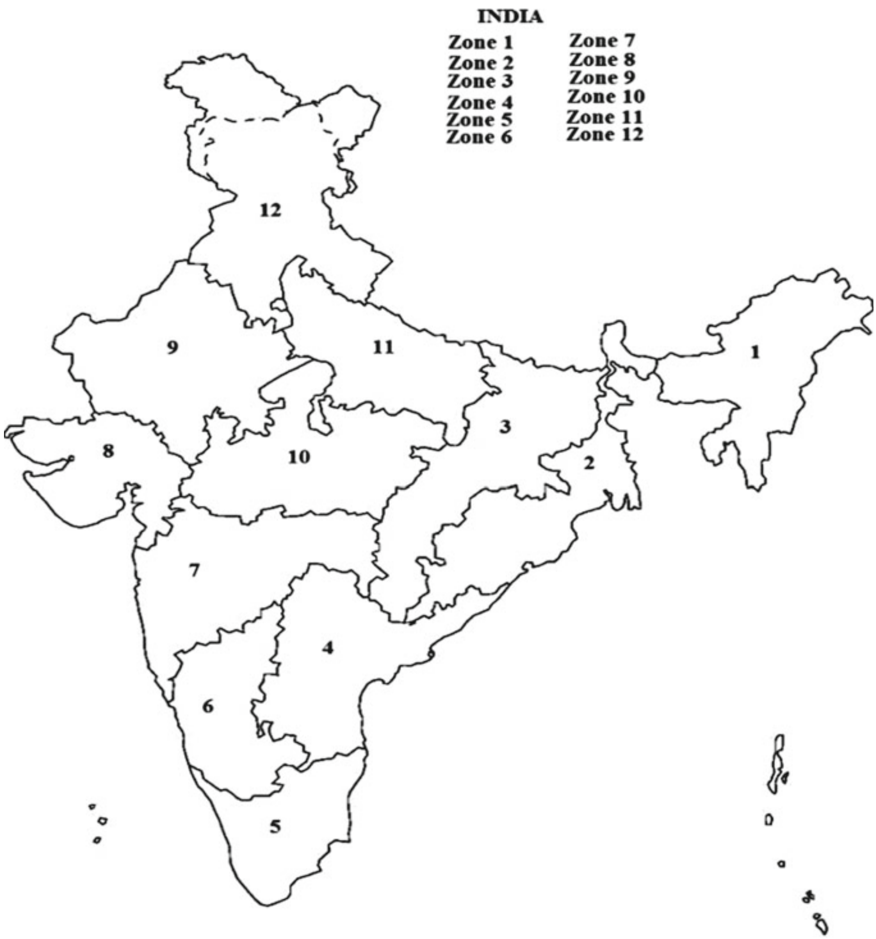


Fig. 17.3 12 zones split of India's terrain

$$\mathrm{CPX}_t = \mathrm{IRC}_t \tag{17.4}$$

$$\mathrm{DEP} = \sum_t \frac{i}{((1+i)^{nn}-1)}(\mathrm{IRC}_t - 0.2 \times \mathrm{IRC}_t) \tag{17.5}$$

Equation 17.1 maximizes the profit NPV involving depreciation, earning, and expenditure where the earning is done from retailing selling and carbon credits (Eq. 17.2). The expenditure involves operating cost (Eq. 17.3) which is distributed across transport, inventory, production, and imports. The infrastructure cost comes under capital cost (Eq. 17.4). The depreciation calculation is done using straight-line technique.

$$\mathrm{FED}_{\mathrm{f,s},t} = \sum_{m} \mathrm{QSM}_{\mathrm{f,s,m,l},t}, \forall f, m \in g, \forall t, \forall l, \forall s, \tag{17.6}$$

$$\mathrm{fed}^{\mathrm{Min}}_{\mathrm{f,s,t}} \times \mathrm{YS}_{\mathrm{s},t} \leq \mathrm{FED}_{\mathrm{f,s},t} \leq \mathrm{fed}^{\mathrm{Max}}_{\mathrm{f,s},t} \times \mathrm{YS}_{\mathrm{s,t}}, \forall f, \forall \mathrm{su}, \forall t \tag{17.7}$$

$$\mathrm{qsm}^{\mathrm{Min}}_{\mathrm{f,s,m,l},t} \times \mathrm{YS}_{\mathrm{s},t} \leq \mathrm{QSM}_{\mathrm{f,s,m,l},t} \leq \mathrm{qsm}^{\mathrm{Max}}_{\mathrm{f,s,m,l},t} \times \mathrm{YM}_{t}, \forall s f_{\mathrm{s,f}}, \forall m, \forall l, \forall t \tag{17.8}$$

Equations 17.6–17.21 shows the material flow, starting with feed (Eq. 17.6) distribution from supplier to manufacturer with limitation on feed capacity (Eq. 17.7) and feed transportation (Eq. 17.8) across the echelon. The feed after reaching manufacturing site is converted into product (Eqs. 17.9, 17.10). There do exists production capacity limitation (Eq. 17.11) to move products across next distribution layer (Eq. 17.12) with transport limitation (Eq. 17.13). At distribution units, imports are moved in (Eq. 17.14) with import capacity (Eq. 17.15) and transport limitation (Eq. 17.16).

$$\sum_{s} \mathrm{QSM}_{\mathrm{f,s,l,m},t} \times \mathrm{COV}_{\mathrm{f,tech,p}} = \mathrm{PMT}_{\mathrm{p,m,tech},t}, \\ \forall \mathrm{tech}, \forall l, \forall m, \forall p, \forall s f_{\mathrm{s,f}}, s \in g, \forall t, \tag{17.9}$$

$$\sum_{\mathrm{tech}} \mathrm{PMT}_{\mathrm{p,m,tech},t} = \mathrm{PM}_{\mathrm{m,p},t}, \forall p, \forall t, \mathrm{tech} \in \mathrm{Tec}, \forall \mathrm{m} \tag{17.10}$$

$$\mathrm{pm}^{\mathrm{Min}}_{\mathrm{m,p},t} \times \mathrm{YM}_{\mathrm{m},t} \leq \mathrm{PM}_{\mathrm{m,p},t} \leq \mathrm{pm}^{\mathrm{Max}}_{\mathrm{m,p},t} \times \mathrm{YM}_{\mathrm{m},t}, \forall p, \forall t, \forall m, \tag{17.11}$$

$$\mathrm{PM}_{\mathrm{m,p},t} = \sum_{\mathrm{d}} \mathrm{QMD}_{\mathrm{p,d,m,l},t}, \forall l, \forall p, d \in g, \forall m, \forall t \tag{17.12}$$

$$\mathrm{qmd}^{\mathrm{Min}}_{\mathrm{p,d,m,l},t} \times \mathrm{YM}_{\mathrm{m},t} \leq \mathrm{QMD}_{\mathrm{p,d,m,l},t} \leq \mathrm{qmd}^{\mathrm{Max}}_{\mathrm{p,d,m,l},t} \\ \times \mathrm{YD}_{\mathrm{d},t}, \forall d, \forall p, \forall l, \forall m, \forall t \tag{17.13}$$

$$\mathrm{IMP}_{\mathrm{i,p},t} = \sum_{d} \mathrm{QID}_{\mathrm{p,i,l,d},t}, \forall p, \forall i, d \in g, \forall l, \forall t \tag{17.14}$$

$$\mathrm{imp}^{\mathrm{Min}}_{\mathrm{i,p},t} \times \mathrm{YI}_{\mathrm{i},t} \leq \mathrm{IMP}_{\mathrm{i,p},t} \leq \mathrm{imp}^{\mathrm{Max}}_{\mathrm{i,p},t} \times \mathrm{YI}_{\mathrm{i,t}}, \forall p, \forall i, \forall t \tag{17.15}$$

$$\mathrm{qid}^{\mathrm{Min}}_{\mathrm{p,i,l,d},t} \times \mathrm{YI}_{\mathrm{i},t} \leq \mathrm{QID}_{\mathrm{p,i,l,d},t} \leq \mathrm{qid}^{\mathrm{Max}}_{\mathrm{p,i,l,d},t} \\ \times \mathrm{YD}_{\mathrm{d},t}, \forall p, \forall i, \forall dd, \forall l, \forall t \tag{17.16}$$

The distributor sites also act as inventory (Eq. 17.17) with input from previous time period, manufacturer, import, and output to next demand retailer happens. There do exists inventory capacity limitation (Eq. 17.18) and transport limitation from inventory to retailer (Eq. 17.19). It is important to mention that there happens linear mixing of imported biofuels with indigenous biofuels (Eq. 17.20). Comparing to the demand of the biofuels at the retailer, the quantity of products supplied to the retailers from distributors are always in larger amount (Eq. 17.21).

$$\mathrm{INV}_{\mathrm{p,d},t} = \mathrm{INV}_{\mathrm{p,d},t-1} + \sum_{m,l} \mathrm{QMD}_{\mathrm{p,d,m,l},t} + \sum_{i,l} \mathrm{QID}_{\mathrm{p,i,l,d},t} - \sum_{\mathrm{r,l}} \mathrm{QDR}_{\mathrm{p,d,r,l},t}, \quad \forall p, r \in g, \forall l, i \in g, \forall t, m \in g, \forall d, \forall t, \tag{17.17}$$

$$\mathrm{inv}^{\mathrm{Min}}_{\mathrm{p,d},t} \times \mathrm{YD}_{\mathrm{d},t} \le \mathrm{INV}_{\mathrm{p,d},t} \le \mathrm{inv}^{\mathrm{Max}}_{\mathrm{p,d},t} \times \mathrm{YD}_{\mathrm{d},t}, \forall p, \forall t, \forall l, \forall d, \tag{17.18}$$

$$\mathrm{qdr}^{\mathrm{Min}}_{\mathrm{p,d,l,r}t} \times \mathrm{YD}_{\mathrm{d},t} \le \mathrm{QDR}_{\mathrm{p,d,l,r},t} \le \mathrm{qdr}^{\mathrm{Max}}_{\mathrm{p,d,l,r},t} \times \mathrm{YR}_{\mathrm{r},t}, \forall p, \forall l, \forall d, \forall r, \forall t \tag{17.19}$$

$$\sum_{i} (\mathrm{IMP}_{\mathrm{p,i},t} \times \mathrm{ipr_p}) + \sum_{d} (\mathrm{INV}_{\mathrm{p,d},t} \times \mathrm{idr_p}) = \mathrm{dr_p} \quad \times (\sum_{\mathrm{i}} \mathrm{IMP}_{\mathrm{p,i},t} + \sum_{\mathrm{d}} \mathrm{INV}_{\mathrm{p,d},t}) \quad \forall t, \forall p, d \in g, i \in g \tag{17.20}$$

$$\sum_{d} \mathrm{QDR}_{\mathrm{p,d,l,r},t} \ge \mathrm{dem}_{\mathrm{p,r},t}, \forall r, \forall l, \forall p, \forall t, d \in g \tag{17.21}$$

Equations 17.22–17.25 represents the pollution calculation in terms of GHGe and its revenue generated as carbon credit. Equation 17.22 calculates the GHGe across the SC life cycle. Equation 17.23 calculates the fossil fuel GHGe to compare with biofuel GHGe to obtain the overall GHGe saving (Eq. 17.24), which is converted into GHGe revenue (Eq. 17.25) also called as carbon credit.

$$\begin{aligned}\mathrm{GHGe}_t = & \sum_{s,f,t} (\mathrm{fbc}_{\mathrm{f},t} \times \mathrm{FED}_{\mathrm{f,s},t}) + \sum_{p,m,t} (\mathrm{fp}_{\mathrm{p},t} \times \mathrm{PM}_{\mathrm{m,p},t}) + \\ & \sum_{p,d,t} (\mathrm{fd}_{\mathrm{p},t} \times \mathrm{INV}_{\mathrm{p,d},t}) + \sum_{s,f,m,l,t} \left(\mathrm{fsm}_{\mathrm{f,l}} \times \mathrm{dsm}_{\mathrm{s,m}} \times \mathrm{QSM}_{\mathrm{f,s,m,l},t} \right) \\ & + \sum_{m,p,l,d,t} \left(\mathrm{fmd}_{\mathrm{p,l}} \times \mathrm{dmd}_{\mathrm{m,d}} \times \mathrm{QMD}_{\mathrm{p,d,m,l},t} \right) \\ & + \sum_{p,i,l,d,t} \left(\mathrm{fid}_{\mathrm{p,l}} \times \mathrm{did}_{i,\mathrm{d}} \times \mathrm{QID}_{\mathrm{p},i,\mathrm{l,d},t} \right) \\ & + \sum_{p,r,d,l,t} \left(\mathrm{fdr}_{\mathrm{p,l}} \times \mathrm{ddr}_{\mathrm{d,r}} \times \mathrm{QDR}_{\mathrm{p,d,l,r},t} \right)\end{aligned} \tag{17.22}$$

$$\mathrm{GHF_t} = \sum_{r,p} \mathrm{dem}_{\mathrm{p,r},t} \times \mathrm{gca_p} \times \mathrm{ged_p} \tag{17.23}$$

$$\mathrm{GHS}_t = \mathrm{GHF}_t - \mathrm{GHGe}_t \tag{17.24}$$

$$\mathrm{GHR}_t = \mathrm{GHS}_t \times \mathrm{cv}_t \tag{17.25}$$

The non-negative physical constraint of feed supply (Eq. 17.26), product manufactured (Eq. 17.27), and product inventory (Eq. 17.28) is kept minimum zero and not below it.

$$\mathrm{FED}_{\mathrm{f,s},t} \geq, s \in g, f \in F, \forall t, \tag{17.26}$$

$$\mathrm{PM}_{\mathrm{m,p},t} \geq 0, m \in g, p \in P, \forall t, \tag{17.27}$$

$$\mathrm{INV}_{\mathrm{p,d},t} \geq 0, d \in g, p \in P, \forall t, \tag{17.28}$$

Equations 17.29–17.34 gives the cost calculation aspect. The total cost (Eq. 17.29) is sum of operating and capital expenditure. The operating cost involves transportation (Eq. 17.30), inventory cost (Eq. 17.31), production cost (Eq. 17.32), import cost (Eq. 17.33), and capital cost involves infrastructure cost (Eq. 17.34).

$$\mathrm{TC}_t = \mathrm{OPX}_t + \mathrm{CPX}_t \tag{17.29}$$

$$\begin{aligned}
\mathrm{TRC}_t = & \sum_{s,f,gc,l} (\mathrm{usm}_{\mathrm{l,f},t} * \mathrm{dsm}_{\mathrm{s,m}} \times \mathrm{QSM}_{\mathrm{f,s,m,l},t}) \\
& + \sum_{m,p,d,l} (\mathrm{umd}_{\mathrm{p,l},t} \times \mathrm{dmd}_{\mathrm{m,d}} \times \mathrm{QMD}_{\mathrm{p,d,m,l},t}) \\
& + \sum_{p,l,d,r} (\mathrm{udr}_{\mathrm{p,l},t} \times \mathrm{ddr}_{\mathrm{d,r}} \times \mathrm{QDR}_{\mathrm{p,d,l,r},t}) \\
& + \sum_{p,i,d,l} (\mathrm{uid}_{\mathrm{p,l},t} \times \mathrm{did}_{\mathrm{i,d}} \times \mathrm{QID}_{\mathrm{p,i,l,d},t}), \\
& p \in P, r \in g, s \in g, \forall t, f \in F, m \in g, l \in L, d \in g, i \in g,
\end{aligned} \tag{17.30}$$

$$\mathrm{IVC}_t = \sum_{p,d} \mathrm{INV}_{\mathrm{p,d},t} \times \mathrm{ud}_{\mathrm{p,d},t}, p \in P, \forall t, d \in g \tag{17.31}$$

$$\mathrm{PRC}_t = \sum_{p,m} \mathrm{cp}_{\mathrm{p},t} \times \mathrm{PM}_{\mathrm{p,m},t}, p \in P, \forall t, m \in g \tag{17.32}$$

$$\mathrm{IMC}_t = \sum_{p,i} \mathrm{ui}_{\mathrm{p},t} \times \mathrm{IMP}_{\mathrm{p,i},t}, p \in P, \forall t, i \in g \tag{17.33}$$

$$\mathrm{IRC}_t = \sum_s (\mathrm{YS}_{\mathrm{s},t} \times \mathrm{cs}_{\mathrm{s},t}) + \sum_m (\mathrm{YM}_{\mathrm{m},t} \times \mathrm{cm}_{\mathrm{m},t}) + \sum_d (\mathrm{YD}_{\mathrm{d},t} \times \mathrm{cd}_{\mathrm{d},t}) + \sum_r (\mathrm{YR}_{\mathrm{r},t} \times \mathrm{cr}_{\mathrm{r},t}) + \sum_i (\mathrm{YI}_{\mathrm{i},t} \times \mathrm{ci}_{\mathrm{i},t}),$$
$$\forall t, d \in g, r \in g, m \in g, s \in g, i \in g \quad (17.34)$$

The model data have been collected are from several authentic government sites and are not shown due to space constraints.

Robust Optimization Implementation

Consider the presentation of optimization in uncertain manner as per Eq. 17.35. Here, the $\boldsymbol{u}$ shows uncertain parameter vector; $\boldsymbol{x}$ shows the decision variable. Both $\boldsymbol{u}$ and $\boldsymbol{x}$ have bounds. g shows the constraints with f as objective function. Equation 17.35 maximizes the objective f. The optimization solver doesn't solve straightforward stochastic equation. First, these stochastic equations are required to be converted into deterministic equivalent as done by Eq. 17.36 which is known as robust counterpart (RC). The RC calculates the infimum (minimum) among all possible sets of uncertain parameter space and then does the maximization. Whether the stochastic parameter follow any statistical distribution or not, the RO can still be applied as the samples are taken from uncertainty space. These uncertainty space can provide with many data points limited by bounds. The reformulation from Eqs. 17.35 to 17.36 is termed as RO with worst-case solution, and Eq. 17.36 can be solved via optimization solvers. Opposite to worst-case supremum (maximum) among all possible sets of uncertain instances followed by maximization gives best-case RO.

$$\max_{\boldsymbol{x}} \{[f(\boldsymbol{x}, \boldsymbol{u})] : g(\boldsymbol{x}, \boldsymbol{u}) \geq 0\} \quad (17.35)$$

$$\max_{x} \left\{ \inf_{u} [f(\boldsymbol{x}, \boldsymbol{u}) : g(\boldsymbol{x}, \boldsymbol{u}) \geq 0] \right\} \quad (17.36)$$

Figure 17.4 explains the worst-case implementation of RO. For the initial value of $\boldsymbol{x}$ and various instances of stochastic parameter $\boldsymbol{u}$**,** objective function and constraints are calculated. Next, constraint satisfaction is checked. If solution obtained is infeasible, then another value of $\boldsymbol{x}$ is chosen by optimizer within parameter bounds. If the solution obtained is feasible, the minimum calculated objective function is sent to optimization solver for feasible solution.

Current paper shows the worst-case formulation of RO for profit NPV maximization of NPV. The uncertainty is incorporated into product demand, biomass feed supply, and externally imported products (Eqs. 17.2, 17.7, 17.21, 17.23, and 17.34). The effects of these uncertain parameters on the considered model have been studied in the further section.

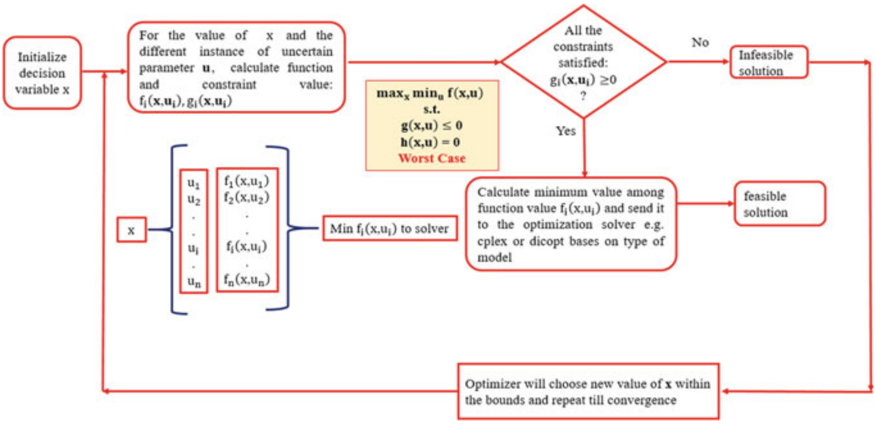


Fig. 17.4 Robust optimization schematic diagram for Max_X Min_u $f(x, u)$ (worst-case: Eq. 17.36)

Computation Analysis

The MILP model described in Section "Model Equations" is solved deterministically and also with uncertainty of standard deviation of $\pm$ 5% where uncertainty has been assumed to vary within $\pm$ 3σ containing varying numbers of samples i (created randomly, i.e., 100, 500, and 1000) to see the impact of sampling on results. The model is solved in GAMS® 24.1.3 integrated development environment.

These sampling is preformed to evaluate the optimization measures, e.g., max, min, etc., with respect to the uncertain variables in the objective functions (inner-loop optimization of Max_x Min_u $f(x, u)$ as given in Fig. 17.4). RO results, when shown for the worst-case Max_x Min_u NPV and the best-case Max_x Max_u NPV, along with second objective of GHGe create Pareto front across time horizon. Here, we have two-product demand, two-product import cost, and eight biofeed raw supply involving stochasticity, leading to $i \times (2 + 2 + 8)$ number of scenarios for time period t (see Eqs. 17.37–17.39 with subscript). The model involves 4,491,309 equations, 611,859 continuous variables, and 255,564 discrete variables. CPU time of 12.69 s and 2×10^9 number of iterations were required for the solution to converge.

$$\text{dem}_{\text{p,r},t,\text{i}} = \text{dem}_{\text{p,r},t} + (\text{stddev} \times \text{randnormal}_\text{i}) \tag{17.37}$$

$$\text{ui}_{\text{p},t,\text{i}} = \text{ui}_{\text{p},t} + (\text{stddev} \times \text{randnormal}_\text{i}) \tag{17.38}$$

$$\text{fed}^{\text{Max}}_{\text{f,s},t,\text{i}} = \text{fed}^{\text{Max}}_{\text{f,s},t} + (\text{stddev} \times \text{randnormal}_\text{i}) \tag{17.39}$$

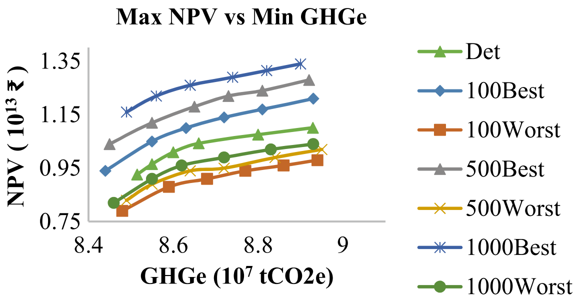


Fig. 17.5 Pareto fronts of profit NPV and GHGe with best and the worst-case instances for stochasticity in import cost, feed supply, and demand

Results and Discussion

Product Demand, Feed and Import Cost Stochasticity with Respect to Deterministic Case

The deterministic Pareto is kept as reference to observe that the best-case instances (100, 500, 1000 data points) are above it and worst case (100, 500, 1000 data points) lies below it. Also, among the 100, 500, and 1000 data points, one can observe that more data points give better NPV (1000 > 500 > 100) as with more sample realizations; the solver gets better option of choosing data point to maximize NPV. Between best, nominal, and worst solutions, the worst solution represents real-time scenario where project can work with minimum profit NPV under critical situations, which is missed by nominal and best-case scenarios. One more point that can be made out of Fig. 17.5 is that, the region lying between best and worst solution provides the project operating all possible zones.

Site Location and Connectivity

It has been found that the number of site location required for all time period, i.e., bioethanol sites 686 and biodiesel sites 922 reduced in number with respect to the deterministic case, i.e., bioethanol sites 714 and biodiesel sties 972 (see Figs. 17.6 and 17.7) and proportionally reduces the infrastructure cost. This happens due the uncertainty handling effect of RO.

Cost Analysis

For cost analysis, one can refer to Figs. 17.8 and 17.9. It can be observed that, the cost for uncertainty case shows higher value for import cost (79.04%) with respect to deterministic case (50.78%) and rest all other, i.e., infrastructure cost, transport

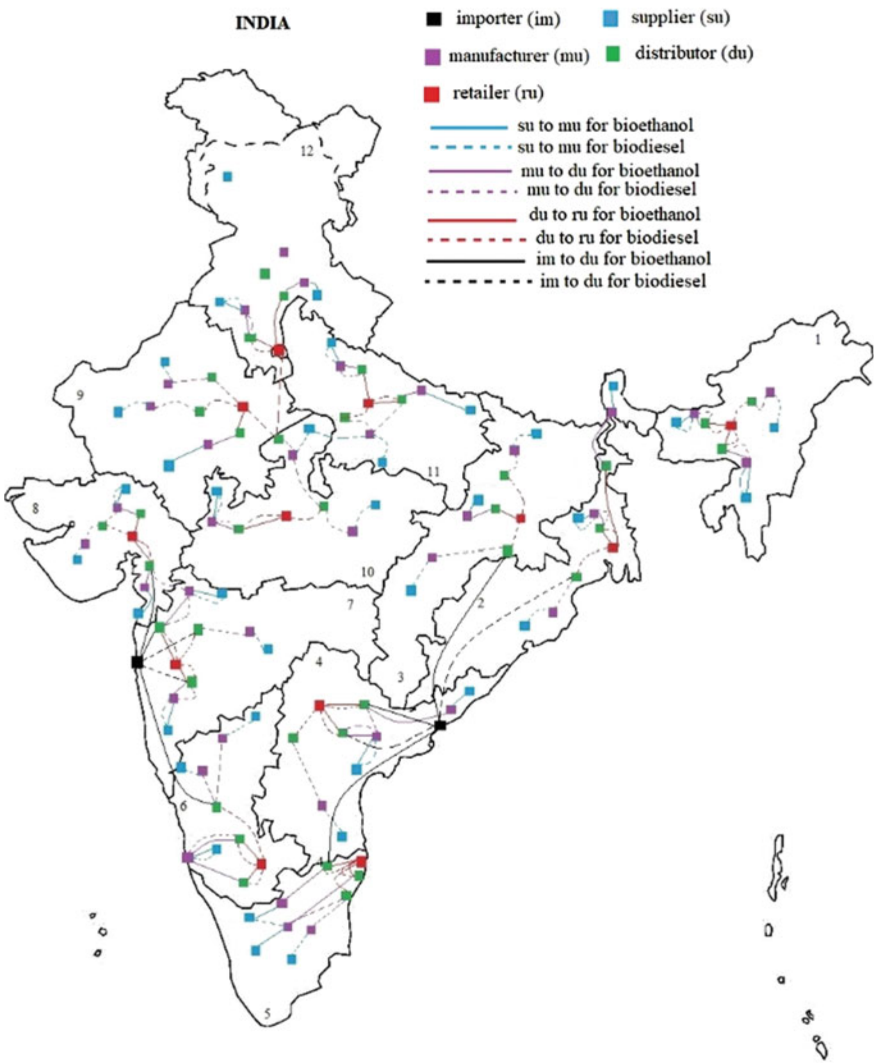


Fig. 17.6 Site location and connectivity among the nodes for deterministic case (@ t_1). Connecting pathways shows chosen sites

cost, import cost, production cost, and inventory costs are lower. Due to RO effect on uncertainty space, less number of sites are chosen across the SC layers providing less infrastructure cost. Less infrastructure cost also indicates less production site and less production cost causing increase in external imports to meet market demand.

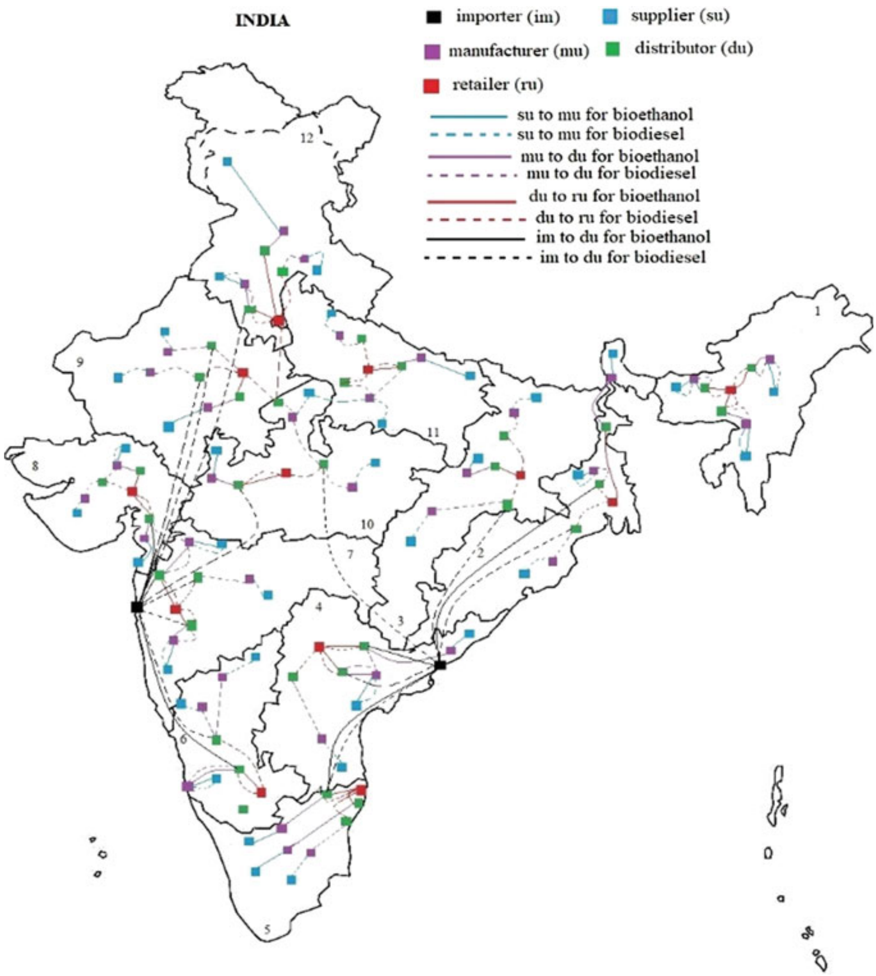


Fig. 17.7 Site location and connectivity among the nodes (lean network) for uncertainty case (@ t_1). Connecting pathways shows chosen sites

Effect of Variability in Data Points

Results with 5% and 10% standard deviations are compared for the best and the worst case using 1000 uncertain sampling points for uncertain case, and it was observed that (Fig. 17.10) the 10% standard deviation gives Pareto front with higher NPV compared to the 5% standard deviation case (worst case). Similar kind of results is obtained for the best case also. As the variability increases, the option to choose from the uncertain data range increases leading to better combination of decision variable and uncertain parameters and thereby better objective functions.

Fig. 17.8 Cost distribution for deterministic case across time horizon

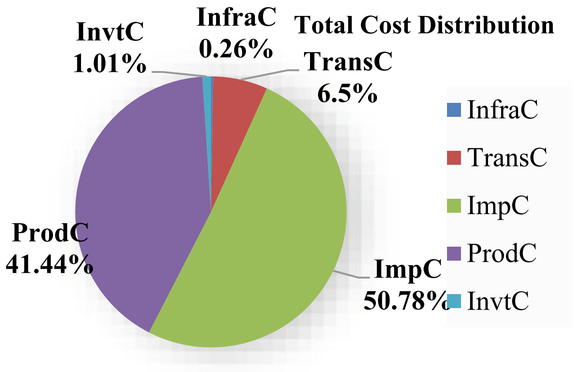
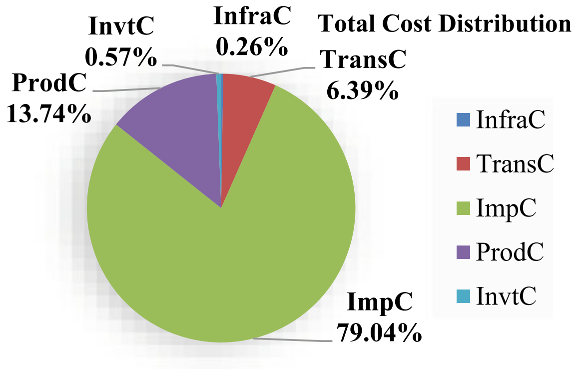


Fig. 17.9 Cost distribution for uncertainty case across time horizon



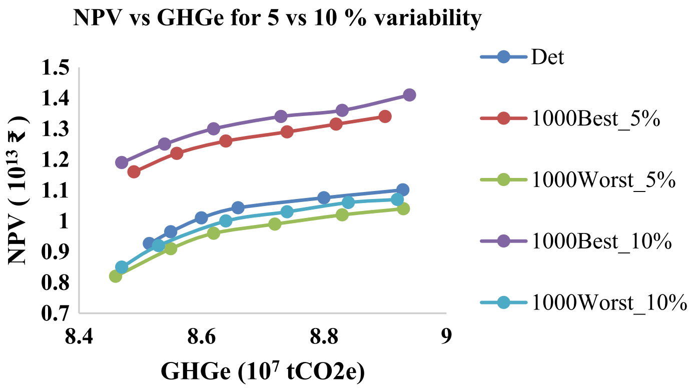


Fig. 17.10 Effect of variability in uncertain data on NPV expressed through different standard deviation values of 5 and 10% based on 1000 uncertain data points

Table 17.1 Feed supply effect on NPV

Deterministic case		Stochastic case	
Feed %	NPV (million ₹)	Feed%	NPV (million ₹)
100	1.10×10^7	100	1.04×10^7
80	8.26×10^6	80	7.98×10^6
60	5.50×10^6	60	5.15×10^6
50	2.12×10^6	50	3.05×10^6
49	$\mathbf{-1.00 \times 10^4}$	**45**	$\mathbf{-1.25 \times 10^4}$
40	-3.03×10^6	40	-9.90×10^5
20	-5.76×10^6	20	-1.72×10^6
0	-8.51×10^6	0	-3.81×10^6

Feed Effecting Profit NPV

NPV value must be positive for project to be economically favorable. Table 17.1 calculates the impact of biomass on profit NPV analysis. The maximum (5%) NPV Pareto point is taken from the worst-case Pareto curve having 1000 sampling data points to compare with deterministic case. The feed is ranged from 100 to 0% to obtain the movement of the NPV toward the negative value. Deterministic case gives negative NPV at ~ 49% feed supply (see Table 17.1). For uncertainty case, negative NPV obtained is at 45% feed supply, respectively. This trend shows that as we increase the number of uncertain parameters from zero to three, the worst-case design of the SCN handles uncertain situation in case of less amount of feed supply and work efficiently within the feasibility range of parameters. This results show that for feed availability 45% and above, uncertain cases give positive NPV. Hence, above 45% feed should be made available, if the project is to run economically under various parametric uncertainty considered in this study. The minimum biomass feed supply of 49% and 45% under worst case scenario is shown in bold font with sign change in NPV column i.e. from positive to negative.

Conclusion and Future Work

A novel techno-economic and environmental SCND model for the biofuels namely biodiesel and bioethanol with stochasticity involved in the price of imported biofuel product, feed of biomass and demand of product, is proposed in current work. The developed model takes care and works under the circumstances of multi-feed, multi-product, multi-technology, multiple time periods following the rules of Indian governments of blending 20% bioethanol and biodiesel with gasoline and diesel, respectively. The model follows multi-objective optimization approach to solve the NPV maximization and GHGe minimization objectives. The revenue is generated from two sources, i.e., firstly from retail selling and secondly from GHGe carbon credits. The GHGe is calculated throughout the LCA using well to tank approach.

To deal with uncertainty, the approach of robust optimization has been used, which takes into consideration of the worst and the best case across various uncertain scenarios. While considering uncertainty, due to effect of RO for the worst case, the number of sites/nodes used in SC echelons has decreased. Also, due to RO effect, it has been found that with increase in the variability in uncertain parameters, i.e., by changing the variance of uncertain parameters from 5 to 10%, the worst-case model provides better NPV value (~ 3% increase). The SC shows economic viability for deterministic case with feed supply above 49 and 45% for stochastic case, indicating that RO is able to handle more number of uncertain parameters by efficient usage of biomass feed across the SC. This also indicated at least 45% and above feed supply is needed for project feasibility under various uncertain scenarios as considered in the study.

From future perspective, current work is not only useful to Indian geography, but to all developing national who have biofuels resources. The approach of biofuel demand fulfilment via mixed integer programming can be used into other domains, e.g., fulfilling medical aids in emergency situations, fulfilling electric vehicle charging via optimal site locations, etc. Apart from RO, other methods of uncertainty handling can be tried and compared. The work on uncertainty handling can be integrated with data-driven machine learning algorithm to enhance the output efficacy.

Nomenclature

Location set is given by g with subsets of supplier sites s_{g}, manufacturer sites m_{g}, distributor sites d_{g}, retailer sites r_{g}, and importer sites i_{g} along with alias copy set gc. Feed material set is given by F, technology by Tech and product by P which is composed of eight types of feed f, eight types of technology tech and two types of product p, respectively. Feed is related to geographic supplier site as $sf_{\mathrm{s,f}}$. Mode of transport l is used along with time period t. Scalar values are used directly in the equations, involving total time period nn, discount factor alpha and i interest rate per annum, Φ_{p} as Goods and Service Tax (GST), RON research octane number of imported biofuel to maintain biofuel quality, $\mathrm{gca_p}$ and $\mathrm{ged_p}$ as biofuel GHGe and biofuel energy density, respectively. Parameter values obtained and derived by the user to be used in equations are carbon credits, infrastructure cost of supplier, manufacturer, distributor, retailer, and import as presented by cv_t, $\mathrm{cs}_{\mathrm{s},t}$, $\mathrm{cm}_{\mathrm{m},t}$, $\mathrm{cd}_{\mathrm{d},t}$, $\mathrm{cr}_{\mathrm{r},t}$ and $\mathrm{ci}_{\mathrm{i},t}$, respectively. Parameter values of unit inventory cost, unit transport cost between supplier, manufacturer, distributor, retailer, and importer presented as $\mathrm{ud}_{\mathrm{p,d},t}$, $\mathrm{usm}_{\mathrm{f,l},t}$, $\mathrm{umd}_{\mathrm{p,l},t}$ $\mathrm{umd}_{\mathrm{p,l},t}$, and $\mathrm{uid}_{\mathrm{p,l},t}$, respectively, are to be provided by the user. Cost of production $\mathrm{cp}_{\mathrm{p},t}$ and purchasing cost of finished products $\mathrm{ui}_{\mathrm{p},t}$ from importer have been derived. Distance between locations $\mathrm{dis}_{\mathrm{g,gc}}$, specifically between supplier to manufacturer $\mathrm{dsm}_{\mathrm{s,m,l}}$, manufacturer to distributor $\mathrm{dmd}_{\mathrm{m,d,l}}$, distributor to retailer $\mathrm{ddr}_{\mathrm{d,r,l}}$, and importer to retailer $\mathrm{did}_{\mathrm{i,d,l}}$ are obtained via standard digital map. Product demand $\mathrm{dem}_{\mathrm{p,r},t}$, selling price $\mathrm{sp}_{\mathrm{p},t}$, feed to product conversion $\mathrm{cov}_{\mathrm{f,tech,p}}$ are parameters to be supplied by a user. To keep the upper and lower bounds on feed ($\mathrm{fed}^{\mathrm{Max}}_{\mathrm{f,s},t}$, $\mathrm{fed}^{\mathrm{Min}}_{\mathrm{f,s},t}$), production ($\mathrm{pm}^{\mathrm{Max}}_{\mathrm{p,m},t}$, $\mathrm{pm}^{\mathrm{Min}}_{\mathrm{p,m},t}$), import ($\mathrm{imp}^{\mathrm{Max}}_{\mathrm{p,i},t}$, $\mathrm{imp}^{\mathrm{Min}}_{\mathrm{p,i},t}$), inventory

($inv^{Max}_{p,d,t}$, $inv^{Min}_{p,d,t}$), transported quantity between the layers: ($qsm^{Max}_{f,s,m,l,t}$, $qsm^{Min}_{f,s,m,l,t}$), ($qmd^{Max}_{p,m,d,l,t}$, $qmd^{Min}_{p,m,d,l,t}$), ($qid^{Max}_{p,i,d,l,t}$, $qid^{Min}_{p,i,d,l,t}$), ($qdr^{Max}_{p,d,l,r,t}$, $qdr^{Min}_{p,d,l,r,t}$) were scaled. To calculate the GHGe, the emission factors are used at various stages as $fbc_{f,t}$ before cultivation, during production $fp_{p,t}$, during storage $fd_{p,t}$, during transport between supplier to manufacturer $fsm_{f,t}$, manufacturer to distributor $fmd_{p,t}$, distributor to retailer $fdr_{p,}$ and importer to distributor fid_{p}.

Integer variables $YS_{s,t}$ are used to check the presence/absence of nodes in suppler chain structures. Similarly, $YM_{m,t}$, $YD_{d,t}$, $YR_{r,t}$, and $YI_{i,t}$ are integer variables used for manufacturing, distributing, retailing, and importing layers, respectively. The same binaries can be used for finding the connections between any two such nodes in the network.

Along with binary, the continuous variables include the quantities to be transported between suppler to manufacturer, manufacturer to distributor, distributor to retailer, import to retailer are given by $QSM_{f,s,m,l,t}$,$QMD_{p,d,m,l,t}$, $QDR_{p,d,l,r,t}$, and $QID_{p,i,d,l,t}$, respectively. The suppler provides the feed $FED_{f,s,t}$ to manufacturer for production with many technologies $PMT_{p,m,tech,t}$ which is summed to give total production $PM_{p,m,t}$. In case of production shortage, imports are ordered $IMP_{p,i,t}$ and kept at distributors for storage $INV_{p,d,t}$. GHF shows the fossil greenhouse pollutants. GHS_t represents the greenhouse gas saving of biofuel with respect to fossil fuels to convert it into income as GHR_t to add into earning EeR_t. Total costing TC_t includes operating and capital expenditures as OPX_t and CPX_t, which further depends on infrastructure IRC_t transport TRC_t, inventory IVC_t, production PRC_t, import IMC_t and infrastructure IFC_t cost, respectively. DEP shows annual depreciation, and net present value is given by NPV.

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Chapter 18
A Circular Economy Approach Toward Managing E-waste in Indian Smart City

Kapil Gumte and Kishalay Mitra

Abstract To mitigate electronic pollution and overcome the imbalance of demand–supply ratio of electronic products, namely mobile phones in the market, design of circular economic-based novel closed-loop supply chain (CLSC) network has been developed for 12-year time horizon (2014–2025) following 5R principle. The target of the mathematical model which is based on mixed-integer nonlinear programing (MINLP) is to find exact site locations of various nodes in different echelons of forward and reverse supply chain to do the techno-economic-environmental analysis. For the first time, an Indian scenario is considered with ten layer pull-based CLSC network for a smart city development involving end-of-life (EOL) and end-of-use (EOU). The economic revenue is obtained not only by selling the retail product but also from electronic waste carbon credits from life cycle assessment (LCA) and Indian government incentive in this domain following government guidelines as per national policy on electronic (NPE) encouraging foreign investment to establish electronic manufacturing in India. The multi-period, multi-feed, multi-product, multi-site, and multi-echelon model has been adapted for the smart city development project of a city in India, namely Pune, which is shown to save 98.49% of electronic waste carbon emissions.

Keywords E-waste recycling · Closed-loop supply chain network · Circular economy · MINLP · E-waste carbon credit · NPV

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Introduction

Electronic products, namely mobile phones, help to facilitate human life, hence are in continuous demand. But, the drawback of such product is shorter life span and high demand which keeps the task of maintaining the demand-supply ratio along the upstream supply chain layers. Apart from demand–supply ratio, due to shorter life span, there is lots of electronics waste (e-waste) generation and environmental threat to future generation. According to published global e-waste monitor 2017 report [1] by United Nations University (UNU), 44.7% waste is generated yearly, out of which only 20% is recycled properly and rest 80% is still unaccounted. These e-waste occupies the landfill and contaminates the land and water bodies with hazardous chemicals.

To tackle these two major challenges, namely electronic pollution and demand-supply, the solution is obtained using the concepts of circular economic closed-loop supply chain (CLSC) network design composed of forward supply chain (FSC) taking care of demand and reverse supply chain (RSC) handling the pollution constraint. The techno-economic novel design of CLSC not only takes care of economic profit, but also takes care of environmental greenhouse gas emissions (GHGe).

To comprehend the current paper work, it is vital to understand the earlier researcher's work and the knowledge gap in this domain. For one of the largest Indian third-party reverse logistics, John et al. [2] developed a multi-echelon, multi-product cyclic mixed-integer nonlinear programming MINLP model for consumer electronic goods, e.g., mobile phone and digital cameras. The main aim was to increase the high product residual value with respect to low product residual value in e-waste recovery to study economic viability of reverse logistics. Amin and Zhang [3] proposed a general mathematical model for CLSC network based on product life cycle and return-recovery pairs. The model follows (MINLP) approach with the aim of maximizing the profit by determining the quantity of parts and products in network along with model extension to secondary market. These type of models miss the time value of money, a major factor to deal in circular economic projects. A multi-echelon reverse logistic network for e-waste has been designed by Wang et al. [4] for Shanghai city to locate the specific spots of collection center, disposal centers and determine the site capacity along with storage period in collection spots. But, the calculation does not involve the forward and backward supply chain aspects together. A pure financial calculation approach to CLSC has been applied by Ramezani et al. [5] incorporating current assets, fixed assets, liabilities, balances of cash, debt, securities, payment delays, and discount for change in equity as objective. Though found effective, this approach, compared to the traditional one, lacks consideration of sustainability issues, e.g., environmental GHGe calculations. In another reverse logistic network with cost minimization for e-waste by Safdar et al. [6], carbon trading was considered to report an increase of 42.6% in profit due to 40% increase in carbon trading. This model also lacked the CLSC aspects with FSC calculations and external imports were not considered to avoid demand stock outs. For end-of-life durable products, Diabat and Jebali [7] developed multi-product and multi-period CLSC with take-back legislation based on reward-penalty mechanism for 100% recovery of waste products. The sensitivity analysis of the model shows the influence of acquisition price, quality of products returned, demand and secondary market, where these industries bear the penalty by their government instead of getting funds and subsidy from government.

Readers can refer to review papers from Islam and Huda [8] and Doan et al. [9] involving diverse aspects of CLSC such as government law, managerial decision, location allocation, transports, informal sectors, climate change, and other major evaluation criteria.

To overcome the lacunas as mentioned in the previous paragraph and Table 18.1, current paper provides with an efficient integrated solution and contributes novelty as follows:

(i). Techno-economic-environmental mathematical MINLP model with implementation for the first time in an Indian smart city through a ten layered close-loop supply chain framework
(ii). Model involving multi-domains, i.e., multi-raw material, multi-product, multi-echelon, multi-node, multi-time period in nature.
(iii). Profit calculation and maximization using net present value (NPV) involving time value of money and depreciation in tandem with circular economics.
(iv). International external import of products in case indigenous production shortfalls.
(v). Four ways of handling e-waste in RSC including end-of use (EOU) and end-of-life (EOL)
(vi). E-waste reduction due to CLSC
(vii). Calculation of pollution reduction in terms of GHGe and conversion of GHGe savings into carbon credit revenue across life cycle assessment (LCA).
(viii). Primary and secondary market impact in conjugation with recycle ratio on profit NPV.

The rest of paper is organized as follows. Section "Materials and Methods" explains the problem statement and model formulation. Section "Model Equations" gives insight of model's computational aspect. Section "Results and Discussion" expounds on results with detailed discussion. Section "Conclusion" concludes with major findings.

Materials and Methods

Problem Statement and Formulation

The CLSC is designed with ten layers with FSC containing supplier, manufacturer, importer, distributor, customer, and RSC containing collection centers, repairing centers, dismantling points, recycling points, and waste disposal points as shown

Table 18.1 Comparing current work with earlier work from literature

Author (year)	Multi-feed, product, echelon, time, site	Primary, Secondary market and recycle ratio	Carbon credits	FSC, RSC, CLSC	Imports	Time value of money	Techno-economic-environment
John et al. [2]	Multi-product, site	Primary, secondary market	Nil	RSC	Nil	Nil	Techno-economic
Amin and Zhang [3]	Multi-product, site,	Primary, secondary market	Nil	CLSC	Nil	Nil	Techno-economic
Wang et al. [4]	Multi-site,	Nil	Nil	CLSC	Nil	Nil	Techno-economic
Ramezani et al. [5]	Multi-time,	Nil	Nil	CLSC	Nil	Nil	Economic
Safdar et al. [6]	Single-time, product	Primary, secondary market	Carbon credits	Nil	Nil	Nil	Economic-environment
Diabat and Jebali [7]	Multi-product, echelon, time	Primary, secondary market	Nil	CLSC	Nil	Nil	Techno-economic-environment
Current paper (2022)	Multi-feed, product, echelon, time, site	Primary, secondary market and recycle ratio	Carbon credits	FSC, RSC, CLSC	Imports	Time value of money	Techno-economic-environment

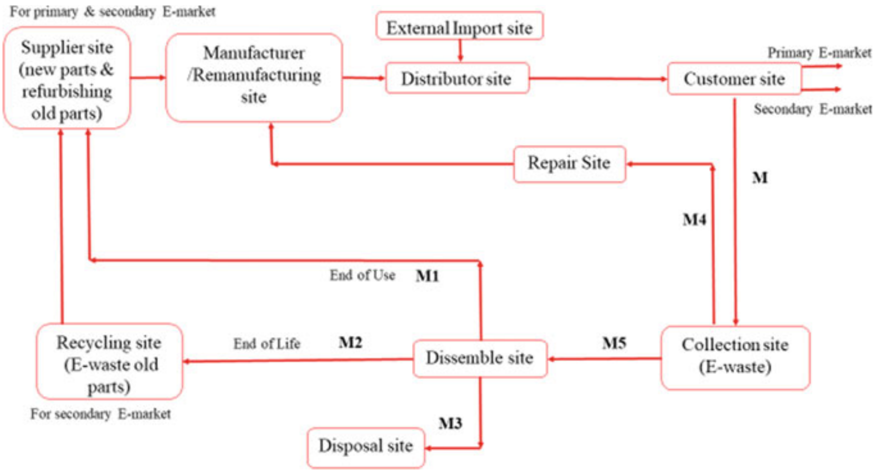

Fig. 18.1 Closed-loop supply chain network design for smartphones e-waste

in Fig. 18.1. The arrows in Fig. 18.1 show the material flow. At supplier site, total four raw material parts $f1$, $f2$, $f3$, $f4$ are used to manufacture two products $p1$ (iPhone), $p2$ (android). For implementation of CLSC network (CLSCN) design, an instance of a smart city development of a developing nation, e.g., Pune City of Indian subcontinent, is chosen. The geography of the city is divided into 5 major zones (see Fig. 18.2). Within each zone, there exist several candidate sites, e.g., each zone can have maximum of 2 suppliers, 2 manufacturers, 2 distributors, 3 demand centers, 2 collection centers, 2 repair centers, 2 dismantling sites, 2 recycling sites, 2 waste disposal sites, except one zone (zone 5), which is the smallest in size. Also, 2 external imports for all distributors are used to avoid product shortage. Overall, there exist potential 87 candidate centers in 5 zones whose selection will be done using profit net present value (NPV) maximization with constraints.

Following assumptions are made while developing the SC model:

1. The selection of site in layers is done using binary integers (1-selected and 0-not selected).
2. Flow of material occurs from one layer to another in sequence as show in Fig. 18.1. There is no jump of material in between layers.
3. The transfer of material can occur across multiple zones. And external imports can be imported in case of product shortage to any zone.
4. There are 3 inventory location, i.e., supplier, distributor, and collection center.
5. The manufacturing and remanufacturing can happen at single site.
6. The lifespan of project is from 2014 to 2025 where single time period is of 3 years indicating total 4 time periods.
7. Within the geography, one person can have either product $p1$ or product $p2$, not both.
8. Within city, all sites are connected via roads.

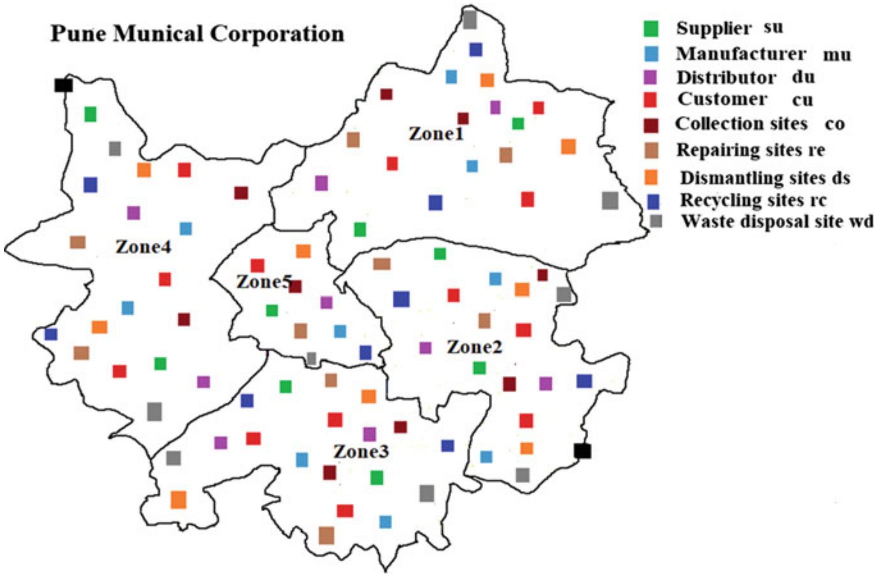

Fig. 18.2 Zonal distribution of smart city Pune for e-waste management supply chain with potential site location. Numbers inside geography show zone numbers

Model Equations

Please go through the nomenclature section in the last before reference section to understanding the equations.

$$\text{NPV} = \sum_t \frac{1}{(1+\text{alpha})^{N-1}}((\text{Earn}_t - \text{Opex}_t - \text{Depr}_t) \times (1 - \Phi_t) + \text{Depr}_t) - \sum_t \frac{1}{(1+\text{alpha})^{N-1}}(\text{Capex}_t),\ t \in T \tag{18.1}$$

$$\text{Earn}_t = \text{Sell}P_t + \text{GHGeRevenue}_t + \text{GovtInct}_t \quad \forall t \tag{18.2}$$

$$\text{Opex}_t = \text{Tran}C_t + \text{Invt}C\text{su}_t + \text{Invt}C\text{du}_t + \text{Invt}C\text{co}_t + \text{Prodn}C_t + \text{Imp}C_t + \text{Infr}MC_t \quad \forall t \tag{18.3}$$

$$\text{Capex}_t = \text{Infra}C_t, \quad \forall t \tag{18.4}$$

$$\text{Depr}_t = \sum_t \frac{i}{((1+i)^N - 1)}(0.5 \times \text{Infra}C_t), \quad \forall t \tag{18.5}$$

Equation 18.1 calculates the objective of NPV maximization involving earning, expenditure, time value of money and depreciation. Equation 18.2 shows the revenue from retail selling, GHGe and Govt. incentives. Equations 18.3 and 18.4 show operating and capital expenditure, respectively, following depreciation in Eq. 18.5. The capital expenditure Eq. 18.4 involves infrastructure cost (Eq. 18.6) which is calculated by establishment cost involved in each layer if selected via binary variable. Once the infrastructure is established, the infrastructure maintenance cost comes into existance as shown by Eq. 18.7. The sites even if are not chosen by binary variables, needs to be taken care as it may be used in next upcoming time periods. Equation 18.8 calculates production cost by multiplying each unit production cost with quantity of products manufactured. In similarly fashion, Eq. 18.9 calculates import cost, and Eqs. 18.10–18.12 calculate inventory cost at supplier, distributor, and collection site.

$$\begin{aligned}
\text{Infra}C_{\text{t}} = & \sum_{f,\text{su}} Y\text{su}_{\text{f,su},t} \times C\text{su}_{\text{f,su},t} + \sum_{p,\text{mu}} Y\text{mu}_{\text{p,mu},t} \times C\text{mu}_{\text{p,mu},t} \\
& + \sum_{p,\text{im}} Y\text{im}_{\text{p,im},t} \times C\text{im}_{\text{p,im},t} + \sum_{p,\text{du}} Y\text{du}_{\text{p,du},t} \times C\text{du}_{\text{p,du},t} \\
& + \sum_{p,\text{cu}} Y\text{cu}_{\text{p,cu},t} \times C\text{cu}_{\text{p,cu},t} + \sum_{p,\text{co}} Y\text{co}_{\text{p,co},t} \times C\text{co}_{\text{p,co},t} \\
& + \sum_{p,\text{re}} Y\text{re}_{\text{p,re},t} \times C\text{re}_{\text{p,re},t} + \sum_{f,\text{ds}} Y\text{ds}_{\text{f,ds},t} \times C\text{ds}_{\text{f,ds},t} \\
& + \sum_{f,\text{rc}} Y\text{rc}_{\text{f,rc},t} \times C\text{rc}_{\text{f,rc},t} + \sum_{\text{f,wd}} Y\text{wd}_{\text{f,wd},t} \times C\text{wd}_{\text{f,wd},t} \\
& \forall t, \text{su, mu, du, im, cu, co, re, ds, rc, wd} \in g,\ f \in F
\end{aligned} \tag{18.6}$$

$$\begin{aligned}
\text{Infra}MC_t = & \sum_{f,\text{su}} (1 - Y\text{su}_{\text{f,su},t}) \times \text{CMsu}_{\text{f,su},t} \\
& \sum_{p,\text{mu}} (1 - Y\text{mu}_{\text{p,mu},t}) \times \text{CMmu}_{\text{p,mu},t} \\
& + \sum_{p,\text{im}} (1 - Y\text{im}_{\text{p,im},t}) \times \text{CMim}_{\text{p,im},t} \\
& + \sum_{p,\text{du}} \left(1 - Y\text{du}_{\text{p,du},t}\right) \times \text{CMdu}_{\text{p,du},t} \\
& + \sum_{p,\text{cu}} \left(1 - Y\text{cu}_{\text{p,cu},t}\right) \times \text{CMcu}_{\text{p,cu},t} \\
& + \sum_{p,\text{co}} \left(1 - Y\text{co}_{\text{p,co},t}\right) \times \text{CMco}_{\text{p,co},t} \\
& + \sum_{p,\text{re}} \left(1 - Y\text{re}_{\text{p,re},t}\right) \times \text{CMre}_{\text{p,re},t}
\end{aligned}$$

$$+\sum_{f,\text{ds}}\left(1-Y\text{ds}_{\text{f,ds},t}\right)\times \text{CMds}_{\text{f,ds},t}$$
$$+\sum_{f,\text{rc}}\left(1-Y\text{rc}_{\text{f,rc},t}\right)\times \text{CMrc}_{\text{f,rc},t}$$
$$+\sum_{f,\text{wd}}\left(1-Y\text{wd}_{\text{f,wd},t}\right)\times \text{CMwd}_{\text{f,wd},t},$$
$$\forall t,\ \text{su, mu, du, im, cu, co, re, ds, rc, wd} \in g,\ f \in F \tag{18.7}$$

$$\text{Prodn}C_t = \sum_{p,\text{mu}} U\text{prod}C_{\text{p},t} \times P\text{mu}_{\text{p,mu},t},$$
$$\forall t,\ p \in P,\ \text{mu} \in g \tag{18.8}$$

$$\text{Imp}C_t = \sum_{p,\text{im}} U\text{Imp}C_{\text{p},t} \times \text{import}_{\text{p,im},t},$$
$$\forall t,\ p \in P,\ \text{im} \in g \tag{18.9}$$

$$\text{Invt}C\text{su}_t = \sum_{f,\text{su}} \text{invtsu}_{\text{f,su},t} \times \text{UICsu}_{\text{f,du},t},$$
$$\forall t,\ f \in F,\ \text{su} \in g \tag{18.10}$$

$$\text{Invt}C\text{du}_t = \sum_{p,\text{du}} \text{invtdu}_{\text{p,du},t} \times \text{UICdu}_{\text{p,du},t},$$
$$\forall t,\ p \in P,\ \text{du} \in g \tag{18.11}$$

$$\text{Invt}C\text{co}_t = \sum_{p,\text{co}} \text{invtdu}_{\text{p,co},t} \times \text{UICco}_{\text{p,co},t},$$
$$\forall t,\ p \in P,\ \text{co} \in g \tag{18.12}$$

Equation 18.13 calculates transport cost by multiplying unit transport cost across SC layers with distance traveled and product quantity transferred in that layers. Apart from retail (Eq. 18.14b), earning happens from government incentive (Eq. 18.14a) and carbon credits (Eq. 18.15). Equations 18.16–18.19 calculates the GHGe saved due to CLSC and compare with FSC to convert into carbon credits.

$$\text{Trans}C_t = \sum_{f,\text{su},\text{mu},l} \text{UTC}f\text{sumu}_{\text{f,l,t}}$$
$$\times\, d\text{sumu}_{\text{su,mu}} \times Qf\text{sumu}l_{\text{f,su,mu,l},t}$$
$$+ \sum_{p,\text{mu},\text{du},l} \text{UTC}p\text{mudu}_{\text{p,l},t} \times d\text{mudu}_{\text{mu,du}}$$

$$
\begin{aligned}
& \times Qp\text{mudu}l_{\text{p,mu,du,l},t} + \sum_{p,\text{im},\text{du},l} \text{UTC}p\text{imdu}_{\text{p,l},t} \\
& \times d\text{imdu}_{\text{im,du}} \times Qp\text{imdu}l_{\text{p,im,du,l},t} \\
& + \sum_{p,\text{du},\text{cu},l} \text{UTC}p\text{ducu}_{\text{p,l},t} \times d\text{ducu}_{\text{du,cu}} \\
& \times Qp\text{ducu}l_{\text{p,du,cu,l},t} + \sum_{p,\text{cu},\text{co},l} \text{UTC}p\text{cuco}_{\text{p,l},t} \\
& \times d\text{cuco}_{\text{cu,co}} \times Qp\text{cuco}l_{\text{p,cu,co,l},t} \\
& + \sum_{p,\text{co},\text{re},l} \text{UTC}p\text{core}_{\text{p,l},t} \times d\text{core}_{\text{co,re}} \\
& \times Qp\text{core}l_{\text{p,co,re,l},t} + \sum_{p,\text{re},\text{mu},l} \text{UTC}p\text{remu}_{\text{p,l},t} \\
& \times d\text{remu}_{\text{re,mu}} \times Qp\text{remu}l_{\text{p,re,mu,l},t} + \sum_{p,\text{co},\text{ds},l} \text{UTC}p\text{cods}_{\text{p,l},t} \\
& \times d\text{cods}_{\text{co,ds}} \times Qp\text{cods}l_{\text{p,co,ds,l},t} + \sum_{f,\text{ds},\text{su},l} \text{UTC}p\text{dssu}_{\text{f,l},t} \\
& \times d\text{dssu}_{\text{ds,su}} \times Qf\text{dssu}l_{\text{f,ds,su,l},t} + \sum_{f,\text{ds},\text{wd},l} \text{UTC}f\text{dswd}_{\text{f,l},t} \\
& \times d\text{dswd}_{\text{ds,wd}} \times Qf\text{dswd}l_{\text{f,ds,wd,l},t} + \sum_{f,\text{ds},\text{rc},l} \text{UTC}f\text{dsrc}_{\text{f,l},t} \\
& \times d\text{dsrc}_{\text{ds,rc}} \times Qf\text{dsrc}l_{\text{f,ds,rc,l},t} + \sum_{f,\text{rc},\text{su},l} \text{UTC}f\text{rcsu}_{\text{f,l},t} \\
& \times d\text{rcsu}_{\text{rc,su}} \times Qf\text{rcmu}l_{\text{f,rc,su,l},t} \\
& \forall t,\ f \in F,\ p \in P,\ l \in L,\ \text{su, mu, du, im, cu, co, re, ds, rc, wd} \in g
\end{aligned} \tag{18.13}
$$

Equation 14a and b calculates earning from government revenue and retail selling, respectively. The government bears 50% of total expenditure whereas retailing involves primary and secondary market selling.

$$\text{GovtInc}t_t = 0.5 \times (\text{Capex}_t + \text{Opex}_t),\ \forall t \tag{14a}$$

$$
\begin{aligned}
\text{Sell}P_t = & \sum_{p,cu} \text{pri} \times \text{Dem}_{\text{p,cu},t} \times \text{priSP}_{\text{p},t} \\
& \sum_{\text{p,cu}} \text{sec} \times \text{Dem}_{\text{p,cu},t} \times \text{secSP}_{\text{p},t}, \\
& \forall t,\ p \in P,\ \text{wd} \in g
\end{aligned} \tag{14b}
$$

Equations 18.15–18.18 calculates GHGe aspect. The revenue generation from carbon credit (Eq. 18.15) due to GHGe saving (Eq. 18.16) as a result of the difference of GHGe in CLSC (Eq. 18.17) and FSC (Eq. 18.18) is calculated. The GHGe CLSC (FSC + RSC) and FSC value is obtained by multiplying unit emission factor with corresponding contribution from each layer of SC starting from supplier feed till recycle from recycle unit.

$$\text{GHGeRevenue}_t = \text{GHGeSaving}_t \times CC\text{val}_t, \forall t \tag{18.15}$$

$$\text{GHGeSaving}_t = \text{GHGeFSC}_t - \text{GHGeCLSC}_t, \forall t \tag{18.16}$$

$$\begin{aligned}
\text{GHGeCLSC}_t = {} & \sum_{f,\text{su}} f\text{su}_f \times P\text{Feed}_{f,su,t} + \sum_{p,\text{mu}} f\text{mu}_\text{p} \times P\text{mu}_{\text{p,mu},t} \\
& \sum_{p,\text{re}} f\text{re}_\text{p} \times Qp\text{re}_{\text{p,re},t} + \sum_{f,\text{ds}} f\text{ds}_\text{f} \times Qf\text{ds}_{\text{f,ds},t} \\
& + \sum_{f,\text{wd}} f\text{wd}_\text{f} \times Qp\text{wd}_{\text{f,wd},t} + \sum_{f,\text{rc}} f\text{rc}_\text{f} \times Qf\text{rc}_{\text{f,rc},t} \\
& + \sum_{f,\text{su}} f\text{su}i_\text{f} \times \text{invtsu}_{\text{f,su},t} + \sum_{p,\text{du}} f\text{du}i_\text{p} \times \text{invtdu}_{\text{p,du},t} \\
& + \sum_{p,\text{co}} f\text{co}i_\text{p} \times \text{invtco}_{\text{p,co},t} + \sum_{f,\text{su},\text{mu},l} f\text{sumu}_\text{f,l} \times d\text{sumu}_\text{su,mu} \\
& \times Qf\text{sumu}l_{\text{f,su,mu,l},t} \sum_{p,\text{mu},\text{du},l} f\text{mudu}_\text{p,l} \times d\text{mudu}_\text{mu,du} \\
& \times Qp\text{mudu}l_{\text{p,mu,du,l},t} + \sum_{p,\text{im},\text{du},l} f\text{imdu}_\text{p,l} \times d\text{imdu}_\text{im,du} \\
& \times Qp\text{imdu}l_{\text{p,im,du,l},t} + \sum_{p,\text{du},\text{cu},l} f\text{ducu}_\text{p,l} \times d\text{ducu}_\text{du,cu} \\
& \times Qp\text{ducu}l_{\text{p,du,cu,l},t} + \sum_{p,\text{cu},\text{co},l} f\text{cuco}_\text{p,l} \times d\text{cuco}_\text{cu,co} \\
& \times Qp\text{cuco}l_{\text{p,cu,co,l},t} + \sum_{p,\text{co},\text{re},l} f\text{core}_\text{p,l} \times d\text{core}_\text{co,re} \\
& \times Qp\text{core}l_{\text{p,co,re,l},t} + \sum_{p,\text{re},\text{mu},l} f\text{remu}_\text{p,l} \times d\text{remu}_\text{re,mu} \\
& \times Qp\text{remu}l_{\text{p,re,mu,l},t} + \sum_{p,\text{co},\text{ds},l} f\text{cods}_\text{p,l} \times d\text{cods}_\text{co,ds} \\
& \times Qp\text{cods}l_{\text{p,co,ds,l},t} + \sum_{f,\text{ds},\text{su},l} f\text{dssu}_\text{p,l} \times d\text{dssu}_\text{ds,su}
\end{aligned}$$

$$\times Qf\text{dssu}l_{\text{f,ds,su,l},t} \sum_{f,\text{ds,wd},l} f\text{dswd}_{\text{f,l}} \times d\text{dswd}_{\text{ds,wd}}$$
$$\times Qf\text{dswd}l_{\text{f,ds,wd,l},t} \sum_{f,\text{ds,rc},l} f\text{dsrc}_{\text{p,l}} \times d\text{dsrc}_{\text{ds,rc}}$$
$$\times Qf\text{dsrc}l_{\text{f,ds,rc,l},t} + \sum_{f,\text{rc,su},l} f\text{rcsu}_{\text{f,l}} \times d\text{rcsu}_{\text{rc,su}}$$
$$\times Qf\text{rcsu}l_{\text{f,rc,su,l},t}$$
$$\forall t,\ f \in F,\ p \in P,\ l \in L,\ \text{su, mu, du, im, cu, co, re, ds, rc, wd} \in g \tag{18.17}$$

$$\begin{aligned}\text{GHGeFSC}_t = {} & \sum_{f,\text{su}} f\text{su}_{\text{f}} \times P\text{Feed}_{\text{f,su},t} + \sum_{p,\text{mu}} f\text{mu}_{\text{p}} \times P\text{mu}_{\text{p,mu},t} \\ & + \sum_{f,\text{su}} f\text{su}i_{\text{f}} \times P\text{Feed}_{\text{f,su},t} + \sum_{p,\text{du}} f\text{du}i_{\text{p}} \times P\text{du}_{\text{p,du},t} \\ & + \sum_{f,\text{su,mu},l} f\text{sumu}_{\text{f,l}} \times d\text{sumu}_{\text{su,mu}} \\ & \times Qf\text{sumu}l_{\text{f,su,mu,l},t} + \sum_{p,\text{mu,du},l} f\text{mudu}_{\text{p,l}} \\ & \times d\text{mudu}_{\text{mu,du}} \times Qp\text{mudu}l_{\text{p,mu,du,l},t} \\ & + \sum_{p,\text{im,du},l} f\text{imdu}_{\text{p,l}} \times d\text{imdu}_{\text{im,du}} \times Qp\text{imdu}l_{\text{p,im,du,l},t} \\ & + \sum_{p,\text{du,cu},l} f\text{ducu}_{\text{p,l}} \times d\text{ducu}_{\text{du,cu}} \times Qp\text{ducu}l_{\text{p,du,cu,l},t} \\ & + \sum_{p,\text{cu,co},l} f\text{waste}_{\text{p}} \times Qp\text{cuco}l_{\text{p,cu,co,l},t} \end{aligned}$$
$$\forall t,\ f \in F,\ p \in P,\ l \in L,\ \text{su, mu, du, im, cu, co,} \in g \tag{18.18}$$

Equations 18.19–18.57 shows the overall material flow (See Fig. 18.1). Equation 18.19–18.33 shows the FSC flow. Feed starting from suppliers (Eq. 18.19) moves to manufacturers with limitation on feed (Eq. 18.20) and transport (Eq. 18.21). The inventory at supplier (Eq. 18.22) takes input from previous time period, recycle unit, disassembly site and sends raw material to manufacturers. The raw material reaches manufacturing site for production (Eq. 18.23) with production limitation (Eq. 18.24) to transfer product to next distribution layer (Eq. 18.25) with transport limitation (Eq. 18.26). At distributor, imports (Eq. 18.27) are included in SC with import (Eq. 18.28) and transport quantity (Eq. 18.29) limitations. The inventory at distributor after taking input from previous time period, manufacturing unit and importers, sends the product to customer retailers (Eq. 18.30) with inventory limitation (Eq. 18.31) and transport quantity limitation (Eq. 18.32). The quantity reaching the demand customer retailers is always kept greater that demand (Eq. 18.33).

$$PFeed_{f,su,t} = \sum_{mu} Qf sumul_{f,su,mu,l,t},$$
$$mu \in g, \forall f, \forall su, \forall l, \forall t \tag{18.19}$$

$$PFeedMin_{f,su,t} \times Ysu_{f,su,t} \leq PFeed_{f,su,t} \leq PFeedMax_{f,su,t} \times Ysu_{f,su,t}, \forall f, \forall su, \forall t \tag{18.20}$$

$$Qf sumulMin_{f,su,mu,l,t} \times Ysu_{f,su,t} \leq Qf sumul_{f,su,mu,l,t} \leq Qf sumulMax_{f,su,mu,l,t} \times Ymu_{p,mu,t} \forall f, \forall p, \forall su, \forall mu, \forall l, \forall t \tag{18.21}$$

$$invtsu_{f,su,t} = invtsu_{f,su,t-1} + \sum_{rc,l} Qf rcsul_{f,rc,su,l,t} + \sum_{ds,l} Qf dssul_{f,ds,su,l,t} - \sum_{mu,l} Qf sumul_{f,su,mu,l,t}$$
$$\forall f, \forall su, \forall t, rc, ds, mu \in g, l \in L \tag{18.22}$$

$$\sum_{f,su,l} Qf sumul_{f,su,mu,l,t} \times qfp_{f,p} = Pmu_{p,mu,t} \quad f \in F, su \in g, l \in L, \forall mu, \forall t \tag{18.23}$$

$$PmuMin_{p,mu,t} \times Ymu_{mu,t} \leq Pmu_{p,mu,t} \leq PmuMax_{p,mu,t} \times Ymu_{mu,t}, \forall p, \forall mu, \forall t \tag{18.24}$$

$$Pmu_{p,mu,t} = \sum_{du} Qpmudul_{p,mu,du,l,t}, \forall p, \forall mu, \forall l, \forall t, du \in g \tag{18.25}$$

$$QpmudulMin_{p,mu,du,l,t} \times Ymu_{p,mu,t} \leq Qpmudul_{p,mu,du,l,t} \leq QpmudulMax_{p,mu,du,l,t} \times Ydu_{p,du,t}, \forall p, \forall mu, \forall du, \forall l, \forall t \tag{18.26}$$

$$import_{p,im,t} = \sum_{du} Qpimdul_{p,im,du,l,t},$$
$$\forall p, \forall im, \forall l, \forall t, du \in g \tag{18.27}$$

$$\text{importMin}_{\text{p,im},t} \times Y\text{im}_{\text{im},t} \leq \text{import}_{\text{p,im},t} \leq \text{importMax}_{\text{p,im},t} \times Y\text{im}_{\text{im},t}, \forall p, \forall \text{im}, \forall t \tag{18.28}$$

$$Qp\text{imdu}l\text{Min}_{\text{p,im,du,l},t} \times Y\text{im}_{\text{p,im},t} \leq Qp\text{imdu}l_{\text{p,im,du,l},t} \leq Qp\text{imdu}l\text{Max}_{\text{p,im,du,l},t} \times Y\text{du}_{\text{p,du},t}, \forall p, \tag{18.29}$$

$$\text{invtdu}_{\text{p,du},t} = \text{invtdu}_{\text{p,du},t-1} + \sum_{\text{mu},l} Qp\text{mudu}l_{\text{p,mu,du,l},t} \sum_{\text{im},l} Qp\text{imdu}l_{\text{p,im,du,l},t} - \sum_{\text{cu},l} Qp\text{ducu}l_{\text{p,du,cu,l},t}, \forall p, \forall \text{du}, \forall t, l \in L, \forall \text{cu}, \forall t, \text{mu, cu, im} \in g \tag{18.30}$$

$$\text{invtduMin}_{\text{p,du},t} \times Y\text{du}_{\text{du},t} \leq \text{invtdu}_{\text{p,du},t} \leq \text{invtduMax}_{\text{p,du},t} \times Y\text{du}_{\text{du},t}, \forall p, \forall \text{du}, \forall t \tag{18.31}$$

$$Qp\text{ducu}l\text{Min}_{\text{p,du,cu,l},t} \times Y\text{du}_{\text{p,du},t} \leq Qp\text{duru}l_{\text{p,du,ru,l},t} \leq Qp\text{ducu}l\text{Max}_{\text{p,du,cu,l},t} \times Y\text{cu}_{\text{p,du,cu}l,t}, \forall p, \forall \text{du}, \forall \text{cu}, \forall l, \forall t \tag{18.32}$$

$$\sum_{\text{du},l} Qp\text{ducu}l_{\text{p,du,cu,l},t} \geq \text{Dem}_{\text{p,cu},t}, \forall p, \forall \text{cu}, l \in L, \forall t, \text{du} \in g \tag{18.33}$$

Till now, it was FSC movement, now from Eq. 18.34 onward RSC starts, to implement circular economy aspect. The (recycle ratio) fraction M of obsolete and damaged product is send back from electronic customer to e-waste collection center (Eq. 18.34), where collection sites act as inventory (Eq. 18.35) with input from previous time period, customer site and output to recycling and dismantling site. There do exists limitation in inventory capacity (Eq. 18.36) and transport to collection sites (Eq. 18.37).

$$M \times \text{Dem}_{\text{p,cu},t} = \sum_{\text{co}} Qp\text{cuco}l_{\text{p,cu,co,l},t}, \forall p, \forall t, \text{cu, co} \in g \tag{18.34}$$

$$\text{invtco}_{\text{p,co},t} = \text{invtco}_{\text{p,co},t-1} + \sum_{\text{cu},l} Qp\text{cuco}l_{\text{p,cu,co,l},t} - \sum_{\text{re},l} Qp\text{core}l_{\text{p,co,re,l},t} - \sum_{\text{ds},l} Qp\text{cods}l_{\text{p,co,ds,l},t}, \quad \forall p, \forall t, \forall \text{co}, \forall t, l \in L, \text{cu, re, ds} \in g \tag{18.35}$$

$$\text{invtcoMin}_{\text{p,co},t} \times Y\text{co}_{\text{co},t} \leq \text{invtco}_{\text{p,co},t} \leq \text{invtcoMax}_{\text{p,co},t} \times Y\text{co}_{\text{co},t}, \quad \forall p, \forall \text{co}, \forall t \tag{18.36}$$

$$Qp\text{cuco}l\text{Min}_{\text{p,cu,co,l},t} \times Y\text{cu}_{\text{p,cu},t} \leq Qp\text{cuco}l_{\text{p,cu,co,l},t} \leq Qp\text{cuco}l\text{Max}_{\text{p,cu,co,l},t} \times Y\text{co}_{\text{p,l},t}, \quad \forall p, \forall \text{cu}, \forall \text{co}, \forall l, \forall t \tag{18.37}$$

From collection site, one path of repairable product is send back to remanufacturing via repairing site (Eqs. 18.38, 18.39) with transport limitations (Eq. 18.40), and another path is send to dismantling site (41). There do exist limitation on repairing site (Eq. 18.42), transport from repairing to remanufacturing sites (Eq. 18.43), and transport from collection site to dismantling site (Eq. 18.44).

$$\sum_{\text{co}} Qp\text{core}l_{\text{p,co,re,l},t} = Qp\text{re}_{\text{p,re},t}, \quad \forall p, \forall \text{re}, \forall l, \forall t, \text{co} \in g \tag{18.38}$$

$$Qp\text{re}_{\text{p,re},t} = \sum_{\text{mu}} Qp\text{remu}l_{\text{p,re,mu,l},t}, \quad \forall p, \forall \text{re}, \forall l, \forall t, \text{mu} \in g \tag{18.39}$$

$$Qp\text{core}l\text{Min}_{\text{p,co,re,l},t} \times Y\text{co}_{\text{p,co},t} \leq Qp\text{core}l_{\text{p,co,re,l},t} \leq Qp\text{core}l\text{Max}_{\text{p,co,re,l},t} \times Y\text{re}_{\text{p,re},t}, \forall p, \forall \text{co}, \forall \text{re}, \forall l, \forall t \tag{18.40}$$

$$\sum_{p,\text{co},l,f} Qp\text{cods}l_{\text{p,co,ds,l},t} \times qfp_{\text{f,p}} = Qf\text{ds}_{\text{f,ds},t}, \forall \text{ds}, \forall t, l \in L, p \in P, f \in F, \text{co} \in g \tag{18.41}$$

$$Qp\text{reMin}_{\text{p,re},t} \times Y\text{re}_{\text{re},t} \leq Qp\text{re}_{\text{p,re},t} \leq Qp\text{reMax}_{\text{p,re},t}$$

$$\times\ Yre_{re,t},\ \forall p,\ \forall re,\ \forall t \tag{18.42}$$

$$QpremulMin_{p,re,mu,l,t} \times Yre_{p,re,t} \leq Qpremul_{p,re,mu,l,t} \leq QpremulMax_{p,re,mu,l,t} \times Ymu_{p,mu,t},\ \forall p,\ \forall re,\ \forall mu,\ \forall l,\ \forall t \tag{18.43}$$

$$QpcodslMin_{p,co,ds,l,t} \times Yco_{p,co,t} \leq Qpcodsl_{p,co,ds,l,t} \leq QpcodslMax_{p,co,ds,l,t} \times Yds_{f,ds,t},\ \forall p,\ \forall f,\ \forall co,\ \forall ds,\ \forall l,\ \forall t \tag{18.44}$$

At dismantling site, the used product is disintegrated into its components where one path goes to supplier (Eq. 18.45), second to recycling unit (Eq. 18.46), and remaining waste to waste disposal sites (Eq. 18.47). The dismantling site do have capacity limitation (Eq. 18.48) along with transport limitation to supplier (Eq. 18.49), recycling (Eq. 18.50), and waste disposal (Eq. 18.51).

$$M1 \times Qfds_{f,ds,t} = \sum_{su} Qfdssul_{f,ds,su,l,t} \tag{18.45}$$

$$M2 \times Qfds_{f,ds,t} = \sum_{rc} Qfdsrcl_{f,ds,rc,l,t} \tag{18.46}$$

$$(1 - M1 - M2) \times Qfds_{f,ds,t} = \sum_{wd,l} Qfdswdl_{f,ds,wd,l,t} \tag{18.47}$$

$$QfdsMin_{f,ds,t} \times Yds_{ds,t} \leq Qfds_{f,ds,t} \leq QfdsMax_{f,ds,t} \times Yds_{ds,t},\ \forall f,\ \forall ds,\ \forall t \tag{18.48}$$

$$QfdssulMin_{f,ds,su,l,t} \times Yds_{f,ds,t} \leq Qfdssul_{f,ds,su,l,t} \leq QfdssulMax_{f,ds,su,l,t} \times Ysu_{f,su,t},\ \forall f,\ \forall ds,\ \forall su,\ \forall l,\ \forall t \tag{18.49}$$

$$QfdsrclMin_{f,ds,rc,l,t} \times Yds_{f,ds,t} \leq Qfdsrcl_{f,ds,rc,l,t} \leq QfdsrclMax_{f,ds,rc,l,t} \times Yrc_{f,rc,t},\ \forall f,\ \forall ds,\ \forall rc,\ \forall dl,\ \forall t \tag{18.50}$$

$$QfdswdlMin_{f,ds,wd,l,t} \times Yds_{f,ds,t} \leq Qfdswdl_{f,ds,wd,l,t} \leq QfdswdlMax_{f,ds,wd,l,t} \times Ywd_{f,wd,t}\ \forall f,\ \forall ds,\ \forall wd,\ \forall l,\ \forall t \tag{18.51}$$

During transport from dismantling site to waste disposal (Eq. 18.52) and recycling site (Eq. 18.53), there do exists respective transport limitations (Eqs. 18.54 and 18.55). The quantity collected at recycling site (Eq. 18.56) is moved back to supplier site for recycling and refurbishing with transport limitation (Eq. 18.57).

$$\sum_{f,\mathrm{ds},l} Qf\mathrm{dswd}l_{\mathrm{f,ds,wd,l},t} = Qf\mathrm{wd}_{\mathrm{f,wd},t}, \quad f \in F, \mathrm{ds} \in g, l \in L, \forall \mathrm{wd}, \forall t \tag{18.52}$$

$$\sum_{f,\mathrm{ds},l} Qf\mathrm{dsrc}l_{\mathrm{f,ds,rc,l},t} = Qf\mathrm{rc}_{\mathrm{f,rc},t}, \quad f \in F, \mathrm{ds} \in g, l \in L, \forall \mathrm{rc}, \forall t \tag{18.53}$$

$$Qf\mathrm{wdMin}_{\mathrm{f,wd},t} \times Y\mathrm{wd}_{\mathrm{wd},t} \le Qf\mathrm{wd}_{\mathrm{f,wd},t} \le Qf\mathrm{wdMax}_{\mathrm{f,wd},t} \times Y\mathrm{wd}_{\mathrm{wd},t}, \quad \forall f, \forall \mathrm{wd}, \forall t \tag{18.54}$$

$$Qf\mathrm{rcMin}_{\mathrm{f,rc},t} \times Y\mathrm{rc}_{\mathrm{rc},t} \le Qf\mathrm{wd}_{\mathrm{f,wd},t} \le Qf\mathrm{wdMax}_{\mathrm{f,wd},t} \times Y\mathrm{rc}_{\mathrm{rc},t}, \forall f, \forall \mathrm{rc}, \forall t \tag{18.55}$$

$$Qf\mathrm{rc}_{\mathrm{f,rc},t} = \sum_{\mathrm{su}} Qf\mathrm{rcsu}l_{\mathrm{f,rc,su,l},t}, \ \mathrm{su} \in g, \forall f, \forall \mathrm{rc}, \forall l, \forall t \tag{18.56}$$

$$Qf\mathrm{rcsu}l\mathrm{Min}_{\mathrm{f,rc,su,l},t} \times Y\mathrm{rc}_{\mathrm{f,rc},t} \le Qf\mathrm{rcsu}l_{\mathrm{f,rc,su,l},t} \le Qf\mathrm{rcsu}l\mathrm{Max}_{\mathrm{f,rc,su,l},t} \times Y\mathrm{su}_{\mathrm{p,su,l},t} \quad \forall f, \forall \mathrm{rc}, \forall \mathrm{su}, \forall l, \forall t, \tag{18.57}$$

Further, the data for modeling have been obtained via several genuine government url, journals, book chapter, books, manuscripts, thesis, conference proceedings and are not shown due to space constraints which can be made available upon asking.

Results and Discussion

The model developed is MINLP in nature and is solved in GAMS® 24.1.3 using DICOPT® solver. The results section has been sectioned into several parts where Section "Site Selection and Connections" explains about site location and connectivity with respect to increasing demand. Section "Techno-economic Cost analysis" elaborates on techno-economic cost analysis followed by Section "Environment GHGe Analysis" telling the effect of RSC on GHGe and carbon credits.

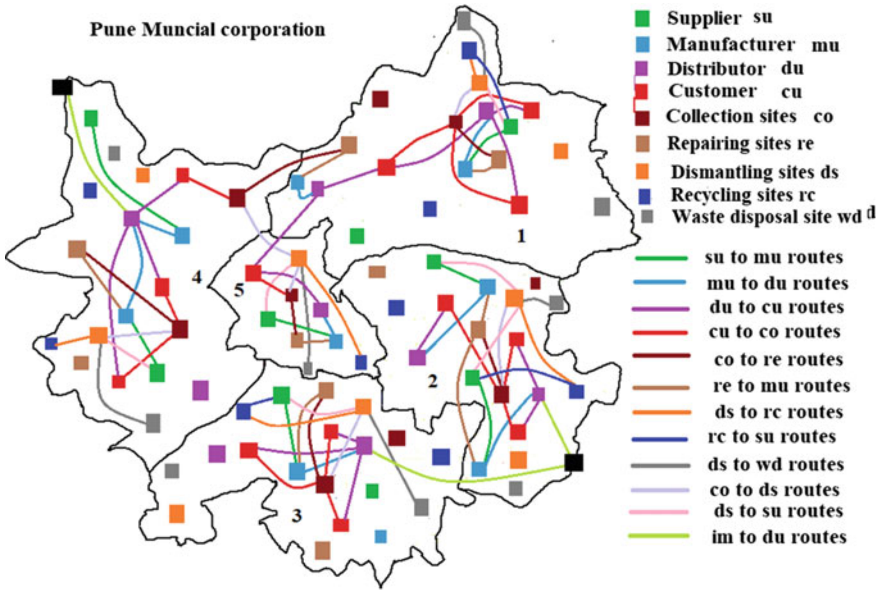

Fig. 18.3 CLSC network design for Pune city for time period $t1$ showing site location and connectivity. Numbers inside geography show the zone number

Site Selection and Connections

With 3 years gap for single time period, there are 4 time period from 2014 to 2025. A total number of 64 sites which are present at time period $t1$ has been observed (see Fig. 18.3) to increase up to 87 sites at the time period $t4$—a growth of 35.9% over the entire time period. From Fig. 18.3, it is clear that not all candidate locations are chosen during model implementation. The choice of these site selection is controlled by objective profit NPV maximization. At the background, infrastructure cost, transportation cost, inventory cost, and the plant or site capacity factor play a vital role, and as a result of that, the sites of echelons with higher capacities and less establishment cost, inter-echelon distance bearing less transport cost and holding cost are observed to be selected. Based on each time period demand, the network structure will also vary indicating dynamic structure.

Techno-economic Cost Analysis

Table 18.2 shows economic and cost analysis of the entire project. There is an increase in the various cost factors across time; the capital expenditure (Capex) due to infrastructure has increased from ₹. 979.69 to 1057.29 million, and the operating expenditure (Opex), i.e., production, transport, inventory, infrastructure maintenance costs, has gone up from ₹. 1176.824 to 1314.175 million with time to accommodate increasing product demand. The noticeable point here is that the import cost has reduced from ₹. 325.5 to 231 million, indicating reduced dependency on imports with more weightage to indigenous production; however, the number of imported product has not reached zero. The revenue is not only from retail, but also from government incentives (50% of Opex plus Capex) and carbon credits. From earning perspective, overall ₹. 954,151.36 million is generated from this project. The direct profit from selling the product contributes ₹. 947,025.44 million (99.25%), government incentives and GHGe revenue contribute ₹. 4511.07 million (0.47%) and ₹. 2614.84 million (0.27%), respectively. The NPV value comes out positive, i.e., ₹. 624,558 clearly indicating the project feasibility.

Environment GHGe Analysis

The effect of adding reverse logistic to handle E-waste can be observed from Fig. 18.4. The GHGe of FSC (blue color) is much higher than that of negligible CLSC (orange color) for all time period. The presence of RSC mitigates the GHGe and corresponding e-waste pollution. Further, the GHGe saved due CLSC with respect to FSC is multiplied by carbon credit value to make the revenue of ₹. 2614.84 million. The overall calculation for all time period form Fig. 18.4 shows that overall 98.48% of GHGe reduced in e-waste due to RSC.

Primary and Secondary Market with Recycle Ration M

In the smartphone market, both primary (new) and secondary products (old) play a vital role which in turn impacts the monetary market. Across the column (see Table 18.3), when the ratio of primary product increases from zero to one and secondary market product ratio decreases from one to zero simultaneously, the NPV value increases for different recycle ratios. Across the rows, when the recycle ratio M is decreased from 1 to 0, one can observe the increase in NPV for particular primary and secondary market product ratio. Hence, only, the objective of NPV maximization leads the primary market to get all the products manufactured with no recycling. However, practically, this is not the sustainable solution due to environmental regulation and resource constraints. This drives the electronic smartphones to be recycled in between 0 and 100% recycle ratio range in tandem with the primary and secondary market (0 to 1 range) money flow (Table 18.3) by simultaneously taking into account the aspect of profit NPV.

Table 18.2 Circular economic analysis of CLSC (in million ₹.)

	*t*1	*t*2	*t*3	*t*4	Total	
Production cost	830.0009	894.8772	991.5452	1054.261	3770.68	
Transport cost	0.023091	0.023954	0.024684	0.026457	0.10	
Inventory cost supplier	0.062104	0.092534	0.114315	0.178327	0.45	
Inventory cost distributor	0.014396	0.024303	0.02526	1.610654	1.67	
Inventory cost collection sites	5.44185	6.091797	6.93	9.765	28.23	
Import cost	325.50	294.00	262.50	231.00	1113.00	
Infra maintenance cost	15.78206	16.5236	17.24973	17.33413	66.89	
Opex	1176.824	1211.633	1278.389	1314.175	4981.02	
Infra cost	979.6933	983.9133	1020.22	1057.297	4041.12	
Capex	979.6933	983.9133	1020.22	1057.297	4041.12	
Depreciation	105.5477	106.0023	109.9138	113.9083	435.37	
Earning	138,556.9	249,876.4	269,503.2	296,214.9	954,151.36	
GHGe revenue	242.6827	544.6332	823.1787	1004.349	2614.84	**0.27**%
Selling revenue	137,118.8	248,187.7	267,446.4	294,272.5	947,025.44	**99.25**%
Govt. incentives	1078.26	1097.77	1149.30	1185.74	4511.07	**0.47**%
NPV	89,948.00	163,620.00	176,710.00	194,280.00	**624,558.00**	

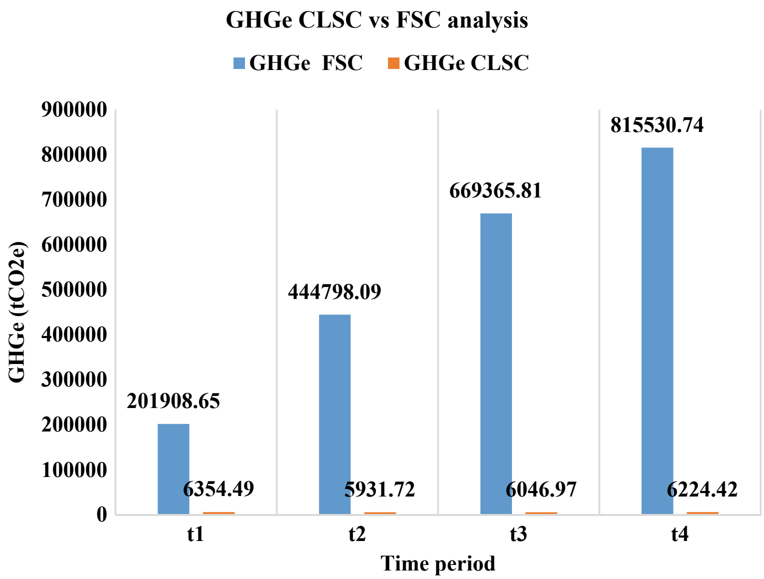

Fig. 18.4 GHGe of CLSC and FSC comparison for time period horizon

Table 18.3 Recycle ratio versus primary and secondary market effect on NPV (Million ₹)

Primary market	Secondary market	NPV, $M = 1$	NPV, $M = 0.8$	NPV, $M = 0.6$	NPV, $M = 0.4$	NPV, $M = 0.2$	NPV, $M = 0$
1	0	663,680	663,830	664,520	665,410	666,120	667,390
0.8	0.2	637,040	637,180	637,880	638,760	639,470	640,750
0.6	0.4	610,390	610,540	611,230	612,120	612,830	614,110
0.4	0.6	583,750	583,900	584,590	585,480	586,190	587,460
0.2	0.8	557,110	557,250	557,940	558,830	559,540	560,820
0	1	530,460	530,610	531,300	532,190	532,900	534,170

Conclusion

The main contribution of this paper is that it shows development and techno-economic-environmental analysis of novel CLSC MINLP model which is implemented for the first time in smart of Pune in developing nation India. Appropriate site location and connection between ten echelons are shown indicating model's capacity to maximize profit NPV even with varying demand and the dynamic network structure. The detailed cost calculation of operating and capital expenditure in CLSC favors indigenous production even though foreign imports are not completely zero as good indicator for Indian economy in mobile manufacturing based on national policy on electronic (NPE). The environmental aspects implementing government's

5R principle mitigate 98.48% of e-waste pollution with positive profit NPV of ₹. 624,558 encouraging investments in e-waste recycling. Also, the effect of primary and secondary market (varying between 0 and 1 ratio) with recycle ratio (varying between 0 and 100%) needs to be balanced in tandem with profit NPV.

Nomenclature

It is highly advised to go through this nomenclature section with explanation to subscripts and relate with model Eqs. 18.1–18.57. Set of all locations are shown by g with subsets of supplier su_g, manufacturer mu_g, distributor du_g, customer cu_g, collection center co_g, repair site re_g, dismantling site ds_g recycling unit rc_g, waste disposal cite wd_g importer im_g along with alias set gc. Sets of all feed F and product P are composed of four raw material f and two product p, respectively. Each raw material is associated with specific product as $qfp_{f,p}$. Set of transport mode l is used along with time period t. Scalar values are directly used in the equations, which involves total time period N, discount factor α and i annual interest rate, Φ_t as Goods and Service Tax (GST). Market types of primary and secondary are pri and sec, respt., with SP as selling price. M shows fraction of flow split from customer to collection site. C represents costing, and CM represents maintenance cost followed by the respective supply chain layer name. U shows unit cost; UIC shows unit inventory cost, and UTC shows unit transport cost followed by their layer name. d shows the distance followed by from to destination supply chain site. f shows the emission factor followed by specific site and also followed by from to destination supply chain site for transport. Q shows the transfer quantity between two layers followed by connecting layer and explicitly indicating minimum Min and maximum quantity Max range. Y indicates binary integers followed site location. GovtInct shows government incentives. Tran shows transport. Invt shows inventory. Prodn shows production. Infra and Infr$M$ show infrastructure and infrastructure maintenance.

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Chapter 19
Strategic Analysis of a Dual Channel Green Supply Chain with Return-Refund Facility

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Abstract Due to the advancement of online marketing, many manufacturers have started to provide a return policy with refund agreements. This paper concerns return policy in a dual-channel supply green chain, wherein customers can buy the products through a traditional retail channel or direct online channel. Under sustainable improvement, we have developed the dual-channel supply chain system with a return strategy including refund via direct online channel. Market demand is dependent on product sales price, green label, and refund amount. Firstly, the supply chain members target to optimize their decision variables under a centralized decision model. Secondly, the entire supply chain members make their decision individually to maximize the overall profit using the non-cooperative Stackelberg game approach. The prime objectives of the paper are to find out the optimal sales price, wholesale price, green label, and refund price so that the profit of the supply chain will be maximized. By solving the game model, we compare the optimal decision under both scenarios and implement sensitivity observation, which helps to reflect the influence of critical parameters.

Keywords Supply chain · Dual channel · Greening decision · Sales price · Centralized · Decentralized

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Introduction

Environment compatibility is a critical aspect of current supply chain management. In the past, unregulated utilization of natural resources and carelessness for the ecosystem positioned are a very difficult situation. The production of green products has been recognized as an essential element in achieving economic performance and environmental sustainability. Green item configuration was initially proposed in Navinchandra [13] as a way to increase an object's compliance with the ecosystem without sacrificing its utility or desirability. Manufacturers provide green commodities and services to their clients in a sustainable supply chain and preserve the environment Zhou et al. [21]. Many companies nowadays, like Walmart, Pepsi, and Adidas, have based their supply chain strategies rely on environmental and greening awareness. The issue of green SCM is growing rapidly among scientists, and it is primarily embedded in each company's important and efficient tasks Mondal and Giri [11]. Srivastava [16] provides a comprehensive overview of the literature on green SCM, which includes a wide range of assessments from supply chain participants' ecological efforts.

Furthermore, businesses favor web-based sales since they are closer to their clients, which benefits the business by lowering costs and expanding share in the market Wu and Wu [19]. As a result, the supply chain works as a dual channel, with traditional e-channels and retail channels. In the dual-channel scenario, the manufacturer sells directly to end customers through an internet website or showrooms like Apple India's Imagine shops, in addition to traditional retail channels. A dual-channel impact on the supply chain is more vital to the manufacturer in order to achieve higher profitability. In a dual-channel distribution chain, an optimum pricing balance is appropriate; nevertheless, the retail market has long been more advantageous than the internet platform Li et al. [10]. In e-marketing, a game theoretic method is required between a manufacturer and a retailer to improve the manufacturer's benefit by lowering the degree of unprofitable price [2]. A few years ago, growing concern about environmental issues prompted many researchers to examine how to express the greenness of things in supply chain. The existence of dual channel is dependent on the loyalty of purchasers and the expense of greening [3, 9, 14]. Ghosh et al. [7] have developed a green supply chain model with single retailer and multi-retailer under the effects of different payment strategies. The selection and evaluation of third-party logistics provider under the condition of sustainability has been explored by [15].

The return policy is one of the many aspects of the dual-channel supply chain structure that has been identified as a key component in helping businesses to improve their profitability in a competitive industry. Consumers are offered legal rights to return their ordered product under a return policy if their needs are not being met for different reasons [12]. Taleizadeh et al. [17] introduced a two-layer supply chain and implemented a return policy from both indirect and direct channel. Yoo [20] investigated the effects of refund issues on the equilibrium decisions of supply chain with a manufacturer and a retailer under Stackelberg game. The ecological compatibility level model and cooperative pricing were provided by Giri and Bardhan [8]. Das et al. [5] explored the optimal ordering and pricing policy with price discount structure in a two-layer supply chain.

Several researchers have examined the competitiveness and potential coordination tactics between dual channels that are held by the same retailer or by distinct businesses. In a manufacturer-retailer two-layer systems, the problem of client returns has also been intensively investigated. To the best of my knowledge, there is no work related to dual-channel return strategy considering the green product. A dual-channel supply chain must use effective pricing strategies to stay ahead of the market, allowing them to adapt to its consumers' ecological and financial concerns at the same time. It is prompted by the premise that buyers can return unwanted products purchased online. The focus of this article is on a dual-channel supply chain system in which the retailer and the manufacturer offer a green item to clients. The dual-channel supply chain advances yielding returns through channels, i.e., the consumers are eligible for a refund through the manufacturer's channel and can be returned acquirement products to the manufacturer directly. In this scenario, the Stackelberg game is used to study the supply chain interactions, and optimum pricing decisions, greening decisions, and refund amount in the dual-channel system are investigated for a single product targeting maximizing the benefits. In this contribution, (1) the mathematical formulation of dual-channel supply chain, (2) return-refund strategy in online marketing, and (3) centralized and decentralized decomposition have been explored in a single platform. In the present work, the self-price elasticity, cross-price elasticity, and cost coefficient of the greenness level, effectiveness coefficient of refund amount have been included with distinct values. The prime objective of this research is to optimize the retail and online selling price, wholesale price, product greenness level, and return amount so that the overall profit of the chain member and the whole system profit will be maximized.

The following is how the rest of the paper is organized: Section "Problem Statement and Mathematical Model Formulation" shows the problem formulation along with the notations utilized throughout the paper. Section "Model Analysis" is concerned with the model's solution approach and analysis. In Section "Numerical Illustration", the numerical illustrations have been presented. Finally, the conclusion and future approach are offered with some future research direction "Conclusion".

Problem Statement and Mathematical Model Formulation

The focus of this research is on a supply chain with a single manufacturer and retailer offering a green goods. The manufacturer develops an item with a certain greenness level and, if necessary, can reduce or enhance that level. We assume the manufacturer distributes the goods through two different channels: traditional retail and online. When consumers receive the goods from the online channel, one of two things can happen: (a) consumers are satisfied with the purchased item, (b) consumers are not satisfied with the purchased item and choose to send the item back requesting a refund. The model's schematic diagram is displayed in Fig. 19.1. The demand at each channel is influenced by the self-price, cross-price, greening

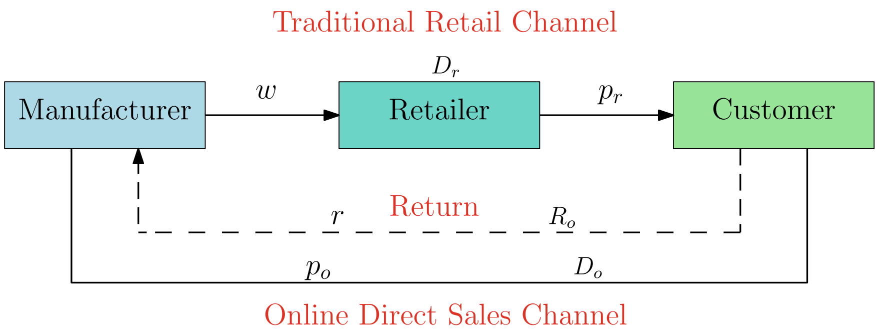


Fig. 19.1 Supply chain framework

level, and the refund amount at the online channel. From Fig. 19.1, at the start of the sales season, consumer demand is D_o on the online channel, and retailer demand is D_r. The manufacturer then produces the required quantity in both channels and delivers the product to the store at a wholesale price w and straight to the consumers at a sales price p_o through the online platform. The client can then purchase things from the offline store at a selling price p_r. Online purchaser returned the unsatisfied product to the manufacturer at a return rate R_o and got a refund amount r.

The notations listed below are employed.

Notations	
c	Manufacturing cost per unit
w	wholesale cost per unit
g	Greenness level
r	Refund amount of returned product
R_o	Return rate
e	Cost coefficient of greenness level
α	Loyalty degree of consumer to the retail channel
$1-\alpha$	Loyalty degree of consumer to the online channel
a	Market potential
D_r	Retail channel demand
D_o	Demand at online channel
p_r	Sales price through retail channel
p_o	Sales price through online channel
b_1	Self-price elasticity parameter
b_2	Cross-price elasticity parameter
v	Effectiveness coefficient of demand in refund amount
ϕ_1	The return quantity factor
ϕ_2	The coefficient of refund amount with respect to return rate
ϕ_3	The coefficient of greenness level with respect to return rate
γ	Expansion effectiveness coefficient of greenness level
Π_M	Total profit of manufacturer
Π_R	Total profit of retailer
Π_SC	Total profit of supply chain

The basic form of our demand is followed the study of [6, 10]. As a result, in this study, the demand functions for the retail channel and direct internet channel are characterized as

$$\begin{cases} D_{\mathrm{r}}(p_{\mathrm{r}}, p_{\mathrm{o}}, g, r) = \alpha a - b_1 p_{\mathrm{r}} + b_2 p_{\mathrm{o}} + \gamma g & (19.1) \\ D_{\mathrm{o}}(p_{\mathrm{r}}, p_{\mathrm{o}}, g, r) = (1-\alpha)a - b_1 p_{\mathrm{o}} + b_2 p_{\mathrm{r}} + \gamma g + vr & (19.2) \end{cases}$$

From Eqs. 19.1 and 19.2, demand is affected inversely by each channel's sales price while also having a direct link with the sales price of the other channel. Also, the greening level of the goods is direct relationship with the demand at both the channels but refund amount of returned item has only influences the online channel demand.

After acquiring an online-purchased goods, the consumer must decide whether pay to keep or return it relying on their refund amount, greening level. The return rate of the online-purchased products is

$$R_{\mathrm{o}}(g, r) = \phi_1 + \phi_2 r - \phi_3 g \tag{19.3}$$

The profit function of the retailer is

$$\Pi_{\mathrm{R}} = (p_{\mathrm{r}} - w)\left\{\alpha a - b_1 p_{\mathrm{r}} + b_2 p_{\mathrm{o}} + \gamma g\right\} \tag{19.4}$$

The profit function of a retailer is represented by Eq. (19.4), where the 1st term signifies the benefit of selling each item and the 2nd term indicates retail channel demand.

Similarly, the profit function of the manufacturer is

$$\begin{aligned} \Pi_{\mathrm{M}} = (w - c)&\left\{\alpha a - b_1 p_{\mathrm{r}} + b_2 p_{\mathrm{o}} + \gamma g\right\} \\ &+ (p_{\mathrm{o}} - c)\left\{(1-\alpha)a - b_1 p_{\mathrm{o}} + b_2 p_{\mathrm{r}} + \gamma g + vr\right\} \\ &- r\left\{\phi_1 + \phi_2 r - \phi_3 g\right\} - e\frac{g^2}{2} \end{aligned} \tag{19.5}$$

The manufacturer benefit function is represented by Eq. (19.5), where the first term represents the profit obtained from supplying the things to the retailer, and the second term represents the revenue generated from selling the items through its own direct online channel. The 3rd term denotes total refund amount of returned products; the 4th term denotes the investment cost against greening items.

Model Analysis

This section analyzes the integration of a dual-channel supply chain system under the centralized and decentralized scenario. In the decentralized scenario, manufacturer-leader Stackelberg's game has been used to develop the model. To attain the goal, we have addressed the following discussions and propositions.

Centralized Scenario

In this scenario, channel members are ready to collaborate and make decisions to maximize total supply chain profit. Both the offline and online channels of the proposed model work together to return items and develop return policies to maximize overall profit. The centralized profit is evaluated by addressing the individual profit of the manufacturer and retailer. The optimization problem under the centralized policy is formulated as follows:

$$\begin{cases} \text{Max } \Pi_{\text{SC}}^{C} \\ \text{subject to constraints} \\ p_{\text{r}}, p_{\text{o}} > w > c > 0, \quad g > 0, \quad p_{\text{o}} > r > 0. \end{cases} \tag{19.6}$$

$$\text{Here,} \quad \Pi_{\text{SC}}^{C} = (p_{\text{r}} - c)\Big\{\alpha a - b_1 p_{\text{r}} + b_2 p_{\text{o}} + \gamma g\Big\} + (p_o - c)\Big\{(1-\alpha)a - b_1 p_o + b_2 p_r + \gamma g + vr\Big\} - r\Big\{\phi_1 + \phi_2 r - \phi_3 g\Big\} - e\frac{g^2}{2} \tag{19.7}$$

In Eq. (19.7), the first and second term indicates the profit through the retail and online channels. The third term indicates the refund amount of returned products, and the fourth term denotes the cost associated with the green level of the products.

Proposition 1 Π_{SC}^{C} *in (19.7) is concave with respect to* p_r, p_o, w, r. *The optimal values come from*

1. $-2b_1 p_{\text{r}} + 2b_2 p_{\text{o}} + \gamma g + \alpha a + b_1 c - b_2 c = 0$
2. $2b_2 p_{\text{r}} - 2b_1 p_{\text{o}} + \gamma g + vr + a - \alpha a + b_1 c - b_2 c = 0$
3. $\gamma p_{\text{r}} + \gamma p_{\text{o}} - eg + \phi_3 r - 2\gamma c = 0$
4. $v p_{\text{o}} + \phi_3 g - 2\phi_2 r - \phi_1 - vc = 0$

Proof At first differentiate Π_{SC}^{C} with respect to $p_{\text{r}}, p_{\text{o}}, w, r$ and equating zero, we get the above linear system of equation. The solution that comes from the above system of equation will be optimal if the principle minor of the corresponding Hessian matrix $(H(\Pi_{SC}^{C}))$ alternate their sign starting with negative. The Hessian matrix of the corresponding profit function Π_{SC}^{C} has been calculated by second-order differentiation of Π_{SC}^{C} with respect to $p_{\mathrm{r}}, p_{\mathrm{o}}, w, r$. Therefore, we get,

$$H(\Pi_{\mathrm{SC}}^{C}) = \begin{bmatrix} \frac{\partial^2 \Pi_{\mathrm{SC}}^{C}}{\partial p_{\mathrm{r}}^2} & \frac{\partial^2 \Pi_{SC}^{C}}{\partial p_{\mathrm{r}} \partial p_{\mathrm{o}}} & \frac{\partial^2 \Pi_{\mathrm{SC}}^{C}}{\partial p_{\mathrm{r}} \partial g} & \frac{\partial^2 \Pi_{\mathrm{SC}}^{C}}{\partial p_{\mathrm{r}} \partial r} \\ \frac{\partial^2 \Pi_{\mathrm{SC}}^{C}}{\partial p_{\mathrm{o}} \partial p_{r}} & \frac{\partial^2 \Pi_{SC}^{C}}{\partial p_{\mathrm{o}}^2} & \frac{\partial^2 \Pi_{\mathrm{SC}}^{C}}{\partial p_{\mathrm{o}} \partial g} & \frac{\partial^2 \Pi_{\mathrm{SC}}^{C}}{\partial p_{\mathrm{o}} \partial r} \\ \frac{\partial^2 \Pi_{\mathrm{SC}}^{C}}{\partial g \partial p_{r}} & \frac{\partial^2 \Pi_{SC}^{C}}{\partial g \partial p_{o}} & \frac{\partial^2 \Pi_{\mathrm{SC}}^{C}}{\partial g^2} & \frac{\partial^2 \Pi_{\mathrm{SC}}^{C}}{\partial g \partial r} \\ \frac{\partial^2 \Pi_{\mathrm{SC}}^{C}}{\partial r \partial p_{\mathrm{r}}} & \frac{\partial^2 \Pi_{SC}^{C}}{\partial r \partial p_{\mathrm{o}}} & \frac{\partial^2 \Pi_{\mathrm{SC}}^{C}}{\partial r \partial w} & \frac{\partial^2 \Pi_{\mathrm{SC}}^{C}}{\partial r^2} \end{bmatrix} = \begin{bmatrix} -2b_1 & 2b_2 & \gamma & 0 \\ 2b_2 & -2b_1 & \gamma & v \\ \gamma & \gamma & -e & \phi_3 \\ 0 & v & \phi_3 & -2\phi_2 \end{bmatrix}$$

Therefore, the determinant of the principle minors is as follows;

$\det[H_{1*1}^{C}] = -2b_1 < 0$

$\det[H_{2*2}^{C}] = 4\left(b_1^2 - b_2^2\right) > 0$ since, $b_1 > b_2$.

$\det[H_{3*3}^{C}] = -4\left(b_1 + b_2\right)\left[e\left(b_1 - b_2\right) - \gamma\right] < 0$ if $e\left(b_1 - b_2\right) > \gamma$ holds.

$\det[H_{4*4}^{C}] = 2\left(b_1^2 + b_2^2\right)\left(2e\phi_2 - \phi_3^2\right) - 2\gamma\left(b_1 + b_2\right)\left(2\gamma\phi_2 + v\phi_3\right) \quad -2vb_1\left(\gamma\phi_3 + ev\right) - 2\gamma b_2\left(2\gamma\phi_2 + v\phi_3\right) - \gamma^2\left(4b_1\phi_2 - v^2\right) > 0$ if $2\left(b_1^2 + b_2^2\right)\left(2e\phi_2 - \phi_3^2\right) + \gamma^2\left(v^2 - 4b_1\phi_2\right) > 2\gamma\left(b_1 + b_2\right)\left(2\gamma\phi_2 + v\phi_3\right) + 2vb_1\left(\gamma\phi_3 + ev\right) + 2\gamma b_2\left(2\gamma\phi_2 + v\phi_3\right)$ holds.

Decentralized Scenario

In this scenario, the channel members have made their own choices independently to optimize decision variables and maximize overall profitability. Here, the manufacturer acts as a leader who optimizes first the online sales price, wholesale price, greenness level, and refund amount on the basis of the best reaction of the retailer. Following the manufacturer's decision, the retailer tries to balance his sales price at the retail channel. In the proposed Stackelberg game model, first, we differentiate the retailer profit function with respect to the offline sales price and utilizing these decision variables manufacturer takes his decision.

The manufacturer and retailer profit function is shown as:

$$\left\{ \begin{aligned} \Pi_{\mathrm{M}}^{DC} &= (w - c)\left\{\alpha a - b_1 p_{\mathrm{r}} + b_2 p_{\mathrm{o}} + \gamma g\right\} \\ &\quad +(p_{\mathrm{o}} - c)\left\{(1 - \alpha)a - b_1 p_{\mathrm{o}} + b_2 p_{\mathrm{r}} + \gamma g + vr\right\} \\ &\quad -r\left\{\phi_1 + \phi_2 r - \phi_3 g\right\} - e\frac{g^2}{2} \qquad (19.8) \\ \Pi_{\mathrm{R}}^{DC} &= (p_{\mathrm{r}} - w)\left\{\alpha a - b_1 p_{\mathrm{r}} + b_2 p_{\mathrm{o}} + \gamma g + vr\right\} \qquad (19.9) \end{aligned} \right.$$

where $p_{\mathrm{r}},\ p_{\mathrm{o}} > w > c > 0, \quad g > 0, \quad p_{\mathrm{o}} > r > 0.$

Proposition 2 *In the decentralized scenario, retailer's profit Π_R^{DC} in eqn. (19.9) is concave on p_r, given by*

$$p_{\rm r} = \frac{b_2 p_{\rm o} + b_1 w + \gamma g + \alpha a}{2b_1} \tag{19.10}$$

Proof Differentiating $\Pi_{\rm R}^{DC}$ from Eq. (19.9) with respect to retailer decision variable $p_{\rm r}$ and equating $\frac{\partial \Pi_{\rm R}^{DC}}{\partial p_{\rm r}} = 0$, we get (19.10). Clearly, $\frac{\partial^2 \Pi_{\rm R}^{DC}}{\partial p_{\rm r}^2} = -2b_1 < 0$. Therefore, $\Pi_{\rm R}^{DC}$ is concave in $p_{\rm r}$.

Proposition 3 *In the decentralized scenario, manufacturer's profit Π_M^{DC} in Eq. (19.8) is concave on p_o, w, g and r.*

Proof Replacing $p_{\rm r}$ in Eq. (19.8), the modified form of the manufacturer's profit function is written as follows,

$$\begin{aligned}\Pi_{\rm M}^{DC} =& \frac{1}{2}(w-c)\Big(\alpha a + b_2 p_{\rm o} + \gamma g - b_1 w\Big)\\ &+ (p_{\rm o} - c)\Big\{a - (1-K_1)\alpha a - (b_1 - b_2 K_1)p_{\rm o}\\ &+ (1+K_1)\gamma g + vr + K_1 b_1 w\Big\} - r\Big\{\phi_1 + \phi_2 r - \phi_3 g\Big\} - e\frac{g^2}{2}\end{aligned} \tag{19.11}$$

where $K_1 = \frac{b_2}{2b_1}$.

Now, in Eq. (19.11), $\Pi_{\rm M}^{DC}$ is a function of decision variable $p_{\rm o}$, w and g. Differentiating (19.11) with respect to $p_{\rm o}$, w and g,

$$\begin{cases} \frac{\partial \Pi_{\rm M}^{DC}}{\partial p_{\rm o}} = -2(b_1 - K_1 b_2)p_{\rm o} + (k_1 b_1 + \frac{b_2}{2})w + (1+K_1)\gamma g + vr \\ \qquad -(1-K_1)\alpha a - a - c(b_1 - K_1 b_2) + \frac{cb_2}{2} & (19.12)\\ \frac{\partial \Pi_{\rm M}^{DC}}{\partial w} = (K_1 b_1 + \frac{b_2}{2})p_{\rm o} - b_1 w + \frac{\gamma}{2}g - K_1 b_1 c + \frac{b_1 c}{2} + \frac{\alpha a}{2} & (19.13)\\ \frac{\partial \Pi_{\rm M}^{DC}}{\partial g} = (1+K_1)\gamma p_{\rm o} + \frac{\gamma}{2}w - eg + \phi_3 r - \gamma c(1+K_1) - \frac{\gamma c}{2} & (19.14)\\ \frac{\partial \Pi_{\rm M}^{DC}}{\partial r} = vp_{\rm o} + \phi_3 g - 2\phi_2 r - \phi_1 & (19.15) \end{cases}$$

The necessary condition for optimal online sales price ($p_{\rm o}$), wholesale price (w), green level (g), and return price (r) is obtained from $\frac{\partial \Pi_{\rm M}^{DC}}{\partial p_{\rm o}} = 0$, $\frac{\partial \Pi_{\rm M}^{DC}}{\partial w} = 0$, $\frac{\partial \Pi_{\rm M}^{DC}}{\partial g} = 0$ and $\frac{\partial \Pi_{\rm M}^{DC}}{\partial r} = 0$. Rearranging the first-order derivative, we get the following system of equations,

$$\begin{cases} A_1 p_o + B_1 w + C_1 g + D_1 r = E_1 & (19.16) \\ A_2 p_o + B_2 w + C_2 g + D_2 r = E_2 & (19.17) \\ A_3 p_o + B_3 w + C_3 g + D_3 r = E_3 & (19.18) \\ A_4 p_o + B_4 w + C_4 g + D_4 r = E_4 & (19.19) \end{cases}$$

where $A_1 = -2(b_1 - K_1 b_2)$, $B_1 = (k_1 b_1 + \frac{b_2}{2})$, $C_1 = (1 + K_1)\gamma$, $D_1 = v$, $E_1 = (1 - K_1)\alpha a - a - c(b_1 - K_1 b_2) + \frac{cb_2}{2}$, $A_2 = (K_1 b_1 + \frac{b_2}{2})$, $B_2 = -b_1$, $C_2 = \frac{\gamma}{2}$, $D_2 = 0$, $E_2 = K_1 b_1 c - \frac{b_1 c}{2} - \frac{\alpha a}{2}$, $A_3 = (1 + K_1)\gamma$, $B_3 = \frac{\gamma}{2}$, $C_3 = -e$, $D_3 = \phi_3$, $E_3 = \gamma c(1 + K_1) + \frac{\gamma c}{2}$, $A_4 = 0$, $B_4 = v$, $C_4 = \phi_3$, $D_4 = -2\phi_2$, $E_4 = \phi_1 + vc$.

The above system of equation has unique optimal solution when $\Delta = \det \begin{pmatrix} A_1 & B_1 & C_1 & D_1 \\ A_2 & B_2 & C_2 & D_2 \\ A_3 & B_3 & C_3 & D_3 \\ A_4 & B_4 & C_4 & D_4 \end{pmatrix} \neq 0.$

Then the solution are $p_o = \frac{1}{\Delta} \det \begin{pmatrix} E_1 & B_1 & C_1 & D_1 \\ E_2 & B_2 & C_2 & D_2 \\ E_3 & B_3 & C_3 & D_3 \\ E_4 & B_4 & C_4 & D_4 \end{pmatrix}$,

$w = \frac{1}{\Delta} \det \begin{pmatrix} A_1 & E_1 & C_1 & D_1 \\ A_2 & E_2 & C_2 & D_2 \\ A_3 & E_3 & C_3 & D_3 \\ A_4 & E_4 & C_4 & D_4 \end{pmatrix}$,

$g = \frac{1}{\Delta} \det \begin{pmatrix} A_1 & B_1 & E_1 & D_1 \\ A_2 & B_2 & E_2 & D_2 \\ A_3 & B_3 & E_3 & D_3 \\ A_4 & B_4 & E_4 & D_4 \end{pmatrix}$, $r = \frac{1}{\Delta} \det \begin{pmatrix} A_1 & B_1 & C_1 & E_1 \\ A_2 & B_2 & C_2 & E_2 \\ A_3 & B_3 & C_3 & E_3 \\ A_4 & B_4 & C_4 & E_4 \end{pmatrix}$.

where all the components are described above. Therefore, the optimum sales price (p_o), wholesale price (w), green level (g), and return price (r) are the simplest form of the mentioned matrix. Due to the complexity, we have illustrated the results in numerically.

Numerical Illustration

In this section, to analyze the behavior of our suggested model, a numerical example with a sensitivity of some significant parameters has been investigated. The numerical datasheets have shown in Table 19.1 from previous literature of [1, 18]. The optimal solutions under the centralized and decentralized decision model are shown in Table 19.2. It is assumed that the manufacturer sells the product at a wholesale price $w = \$120$ for a unit item in the centralized model.

Table 19.1 Key parameters

Parameters	c	a	b_1	b_2	e	α	γ	v	ϕ_1	ϕ_2	ϕ_3
Values	50	700	1.8	0.15	5	0.5	1.0	0.9	0.2	0.5	0.18

Table 19.2 Optimal results

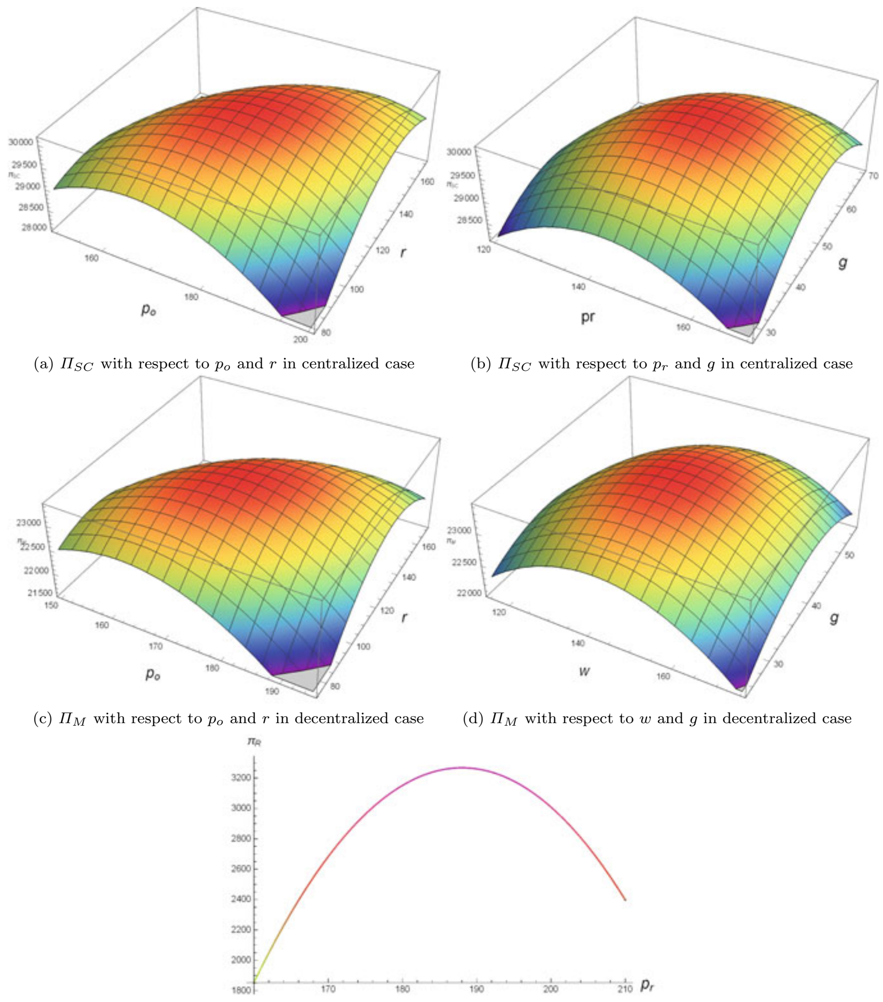
Datasets	p_r^*	p_o^*	w^*	r^*	g^*	Π_R	Π_M	Π_{SC}
Centralized policy	148.66	177.07	–	123.08	49.58	4544	25635	30180
Decentralized Policy	188.02	172.42	145.41	117.05	39.26	3267	23425	26693

From Table 19.2, it has been seen that the overall profitability of the supply chain system is higher in the centralized model compared to the decentralized model. The centralized model provides a higher green label of the products with a high amount of refund than the decentralized model. A comparison of the centralized and decentralized model results shows that offline retail channel sales price is higher, and online channel sales price is lower in decentralized model. It has been demonstrated that in the decentralized scenario, retail sales prices are substantially higher than in the centralized scenario. Still, the price of online sales in a centralized system is higher than in a decentralized system. The outcomes of the numerical results claim that centralized market decision is more beneficial for the overall supply chain than the decentralized decision model.

The concavity graph of the decision variables is shown Fig. 19.2a–e for both centralized and decentralized case. Figure 19.2a shows the concavity of Π_{SC}^C in (p_o, r), if the manufacturer sells the product at \$177.07 at through online channel, and refund amount is \$123.08, and the supply chain system achieves its highest profit 30, 180. Figure 19.2b shows the concavity of Π_{SC}^C in (p_r, g), if the retailer sells the product at a sales price \$148.66 and greening level is 49.58 units, and the supply chain system achieves its highest profit \$30, 180. Figure 19.2c shows the concavity of Π_M in (p_o, r) in decentralized case, if the online sales price is \$172.42 and refund amount is \$117.05, the manufacturer achieves its highest profit \$23, 425 in decentralized model. Figure 19.2d shows the concavity of Π_M in (w, g) in decentralized case, if wholesale price is \$145.41 and greening level is 39.26 units and the manufacturer achieves its highest profit \$23, 425 in decentralized model. Figure 19.2e shows the concavity of Π_R in (p_r); when \$188.02, the retailer achieves its highest profit \$3267.

Sensitivity Analysis and Managerial Implications

Sensitivity analysis of some key parameters $(\alpha, \gamma, \phi_2, \phi_3)$ is carried out to study its influence on the optimal results and profit depicted in Table 19.3 by changing the value from −30% to +30% and keeping other parameters constant.



(a) Π_{SC} with respect to p_o and r in centralized case

(b) Π_{SC} with respect to p_r and g in centralized case

(c) Π_M with respect to p_o and r in decentralized case

(d) Π_M with respect to w and g in decentralized case

(e) Π_R with respect to p_r in decentralized case

Fig. 19.2 Concavity curve with respect to decision variables

Impact of customers loyalty degree (α): With the increasing value of α, the benefit of the retailer increases, manufacturer benefit decreases, and total profit of the system reduces in both centralized and decentralized models. For the higher values of α; the greening level is low together with the refund amount of returned products seen in Table 19.3. Second, the consumer's online demand is smaller for lower limits of α. As a result, manufacturers lower their selling prices through the direct internet channel, while offline retailers raise their prices for a high-profit margin. Manufacturer profit is higher for lower α values. The retail channel is in high demand for high values of α. As a result, with greater values of α, the overall system profit is more significant.

Impact of demand sensitivity parameter for green level (γ): The consumer green preference will be induced for higher values of γ. From Table 19.3, in both centralized and decentralized model optimum sales price, greening level, refund amount, individual channel member profit, and overall supply chain profit increase with the increasing value of γ. For the responsibility of environmental protection, in recent days, many consumers have been gravitating toward green items, and they would like to spend more money on green goods than on regular goods. From this consumer's viewpoint, the manufacturer should produce eco-friendly green products according to the buyer's preference and minimize environmental damage. To motivate customers to choose things that are more ecologically friendly, the supply chain manager should concentrate on refund amount.

Impact of return rate elasticity parameters (ϕ_2, ϕ_3): From Table 19.3, with the increasing value of refund amount elasticity parameter (ϕ_2), both offline and online sales price of the item in decentralized and centralized model, wholesale price, greenness level of the product, refund amount, individual channel member profit, and overall supply chain profit are all decrease simultaneously. Managers of a dual-channel supply chain system should spend less on the refund of returned items when the sensitivity of demand to refund product amount is larger because increasing ϕ_2 increases the return rate. By employing this strategy, the demand will decrease, and online distributors will reduce their sales price greenness level.

An opposite phenomenon is observed with the increasing value of ϕ_3. The coefficient of greenness level negatively impacts the return rate of the product. But increasing ϕ_3 positively affects the decision variables, individual channel member profit, and overall system benefit. When consumers are highly sensitive to return rate, they will be more attentive to the greenness level of the product. Managers should look for a flexible return policy, such as increasing the greening level and refunding the amount of returned items.

Conclusion

This work investigates a mathematical model for designing a dual-channel supply chain system with a product return policy. The manufacturer makes a green product and distributes it to consumers through a traditional retailer and a direct internet channel. We have incorporated the return-refund strategy in the online platform to attract more customers. The customer demand in the retail channel is influenced by sales price, the product's green label, but in the online channel, demand also influences the return price of the products. This model mainly focuses on balancing pricing policy, greening decisions in the dual-channel structure of the supply chain. This study aims to maximize the supply chain profit by determining the optimum sales price, greening level, and refund amount for returned items.

The accompanying outcomes relating to the coordinating contracts are something we have noticed. Through integration (centralized model), the supply chain can gain

Table 19.3 Sensitivity analysis

Parametric changes		Centralized							Decentralized							
	%	p_r	p_o	r	g	Π_R	Π_M	Π_{SC}	p_r	p_o	w	r	g	Π_R	Π_M	Π_{SC}
α	−30	123.48	213.43	156.43	53.01	375	36,018	36393	150.23	210.27	121.27	152.32	46.00	1509	33273	34782
	−15	136.07	195.25	139.76	51.30	2140	30338	32479	169.15	191.34	133.34	134.68	42.63	2304	27714	30019
	0%	148.66	177.07	123.08	49.58	4544	25635	30180	188.02	172.42	145.41	117.05	39.26	3267	23425	26693
	+15	161.25	158.90	106.42	47.86	7588	21908	29496	206.91	153.50	157.48	99.41	35.89	4397	20405	24803
	+30	173.85	140.71	89.75	46.14	11271	19156	30428	225.80	134.58	169.55	81.77	32.52	5695	18653	24349
γ	−30	140.32	165.65	109.80	32.78	2950	24587	24538	178.70	163.61	138.95	106.84	26.64	2843	21766	24609
	−10	143.96	170.66	115.71	40.65	3618	25075	28694	182.80	167.52	141.80	111.44	32.62	3026	22501	25527
	0%	148.66	177.07	123.08	49.58	4544	25635	30180	188.02	172.42	145.41	117.05	39.26	3267	23425	26693
	+15	154.73	185.27	132.34	59.96	5843	26244	32088	194.57	178.54	149.95	123.90	46.75	3583	24581	28165
	+30	162.60	195.85	144.09	72.38	7701	26851	34553	202.80	186.20	155.65	132.34	55.38	4002	26028	30030
ϕ_2	−30	151.83	194.24	183.77	55.83	5146	27748	32894	191.76	188.65	148.36	174.58	45.00	3390	25877	29268
	−15	149.93	183.96	147.43	52.08	4783	26485	31269	189.52	178.93	146.60	140.14	41.56	3316	24409	27726
	0%	148.66	177.07	123.08	49.58	4544	25635	30180	188.02	172.42	145.41	117.05	39.26	3267	23425	26693
	+15	147.75	172.13	105.64	47.78	4375	25024	29400	186.94	167.75	144.56	100.48	37.60	3232	22719	25952
	+30	147.07	168.42	92.53	46.43	4249	24564	28813	186.12	164.24	143.92	88.03	36.36	3206	22189	25395
ϕ_3	−30	148.00	175.37	118.64	47.66	4411	25451	29863	187.18	171.00	144.81	113.42	37.54	3230	23186	26417
	−15	148.32	176.20	120.82	48.60	4476	25542	30018	187.60	171.70	145.11	115.20	38.38	3248	23303	26552
	0%	148.66	177.07	123.08	49.58	4544	25635	30180	188.02	172.42	145.41	117.05	39.26	3267	23425	26693
	+15	149.02	177.97	125.50	50.60	4616	25732	30348	188.45	173.18	145.72	118.98	40.16	3286	23552	26838
	+30	149.40	178.91	127.90	51.64	4690	25832	30523	188.91	173.97	146.05	121.00	41.09	3306	23683	26990

better benefits and reduce the negative effects of supply chain members' conflicts on the supply chain's performance. It is clear that the the individual profit of the manufacturer, retailer, and overall supply chain is higher than the decentralized scenario. In the centralized model, the item with the highest green level is always created, and the refund amount is also larger. Sensitivity analysis also shows the impact of some key parameters on optimum choices.

In future research, the present model can be extended by addressing a real case study with current industrial data. The proposed model can be expanded by considering stochastic demand. Other operational phenomena, such as retailer's inventory manufacturer's inventory [4], can be incorporated in future research more realistically.

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Chapter 20
Vector Variational-Like Inequalities on the Space of Real Square Matrices

Sandip Chatterjee, S. K. Mishra, and Sudipta Roy

Abstract In this paper, "Vector Variational-like Inequalities" and "Vector Optimization Problems" have been discussed over the space of real square matrices. Under the assumption of invexity, defined suitably in such space, the relationship between such problems has also been established.

Keywords Variational-like Inequalities · Vector Optimization Problem · Pseudoinvex function

Mathematics Subject Classification 49J52 · 58E17 · 58E35

Introduction

"Vector Variational Inequalities" (VVI) and "Vector Optimization Problems" (VOP) have been widely studied [1, 2, 6, 11–13] over the n-dimensional Euclidean space in the last few decades. Many valuable resources regarding vector optimization are available in [2]. It has been observed that many important optimization problems involve matrices as variables. Such problems arise from a wide range of diverse applications, e.g. image processing, graph optimization, nearest correlation matrix problem, etc. In this paper, "Vector Variational-like Inequalities" (VVLI) and VOP have been introduced over the space of real square matrices. The new class of VVLI problems defined on the space of real square matrices has been referred as "Matrix-Vector Variational-like Inequality Problems" (MVVLIP). Such class of problems

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is a generalized form of VVLI problems studied by Ruiz-Garzón et al. [13]. The new class of VOP defined on the space of real square matrices has been referred as "Matrix-Vector Optimization Problem" (MVOP).

Recently, researchers have found some applications of the theoretical developments of the concept of invexity and their generalizations. For example, Dinuzzo et al. [7] got a kernel function in machine learning which is neither convex nor quasiconvex but invex.

In this paper, the concept of invexity has also been suitably generalized on the space of real square matrices. It has been obtained that under the assumption of invexity, the solutions of the MVVLIP are the efficient points of the corresponding MVOP. Similar result has already been proved in the n-dimensional Euclidean space and hence validates the generalization proposed in this paper. Mishra et al. [12] considered weak VVI and established certain results related weak vector minimal points. In this paper, it has further been observed that under the assumption of pseudoinvexity, the weak MVVLIP and the corresponding weak MVOP admit the same set of solutions.

The overall contribution of the paper is the representation of VVLI and VOP in a completely newer space structure than what can be found in the existing literature. Interestingly, the widely discussed equality of solutions of such class of problems has been proved to be true in the proposed structure as well. In that sense, the work in this paper can be termed as an extension of many related existing works.

In Section "Matrix-Vector Variational-Like Inequality Problem", matrix derivative [10] and MVVLI have been discussed. In Section "Matrix-Vector Optimization Problem", MVOP has been defined with suitable example. The relationship between MVVLIP and MVOP has been established in Section "Solutions of MVVLIP and MVOP". Future direction of research in the proposed structure has been discussed briefly in the concluding section, i.e. Section "Conclusion".

Matrix-Vector Variational-Like Inequality Problem

Let us consider the following notations to be used in this paper:

$\mathbb{R}_t^{n\times n}$:= The set of all $n \times n$ matrices with elements as real valued functions of a real variable t.

$$M = \left\{ F : F = \begin{pmatrix} f_{11}(t) & f_{12}(t) & \dots & f_{1n}(t) \\ f_{21}(t) & f_{22}(t) & \dots & f_{2n}(t) \\ & \dots & & \\ f_{n1}(t) & f_{n2}(t) & \dots & f_{nn}(t) \end{pmatrix} \in \mathbb{R}_t^{n\times n}, t \in \mathbb{R}, f_{ij} : \mathbb{R} \to \mathbb{R} \right\}.$$

Let $A, B \in \mathbb{R}_t^{n\times n}$ the Frobenius inner product is defined as $\langle A, B\rangle_F = Tr(A^T B)$ and Frobenius norm is defined as $\|AB\|_F = \sqrt{Tr(A^T B)}$.

Definition 20.2.1 Let $F = (f_{st})$ be an $n \times n$ matrix function of a scalar t, then

$$\frac{\delta F(t)}{\delta t} = \begin{pmatrix} \frac{\partial f_{11}(t)}{\partial t} & \frac{\partial f_{12}(t)}{\partial t} & \dots & \frac{\partial f_{1n}(t)}{\partial t} \\ \frac{\partial f_{21}(t)}{\partial t} & \frac{\partial f_{22}(t)}{\partial t} & \dots & \frac{\partial f_{2n}(t)}{\partial t} \\ \dots \\ \frac{\partial f_{n1}(t)}{\partial t} & \frac{\partial f_{n2}(t)}{\partial t} & \dots & \frac{\partial f_{nn}(t)}{\partial t} \end{pmatrix}.$$

which has the same dimension as F.

Let M be a subset of $\mathbb{R}_t^{n\times n}$, $\mu : M \times M \to \mathbb{R}_t^{n\times n}$ be a matrix-valued function and $G : M \subset \mathbb{R}_t^{n\times n} \to \mathbb{R}_t^{n\times n}$ be differentiable. Let us consider the two definitions given below:

Definition 20.2.2 An MVVLIP is a problem that finds an $F(\bar{t}) \in M$, such that $\forall F(x) \in M$,

$$\left\langle \frac{\delta G(F(\bar{t}))}{\delta \bar{t}}, \mu(F(x), F(\bar{t})) \right\rangle_F > 0. \qquad \text{(MVVLIP)}$$

Definition 20.2.3 A weak MVVLIP is a problem that finds an $F(\bar{t}) \in M$ such that, $\forall F(x) \in M$,

$$\left\langle \frac{\delta G(F(\bar{t}))}{\delta \bar{t}}, \mu(F(x), F(\bar{t})) \right\rangle_F \geq 0. \qquad \text{(WMVVLIP)}$$

Definition 20.2.4 A "Matrix-Vector Variational Inequality Problem" (MVVIP) is defined as a problem that finds an $F(\bar{t}) \in M$ such that, $\forall F(x) \in M$,

$$\left\langle \frac{\delta G(F(\bar{t}))}{\delta \bar{t}}, F(x) - F(\bar{t}) \right\rangle_F > 0. \qquad \text{(MVVIP)}$$

The weak MVVIP can also be defined in a similar way by weakening the inequality.

Matrix-Vector Optimization Problem

The following definitions are motivated from the related concepts discussed in many existing literatures which can collectively be found in [2, 3].

Definition 20.3.1 Let M be an open subset of $\mathbb{R}_t^{n\times n}$ and $G : M \subset \mathbb{R}_t^{n\times n} \to \mathbb{R}_t^{n\times n}$ be differentiable. A matrix $F(\bar{t})$ is called "efficient" if

$$\|G(F(x))\|_F - \|G(F(\bar{t}))\|_F > 0, \forall F(x) \in M.$$

$E(G, M)$ denotes the set of all such "efficient" matrices.

Given M and $G : M \subset \mathbb{R}_t^{n\times n} \to \mathbb{R}_t^{n\times n}$, think about finding $E(G, M)$ for the following MVOP:

$$\begin{aligned}&\text{Minimize } G(F(t)),\\&\text{subject to } F(t) \in M.\end{aligned}$$

Definition 20.3.2 Consider M to be an open subset of $\mathbb{R}_t^{n\times n}$ and $G : M \subset \mathbb{R}_t^{n\times n} \to \mathbb{R}_t^{n\times n}$ differentiable. A matrix $F(\bar{t})$ is defined as "weakly efficient" if

$$\|G(F(x))\|_F - \|G(F(\bar{t}))\|_F \geq 0, \forall F(x) \in M.$$

The set of "weakly efficient" points is written as $WE(G, M)$.

Assume M to be an open subset of $\mathbb{R}_t^{n\times n}$ and $G : M \subset \mathbb{R}_t^{n\times n} \to \mathbb{R}_t^{n\times n}$ be differentiable. Now, look into the problem of finding $WE(G, M)$ for the following WMVOP:

$$\begin{aligned}&\text{Minimize } G(F(t)),\\&\text{subject to } F(t) \in M.\end{aligned}$$

Example 20.3.1 Let us discuss the problem given below:

$$\begin{aligned}&\text{Minimize } G(F(t)),\\&\text{subject to } F(t) \in M.\end{aligned}$$

where $M = \left\{F(t) : F(t) = \begin{pmatrix} t & -1 \\ 0 & t \end{pmatrix}, t \in (-\infty, -1] \cup [1, \infty)\right\}$. $G : M \to \mathbb{R}_t^{n\times n}$ is defined as $G(F(t)) = -F(t) = \begin{pmatrix} -t & 1 \\ 0 & -t \end{pmatrix}, t \in (-\infty, -1] \cup [1, \infty)$.

It is clear that for every $F(t) \in M$, $\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$ is a "weakly efficient solution" to the corresponding WMVOP.

According to the following example, an "efficient solution" of MVOP is not necessarily a solution of MVVIP.

Example 20.3.2

$$\begin{aligned}&\text{Minimize } G(F(t)),\\&\text{subject to } F(t) \in M.\end{aligned}$$

where $M = \left\{ F(t) : F(t) = \begin{pmatrix} 1 & t & t^3 \\ t & 2 & t^2 \\ t^2 & t & 1 \end{pmatrix}, t \in [-1, 1] \right\}$. $G : M \to \mathbb{R}_t^{n\times n}$ is defined as $G(F(t)) = (F(t))^T = \begin{pmatrix} 1 & t & t^2 \\ t & 2 & t \\ t^3 & t^2 & 1 \end{pmatrix}, t \in [-1, 1]$.

Obviously, for every $F(t) \in M$, $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ is an efficient solution.

Now, $\frac{\delta G(F(t))}{\delta t} = \begin{pmatrix} 0 & 1 & 2t \\ 1 & 0 & 1 \\ 3t^2 & 2t & 0 \end{pmatrix}$ and at $t = 0$, $\langle \frac{\delta G(F(t))}{\delta t}, F(x) - F(t) \rangle_F = 2x + x^2$.

When x is in $[-1, 0]$, $F(t) = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ is not a solution to the corresponding MVVIP.

Definition 20.3.3 Let M be an open subset of $\mathbb{R}_t^{n\times n}$ and $G : M \subset \mathbb{R}_t^{n\times n} \to \mathbb{R}_t^{n\times n}$ be a differentiable function. Then, G is defined as *invex* if $\exists\ \mu : M \times M \to \mathbb{R}_t^{n\times n}$ such that,

$$\|G(F(x))\|_F - \|G(F(t))\|_F \geq \left\langle \frac{\delta G(F(t))}{\delta t}, \mu(F(x), F(t)) \right\rangle_F, \ \forall F(t), F(x) \in M.$$

Furthermore, G is defined as *strictly invex* if $\exists\ \mu : M \times M \to \mathbb{R}_t^{n\times n}$ such that,

$$\|G(F(x))\|_F - \|G(F(t))\|_F > \left\langle \frac{\delta G(F(t))}{\delta t}, \mu(F(x), F(t)) \right\rangle_F, \quad \forall F(t), F(x) \in M,\ F(t) \neq F(x)$$

and G is defined as *pseudoinvex* if $\exists\ \mu : M \times M \to \mathbb{R}_t^{n\times n}$ such that,

$$\|G(F(x))\|_F - \|G(F(t))\|_F < 0 \Rightarrow \left\langle \frac{\delta G(F(t))}{\delta t}, \mu(F(x), F(t)) \right\rangle_F < 0, \quad \forall F(t), F(x) \in M.$$

Solutions of MVVLIP and MVOP

In this section, we shall extend the results of [6] to the case of functions on the space of real square matrices.

Theorem 20.4.1 Let M be an open set and $G : M \subset \mathbb{R}_t^{n\times n} \to \mathbb{R}_t^{n\times n}$ differentiable on M. If G is invex with respect to μ and $F(\bar{t})$ solves (MVVLIP) with respect to same μ, then $F(\bar{t})$ is an efficient solution to (MVOP).

Proof Let us assume the contrary. Let $F(\bar{t})$ is not an efficient point to (MVOP) which implies that $\exists F(x) \in M$, such that,

$$\|G(F(x))\|_F - \|G(F(\bar{t}))\|_F \le 0. \tag{20.1}$$

Since G is invex with respect to the same μ, (20.1) implies that there exist $F(x) \in M$ such that,

$$\left\langle \frac{\delta G(F(t))}{\delta t}, \mu(F(x), F(t)) \right\rangle_F \le 0. \tag{20.2}$$

The above inequality is a clear contradiction that proves the theorem. □

In the following example, it has been shown that a solution of the (MVOP) will be the solution of (MVVLIP) with respect to μ though G is not invex with respect same to μ.

Example 20.4.1 Let us think about the MVOP given below:

$$\begin{aligned}&\text{Minimize } G(F(x)),\\ &\text{subject to } F(x) \in M.\end{aligned}$$

where $M = \left\{ F(t) : F(t) = \begin{pmatrix} t & t^2 \\ 1 & t \end{pmatrix}, t \in \mathbb{N} \right\}$. $G : M \to \mathbb{R}_t^{n\times n}$ is defined as $G(F(t)) = (F(t))^T$.

It is clear that for every $F(t) \in M$, $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ is an "efficient solution".

Now, $\frac{\delta G(F(t))}{\delta t} = \begin{pmatrix} 1 & 0 \\ 2t & 1 \end{pmatrix}$ and at $t = 1$, $\langle \frac{\delta G(F(t))}{\delta t}, F(x) + F(t) \rangle_F \ge 0$.

Hence, $F(t) = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ is a solution to the corresponding MVVLIP but G is not invex with respect to same $\mu = F(x) + F(t)$.

The following theorem provides us with the conditions under which the solutions of WMVVLIP can be identified with the "weakly efficient points".

Theorem 20.4.2 If $F(\bar{t})$ is "weakly efficient" for (WMVOP) and G is invex then $F(\bar{t})$ solves (WMVVLIP). Moreover, if G is pseudoinvex then the converse is also true.

Proof Let $F(\bar{t})$ be the solution to (WMVOP). Then, $\nexists F(x) \in M$ such that,

$$\|G(F(x))\|_F - \|G(F(\bar{t}))\|_F < 0 \tag{20.3}$$

Now, since G is invex,

$$\left\langle \frac{\delta G(F(\bar{t}))}{\delta \bar{t}}, \mu(F(x), F(\bar{t})) \right\rangle_F \leq \|G(F(x))\|_F - \|G(F(\bar{t}))\|_F. \tag{20.4}$$

From (20.3) and (20.4), we see that $\nexists F(x) \in M$ such that $\langle \frac{\delta G(F(\bar{t}))}{\delta \bar{t}}, \mu(F(x), F(\bar{t})) \rangle_F <$ 0, i.e.

$$\left\langle \frac{\delta G(F(\bar{t}))}{\delta \bar{t}}, \mu(F(x), F(\bar{t})) \right\rangle_F \geq 0, \ \forall F(x) \in M \tag{20.5}$$

Conversely, if $F(\bar{t})$ is not an "weakly efficient point", then $\exists F(x) \in M$ such that,

$$\|G(F(x))\|_F - \|G(F(\bar{t}))\|_F < 0 \tag{20.6}$$

By the pseudoinvexity of F, (20.6) implies the existence $F(x) \in M$ such that,

$$\left\langle \frac{\delta G(F(\bar{t}))}{\delta \bar{t}}, \mu(F(x), F(\bar{t})) \right\rangle_F < 0 \tag{20.7}$$

which contradicts that $F(\bar{t})$ is a solution of WMVVLIP. □

Thus, if G is pseudoinvex, the (WMVVLIP) and the corresponding (WMVOP) admit the same set of solutions.

According to the following example, if G is pseudoinvex and if $F(t)$ is not a solution of (WMVOP), then it will not be a solution of (WMVVLIP).

Example 20.4.2 Let us consider the MVOP given below:

$$\begin{aligned} &\text{Minimize } G(F(x)), \\ &\text{subject to } F(x) \in M. \end{aligned}$$

where $M = \left\{ F(t) : F(t) = \begin{pmatrix} t & -1 \\ 0 & t \end{pmatrix}, t \in [0, 1] \right\}$.

$G : M \to \mathbb{R}_t^{n\times n}$ is defined as $G(F(t)) = -F(t)$ and for $x < t$; $x, t \in [0, 1]$, G is pseudoinvex with respect to $\mu(F(t), F(x)) = F(t) + F(x)$.

Now, $\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$ neither a solution of WMVOP nor a solution of WMVVLIP with respect to same μ.

The following theorem tells us that under certain assumptions, a solution of the WMVOP will be a solution of the MVOP. It is to be noted that the converse is unconditionally true.

Theorem 20.4.3 Let $G : M \subset \mathbb{R}_t^{n\times n} \to \mathbb{R}_t^{n\times n}$ be differentiable and strictly invex with respect to μ. If $F(\bar{t})$ solves WMVOP, then $F(\bar{t})$ also solves MVOP.

Proof Let us assume that $F(\bar{t})$ solves WMVOP but does not solve MVOP. Then, $\exists F(x) \in M$ such that,

$$\|G(F(x))\|_F - \|G(F(\bar{t}))\|_F \leq 0.$$

Now, since G is strictly invex with respect to μ, we have,

$$0 \geq \|G(F(x))\|_F - \|G(F(\bar{t}))\|_F > \left\langle \frac{\delta G(F(\bar{t}))}{\delta \bar{t}}, \mu(F(x), F(\bar{t})) \right\rangle_F . \tag{20.8}$$

which implies that $\exists F(x) \in M$ such that $\langle \frac{\delta G(F(\bar{t}))}{\delta \bar{t}}, \mu(F(x), F(\bar{t})) \rangle_F < 0$ and consequently $F(\bar{t})$ does not solve the (WMVVLIP). According to Theorem 20.4.2, this in turn implies that $F(\bar{t})$ does not solve the (WMVOP), which is a contradiction. □

Conclusion

In this brief article, a widely discussed equality of solutions of two specific class of problems related to optimization has been discussed in a completely new kind of space structure which makes the work a novel one. There are many aspects of the proposed class of optimization problems which can further be discussed, for example, the concept of duality of the proposed MVOP. Matrices and optimization problems are pivotal in several areas like Machine Learning, Epidemiological Modelling, etc. Once the proposed representation is further developed, there may be a scope of application of the same in those areas.

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Chapter 21
Circularity Tactic Comport Sustainable Development—Review

Monika Vyas, Gunjan Yadav, and Sunil Pipleya

Abstract Circular economy and circular supply chain management is demand for preserving natural resources for future generation globally. This circular supply chain and its management emphasizes on 9R framework, which initially propagated from 3R framework with clear vision toward environmental education and awareness, resource utilization, waste of one can be input to other vision. Further circular economy promotes due to technology revolution with IR4, advancement due to growing virtual world with ICT along with digitization needs to lend toward any one or all three pillars of sustainability and sustainable development goals in organizations and industry. Social, environmental, and economical well-being is partially considered together with technology advancement toward sustainability; thus aim of this paper is to study all under one umbrella. Every field and society demands a quality product in competitive era that too at reasonable price. Sustainable growth ensures higher productivity by using advanced tools, techniques and intelligent framework which leads to improved industry performance. Selected research papers from Web of science, analysis provide a vision on application of circularity, affecting processes, technology parameters in manufacturing industry. Thus a major path way for sustainable development goal (SDG) 12, responsible consumption and production, is the heart of the circular economy. In this review paper, Web of science source is used for analysis from circularity view point of an industry and concluded major affecting parameters way for sustainable development along with living standards and better environment is through digital circularity.

Keywords Circular economy · Digitization · Sustainable development goals

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Introduction

Circular economy is time demand; based on type of industry, it is beneficial with expected application barriers. Industry 4.0 revolution coined by Germany, advancement in technologies like IoT, cloud computing, big data, additive manufacturing, and many other technologies change the world with few pros and cons with different impact in developed and developing countries [1]. Digitalization which includes digitization, i.e., use of technologies by an organization and industry possess tremendous practical gain with multiple challenges on other side Digitalization pace up globally, where as in developing country like India adoption started from 2015 by Digital Bharat Abhyan. Digital technologies emerged as one of circular economy (CE) promoter in present era; CE accelerates with innovative circular business model of an organization, along with efficient resource flow and value creation techniques [2]. Furthermore, cross-sector digital markets need well execution of circular supply chains with proper intra-connections. Gross domestic product ascertains economic growth of any country, to support this United Nation General Assembly laid down 17 Sustainable development goals in 2015, where 193 countries adopted this agenda 2030 for betterment of world. Developed countries and developing countries have different level of achievements of their goals, attainment leads to social, economic, and environmental benefits leading to prosperity [3]. Sustainable development goals (SDGs) for developing country like India are incorporated by three different economic sectors, primary, secondary, and tertiary leading to some specific and few common SDG goals. Author's main aim in this paper is to study circular economy of industry in digital era comport/leading toward sustainable development goals. Overall integration of above domains leads to three research questions as

RQ 1 How technologies promote circular economy?

IR4 introduced many technologies in decade, whereas global pandemic put forth more technology adoption with changes in organization model in industry. CE is paced due to resource management, waste reduction with improved quality and safety.

RQ 2 Why CE principles and circularity metrics are applied to industry?

CE principles are R imperatives; it is economic regenerative and restorative model composed of business models, leading to different supply chain based on industry. It is used highly in manufacturing, automobile, textile, construction, and other. Output always has to be measured; thus indices exist at micro-level, material, and product level which are researched more as compared to other.

RQ 3 How digitized CE adopted industry achieves SDG goals?

17 SDG goals are adopted by 193 countries, including India. Different goals can be applicable to varied sectors, and few are common mandatory social goals not much

focused in industry study. Industrial sector has to be sustainable; applications of circular supply chain, business model, cross-sector digital inter- and intra-linkages, waste of one can be resource for other, are focused in research articles with many underlying barriers to overcome for better result.

Paper is structured as Literature methodology adopted in Section "Background of Literature Method", followed with classification framework in Section "Adopted Research Methodology"; Summary and discussion in Section "Summary and Discussion" followed with conclusion and further research in Section "Conclusion".

Background of Literature Method

Here, history of SDG, technology, and CE is subdivides in Sections "History of Sustainable Development Goals in India" and "Background of CE and Digitization". Followed by analysis of review articles and research articles in Section "Existing Literature Review".

History of Sustainable Development Goals in India

India and their SDG's lead towards better world with consideration for economic, environment, and social aspect. All SDG's leads toward better world with consideration either for economic, environment, and social aspect. Below-mentioned table clarifies SDG 12, is applicable to all sectors, additionally social well-being in terms of no poverty, zero hunger, gender equality, good health, and well-being are applicable everywhere, but yet less researched [4]. This is based on data of 2018–2019, to restrict data variation due to pandemic in past two year. This is to focus the study toward secondary or industrial segment of application of CE toward SDG 12.

Classification of India and their SDG's based on sector before industrial boom in last two years as shown in Fig. 21.1 [11].

Along with further technology boom in global pandemic is highly remarkable, it effects in terms of output is also considered. From research article, it can be concluded in spite of technology up-gradation industrial sectors are yet to prosper in terms of employment, technology applications, skills, and many other aspects.

Background of CE and Digitization

CE operation can be executed at three levels, namely micro-level (product level, company, and consumers), meso level (eco industrial parks), and macro-level (city, region, nation, and beyond). It is economic model based on business model by

Primary sector: (Agricultural sector)
Consists of: forest -farming and mining
Employment approximately: (47%) and GDP: 15.87% (2018-2019)
SDG's goal: 12, 6, 7, 14 and 15
Notable point: World's second largest producer of agricultural products and provides raw material for secondary sector

Secondary sector (Industrial sector)
Consists of: Manufacturing, construction, textile, electricity, gas, water supply and many other utility services and industries
Employment approximately: (22%) and GDP: 29.73% (2018-2019)
SDG's goal: 12, 2, 3 and 9
Notable point: Crucial sector GDP and employment below its potential

Tertiary sector (Service sector: facilities to general people)
Consists of: Banking, health and insurance, tourism, defense, transport, public administration and many other public services
Employment approximately: (31%) and GDP 45.50% (2018-2019)
SDG's goal: 12, 4, 6, 7 and 9

Fig. 21.1 Indian economy classification to achieve sustainable development goals

replacing end of life by reusing, recycling, and recovery of materials in production and consumption. Adoption is challenge in itself; various companies adopted CE partially or completely yet it is growing field. Global pandemic had boom technology at personal and professional aspect, still underlying hurdles of lack of knowledge, lack of training, skills, selection of right techniques, issues of security, governance, etc., persists when applied to industry. Uncertainty, market risk leads to extended or interconnected supply chain network [1] (Table 21.1).

Existing Literature Review

From above literature review of articles and research papers, it is evident that very few papers have a combine approach all three aspects together. Additionally, majorly environmental aspect is focused yet more research is to be done for CE toward economic and society benefit.

Table 21.1 Review of review articles and research articles

S. No.	Article title and reference	Type of paper	Study focus	Decision-making in research articles/articles analyzed in review paper	Circular economy	Digitization	SDG focused	Industry type
1	A review of internet of things (IoT) embedded sustainable supply chain for Industry 4.0 requirements	Literature review [5]	ERP, IoT, CE, and CSC	NA	Sustainable supply chain	Yes	Sustainable economic and environmental	Manufacturing
2	Toward a conceptual development of Industry 4.0, servitization, and circular economy	Literature review [6]	I 4.0 towards CE	NA	Circular business model	Yes	–	Manufacturing
3	The circular economy in the textile and apparel industry	Literature review [7]	Circularity practices relationship building, dynamic supply chain execution, and performance	109 papers analyzed	CSC	–	Economic and Environmental	Textile
			Measurement indicators					
4	A review on energy, environment and economic assessment in remanufacturing based on life cycle assessment method	Literature review [8]	Life cycle assessment, remanufacturing and cost benefit	NA	CBM	–	Economic and Environmental	Manufacturing
5	Human–robot collaboration in industrial environments: a literature review on non-destructive disassembly	Literature review [9]	Circular economy business model	NA	Circular business model	Yes	Environmental and economic sustainability	Manufacturing

(continued)

Table 21.1 (continued)

S. No.	Article title and reference	Type of paper	Study focus	Decision-making in research articles/articles analyzed in review paper	Circular economy	Digitization	SDG focused	Industry type
6	Smart remanufacturing: a review and research framework	Literature review [10]	Accelerate CE by digitization	329	Remanufacturing	Yes	Environmental	Manufacturing
7	An integrated multi-objective optimization modeling for sustainable development goals of India	Research [11]	Sustainable goals	Decision-making goal theory	Optimization	–	SDG and sectors	Indian Industry
8	A hybrid pythagorean fuzzy AHP–CoCoSo framework to rank the performance outcomes of circular supply chain due to adoption of its enablers	Research [12]	CSC enablers in Indian manufacturing industries	Hybrid	Environment regulatory followed by strategic enablers	No	Reduce carbon foot print and material and resource efficiency	Indian manufacturing industry
9	Industry 4.0-based dynamic social organizational life cycle assessment to target the social circular economy in manufacturing	Research [13]	Social metric measure	SO-LCA social organization life cycle assessment	CE yet to explore more in social terms	ERP	SDG 8, 9, 12 and 13	Tiles manufacturing
10	Industry 4.0 to accelerate the circular economy: a case study of electric scooter sharing	Research [14]	Product life and geographic value chain	Interpretive structural modeling (ISM)	Sharing economy	CE with IR 4, real-time planning with reduced waste	Zero emission	Electric scooter and manufacturing

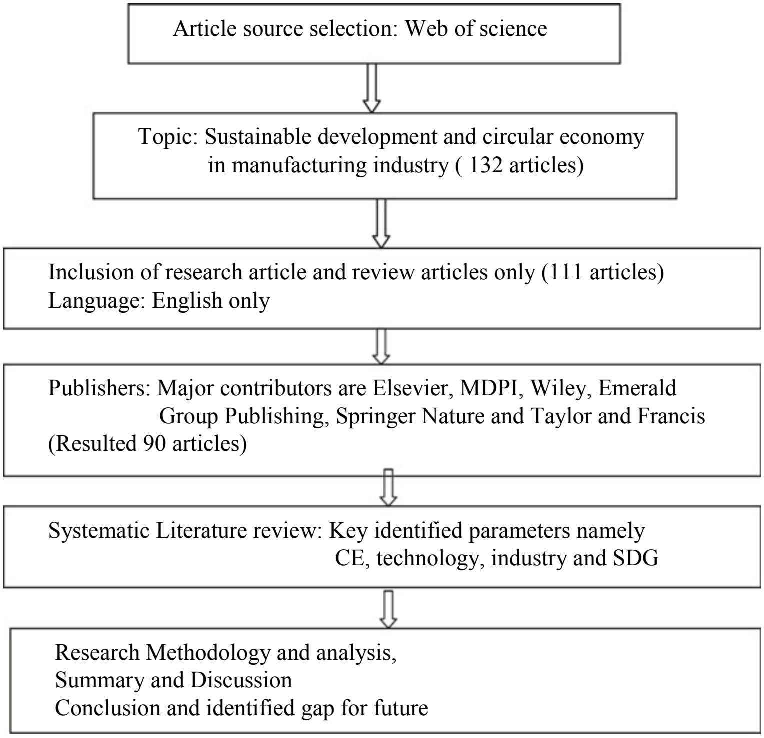


Fig. 21.2 Research methodology framework

Adopted Research Methodology

Article Selection

Research papers from Web of science are considered for study with few inclusion and exclusion. Here, title is search as circular economy and sustainable development in industry, digitization, and sustainable development goals in India. Total 99 papers are considered for study related to all of above. Study on environment science, decision-making, engineering, multidisciplinary, and social science more focused (Fig. 21.2).

Article Classification

Circular economy principles prosper from 3R to 6R, 9R, and 10R framework with span. Technical cycle and biological cycle constituents CE, slowing and narrowing loops and closing loops exist. Only, technical cycles are considered. Applications and advancement of technologies are considered on few selected industry. Measurement is vital; thus circularity is focused with sustainability.

Article Analysis

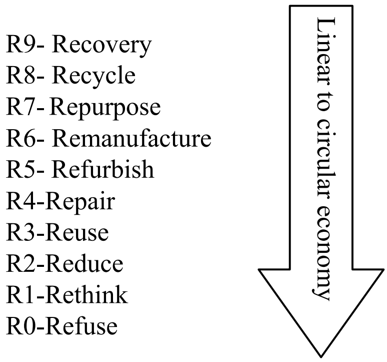
Three broad domains interconnection is authors aim. Thus various parameters are discuss related to CE, IR4, and SDG 12 in next session

Summary and Discussion

Integration of digital technologies promoting CE toward sustainability is shared in below figure. This is integration of adoption of various technologies like CPS, IoT, big data, VR/AR, cloud computing, intelligent robotics, industrial artificial intelligence (IAI), additive manufacturing, and many other in various industries primarily in manufacturing, textile, automobile, electric vehicle, etc. [15]. This leads to grow 10R framework of CE with its innovation in business model results to meet SDG12 responsible consumption and production with more environmental aspect followed by economic and then social [16].

From review, various advancements are in terms of environmental protection reduce carbon foot prints, optimization of material use, waste management, quality improvement [17], reliability with automated tools reduction in time, digitize supply chain, improved resource efficiency, and lastly very few research share a social dimension, but AR/VR technology and simulation models can improve human training which is beneficial [18]. Thus authors main aim is achieved can be observed form Fig. 21.3.

Fig. 21.3 Framework 9R for circular economy



Conclusion

Traditional economy changes to circular economy to save natural resources for future generations, restrict global warming, and protect planet by renewable energy.

Every industry component if properly nurture leads to gain advancement, 6M, namely money, man, management, machine, method, and material optimized leads to sustainable development. Nothing is free of cost; accounted cost and monetary gains can be avail at long run span. Government policies and standardization are yet to be stringent in developing countries.

Product service system, sharing economy builds on trust building. CE principles applied in varied industries can be either of below R framework [18]. Advancement from 3R to 9R increased circularity. R0, R1, and R2 are smart product use and manufacture, whereas R3 to R7 are based on extended life span of products, and its parts and R8 and R9 are useful application of material [19].

Digitization imparts flexibility in the supply chain are of prime value in gaining sustainable development of the supply chain [20]. The aim of current paper is to carry out and systematic literature review of the latest research and applications. From WOS database, 99 research articles emerge out with article title, various decision-making and optimization techniques. Direction and efforts of authors is to combine various subparts under a common umbrella stated by different authors are included in this paper.

Practical Implications

Systematic literature review and analysis of selected articles provide re-search trend followed by future research work identification. Mentioned below are research implications that can be adopted for managers, customers, organization owners, researchers, and practitioners.

Data collection is authentic, still many other sources can be adopted, leading publishers are only considered.

Digital technologies drive CE more efficiently.

For managers, practitioners and researchers a decision-making tool consists of nurturing, perfectly using, 6M to promote CSC. They are man, material, method, machine, money, and management in proper terms to achieve.

This review can be adopted as a basic tool for checking CE with technology advancement toward sustainable development.

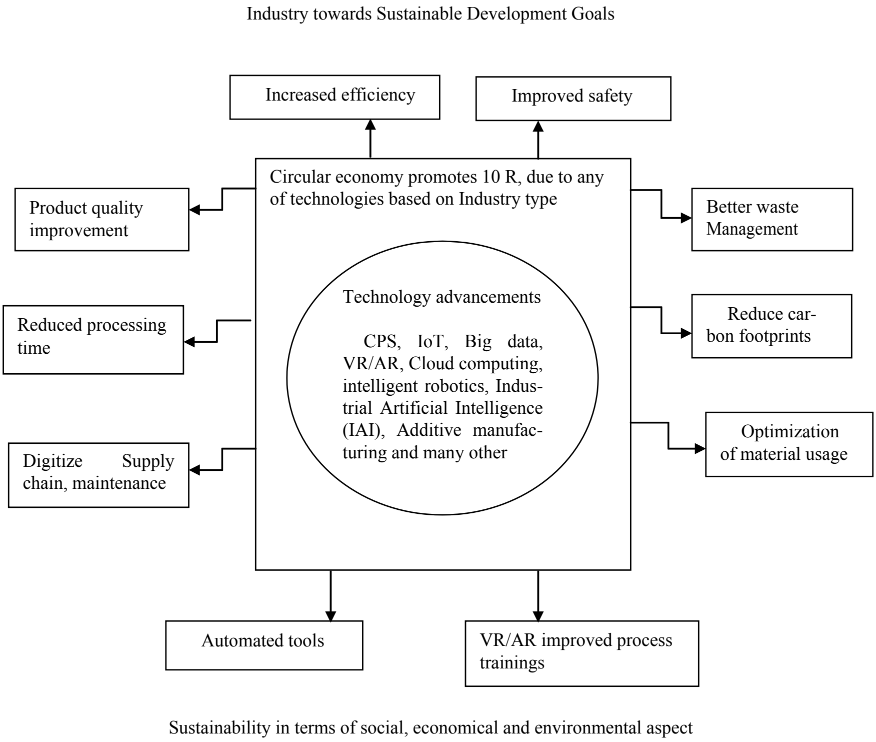


Fig. 21.4 Overall integration of CE, technology, and SDG

Study Limitation and Future Gap

Different performance metrics, detail of all SDGs applicable in all sectors, any industry can be considered. Different electronic data sources can be considered; authors contribution and publishers can be plotted. Varied keywords combination can be adopted as it is very vast field. Individual industry can be reviewed will provide greater insights of CE in digital times toward sustainability (Fig. 21.4).

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Chapter 22
Symmetric Duality for a Multiobjective Fractional Programming with Cone Objectives as Well as Constraints

Balram, Ramu Dubey, and Lakshmi Narayan Mishra

Abstract In the present article, we study naturally K-pseudoconvex and strongly K-pseudoconvex definition and also give existing numerical examples of functions of this kind, and under cones functions, we develop a novel type of non-differentiable multiobjective fractional symmetric dual programming of the Mond–Weir type and prove duality relations involving strongly K-pseudoinvexity assumptions. Our results generalize a number of previous findings in the literature.

Keywords Non-differentiable · Strong duality · Fractional programming · Mond–Weir · Strongly K-pseudoinvexity · Cones functions

Introduction

As illustrated in [3, 16], multiobjective programming, also known as vector optimization, has advanced greatly in several areas in the realms of Pareto optimality, equilibria, game theory, and variational inequalities. In vector optimization, convex functions are crucial. As a result, applications of convexity and its many expansions have improved it. In management, engineering, economics, and optimization, convex functions and extended convex functions have a wide range of applications. In the literature, several generalizations of convex functions have been examined. Weir et al. defined the cone-convex function in [18]. Cambini [2] suggested many classes of concave vector-valued functions that are possible expansions of scalar generalized concavity based on order relations established by a cone, its interior, or a cone with-

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out an origin. Suneja et al. [17] defined second-order cone-convex, pseudoconvex, highly pseudoconvex, and quasiconvex functions. You may work in management, military affairs, science, or engineering design, to name a few sectors. The convex function is one of them. Because of its importance in today's environment, the most often used concept is modern mathematics.

In recent years, several researchers have worked on fractional optimization problems with multiple objective functions, resulting in various optimality and duality conclusions. The following is a model for a generic k-objective fractional problem:

$$\textbf{(MFP)}\ \text{Minimize}\ G(u) = \left(\frac{f_1(u)}{g_1(u)}, \frac{f_2(u)}{g_2(u)}, \frac{f_3(u)}{g_3(u)}, \ldots, \frac{f_k(u)}{g_k(u)}\right)$$
$$\text{subject to}\quad u \in X^0 = \{u \in X \subset \mathbb{R}^n : h_j(u) \leq 0,\ \ j = 1, 2, 3, \ldots, m\},$$

where $g = (g_1, g_2, g_3, \ldots, g_k) : X \to \mathbb{R}^k$, $h = (h_1, h_2, \ldots, h_m) : X \to \mathbb{R}^m$ and $f = (f_1, f_2, f_3, \ldots, f_k) : X \to \mathbb{R}^k$ are differentiable on X.

(Egudo [8]) Considering the above multiobjective fractional programming problems, the idea of efficiency (Pareto optimal) is employed to construct duality theorem and derived their results under generalized assumption. Kharbanda and Agarwal [12] summed up class of higher (F, α, ρ, d)-V-type I work and presented for a non-smooth multiobjective fractional programming problem including support capacities. Mishra et al. [15] for solving unconstrained optimization problems. Jayswal and Jha [9] prove weak, strong and converse duality theorems under second-order F-convexity assumptions by employing a pair of second-order fractional symmetric variational programmes over cone constraints. Later, Sharma and Kaur [11] proved weak, strong, and converse duality results under (ϕ, ρ) convexity suppositions. Several researchers are also working in this area (for more informations, see [1, 4–7, 10, 13, 14]).

In this article, we create a new class of naturally K-pseudoinvex functions and strongly K-pseudoinvex functions for a mathematical programming problem. We also present constraint and derived duality solutions under extended conditions for a non-differentiable multiobjective fractional programming problem with a cone objective. Also, for existing similar functions, create non-trivial numerical examples.

Notations and Definitions

In this paper, we used $\mathbb{R}^n$ for n-dimensional Euclidean space and $\mathbb{R}^n_+$ for non-negative orthant. Also, here C_1 and C_2 used for closed convex cones in $\mathbb{R}^n$ and $\mathbb{R}^m$, respectively, with non-void interiors. For a real-valued twice differentiable function $g(a, b)$ described on an open set in $\mathbb{R}^n \times \mathbb{R}^m$ is indicated by $\nabla_a g(\bar{a}, \bar{b})$ the gradient vector of g w.r.t. a an at $(\bar{a}, \bar{b})$, $\nabla_{aa} g(\bar{a}, \bar{b})$ the Hessian matrix w.r.t. a an at $(\bar{a}, \bar{b})$. Let us assume throughout the paper $\tilde{N} = \{1, 2, \ldots, k\}$.

K is used for pointed convex cone with non-void interiors in $\mathbb{R}^k$. Then, for $b, c \in \mathbb{R}^k$. Now, we specify cone orders w.r.t. K as follows:

$$b \leq c \iff c - b \in K; \;\; b \leq c \iff c - b \in K \backslash \{0\}; \;\; b < c \iff c - b \in \operatorname{int} K.$$

Definition 1 Let D be 1 convex compact set in $\mathbb{R}^n$. The support function of D is defined by

$$s(l|D) = \max \left\{ l^T m : m \in D \right\},$$

there exists $n \in \mathbb{R}^n$ s.t.

$$s(m|D) \geq s(l|D) + n^T (m - l), \;\; \forall\, m \in D.$$

The subdifferential of $s(l|D)$ is given by

$$\partial s(l|D) = \left\{ n \in D : n^T a = s(l|D) \right\}$$

For any set $S \subset \mathbb{R}^n$, the normal cone to S at 1 point $l \in S$ is defined by

$$N_S(l) = \left\{ m \in \mathbb{R}^n : m^T (n - l) \leq 0, \;\; \forall n \in S \right\}.$$

Obviously, for 1 compact convex set D, m is in $N_D(l)$ if and only if $s(m|D) = l^T b$, or equivalently, m is in $\partial s(m|D)$. Consider the multiobjective programming problem with cone function as follows:

(MP) K-min $f(a) = \left(f_1(a), f_2(a), f_3(a), \ldots, f_k(a) \right)$, subject to $a \in X$,
where $f = (f_1, f_2, \ldots, f_n) : \mathbb{R}^n \to \mathbb{R}^k$ and $X \subseteq \mathbb{R}^n$.

Definition 2 Let $C \subseteq \mathbb{R}^s$ be a cone and C^* positive polar cone is defined by

$$C^* = \left\{ b \in \mathbb{R}^s : a^T b \geq 0 \right\}.$$

Definition 3 An feasible point $\bar{a} \in X$ is a weak efficient solution of (MP) if, $\nexists$ $a \in X$ s.t.

$$f(\bar{a}) - f(a) \in \operatorname{int} K.$$

Definition 4 An feasible point $\bar{a} \in X$ is an efficient solution of (MP) if, $\nexists$ $a \in X$ s.t.

$$f(\bar{a}) - f(a) \in K \backslash \{0\}.$$

Example 1 Let $K = \{(x_1, x_2); x_2 \geq 0$ and $x_1 \geq x_2\}$ and define a function $f = (f_1, f_2) : K \to \mathbb{R} \times \mathbb{R}$ where

$$f_1(x_1) = \sin^2 x_1 \text{ and } f_2(x_2) = x_2.$$

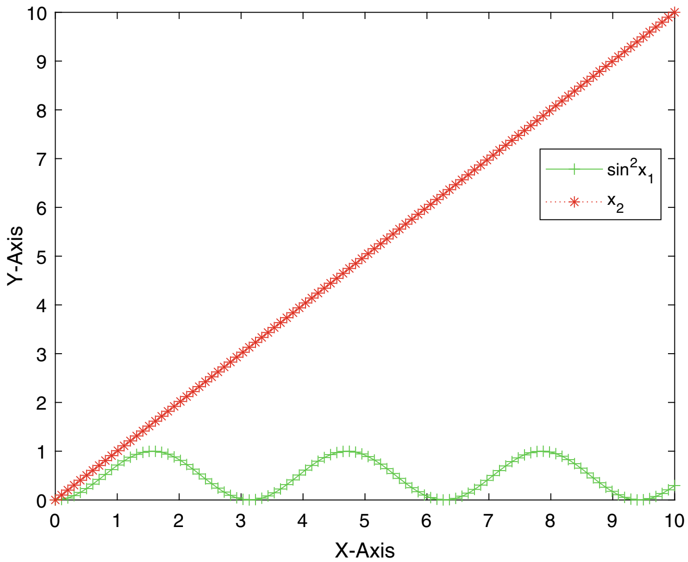

Fig. 22.1 $(f_1, f_2) = (\sin^2 x_1, x_2)$

From the defined functions f_1 and f_2, it is clear from the definition and above Fig. 22.1 that the weak efficient solutions are uncountable, i.e. $(0, x_2) \cup (x_1, 0)$, where $x = (x_1, x_2) \in K$, and efficient solution is $(0, 0)$.

Now, ω is differentiable function $\omega = (\omega_1, \omega_2, \ldots, \omega_k) : C \to \mathbb{R}^k$.

Definition 5 Let $\eta : C \times C \to \mathbb{R}$,

(a) ω is naturally K-pseudoinvex w.r.t. η at $\bar{a} \in C$ if $\forall\, a \in C$ s.t.

$$\eta^T(a, \bar{a})\left\{\nabla\omega_1(\bar{a}),\ \nabla\omega_2(\bar{a}), \ldots, \nabla\omega_k(\bar{a})\right\} \in K$$
$$\Rightarrow \{\omega_1(a) - \omega_1(\bar{a}),\ \omega_2(a) - \omega_2(\bar{a}), \ldots, \omega_k(a) - \omega_k(\bar{a})\right\} \in K.$$

(b) ω is strongly K-pseudoinvex w.r.t. η at $\bar{a} \in C$, if

$$-\eta^T(a, \bar{a})\{\nabla\omega_1(\bar{a}), \nabla\omega_2(\bar{a}), \ldots, \nabla\omega_k(\bar{a})\} \notin \operatorname{int} K$$
$$\Rightarrow \{\omega_1(a) - \omega_1(\bar{a}),\ \omega_2(a) - \omega_2(\bar{a}), \ldots, \omega_k(a) - \omega_k(\bar{a})\} \in K, \quad \forall\, a \in C.$$

(c) ω in naturally K-pseudoincave w.r.t. η at $\bar{a} \in C$ if $\forall a \in C$ s.t.

$$\eta^T(a, \bar{a})\left\{\nabla\omega_1(\bar{a}), \nabla\omega_2(\bar{a}), \ldots, \nabla\omega_k(\bar{a})\right\} \in -K$$
$$\Rightarrow \left\{\omega_1(a) - \omega_1(\bar{a}),\ \omega_2(a) - \omega_2(\bar{a}), \ldots, \omega_k(a) - \omega_k(\bar{a})\right\} \in -K.$$

(d) ω is strongly K-pseudoincave w.r.t. η at $\bar{a} \in C$, if

$$-\eta^T(a,\bar{a})\{\nabla\omega_1(\bar{a}),\nabla\omega_2(\bar{a}),\ldots,\nabla\omega_k(\bar{a})\}\notin -\text{int}\,K$$
$$\Rightarrow\{\omega_1(a)-\omega_1(\bar{a}),\omega_2(a)-\omega_2(\bar{a}),\ldots,\omega_k(a)-\omega_k(\bar{a})\}\in -K,\quad\forall\, a\in C.$$

All the above definitions are generalized in two variables as follows:

A differentiable function $\omega=(\omega_1,\omega_2,\ldots,\omega_k):X\times Y\to\mathbb{R}^k$, $\eta_1:X\times X\to\mathbb{R}^n$ and $\eta_2:Y\times Y\to\mathbb{R}^n$.

(a1) ω is naturally K-pseudoinvex at $s\in X$ with fixed $t\in Y$ w.r.t. η_1, if

$$\eta_1^T(a,s)\left\{\nabla_a\omega_1(s,t),\nabla_a\omega_2(s,t),\ldots,\nabla_a\omega_k(s,t)\right\}\in K$$
$$\Rightarrow\left\{\omega_1(a,t)-\omega_1(s,t),\omega_2(a,t)-\omega_2(s,t),\ldots,\omega_k(a,t)-\omega_k(s,t)\right\}\in K,\forall\, a\in X$$

and ω is naturally $K-$ pseudoinvex at $t\in Y$ with fixed $s\in X$ w.r.t. η_2, if

$$\eta_2^T(b,t)\left\{\nabla_b\omega_1(s,t),\nabla_b\omega_2(s,t),\ldots,\nabla_b\omega_k(s,t)\right\}\in K,$$
$$\Rightarrow\left\{\omega_1(s,b)-\omega_1(s,t),\omega_2(s,b)-\omega_2(s,t),\ldots,\omega_k(s,b)-\omega_k(s,t)\right\}\in K,\forall a\in X.$$

(a2) ω is strongly $K-$ pseudoinvex at $s\in X$ with fixed $t\in Y$ w.r.t. η_1, if

$$-\eta_1^T(a,s)\left\{\nabla_a\omega_1(s,t),\nabla_a\omega_2(s,t),\ldots,\nabla_a\omega_k(s,t)\right\}\notin\text{int}\,K,$$
$$\Rightarrow\left\{\omega_1(a,t)-\omega_1(s,t),\omega_2(a,t)-\omega_2(s,t),\ldots,\omega_k(a,t)-\omega_k(s,t)\right\}\notin K,\quad\forall a\in X$$

and ω is strongly $K-$ pseudoinvex at $t\in Y$ with fixed $s\in X$ w.r.t. η_2, if

$$\eta_2^T(b,t)\left\{\nabla_b\omega_1(s,t),\nabla_b\omega_2(s,t),\ldots,\nabla_b\omega_k(s,t)\right\}\notin\text{int}\,K,$$
$$\Rightarrow\left\{\omega_1(s,b)-\omega_1(s,t),\omega_2(s,b)-\omega_2(s,t),\ldots,\omega_k(s,b)-\omega_k(s,t)\right\}\notin\text{int}K,\forall a\in X.$$

(a3) ω is naturally K-pseudoincave at $s\in X$ with fixed $t\in Y$ w.r.t. η_1, if

$$-\eta_1^T(a,s)\left\{\nabla_a\omega_1(s,t),\nabla_a\omega_2(s,t),\ldots,\nabla_a\omega_k(s,t)\right\}\in -K$$
$$\Rightarrow\left\{\omega_1(a,t)-\omega_1(s,t),\omega_2(a,t)-\omega_2(s,t),\ldots,\omega_k(a,t)-\omega_k(s,t)\right\}\in -K,\forall a\in X.$$

and ω is naturally K-pseudoincave at $t\in Y$ with fixed $s\in X$ w.r.t. η_2, if

$$\eta_2^T(b,t)\left\{\nabla_b\omega_1(s,t),\nabla_b\omega_2(s,t),\ldots,\nabla_b\omega_k(s,t)\right\}\in -K$$
$$\Rightarrow\left\{\omega_1(s,b)-\omega_1(s,t),\omega_2(s,b)-\omega_2(s,t),\ldots,\omega_k(s,b)-\omega_k(s,t)\right\}\in -K,\forall a\in X.$$

(a4) ω is strongly K-pseudoincave at $s\in X$ with fixed $t\in Y$ w.r.t. η_1, if

$$-\eta_1^T(a,s)\left\{\nabla_a\omega_1(s,t),\nabla_a\omega_2(s,t),\ldots,\nabla_a\omega_k(s,t)\right\}\notin -\text{int}\,K$$
$$\Rightarrow\left\{\omega_1(a,t)-\omega_1(s,t),\omega_2(a,t)-\omega_2(s,t),\ldots,\omega_k(a,t)-\omega_k(s,t)\right\}\in -K,\forall a\in X$$

and ω is strongly K-pseudoincave at $t\in Y$ with fixed $s\in X$ w.r.t. η_2, if

$$\eta_2^T(b,t)\{\nabla_b\omega_1(s,t), \nabla_b\omega_2(s,t), \ldots, \nabla_b\omega_k(s,t)\} \notin -\text{int}\,K$$
$$\Rightarrow \{\omega_1(s,b)-\omega_1(s,t), \omega_2(s,b)-\omega_2(s,t), \ldots, \omega_k(s,b)-\omega_k(s,t)\} \in -K, \forall a \in X.$$

Example 2 Let $K = \{(a_1, a_2); a_2 \geq 0$ and $a_1 \geq a_2\}$ and define a function $\psi = (\psi_1, \psi_2) : \mathbb{R}^2 \to \mathbb{R}$ and $\eta : C \times C \to \mathbb{R}$ where

$$\psi_1(a) = e^a,\ \psi_2(a) = \sin a,\ \eta^T(a,\bar{a}) = a - \bar{a} \text{ and } a \in C \geq 0.$$

In the above example, we will try to show that ψ is naturally K-pseudoinvex as well as ψ is strongly K-pseudoinvex w.r.t. same η at the point $\bar{a} = 0$.

Firstly, we show that ω is naturally K-pseudoinvex w.r.t. η at $\bar{a} \in C$ if $\forall a \in C$ s.t.

$$\eta^T(a,\bar{a})\{\nabla\omega_1(\bar{a}),\ \nabla\omega_2(\bar{a})\} \in K \Rightarrow \{\omega_1(a)-\omega_1(\bar{a}),\ \omega_2(a)-\omega_2(\bar{a})\} \in K.$$

Next, we consider

$$\psi_1 = \eta^T(a,\bar{a})\{\nabla\omega_1(\bar{a}),\ \nabla\omega_2(\bar{a})\} \text{ and } \omega_2 = \{\omega_1(a)-\omega_1(\bar{a}),\ \omega_2(a)-\omega_2(\bar{a})\},$$

$$\psi_1 = (a-\bar{a})(e^{\bar{a}}, \cos\bar{a}).$$

At $\bar{a} = 0$, we get

$$\psi_1 = (a, a) \in K.$$

This implies

$$\psi_2 = (e^a - e^{\bar{a}}, \sin a - \sin\bar{a}).$$

At $\bar{a} = 0$, we get

Clearly, from the above Fig. 22.2

$$\omega_2 = (e^a - 1, \sin a) \in K$$

Now, we show that strongly K-pseudoinvex w.r.t. η at $\bar{a} \in C$, if

$$-\eta^T(a,\bar{a})\{\nabla\omega_1(\bar{a}), \nabla\omega_2(\bar{a})\} \notin \text{int}K \Rightarrow \{\omega_1(a)-\omega_1(\bar{a}),\ \omega_2(a)-\omega_2(\bar{a})\} \in K, \forall\, a \in C.$$

Again, let

$$\Phi_1 = -\eta^T(a,\bar{a})\{\nabla\omega_1(\bar{a}), \nabla\omega_2(\bar{a})\} \text{ and } \Phi_2 = \{\omega_1(a)-\omega_1(\bar{a}),\ \omega_2(a)-\omega_2(\bar{a})\}.$$

$$\Phi_1 = -(a-\bar{a})(e^{\bar{a}},\ \cos\bar{a}).$$

At $\bar{a} = 0$, we get

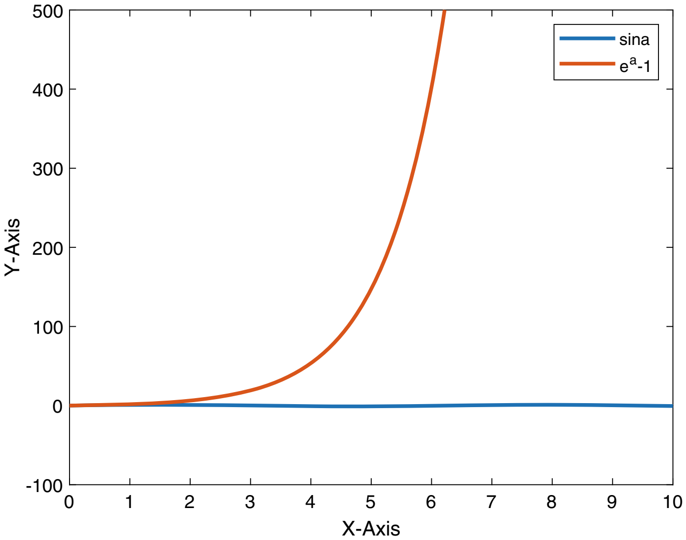

Fig. 22.2 $\omega_2 = (e^a - 1, \sin a)$

$$\Phi_1 = (-a, -a) \notin \operatorname{int} K.$$

Next,

$$\Phi_2 = (e^a - e^{\bar{a}}, \sin a - \sin \bar{a}).$$

At $\bar{a} = 0$, we get

$$\Phi_2 = (e^a - 1, \sin a) \in K.$$

Non-differentiable Multiobjective Symmetric Duality Model over Cone

Let $E=\{E_1, E_2, E_3, \ldots, E_k\}$, $B = \{B_1, B_2, B_3, \ldots, B_k\}$, $D=\{D_1, D_2, D_3, \ldots, D_k\}$ and $F = \{F_1, F_2, F_3, \ldots, F_k\}$. In this section, we formulate non-differentiable multiobjective symmetric fractional duality model with cone functions as follows:

Primal Problem (FPP)

$$\text{K-min.}\ \left(\frac{f_1(a,b)+s(a|E_1)-b^Tc_1}{g_1(a,b)-s(a|B_1)+b^Tr_1}, \frac{f_2(a,b)+s(a|E_2)-b^Tc_2}{g_2(a,b)-s(a|B_2)+b^Tr_2}, \ldots, \frac{f_k(a,b)+s(a|E_k)-b^Tc_k}{g_k(a,b)-s(a|B_k)+b^Tr_k}\right),$$

subject to

$$-\sum_{i=1}^{k}\lambda_i\left[\nabla_b f_i(a,b)-c_i-\frac{f_i(a,b)+s(a|E_i)-b^Tc_i}{g_i(a,b)-s(a|B_i)+b^Tr_i}\big(\nabla_b g_i(a,b)+r_i\big)\right]\in C_2^*, \tag{22.1}$$

$$b^T\sum_{i=1}^{k}\lambda_i\left[\nabla_b f_i(a,b)-c_i-\frac{f_i(a,b)+s(a|E_i)-b^Tc_i}{g_i(a,b)-s(a|B_i)+b^Tr_i}\big(\nabla_b g_i(a,b)+r_i\big)\right]\geq 0, \tag{22.2}$$

$$a\in C_1, \lambda\in \text{int}K^*, c_i\in D_i,\ r_i\in F_i,\ 1\leq i\leq k. \tag{22.3}$$

Dual Problem (FDP)

$$\text{K-max.}\ \left(\frac{f_1(p,q)-s(q|D_1)+p^Tw_1}{g_1(p,q)+s(q|F_1)-p^Tt_1}, \frac{f_2(p,q)-s(q|D_2)+p^Tw_2}{g_2(p,q)+s(q|F_2)-p^Tt_2}, \ldots, \frac{f_k(p,q)-s(q|D_k)+p^Tw_k}{g_k(p,q)+s(q|F_k)-p^Tt_k}\right)$$

subject to:

$$\sum_{i=1}^{k}\lambda_i\left[\nabla_p f_i(p,q)+w_i-\frac{f_i(p,q)-s(q|D_i)+p^Tw_i}{g_i(p,q)+s(q|F_i)-p^Tt_i}\big(\nabla_p g_i(p,q)-t_i\big)\right]\in C_1^*, \tag{22.4}$$

$$p^T\sum_{i=1}^{k}\lambda_i\left[\nabla_p f_i(p,q)+w_i-\frac{f_i(p,q)-s(q|D_i)+p^Tw_i}{g_i(p,q)+s(q|F_i)-p^Tt_i}\big(\nabla_p g_i(p,q)+t_i\big)\right]\leq 0, \tag{22.5}$$

$$q\in C_2, \lambda\in \text{intK}^*, w_i\in E_i,\ b_i\in B_i,\ 1\leq i\leq k, \tag{22.6}$$

where for $i\in\tilde{N}$,

(i) $f_i, g_i : X\times Y\to\mathbb{R}$ are twice differentiable functions, where $X\subseteq\mathbb{R}^n$ and $Y\subseteq\mathbb{R}^m$,

(ii) D_i, F_i are convex compact sets in $\mathbb{R}^m$, and C_i, E_i are convex compact sets in $\mathbb{R}^n$,

(iii) C_1^*, C_2^* and K^* are the positive polar cones of C_1, C_2 and K, respectively,
(iv) $s(a|E_i),\ s(a|B_i),\ s(a|D_i) and\ s(a|F_i)$ are the support function of $E_i,\ B_i,\ D_i$ and F_i, respectively,
(v) In the feasible regions, numerator is non-negative and denominator in the positive of the objective functions.

Under the aforesaid assumptions, we now discuss duality theorems. Let $c = (c_1, c_2, \ldots, c_k)$, $w = (w_1, w_2, \ldots, w_k)$, $r = (r_1, r_2, \ldots, r_k)$ and $t = (t_1, t_2, \ldots, t_k)$. Let T^0 and Q^0 be the set of feasible solutions of (FPP) and (FDP), respectively.

Theorem 1 (Weak duality) *Let* $(a, b, \lambda, c, r) \in T^0$ *and* $(p, q, \lambda, w, t) \in Q^0$. *Let*

(i) $$\left\{ f_1(.,q) + (.)^T w_1 - \frac{f_1(p,q) - s(q|D_1) + p^T w_1}{g_1(p,q) + s(q|F_1) - p^T t_1}\left(g_1(.,q) - (.)^T t_1\right), \ldots, f_k(.,q) + (.)^T w_k - \frac{f_k(p,q) - s(q|D_k) + p^T w_k}{g_k(p,q) + s(q|F_k) - p^T t_k}\left(g_k(.,q) - (.)^T t_k\right) \right\}$$
be naturally K-pseudoinvex w.r.t. η_1 *at p,*

(ii) $$\left\{ f_1(a,.) - (.)^T c_1 - \frac{f_1(a,b) + s(a|E_1) - b^T c_1}{g_1(a,b) - s(a|B_1) + b^T r_1}(g_1(a,.) + (.)^T r_1), \ldots, f_k(a,.) - (.)^T c_k - \frac{f_k(a,b) + S(a|E_k) - b^T c_k}{g_k(a,b) - s(a|B_k) + b^T r_k}\left((g_k(a,.) + (.)^T r_k)\right) \right\}$$
be naturally K-pseudoincave w.r.t. η_2 *at b,*

(iii) $\eta_1(a, p) + p \in C_1$,
(iv) $\eta_2(q, b) + b \in C_2$.

Then,

$$\left[\left(\frac{f_1(p,q) - s(q|D_1) + p^T w_1}{g_1(p,q) + s(q|F_1) - p^T t_1}, \frac{f_2(p,q) - s(q|D_2) + p^T w_2}{g_2(p,q) + s(q|F_2) - p^T t_2}, \ldots, \frac{f_k(p,q) - s(q|D_k) + p^T w_k}{g_k(p,q) + s(q|F_k) - p^T t_k}\right) - \left(\frac{f_1(a,b) + s(a|E_1) - b^T c_1}{g_1(a,b) - s(a|B_1) + b^T r_1}, \frac{f_2(a,b) + s(a|E_2) - b^T c_2}{g_2(a,b) - s(a|B_2) + b^T r_2}, \ldots, \frac{f_k(a,b) + s(a|E_k) - b^T c_k}{g_k(a,b) - s(a|B_k) + b^T r_k}\right)\right] \notin K \backslash \{0\}.$$

Proof From hypothesis (i), we have

$$\eta^T(a,p)\left\{ \nabla_p f_1(p,q) + w_1 - \frac{f_1(p,q) - s(q|D_1) + p^T w_1}{g_1(p,q) + s(q|F_1) - p^T t_1}\left(\nabla_p g_1(p,q) - t_1\right), \ldots, \nabla_p f_k(p,q) + w_k - \frac{f_k(p,q) - s(q|D_k) + p^T w_k}{g_k(p,q) + s(q|F_k) - p^T t_k}\left(\nabla_p g_k(p,q) - t_k\right) \right\} \in K.$$

This implies

$$\Bigg\{\bigg(f_1(a,q)+a^Tw_1-\frac{f_1(p,q)-s(q|D_1)+p^Tw_1}{g_1(p,q)+s(q|F_1)-p^Tt_1}\big(g_1(a,q)+s(q|F_1)-a^Tt_1\big)\bigg)$$

$$-\bigg(f_1(p,q)+p^Tw_1-\frac{f_1(p,q)-s(q|D_1)+p^Tw_1}{g_1(p,q)+s(q|F_1)-p^Tt_1}\big(g_1(p,q)+s(q|F_1)-p^Tt_1\big)\bigg),\ldots,$$

$$\bigg(f_k(a,q)+a^Tw_k-\frac{f_k(p,q)-s(q|D_k)+p^Tw_k}{g_k(p,q)+s(q|F_k)-p^Tt_k}\big(g_k(a,q)+s(q|F_k)-a^Tt_k\big)\bigg)$$

$$-\bigg(f_k(p,q)+p^Tw_k-\frac{f_k(p,q)-s(q|D_k)+p^Tw_k}{g_k(p,q)+s(q|F_k)-p^Tt_k}\big(g_k(p,q)+s(q|F_k)-p^Tt_k\big)\bigg)\Bigg\}\in K.$$

Since $\lambda\neq 0$ and $\lambda\in int K^*$, it follows that

$$\eta^T(a,p)\sum_{i=1}^{k}\lambda_i\bigg[\nabla_p f_i(p,q)+w_i-\frac{f_i(p,q)-s(q|D_i)+p^Tw_i}{g_i(p,q)+s(q|F_i)-p^Tt_i}\big(\nabla_p g_i(p,q)-t_i\big)\bigg]$$
$$\geq 0.$$

This implies

$$\sum_{i=1}^{k}\lambda_i\bigg[f_i(a,q)+a^Tw_i-\frac{f_i(p,q)-s(q|D_i)+p^Tw_i}{g_i(p,q)+s(q|F_i)-p^Tt_i}\big(g_i(a,q)+s(q|F_i)-a^Tt_i\big)\bigg]$$
$$\geq\sum_{i=1}^{k}\lambda_i\bigg[f_i(p,q)+a^Tw_i-\frac{f_i(p,q)-s(q|D_i)+p^Tw_i}{g_i(p,q)+s(q|F_i)-p^Tt_i}\big(g_i(p,q)+s(q|F_i)-p^Tt_i\big)\bigg]. \tag{22.7}$$

Using hypothesis (iii) and dual constraint (22.4), we get

$$(\eta^T(a,p)+p)^T\sum_{i=1}^{k}\lambda_i\bigg(\nabla_p f_i(p,q)+w_i-\frac{f_i(p,q)-s(q|D_i)+p^Tw_i}{g_i(p,q)+s(q|F_i)-p^Tt_i}\big(\nabla_p g_i(p,q)-t_i\big)\bigg)\geq 0,$$

or

$$\eta_1^T(a,p)\sum_{i=1}^{k}\lambda_i\big[\nabla_p f_i(p,q)+w_i-\frac{f_i(p,q)-s(q|D_i)+p^Tw_i}{g_i(p,q)+s(q|F_i)-p^Tt_i}(\nabla_p g_i(p,q)-t_i)\big]$$
$$\geq -p^T\sum_{i=1}^{k}\lambda_i\big[\nabla_p f_i(p,q)+w_i-\frac{f_i(p,q)-s(q|D_i)+p^Tw_i}{g_i(p,q)+s(q|F_i)-p^Tt_i}(\nabla_p g_i(p,q)-t_i)\big].$$

From dual constraint (22.5), it follows that

$$\eta_1^T(a,p)\sum_{i=1}^{k}\lambda_i\left[\nabla_p f_i(p,q)+w_i-\frac{f_i(p,q)-s(q|D_i)+p^Tw_i}{g_i(p,q)+s(q|F_i)-p^Tt_i}(\nabla_p g_i(p,q)-t_i)\right]\geq 0. \tag{22.8}$$

Again, using inequalities (22.7) and (22.8), after simplifying, we get

$$\sum_{i=1}^{k}\lambda_i\left[f_i(a,q)+a^Tw_i-s(q|D_i)-\frac{f_i(p,q)-s(q|D_i)+p^Tw_i}{g_i(p,q)+s(q|F_i)-p^Tt_i}(g_i(p,q)+s(q|F_i)-a^Tt_i)\right]\geq 0. \tag{22.9}$$

Using $a^Tw_i \leq S(a|E_i),\ w_i \in E_i,\ i \in \tilde{N}$, (22.6) and (22.9), we get

$$\sum_{i=1}^{k}\lambda_i\left[f_i(a,q)+s(a|E_i)-s(q|D_i)-\frac{f_i(p,q)-s(q|D_i)+p^Tw_i}{g_i(p,q)+s(q|F_i)-p^Tt_i}(g_i(a,q)+s(q|F_i)-(a^Tt_i)\right]\geq 0. \tag{22.10}$$

Next, using $a^Tt_i \leq s(a|E_i), t_i \in E_i, q^Tr_i \leq s(q|F_i), r_i \in F_i$, it is mentioned that $f_i(p,q)-s(q|D_i)+p^Tw_i \geq 0, g_i(p,q)+s(q|F_i)-p^Tt_i > 0$ and also by (22.10), we have

$$\sum_{i=1}^{k}\lambda_i\left[f_i(a,q)+s(a|E_i)-s(q|D_i)-\frac{f_i(p,q)-s(q|D_i)+p^Tw_i}{g_i(p,q)+s(q|F_i)-p^Tt_i}(g_i(a,q)-s(a|B_i)+q^Tr_i)\right]\geq 0. \tag{22.11}$$

Similarly, using hypotheses (ii), (iv) and primal constraints, we have

$$\sum_{i=1}^{k}\lambda_i\left[f_i(a,q)+s(a|E_i)-s(q|D_i)-\frac{f_i(a,b)+s(a|E_i)-b^Tc_i}{g_i(a,b)-s(a|B_i)+b^Tr_i}(g_i(a,q)-s(a|B_i)+q^Tr_i)\right]\leq 0. \tag{22.12}$$

From inequalities (22.11) and (22.12), it gives that

$$\sum_{i=1}^{k}\lambda_i\left[\left(\frac{f_i(a,b)+s(a|E_i)-b^Tc_i}{g_i(a,b)-s(a|E_i)+b^Tr_i}\right)-\left(\frac{f_i(p,q)-s(q|D_i)+p^Tw_i}{g_i(a,q)+s(q|F_i)-p^Tt_i}\right)(g_i(a,q)-s(a|F_i)+q^Tr_i)\right]\geq 0.$$

Since, $\lambda \in \operatorname{int} K^* \Rightarrow \lambda > 0$ and $g_i(a, q) - s(q|B_i) + q^T r_i > 0, \quad i \in \tilde{N}$ above inequality follows that

$$\Bigg[\Bigg(\frac{f_1(p,q) - s(q|D_1) + p^T w_1}{g_1(p,q) + s(q|F_1) - p^T t_1}, \frac{f_2(p,q) - s(q|D_2) + p^T w_2}{g_2(p,q) + s(q|F_2) - p^T t_2}, \ldots,$$

$$\frac{f_k(p,q) - s(q|D_k) + p^T w_k}{g_k(p,q) + s(q|F_k) - p^T t_k}\Bigg) - \Bigg(\frac{f_1(a,b) + (a|E_1) - b^T c_1}{g_1(a,b) - (a|B_1) + b^T r_1},$$

$$\frac{f_2(a,b) + (a|E_2) - b^T c_2}{g_2(a,b) - s(a|B_2) + b^T r_2}, \ldots, \frac{f_k(a,b) + (a|E_k) - b^T c_k}{g_k(a,b) - (a|B_k) + b^T r_k}\Bigg)\Bigg] \notin K \backslash \{0\}.$$

Hence, complete the theorem. □

Theorem 2 (Weak duality) *Let* $(a, b, \lambda, c, r) \in T^0$ *and* $(p, q, \lambda, w, t) \in Q^0$. *Let*

(i) $$\Bigg\{f_1(.,q) + (.)^T w_1 - \frac{f_1(p,q) - s(q|D_1) + p^T w_1}{g_1(p,q) + s(q|F_1) - p^T t_1}\Big(g_1(.,q) - (.)^T t_1\Big), \ldots,$$
$$f_k(.,q) + (.)^T w_k - \frac{f_k(p,q) - s(q|D_k) + p^T w_k}{g_k(p,q) + s(q|F_k) - p^T t_k}\Big(g_k(.,q) - (.)^T t_k\Big)\Bigg\}$$
be strongly K*-pseudoinvex w.r.t.* η_1 *at* p,

(ii) $$\Bigg\{f_1(a,.) - (.)^T c_1 - \frac{f_1(a,b) + s(a|E_1) - b^T c_1}{g_1(a,b) - s(a|B_1) + b^T r_1}(g_1(a,.) + (.)^T r_1), \ldots,$$
$$f_k(a,.) - (.)^T c_k - \frac{f_k(a,b) + S(a|E_k) - b^T c_k}{g_k(a,b) - s(a|B_k) + b^T r_k}\Big((g_k(a,.) + (.)^T r_k)\Big)\Bigg\}$$
be strongly K*-pseudoincave w.r.t.* η_2 *at* b,

(iii) $\eta_1(a, p) + p \in C_1$,

(iv) $\eta_2(q, b) + b \in C_2$.

Then,

$$\Bigg[\Bigg(\frac{f_1(p,q) - s(q|D_1) + p^T w_1}{g_1(p,q) + s(q|F_1) - p^T t_1}, \frac{f_2(p,q) - s(q|D_2) + p^T w_2}{g_2(p,q) + s(q|F_2) - p^T t_2}, \ldots,$$

$$\frac{f_k(p,q) - s(q|D_k) + p^T w_k}{g_k(p,q) + s(q|F_k) - p^T t_k}\Bigg) - \Bigg(\frac{f_1(a,b) + s(a|E_1) - b^T c_1}{g_1(a,b) - s(a|B_1) + b^T r_1},$$

$$\frac{f_2(a,b) + s(a|E_2) - b^T c_2}{g_2(a,b) - s(a|B_2) + b^T r_2}, \ldots, \frac{f_k(a,b) + s(a|E_k) - b^T c_k}{g_k(a,b) - s(a|B_k) + b^T r_k}\Bigg)\Bigg] \notin \textit{int}\, K.$$

Proof Theorem 1 proof follows the same approach as proof. □

Theorem 3 (Strong duality) *Let* $(\bar{a}, \bar{b}, \bar{\lambda}, \bar{c}, \bar{r})$ *be an efficient solution to (FPP) and fix* $\lambda = \bar{\lambda}$ *in (FDP). Suppose that all the conditions in Theorem 1 or 2 are fulfilled. Furthermore, suppose that*

(i) $\sum_{i=1}^{k} \bar{\lambda}_i \Big[\nabla_{bb} f_i(\bar{a}, \bar{b}) - \Big(\frac{\nabla_b f_i(\bar{a}, \bar{b}) - \bar{c}_i}{g_i(\bar{a}, \bar{b}) - s(\bar{a}|B_i) + \bar{b}^T r_i}$
$- \frac{(\nabla_b g_i(\bar{a},\bar{b}) + r_i)(f_i(\bar{a},\bar{b}) + s(\bar{a}|E_i) - \bar{b}^T \bar{c}_i)}{(g_i(\bar{a},\bar{b}) - s(\bar{a}|B_i) + \bar{b}^T \bar{r}_i)^2} \Big) (\nabla_b g_i(\bar{a}, \bar{b}) + \bar{r}_i)^T$
$- \frac{f_i(\bar{a}, \bar{b}) + s(\bar{a}|E_i) - \bar{b}^T \bar{c}_i}{g_i(\bar{a}, \bar{b}) - s(\bar{a}|B_i) + \bar{b}^T \bar{r}_i} (\nabla_{bb} g_i(\bar{a}, \bar{b})) \Big]$ *is positive or negative definite,*

(ii) $\left\{ \nabla_b f_i(\bar{a}, \bar{b}) - \bar{c}_i - \frac{f_i(\bar{a}, \bar{b}) + s(\bar{a}|E_i) - \bar{b}^T \bar{c}_i}{g_i(\bar{a}, \bar{b}) - s(\bar{a}|B_i) + \bar{b}^T \bar{r}_i} (\nabla_b g_i(\bar{a}, \bar{b}) + \bar{r}_i) \right\}_{i=1}^{k}$ *is linearly independent,*

(iii) *where K is a closed convex pointed cone with $\mathbb{R}^k_+ \subseteq K$.*

Then, there exist $\bar{w}_i \in E_i$ and $t_i \in E_i, i \in \tilde{N}$ s.t. $(\bar{a}, \bar{b}, \bar{\lambda}, \bar{w}, \bar{t})$ is an efficient of (FDP).

Proof Since $(\bar{a}, \bar{b}, \bar{\lambda}, \bar{c}, \bar{r})$ is an efficient solution of (FPP). By Fritz John, necessary optimality [11] conditions $\alpha \in K^*,\ \beta \in C_2,\ \eta \in \mathbb{R}_+$ s.t.

$$\Bigg\{ \sum_{i=1}^{k} \Bigg(\alpha_i \{ \nabla_a f_i(\bar{a}, \bar{b}) - \bar{w}_i - \frac{f_i(\bar{a}, \bar{b}) + s(\bar{a}|E_i) - \bar{b}^T \bar{c}_i}{g_i(\bar{a}, \bar{b}) - s(\bar{a}|B_i) + \bar{b}^T \bar{r}_i} (\nabla_a g_i(\bar{a}, \bar{b}) - \bar{t}_i) \Bigg) -$$

$$\Bigg(\frac{\nabla_a f_i(\bar{a}, \bar{b}) - \bar{w}_i}{g_i(\bar{a}, \bar{b}) - s(\bar{a}|B_i) + \bar{b}^T r_i)} - \frac{(\nabla_a g_i(\bar{a}, \bar{b}) - \bar{t}_i)(f_i(\bar{a}, \bar{b}) + s(\bar{a}|E_i) - \bar{b}^T c_i)}{g_i(\bar{a}, \bar{b}) - s(\bar{a}|B_i) + \bar{b}^T r_i)^2} \Bigg)$$

$$(\nabla_b g_i(\bar{a}, \bar{b}) + \bar{r}_i)^T \frac{f_i(\bar{a}, \bar{b}) + s(\bar{a}|E_i) - \bar{b}^T \bar{c}_i}{g_i(\bar{a}, \bar{b}) - s(\bar{a}|B_i) + \bar{b}^T \bar{r}_i} \nabla_{yx} g_i(\bar{a}, \bar{b}) \Bigg\} (a - \bar{a}) \geq 0,\ \forall a \in C_1, \tag{22.13}$$

$$\sum_{i=1}^{k} (\alpha_i - \eta \bar{\lambda}_i) \Bigg[\nabla_b f_i(\bar{a}, \bar{b}) - \bar{c}_i - \frac{f_i(\bar{a}, \bar{b}) + s(\bar{a}|E_i) - \bar{b}^T \bar{c}_i}{g_i(\bar{a}, \bar{b}) - s(\bar{a}|B_i) + \bar{b}^T \bar{r}_i} (\nabla_b g_i(\bar{a}, \bar{b}) + \bar{r}_i) \Bigg] + (\beta - \eta \bar{b})^T$$

$$\sum_{i=1}^{k} \bar{\lambda}_i \Bigg[\nabla_{bb} f_i(\bar{a}, \bar{b}) - \Bigg(\frac{\nabla_a f_i(\bar{a}, \bar{b}) - \bar{c}_i}{g_i(\bar{a}, \bar{b}) - s(\bar{a}|B_i) + \bar{b}^T r_i)}$$

$$- \frac{(\nabla_a g_i(\bar{a}, \bar{b}) + \bar{r}_i)(f_i(\bar{a}, \bar{b}) + s(\bar{a}|E_i) - \bar{b}^T c_i)}{g_i(\bar{a}, \bar{b}) - S(\bar{a}|B_i) + \bar{b}^T r_i)^2} \Bigg)$$

$$(\nabla_b g_i(\bar{a}, \bar{b}) + \bar{r}_i)^T - \frac{f_i(\bar{a}, \bar{b}) + S(\bar{a}|E_i) - \bar{b}^T \bar{c}_i}{g_i(\bar{a}, \bar{b}) - S(\bar{a}|B_i) + \bar{b}^T \bar{r}_i} \nabla_{bb} g_i(\bar{a}, \bar{b}) \Bigg] = 0, \tag{22.14}$$

$$(\beta - \eta\bar{b})^T\left[\nabla_b f_i(\bar{a},\bar{b}) - \bar{c}_i - \frac{f_i(\bar{a},\bar{b}) + s(\bar{a}|E_i) - \bar{b}^T\bar{c}_i}{g_i(\bar{a},\bar{b}) - s(\bar{a}|B_i) + \bar{b}^T\bar{r}_i}\ (\nabla_b g_i(\bar{a},\bar{b}) + \bar{r}_i)\right](\lambda_i - \bar{\lambda}_i) \geq 0,$$

$$\forall\lambda \in \text{int}K^*, i \in \tilde{N}, \tag{22.15}$$

$$\beta^T \sum_{i=1}^{k} \bar{\lambda}_i\left[\nabla_b f_i(\bar{a},\bar{b}) - \bar{c}_i - \frac{f_i(\bar{a},\bar{b}) + s(\bar{a}|E_i) - \bar{b}^T\bar{c}_i}{g_i(\bar{a},\bar{b}) - s(\bar{a}|B_i) + \bar{b}^T\bar{r}_i}(\nabla_b g_i(\bar{a},\bar{b}) + \bar{r}_i)\right] = 0, \tag{22.16}$$

$$\eta\bar{b}^T \sum_{i=1}^{k} \bar{\lambda}_i\left[\nabla_b f_i(\bar{a},\bar{b}) - \bar{c}_i - \frac{f_i(\bar{a},\bar{b}) + s(\bar{a}|E_i) - \bar{b}^T\bar{c}_i}{g_i(\bar{a},\bar{b}) - s(\bar{a}|B_i) + \bar{b}^T\bar{r}_i}(\nabla_b g_i(\bar{a},\bar{b}) + \bar{r}_i)\right] = 0,$$

$$\alpha_i\bar{b} + \lambda_i(\beta - \eta\bar{b}) \in N_{D_i}(c_i), i \in \tilde{N}, \tag{22.17}$$

$$\alpha_i\bar{b}\left(\frac{f_i(\bar{a},\bar{b}) + s(\bar{a}|E_i) - \bar{b}^T\bar{c}_i}{g_i(\bar{a},\bar{b}) - s(\bar{a}|B_i) + \bar{b}^T\bar{r}_i}\right) - \bar{\lambda}_i(\beta - \eta\bar{b})\left(\frac{f_i(\bar{a},\bar{b}) + s(\bar{a}|E_i) - \bar{b}^T\bar{c}_i}{g_i(\bar{a},\bar{b}) - S(\bar{a}|B_i) + \bar{b}^T\bar{r}_i}\right)$$

$$\left[(\nabla_b g_i(\bar{a},\ \bar{b}) + \bar{r}_i)\bar{b} - 1\right] \in N_{F_i}(r_i),\ i \in \tilde{N}, \tag{22.18}$$

$$\bar{t}_i \in B_i, \bar{w}_i \in E_i,\ i \in \tilde{N}, \tag{22.19}$$

$$\bar{a}^T\bar{w}_i^T = S(\bar{a}|E_i),\ \bar{t}_i \in B_i,\ \bar{w}_i \in E_i,\ i \in \tilde{N}, \tag{22.20}$$

$$(\alpha,\beta,\eta) \geq 0,\ (\alpha,\beta,\eta) \neq 0. \tag{22.21}$$

Inequality (22.15) can be rewritten as

$$(\beta - \eta\bar{b})^T\left[\nabla_b f_i(\bar{a},\bar{b}) - \bar{c}_i - \frac{f_i(\bar{a},\bar{b}) + S(\bar{a}|E_i) - \bar{b}^T\bar{c}_i}{g_i(\bar{a},\bar{b}) - s(\bar{a}|B_i) + \bar{b}^T\bar{r}_i}\right.$$

$$\left.(\nabla_b g_i(\bar{a},\bar{b}) + \bar{r}_i)\right] = 0, \quad i \in \tilde{N}. \tag{22.22}$$

Multiplying (22.14), by $(\beta - \eta\bar{b})^T$ from left and by (22.22), we have

$$(\beta - \eta\bar{b})^T \sum_{i=1}^{k} \bar{\lambda}_i\left[\nabla_{bb} f_i(\bar{a},\bar{b}) - \left(\frac{\nabla_b f_i(\bar{a},\bar{b}) - \bar{c}_i}{g_i(\bar{a},\bar{b}) - s(\bar{a}|B_i) + \bar{b}^T r_i}\right.\right.$$
$$\left. - \frac{(\nabla_b g_i(\bar{a},\bar{b}) + r_i)(f_i(\bar{a},\bar{b}) + s(\bar{a}|E_i) - \bar{b}^T\bar{c}_i)}{(g_i(\bar{a},\bar{b}) - s(\bar{a}|B_i) - \bar{b}^T\bar{r}_i)^2}\right)(\nabla_b g_i(\bar{a},\bar{b}) + \bar{r}_i)^T$$
$$\left. - \frac{f_i(\bar{a},\bar{b}) + s(\bar{a}|E_i) - \bar{b}^T\bar{c}_i}{g_i(\bar{a},\bar{b}) - s(\bar{a}|B_i) - \bar{b}^T\bar{r}_i}\ (\nabla_{bb} g_i(\bar{a},\bar{b}))\right](\beta - \eta\bar{b}) = 0.$$

By using assumption (i), it follows that

$$\beta = \eta \bar{b}. \tag{22.23}$$

From (22.14) and (22.19), we get

$$\sum_{i=1}^{k}(\alpha - \eta\bar{\lambda}_i)\left[\nabla_b f_i(\bar{a},\bar{b}) - \bar{c}_i - \frac{f_i(\bar{a},\bar{b}) + s(\bar{a}|E_i) - \bar{b}^T\bar{c}_i}{g_i(\bar{a},\bar{b}) - s(\bar{a}|B_i) + \bar{b}^T\bar{r}_i}(\nabla_b g_i(\bar{a},\bar{b}) + \bar{r}_i)\right] = 0. \tag{22.24}$$

According to hypothesis (ii) and Eq. (22.24), it gives that

$$\alpha_i = \eta\,\bar{\lambda}_i, \quad i \in \tilde{N}. \tag{22.25}$$

If $\eta = 0$, then $\alpha_i = 0, i \in \tilde{N}$. From (22.23), $\beta = 0$. Thus, $(\alpha, \beta, \eta) = 0$, which contradicts the fact (22.21).

Hence, $\eta > 0$. From (22.25) and hypothesis (iii), since, $\mathbb{R}^k_+ \subseteq K \Rightarrow K^* \subseteq \mathbb{R}^k_+ \Rightarrow \text{int}K^* \subseteq \text{int}(\mathbb{R}^k_+)$. As $\bar{\lambda} \in \text{int}K^* \Rightarrow \bar{\lambda} > 0$, we have $\alpha_i > 0, i \in \tilde{N}$. By using (22.13), (22.23) and (22.25), we obtain

$$(a-\bar{a})^T\sum_{i=1}^{k}\bar{\lambda}_i\left[\nabla_a f_i(\bar{a},\bar{b}) - \bar{w}_i - \frac{f_i(\bar{a},\bar{b}) + s(\bar{a}|E_i) - \bar{b}^T\bar{c}_i}{g_i(\bar{a},\bar{b}) - s(\bar{a}|B_i) + \bar{b}^T\bar{c}_i}\right.$$
$$\left.(\nabla_b g_i(\bar{a},\bar{b}) - \bar{t}_i)\right] \geq 0, \quad \forall a \in C_1. \tag{22.26}$$

Substituting (22.25) in (22.26) and using the fact that $\eta > 0$, we get

$$(a-\bar{a})^T\sum_{i=1}^{k}\bar{\lambda}_i\left[\nabla_a f_i(\bar{a},\bar{b}) + \bar{w}_i - \frac{f_i(\bar{a},\bar{b}) + S(\bar{a}|E_i) - \bar{b}^T\bar{c}_i}{g_i(\bar{a},\bar{b}) - S(\bar{a}|B_i) + \bar{b}^T\bar{c}_i}\right.$$
$$\left.(\nabla_b g_i(\bar{a},\bar{b}) - \bar{t}_i)\right] \geq 0, \quad \forall a \in C_1. \tag{22.27}$$

Let $a \in C_1$. Then, $\bar{a} + a \in C_1$ as C_1 is a closed convex cone, and so from (22.27), it yields

$$a^T\sum_{i=1}^{k}\bar{\lambda}_i\left[\nabla_a f_i(\bar{a},\bar{b}) + \bar{w}_i - \frac{f_i(\bar{a},\bar{b}) + s(\bar{a}|E_i) - \bar{b}^T\bar{c}_i}{g_i(\bar{a},\bar{b}) - s(\bar{a}|B_i) + \bar{b}^T\bar{c}_i}(\nabla_b g_i(\bar{a},\bar{b}) - \bar{t}_i)\right] \geq 0,$$
$$\forall a \in C_1,$$

which follows that

$$\sum_{i=1}^{k} \bar{\lambda}_i \Bigg[\nabla_a f_i(\bar{a}, \bar{b}) + \bar{w}_i - \frac{f_i(\bar{a}, \bar{b}) + s(\bar{a}|E_i) - \bar{b}^T \bar{c}_i}{g_i(\bar{a}, \bar{b}) - s(\bar{a}|B_i) + \bar{b}^T \bar{c}_i} (\nabla_b g_i(\bar{a}, \bar{b}) - \bar{t}_i) \Bigg] \geq 0, \quad \forall a \in C_1^*. \tag{22.28}$$

Now, taking $a = 0$ and $a = 2\bar{a}$, simultaneously in (22.27), we get

$$a^T \sum_{i=1}^{k} \bar{\lambda}_i \Bigg[\nabla_a f_i(\bar{a}, \bar{b}) + \bar{w}_i - \frac{f_i(\bar{a}, \bar{b}) + s(\bar{a}|E_i) - \bar{b}^T \bar{c}_i}{g_i(\bar{a}, \bar{b}) - s(\bar{a}|B_i) + \bar{b}^T \bar{c}_i} (\nabla_b g_i(\bar{a}, \bar{b}) - \bar{t}_i) \Bigg] = 0. \tag{22.29}$$

Also, from the expression $\beta = \eta\, \bar{b}$, we get

$$\bar{b} = \frac{\beta}{\eta} \in C_2, \text{ as } \eta > 0. \tag{22.30}$$

From (22.19), (22.28) and (22.29), we obtain

$$\bar{b}\, \bar{c}_i = s(\bar{b}|D_i), \bar{b}^T \bar{r}_i = s(\bar{b}|F_i), \quad i \in \tilde{N}. \tag{22.31}$$

Thus, $(\bar{a}, \bar{b}, \bar{\lambda}, \bar{w}, \bar{t}) \in Q^0$ and the value objective functions of (FPP) and (FDP) are same. Obviously, $(\bar{a}, \bar{b}, \bar{\lambda}\bar{w}, \bar{t})$ is an efficient solution of (FDP). Hence, it completes the result. □

Theorem 4 (Converse duality) *Let $(\bar{p}, \bar{q}, \bar{\lambda}, \bar{w}, \bar{t})$ be an efficient solution to (FDP) and fix $\lambda = \bar{\lambda}$ in (FPP). Assume that all of the criteria in Theorems 1 and 2 are fulfilled. Consider the following:*

(i) $\sum_{i=1}^{k} \bar{\lambda}_i \Bigg[\nabla_{qq} f_i(\bar{p}, \bar{q}) - \Bigg(\frac{\nabla_q f_i(\bar{p}, \bar{q}) + \bar{w}_i}{g_i(\bar{p}, \bar{q}) + s(\bar{p}|B_i) - \bar{q}^T t_i} - \frac{(\nabla_q g_i(\bar{p},\bar{q}) + t_i)(f_i(\bar{p},\bar{q}) + s(\bar{p}|E_i) + \bar{q}^T \bar{w}_i)}{(g_i(\bar{p},\bar{q}) + s(\bar{p}|B_i) - \bar{q}^T \bar{t}_i)^2} \Bigg) (\nabla_q g_i(\bar{p}, \bar{q}) + \bar{t}_i)^T - \frac{f_i(\bar{p}, \bar{q}) - s(\bar{p}|E_i) + \bar{q}^T \bar{w}_i}{g_i(\bar{p}, \bar{q}) + s(\bar{p}|B_i) - \bar{q}^T \bar{t}_i} (\nabla_{qq} g_i(\bar{p}, \bar{q})) \Bigg]$ *is positive or negative definite,*

(ii) $\left\{ \nabla_q f_i(\bar{p}, \bar{q}) - \bar{w}_i - \frac{f_i(\bar{p}, \bar{q}) + s(\bar{p}|E_i) - \bar{q}^T \bar{w}_i}{g_i(\bar{p}, \bar{q}) - s(\bar{p}|B_i) + \bar{q}^T \bar{t}_i} (\nabla_q g_i(\bar{p}, \bar{q}) + \bar{t}_i) \right\}_{i=1}^{k}$ *is linearly independent,*

(iii) where K is a closed convex pointed cone with $\mathbb{R}_+^k \subseteq K$.

Then, there exist $\bar{w}_i \in E_i$ *and* $t_i \in B_i,\ i \in \tilde{N}$ *s.t.* $(\bar{p}, \bar{q}, \bar{\lambda}, \bar{w}, \bar{t})$ *is an efficient of (FPP).*

Proof The primal-dual model present in the paper is symmetric. So, Theorem 3 proof follows the same approach as proof. □

Conclusions

In this article, we have discussed a new type of non-differentiable multiobjective fractional programming problem over arbitrary cone functions and derived duality theorems under the aforesaid assumptions. It will be interesting to see whether the above mathematical programming problems developed over second and higher order. This would be the task of some of our forthcoming works.

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Chapter 23
Primary Health Care Facility Location and Telemedicine

Kaushal Kumar

Abstract A major challenge for achieving universal health care is to sustain balanced primary care access in society. In highly populated developing countries like India, a low-cost approach is inevitable. Telemedicine can be highly beneficial to address this concern. Moreover, the growing demand for health care in epidemics like COVID-19 has also promoted the usage of teleconsultation. This article presents a framework for incorporating telemedicine services to strengthen the existing primary health care network. An integer linear programming-based optimization model has been developed for locating primary care facilities in the presence of functioning telemedicine services. The objective would be to find a trade-off between the location of new primary health care facilities and the usage of telemedicine services for primary health care. Numerical experiments and a case study have been carried out to explain the working of the model and to derive policy directions. While performing sensitivity analysis concerning budget, it was observed that teleconsultation services outweigh new primary care facilities. Moreover, increasing the levels of capacity by 50% over existing capacity levels resulted in complete coverage. By conducting a case study in the northeast district of Delhi, it was found that complete coverage could not be achieved and therefore, new sites for primary care centers along with new teleconsultation services are required to be identified.

Keywords Primary health care · Location-allocation · Telemedicine · Teleconsultation · Maximal covering location model · COVID-19

Introduction

Universal health coverage requires a robust health care system in place. The role of primary health care is immeasurable in ensuring satisfactory functioning of a health care system [26]. Reinforcing primary health care is the most efficient and cost-

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effective approach to strengthen overall health system [4]. Success of primary health care depends on accessibility of primary health care services [25]. If the access to primary health care services is not convenient and comfortable, it tends to scale down the effectiveness of primary health care. A useful approach to address this concern is to consider patient preferences while designing a primary health care system.

The demand for primary health care services becomes manifold during pandemics like COVID-19. Moreover, the closing of health facilities in many countries has resulted in primary care services cutback [15]. As a result, delivery of primary care services becomes an enormous challenge [17]. Expansion of existing primary care services is therefore needed to meet the increased demand [3]. Virtual health care practices like teleconsultation can be highly beneficial in these situations [18, 28].

Teleconsultation services assist in the smooth delivery of primary care and become highly significant during epidemics [23]. Virtual health practices like teleconsultation have the potential to regulate congestion at health facilities and thereby improve their utilization [30]. Moreover, teleconsultation is not only a cost-effective approach, but it can also cater to a higher population in comparison with traditional health systems [12]. Therefore, in addition to improving access to care, teleconsultation also lifts the quality of health care delivery [22]. Kadir [17] discussed the importance of teleconsultation services for primary care during the COVID-19 outbreak. Jnr [16] presented a discussion on the role of digital health care practices like telemedicine in handling pandemic situations like COVID-19.

Leventer-Roberts et al. [20] conducted a cross-sectional study in Israel and found teleconsultation feasible for pediatrician after-hours services. Vosburg and Robinson [28] in their study reported higher patient and provider satisfaction levels associated with teleconsultation services in comparison with in-person visits. Bressman et al. [5] through their findings established the ability of teleconsultation services to improve post-discharge primary care access. Gomez et al. [11] performed a qualitative study on physicians and found teleconsultation services convenient for patients and helpful in improving access to primary care.

Extensive research is available on location-allocation problems in health care. The majority of the location-allocation literature caters to an emergency-free environment [12]. Emergencies like pandemics result in a sudden increase in health care demand, and prompt location-allocation decision-making of health services is required in such situations [29]. Moreover, location-allocation decision-making is inevitable for establishing an emergency response system [21]. In recent times, the outburst of COVID-19 calls for making teleconsultation an inherent part of a health system to deal with emergencies [24].

Gulzari and Tarakci [12] presented a location-allocation framework involving teleconsultation for assisting relief activities during an earthquake emergency. Gao et al. [9] presented a location-allocation model for locating temporary health centers to handle earthquake-emergency operations. This study presents a location-allocation model that incorporates teleconsultation services in addition to finding the location of new health centers. Demirbas and Ertem [8] presented a mixed-integer programming-based location-reallocation model for disaster relief operations. Geng et al. [10] introduced a location-allocation emergency relief model based on victims' pain perception cost.

Maximal covering location model given by Church and ReVelle [7] is concerned with maximizing the coverage of the target population. Due to this important objective, it finds applications in various service domains, viz. health care, banking, retail, hotel, restaurant, etc. [6]. Moreover, the significance of coverage maximization becomes immense in emergencies for health care. Many contributions illustrate the utilization of the maximal covering model in health emergencies. Zhang et al. [31] utilized maximal covering location model to develop an optimization framework to deal with uncertainty during emergencies. Alizadeh et al. [1] presented a multi-period version of the maximal covering location model for the location of relief centers for calamities. Hassan et al. [13] suggested a nonlinear version of the maximal coverage model for locating field hospitals to tackle the COVID-19 pandemic. Taiwo [27] used maximal covering location model for locating supplementary COVID-19 testing laboratories in Nigeria. This study presents a novel expansion of the maximal covering location model that considers engagement of teleconsultation services and locating new facilities simultaneously for the first time.

This study extends the maximal covering location model to incorporate teleconsultation services in the health care optimization modeling framework with a view to achieving universal basic health care. The proposed model tries to find a trade-off between locating new primary care centers and using teleconsultation services for primary care. The objective is the maximization of overall patient coverage. The model considers patient choice for locating new primary care facilities and also ensures equitable distribution of these facilities by means of anti-covering restrictions. Moreover, the model also takes care of congestion and places capacity restrictions for both primary care centers and teleconsultation services.

In the next section, the optimization model is presented. Numerical illustrations of the model are provided in Section "Numerical Experiment". A real-life case, where the model is implemented is given in Section "Case Study". Section "Conclusion" presents conclusion.

Optimization Model

The success of public health care programs depends on the effectiveness of strategic decision-making, e.g., location-allocation problems in health care. For effective application of location-allocation problems in health care, the preference of patients holds very high importance. Moreover, the inclusion of teleconsultation services in the location-allocation modeling framework will make it more practical and robust. This study presents next a discrete choice framework-based expansion of the maximal covering location model that incorporates teleconsultation services. Assumptions of the proposed optimization model are listed below:

1. Patients depend on primary care centers and teleconsultation services for their primary care needs.
2. The model is based on a discrete choice framework.
3. Physical facilities are separated by *minimum separation distance* to ensure their equitable distribution.
4. Policy maker has limited budget for location-allocation decision-making.
5. Primary care facilities and teleconsultation-based facilities have limited capacities.

Teleconsultation-Based Maximal Coverage Model

Consider a territory where teleconsultation program is required to be setup alongside locating new primary care centers. The objective is to satisfy the primary care demand of the target population concentrated in various neighborhoods. Consider, D_i be the demand for primary health care in locality $i \in I$; $j_1 \in J_1$ represents a likely location for a new primary care facility and $j_2 \in J_2$ represents a health care facility providing primary care services through teleconsultation.

Consider $a_{ij}, i \in I, j \in J_1$, be a binary parameter capturing the preference of patients in locality i for new primary care facility at $j \in J_1$. When patients at i make use of primary care facility $j \in J_1$, a_{ij} is 1, otherwise it is 0. Further, assume $b_{ij}, i \in I, j \in J_2$, be a binary parameter capturing the choice of patients in locality i for teleconsultation service provided by $j \in J_2$. The objective of providing primary health care services to maximum population in the territory jointly by new facilities and teleconsultation can be achieved by extending the *maximal covering location model* [7] as given below.

Mathematical model

$$\textbf{Maximize} \sum_{i \in I} \sum_{j \in J_1} D_i \ X_{ij} + \sum_{i \in I} \sum_{j \in J_2} D_i \ Y_{ij} \tag{23.1}$$

subject to

$$\sum_{j \in J_1} a_{ij} X_{ij} + \sum_{j \in J_2} b_{ij} Y_{ij} = 1 \qquad \forall i \in I \tag{23.2}$$

$$\sum_{j \in J_1} C_{1j} W_j + \sum_{j \in J_2} C_{2j} Z_j \leq B \tag{23.3}$$

$$\sum_{i \in I} D_i X_{ij} \leq \text{Cap}_j^1 \qquad \forall j \in J_1 \tag{23.4}$$

$$\sum_{i \in I} D_i Y_{ij} \leq \text{Cap}_j^2 \qquad \forall j \in J_2 \tag{23.5}$$

$$\alpha(1 - W_j) \geq \sum_{k \in S_j} W_k \qquad \forall j \in J_1 \tag{23.6}$$

$$X_{ij} \leq W_j \qquad \forall i \in I, j \in J_1 \tag{23.7}$$

$$Y_{ij} \leq Z_j \qquad \forall i \in I, j \in J_2 \tag{23.8}$$

$$X_{ij} \in [0, 1] \qquad \forall i \in I, j \in J_1$$

$$Y_{ij} \in [0, 1] \qquad \forall i \in I, j \in J_2$$

$$W_j \in \{0, 1\} \qquad \forall j \in J_1$$

$$Z_j \in \{0, 1\} \qquad \forall j \in J_2$$

In this model, C_{1j} is cost of locating new primary care facility at $j \in J_1$; C_{2j} is cost to the government for bringing health facility $j \in J_2$ into teleconsultation network; W_j represents a binary variable taking value 1 if a new primary care facility gets established at $j \in J_1$; Z_j also represents a binary variable taking value 1 if health care facility $j \in J_2$ gets included in teleconsultation network; X_{ij} represents the proportion of primary care demand at locality i serviced by primary care facility at $j \in J_1$, and Y_{ij} represents the proportion of primary care demand at locality i served by the facility $j \in J_2$ through teleconsultation; Cap_j^1 represents capacity of new primary care facility at $j \in J_1$, and Cap_j^2 represents capacity of facility $j \in J_2$ for teleconsultation services; B represents available budget; $S_j = \{k \in J_1\text{: for } k \neq j, d_{kj} \leq D\}$; D is radius of exclusion zone for new primary care facilities and d_{kj} is distance between "k" and "j"; α represents a large number.

The objective function (23.1) is maximizing population coverage through new primary care centers and teleconsultation services. Constraint (23.2) represents the covering constraint and ensures that demand at locality i is covered either through new primary care centers or teleconsultation. (23.3) represents the budget limitation. (23.4) and (23.5) are capacity constraints. Constraint (23.6) is the anti-covering restriction for new primary care facilities to ensure equitable distribution of these facilities. (23.7) and (23.8) represent contingency conditions.

Numerical Experiment

In this section, we explain the working of the model discussed in Section "Optimization Model" through a small example. Consider a territory with 15 localities, named L1, L2, . . ., L15. There are five sites identified for locating new primary care facilities (H1, H2, . . ., H5), and five health care facilities identified for teleconsultation services (H6, H7, . . ., H10). Tables 23.1, 23.2 and 23.3 provide the parameter values required for the model. For each locality, preferences for new primary care facilities and teleconsultation services are listed in Table 23.1. For each site for a physical primary care facility, the primary care facilities lying at sites in the anti-covering exclusion zone are provided in Table 23.2.

Table 23.1 Population characteristics

Locality	Demand for primary care	Choice for primary care	
		New facility	Teleconsultation
L1	72	H1 H3 H5	H6 H7 ··· H10
L2	65	H1 H3 H4	H6 H7 ··· H10
L3	56	H1 H3 H4 H5	H6 H7 ··· H10
L4	69	H1 H2 H4	H6 H7 ··· H10
L5	84	H1 H4 H5	H6 H7 ··· H10
L6	92	H1 H4	H6 H7 ··· H10
L7	75	H1 H3 H5	H6 H7 ··· H10
L8	54	H2 H5	H6 H7 ··· H10
L9	45	H2 H5	H6 H7 ··· H10
L10	52	H2 H3	H6 H7 ··· H10
L11	68	H2 H3 H5	H6 H7 ··· H10
L12	64	H2 H3	H6 H7 ··· H10
L13	77	H2 H4 H5	H6 H7 ··· H10
L14	66	H2 H3 H4	H6 H7 ··· H10
L15	57	H2 H3 H5	H6 H7 ··· H10

Table 23.2 New facility-site characteristics

Site	H1	H2	H3	H4	H5
Capacity	150	150	150	150	150
Cost of setting up	45	40	54	62	65
Sites in exclusion zone	H2	H1	H4 H5	H3	H3

Table 23.3 Teleconsultation facility characteristics

Facility	H6	H7	H8	H9	H10
Capacity	50	50	50	50	50
Cost of teleconsultation contracts	20	20	20	20	20

The model presented in section "Optimization Model" was applied on the data given in Tables 23.1, 23.2 and 23.3. Initially, the solution was observed at different levels of budget. Table 23.4 presents the experimental results. With increase in budget, coverage increases as expected (Fig. 23.1). A change in combinations of new sites and teleconsultation is observed, and no definite trend was detected (Fig. 23.2). For higher budget levels, number of teleconsultation contracts outweigh the number of new primary care centers. It might be happening due to anti-covering restrictions for new primary care centers. As seen in Table 23.4, a maximum coverage of 70.28% could be

Table 23.4 Sensitivity analysis_budget variations

Budget	Coverage (%)	Teleconsultation	Tele coverage (%)	New facilities	New facilities coverage (%)
20	5.02	H6	5.02	–	0
40	15.06	–	0	H2	15.06
50	15.06	–	0	H1	15.06
60	20.08	H10	5.02	H2	15.06
80	25.10	H7 H8	10.04	H2	15.06
100	30.12	H8 H9 H10	15.06	H2	15.06
120	35.14	H10	5.02	H2 H3	30.12
140	40.16	H6 H7 H8 H9 H10	25.10	H2	15.06
150	40.16	H6 H7 H8 H9 H10	25.10	H2	15.06
200	55.22	H6 H7 H8 H9 H10	25.10	H2 H3	30.12
250	65.26	H6 H7 H9 H10	20.08	H2 H4 H5	45.18
300	70.28	H6 H7 H8 H9 H10	25.10	H1 H4 H5	45.18
350	70.28	H6 H7 H8 H9 H10	25.10	H1 H4 H5	45.18

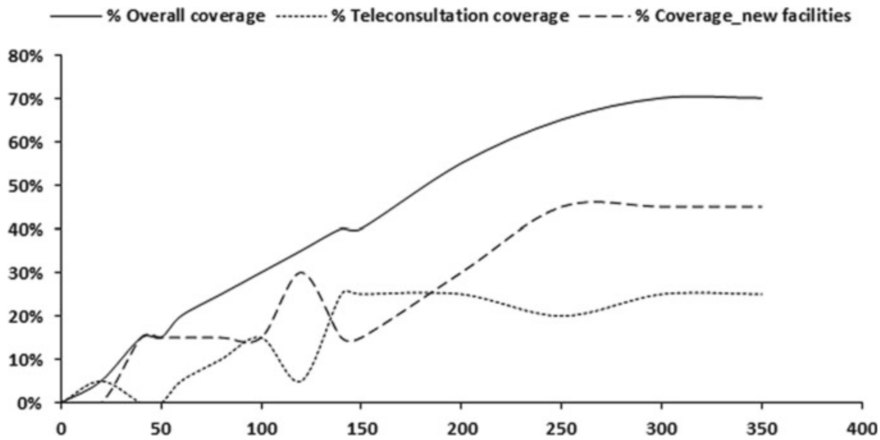

Fig. 23.1 Experiment_budget versus coverage

achieved in the existing setup. To increase the coverage further, either additional sites for new primary care facilities or new facilities providing teleconsultation services are to identified.

The model was further observed at different capacity levels. Capacities of both new primary care centers and teleconsultation services were increased simultane-

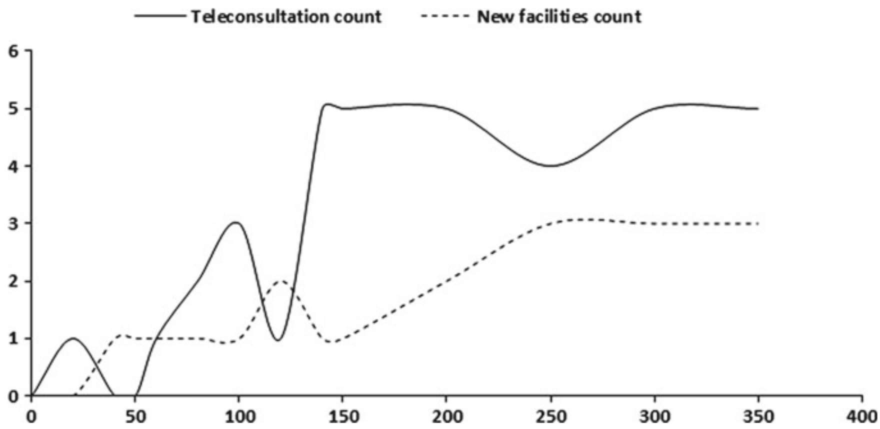

Fig. 23.2 Experiment_budget versus teleconsultation and new facilities count

Table 23.5 Sensitivity analysis_capacity variations

Capacity new facility	Capacity teleconsultation	Overall coverage (%)	Teleconsultation coverage (%)	New facilities coverage (%)
90	30	42.17	15.06	27.11
105	35	49.20	17.57	31.63
120	40	56.22	20.08	36.14
135	45	63.25	22.59	40.66
150	50	70.28	25.10	45.18
165	55	77.31	27.61	49.70
180	60	84.34	30.12	54.22
195	65	91.37	32.63	58.73
210	70	98.39	35.14	63.25
225	75	100.00	37.65	62.35

ously in the same proportion. Budget was fixed at 300 for this experiment. Table 23.5 shows the results of the experiment. Again, overall coverage increased with increase in the levels of capacity. Increasing trend is observed in the coverage through teleconsultation and new facilities. At capacity level (225, 75), the entire population gets covered, and teleconsultation services are utilized to their full capacity and remaining coverage corresponds to new primary care centers. This complete coverage is achieved by making a 50% increase in existing capacity levels, viz. (150, 50).

Later, a combined experiment was performed to see the effect of changing budget and capacity simultaneously. Results are presented in Table 23.6. At low budget and capacity levels, the coverage was attributed equally to teleconsultation and new primary care centers, while at intermediary budget levels, coverage by new facilities exceeds teleconsultation coverage slightly. When capacity levels are very large, cov-

Table 23.6 Sensitivity analysis_budget and capacity variations

Budget	Capacity new facility	Capacity telecon-sultation	Overall coverage (%)	Tele count	Tele coverage (%)	New facilities count	New facilities coverage (%)
100	90	30	18.08	3	9.04	1	9.04
100	150	50	30.12	3	15.06	1	15.06
100	210	70	42.17	0	0	2	42.17
200	90	30	33.13	5	15.06	2	18.07
200	150	50	55.22	5	25.10	2	30.12
200	210	70	77.31	5	35.14	2	42.17
350	90	30	42.17	5	15.06	3	27.11
350	150	50	70.28	5	25.10	3	45.18
350	210	70	98.39	5	35.14	3	63.25

erage by new primary care centers exceeds teleconsultation coverage by a significant amount.

Case Study

A case study was conducted to comprehend the usefulness of proposed optimization model. The model was applied on a data set pertaining to economically weaker people[1] residing in northeast district of Delhi. The analysis was performed on twenty nine localities of disadvantaged population. Locations of twenty-one *Mohalla clinics* (viz. M1, M2, ..., M21) have been considered as potential candidates for locating new primary care centers, while ten big hospitals have been considered for teleconsultation.[2] Table 23.7 presents the population of twenty nine neighborhoods considered for the study and number of *Mohalla clinics* located within a radius of 300 m, 500 m and 1 km of these twenty nine neighborhoods. Spatial distribution of twenty nine population centers and twenty-one *Mohalla clinics* can be found in Fig. 23.3.

The proposed optimization model is aimed at maximizing the coverage of the population seeking primary care. In this study, coverage of 19,828 households residing in twenty nine localities is to be maximized through new primary care centers and teleconsultation services. For locating new primary care facilities, choice of population is considered. It was assumed that a patient visits a new primary care center if it is located within 500 m from the residence. Moreover, it is also assumed that every patient prefers to use teleconsultation-based primary care services through

[1] https://delhishelterboard.in/main/.

[2] http://health.delhigovt.nic.in/wps/wcm/connect/doit_health/Health/Home/Directorate+General+of+Health+Services/Aam+Aadmi+Mohalla+Clinics/.

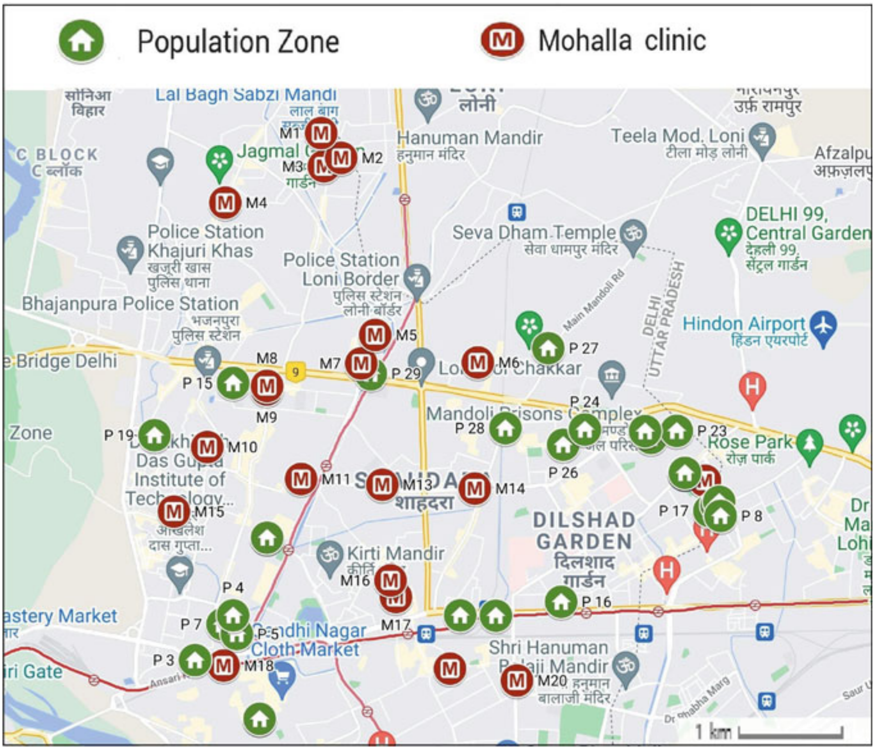

Fig. 23.3 Spatial distribution of localities and Mohalla clinics

any available facility. The anti-covering distance was assumed to be 1.5 km for primary care facilities. Cost of locating a new primary care center was assumed to be INR 2,000,000, whereas cost incurred on establishing teleconsultation services in a hospital is considered to be INR 500,000. Capacity of a new primary care center is assumed to be 2000 households, while capacity of a single teleconsultation platform is considered to be 500 households. Table 23.8 presents the implementation of the proposed model at different levels of budget. It was observed that teleconsultation services are prevalent at low budget levels. With increase in budget, new primary care facilities are located. As seen in Fig. 23.4, patient coverage increases with increase in budget and a maximum coverage of 67.35% could be achieved through existing setup of primary care centers and teleconsultation services (Table 23.8). It implies that there is a need to consider new set of sites for primary care centers, and additional hospitals should be included into teleconsultation network.

Table 23.7 Target localities and Mohalla clinics in northeast Delhi

Zone	Population size	Mohalla clinics within		
		300 m	500 m	1 km
p1	3943	0	0	1
p2	5000	0	0	1
p3	151	0	1	1
p4	348	0	0	1
p5	254	0	1	1
p6	1000	1	1	2
p7	107	0	0	1
p8	175	0	1	1
p9	313	0	1	1
p10	157	0	1	1
p11	129	1	1	1
p12	110	1	1	1
p13	320	0	1	1
p14	444	1	1	1
p15	488	0	2	3
p16	841	0	0	0
p17	1107	0	1	1
p18	123	0	0	3
p19	223	0	0	2
p20	557	0	0	2
p21	1073	1	1	1
p22	999	0	1	1
p23	188	0	0	1
p24	93	0	0	0
p25	290	0	0	1
p26	69	0	0	0
p27	99	0	0	1
p28	492	0	0	2
p29	735	0	0	1

Conclusion

To achieve universal health care, standards of primary care require improvement. Moreover, pandemics like COVID-19 increase the demand for primary health care by an immense amount. To meet the rising demand in these situations, finding locations of new primary care facilities is helpful but not sufficient. Teleconsultation services can help to a great extent in these situations. Therefore, policymakers should consider

Table 23.8 Case study

Budget	Coverage (%)	Teleconsultation	Tele coverage (%)	New facilities	New facilities coverage (%)
1,000,000	5.04	2	5.04	–	0
2,000,000	10.09	4	10.09	–	0
4,000,000	20.18	4	10.09	1	10.09
5,000,000	25.22	6	15.13	1	10.09
6,000,000	30.26	8	20.17	1	10.09
8,000,000	40.34	8	20.17	2	20.17
10,000,000	49.72	8	20.17	3	29.55
15,000,000	65.72	10	25.22	5	40.50
20,000,000	67.35	10	25.22	7	42.13
25,000,000	67.35	10	25.22	7	42.13

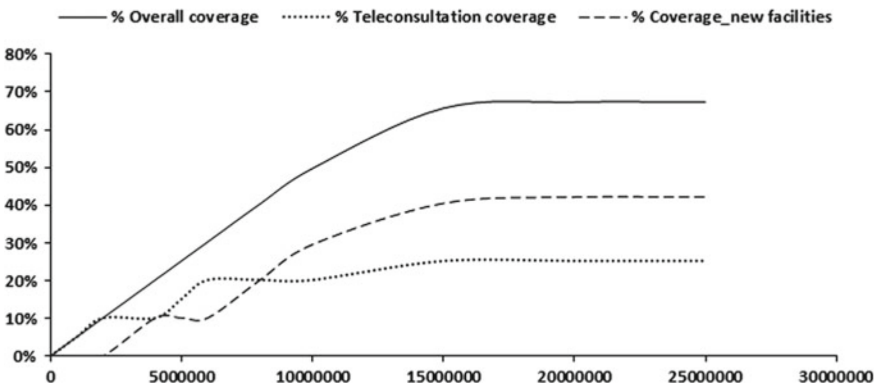

Fig. 23.4 Case study_budget versus coverage

the possibility of starting teleconsultation services along with locating new primary care facilities.

In this article, a location optimization model is proposed that incorporates teleconsultation services in the location-allocation optimization framework. Numerical experiments along with a case study were carried out to explain the working of the model. The proposed optimization model was solved by varying model parameters. Teleconsultation services surpassed new primary care facilities at a higher budget level. Moreover, increasing the levels of capacity by 50% over existing capacity levels resulted in complete coverage. A case study in the northeast district of Delhi has discovered the need to identify new sites for primary care centers and additional teleconsultation services.

Pandemics like COVID-19 enforce the need for extensive health care services instantly. Giving due consideration to primary health care is required to meet rising health care demand in such situations. Locating new makeshift primary health care centers will be helpful but not sufficient. Therefore, telemedicine services should be inherent to the primary health care system as a future medical practice. Keeping this motivation in mind, this article presents a location-allocation-based decision-making framework for incorporating telemedicine services in the existing primary health care network. Implementation of the model will help in strengthening the existing primary health care network.

Parameters involved in proposed optimization model are assumed to be known and deterministic. However, status of health changes dynamically in real life. Zhang and Atkins [32] presented probabilistic choice models to design a network of medical care facilities. Bagherinejad and Shoeib [2] considered dynamic capacities in maximal covering location problem and described a dynamic maximal coverage model. The proposed model can be more practical if it involves dynamic parameters.

Integer linear programming (ILP) problems are generally very complex and hard to solve, and therefore, efficient solution algorithms should be developed for these problems. The model proposed in this paper is an integer programming formulation and requires a solution algorithm for dealing with larger instances of the problem. There are many contributions illustrating the importance and utilization of solution algorithms in facility location problems. Some of the recent contributions are given as follows: [2, 14, 19, 32]. We are also currently working on designing a solution algorithm for solving larger dimensions of the problem more quickly and efficiently.

The proposed model considers locating facilities of same type that can be extended to location of facilities of multiple types. Heyns and van Vuuren [14] proposed a location model for locating multiple types of facilities in multiple areas. Kumar and Bardhan [19] proposed an optimization model for locating two types of primary health care centers. A more general location modeling approach would be to develop a hierarchical facility network. A three-tier health care system may be developed to satisfy health care needs of the population.

This paper also presents a case study concentrated on the northeast district of Delhi. The study has brought some interesting and important insights useful for policymaking. This study can be extended to other regions of the city to draw more general policy directives. The implementation of the proposed model in rural areas is expected to produce more useful results and observations.

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Chapter 24
Modified Round Robin CPU Scheduling: A Fuzzy Logic-Based Approach

Rajeev Sharma, Atul Kumar Goel, M. K. Sharma, Nitesh Dhiman, and Vishnu Narayan Mishra

Abstract Multiprogramming plays an essential role more effectively in resources utilization. In the context of multiprogramming, CPU scheduling plays a key role. Several algorithms have already been introduced to achieve the objectives of CPU scheduling. Among these, Round Robin is one of the important CPU scheduling algorithms, but time quantum and unnecessary context switching increase the waiting time, response time, and turnaround time in the scheduling of processes in this algorithm. In this work, we planned a new procedure based on fuzzy logic method to overcome the drawbacks of Round Robin scheduling policy in order to optimize the waiting time, response time, and turnaround time. We modified the Round Robin algorithm to improve the effectiveness of CPU scheduling in a multiprogramming environment. We also compared the outcome of proposed algorithm with exiting scheduling algorithms like; Round Robin and FCFS. We also gave the future aspects along with the particle application and limitations of proposed fuzzy Round Robin technique.

Keywords Fuzzy logic · CPU scheduling · First come first serve (FCFS) · Time quantum (TQ) · Round robin (RR)

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Introduction

In the present scenario, multiprogramming [1] plays an essential task in the modern technology. In multiprogramming situation, several processes are assigned to run on CPU. When multiple processes reached in ready queue to run and wait for the allocation of CPU, then operating system (OS) [2, 3] plays a key role in deciding the execution time for a process [2, 4] through scheduler and will assign a process to CPU. The assignment of a process from the number of processes in ready queue is carried out by a unit which is known as CPU scheduler or short-term scheduler [5]. The algorithm used to make such types of decision is called scheduling algorithm (SA) [6]. Numbers of scheduling algorithms are available for the execution of processes. They are differentiating with their properties. For the assessment of CPU scheduling algorithms and choosing the optimal algorithm, a number of constraints have been recommended. Some of the important constraints are as follows: Fairness, CPU utilization, Throughput, Turnaround time, Waiting time, Response time, etc. CPU scheduling is essential in multiprogramming environment since it has a huge impact on the resource sharing and other performance parameters.

To achieve the maximum use of CPU [7, 8] and make system efficient, fast, and fair, many scheduling algorithms exist. Some of them are as follow: FCFS: In this algorithm, the job scheduled according their arrival time means the process reached first will execute first [9, 10]. Next process is selected for the execution and after completion of the previous one. SJF: According to shortest job first algorithm, the jobs will be scheduled according to their burst time [9, 11] means the job which has minimum burst time will execute first among others. SRTF: According to shortest remaining time first, a job scheduled for execution that has very less remaining time after execution [9, 12]. Priority scheduling: In priority-based scheduling, the process is scheduled for execution that has very high priority compare to other jobs [9, 13]. Multilevel queue: Here, the ready queue is divided in numerous queues and the allotment of jobs to one queue depend upon some parameters such as: (1) Memory size (2) Precedence of Jobs (3) Types of Jobs, [14]. HRRN: In highest response ratio, the processes are based upon the response ratio. To schedule the processes, response ratio of each process is calculated, and the process which has highest response ratio is scheduled first for completing [15]. Round Robin algorithm: In this approach, scheduler is assigned for the process from ready queue to the CPU in such a manner that no processes will get CPU more than one as given value of TQ in one cycle [9, 16, 17].

The static value of TQ for the execution of each process from the ready queue is an issue because the value is too large; then, the response time increases for every process, and it will work like FCFS, but if it is too less, then it creates an unnecessarily frequent context switching which decreases the throughput of algorithm due to more overheads. Another problem in Round Robin is that after execution of each process in one cycle with its arrival time, it uses similar approach to run the execution for the process processes in next cycle which increases the turnaround time and waiting time.

Inside of this work, a new scheduling technique is proposed based upon fuzzy logic [18] to choose the value of TQ neither large nor less. Such that each process got minimum response time and prevent unnecessary context switching. There are many real-life applications of fuzzy logic [19, 20], and its extension [21–23] exists in literatures. The proposed algorithm makes a decision for the execution of executed processes after completion of one cycle that will be execute and how longtime it will in next cycle to accomplish the maximum throughput and least waiting time and turnaround time. A generalized fuzzy decision-making process based on hesitant fuzzy sets [24] in the context of drug assortment to treat the slight symptoms of COVID-19 has been given in the existing literature. Fuzzy logic controller has also been used in aided expert relaying mechanism system [25]. The novelty and the objective of the proposed work is illustrated by the following points;

(1) Round Robin is one of the essential CPU scheduling algorithm, but the response time got increase, due to TQ and unnecessary context switching. In this work, we will build up a novel approach based on fuzzy logic approach to defeat the drawbacks of Round Robin
(2) Proposed algorithm will help us to optimize the waiting time response time and turnaround time.
(3) We will generalize the well-known Round Robin algorithm to enhance the usefulness of CPU scheduling for multiprogramming prospective.
(4) We will give a numerical example to illustrate the applicability of proposed methodology.
(5) We will also give a comparative study in between the results of existing approach and proposed approach.
(6) We will also give some practical applications of the developed modified Round Robin approach.

The present work is alienated into seven sections. In Section "Basic Concepts Related to Proposed Work", we gave some basic concept of fuzzy logic, fuzzy inference system, and CPU scheduling algorithm and multiprocessor scheduling. In Section "Proposed Algorithm", we defined the structure of proposed algorithm. Architecture of the developed algorithm is also defined in this section. In Section "FIS for Finding Time Quantum", we used the fuzzification process for including factors (that is the input factors: number of jobs, burst time, and output factor: time quantum). In Section "Results and Comparison", we compared the result of proposed algorithm with existing FCFS and Round Robin algorithms. In Section "Practice Importance and limitations", we gave the practical applications and limitations of the proposed algorithm. In Section "Conclusion and Future Work", we draw the conclusion of entire work and gave the interpretation of our results.

Basic Concepts Related to Proposed Work

CPU Scheduling

CPU scheduling [26] is a technique used to determining the execution of one process over another one. The main objective of CPU scheduling is to make sure that when the CPU becomes idle, the operating system (OS) must select at least one of the process available in the queue for execution process. The selection process follows the CPU scheduler, the scheduler chooses from out of the process in memory that are prepared to accomplish.

Scheduling Criteria

Various scheduling algorithms have different properties and may favor one class of procedures over additional. To select a particular algorithm according to the requirement, we must follow the properties of numerous algorithms. The criteria used for the selection of best algorithm include CPU throughput, utilization, waiting time, turnaround time, and response time.

Scheduling Algorithms

CPU scheduling deals with the decision-making to find the process in the ready queue for allocation. There are various CPU scheduling algorithms exist like; first come, first severed (FCFS), shortest job first (SJF), priority scheduling, Round Robin scheduling, multilevel queue scheduling, and multilevel feedback queue scheduling [27].

Multiple-processor Scheduling

Multiple scheduling problem [28] exists in the system when systems have numerous CPUs; in such cases, the scheduling method became more compound. Many systems exist where the processor is homogeneous, or it may in heterogeneous form in terms of their functionality. Apart from these, many symmetric and asymmetric multiprocessing-based system are also applied on the existing techniques [29–32]. But still, there are some limitation of using such multiprocessor systems.

To avoid such schemes, where we have more than one scheduling approaches, the fuzzy logic technique plays a vital role, to handle all the uncertain factors involve in the problem. In this work, we gave a generalized CPU scheduling [33] scheme based on fuzzy logic approach to handle the problem having single or more than one processor-based techniques.

Fuzzy Logic

The term fuzzy logic plays an important role when the things are not defined in crisp environment. In practical situations, the researchers face often a difficulty when the state cannot represent by true or 1 and false or 0. In order to overcome such types of difficulty, fuzzy logic provides an excellent tool for reasoning. By using this concept, the fuzziness present in states may be characterized in effetely way. In Boolean system, true and false values are represented by 1 and 0, respectively, but in circumstance of fuzzy logic, there is an intermediate value lies between 0 and 1 which represents the situation of partially true or partially false. For example, suppose one have to allot subjects to some students who are under counseling and then one has to study their concentration, ability, independent thinking, logical reasoning, memory power, etc., which are fuzzy in nature. In this situation, the theory of crisp sets cannot be used. But, this difficulty can be solved by concept of fuzzy sets. So, the theory of fuzzy sets is more interesting, and many mathematicians are attracted by its fascination and usefulness.

Fuzzy Set

Let us consider a universal set U and a non-empty subset F of U, i.e., $F \subseteq U$, then F is called a fuzzy set on U if it is represented as $F = \{(u, \mu_{\mathrm{F}}(u)) : u \in U\}$, where $\mu_{\mathrm{F}} : U \to [0, 1]$ called membership function. For example, a triangular fuzzy set (as shown in Fig. 24.1a) can be written as $F = (a_1, a_2, a_3)$ or can be defined as

$$F = \begin{cases} \frac{u-a_1}{a_2-a_1}, & \text{if } a_1 \le u \le a_2 \\ \frac{a_3-u}{a_3-a_2}. & \text{if } a_2 \le u \le a_3 \end{cases}$$

Fuzzy Inference Systems (FISs)

FIS is moreover known by another name which is fuzzy rule-based system, one of the most popular computing frameworks which based on fuzzy rules. FIS takes the crisp

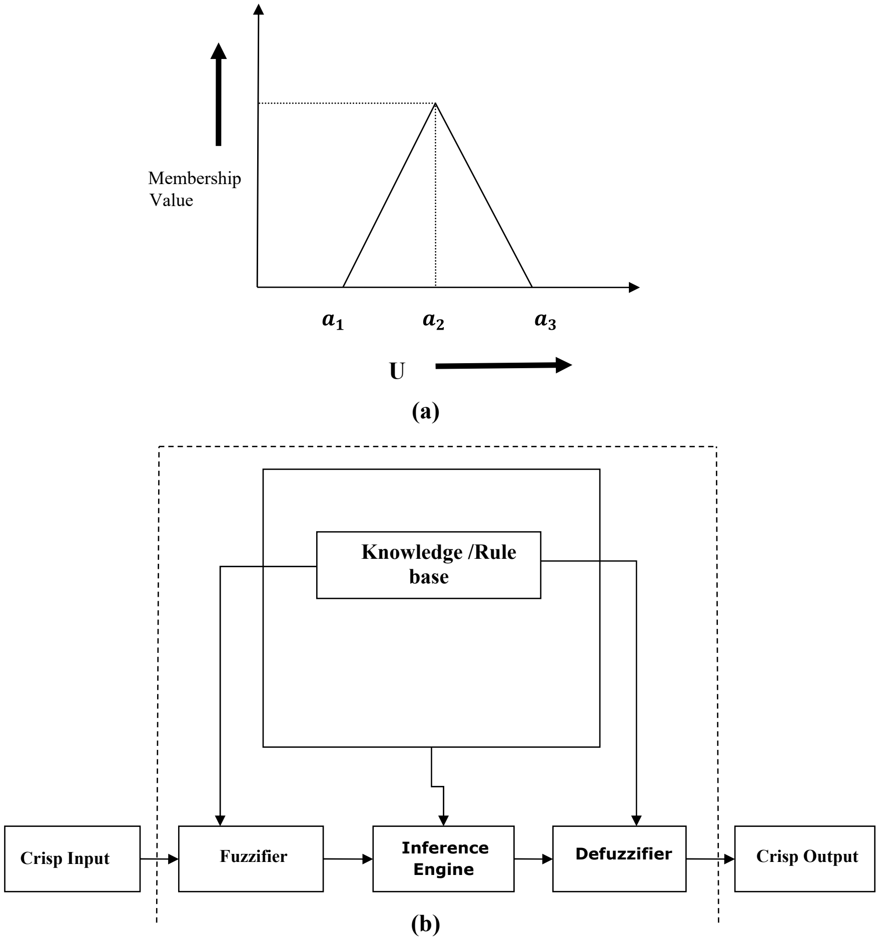


Fig. 24.1 **a** Triangular membership function. **b** Structure of FIS

input, fuzzify the input, processes the fuzzify input with knowledge base and produce the fuzzy output and for final crisp output defuzzify the output. Along with fuzzy logic, fuzzy reasoning, it uses the "IF-THEN" rules for drawing essential decision rules.

FIS has three components which are as follows:

- Fuzzification
- Inference Engine (Knowledge Base)
- Defuzzification.

Block diagram of FIS has been exposed in Fig. 24.1b. FIS makes the judgment from a knowledgebase by using inference engine. An FIS includes three parts as shown in Fig. 24.1b, namely as input, processing, and output. In the first step, FIS takes a crisp value as an input and produce the fuzzy output for processing stage which process the fuzzy value according to fuzzy rules and fuzzy knowledge base. After this output stage it takes fuzzy value as an input and produces the crisp output.

FIS is based on five steps. They are like:

- Fuzzification
- Relevant fuzzy operators
- Fuzzy implication
- Aggregation
- Defuzzification for crisp output.

Figure 24.1b describes the structure of FIS. Three fuzzy inference systems [34–36] have been used for decision-making. The FIS is called Mamdani's fuzzy inference developed by Ebrahim Mamdani et al. in 1975 [34]; the second inference system is Takagi-Sugeno-Kang fuzzy inference method introduced in 1985 [35].

Proposed Algorithm

In Round Robin [37] algorithm, the value of TQ remains fixed, and then, processes are schedule in such manner that no process gets CPU more than the value of TQ in one cycle. Big amount value of TQ increases the response time, and less amount value increases the unnecessarily frequent context switching which result is less throughput, so to overcome of this issue, we have used the fuzzy logic which provides a value for TQ which is not very large and no very less. Another issue with Round Robin is remaining burst time of executed processes because each process executed with their allocated TQ in one cycle, but if the remaining burst time of the process is additional to the given TQ, then another cycle is required. For the execution of the executed processes in next cycle, Round Robin used similar approach which increased turnaround time and waiting time, so to overcome this issue; we have proposed a selection procedure in next cycle that the process which have shortest remaining burst time will accomplish first and repeat this procedure until the execution of all process. The planned algorithm will be accomplished in three stages which assist to reduce the average waiting, average, average turnaround time and response time. The phases are as follow:

Phase 1:

Step 1-Select n number of processes from ready queue.

Step 2-Calculate the average burst time (ABT) of the selected processes.

Step 3-For finding the value of TQ, allot entire process and ABT to the FIS.

Step 4-Take output value as TQ from FIS.

Step 5-By applying Round Robin algorithm, allocate every process to CPU with find TQ.

Phase 2:

Subsequently finishing of first cycle, carry out the following steps:

Step 6-Choose the minimal process from the waiting queue according to their remaining burst time and allocate to CPU.

Step 7-After step 6, select another minimal process for execution by exclusive of the previously executed process in this stage.

Phase 3: Till the whole execution of whole processes, repeat phase 2.

Block Diagram for Proposed Technique

The architecture of the planned algorithm is represented in Chart 24.1.

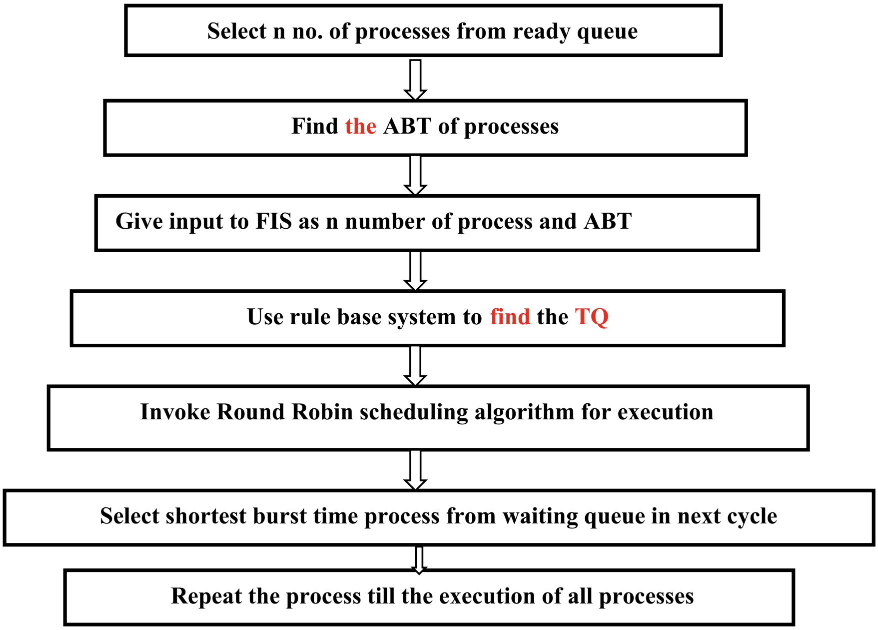


Chart 24.1 Flowchart of proposed algorithm

FIS for Finding Time Quantum

For finding the TQ by FIS, we shall take two inputs and one output as result [14]. The primary input is number of process/jobs in the system which can be represented by N, and the second input is ABT of the process in the ready queue. TQ is the result of FIS. Here, we have used three triangular membership functions, numbers of process/jobs, ABT as input and TQ as output which are as

Fuzzy Membership Function for Number of Process/Jobs (N)

Membership function (MF) type: Triangular, range: 0–10. We have divided N by three different triangular MF which are low (Lw)—[1, 2.5, 4], medium (Md)—[3, 5, 7], high (Hg)—[6, 7.5, 9]

Membership functions of Lw, Md, and Hg for number of jobs:

$$\mu_{\mathrm{Lw}}(x) = \begin{Bmatrix} \frac{x-1}{1.5} \text{if } 1 \le x \le 2.5 \\ \frac{4-x}{1.5} \text{if } 2.5 \le x \le 4 \\ 0 \text{ if } x \le 1 \text{ and } x \ge 4 \end{Bmatrix}$$

$$\mu_{\mathrm{Md}}(x) = \begin{Bmatrix} \frac{x-3}{2} \text{if } 3 \le x \le 5 \\ \frac{7-x}{2} \text{if } 5 \le x \le 7 \\ 0 \text{ if } x \le 3 \text{ and } x \ge 7 \end{Bmatrix}$$

$$\mu_{\mathrm{Hg}}(x) = \begin{Bmatrix} \frac{x-6}{1.5} \text{if } 6 \le x \le 7.5 \\ \frac{9-x}{1.5} \text{if } 7.5 \le x \le 9 \\ 0 \text{ if } x \le 6 \text{ and } x \ge 9 \end{Bmatrix}$$

To fuzzify number of process/jobs, we have used the technique which is illustrated in Fig. 24.2. We have separated it in three triangular ranges which are Lw, Md, and Hg. The outcome will lie in these proposed ranges with value from 0 to 1 depending upon the number of process/jobs.

Membership Function for the ABT

Type—triangular, range: 0–10, Lw—[−3, 0, 3], Md—[2, 4, 6], Hg—[5, 10, 15]

Membership functions of Lw, Md, and Hg for ABT:

$$\mu_{\mathrm{Lw}}(x) = \begin{Bmatrix} \frac{x+3}{3} \text{if} - 3 \le x \le 0 \\ \frac{3-x}{3} \text{if } 0 \le x \le 3 \\ 0 \text{ if } x \le 0 \text{ and } x \ge 3 \end{Bmatrix}$$

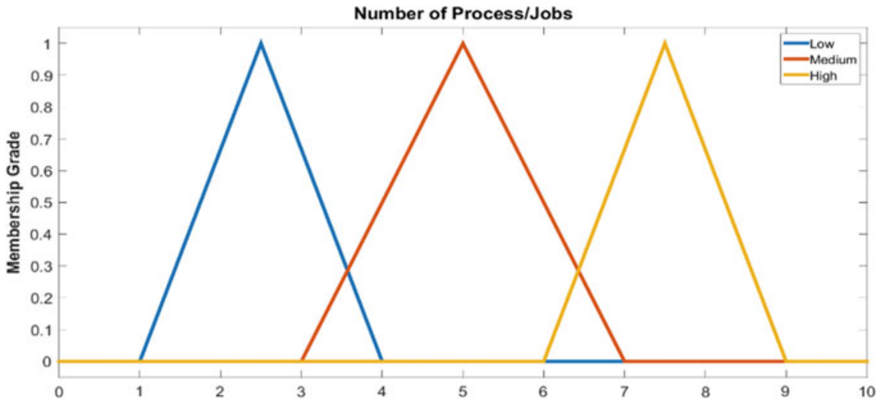


Fig. 24.2 Memberships functions for jobs

$$\mu_{\mathrm{Md}}(x) = \left\{ \begin{array}{c} \frac{x-2}{2}\text{if } 2 \le x \le 4 \\ \frac{6-x}{2}\text{if } 4 \le x \le 6 \\ 0 \text{ if } x \le 2 \text{ and } x \ge 6 \end{array} \right\}$$

$$\mu_{\mathrm{Hg}}(x) = \left\{ \begin{array}{c} \frac{x-5}{5}\text{if } 5 \le x \le 10 \\ \frac{15-x}{5}\text{if } 10 \le x \le 15 \\ 0 \text{ if } x \le 5 \text{ and } x \ge 15 \end{array} \right\}$$

To fuzzifying ABT, we have used the technique which we have illustrated in Fig. 24.3. We have also separated it in three triangular ranges Lw, Md, and Hg. We have cover up the burst times in the range of 1–10 starting from least to highest burst time support depending upon ABT outcome will lies in these ranges with value from 0 to 1.

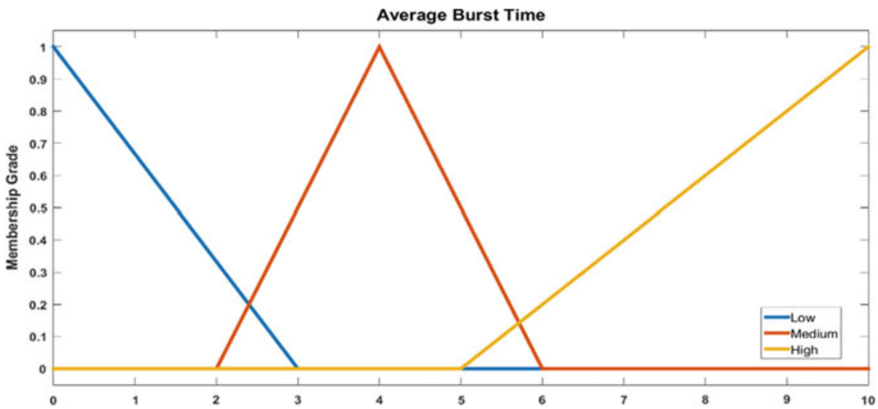


Fig. 24.3 Membership function for ABT

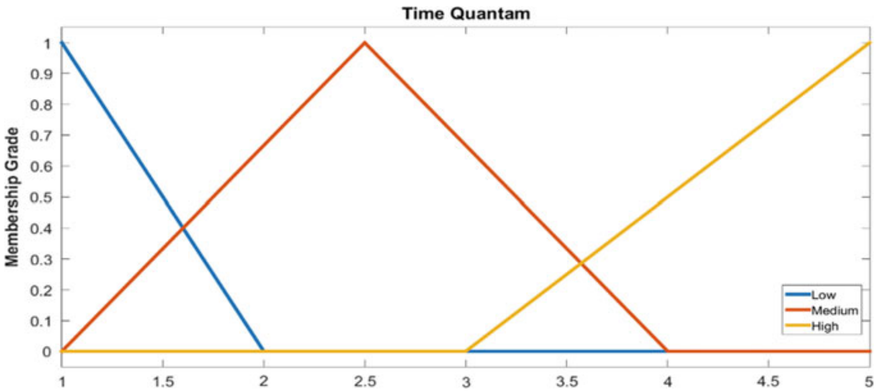


Fig. 24.4 Membership function for time quantum

Membership Function for Time Quantum

Type—triangular, range: 1–5, Lw—[0, 1, 2], Md—[1, 2.5, 4], Hg—[3, 5, 7]

Membership functions of Lw, Md, and Hg:

$$\mu_{\mathrm{Lw}}(x) = \left\{ \begin{array}{c} \frac{x-0}{1}\text{if} - 0 \le x \le 1 \\ \frac{2-x}{1}\text{if } 1 \le x \le 2 \\ 0 \text{ if } x \le 1 \text{ and } x \ge 2 \end{array} \right\}$$

$$\mu_{\mathrm{Md}}(x) = \left\{ \begin{array}{c} \frac{x-1}{1.5}\text{if } 1 \le x \le 2.5 \\ \frac{4-x}{1.5}\text{if } 2.5 \le x \le 4 \\ 0 \text{ if } x \le 1 \text{ and } x \ge 4 \end{array} \right\}$$

$$\mu_{\mathrm{Hg}}(x) = \left\{ \begin{array}{c} \frac{x-3}{2.5}\text{if } 3 \le x \le 5.5 \\ \frac{8-x}{3.5}\text{if } 5.5 \le x \le 8 \\ \text{if } x \le 3 \text{ and } x \ge 8 \end{array} \right\}$$

To fuzzifying TQ, we have also separated it into three categories as exposed in Fig. 24.4. We have covered the given TQ between 1 and 5 ranges and measured the estimated TQ, based on the rule base table as deliberated in next section.

Rule Base of FIS to Find the TQ

See Table 24.1.

Fuzzy inference system can use rule base which is shown in Table 24.1 for calculate estimated time quantum.

Table 24.1 Fuzzy rule for linguistic terms

S. No.	Number of process/jobs	Average burst time (ABT)	Time quantum (TQ)
1	Lw	Lw	Lw
2	Lw	Md	Md
3	Lw	Hg	Hg
4	Md	Lw	Md
5	Md	Md	Md
6	Md	Hg	Md
7	Hg	Lw	Lw
8	Hg	Md	Lw
9	Hg	Hg	Md

Table 24.2 Outcome of planned scheme with first experimental data

Name	Arrival time	Burst time	Response time	Complete time	Turnaround time	Waiting time
P1	0	53	0	160	160	107
P2	1	30	25	107	106	76
P3	2	27	50	102	100	73
P4	3	68	75	178	175	107

Results and Comparison

We have designed the FIS to generate the suitable TQ which play a key role to draw the result. Here, the given TQ was 20, but to find the new fuzzy-based time quantum, we have assigned (a) No of process = 4, (b) Average burst time = 40 as input to FIS, and appropriate time quantum = 25 as the output. Table 24.2 is illustrating the result of proposed algorithm like response time, complete time, waiting time, and turnaround time. Table 24.3 demonstrate the comprasion result of planned algorithm with existing FCFS and Round Robin scheduling schemes. Table 24.4 shows the result of proposed scheme with second experimental data.

Table 24.3 Comprasion of result with existing algorithms using first experimental data

Scheduling techniques	Average response time	Average turnaround time	Average waiting time
FCFS	61.5	104.25	60.25
Round robin	30	137.25	92.75
Proposed scheme	37.5	135.25	90.75

Gantt chart for modified Round Robin scheduling

Table 24.4 Result of proposed scheme with second experimental data

Name	Arrival time	Burst time	Response time	Complete time	Turnaround time	Waiting time
P1	0	30	0	125	125	95
P2	2	60	25	200	198	138
P3	4	40	50	165	161	121
P4	6	50	75	190	184	134

P1			P2		P3		P4		P3	P2	P1	P4	P1	P4
0	25	50	75	100	102	107	132	157	160	178				

"Average Response Time = 37.5"

"Average Turnaround Time = 135.25"

"Average Waiting Time = 90.75"

Gantt chart for Modified Round Robin scheduling

P1		P2	P3	P4	P5	P1	P3	P4	P2	P2
0	25	50	75	100	120	125	140	165	190	200

"Average Response Time = 50"

"Average Turnaround Time = 146"

"Average Waiting Time = 106"

From the above results as shown in Tables 24.3 and 24.4, we obtain that the proposed algorithm decreased the average response time, average waiting time, and average turnaround time. Furthermore, the planned algorithm is more competent with existing scheduling algorithms (Table 24.5).

Table 24.5 Comprasion of result with existing algorithms using second experimental data

Scheduling techniques	Average response time	Average turnaround time	Average waiting time
FCFS	86	122	82
Round robin	50	150	116
Proposed scheme	50	146	106

Practice Importance and Limitations

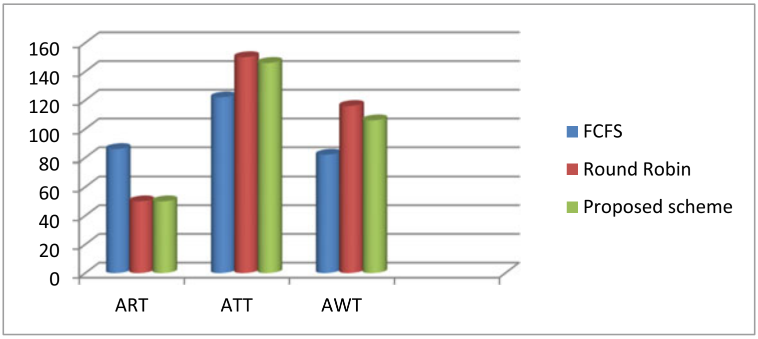
We can apply the proposed algorithm in multiprocessing. The proposed policy can be applied in real-time systems, which respond to the occurrence within a precise time limit. The proposed algorithm schedules the tasks with an optimal way in cloud computing environment. It can minimize the resources starvation. Packets scheduling in network is a big task, so the proposed algorithm utilized the propagation of packets in network. It can be used in load balancing when a client request to data from the remote located server to distribute the request across set of servers. Apart for this, there are few limitations of proposed algorithm; we cannot assign the priority of processes; small TQ value may increase the region of Gantt chart. Furthermore, the small slicing time may decrease the processor output, and the outcomes of the execution of the processes depend on TQ.

Conclusion and Future Work

The key aim of planned algorithm is to eliminate the drawbacks of traditional Round Robin and improve the performance by using the fuzzy logic in which fuzzy set theory deals with the impreciseness present in selection of task. The planned algorithm provides an innovative way of selection of executed jobs from waiting queue in second cycle which reduces the waiting time, response time, and turnaround time. The entire work done in this research paper illustrated following points;

- FIS is used to discover an optimum TQ which can avoid the unnecessary context switching of jobs and generalization the RR technique.
- The result of selection shortest job from waiting queue in second execution cycle has been minimize the waiting time, response time, and turnaround time of traditional Round Robin.
- The throughput of proposed algorithm is high, gave comprehensive and optimal solutions than the existing approaches.
- Charts 24.2 and 24.3 shown the comprasion result between FCFS, Round robin, and proposed algorithm.

In short, we can say that proposed algorithm also opens a novel method to create a new hybrid scheduling algorithm by merging it with other developed algorithms.



"ART: Average Response Time" "ATT: Average Turnaround Time"

Chart 24.2 Comprasion chart of obtained result with existing algorithms

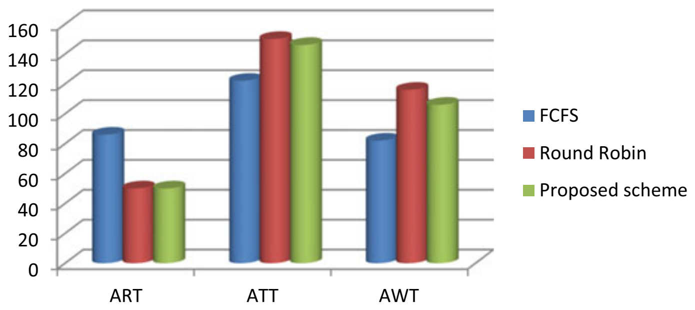


Chart 24.3 Comprasion chart of obtained result with existing algorithms

Declaration of Competing Interest The authors declare that they have no personal relationships or known competing financial interests that could have reflected to influence the work reported in this article.

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Chapter 25
Modeling of Fourier–Motzkin Elimination Technique for Separable Programming Problem

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Abstract Separable programming problem (SPP) is an important subclass of nonlinear programming problem (NLPP). SPP deals with the programming problems of optimizing the linear (or/and) nonlinear objective function, along with subject to a set of constraints having linear (or/and) nonlinear inequalities in nature. In this research paper, an attempt has been made to solve SPP by using Fourier–Motzkin elimination technique. For this, first, we convert SPP to LPP by approximating each separable function to a piecewise linear function. Then, the solution of programming problem thus obtained can be recovered by Fourier–Motzkin elimination (FM-E) technique, which is used to solve the system of linear inequalities.

Keywords Separable programming problem (SPP) · Fourier–Motzkin elimination (FM-E) technique · Objective function · Breaking points · Inequalities

Introduction

There may be nonlinear objective function or constraints of nonlinear nature or both in an NLPP. In this research paper, SPP is being discussed. Separable programming is one of the indirect methods to solve an NLPP. One can solve an NLPP by using these methods which deal with one or more linear problems that are extracted from the original problem. Our aim in this proposed research is to reduce the computation time of the problem under consideration, in which objective function is treated as constraint in nature. One can easily see these types of constraints in design of water supply system, management of water supply system, logistics, analysis of electrical

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network, econometric data fitting. Various methods or techniques are available so far in the literature to solve both linear as well as nonlinear programming problems. Earlier Jain [1] and Jain and Singh [3] presented the solution of SPP by applying Gauss elimination technique and modified Fourier elimination technique, respectively. Jain et al. [2] presented the solution of multi-objective linear programming problem (MOLPP) by Fourier–Motzkin elimination technique. After that, Tak et al. [4] solved the linear fractional programming problem (LFPP) by FM-E technique.

In this research paper, we are proposing an attempt to find an optimal solution of SPP by FM-E technique. Firstly, we formulate the given SPP into LPP by approximating it into piecewise linear functions and with the help of breaking points. Then, an optimal solution of this LPP can be recovered by FM-E technique. We have solved three different type of programming problems in which nonlinearity occurs only in an objective function, in constraints or in both. We tried to solve different types of SPPs by FM-E technique for the first time in the literature. We are not claiming that it is somehow better than Gauss elimination and modified Fourier elimination techniques, but it is obvious from this paper that one can also solve SPP by FM-E technique.

The paper unfold as follows: A brief taxonomy of Fourier–Motzkin elimination (F-ME) technique, which is applied to find an optimal solution of SPP, is described in Section "Fourier–Motzkin Elimination Technique". The problem under consideration which is to be solved by proposed F-ME technique is discussed in Section "The Problem". The solution of different types of nonlinear programming problems by F-ME technique is presented in Section "Numerical Example".

Fourier–Motzkin Elimination Technique

Initially, F-ME technique was applied on equations, but here, in this research work, we applied this technique on inequalities to find an optimal solution of separable nonlinear programming problem. The basic requirement to deal with this technique is that the system of inequalities must be of the same nature either ($\leq$) or ($\geq$). This technique consists of three different classes as far as variables (x_i) are concerned, which are mentioned below:

First Class: The inequalities, in which coefficient of the variable x_i is $+$ 1, are included in the first class.

Second Class: The inequalities, in which coefficient of the variable x_i is (-1), are included in the second class.

Third Class: The inequalities, in which coefficient of the variable x_i is 0, are included in the third class.

Now, the process of elimination of the variables is carried out in such a way that the variables reduced one by one in each iteration. Finally, there remains a single variable with bounded values at the end of the process. Accordingly, one can find an optimal value of that particular variable. By applying the process of back substitution, one can get the values of decision variables. FM-E technique takes least time to achieve an optimal solution of programming problem as compared to other existing techniques available in the literature so far. It is easy to understand, and least computation time is required in this technique.

The Problem

Consider the generalized form of SPP:

$$\begin{aligned} &\text{Max./Min. } Z = \sum_{i=1}^{n} f_i(x_i) \\ &\sum_{i=1}^{n} g_{ij}\left(x_j\right) \leq b_j, \text{ for } j = 1, 2, \ldots, m \\ &x_i \geq 0 \qquad\qquad \text{for } i = 1, 2, \ldots, n \end{aligned}$$

where some or all $g_{ij}, x_{ij}, f\left(x_j\right)$ are nonlinear. In the next step, with the help of arbitrary break points, we can approximate the sub-function g_{ij}. Let us assume that K_j denotes the number of breaking points for the *j*th variable, breaking value is a_{jk}, and the weight associated with the *K*th breaking point of *j*th variable be w_{jk}. Now, the reduced SPP is as:

$$\begin{aligned} &\text{Max./Min. } Z = \sum_{i=1}^{n} \sum_{k=0}^{r_i} f(x_{ik})\lambda_{ik} \\ &\sum_{i=1}^{n} \sum_{k=0}^{r_i} g_{jk}(x_{ik})\lambda_{ik} \leq b_j, \text{ for } j = 1, 2, \ldots, m \\ &x_i \geq 0, \qquad\qquad \text{for } i = 1, 2, \ldots, n \end{aligned}$$

where $\sum_{k=0}^{r_i} \lambda_{ik} = 1$ and $0 \leq \lambda_{ik} \leq 1$.

Numerical Example

Here, we are presenting the detailed solution procedure of different types of nonlinear programming problems one by one in which either nonlinearity occurs only in objective function of the problem under consideration or only nonlinear constraints involved in the problem, or nonlinearity occurs in both objective function as well as in constraints involved in the problem.

(a) Consider the following NLPP in which nonlinearity occurs only in objective function [3]

$$\begin{aligned} \text{Max} \quad & Z = x_1^2 - 2x_1 + x_2 \\ \text{subject to } & x_1 + 2x_2 \le 5 \\ & 2x_1 + x_2 \le 6 \\ \text{and} \quad & x_1, x_2 \ge 0 \end{aligned}$$

The objective function of the above problem is already in the maximization form. Now, consider the following separable functions:

$$f_1(x_1) = x_1^2 - 2x_1, \quad f_2(x_2) = x_2$$
$$g_{11}(x_1) = x_1, \quad g_{12}(x_2) = 2x_2, \quad g_{21}(x_1) = 2x_1, \quad g_{22}(x_2) = x_2$$

Here, $f_1(x_1)$ is present in nonlinear form; therefore, we have to approximate it into piecewise linear function. By inspection, the above problem suggests that $0 \le x_1 \le 2$. To obtain the approximate LPP, for the given NLPP; divide the interval $0 \le x_1 \le 2$ into three breaking points x_{ik} ($k = 0, 1, 2$) such that:

$$x_{10} = 0, \quad x_{11} = 1, \quad x_{12} = 2$$

We have, $f_1(x_1) = \sum_{k=0}^{2} \lambda_{1k} f_1(x_{1k}) = 0\lambda_{10} + (-1)\lambda_{11} + 0\lambda_{12} = -\lambda_{11}$ along with $\lambda_{10} + \lambda_{11} + \lambda_{12} = 1$.

Looking the above transformations, above NLPP reduces to LPP as follows:

$$\begin{aligned} & Z + \lambda_{11} - x_2 \le 0 \\ & x_1 + 2x_2 \le 5 \\ & 2x_1 + x_2 \le 6 \\ & -x_1 \le 0 \\ & -x_2 \le 0 \\ & -\lambda_{10} \le 0 \\ & -\lambda_{11} \le 0 \\ & -\lambda_{12} \le 0 \\ & \lambda_{10} + \lambda_{11} + \lambda_{12} \le 1 \end{aligned}$$

The system of inequalities after elimination of x_2 will be:

$$\begin{aligned} & x_1 + 2\lambda_{11} + 2Z \le 5 \\ & x_1 \le 5 \\ & x_1 + \frac{Z}{2} + \frac{\lambda_{11}}{2} \le 3 \\ & x_1 \le 3 \\ & -x_1 \le 0 \\ & -\lambda_{10} \le 0 \end{aligned}$$

$$-\lambda_{11} \leq 0$$
$$-\lambda_{12} \leq 0$$
$$\lambda_{10} + \lambda_{11} + \lambda_{12} \leq 1$$

The system of inequalities after elimination of x_1 will be:

$$\lambda_{11} + Z \leq \frac{5}{2}$$
$$\lambda_{11} + Z \leq 6$$
$$-\lambda_{10} \leq 0$$
$$-\lambda_{11} \leq 0$$
$$-\lambda_{12} \leq 0$$
$$\lambda_{10} + \lambda_{11} + \lambda_{12} \leq 1$$

The system of inequalities after elimination of λ_{11} will be:

$$Z \leq \frac{5}{2}$$
$$Z \leq 6$$
$$-\lambda_{10} \leq 0$$
$$-\lambda_{12} \leq 0$$
$$\lambda_{10} + \lambda_{11} + \lambda_{12} \leq 1$$

The system of inequalities after elimination of λ_{10} will be:

$$Z \leq \frac{5}{2}$$
$$Z \leq 6$$
$$\lambda_{12} \leq 1$$
$$-\lambda_{12} \leq 0$$

The system of inequalities after elimination of λ_{12} will be:

$$Z \leq \frac{5}{2}$$
$$Z \leq 6$$

Out of these bounded values of Z, $Z = \frac{5}{2}$ satisfies all the inequalities altogether. Hence, $Z = \frac{5}{2}$ is an optimal solution. Now, one can easily get the values of decision variables x_1 and x_2 as 0 and $\frac{5}{2}$, respectively, by the process of back substitution.

(b) Consider the following NLPP in which nonlinearity occurs only in constraints [3]

$$\begin{aligned} &\text{Max} && Z = -3x_1 + 5x_2 \\ &\text{subject to} && x_1^3 + 7x_2 \le 8 \\ &\text{and} && x_1, x_2 \ge 0 \end{aligned}$$

The objective function of the above problem is already in the maximization form. Now, consider the following separable functions:

$$f_1(x_1) = -3x_1, \quad f_2(x_2) = 5x_2$$
$$g_{11}(x_1) = x_1^3, \quad g_{12}(x_2) = 7x_2$$

Here, $g_{11}(x_1)$ is present in nonlinear form; therefore, we have to approximate it into piecewise linear function. By inspection, the above problem suggests that $0 \le x_1 \le 2$. To obtain the approximate LPP, for the given NLPP, divide the interval $0 \le x_1 \le 2$ into three breaking points x_{ik} $(k = 0, 1, 2)$ such that:

$$x_{10} = 0, \quad x_{11} = 1, \quad x_{12} = 2$$

We have, $g_{11}(x_1) = \sum_{k=0}^{2} \lambda_{1k} g_{11k}(x_{1k}) = 0\lambda_{10} + (2)\lambda_{11} + (16)\lambda_{12}$ along with $\lambda_{10} + \lambda_{11} + \lambda_{12} = 1$.

Looking the above transformations, above NLPP reduces to LPP as follows:

$$\begin{aligned} & Z + 3x_1 - 5x_2 \le 0 \\ & 2\lambda_{11} + 16\lambda_{12} + 7x_2 \le 8 \\ & -x_1 \le 0 \\ & -x_2 \le 0 \\ & -\lambda_{10} \le 0 \\ & -\lambda_{11} \le 0 \\ & -\lambda_{12} \le 0 \\ & \lambda_{10} + \lambda_{11} + \lambda_{12} \le 1 \end{aligned}$$

The system of inequalities after elimination of λ_{11} will be:

$$\begin{aligned} & Z + 3x_1 - 5x_2 \le 0 \\ & \lambda_{12} + \frac{7}{16}x_2 \le \frac{1}{2} \\ & -x_1 \le 0 \\ & -x_2 \le 0 \\ & -\lambda_{10} \le 0 \end{aligned}$$

$$-\lambda_{12} \le 0$$
$$\lambda_{10} + \lambda_{12} \le 1$$

The system of inequalities after elimination of λ_{12} will be:

$$Z + 3x_1 - 5x_2 \le 0$$
$$-x_1 \le 0$$
$$-x_2 \le 0$$
$$-\lambda_{10} \le 0$$
$$x_2 \le \frac{8}{7}$$
$$\lambda_{10} \le 1$$

The system of inequalities after elimination of x_1 will be:

$$Z - 5x_2 \le 0$$
$$-x_2 \le 0$$
$$-\lambda_{10} \le 0$$
$$x_2 \le \frac{8}{7}$$
$$\lambda_{10} \le 1$$

The system of inequalities after elimination of x_2 will be:

$$Z \le \frac{40}{7}$$
$$-\lambda_{10} \le 0$$
$$\lambda_{10} \le 1$$

After eliminating λ_{10}, there remains only Z. Hence, $Z = \frac{40}{7}$ is an optimal solution. Now, one can easily get the values of decision variables x_1 and x_2 as 0 and $\frac{8}{7}$, respectively, by the process of back substitution.

(iii) Consider the following NLPP in which nonlinearity occurs both in objective function as well as in constraints [3]

$$\begin{aligned} &\text{Max} && Z = x_1^2 + x_2 \\ &\text{subject to} && 2x_1^2 + 4x_2 \le 8 \\ &\text{and} && x_1, x_2 \ge 0 \end{aligned}$$

The objective function of the above problem is already in the maximization form. Now, consider the following separable functions:

$$f_1(x_1) = x_1^2, \quad f_2(x_2) = x_2$$
$$g_{11}(x_1) = 2x_1^2, \quad g_{12}(x_2) = 4x_2$$

Here, $f_1(x_1)$ and $g_{11}(x_1)$ both present in nonlinear form; therefore, we have to approximate these into piecewise linear functions. By inspection, the above problem suggests that $0 \leq x_1 \leq 2$. To obtain the approximate LPP, for the given NLPP, divide the interval $0 \leq x_1 \leq 2$ into three breaking points x_{ik} $(k = 0, 1, 2)$ such that:

$$x_{10} = 0, \quad x_{11} = 1, \quad x_{12} = 2$$

We have, $f_1(x_1) = \sum_{k=0}^{2} \lambda_{1k} f_1(x_{1k}) = 0\lambda_{10} + 1\lambda_{11} + 4\lambda_{12}$ and $g_{11}(x_1) = \sum_{k=0}^{2} \lambda_{1k} g_{11k}(x_{1k}) = 0\lambda_{10} + 2\lambda_{11} + 8\lambda_{12}$ along with $\lambda_{10} + \lambda_{11} + \lambda_{12} = 1$.

Looking the above transformations, above NLPP reduces to LPP as follows:

$$\begin{aligned}
&Z - \lambda_{11} - 4\lambda_{12} - x_2 \leq 0 \\
&2\lambda_{11} + 8\lambda_{12} + 4x_2 \leq 8 \\
&\lambda_{10} + \lambda_{11} + \lambda_{12} \leq 1 \\
&-\lambda_{10} \leq 0 \\
&-\lambda_{11} \leq 0 \\
&-\lambda_{12} \leq 0 \\
&-x_2 \leq 0
\end{aligned}$$

The system of inequalities after elimination of λ_{11} will be:

$$\begin{aligned}
&x_2 + Z \leq 4 \\
&\lambda_{10} - 3\lambda_{12} - x_2 + Z \leq 1 \\
&4\lambda_{12} + 2x_2 \leq 4 \\
&\lambda_{10} + \lambda_{12} \leq 1 \\
&-\lambda_{10} \leq 0 \\
&-\lambda_{12} \leq 0 \\
&-x_2 \leq 0
\end{aligned}$$

The system of inequalities after elimination of x_2 will be:

$$\begin{aligned}
&-\lambda_{12} + \frac{\lambda_{10}}{3} + \frac{2Z}{3} \leq \frac{5}{3} \\
&-\lambda_{12} + \lambda_{10} + Z \leq 3 \\
&\lambda_{12} \leq 2 \\
&\lambda_{12} + \lambda_{10} \leq 1 \\
&-\lambda_{10} \leq 0
\end{aligned}$$

$$-\lambda_{12} \leq 0$$
$$Z \leq 4$$

The system of inequalities after elimination of λ_{12} will be:

$$\lambda_{10} + 2Z \leq 11$$
$$\lambda_{10} + \frac{1}{2}Z \leq 4$$
$$\lambda_{10} + Z \leq 5$$
$$\lambda_{10} + \frac{1}{2}Z \leq 2$$
$$\lambda_{10} \leq 1$$
$$-\lambda_{10} \leq 0$$
$$Z \leq 4$$

The system of inequalities after elimination of λ_{10} will be:

$$Z \leq 5.5$$
$$Z \leq 8$$
$$Z \leq 5$$
$$Z \leq 4$$
$$0 \leq 1$$

Out of these bounded values of Z, $Z = 4$ satisfies all the inequalities altogether. Hence, $Z = 4$ is an optimal solution. Now, one can easily get the values of decision variables x_1 and x_2 as 2 and 0, respectively, by the process of back substitution.

Conclusion

This research paper describes an approach to find an optimal solution of SPP by using FM-E technique. By FM-E technique, one can easily solve SPP as compared to traditional simplex method meant for LPP. We can solve these types of programming problems by various elimination techniques such as Gauss elimination technique, modified Fourier elimination technique, and FM-E technique. These techniques are quite easy to understand as compared to the methods already available in the literature so far because it involves simple calculations and takes least time for computation.

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Chapter 26
Optimizing EOQ Model for Carbon Emission Under Inflation for Expiring Items

Chaman Singh and Gurudatt Rao Ambedkar

Abstract Nowadays, ventures like the supermarket industry are confronting numerous difficulties to meet the objectives of sustainable development. The client's choices are changing at a quicker speed because of new advances and quickly changing the general climate. It is seen that demand of the items is straightforwardly associated with the stock level however keeping a colossal stock reason for increment in holding cost and crumbling effect. It is a major test to sell the whole lot before its lapse with the goal that its impact on climate can be diminished. In the changing environment, storage and managing of inventory items in the supermarket also play a major contribution in green house gas emission; consequently, supermarkets are enforced to make efforts to reduce the emission which also affects the behaviour of an inventory. Deterioration and inflation also have a significant effect on any model, thus cannot be ignored. This study optimizes and EOQ model for expiring items with stock, selling cost and lifetime-dependent demand which gives an ideal yield to acquire the most extreme benefit under inflationary conditions with carbon emission. The model is tackled mathematically to validate the analytical results, which would expand the all-out benefit. Sensitivity analysis strengthens the applicability of the model.

Keywords Optimization · Expiring items · Carbon emission · Stock selling cost and lifetime-dependent demand · Inflation

Introduction

In the recent time, numerous analyses have been carried out around the advancement of an inventory model in realistic environment. Many scientists have shown their

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interest in economic order quantity model (EOQ) excessively joined for real-life day-to-day exercise. We have seen a trend that customers always inclined towards a big quality stock and this behaviour helps in making the greater levels of popularity/demand of the item. In the present time when competition level is very high for any business sector, it becomes more important to analyse and differentiate the demand pattern of the clients. It has been seen that to reach at the satisfactory customer uphold level, it is always beneficial to have large stock however only large number of things causes endless supply of utility, disappearing, debasement and deterioration taking account the dispatch of new progress or the substitution like plan and periodic items, electronic stuff, etc. The fundamental method to increase productivity for green items is assumed as deciding cost and demand. In the present scenario, the freshness of the items is a major concern. Expiration date plays the most significant role in deciding the freshness of items and could essentially affect its demand. Beside this, in the green industry, it is necessary to lower the carbon emission which we produce while using advance technologies to remain the item fresh for long time. In this way, in the present situation, managing the carbon emission for stock-dependent demand model for expiry items is more reliable.

Various examinations have done on EOQ model with stock-dependent demand but very few study are there with carbon emission effect. Levin et al. [1] explain that large stock of inventory attracts the customer to buy more. Datta and Pal [2] proposed and inventory model for stock level-dependent demand. Ghare and Schrader [3] proposed an inventory model for deteriorating items with a constant deteriorating rate. Hammami et al. [4] considered emissions due to transportation, process set-up, production and storage. Mishra et al. [5] analyse a sustainable supply chain model to explain the effect of carbon emissions and control the deterioration. An EOQ model was developed for deteriorating item with lifetime-dependent demand by [6]. Singh et al. [7] presented an EOQ model for perishable product with stock-dependent demand with trade credit period and preservation technologies. Shaikh et al. [8] proposed a non-instantaneous inventory model for fully backlogged shortage under inflation. Hovelaque and Bironneau [9] propose an EOQ model with price and carbon emission-dependent demand. Chen et al. [10] provide a solution to reduce carbon emission using modified order quantity. Mashud [11] proposed a solution algorithm to reduce the CO_2 emission using an inventory model with price-dependent demand. Wu et al. [12] study the effect of carbon emission cost on joint production and location decisions for manufacturer. Li and Hai [13] study the one warehouse multi-retailer system (OWMR system) and proposed a model to calculate the carbon emission based on the frequency of inventory replenishment activities. Gurtu et al. [14] study the EOQ models to demonstrate the impact of increasing transport cost due to change in fuel price and carbon tax on inventory policies. Chung [15] proposed stock model with the use of a power form stock-dependent demand. Pervin et al. [16] study an optimum order quantity model to minimize the cost function for deteriorating items with stock-dependent demand and variable holding cost. Urban and Baker [17] modified an EOQ model for deterministic demand which is a multivariate function of price, time and level of stock. Alfares and Ghaithan [18] presented an EOQ model for price-based demand and time-dependent holding cost with quantity discount. Ruidas et al. (2019) emphasized a theory for the production model with stock and price-dependent demand for perishable things where particular cost limits considered as range numbers and find a function for profit. Manna et al. [19] used metaheuristic algorithms to develop a two-plant production model with demand-dependent warranty period of the product and carbon emission. Chung and Cardenas-Barron [20] proposed a solution for perishable items with stock-dependent demand. Ghosh et al. [21, 22] used a game theoretic approach to develop a supply chain coordination model for green product with different payment methods. Ghosh et al. [21, 22] also investigate an economic order quantity model with complete backorder for deteriorating inventory items under different payment methods. Sarker et al. [23] developed an EOQ model for weakening items under admissible delay of instalment and suitable shortage.

Research Gap and Contribution

Reviewing a brief study of various literature, a number of researchers, viz. Chen et al. [10], Chung and Cardenas-Barron [20], Singh et al. [7], Alfares and Ghaithan [18], Pervin [16], Shaikh [8], Manna [19], Ghosh et al. [21, 22], studied and established different EOQ model for different demand pattern and situations but to the best of our knowledge, none of them has taken into account a demand dependent on stock, selling cost and lifetime all together with carbon emission under inflation. This research gap has been removed in our work. A comparative review of the related literature to the proposed model has been given in Table 26.1.

In this paper, an EOQ model is produced for green things with stock, selling cost and lifetime-dependent demand. The model is analysed for deterioration impact and inflationary condition with carbon emission. At that point, the model is tackled

Table 26.1 Contribution of different researchers related to the proposed model

Authors	Constant demand	Stock selling cost and lifetime-dependent demand	Deterioration	Inflation	Carbon emission
Chunga and Cárdenas-Barrón [20]	✓		✓		
Singh et al. [7]			✓		✓
Shaikh et al. [8]	✓		✓	✓	
Pervin et al. [16]	✓		✓		
Ghosh et al. [21, 22]	✓	✓			
Mashud et al. [11]	✓		✓		✓
Mishra et al. [5]		✓	✓		✓
Manna et al. [19]	✓		✓		✓
Our work	✓	✓	✓	✓	✓

mathematically to get the ideal process duration, which would boost the absolute benefit. Eventually, a sensitivity investigation is performed to check the strength of the model.

Notations and Assumptions

Following notations and assumptions are utilized all through this paper to build up the model.

Notations

a	Scale demand, $a > 0$
b	Mark up, $b > 0$
n	Price elasticity, $n > 0$
M	Expiration date (in months)
Q	The Order Quantity at $t = 0$
c	Purchase cost per unit per dollar
A	Ordering cost per order
θ	Deterioration rate
r	Inflation rate
C_H	Unit holding cost per unit time (in dollar)
C_D	Per unit time deterioration cost (in dollar)
C_S	Average of carbon emission cost due to the storage
C_E	Average of carbon emission cost due to the deterioration
T	Cycle Time (in months)
p	Selling cost per unit, selling cost is always greater than purchasing cost ($p > c$)
$I(t)$	Inventory level at time t where $t \in [0, T]$
TP	Total profit in dollar

Assumptions

1. The demand rate is assumed to be price, stock and lifetime which is given by $D\{p, I(t)\} = \{a + bI(t)\}p^{(-n)}\frac{M-t}{M} \quad 0 < t < M$
2. The inventory cycle is lower than the maximum lifetime of the product.
3. Shortage is not allowed.

4. Deterioration rate is constant.
5. Carbon emission due to storage and deterioration is also considered in this model.
6. Time horizon is infinite.
7. Inflation is also considered in our model.

Model Formulation

The inventory system starts with an order quantity of Q unit at $t = 0$. At that point, because of client interest and decay impact of item, inventory level beginning diminishing. At time $t = T$, inventory level reaches to zero. We are assuming a constant deterioration rate (Fig. 26.1).

At any time $t \geq 0$, the instantaneous inventory level is defined by $I(t)$. The rate of change in inventory level is dependent on the demand and deterioration rate of an item. Then inventory level at any time t with constant deterioration rate θ is satisfied by the following differential equation.

$$\frac{\mathrm{d}I(t)}{\mathrm{d}t} = -\theta I(t) - (a + bI(t))p^{-n}\left(\frac{M-t}{M}\right), \tag{26.1}$$

$$I(t) = 0 \quad \text{when } t = T \text{ and } I(t) = Q \quad \text{when } t = 0 \tag{26.2}$$

$$\frac{\mathrm{d}I(t)}{\mathrm{d}t} + \left(\theta + bp^{-n}\left(1 - \frac{t}{M}\right)\right)I(t) = -ap^{-n}\left(1 - \frac{t}{M}\right)$$

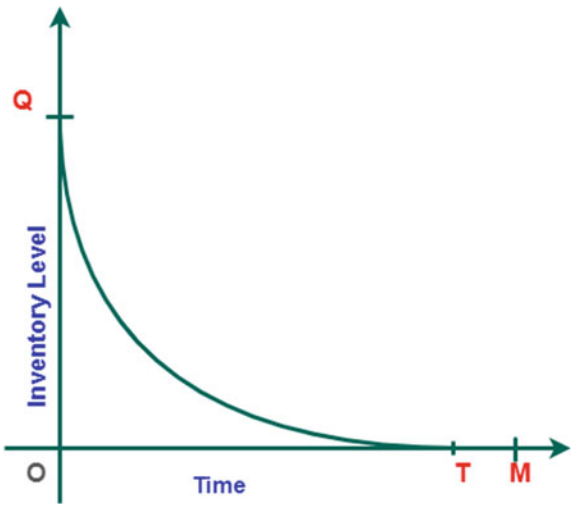

Fig. 26.1 Inventory with respect to time

$$I(t) = \mathrm{e}^{-\left(\theta t + bp^{-n}\left(t - \frac{t^2}{2M}\right)\right)} ap^{-n}\left\{(T-t) + \left(\theta + bp^{-n} - \frac{1}{M}\right)\left(\frac{T^2 - t^2}{2}\right) - \left(\frac{3bp^{-n}}{2M} + \frac{\theta}{M}\right)\left(\frac{T^3 - t^3}{3}\right) + bp^{-n}\left(\frac{T^4 - t^4}{8M^2}\right)\right\} \tag{26.3}$$

And

Now since $I(0) = Q$, so

$$Q = ap^{-n}\left\{T + \left(\theta + bp^{-n} - \frac{1}{M}\right)\frac{T^2}{2} - \left(\frac{3bp^{-n}}{2M} + \frac{\theta}{M}\right)\frac{T^3}{3} + bp^{-n}\frac{T^4}{8M^2}\right\} \tag{26.4}$$

Holding Cost: The cost incurred to store the inventory in warehouse is defined as holding cost. For a longer freshness of the item, preservation equipment is used which generate carbon emission. So, the carbon emission cost due to storage also taken into account to calculate the holding cost.

Total inventory holding cost with inflation rate r is given by

$$\begin{aligned}
\mathrm{HC} &= (C_H + C_S)\int_0^T \mathrm{e}^{-rt} I(t)\mathrm{d}t \\
&= (C_H + C_S)\int_0^T \mathrm{e}^{-\left(rt + \theta t + bp^{-n}\left(t - \frac{t^2}{2M}\right)\right)} ap^{-n}\left\{(T-t) + \left(\theta + bp^{-n} - \frac{1}{M}\right)\left(\frac{T^2 - t^2}{2}\right)\right. \\
&\quad \left. - \left(\frac{3bp^{-n}}{2M} + \frac{\theta}{M}\right)\left(\frac{T^3 - t^3}{3}\right) + bp^{-n}\left(\frac{T^4 - t^4}{8M^2}\right)\right\}\mathrm{d}t \\
&= ap^{-n}(C_H + C_S)\left\{\int_0^T \mathrm{e}^{-\left(rt + \theta t + bp^{-n}\left(t - \frac{t^2}{2M}\right)\right)}\left\{T + \left(\theta + bp^{-n} - \frac{1}{M}\right)\left(\frac{T^2}{2}\right)\right.\right. \\
&\quad \left. - \left(\frac{3bp^{-n}}{2M} + \frac{\theta}{M}\right)\left(\frac{T^3}{3}\right) \times bp^{-n}\left(\frac{T^4}{8M^2}\right)\right\}\mathrm{d}t \\
&\quad - \int_0^T e^{-\left(rt + \theta t + bp^{-n}\left(t - \frac{t^2}{2M}\right)\right)}\left\{t + \left(\theta + bp^{-n} - \frac{1}{M}\right)\left(\frac{t^2}{2}\right)\right. \\
&\quad \left.\left. - \left(\frac{3bp^{-n}}{2M} + \frac{\theta}{M}\right)\left(\frac{t^3}{3}\right) + bp^{-n}\left(\frac{t^4}{8M^2}\right)\right\}\mathrm{d}t\right\} \\
&= ap^{-n}(C_H + C_S)\left\{\left\{\frac{T^2}{2} + \left(\theta + bp^{-n} - \frac{1}{M}\right)\left(\frac{T^3}{3}\right)\right.\right. \\
&\quad \left. - \left(\frac{3bp^{-n}}{2M} + \frac{\theta}{M}\right)\left(\frac{T^4}{4}\right) + \frac{4bp^{-n}}{8M^2}\left(\frac{T^5}{5}\right)\right\} - 2(r + \theta)\frac{T^3}{3} \\
&\quad - bp^{-n}\left(\frac{T^3}{6} - \frac{T^4}{24M}\right) - \left(\theta + bp^{-n} - \frac{1}{M}\right)\left(3(r + \theta)\frac{T^4}{8} + bp^{-n}\frac{T^4}{8} - bp^{-n}\frac{T^5}{30M}\right)
\end{aligned}$$

$$
\begin{aligned}
&+\left(\frac{3bp^{-n}}{2M}+\frac{\theta}{M}\right)\left(4(r+\theta)\frac{T^5}{15}+bp^{-n}\frac{T^5}{10}-bp^{-n}\frac{T^6}{36M}\right)\\
&-\frac{bp^{-n}}{8M^2}\left(5(r+\theta)\frac{T^6}{6}+bp^{-n}\frac{T^6}{3}-2bp^{-n}\frac{T^7}{21M}\right)\Bigg\}\\
&=ap^{-n}(C_H+C_S)\Bigg\{\left\{\frac{T^2}{2}-2(r+\theta)\frac{T^3}{3}-bp^{-n}\left(1-\frac{T}{4M}\right)\frac{T^3}{6}\right\}\\
&+\left(\theta+bp^{-n}-\frac{1}{M}\right)\left\{\frac{T^3}{3}-3(r+\theta)\frac{T^4}{8}-bp^{-n}\left(\frac{1}{8}-\frac{T}{30M}\right)T^4\right\}\\
&-\left(\frac{3bp^{-n}}{2M}+\frac{\theta}{M}\right)\left\{\frac{T^4}{4}-4(r+\theta)\frac{T^5}{15}-bp^{-n}\left(\frac{1}{10}-\frac{T}{36M}\right)T^5\right\}\\
&+\frac{bp^{-n}}{8M^2}\left(4T^5-5(r+\theta)\frac{T^6}{6}-bp^{-n}\left(1-\frac{2T}{7M}\right)\frac{T^6}{3}\right)\Bigg\}
\end{aligned}
\tag{26.5}
$$

Deterioration Cost: Deterioration can be understood as the change, damage or decay. Loss of value or loss of usefulness in goods results in the decreasing usefulness from the original one.

The traditional deteriorating cost (C_D) and carbon emission cost (C_E) generated by deterioration items both can be included in consumer's deteriorating cost per unit time as:

$$
\begin{aligned}
\mathrm{DC}&=(C_D+C_E)\int_0^T \theta e^{-rt} I(t)\mathrm{d}t\\
&=(C_D+C_E)\theta\int_0^T e^{-\left(rt+\theta t+bp^{-n}\left(t-\frac{t^2}{2M}\right)\right)}ap^{-n}\{(T-t)\\
&\quad+\left(\theta+bp^{-n}-\frac{1}{M}\right)\left(\frac{T^2-t^2}{2}\right)-\left(\frac{3bp^{-n}}{2M}+\frac{\theta}{M}\right)\left(\frac{T^3-t^3}{3}\right)\\
&\quad+bp^{-n}\left(\frac{T^4-t^4}{8M^2}\right)\Bigg\}\mathrm{d}t\\
&=ap^{-n}\theta(C_D+C_E)\Bigg\{\left\{\frac{T^2}{2}-2(r+\theta)\frac{T^3}{3}-bp^{-n}\left(1-\frac{T}{4M}\right)\frac{T^3}{6}\right\}\\
&\quad+\left(\theta+bp^{-n}-\frac{1}{M}\right)\left\{\frac{T^3}{3}-3(r+\theta)\frac{T^4}{8}-bp^{-n}\left(\frac{1}{8}-\frac{T}{30M}\right)T^4\right\}\\
&\quad-\left(\frac{3bp^{-n}}{2M}+\frac{\theta}{M}\right)\left\{\frac{T^4}{4}-4(r+\theta)\frac{T^5}{15}-bp^{-n}\left(\frac{1}{10}-\frac{T}{36M}\right)T^5\right\}\\
&\quad+\frac{bp^{-n}}{8M^2}\left(4T^5-5(r+\theta)\frac{T^6}{6}-bp^{-n}\left(1-\frac{2T}{7M}\right)\frac{T^6}{3}\right)\Bigg\}
\end{aligned}
\tag{26.6}
$$

Per order Ordering Cost (OC) is characterized by A. Since c is the purchasing cost per unit, so total purchasing cost per order is given by.

Purchasing Cost (PC) $= cQ$

$$\mathrm{PC} = \mathrm{cap}^{-n}\left\{T + \left(\theta + bp^{-n} - \frac{1}{M}\right)\frac{T^2}{2} - \left(\frac{3bp^{-n}}{2M} + \frac{\theta}{M}\right)\frac{T^3}{3} + \frac{bp^{-n}}{8M^2}T^4\right\} \tag{26.7}$$

Sales Revenue: The income generated by a manufacturer or seller from its sale of item is defined as sales revenue which is directly dependent on the selling price of an item and demand of the item. Selling price per unit is p so total sales revenue collected is given by.

$$\text{Sales Revenue (SR)} = \int_0^T (a + bI(t))p^{-n+1}\frac{M-t}{M}\mathrm{d}t$$

$$\begin{aligned}
\mathrm{SR} &= ap^{-n+1}\left\{T + \left(\theta + bp^{-n} - \frac{1}{M}\right)\frac{T^2}{2} - \left(\frac{3bp^{-n}}{2M} + \frac{\theta}{M}\right)\frac{T^3}{3} + \frac{bp^{-n}}{8M^2}T^4\right\} \\
&= ap^{-n+1}\left(T - \frac{T^2}{2M}\right) + abp^{-n+1}\left\{\left\{\left(\frac{T^2}{2} - \frac{T^3}{6M}\right)\right.\right. \\
&\quad + \left(\theta + bp^{-n} - \frac{1}{M}\right)\left(\frac{T^3}{3} - \frac{T^4}{8M}\right) - \left(\frac{\theta}{M} + \frac{3bp^{-n}}{2M}\right)\left(\frac{T^4}{4} - \frac{T^5}{10M}\right) \\
&\quad \left. + \frac{bp^{-n}}{8M^2}\left(\frac{T^5}{5} - \frac{T^6}{3M}\right)\right\} - \left(\theta + bp^{-n}\right)\left\{\left(\frac{T^3}{3} - \frac{T^4}{12M}\right)\right. \\
&\quad \times \left(\theta + bp^{-n} - \frac{1}{M}\right)\left(\frac{T^4}{8} - \frac{T^5}{15M}\right) \\
&\quad \left. - \left(\frac{\theta}{M} + \frac{3bp^{-n}}{2M}\right)\left(\frac{T^5}{10} - \frac{T^6}{18M}\right) + \frac{bp^{-n}}{8M^2}\left(\frac{T^6}{3} - \frac{4T^7}{21M}\right)\right\} \\
&\quad + \frac{bp^{-n}}{M}\left\{\left(\frac{T^4}{12} - \frac{T^5}{24M}\right) + \left(\theta + bp^{-n} - \frac{1}{M}\right)\left(\frac{T^5}{15} - \frac{T^6}{24M}\right)\right. \\
&\quad \left.\left. - \left(\frac{\theta}{M} + \frac{3bp^{-n}}{2M}\right)\left(\frac{T^6}{18} - \frac{T^7}{28M}\right) + \frac{bp^{-n}}{8M^2}\left(\frac{4T^7}{21} - \frac{T^8}{8M}\right)\right\}\right\}
\end{aligned} \tag{26.8}$$

Total profit per order is given by

$$\mathrm{TP} = \frac{1}{T}(\mathrm{SR} - \mathrm{PC} - \mathrm{HC} - \mathrm{DC} - \mathrm{OC})$$

$$\mathrm{TP} = \frac{1}{T}\left[ap^{-n+1}\left(T - \frac{T^2}{2M}\right) + abp^{-n+1}\left\{\left\{\left(\frac{T^2}{2} - \frac{T^3}{6M}\right)\right.\right.\right.$$

$$+\left(\theta+bp^{-n}-\frac{1}{M}\right)\left(\frac{T^3}{3}-\frac{T^4}{8M}\right)-\left(\frac{\theta}{M}+\frac{3bp^{-n}}{2M}\right)\left(\frac{T^4}{4}-\frac{T^5}{10M}\right)$$
$$+\frac{bp^{-n}}{8M^2}\left(\frac{T^5}{5}-\frac{T^6}{3M}\right)\Bigg\}-\left(\theta+bp^{-n}\right)\Bigg\{\left(\frac{T^3}{3}-\frac{T^4}{12M}\right)$$
$$\times\left(\theta+bp^{-n}-\frac{1}{M}\right)\left(\frac{T^4}{8}-\frac{T^5}{15M}\right)-\left(\frac{\theta}{M}+\frac{3bp^{-n}}{2M}\right)\left(\frac{T^5}{10}-\frac{T^6}{18M}\right)$$
$$+\frac{bp^{-n}}{8M^2}\left(\frac{T^6}{3}-\frac{4T^7}{21M}\right)\Bigg\}$$
$$+\frac{bp^{-n}}{M}\Bigg\{\left(\frac{T^4}{12}-\frac{T^5}{24M}\right)+\left(\theta+bp^{-n}-\frac{1}{M}\right)\left(\frac{T^5}{15}-\frac{T^6}{24M}\right)$$
$$-\left(\frac{\theta}{M}+\frac{3bp^{-n}}{2M}\right)\left(\frac{T^6}{18}-\frac{T^7}{28M}\right)$$
$$+\frac{bp^{-n}}{8M^2}\left(\frac{4T^7}{21}-\frac{T^8}{8M}\right)\Bigg\}\Bigg\}$$
$$-ap^{-n}c\left\{T+\left(\theta+bp^{-n}-\frac{1}{M}\right)\frac{T^2}{2}-\left(\frac{3bp^{-n}}{2M}+\frac{\theta}{M}\right)\frac{T^3}{3}+\frac{bp^{-n}}{8M^2}T^4\right\}$$
$$-ap^{-n}(C_H+C_S+\theta C_D+\theta C_E)$$
$$\left\{\left\{\frac{T^2}{2}-2(r+\theta)\frac{T^3}{3}-bp^{-n}\left(1-\frac{T}{4M}\right)\frac{T^3}{6}\right\}\right.$$
$$+\left(\theta+bp^{-n}-\frac{1}{M}\right)\left\{\frac{T^3}{3}-3(r+\theta)\frac{T^4}{8}-bp^{-n}\left(\frac{1}{8}-\frac{T}{30M}\right)T^4\right\}$$
$$-\left(\frac{3bp^{-n}}{2M}+\frac{\theta}{M}\right)\left\{\frac{T^4}{4}-4(r+\theta)\frac{T^5}{15}-bp^{-n}\left(\frac{1}{10}-\frac{T}{36M}\right)T^5\right\}$$
$$\left.+\frac{bp^{-n}}{8M^2}\left(4T^5-5(r+\theta)\frac{T^6}{6}-bp^{-n}\left(1-\frac{2T}{7M}\right)\frac{T^6}{3}\right)\right\}-A] \quad (26.9)$$

Numerical Examples

To illustrate and validate the developed inventory model, we have chosen a random data and validate our model using software Wolfram Mathematica 12.

The values of various used parameters can be taken as follows:

$A = \$150$, $a = 600$, $b = 3$, $n = 1.2$, $M = 10$ months, $c = \$10$/unit, $r = 8\%$, $\theta = 0.1$, $C_H = \$1.4$/unit, $12C_S = \$0.25$/unit, $C_D = \$1.2$/unit and $C_E = \$0.40$/unit.

After using wolfram Mathematica 12, we are getting an optimum cycle length $T = 8.4969$ months and selling price per unit $p = \$22.3289$ with maximum total profit TP = \$209.676 (Fig. 26.2).

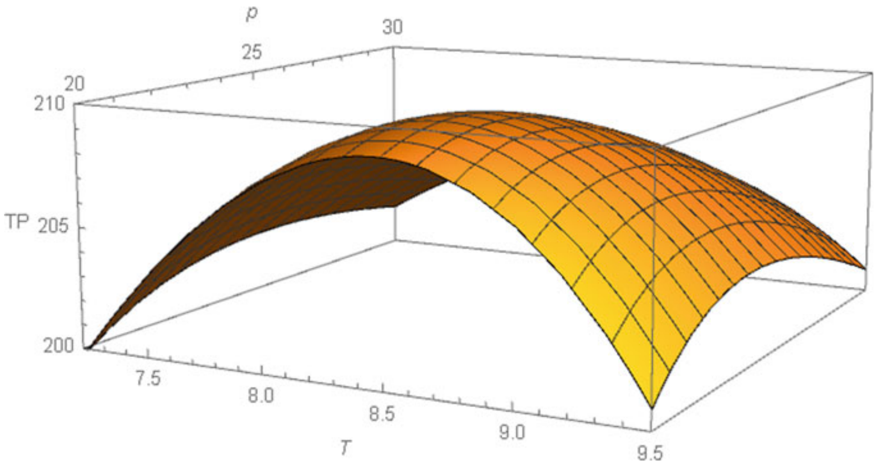

Fig. 26.2 Concavity of total profit function

Sensitivity Analysis

Sensitivity analysis is completed concerning change in some info boundaries of the created stock model. Various levels have been contemplated, for example, −20, −10, 10 and 20 separately for various boundaries. Tables 26.2, 26.3, 26.4, 26.5, 26.6, 26.7, 26.8, 26.9, 26.10, 26.11, 26.12 and 26.13 give the change in *p*, *T* and TP as for various boundaries.

Table 26.2 Sensitivity analysis for the ordering cost *A*

Change in parameter (*A*) (%)	Value of parameter (*A*)	*T*	*p*	TP
− 20	120	8.4588	22.3243	213.214
− 10	135	8.4780	22.326	211.443
0	150	8.4969	22.3289	209.676
10	165	8.51551	22.333	207.912
20	180	8.53387	22.3384	206.153

Table 26.3 Sensitivity analysis for the scale demand *a*

Change in parameter (*a*) (%)	Value of parameter (*a*) (%)	*T*	*p*	TP
− 20	480	8.54295	22.3414	164.22
− 10	540	8.51757	22.3336	186.945
0	600	8.4969	22.3289	209.676
10	660	8.47973	22.3262	232.411
20	720	8.46523	22.3247	255.148

Table 26.4 Sensitivity analysis for the mark up *b*

Change in parameter (*b*) (%)	Value of parameter (*b*)	*T*	*p*	TP
− 20	2.4	8.80716	21.3694	209.569
− 10	2.7	8.64318	21.9271	209.436
0	3	8.4969	22.3289	209.676
10	3.3	8.36528	22.5987	210.198
20	3.6	8.24592	22.7553	210.944

Table 26.5 Sensitivity analysis for the price elasticity *n*

Change in parameter (*n*) (%)	Value of parameter (*n*)	*T*	*p*	TP
− 10	1.08	5.24823	78.3437	327.916
0	1.2	8.4969	22.3289	209.676
5	1.26	8.75372	19.6307	171.894
10	1.32	8.97648	17.633	142.696
20	1.44	9.36194	14.8513	101.597

Table 26.6 Sensitivity analysis for the expiry time *M*

Change in parameter (*M*) (%)	Value of parameter (*M*)	*T*	*p*	TP
− 20	8	5.14235	44.3774	165.132
− 10	9	7.1984	29.7466	182.148
−5	9.5	7.92383	25.5499	194.471
0	10	8.4969	22.3289	209.676
10	11	9.34037	16.6758	251.704

Table 26.7 Sensitivity analysis for the purchasing cost *c*

Change in parameter (*c*) (%)	Value of parameter (*c*)	*T*	*p*	TP
− 20	8	8.18354	15.9433	241.062
− 10	9	8.37461	19.1706	223.5
0	10	8.4969	22.3289	209.676
10	11	8.5648	25.5989	198.36
20	12	8.5797	29.1339	188.894

Table 26.8 Sensitivity analysis for the inflation rate *r*

Change in parameter (*r*) (%)	Value of parameter (*r*)	*T*	*p*	TP
− 20	0.064	8.20254	27.0774	195.488
− 10	0.072	8.37812	24.5363	202.052
0	0.08	8.4969	22.3289	209.676
10	0.088	8.57319	20.344	218.443
20	0.096	8.6151	18.5115	228.486

Table 26.9 Sensitivity analysis for the scale deterioration rate θ

Change in parameter (θ) (%)	Value of parameter (θ)	T	p	TP
− 20	0.08	7.881	29.4601	182.303
− 10	0.09	8.26481	25.5151	194.639
0	0.1	8.4969	22.3289	209.676
10	0.11	8.6329	19.5262	227.894
20	0.12	8.69405	16.9271	250.082

Table 26.10 Sensitivity analysis for the holding cost C_H

Change in parameter (C_H) (%)	Value of parameter (C_H)	T	p	TP
− 20	1.12	8.29957	24.8772	198.896
− 10	1.26	8.4212	23.4718	204.104
0	1.4	8.4969	22.3289	209.676
10	1.54	8.54247	21.3533	215.552
20	1.68	8.56695	20.491	221.694

Table 26.11 Sensitivity analysis for average of carbon emission cost C_S due to storage

Change in parameter (C_S) (%)	Value of parameter (C_S)	T	p	TP
− 20	0.2	8.47395	22.7141	207.648
− 10	0.225	8.48593	22.5187	208.657
0	0.25	8.4969	22.3289	209.676
10	0.275	8.50692	22.1443	210.704
20	0.3	8.51606	21.9646	211.742

Table 26.12 Sensitivity analysis for the per unit time deterioration cost C_D

Change in parameter (C_D) (%)	Value of parameter (C_D)	T	p	TP
− 20	0.96	8.48639	22.511	208.696
− 10	1.08	8.49175	22.4193	209.186
0	1.2	8.4969	22.3289	209.676
10	1.32	8.50182	22.2397	210.168
20	1.44	8.50654	22.1516	210.663

Table 26.13 Sensitivity analysis for the average of carbon emission cost C_E due to deterioration

Change in parameter (C_E) (%)	Value of parameter (C_E)	T	p	TP
− 20	0.32	8.49349	22.389	209.349
− 10	0.36	8.49521	22.3589	209.512
0	0.4	8.4969	22.3289	209.676
10	0.44	8.49856	22.299	209.84
20	0.48	8.5002	22.2693	210.004

Graphical representation of sensitivity analysis (Figs. 26.3, 26.4 and 26.5).

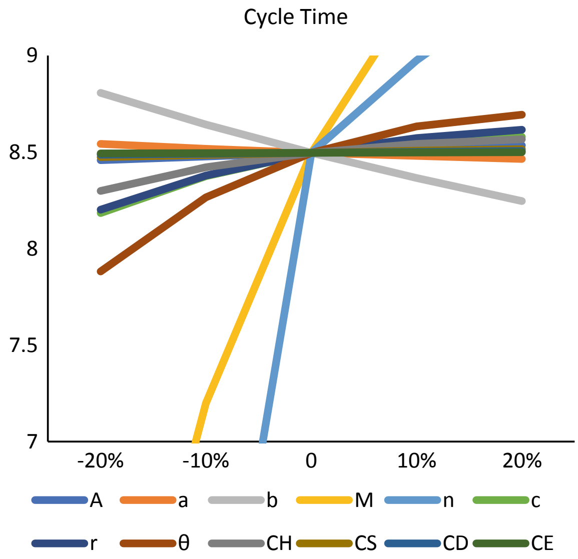

Fig. 26.3 Change in cycle time with respect to change in different parameter

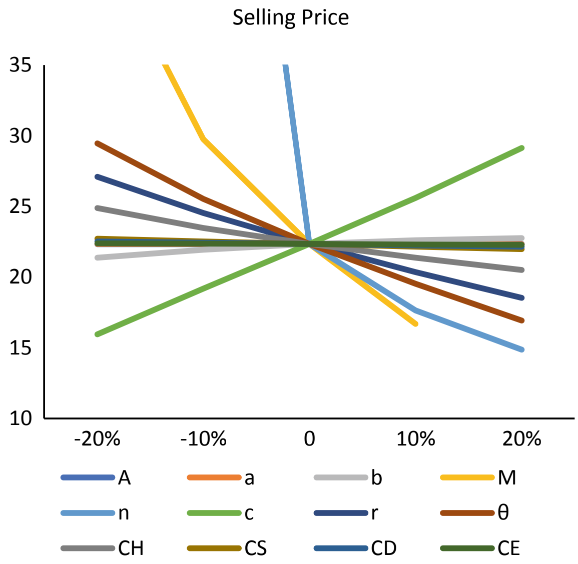

Fig. 26.4 Change in selling price with respect to change in different parameter

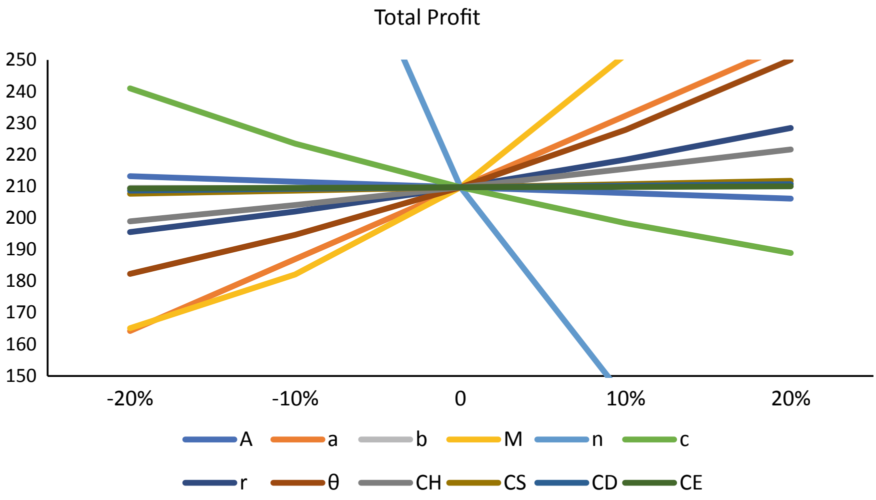

Fig. 26.5 Effect on total profit with respect to change in different parameter

Observations

(a) Table scale demand shows reversible effect on price and cycle time.
(b) Selling price increases as the ordering cost and mark up increases.
(c) Selling price decreases as the price elasticity, holding cost and inflation rate increases.
(d) Purchasing cost, price elasticity and ordering cost have reversible effect on total profit.
(e) Table 26.2 gives that selling price and cycle time increase while total profit decreases as the ordering cost increases.
(f) Table 26.8 gives that cycle time and total profit increase while selling price decreases as the inflation rate increases.
(g) Tables 26.9 and 26.10 give that cycle time and total profit increase and selling price decreases as the deterioration rate and holding cost increase.
(h) Tables 26.11 and 26.13 give that the cycle time and total profit increase while selling price decreases as the carbon emission cost due to storage and deterioration increase.

Managerial Insight

The findings are obtained from the recommended model and economically can be applied by the managers to accomplish their targets. A greenhouse farm manager effectively lessens how much carbon produced from the chain by utilizing conservation innovation yet he ought to likewise notice the expense expected to apply this innovation and the impact of this expense in selling cost and by and large benefit. From this proposed model, a manager can easily understand the effect of freshness of the item in any business. This model gives a plan to settle on a choice for necessity of safeguarding innovation for the newness of the item and effect of related cost on the all-out benefit. From the industrial point of view, it is necessary to analyse the demand pattern based on the lifetime of the product, selling price and carbon emission cost under inflation. This proposed model also provided and informed decision to finalize the selling price of the item under assumed conditions to obtain maximum profit and to sell out the complete stock before its expiry which is necessary to obtain the goal of sustainable development. This model would be more applicable to any industry if a built-in function can be converted for all these result by the manager.

Conclusion

This paper proposed an integrated inventory model for green items with stock, selling price and lifetime-dependent demand with constant deterioration rate under an inflationary environment. The overall impact of carbon emission cost is taken into consideration. Ordering cost, purchasing cost and holding cost, which are required for any inventory model, are also considered. The main objective of this inventory model is to understand the effect of carbon emission cost and expiry date on customer purchasing decision. It is observed that green items have short life and they cannot be sold after expiration date so to achieve the success in any business of green item, it is important to manage such type of perishable items. On the other hand, it is also important to make a strategy to reduce the carbon emission for achieving the goal of sustainable development for a better and safe future. The objective of this study is to maximize the profit function.

Sensitivity analysis has been tested for the proposed inventory model, and a numerical illustration is also performed. The result of this study gave an insight to decision maker to develop an optimum decision for deteriorating green items under inflationary conditions that will help to achieve the goal sustainable development.

This study can be further prolonged for product having lifetime less than cycle time with variable deterioration rate. Further study can also include concept of trade credits and greening efforts.

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Chapter 27
New Class of Multiobjective Fractional Symmetric Programming with Cone Functions Under Generalized Assumptions

Jyoti, Ramesh Kumar, Chetan Swarup, Vishnu Narayan Mishra, and Ramu Dubey

Abstract In this chapter, a pair of nondifferentiable multiobjective symmetric fractional duality models with cone function are formulated in a vector optimization problem, where each component of the objective function contains support function of a compact convex set. The K-(C, ρ)-convexity and K-(C, ρ)-quasiconvexity functions are defined, and also, we constructed concrete numerical examples for existing such type of function. The duality results are established using these aforesaid assumptions.

Keywords Symmetric duality · Nondifferentiable · Multiobjective fractional programming · Support function · Efficient solution

Introduction

Duality in optimization algorithms has been used not just to numerous mathematical and computer breakthroughs in optimization algorithms, but to economics, control theory, business challenges, and a variety of many other domains. A combination of

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primal and dual issues is said to be symmetrical in mathematical programming if the dual problem's dual is the primal problem; i.e., if the dual problem is represented as the primal problem, then dual of the dual problem is the primal problem. Unlike linear programming, bulk of dual formulas is not symmetrical in the nonlinear programming problem. Nondifferentiable multiobjective fractional programming problems with cone constraints over arbitrary closed convex cones are introduced by Kim et al. [14] in which each component of the objective function incorporates a term involving the support function of a compact convex set. Wolfe dual and Mond–Weir dual are introduced for this problem. Under the premise of suitable (V, ρ)-invexity assumptions.

Gupta and Kailey [9] provide a set of Wolfe-type second-order multiobjective symmetric dual problems with nondifferentiable functions. The concept of second-order F-convexity assumptions is then used to prove duality theorems [10]. The appropriate functions are locally Lipschitz in a family of multiobjective fractional programming problems (MFP). To arrive at our main conclusion, they introduce the notion of (p, r)-ρ-(η, θ) -invex class in relation to the Clarke generalized gradient. Sufficient criteria for optimality are stated under the aforesaid invexity assumption. Finally, relevant dual theorems are established for three categories of dual problems corresponding to (MFP) and for minimax fractional programming problem that is nondifferentiable.

Recently, Garg and Kumar [15] utilize the second-order (C, α, ρ, d)-V-type-I convex function definition for a differentiable function. Furthermore, under aforesaid assumptions, a Mond–Weir type dual has been defined for this problem, and corresponding duality findings have been shown. Under second-order F-convexity assumptions. Gulati and Geeta [8] provide a pair of second-order symmetric dual multiobjective problems of the Mond–Weir type over arbitrary cones. Duality theorems are established using K-F -convexity/pseudoinvexity assumptions. Recently, Kaur and Sharma [12] studied a pair of nondifferentiable multiobjective symmetric higher-order fractional programming problem over cone constraint is formulated. (ϕ, ρ)-cone convex function are defined. Duality results are deduced using properties of aforesaid functions. Some concrete examples are discussed to prove existence and of constructed model and uniqueness of higher-order cone-invex function. Aside from them, several researchers are also working in this area (for more information's, see [1–7, 11, 13, 16]).

In the present paper, we introduce a new class of K-(C, ρ)-convex functions as well as K-(C, ρ)-quasiconvex functions for a mathematical programming problem. We create a variety of numerical examples for existing such type of functions. Also, we establish a nondifferentiable multiobjective fractional programming problem with a cone objective as well as a constraint and derived duality solutions under extended circumstances.

Notations and Definitions

In this paper, we used $\mathbb{R}^n$ for n-dimensional Euclidean space and $\mathbb{R}^n_+$ for semi-positive orthant. Further, here C_1 and C_2 used for closed convex cone in $\mathbb{R}^n$ and $\mathbb{R}^m$, respectively, with non-void interiors. For a real-valued twice differentiable function $g(\tau, \kappa)$ described on an open set in $\mathbb{R}^n \times \mathbb{R}^m$, indicate by $\nabla_\tau g(\bar{\tau}, \bar{\kappa})$ the gradient vector of g with respect to τ at $(\bar{\tau}, \bar{\kappa})$, $\nabla_{\tau\tau} g(\bar{\tau}, \bar{\kappa})$ the Hessian matrix with respect to τ an at $(\bar{\tau}, \bar{\kappa})$. Let assume throughout the paper $\tilde{N} = \{1, 2, \ldots, k\}$.

K is used for pointed convex cone with non-void interiors in $\mathbb{R}^k$. Then, for $\kappa, c \in \mathbb{R}^k$. Now, we specify three cone orders with respect to K as follows:

$$\kappa \leq c \iff c - \kappa \in K; \quad \kappa \leq c \iff c - \kappa \in K \backslash \{0\}; \quad \kappa < c \iff c - \kappa \in \operatorname{int} K.$$

Consider the following (MP) with cone functions as:

$$\text{(MPP)}\ K\text{-min}\ \varphi(\tau) = \Big(\varphi_1(\tau), \varphi_2(\tau), \varphi_3(\tau), \ldots, \varphi_k(\tau)\Big)$$
$$\text{subject to } \tau \in Z^0 = \Big\{\tau \in S : -g(\tau) \in Q\Big\},$$

where $S \subseteq R^n$, $\varphi : S \to \mathbb{R}^k$ and $g : S \to \mathbb{R}^m$, K is convex pointed cone in R^k with $\operatorname{int} K \neq \phi$ and Q is a closed convex cone with a non-empty interior in R^m.

Definition 1 $\bar{\tau} \in Z^0$ is a weak efficient solution (MPP), $\nexists\ \tau \in Z^0$ such that

$$\varphi(\bar{\tau}) - \varphi(\tau) \in \operatorname{int} K.$$

Definition 2 $\bar{\tau} \in Z^0$ is an efficient solution of (MPP), $\nexists\ \tau \in Z^0$ such that

$$\varphi(\bar{\tau}) - \varphi(\tau) \in K \setminus \{0\}.$$

Definition 3 Let $C : Z \times Z \times R^n \to R$ $(Z \subseteq R^n)$ be a function which satisfies $C_{\tau,\mu}(0) = 0,\ \forall\ (\tau, \mu) \in Z \times Z$. Then, the function C is convex on R^n w.r.t. third position iff for $(\tau, \mu) \in Z \times Z$,

$$C_{\tau,\mu}(\lambda\tau_1 + (1-\lambda)\tau_2) \leq \lambda C_{\tau,\mu}(\tau_1) + (1-\lambda) C_{\tau,\mu}(\tau_2),\ \ \forall\ \lambda \in (0, 1),\ \forall\ \tau_1,\ \tau_2 \in R^n.$$

Let C_1 and C_2 be closed convex cones with non-empty interiors in R^n and R^m, respectively.

Definition 4 The positive polar cone C_i^* of a cone $C_i \subseteq \mathbb{R}^s$ $(i = 1, 2)$ is defined as

$$C_i^* = \Big\{z : \tau^T z \geq 0, \forall \tau \in C_i\Big\}.$$

Suppose that $S_1 \subseteq R^n$ and $S_2 \subseteq R^m$ are open sets in such that $C_1 \times C_2 \subset S_1 \times S_2$.

Definition 5 A function φ is K-(C, ρ)-convex at $\mu \in X$ $(\subseteq R^n)$, if

$$\{\varphi_1(\tau) - \varphi_1(\mu) - C_{\tau,\mu}(\nabla\varphi_1(\mu)) - \rho_1\|\tau - \mu\|^2, \varphi_2(\tau) - \varphi_2(\mu) - C_{\tau,\mu}(\nabla\varphi_2(\mu)) \\ - \rho_2\|\tau - \mu\|^2, \ldots, \varphi_k(\tau) - \varphi_k(\mu) - C_{\tau,\mu}(\nabla\varphi_k(\mu)) - \rho_k\|\tau - \mu\|^2\} \in K.$$

Remark 1 If we replace K by $-K$, then function is called φ is K-(C, ρ)-concave at $\mu \in X$.

Definition 6 A function φ is K-(C, ρ)-quasiconvex at $\mu \in X$, if

$$\{-\varphi_1(\tau) + \varphi_1(\mu), -\varphi_2(\tau) + \varphi_2(\mu), \ldots, -\varphi_k(\tau) + \varphi_k(\mu)\} \in K \Rightarrow \{-C_{\tau,\mu}(\nabla\varphi_1(\mu)) \\ - \rho_1\|\tau - \mu\|^2, -C_{\tau,\mu}(\nabla\varphi_2(\mu)) - \rho_2\|\tau - \mu\|^2, \ldots, -C_{\tau,\mu}(\nabla\varphi_k(\mu)) - \rho_k\|\tau - \mu\|^2\} \in K.$$

Remark 2 If we replace K by $-K$, then function φ is K-(C, ρ)-quasiconcave at $\mu \in X$.

Example 1 Let $X = [0, 2]$ and $K = \{(\tau, \kappa) : \tau \leq 0, \tau \geq \kappa\}$. Consider the function $\varphi : X \to \mathbb{R}^2$ defined by

$$\varphi(\tau) = \Big(\varphi_1(\tau), \varphi_2(\tau)\Big),$$

where

$$\varphi_1 = \sin\tau, \quad \varphi_2 = \tau$$

and

$$C_{\tau,\mu}(a) = (\tau^2 + \mu)|a| \text{ and } \rho_1, \ \rho_2 \in R.$$

Proof We have to claim that function φ is $K - (C, \rho)$-quasiconvex at $\mu \in X$,, i.e., $\{-\varphi_1(\tau) + \varphi_1(\mu), \quad -\varphi_2(\tau) + \varphi_2(\mu)\} \in K \Rightarrow \{-C_{\tau,\mu}(\nabla_\tau\varphi_1(\mu)) - \rho_1\|\tau - \mu\|^2,$ $-C_{\tau,\mu}(\nabla_\tau\varphi_2(\mu)) - \rho_2\|\tau - \mu\|^2\} \in K$. Consider,

$$\Phi^1 = \{-\varphi_1(\tau) + \varphi_1(\mu), \ -\varphi_2(\tau) + \varphi_2(\mu)\}.$$

Putting the values of φ_1 and φ_2, we get

$$\Phi^1 = \{-\sin\tau + \sin\mu, -\tau + \mu\}$$

The value of expression at the point $\mu = 0$, we have

$$\Phi^1 = \{-\sin\tau, \ -\tau\}.$$

Obviously,

$$\Phi^1 = \{-\sin\tau, -\tau\} \in K.$$

Let,

$$\Phi^2 = \{-C_{\tau,\mu}(\nabla_\tau \varphi_1(\mu)) - \rho_1\|\tau - \mu\|^2,\ -C_{\tau,\mu}(\nabla_\tau \varphi_2(\mu)) - \rho_2\|\tau - \mu\|^2\}.$$

Substituting the values of $\varphi_1,\ \varphi_2,\ \rho_1,\ \rho_2$, and $C_{\tau,\mu}$, we obtain

$$\Phi^2 = \{-(\tau^2 + \mu)|\cos\mu| - \rho_1\|\tau - \mu\|^2,\ -(\tau^2 + \mu) - \rho_2\|\tau - \mu\|^2\}.$$

At $\mu = 0,\ \rho_1 = 1$ and $\rho_2 = 1$, we obtain

$$\Phi^2 = \{-2\tau^2, -2\tau^2\}.$$

Naturally,

$$\Phi^2 = \{-2\tau^2, -2\tau^2\} \in K.$$

Now we show that above example is not K-(C, ρ)-convex, i.e.,

$$\{\varphi_1(\tau) - \varphi_1(\mu) - C_{\tau,\mu}(\nabla\varphi_1(\mu)) - \rho_1\|\tau - \mu\|^2,$$

$$\varphi_2(\tau) - \varphi_2(\mu) - C_{\tau,\mu}(\nabla\varphi_2(\mu)) - \rho_2\|\tau - \mu\|^2\} \notin K.$$

Let,

$$\Phi^3 = \{\varphi_1(\tau) - \varphi_1(\mu) - C_{\tau,\mu}(\nabla\varphi_1(\mu)) - \rho_1\|\tau - \mu\|^2,$$

$$\varphi_2(\tau) - \varphi_2(\mu) - C_{\tau,\mu}(\nabla\varphi_2(\mu)) - \rho_2\|\tau - \mu\|^2\}.$$

Put $\mu = 0,\ \rho_1 = 1$ and $\rho_2 = 1$, we get

$$\Phi^3 = \{\sin\tau - 2\tau^2, \tau - 2\tau^2\}.$$

It is clear that

$$\Phi^3 = \{\sin\tau - 2\tau^2, \tau - 2\tau^2\} \notin K.$$

Hence, completes the results. □

Generalized all the above definitions in two variables as follows:

A differentiable function $\phi = (\varphi_1, \varphi_2, \ldots, \varphi_k) : X \times Y \to \mathbb{R}^k$ and $\rho_i \in \mathbb{R}^k,\ i \in \tilde{N}$.

(A1.) ϕ is K-(C, ρ) -convex at $\mu \in X$ with fixed $\vartheta \in Y$, if

$$\begin{aligned}&\{\varphi_1(\tau, \vartheta) - \varphi_1(\mu, \vartheta) - C_{\tau,\mu}(\nabla_\tau \varphi_1(\mu, \vartheta)) - \rho_1\|\tau - \mu\|^2, \varphi_2(\tau, \vartheta)\\ &- \varphi_2(\mu, \vartheta) - C_{\tau,\mu}(\nabla_\tau \varphi_2(\mu, \vartheta)) - \rho_2\|\tau - \mu\|^2, \ldots, \varphi_k(\tau, \vartheta) - \varphi_k(\mu, \vartheta)\\ &- C_{\tau,\mu}(\nabla_\tau \varphi_k(\mu, \vartheta)) - \rho_k\|\tau - \mu\|^2\} \in K\end{aligned}$$

and ϕ is K-(C,ρ)-convex at $\vartheta \in Y$ with fixed $\mu \in X$, if

$$\{\varphi_1(\kappa,\vartheta) - \varphi_1(\mu,\vartheta) - C_{\tau,\mu}(\nabla_\kappa \varphi_1(\mu,\vartheta)) - \rho_1\|\kappa-\mu\|^2, \varphi_2(\kappa,\vartheta) \\ - \varphi_2(\mu,\vartheta) - C_{\tau,\mu}(\nabla_\kappa \varphi_2(\mu,\vartheta)) - \rho_2\|\kappa-\mu\|^2, \ldots, \varphi_k(\kappa,\vartheta) - \varphi_k(\mu,\vartheta) \\ - C_{\tau,\mu}(\nabla_\kappa \varphi_k(\mu,\vartheta)) - \rho_k\|\kappa-\mu\|^2\} \in K.$$

Remark 3 If we replace K by $-K$, then function is called K-(C,ρ) -concave at $\mu \in X$ with fixed $\vartheta \in Y$ / $K-(C,\rho)$-concave at $\vartheta \in Y$ with fixed $\mu \in X$.

(A2.) ϕ is K-(C,ρ)-quasiconvex at $\mu \in X$ with fixed $\vartheta \in Y$, if

$$\{-\varphi_1(\tau,\vartheta) + \varphi_1(\mu,\vartheta), -\varphi_2(\tau,\vartheta) + \varphi_2(\mu,\vartheta), \ldots, -\varphi_k(\tau,\vartheta) + \varphi_k(\mu,\vartheta)\Big\} \in K \\ \Rightarrow \Big\{ - C_{\tau,\mu}(\nabla_\tau \varphi_1(\mu,\vartheta)) - \rho_1\|\tau-\mu\|^2, -C_{\tau,\mu}(\nabla_\tau \varphi_2(\mu,\vartheta)) \\ - \rho_2\|\tau-\mu\|^2, \ldots, -C_{\tau,\mu}(\nabla_\tau \varphi_k(\mu,\vartheta)) - \rho_k\|\tau-\mu\|^2\} \in K$$

and ϕ is K-(C,ρ)-quasiconvex at $\vartheta \in Y$ with fixed $\mu \in X$, if

$$\{-\varphi_1(\kappa,\vartheta) + \varphi_1(\mu,\vartheta), -\varphi_2(\kappa,\vartheta) + \varphi_2(\mu,\vartheta), \ldots, -\varphi_k(\kappa,\vartheta) + \varphi_k(\mu,\vartheta)\} \in K \\ \Rightarrow \Big\{ - C_{\tau,\mu}(\nabla_\kappa \varphi_1(\mu,\vartheta)) - \rho_1\|\kappa-\mu\|^2, -C_{\tau,\mu}(\nabla_\kappa \varphi_2(\mu,\vartheta)) \\ - \rho_2\|\kappa-\mu\|^2, \ldots, -C_{\tau,\mu}(\nabla_\kappa \varphi_k(\mu,\vartheta)) - \rho_k\|\kappa-\mu\|^2\} \in K.$$

Remark 4 If we replace K by $-K$, then function K-(C,ρ)-quasiconcave at $\mu \in X$ with fixed $\vartheta \in Y$ / $K-(C,\rho)$-quasiconcave at $\vartheta \in Y$ with fixed $\mu \in X$.

Nondifferentiable Multiobjective Symmetric Duality Model over Cones

In this part, we will formulate nondifferentiable multiobjective symmetric fractional duality model with cone functions as follows:

Primal Problem (FPP)

$$\text{K-min.}\Big(\frac{\varphi_1(\tau,\kappa) + s(\tau|E_1) - \kappa^T z_1}{\psi_1(\tau,\kappa) - s(\tau|B_1) + \kappa^T r_1}, \frac{\varphi_2(\tau,\kappa) + s(\tau|E_2) - \kappa^T z_2}{\psi_2(\tau,\kappa) - s(\tau|B_2) + \kappa^T r_2}, \\ \ldots, \frac{\varphi_k(\tau,\kappa) + s(\tau|E_k) - \kappa^T z_k}{\psi_k(\tau,\kappa) - s(\tau|B_k) + \kappa^T r_k}\Big)$$

subject to

$$-\sum_{i=1}^{k}\lambda_i\left[\nabla_\kappa\varphi_i(\tau,\kappa)-z_i-\frac{\varphi_i(\tau,\kappa)+s(\tau|E_i)-\kappa^T z_i}{\psi_i(\tau,\kappa)-s(\tau|B_i)+\kappa^T r_i}\big(\nabla_\kappa\psi_i(\tau,\kappa)+r_i\big)\right]\in C_2^*, \tag{27.1}$$

$$\kappa^T\sum_{i=1}^{k}\lambda_i\left[\nabla_\kappa\varphi_i(\tau,\kappa)-z_i-\frac{\varphi_i(\tau,\kappa)+s(\tau|E_i)-\kappa^T z_i}{\psi_i(\tau,\kappa)-s(\tau|B_i)+\kappa^T r_i}\big(\nabla_\kappa\psi_i(\tau,\kappa)+r_i\big)\right]\geq 0, \tag{27.2}$$

$$\tau\in C_1,\ \lambda\in \operatorname{int}K,\ r_i\in F_i,\ z_i\in D_i,\ i\in\tilde{N}, \tag{27.3}$$

$$\lambda^T e_k=1. \tag{27.4}$$

Dual Problem (FPD)

$$\text{K-max.}\ \left(\frac{\varphi_1(\mu,\vartheta)-s(\vartheta|D_1)+\mu^T w_1}{\psi_1(\mu,\vartheta)+s(\vartheta|F_1)-\mu^T t_1},\frac{\varphi_2(\mu,\vartheta)-s(\vartheta|D_2)+\mu^T w_2}{\psi_2(\mu,\vartheta)+s(\vartheta|F_2)-\mu^T t_2},\right.$$
$$\left.\ldots,\frac{\varphi_k(\mu,\vartheta)-s(\vartheta|D_k)+\mu^T w_k}{\psi_k(\mu,\vartheta)+s(\vartheta|F_k)-\mu^T t_k}\right)$$

subject to:

$$\sum_{i=1}^{k}\lambda_i\left[\nabla_\mu\varphi_i(\mu,\vartheta)+w_i-\frac{\varphi_i(\mu,\vartheta)-s(\vartheta|D_i)+\mu^T w_i}{\psi_i(\mu,\vartheta)+s(\vartheta|F_i)-\mu^T t_i}\big(\nabla_\mu\psi_i(\mu,\vartheta)-t_i\big)\right]\in C_1^*, \tag{27.5}$$

$$\mu^T\sum_{i=1}^{k}\lambda_i\left[\nabla_\mu\varphi_i(\mu,\vartheta)+w_i-\frac{\varphi_i(\mu,\vartheta)-s(\vartheta|D_i)+\mu^T w_i}{\psi_i(\mu,\vartheta)+s(\vartheta|F_i)-\mu^T t_i}\big(\nabla_\mu\psi_i(\mu,\vartheta)+t_i\big)\right]\leq 0, \tag{27.6}$$

$$\vartheta\in C_2,\ \lambda\in \operatorname{int}K,\ w_i\in E_i,\ y_i\in B_i,\ i\in\tilde{N}, \tag{27.7}$$

$$\lambda^T e_k=1. \tag{27.8}$$

Under the aforementioned assumptions, we now consider weak, strong, and converse duality theorems. Let T^0 and Q^0 be the set of feasible solutions of (FPP) and (FDP), respectively.

Where, for $i\in\tilde{N}$,

- $\varphi_i,\ \psi_i : X\times Y\to\mathbb{R}$ are twice differentiable functions,
- $D_i,\ F_i$ are compact convex sets in $\mathbb{R}^m$ and $C_i,\ E_i$ are compact convex sets in $\mathbb{R}^n$,
- $K^*,\ C_1^*$, and C_2^* are the positive polar cones of $K, C_1,$ and C_2, respectively,
- $s(\tau|E_i),\ s(\tau|B_i),\ s(\tau|D_i),$ and $s(\tau|F_i)$ are the support function of $E_i,\ B_i,\ D_i,$ and F_i, respectively,

- In the feasible region, we assumed that numerator is non-negative and denominator is the positive of the objective function.
  Let $c = (c_1, c_2, \ldots, c_k)$, $w = (w_1, w_2, \ldots, w_k)$, $r = (r_1, r_2, \ldots, r_k)$ and $t = (t_1, t_2, \ldots, t_k)$. Let T^0 and Q^0 be the set of feasible solutions of (FPP) and (FDP), respectively.

Theorem 1 (Weak duality) *Let* $(\tau, \kappa, \lambda, z, r) \in T^0$ *and* $(\mu, \vartheta, \lambda, w, t) \in Q^0$. *Suppose that*

(i) $$\Big\{\varphi_1(., \vartheta) + (.)^T w_1 - \frac{\varphi_1(\mu, \vartheta) - s(\vartheta|D_1) + \mu^T w_1}{\psi_1(\mu, \vartheta) + s(\vartheta|F_1) - \mu^T t_1}\Big(\psi_1(., \vartheta) - (.)^T t_1\Big), \ldots, \varphi_k(., \vartheta) + (.)^T w_k - \frac{\varphi_k(\mu, \vartheta) - s(\vartheta|D_k) + \mu^T w_k}{\psi_k(\mu, \vartheta) + s(\vartheta|F_k) - \mu^T t_k}\Big(\psi_k(., \vartheta) - (.)^T t_k\Big)\Big\}$$
be K-(C, ρ)-*convex at* μ,

(ii) $$\Big\{\varphi_1(\tau, .) - (.)^T z_1 - \frac{\varphi_1(\tau, \kappa) + s(\tau|E_1) - \kappa^T z_1}{\psi_1(\tau, \kappa) - s(\tau|B_1) + \kappa^T r_1}(\psi_1(\tau, .) + (.)^T r_1), \ldots, \varphi_k(\tau, .) - (.)^T z_k - \frac{\varphi_k(\tau, \kappa) + s(\tau|E_k) - \kappa^T z_k}{\psi_k(\tau, \kappa) - s(\tau|B_k) + \kappa^T r_k}(\psi_k(\tau, .) + (.)^T r_k)\Big\}$$
be K-(C, ρ)-*concave at* κ,

(iii) $C_{\tau,\mu}(a) + a^T \mu \geq 0, \ \ \forall\, a \in C_1^*$,

(iv) $C_{\vartheta,\kappa}(b) + b^T \kappa \geq 0, \ \ \forall\, b \in C_2^*$.

Then,

$$\Big[\Big(\frac{\varphi_1(\mu, \vartheta) - s(\vartheta|D_1) + \mu^T w_1}{\psi_1(\mu, \vartheta) + s(\vartheta|F_1) - \mu^T t_1}, \frac{\varphi_2(\mu, \vartheta) - s(\vartheta|D_2) + \mu^T w_2}{\psi_2(\mu, \vartheta) + s(\vartheta|F_2) - \mu^T t_2}, \ldots, \frac{\varphi_k(\mu, \vartheta) - s(\vartheta|D_k) + \mu^T w_k}{\psi_k(\mu, \vartheta) + s(\vartheta|F_k) - \mu^T t_k}\Big) - \Big(\frac{\varphi_1(\tau, \kappa) + s(\tau|E_1) - \kappa^T z_1}{\psi_1(\tau, \kappa) - s(\tau|B_1) + \kappa^T r_1}, \frac{\varphi_2(\tau, \kappa) + s(\tau|E_2) - \kappa^T z_2}{\psi_2(\tau, \kappa) - s(\tau|B_2) + \kappa^T r_2}, \ldots, \frac{\varphi_k(\tau, \kappa) + s(\tau|E_k) - \kappa^T z_k}{\psi_k(\tau, \kappa) - s(\tau|B_k) + \kappa^T r_k}\Big)\Big] \notin K \backslash \{0\}.$$

Proof of Theorem From hypothesis (i), we have

$$\Big\{\varphi_1(\tau, \vartheta) + \tau^T w_1 - \frac{\varphi_1(\mu, \vartheta) - s(\vartheta|D_1) + \mu^T w_1}{\psi_1(\mu, \vartheta) + s(\vartheta|F_1) - \mu^T t_1}(\psi_1(\tau, \vartheta) + s(\vartheta|F_1) - \tau^T t_1)$$
$$- \big(\varphi_1(\mu, \vartheta) + \mu^T w_1 - \frac{\varphi_1(\mu, \vartheta) - s(\vartheta|D_1) + \mu^T w_1}{\psi_1(\mu, \vartheta) + s(\vartheta|F_1) - \mu^T t_1}(\psi_1(\mu, \vartheta) + s(\vartheta|F_1) - \mu^T t_1)$$
$$- C_{\tau,\mu}[\nabla_\mu \varphi_1(\mu, \vartheta) + w_1 - \frac{\varphi_1(\mu, \vartheta) - s(\vartheta|D_1) + \mu^T w_1}{\psi_1(\mu, \vartheta) + s(\vartheta|F_1) - \mu^T t_1}(\nabla_\mu \psi_1(\mu, \vartheta) - t_1)],$$
$$\ldots, \varphi_k(\tau, \vartheta) + \tau^T w_k - \frac{\varphi_k(\mu, \vartheta) - s(\vartheta|D_k) + \mu^T w_k}{\psi_k(\mu, \vartheta) + s(\vartheta|F_k) - \mu^T t_k}(\psi_k(\tau, \vartheta) + s(\vartheta|F_k) - \tau^T t_k)$$
$$- \big(\varphi_k(\mu, \vartheta) + \mu^T w_k - \frac{\varphi_k(\mu, \vartheta) - s(\vartheta|D_k) + \mu^T w_k}{\psi_k(\mu, \vartheta) + s(\vartheta|F_k) - \mu^T t_k}(\psi_k(\mu, \vartheta) + s(\vartheta|F_k) - \mu^T t_k)$$
$$- C_{\tau,\mu}[\nabla_\mu \varphi_k(\mu, \vartheta) + w_k - \frac{\varphi_k(\mu, \vartheta) - s(\vartheta|D_k) + \mu^T w_k}{\psi_k(\mu, \vartheta) + s(\vartheta|F_k) - \mu^T t_k}(\nabla_\mu \psi_k(\mu, \vartheta) - t_k)]\Big\} \in K.$$

Since $\lambda \in \text{int}K^*$, we get

$$\sum_{i=1}^{k}\lambda_i\Big[\varphi_i(\tau,\vartheta)-s(\vartheta|D_i)+\tau^T w_i-\frac{\varphi_i(\mu,\vartheta)-s(\vartheta|D_i)+\mu^T w_i}{\psi_i(\mu,\vartheta)+s(\vartheta|F_i)-\mu^T t_i}(\psi_i(\tau,\vartheta)$$
$$+s(\vartheta|F_i)-\tau^T t_i)\Big]\geq\sum_{i=1}^{k}\lambda_i C_{\tau,\mu}\Big[\nabla_\mu\varphi_i(\mu,\vartheta)+w_i$$
$$-\frac{\varphi_i(\mu,\vartheta)-s(\vartheta|D_i)+\mu^T w_i}{\psi_i(\mu,\vartheta)+s(\vartheta|F_i)-\mu^T t_i}(\nabla_\mu\psi_i(\mu,\vartheta)-t_i)\Big].$$

Using $\lambda \in \text{int}K^*$, it follows that

$$\sum_{i=1}^{k}\lambda_i\big[\varphi_i(\tau,\vartheta)-s(\vartheta|D_i)+\tau^T w_i-\frac{\varphi_i(\mu,\vartheta)-s(\vartheta|D_i)+\mu^T w_i}{\psi_i(\mu,\vartheta)+s(\vartheta|F_i)-\mu^T t_i}$$
$$(\psi_i(\tau,\vartheta)+s(\vartheta|F_i)-\tau^T t_i)\big]\geq\sum_{i=1}^{k}\lambda_i C_{\tau,\mu}\Big[\{\nabla_\mu\varphi_i(\mu,\vartheta)+w_i$$
$$-\frac{\varphi_i(\mu,\vartheta)-s(\vartheta|D_i)+\mu^T w_i}{\psi_i(\mu,\vartheta)+s(\vartheta|F_i)-\mu^T t_i}(\nabla_\mu\varphi_i(\mu,\vartheta)-t_i)\}\Big]. \tag{27.9}$$

Using convexity assumption with $\lambda^T e_k = 1$, it yields that

$$\sum_{i=1}^{k}\lambda_i C_{\tau,\mu}\Big[\{\nabla_\mu\varphi_i(\mu,\vartheta)+w_i-\frac{\varphi_i(\mu,\vartheta)-s(\vartheta|D_i)+\mu^T w_i}{\psi_i(\mu,\vartheta)+s(\vartheta|F_i)-\mu^T t_i}$$
$$(\nabla_\mu\varphi_i(\mu,\vartheta)-t_i)\}\Big]=C_{\tau,\mu}\Big[\sum_{i=1}^{k}\lambda_i\{\nabla_\mu\varphi_i(\mu,\vartheta)+w_i$$
$$-\frac{\varphi_i(\mu,\vartheta)-s(\vartheta|D_i)+\mu^T w_i}{\psi_i(\mu,\vartheta)+s(\vartheta|F_i)-\mu^T t_i}(\nabla_\mu\varphi_i(\mu,\vartheta)-t_i)\}\Big]. \tag{27.10}$$

Again, using hypothesis (iii), we obtain

$$C_{\tau,\mu}\Big[\sum_{i=1}^{k}\lambda_i\{\nabla_\mu\varphi_i(\mu,\vartheta)+w_i-\frac{\varphi_i(\mu,\vartheta)-s(\vartheta|D_i)+\mu^T w_i}{\psi_i(\mu,\vartheta)+s(\vartheta|F_i)-\mu^T t_i}$$
$$(\nabla_\mu\varphi_i(\mu,\vartheta)-t_i)\}\Big]\geq-\mu^T\sum_{i=1}^{k}\lambda_i\Big\{\nabla_\mu\varphi_i(\mu,\vartheta)+w_i$$
$$-\frac{\varphi_i(\mu,\vartheta)-s(\vartheta|D_i)+\mu^T w_i}{\psi_i(\mu,\vartheta)+s(\vartheta|F_i)-\mu^T t_i}(\nabla_\mu\varphi_i(\mu,\vartheta)-t_i)\Big\}.$$

By dual constraint (27.6), we find that

$$C_{\tau,\mu}\Bigg[\sum_{i=1}^{k}\lambda_i\{\nabla_\mu\varphi_i(\mu,\vartheta)+w_i-\frac{\varphi_i(\mu,\vartheta)-s(\vartheta|D_i)+\mu^T w_i}{\psi_i(\mu,\vartheta)+s(\vartheta|F_i)-\mu^T t_i}$$
$$(\nabla_\mu\varphi_i(\mu,\vartheta)-t_i)\}\Bigg]\geq 0. \tag{27.11}$$

Again, (27.9)–(27.11) and using the fact that $w_i \in C_i, \tau^T w_i \leq s(\tau|C_i), i \in \tilde{N}$, we get

$$\sum_{i=1}^{k}\lambda_i\Big[\varphi_i(\tau,\vartheta)+s(\vartheta|C_i)-s(\vartheta|D_i)-\frac{\varphi_i(\mu,\vartheta)-s(\vartheta|D_i)+\mu^T w_i}{\psi_i(\mu,\vartheta)+s(\vartheta|F_i)-\mu^T t_i}$$
$$\big(\psi_i(\tau,\vartheta)+s(\vartheta|F_i)-\tau^T t_i\big)\Big]\geq 0. \tag{27.12}$$

Similarly, hypotheses (ii), (iv), primal constraints, using the fact $\vartheta^T z_i \leq s(\vartheta|D_i),\ z_i \in D_i,\ \tau^T t_i \leq s(\tau|B_i), t_i \in B_i,\ \vartheta^T r_i \leq s(\vartheta|F_i),\ r_i \in F_i,\ i \in \tilde{N}$, numerator is non-negative and denominator is positive, we obtain

$$\sum_{i=1}^{k}\lambda_i\big[\varphi_i(\tau,\vartheta)+s(\tau|C_i)-s(\vartheta|D_i)-\frac{\varphi_i(\tau,\kappa)+s(\tau|C_i)-\kappa^T z_i}{\psi_i(\tau,\kappa)-s(\tau|F_i)+\kappa^T r_i}$$
$$(\psi_i(\tau,\vartheta)+s(\vartheta|F_i)-\tau^T t_i)\big]\leq 0. \tag{27.13}$$

Using (27.12) and (27.13), we get

$$\sum_{i=1}^{k}\lambda_i\left[\frac{\varphi_i(\tau,\kappa)+s(\tau|C_i)-\kappa^T z_i}{\psi_i(\tau,\kappa)-s(\tau|F_i)+\kappa^T r_i}-\frac{\varphi_i(\mu,\vartheta)-s(\vartheta|D_i)+\mu^T w_i}{\psi_i(\mu,\vartheta)+s(\vartheta|F_i)-\mu^T t_i}\right]$$
$$\big(\psi_i(\tau,\vartheta)+s(\vartheta|F_i)-\tau^T t_i\big)\geq 0. \tag{27.14}$$

Since $\Big(\psi_i(\tau,\vartheta)+s(\vartheta|F_i)-\tau^T t_i\Big)>0,\ \ i\in\tilde{N}$, we obtain

$$\sum_{i=1}^{k}\lambda_i\left[\frac{\varphi_i(\tau,\kappa)+s(\tau|C_i)-\kappa^T z_i}{\psi_i(\tau,\kappa)-s(\tau|F_i)+\kappa^T r_i}-\frac{\varphi_i(\mu,\vartheta)-s(\vartheta|D_i)+\mu^T w_i}{\psi_i(\mu,\vartheta)+s(\vartheta|F_i)-\mu^T t_i}\right]\geq 0. \tag{27.15}$$

Since $\lambda \neq 0$ and $\lambda \in \text{int}K^* \Rightarrow \lambda > 0$, we find the following expression

$$\Bigg[\Bigg(\frac{\varphi_1(\mu,\vartheta)-s(\vartheta|D_1)+\mu^T w_1}{\psi_1(\mu,\vartheta)+s(\vartheta|F_1)-\mu^T t_1},\frac{\varphi_2(\mu,\vartheta)-s(\vartheta|D_2)+\mu^T w_2}{\psi_2(\mu,\vartheta)+s(\vartheta|F_2)-\mu^T t_2},\ldots,$$
$$\frac{\varphi_k(\mu,\vartheta)-s(\vartheta|D_k)+\mu^T w_k}{\psi_k(\mu,\vartheta)+s(\vartheta|F_k)-\mu^T t_k}\Bigg),\ldots,\Bigg(\frac{\varphi_1(\tau,\kappa)+s(\tau|C_1)-\kappa^T z_1}{\psi_1(\tau,\kappa)-s(\tau|F_1)+\kappa^T r_1},$$

$$\left.\left.\frac{\varphi_2(\tau,\kappa)+s(\tau|C_2)-\kappa^T z_2}{\psi_2(\tau,\kappa)-s(\tau|F_2)+\kappa^T r_2},\ldots,\frac{\varphi_k(\tau,\kappa)+s(\tau|C_k)-\kappa^T z_k}{\psi_k(\tau,\kappa)-s(\tau|F_k)+\kappa^T r_k}\right)\right]\notin K\backslash\{0\}.$$

Hence, we can get the required result. □

Remark 5 Since every quasiconvex is pseudoconvex, the following results can be done on the lines of Theorem 1.

Theorem 2 (Weak duality) *Let* $(\tau,\kappa,\lambda,z,r)\in T^0$ *and* $(\mu,\vartheta,\lambda,w,t)\in Q^0$. *Suppose that*

(i) $$\Big\{\varphi_1(.,\vartheta)+(.)^T w_1-\frac{\varphi_1(\mu,\vartheta)-s(\vartheta|D_1)+\mu^T w_1}{\psi_1(\mu,\vartheta)+s(\vartheta|F_1)-\mu^T t_1}\Big(\psi_1(.,\vartheta)-(.)^T t_1\Big),$$
$$\ldots,\varphi_k(.,\vartheta)+(.)^T w_k-\frac{\varphi_k(\mu,\vartheta)-s(\vartheta|D_k)+\mu^T w_k}{\psi_k(\mu,\vartheta)+s(\vartheta|F_k)-\mu^T t_k}\Big(\psi_k(.,\vartheta)-(.)^T t_k\Big)\Big\}$$
be K-(C,ρ)-*quasiconvex at* μ,

(ii) $$\Big\{\varphi_1(\tau,.)-(.)^T z_1-\frac{\varphi_1(\tau,\kappa)+s(\tau|E_1)-\kappa^T z_1}{\psi_1(\tau,\kappa)-s(\tau|B_1)+\kappa^T r_1}(\psi_1(\tau,.)+(.)^T r_1),$$
$$\ldots,\varphi_k(\tau,.)-(.)^T z_k-\frac{\varphi_k(\tau,\kappa)+s(\tau|E_k)-\kappa^T z_k}{\psi_k(\tau,\kappa)-s(\tau|B_k)+\kappa^T r_k}(\psi_k(\tau,.)+(.)^T r_k)\Big\}$$
be K-(C,ρ)-*quasiconcave at* κ,

(iii) $C_{\tau,\mu}(a)+a^T\mu\geq 0,\quad\forall\, a\in C_1^*$,

(iv) $C_{\vartheta,\kappa}(b)+b^T\kappa\geq 0,\quad\forall\, b\in C_2^*$.

Then,

$$\Big[\Big(\frac{\varphi_1(\mu,\vartheta)-s(\vartheta|D_1)+\mu^T w_1}{\psi_1(\mu,\vartheta)+s(\vartheta|F_1)-\mu^T t_1},\frac{\varphi_2(\mu,\vartheta)-s(\vartheta|D_2)+\mu^T w_2}{\psi_2(\mu,\vartheta)+s(\vartheta|F_2)-\mu^T t_2},$$
$$\ldots,\frac{\varphi_k(\mu,\vartheta)-s(\vartheta|D_k)+\mu^T w_k}{\psi_k(\mu,\vartheta)+s(\vartheta|F_k)-\mu^T t_k}\Big)-\Big(\frac{\varphi_1(\tau,\kappa)+s(\tau|E_1)-\kappa^T z_1}{\psi_1(\tau,\kappa)-s(\tau|B_1)+\kappa^T r_1},$$
$$\frac{\varphi_2(\tau,\kappa)+s(\tau|E_2)-\kappa^T z_2}{\psi_2(\tau,\kappa)-s(\tau|B_2)+\kappa^T r_2},\ldots,\frac{\varphi_k(\tau,\kappa)+s(\tau|E_k)-\kappa^T z_k}{\psi_k(\tau,\kappa)-s(\tau|B_k)+\kappa^T r_k}\Big)\Big]\notin K\backslash\{0\}.$$

Proof of Theorem The above prove will adopt the same lines as Theorem 3. □

Theorem 3 (Strong duality) *Let* $(\bar\tau,\bar\kappa,\bar\lambda,\bar z,\bar r)$ *be an efficient solution to* (FPP) *and fix* $\lambda=\bar\lambda$ *in* (FDP). *Assume that all of the criteria in Theorems 1 and 2 are met. Consider the following case*

(i) $$\sum_{i=1}^{k}\bar\lambda_i\Big[\nabla_{\kappa\kappa}\varphi_i(\bar\tau,\bar\kappa)-\Big(\frac{\nabla_\kappa\varphi_i(\bar\tau,\bar\kappa)-\bar z_i}{\psi_i(\bar\tau,\bar\kappa)-s(\bar\tau|B_i)+\bar\kappa^T r_i}$$
$$-\frac{(\nabla_\kappa\psi_i(\bar\tau,\bar\kappa)+r_i)(\varphi_i(\bar\tau,\bar\kappa)+s(\bar\tau|E_i)-\bar\kappa^T\bar z_i)}{(\psi_i(\bar\tau,\bar\kappa)-s(\bar\tau|B_i)+\bar\kappa^T\bar r_i)^2}\Big)-(\nabla_\kappa\psi_i(\bar\tau,\bar\kappa)+\bar r_i)^T$$
$$-\frac{\varphi_i(\bar\tau,\bar\kappa)+s(\bar\tau|E_i)-\bar\kappa^T\bar z_i}{\psi_i(\bar\tau,\bar\kappa)-s(\bar\tau|B_i)+\bar\kappa^T\bar r_i}\,(\nabla_{\kappa\kappa}\psi_i(\bar\tau,\bar\kappa))\Big]$$ *is positive or negative definite,*

(ii) $\left\{\nabla_\kappa \varphi_i(\bar\tau,\bar\kappa) - \bar z_i - \frac{\varphi_i(\bar\tau,\bar\kappa) + s(\bar\tau|E_i) - \bar\kappa^T \bar z_i}{\psi_i(\bar\tau,\bar\kappa) - s(\bar\tau|B_i) + \bar\kappa^T \bar r_i} (\nabla_\kappa \psi_i(\bar\tau,\bar\kappa) + \bar r_i)\right\}_{i=1}^{k}$ *is linearly independent,*

(iii) $\mathbb{R}^k_+ \subseteq K$.

Then, $\exists\, r_i \in B_i$ *and* $\bar z_i \in E_i, i \in \tilde N$ *s.t.* $(\bar\tau, \bar\kappa, \bar\lambda, \bar z, \bar t)$ *is an efficient of (FDP)*

Proof of Theorem Since $(\bar\tau, \bar\kappa, \bar\lambda, \bar z, \bar r)$ is an efficient solution of (FPP). By Fritz John necessary optimality [12] conditions $\alpha \in K^*,\ \beta \in C_2,\ \eta \in \mathbb{R}_+$ such that

$$\left\{\sum_{i=1}^{k}\left(\alpha_i\{\nabla_\tau \varphi_i(\bar\tau,\bar\kappa) - \bar w_i - \frac{\varphi_i(\bar\tau,\bar\kappa) + s(\bar\tau|E_i) - \bar\kappa^T \bar z_i}{\psi_i(\bar\tau,\bar\kappa) - s(\bar\tau|B_i) + \bar\kappa^T \bar r_i} (\nabla_\tau \psi_i(\bar\tau,\bar\kappa) - \bar t_i)\right)\right.$$
$$- \left(\frac{\nabla_\tau \varphi_i(\bar\tau,\bar\kappa) - \bar w_i}{\psi_i(\bar\tau,\bar\kappa) - s(\bar\tau|B_i) + \bar\kappa^T r_i)} - \frac{(\nabla_\tau \psi_i(\bar\tau,\bar\kappa) - \bar t_i)(\varphi_i(\bar\tau,\bar\kappa) + s(\bar\tau|E_i) - \bar\kappa^T z_i)}{\psi_i(\bar\tau,\bar\kappa) - s(\bar\tau|B_i) + \bar\kappa^T r_i)^2}\right)$$
$$(\nabla_\kappa \psi_i(\bar\tau,\bar\kappa) + \bar r_i)^T\ \frac{\varphi_i(\bar\tau,\bar\kappa) + s(\bar\tau|E_i) - \bar\kappa^T \bar z_i}{\psi_i(\bar\tau,\bar\kappa) - s(\bar\tau|B_i) + \bar\kappa^T \bar r_i}$$
$$\left.\nabla_{\kappa\tau} \psi_i(\bar\tau,\bar\kappa)\right\}(\tau - \bar\tau) \geqq 0,\ \forall \tau \in C_1, \tag{27.16}$$

$$\sum_{i=1}^{k}(\alpha_i - \eta\bar\lambda_i)\left[\nabla_\kappa \varphi_i(\bar\tau,\bar\kappa) - \bar z_i - \frac{\varphi_i(\bar\tau,\bar\kappa) + s(\bar\tau|E_i) - \bar\kappa^T \bar z_i}{\psi_i(\bar\tau,\bar\kappa) - s(\bar\tau|B_i) + \bar\kappa^T \bar r_i} (\nabla_\kappa \psi_i(\bar\tau,\bar\kappa) + \bar r_i)\right]$$
$$+ (\beta - \eta\bar\kappa)^T \sum_{i=1}^{k} \bar\lambda_i\left[\nabla_{\kappa\kappa} \varphi_i(\bar\tau,\bar\kappa) - \left(\frac{\nabla_\tau \varphi_i(\bar\tau,\bar\kappa) - \bar z_i}{\psi_i(\bar\tau,\bar\kappa) - s(\bar\tau|B_i) + \bar\kappa^T r_i)}\right.\right.$$
$$\left. - \frac{(\nabla_\tau \psi_i(\bar\tau,\bar\kappa) + \bar r_i)(\varphi_i(\bar\tau,\bar\kappa) + s(\bar\tau|E_i) - \bar\kappa^T z_i)}{\psi_i(\bar\tau,\bar\kappa) - s(\bar\tau|B_i) + \bar\kappa^T r_i)^2}\right)$$
$$\left.(\nabla_\kappa \psi_i(\bar\tau,\bar\kappa) + \bar r_i)^T - \frac{\varphi_i(\bar\tau,\bar\kappa) + s(\bar\tau|E_i) - \bar\kappa^T \bar z_i}{\psi_i(\bar\tau,\bar\kappa) - s(\bar\tau|B_i) + \bar\kappa^T \bar r_i} \nabla_{\kappa\kappa} \psi_i(\bar\tau,\bar\kappa)\right](\tau - \bar\tau) = 0, \tag{27.17}$$

$$(\beta - \eta\bar\kappa)^T\left[\nabla_\kappa \varphi_i(\bar\tau,\bar\kappa) - \bar z_i - \frac{\varphi_i(\bar\tau,\bar\kappa) + s(\bar\tau|E_i) - \bar\kappa^T \bar z_i}{\psi_i(\bar\tau,\bar\kappa) - s(\bar\tau|B_i) + \bar\kappa^T \bar r_i}\right.$$
$$\left.(\nabla_\kappa \psi_i(\bar\tau,\bar\kappa) + \bar r_i)\right](\lambda_i - \bar\lambda_i) \geq 0,$$
$$\forall\ \lambda \in \text{int}K^*, i \in \tilde N, \tag{27.18}$$

$$\beta^T \sum_{i=1}^{k} \bar\lambda_i\big[\nabla_\kappa \varphi_i(\bar\tau,\bar\kappa) - \bar z_i - \frac{\varphi_i(\bar\tau,\bar\kappa) + s(\bar\tau|E_i) - \bar\kappa^T \bar z_i}{\psi_i(\bar\tau,\bar\kappa) - s(\bar\tau|B_i) + \bar\kappa^T \bar r_i}$$

$$(\nabla_\kappa \psi_i(\bar{\tau},\bar{\kappa})+\bar{r}_i)\Big]=0, \tag{27.19}$$

$$\eta\bar{\kappa}^T\sum_{i=1}^{k}\bar{\lambda}_i\Big[\nabla_\kappa\varphi_i(\bar{\tau},\bar{\kappa})-\bar{z}_i-\frac{\varphi_i(\bar{\tau},\bar{\kappa})+s(\bar{\tau}|E_i)-\bar{\kappa}^T\bar{z}_i}{\psi_i(\bar{\tau},\bar{\kappa})-s(\bar{\tau}|B_i)+\bar{\kappa}^T\bar{r}_i}(\nabla_\kappa\psi_i(\bar{\tau},\bar{\kappa})+\bar{r}_i)\Big]=0,$$

$$\alpha_i\bar{\kappa}+\lambda_i(\beta-\eta\bar{\kappa})\in N_{D_i}(z_i),\, i\in\tilde{N}, \tag{27.20}$$

$$\alpha_i\bar{\kappa}\left(\frac{\varphi_i(\bar{\tau},\bar{\kappa})+s(\bar{\tau}|E_i)-\bar{\kappa}^T\bar{z}_i}{\psi_i(\bar{\tau},\bar{\kappa})-s(\bar{\tau}|B_i)+\bar{\kappa}^T\bar{r}_i}\right)-\bar{\lambda}_i(\beta-\eta\bar{\kappa})\left(\frac{\varphi_i(\bar{\tau},\bar{\kappa})+s(\bar{\tau}|E_i)-\bar{\kappa}^T\bar{z}_i}{\psi_i(\bar{\tau},\bar{\kappa})-s(\bar{\tau}|B_i)+\bar{\kappa}^T\bar{r}_i}\right)$$

$$\big[(\nabla_\kappa\psi_i(\bar{\tau},\ \bar{\kappa})+\bar{r}_i)\bar{\kappa}-1\big]\in N_{F_i}(r_i),\ i\in\tilde{N}, \tag{27.21}$$

$$\bar{\delta}^T(\bar{\lambda}^T e_k-1)=0 \tag{27.22}$$

$$\bar{t}_i\in B_i,\bar{w}_i\in E_i,\ i\in\tilde{N}, \tag{27.23}$$

$$\bar{\tau}^T\bar{w}_i^T=s(\bar{\tau}|E_i),\ \bar{t}_i\in B_i,\ \bar{w}_i\in E_i,\ i\in\tilde{N}. \tag{27.24}$$

$$(\alpha,\beta,\eta,\delta)\geq 0,\ (\alpha,\beta,\eta,\delta)\neq 0. \tag{27.25}$$

The rest of the verification is given along the lines of [17]. □

Theorem 4 (Converse duality) *Let $(\bar{\mu},\bar{\vartheta},\bar{\lambda},\bar{w},\bar{t})$ be an efficient solution to (FDP) and fix $\lambda=\bar{\lambda}$ in (FPP). Assume that all of the criteria in Theorems 1 and 2 are met. Consider the following case*

(i) $\sum_{i=1}^{k}\bar{\lambda}_i\Bigg[\nabla_{\vartheta\vartheta}\varphi_i(\bar{\mu},\bar{\vartheta})-\Bigg(\frac{\nabla_\vartheta\varphi_i(\bar{\mu},\bar{\vartheta})-\bar{w}_i}{\psi_i(\bar{\mu},\bar{\vartheta})-s(\bar{\mu}|B_i)+\bar{\vartheta}^T t_i}-\frac{(\nabla_\vartheta\psi_i(\bar{\mu},\bar{\vartheta})+t_i)(\varphi_i(\bar{\mu},\bar{\vartheta})+s(\bar{\mu}|E_i)-\bar{\vartheta}^T\bar{w}_i)}{(\psi_i(\bar{\mu},\bar{\vartheta})-s(\bar{\mu}|B_i)+\bar{\vartheta}^T\bar{t}_i)^2}\Bigg)-(\nabla_\vartheta\psi_i(\bar{\mu},\bar{\vartheta})+\bar{t}_i)^T-\frac{\varphi_i(\bar{\mu},\bar{\vartheta})+s(\bar{\mu}|E_i)-\bar{\vartheta}^T\bar{w}_i}{\psi_i(\bar{\mu},\bar{\vartheta})-s(\bar{\mu}|B_i)+\bar{\vartheta}^T\bar{t}_i}\ (\nabla_{\vartheta\vartheta}\psi_i(\bar{\mu},\bar{\vartheta}))\Bigg]$ *is positive or negative definite,*

(ii) $\left\{\nabla_\vartheta\varphi_i(\bar{\mu},\bar{\vartheta})-\bar{w}_i-\frac{\varphi_i(\bar{\mu},\bar{\vartheta})+s(\bar{\mu}|E_i)-\bar{\vartheta}^T\bar{w}_i}{\psi_i(\bar{\mu},\bar{\vartheta})-s(\bar{\mu}|B_i)+\bar{\vartheta}^T\bar{t}_i}\ (\nabla_\vartheta\psi_i(\bar{\mu},\bar{\vartheta})+\bar{t}_i)\right\}_{i=1}^{k}$ *is linearly independent,*

(iii) $\mathbb{R}^k_+\subseteq K$.

Then, there exist $t_i\in B_i$ and $\bar{w}_i\in E_i,\ i\in\tilde{N}$ s.t. $(\bar{\mu},\bar{\vartheta},\bar{\lambda},\bar{w},\bar{t})$ is an efficient of (FPP).

Proof of Theorem Because of the symmetric nature, the proof is similar to Theorem 3. □

Conclusion

In this chapter, we have considered new kinds of nondifferentiable fractional symmetric dual projects with cone objectives as well as constraints and talked about duality hypotheses under summed up suspicions. The current work can additionally be reached out to nondifferentiable higher-order symmetric fractional programming over cones. This will situate the future undertaking of the scientists.

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Chapter 28
The Drivers and Challenges for Customer Satisfaction in E-commerce Industry for Urban and Rural India: A Key Stakeholders' Perspectives

Anurag Mishra, Pankaj Dutta, Siva Prasad Reddy, and Teja Praneet

Abstract The Indian e-commerce market is one of the fastest-growing segments and is yet to achieve its full potential, with customer satisfaction being the prime focus for e-commerce firms. The purpose of this paper is firstly to provide a thorough understanding of customer satisfaction in the Indian e-commerce industry both in urban and rural areas and thereby determine the key parameters which are crucial for providing top-class customer satisfaction. Secondly, evolve the interrelationship among these parameters and identify the difference of opinion between consumers and professionals working in e-commerce firms. We intend to address this gap toward improving customer satisfaction from rural and urban perspectives. The Interpretive Structural Modeling (ISM)-based approach has been employed to analyze parameter relationships. Using fuzzy MICMAC analysis, these parameters' driving and dependent power have been identified to shed light on correlation with customer satisfaction. The main finding of this research is the identification of key drivers affecting each other and thus implicitly affecting customer satisfaction. The study also highlights the preferences of rural and urban customers and how they differ. Results for improvement in customer satisfaction measures are stated concerning overall improvement in trust, on-time delivery, reliability, competitive price, and quality.

Keywords E-commerce · Customer satisfaction · Interpretive structural modeling · Fuzzy MICMAC · E-retailing business models · Urban and rural India

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Introduction

The Indian e-commerce market was valued at $33 billion in 2017, with a growth rate of 19.1%. The Indian e-commerce industry has grown predominantly over the years, with Gross Merchandise Value growing day by day. However, it has yet to achieve its full potential due to better mobile connectivity, rising internet penetration, a drop in data costs, and a constant shift to smartphones. E-commerce has also made it easier for many global brands to enter the Indian market and deliver directly to end consumers, thereby improving cross-border trade. Technology is driving e-commerce with innovations in digital payments, analytics, and logistics support, which has even encouraged cross-border e-commerce [62]. Customer satisfaction, one of the critical success factors in any business, including e-commerce, is an important research topic in marketing and information management systems [4]. Customer satisfaction is also influenced by smooth return management [17, 25]. Research suggests that acquiring new customers is significantly more expensive than retaining existing customers [63]. Thus, companies pay attention to increasing their market share by continuously updating and optimizing customer retention [15]. With the increasing modes of communication and ever-changing tastes, satisfaction alone may not be adequate to retain a long-term relationship [34]. As e-commerce consumers often share their shopping experiences by providing direct feedback, communicating with friends, or participating in social networks, adequate customer satisfaction would magnify the recommendation either way. Therefore, customer value is presumed as an impression that accounts for several heterogeneous components [56]. Hence, to understand the complex relationship between diverse parameters governing customer satisfaction and to derive actionable insights, we explore customer satisfaction through ISM methodology. We investigate the study for urban and rural India from two different perspectives: customer and professional preferences. Knowing that one tries to cope with e-commerce complexity, this research aims to bridge the gap between organizations' perception of customers and what customers want by defining, describing, and modeling better actual customer satisfaction parameters.

The remainder of the paper is organized as follows. Section "Literature Review" provides the literature review on the e-commerce industry, focusing on customer satisfaction and observing key parameters. Section "Problem Statement" provides the problem statement and research objective and the detailed description of crucial parameters for customer satisfaction. The research methodology is described in Section "Research Methodology", and the results and analysis are presented in Section "Results and Analysis". Section "Managerial Discussions and Recommendations" discusses the findings of this research and their implications. Section "Conclusions" concludes the paper with future research scope.

Literature Review

Corresponding studies often leave out key parameters; we suggest that more analysis is required to clearly understand the consumers' mindset on different parameters that directly and indirectly affect their satisfaction [55]. Considerable research has been conducted into classifying business models over the past decade through a transaction-based framework for business models, transactions in different phases of the purchase, transaction cost economics, and consumer behavior research are highlighted in their work [64]. Product delivery and service provider's infrastructure can be considered for critical selection criteria [31]. Ho et al. [26] studied the impact and importance of technology in e-commerce in line with the market trends to evaluate how technology will improve the feasibility for business models. The hybrid growth theory approach is considered for initial level analysis. Their findings showed that endogenous variables like internet user penetration, telecommunications, etc., and exogenous variables like international openness drive e-commerce revenues over time. Chan and Al-Hawamdeh [11] researched the government's role in establishing the e-commerce infrastructure in Singapore and its impact in transforming the information society there. Halawani et al. [22] and Enz and James [19] established that e-commerce has excellent potential for business opportunities even in the tourism industry. Multiple scenarios of the hotel industry, travel industry, and mix of online travel agencies were studied to understand the essence of e-commerce toward modern-day requirements and merchant versus retail pricing model to understand the impact of price as an essential parameter. Kawa [35] and Beranek [6] studied the importance of return management in e-commerce and how it affects customer loyalty and retention.

Kao and Decou [33] had developed a model that focused on the critical aspects of planning in e-commerce which consists of finance, legal issues, logistics, marketing, operations, security, and technology. Ho et al. [26] identified technological evolutions in the e-commerce industry, game changers to understand the effect of technology adoption among consumers and the organizations. As e-commerce is a novel concept in developing countries, we have studied its penetration and adoption from consumers' perspectives in detail to understand consumer satisfaction in rural areas, and Almousa [2] and Chaparro-Peláez et al. [12] have made a critical analysis of the drivers, inhibitors, enablers, and success factors for e-commerce growth and its clear-cut projection for e-commerce adoption. Colla and Lapoule [13] explored e-commerce as a knowledge management domain that mainly focuses on customers' requirements, interests, and buying patterns, thereby devising a personalization strategy for individual customers and socially constructed models toward creating a loyal and receptive customer's customer base. Through this, businesses and customers can interact and share their interests mutually and thereby a win–win situation for all. In addition, issues like convenience, quality, website features, and usefulness were relevant to McLean and Blackie [41].

Different models to understand customer satisfaction and competitiveness, such as structural equation modeling, asset process performance (APP), competitive theoretical framework, are studied in detail to improve customer satisfaction for e-commerce core competencies [55]. Few authors concentrated on selected parameters like last-mile connectivity. They used ISM for understanding the correlation [16], but we have identified a varied set of parameters, on which customer satisfaction may hugely rest upon. Valmohammadi and Dashti [59] used a specific analytical technique in the e-commerce of Iran Khodro industrial group, a leading Iranian automotive company, to specify the interactions and henceforth calculate the ranking of the barriers. Their methodology first used ISM and the fuzzy analytical network process (FANP). In total, thirteen barriers and challenges to implementing e-commerce were determined and divided into four main factors: technical, organizational, individual, and environmental.

The growth of e-commerce is unparalleled. Despite the apparent centrality of customers in terms of business profitability and revenue growth, there is still a dearth of published work on identifying the need for changing customer preferences that organizations should look out for.

Problem Statement

Research Objectives

The analysis of the e-commerce sector has been done many times from different perspectives. The discussion often centers on profitability, growth, sustainability, and prospects. However, customer satisfaction is an area that we believe needs a more thorough analysis as it impacts various crucial business outcomes, including customer retention, customer lifetime values, and others. In this regard, the primary objectives of this research are:

To take up a thorough study of diverse parameters affecting customer satisfaction in the e-commerce industry in India.

To establish a framework and develop an understanding of the interaction between the various parameters or factors affecting customer satisfaction and the order and strength of relationship with each other.

To analyze and identify the differences between the views of customers and e-commerce professionals on determinants of customer satisfaction for both Urban and Rural India.

To discuss managerial insights of this research toward customer satisfaction and thereby taking the e-commerce business forward in a growth trajectory.

Identification of Parameters for Customer Satisfaction

We have identified instrumental parameters for customer satisfaction in urban and rural areas by analyzing different publications based on e-tailing on e-commerce, customer satisfaction, pricing, competitiveness, ISM approaches, and the latest news articles from e-commerce experts. Analysis from developed markets and other developing markets cannot be directly implemented here. Instead, the Indian market needs its analysis and execution. We have analyzed different frameworks and models, viz.: customer satisfaction and competitiveness in e-retailing; determinants of e-commerce customer satisfaction, trust, and loyalty; factors influencing e-commerce development; implications for the developing countries; effect of the online privacy policy; exploratory and confirmatory factors.

The study performed a systematic review of the literature and drew 41 factors contributing to customer satisfaction. In order to validate the factors and remove the researcher's bias, the authors then contacted experts from industry and academia who are actively involved in similar work and studies. The task allotted to the reviewers was to assess and make modifications wherever required to make the list of variables more robust and free of errors and duplication. The reviewers explained the study's objective and were asked to manually club some parameters if required and then rank them based on their importance. The study finalized 24 parameters which were then used for performing further analysis. The table below denotes the list of references used to draw the final set of parameters. Table 28.1 gives an overview of the same.

Research Methodology

As we aim to provide a thorough understanding of customer satisfaction in the Indian e-commerce sector both for urban and rural segments, we analyze the ISM models and fuzzy MICMAC analysis for each of the four modules or cases: which are analysis for urban India: customer preferences, analysis for urban India: professional preferences, analysis for rural India: customer preferences, and analysis for rural India: professional preferences.

Introduction to ISM

ISM is a process based on interactive learning. The method is interpretive since the judgment of a group of people decides if and how items are related; it gives a structural relationship among various elements of a system [54]. Any situation can be more accurately explained by the relationships between the factors rather than the individual factor taken in isolation. Therefore, ISM will lead to collective

Table 28.1 List of parameters with category and references

Parameters	Urban (U)/rural (R)	Characteristics
Product information	U	It includes product descriptions like color, size, shape, and other specifications with the user manual [40]
Product variety	U	It includes alternative products with different prices, specifications, and different sellers selling the product on the platform [7, 30]
Trust	U, R	Money-back guarantees-assured product returns on-time deliveries form the crux of trust-building (Bauman and Bachmann 2017) [37]
Perceived complexity	U	The easier it is for the customer to shop on the e-commerce platform with little time consumption, the better the experience for customers [51]
Packaging	U, R	The more secured the packaging and more secured the product, the better it is for customer satisfaction [24, 46]
Special promotions	U, R	Sales' activities like price discounts, loyalty discounts to regular customers, lure more customers in buying [10, 42]
User interface	U	The interaction of the customer with the online system which includes the software-based mechanism [39]
Price	U, R	Indian customers are sensitive to price, and showcasing the brand as value for money will improve customer experience [50, 38]
Quality	U, R	It includes both customer and service quality [27, 55]
Responsiveness	U, R	It signifies the way of acting swiftly and positively [49]
Convenience	U	It includes reducing time and effort in shopping and convenience in product selection, ordering, payment solutions [8]
Delivery options	U, R	Options like free delivery, one-day delivery, and last-mile connectivity are preferred options for customers [36, 52]
Reliability	U, R	Focus on consistency and dependability of the e-commerce site [29, 45]
Problem resolution	U, R	It is the art of resolving the customer's issue [21, 14]
Product reviews	U	It is the textual assessment of a product, its features, and delivery written by the customer [57, 28]

(continued)

Table 28.1 (continued)

Parameters	Urban (U)/rural (R)	Characteristics
Product returns	U, R	It is one of the most critical parameters for customer retention, and at the same time, a significant challenge by firms worldwide [32, 58]
On-time delivery	U, R	Is the timely delivery of the product, i.e., within the time indicated? Supply chain and process efficiency are key to on-time delivery [18]
Regional language support	R	Penetration of tier II and tier III cities will be aided and accelerated by the provision of an interface with regional language [44, 60]
Buying assistance	R	It is a crucial feature, especially needed in the rural area as the customers in rural areas may not be able to explore the features of the site [47, 1]
Delivery charges	R	The customers pay an additional fee over the price of the purchased product. Customers perceive this as an additional cost, and therefore, it is an essential parameter for customer satisfaction [61, 48]
Usability	R	The convenience determines it, and customer interaction with the organization also includes lesser disruptions personalization awareness of the services [20]
Payment options	R	Payment options must have speed, security, efficiency, and quality of services. By providing multiple payment options, customers are empowered to go for the payment method they trust and are comfortable with [3]
Lite website availability	R	Especially, for the rural Indian context where penetration is in the early stage and higher data charges, lite websites provide the customers with all the necessary experience of e-commerce [23, 5]
Logistics partner	R	Logistics partner determines the efficiency of the organizations and on-time delivery assurance, especially in towns and villages where last-mile connectivity is still a significant challenge [55, 63]

Source Author

understandings of these relationships. The ISM methodology is explained in the following steps illustrated by various studies [9, 53]:

Identify relevant elements by literature review and ideation based on the primary and secondary research.

Finding out the contextual relationship between various elements through expert opinion and surveys.

Developing a structural self-interaction matrix (SSIM) shows the pairwise relationship between system elements.

The symbols are used to indicate the degree of association between the pairs of the parameters denoted by '*i*' and '*j*' (referring to a serial number of a parameter in row and column, respectively).

V—parameter '*i*' engenders '*j*',

A—parameter '*j*' engenders '*i*',

X—both parameters '*i*' and '*j*' engender each other and

O—parameters '*i*' and '*j*' are independent and does not lead to each other.

Development of reachability matrix from SSIM.

Check transitivity and develop a final reachability matrix after correcting errors of transitivity.

Partition reachability matrix into different levels through iterations.

Draw a directed graph (digraph) based on the relationships given in the final reachability matrix and remove transitive links.

Replace the element nodes with statements and convert the resulting digraph into an ISM.

Review and finalize the ISM model by checking for conceptual inconsistency and making required changes, if any.

Data Collection

ISM among rural parameters is collected separately from rural consumers such as college-going teenagers, government and corporate employees from towns, and tier-3 localities in India. ISM among urban parameters is collected from all demographics residing in metros and tier-1 cities who regularly purchase goods online. Expert data for ISM among urban and rural parameters are collected from professionals working in e-commerce companies across India in different divisions. Data are collected through online surveys and offline questionnaires with direct interaction with the consumers to understand their opinion on present e-commerce portals. Data are collected from a frequent purchasing pool of 55 urban consumers, 55 rural consumers, and 21 professionals working in e-commerce majors in different parts of India.

Results and Analysis

ISM Model and Fuzzy MICMAC Analysis for Urban India: Customer Preferences

Interpretive Logic Matrix

The data gathered from the general questionnaires, surveys, and VAXO marking sheets are condensed into a matrix for further analysis. When more than 60% of respondents affirmed the relation, we took it as 'V' or 'A' depending on the direction. Moreover, in the case of the mutual dependence questions, we went with 'X.' In the remaining cases, where the response is less than 60% or the negative regarding the relationships, we choose to represent it as 'O.' Apart from this, where there is a lack of clarity in the survey results or some complexities in the relation, which were not apparent to customers, the relationship is based on experts' opinions. Based on the results from surveys and expert opinion, we recognized the relationships to formulate SSIM matrix as shown in Table 28.2. For example, the following statements explain symbols V, A, X, and O in SSIM.

(1) Product variety (V2) helps achieve or influences price (V8)—(V)
(2) Perceived complexity (V4) is impacted or influenced by user interface (V7)—(A)
(3) Special promotions (V6) and price (V8) help achieve or influence each other (X)
(4) Packaging (V5) and responsiveness (V10) are unrelated (O).

It helps design an upper triangular matrix with a diagonal running from the top-left corner to the bottom-right corner. The other half of the matrix is intentionally left blank as the results are always represented from *i* to *j* and not vice versa.

Reachability Matrix

The next ISM step is to complete the remaining half of the matrix. This operation is performed by converting the letter notations to binary values. Each of the four notations: V, A, X, and O will represent some combination of 1 0 in some specific order. The SSIM is then converted into a binary matrix, called the initial reachability matrix.

The substitution of the rows and columns is done using a specific order, and it is as follows:

If entry (i, j) in SSIM = 'V,' enter element (i, j) as '1' and $(j. i)$ as '0.'
If entry (i, j) in SSIM = 'A,' enter element (i, j) as '0' and $(j. i)$ as '1.'
If entry (i, j) in SSIM = 'X,' enter element (i, j) as '1' and $(j. i)$ as '1.'
If entry (i, j) in SSIM = 'O,' enter element (i, j) as '0' and $(j. i)$ as '0.'

Table 28.2 SSIM matrix with key parameters: customer preferences

	V1	V2	V3	V4	V5	V6	V7	V8	V9	V1	V1	V1	V1	V1	V1	V1	V1	V1
Product information (V1)		O	V	O	O	O	A	O	O	O	O	O	V	O	A	O	O	V
Product variety (V2)			O	O	O	O	O	V	O	O	O	O	O	O	O	O	O	V
Trust (V3)				A	A	O	A	A	A	A	O	A	X	A	A	A	A	V
Perceived complexity (V4)					O	O	A	O	O	O	A	O	O	O	O	O	O	V
Packaging (V5)						O	O	O	O	O	O	O	V	O	O	V1	O	V
Special promotions (V6)							O	X	O	O	O	O	O	O	O	O	O	V
User interface (V7)								O	O	O	V	O	V	O	V	O	O	V
Price (V8)									X	O	O	A	O	O	O	O	O	V
Quality (V9)										O	O	O	V	O	O	V	O	V
Responsiveness (V10)											O	O	V	A	O	V	O	V
Convenience (V11)												A	O	O	O	O	O	V
Delivery options (V12)													V	O	O	O	V	V
Reliability (V13)														A	A	O	A	V
Problem resolution (V14)															O	O	O	V
Product reviews (V15)																O	O	V
Product returns (V16)																	O	V
On-time delivery (V17)																		V
Customer satisfaction (V18)																		

Source Author

The resulting reachability matrix is given in Table 28.3. Transitivity errors are present in the initial reachability matrix, which we corrected using the transitivity principle [47]. If '*x*' helps achieve or influences '*y*' and '*y*' helps achieve or influences '*z*,' then '*x*' necessarily helps achieve or influences '*z*.' 1* entries are put to include transitivity to fill the gap to develop the final reachability matrix. The factors are ranked using their driving power and dependence power as calculated in Table 28.4.

Level Partitions and Formulation of ISM

After the inconsistencies are removed, we obtain the final reachability matrix, based on which we identify levels to develop a digraph. First, we compute each driver's antecedent and reachability set, as shown in Table 28.5. The reachability set consists of the driver itself and the other drivers, which may impact or influence. In contrast, the antecedent set consists of the driver and the other drivers, impacting or influencing it. The drivers for which the reachability set and intersection set are the same, are assigned the top level in the ISM hierarchy. The top-level elements for customer satisfaction will not engender the other parameters above their level. Once the top-level driver is identified, it is eliminated from the further hierarchical analysis, and this process is continued until the level of each element is found. For example, trust (V3) and reliability (V13) are found in level I. Thus, they will be positioned at the top of the hierarchy (Fig. 28.1).

Fuzzy MICMAC Analysis

In the previous section, while defining the interactive relationship among the variables, we used binary numbers, i.e., '1' if there is a relation between the parameters and '0' if there is no relation. There is no scope to assign the weight of the relationship. A fuzzy MICMAC (Matriced' Impacts Croise's Multiplication Appliquée a un Classement) analysis is carried out to overcome this relation. This approach defined the relationships between the variables in qualitative languages like fragile, weak, strong, robust, complete, and no relation. We then quantified them using some present values [43]. It is to note that while preparing our initial VAXO matrix (Table 28.3), we converted the opinions into a binary number {0, 1} if the majority of the respondents affirmed/negated the relation. However, under ISM–fuzzy MICMAC analysis, we use the scale between 0, 0.2, 0.4, 0.6, 0.8, and 1.

A fuzzy direct relationship matrix (FDRM) for critical variables influencing customer satisfaction is presented in Table 28.6. Based on the parameters' revised dependence and driving power, they are classified into four categories—autonomous, dependent, linkage, and independent factors. We depict the fuzzy MICMAC graph as shown in Fig. 28.2. In brief, autonomous variables have weak drive power and weak dependence power. They are relatively disengaged from the system; hence, they do not influence it. Dependent variables have weak drive power but strong dependence power, whereas the linkage variables have strong and strong power. These factors are

Table 28.3 Reachability matrix with key parameters

	V1	V2	V3	V4	V5	V6	V7	V8	V9	V10	V11	V12	V13	V14	V15	V16	V17	V18
V1	1	0	1	0	0	0	0	0	0	0	0	0	1	0	0	0	0	1
V2	0	1	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	1
V3	0	0	1	0	0	0	0	0	0	0	0	0	1	0	0	0	0	1
V4	0	0	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	1
V5	0	0	1	0	1	0	0	0	0	0	0	0	1	0	0	1	0	1
V6	0	0	0	0	0	1	0	1	0	0	0	0	0	0	0	0	0	1
V7	1	0	1	1	0	0	1	0	0	0	1	0	1	0	1	0	0	1
V8	0	0	1	0	0	1	0	1	1	0	0	0	0	0	0	0	0	1
V9	0	0	1	0	0	0	0	1	1	0	0	0	1	0	0	1	0	1
V10	0	0	1	0	0	0	0	0	0	1	0	0	1	0	0	1	0	1
V11	0	0	0	1	0	0	0	0	0	0	1	0	0	0	0	0	0	1
V12	0	0	1	0	0	0	0	1	0	0	1	1	1	0	0	0	1	1
V13	0	0	1	0	0	0	0	0	0	0	0	0	1	1	0	0	0	1
V14	0	0	1	0	0	0	0	0	0	1	0	0	1	0	0	0	0	1
V15	1	0	1	0	0	0	0	0	0	0	0	0	1	0	1	0	0	1
V16	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	1	0	1
V17	0	0	1	0	0	0	0	0	0	0	0	0	1	0	0	0	1	1
V18	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1

Source Author

Table 28.4 Final reachability matrix with transitivity (*)

	V1	V2	V3	V4	V5	V6	V7	V8	V9	V10	V11	V12	V13	V14	V15	V16	V17	V18	Driving power
V1	1	0	1	0	0	0	0	0	0	0	0	0	1	0	0	0	0	1	4
V2	0	1	0	0	0	0	0	1	1*	0	0	0	1*	0	0	1*	0	1	6
V3	0	0	1	0	0	0	0	0	0	0	0	0	1	0	0	0	0	1	3
V4	0	0	1	1	0	0	0	0	0	0	0	0	1*	0	0	0	0	1	4
V5	0	0	1	0	1	0	0	0	0	0	0	0	1	0	0	1	0	1	5
V6	0	0	1*	0	0	1	0	1	1*	0	0	0	1*	0	0	1*	0	1	7
V7	1	0	1	1	0	0	1	0	0	0	1	0	1	0	1	0	0	1	8
V8	0	0	1	0	0	1	0	1	1	0	0	0	1*	0	0	1*	0	1	7
V9	0	0	1	0	0	1*	0	1	1	0	0	0	1	0	0	1	0	1	7
V10	0	0	1	0	0	0	0	0	0	1	0	0	1	0	0	1	0	1	5
V11	0	0	1*	1	0	0	0	0	0	0	1	0	1*	0	0	0	0	1	5
V12	0	0	1	1*	0	0	0	1	0	0	1	1	1	0	0	0	1	1	8
V13	0	0	1	0	0	0	0	0	0	0	0	0	1	1	0	0	0	1	3
V14	0	0	1	0	0	0	0	0	0	1	0	0	1	0	0	1*	0	1	6
V15	1	0	1	0	0	0	0	0	0	0	0	0	1	0	1	0	0	1	5
V16	0	0	1	0	0	0	0	0	0	0	0	0	1*	0	0	1	0	1	4
V17	0	0	1	0	0	0	0	0	0	0	0	0	1	0	0	0	1	1	4
V18	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1
Dependence	3	1	16	4	1	3	1	5	4	2	3	1	17	1	2	8	2	18	

Source Author

Table 28.5 Level partition analysis on reachability matrix

Parameters	Antecedent set (AS)	Reachability set (RS)	RS ∩ AS	Level
V1	1, 7, 15	1, 3, 13, 18	1	Third
V2	2	2, 8, 9, 13, 16, 18	2	Seventh
V3	1, 3, 5, 6, 7, 8, 9, 10, 12, 13, 14, 15, 17	3, 13, 18	3, 13	Second
V4	4, 7, 11, 12	3, 4, 13, 18	4	Third
V5	5	3, 5, 13, 16, 18	5	Fourth
V6	6, 8, 9	3, 6, 8, 9, 13, 16, 18	6, 8, 9	Sixth
V7	7	1, 3, 4, 7, 11, 13, 15, 18	7	Fifth
V8	2, 6, 8, 9, 12	3, 6, 8, 9, 13, 16, 18	6, 8, 9	Sixth
V9	2, 6, 8, 9	3, 6, 8, 9, 13, 16, 18	6, 8, 9	Sixth
V10	10, 14	3, 10, 13, 16, 18	10	Fourth
V11	7, 11, 12	3, 4, 11, 13, 18	11	Fourth
V12	12	3, 4, 8, 11, 12, 13, 17, 18	12	Seventh
V13	1, 2, 3, 5, 6, 7, 8, 9, 10, 12, 13, 14, 15, 17	3, 13, 18	3, 13	Second
V14	14	3, 10, 13, 14, 16, 18	14	Fifth
V15	7, 15	1, 3, 13, 15, 18	15	Fourth
V16	2, 5, 6, 8, 9, 10, 14, 16	3, 13, 16, 18	16	Third
V17	12, 17	3, 13, 17, 18	17	Third
V18	1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18	18	18	First

Source Author

unsteady, and any action on these drivers would affect others and give feedback on themselves. Independent variables are called driver parameters. They help achieve the other drivers, and management needs to address these drivers cautiously and may be treated as route cause of all other parameters.

We develop the ISM model, which can be further utilized to understand the relationship between various customer satisfaction elements and improve it by influencing key parameters. Also, further analysis can be done with one specific part of e-commerce, like product returns, etc., to arrive at a more specific and relevant outcome.

Observation 1: As from the analysis of the urban scenario, we have the parameters trust and reliability as the dependent variables. These have high dependency power, so it is prudent to influence these through other parameters. From the customers' viewpoint, quality, responsiveness, and user interface are the independent parameters with high driving power and low dependency. These are the ideal parameters on which companies can focus, thereby improving customer satisfaction. The parameters such as product information and packaging are autonomous variables.

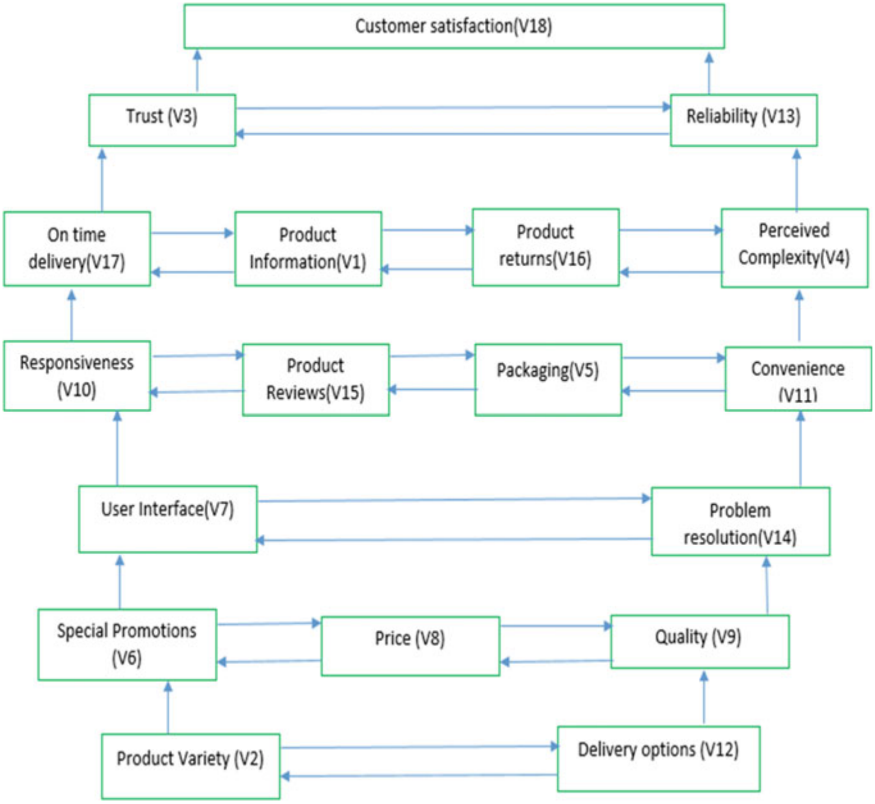

Fig. 28.1 ISM framework for key parameters influencing customer satisfaction (*Source* Authors)

Urban India: Professional Preferences

As we aim to determine and analyze the key drivers for customer satisfaction in major e-commerce organizations in India from both customer and professional perspectives, in this section, we present our integrated ISM and fuzzy MICMAC analysis for urban India from professional perspectives. Please note that we repeated the entire research methodology for professional preferences by constructing the VAXO matrix to the ISM digraph to the subsequent fuzzy MICMAC analysis. Therefore, we present only the fuzzy MICMAC graph (Fig. 28.3) professionally.

Observation 2: From the professionals' standpoint, we can also see that trust and reliability are dependent parameters. However, convenience and perceived complexity are considered to be dependent variables per professionals' perception. Also, professionals believe that quality will be an independent parameter that can be acted upon

Table 28.6 Fuzzy reachability matrix of urban: customer preferences (*Source* Authors)

Fuzzy reachability matrix for customer satisfaction of urban customers

Parameters	Product information	Product variety	Trust	Perceived complexity	Packaging	Special promotions	User interface	Price	Quality	Responsiveness	Convenience	Delivery options	Reliability	Problem resolution	Product reviews	Product returns	On-time delivery	Customer satisfaction	Driving power
Product information	1	0	1	0	0	0	0	0	0	0	0	0	1	0	0	0	0	1	4
Product variety	0	0.6	0	0	0	0	0	1	1	0	0	0	1	0	0	1	0	1	5.6
Trust	0	0	1	0	0	0	0	0	0	0	0	0	1	0	0	0	0	1	3
Perceived complexity	0	0	1	1	0	0	0	0	0	0	0	0	1	0	0	0	0	1	4
Packaging	0	0	1	0	0.4	0	0	0	0	0	0	0	1	0	0	1	0	1	4.4
Special promotions	0	0	1	0	0	1	0	0.4	1	0	0	0	1	0	0	1	0	1	6.4
User interface	0.4	0.4	1	0.9	0.3	0.3	1	0.4	0.4	0.4	1	0.4	1	0.4	1	0.4	0.4	1	11.1
Price	0	0	1	0	0	1	0	1	0.8	0	0	0	1	0	0	1	0	1	6.8
Quality	0.4	0.4	1	0.4	0.4	1	0.4	0.9	1	0.4	0.4	0.4	1	0.4	0.4	1	0.4	1	11.3
Responsiveness	0	0	1	1	0	0	0	0	0	1	1	0	1	1	0	1	0	1	8
Convenience	0	0	1	1	0	0	0	0	0	0	1	0	1	0	0	0	0	1	5
Delivery options	0	0	0.2	1	0	0	0	0	0	0	0.8	0.6	1	0	0	0	1	1	6.6
Reliability	0	0	1	0	0	0	0	0	0	0	0	0	1	0	0	0	0	1	3
Problem resolution	0	0	1	0	0	0	0	0	0	0.9	0	0	1	1	0	0	0	1	5.9
Product reviews	1	0	1	0	0	0	0	0	0	0	0	0	1	0	0.2	0	0	1	4.2
Product returns	0	0	1	0	0	0	0	0	0	0	0	0	1	0	0	1	0	1	4
On-time delivery	0	0	1	0	0	0	0	0	0	0	0	0	0.8	0	0	0	1	1	3.8
Customer satisfaction	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1
Dependence	2.8	1.4	15.2	5.3	1.1	3.3	1.4	4.7	4.2	2.7	4.2	1.4	16.8	2.8	1.6	8.4	2.8	18	

Source Author

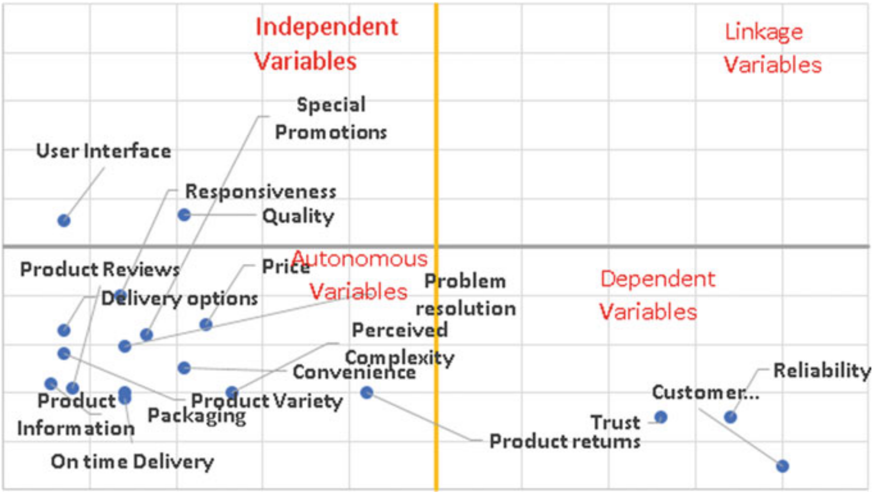

Fig. 28.2 Fuzzy MICMAC graph of urban customers (*Source* Authors)

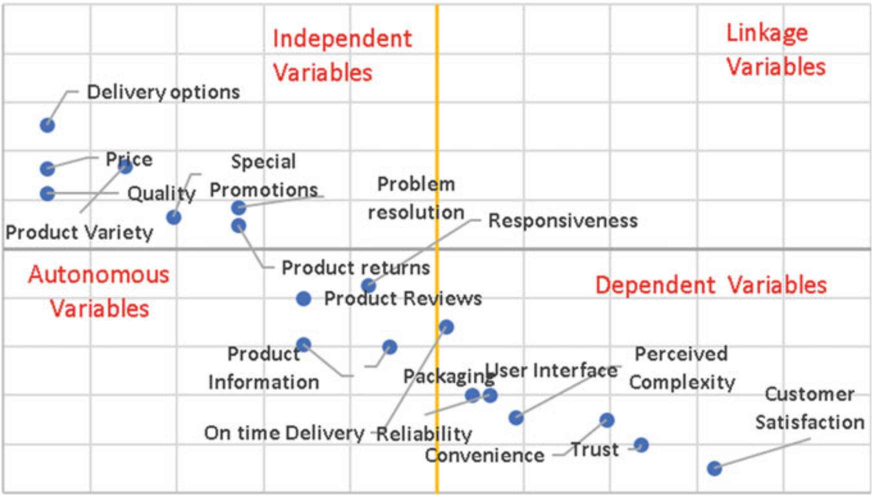

Fig. 28.3 Fuzzy MICMAC analysis of urban: professionals' preferences (*Source* Authors)

to improve customer satisfaction consistent with the customer's viewpoint. Nevertheless, professionals consider price, special promotions, and delivery options to be independent and actionable parameters to improve customer satisfaction.

Analysis of Data Collected from Rural India

Rural India: Customer Preferences

In Section “Research Methodology”, we investigated that to meet the rural customer’s satisfaction, e-commerce companies need to put in the additional effort as rural customers’ tastes and expectations are different from urban customers. We found the 18 variables including ‘customer satisfaction’ suitable for ISM analysis from a rural perspective, among which 11 are common (trust, packaging, special promotions, price, quality, responsiveness, payment options, delivery options, reliability, problem resolution, product returns, and on-time delivery) with urban analysis. Regional language support, buying assistance, delivery charges, usability, payment options, lite website, and logistics partner are significant factors for rural customers. We repeated the entire ISM and fuzzy MICMAC analysis for customers’ viewpoints, and the final fuzzy MICMAC graph results are presented in Fig. 28.4.

Observation 3: As per analysis, customers, we believe trust, reliability, and quality to be dependent parameters having less driving power and high dependence power. It is observed that they consider some parameters like price and on-time delivery dependent on other parameters. Also, they perceive special promotions, lite website, regional languages, and problem resolution to be independent parameters that can positively impact customer satisfaction and other dependent parameters. Also, from rural customers’ viewpoint, all parameters are interlinked as there are barely any parameters in the autonomous category.

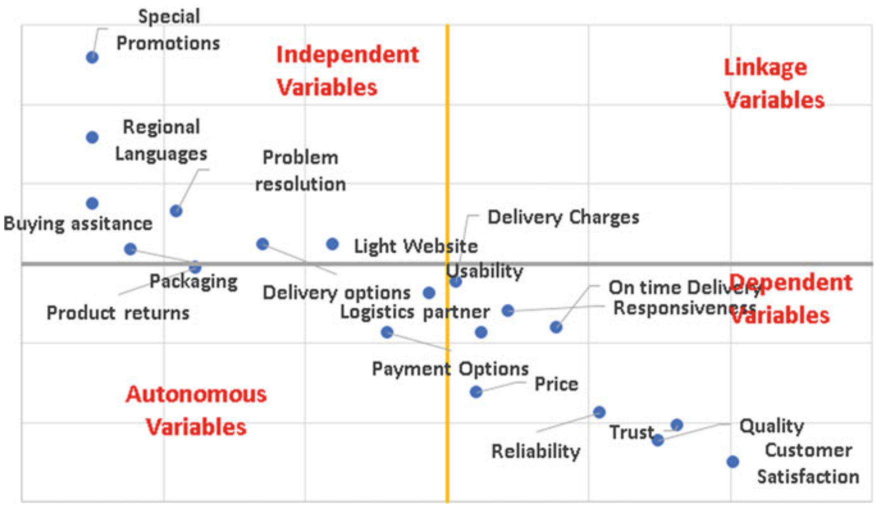

Fig. 28.4 Fuzzy MICMAC analysis of rural: customers’ preferences (*Source* Authors)

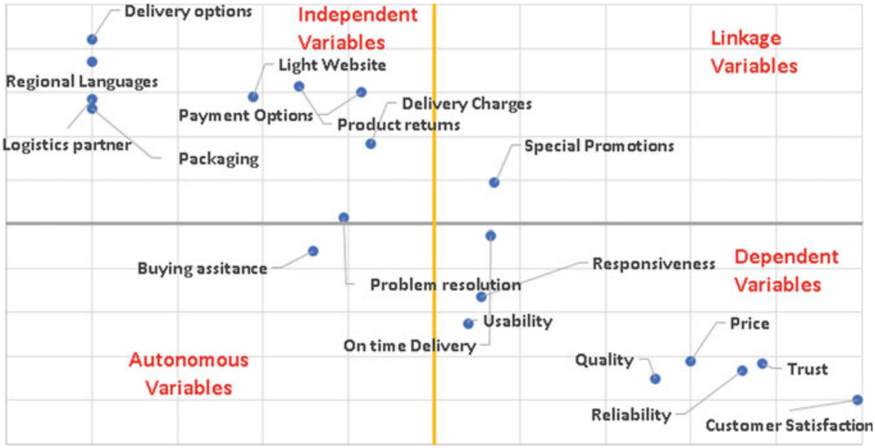

Fig. 28.5 Fuzzy MICMAC analysis of rural: professionals' preferences (*Source* Authors)

Rural India: Professional Preferences

Here, we present the results of professional preferences for the rural segment of India. Figure 28.5 depicts the fuzzy MICMAC analysis.

Observation 4: From the professionals' viewpoint, trust, reliability, and quality are dependent parameters that can be better influenced through other parameters. We can also observe that professionals are starting to recognize the importance of regional languages, light websites, packaging, and delivery options. However, we can see that they are underestimating the importance of buying assistance and problem resolution, and at the same time, overestimating the importance of delivery options and payment options to some extent.

Managerial Discussions and Recommendations

Our analysis identified three critical parameters for the success of e-commerce in India, which is nascent today and yet to evolve even more extensively and more significantly every day. Our study found that e-commerce has not entered all (retail) segments in India, which is also a growth opportunity in the future. Three parameters for the success of e-commerce are on-time delivery (speed), price (cost), and selection (wider choices to choose from). These three parameters cumulatively improve the net promoter score of an e-commerce organization in India and enable its growth day to day. As per our analysis, key drivers of customer satisfaction are majorly trust and quality. Customers who associate a brand with higher quality also give higher net promoter scores. Reliability, price, on-time delivery, and responsiveness are others.

Key Differences Between Customer and Professionals

Analysis of both customers' and professionals' points of view provides common parameters of prime importance from both perspectives. There is a considerable difference of opinion in customers from professionals.

Customers felt that providing buying assistance, regional language support in dealings, and regular special promotions would make them more inclined toward online shopping while retaining the medium and brand they want to shop with.

Customers felt that immediate drivers for customer satisfaction are quality and trust.

Rural consumers are very averse to understanding the intricacies involved in the business operations of e-commerce. Henceforth, getting something through a channel where there would not be face-to-face interaction and regular interaction with the same person is completely strange to them.

Rural Indians are averse to too many price fluctuations, and any price difference from local markets might affect their interests.

Except for the hierarchy, overall level-1 and level-2 drivers from both customer's and professional's perspectives are mostly the same, with trust, price, reliability, and quality being the prime drivers to customer satisfaction. The above-identified parameters would also have an indirect impact on the market share of the companies.

Key Recommendations

It is recommended that 'on-time delivery,' i.e., speed, be improved in the rural scenario. Therefore, the product should reach the customer within a couple of days for early penetration of e-commerce into rural India.

For urban customers, we recommend improved selection and well-planned sales in advance. Urban customers have many options to shop for. Providing a better selection will enable them to shop online and have better customer retention.

Also, as per the analysis, we can see that to maintain trust and reliability on-time delivery, responsiveness plays a key role and is placed at the top hierarchy level.

The price and quality of the product influence the other parameters and thus play a key role in achieving customer satisfaction.

Special promotions that might appear just as a marketing stunt are key parameters building trust and encouraging customers to shop online and retain customers.

Similarly, in the rural context, buying assistance, regional languages, and special promotions will be crucial in improving customer satisfaction from the customer perspective. Delivery options, logistics partners, regional languages, and packaging from a professional's perspective in a rural context will act as key levers.

Conclusions

This paper provides a thorough understanding of customer satisfaction in the Indian e-commerce industry, both rural and urban areas. Identifying the crucial parameters for providing top-class customer satisfaction is done by referring multiple citations on customer satisfaction, e-commerce industry, and marketing insights on consumer behavior. Difference of opinion between consumers and professionals working in e-commerce is identified concerning the parameters toward improving the customer satisfaction of rural and urban consumers, and inputs are taken from professionals working in significant e-commerce organizations in India from rural and urban perspectives. Interpretive structural modeling, MICMAC analysis, and fuzzy MICMAC analysis are used to understand every parameter's correlation along with the to and fro effect, there by understand the relation with customer satisfaction. Results and findings for customer satisfaction measures are stated concerning the overall improvement in business performance, customer retention, and the creation of a reliable and trustworthy environment in the sector. The paper identifies the parameters affecting customer satisfaction in the e-commerce industry in India. The uniqueness of this research is the most influencing driver, especially for rural India, and gap identification between the 'customers need' and 'professionals view' toward customer satisfaction.

This study's future scope expands to identify the most significant customer satisfaction parameters in various e-commerce-related sectors in India or abroad. The proposed modeling and analysis are based on the Indian context, but the implication of this research can be applied to other developing countries as well. Therefore, a cross-country comparison of such frameworks and subsequent observations would be an interesting future extension, especially for rural segments.

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Chapter 29
Investigation of Reliability Measures of Complex Structure via Linear Differential Equation

Hemlata Thakur, Pradeep K. Joshi, and Chitaranjan Sharma

Abstract In this research paper, we investigate a complex system with two subsystems. Subsystem 1 has two similar units in parallel connection, whereas subsystem 2 has single unit which is connected to subsystem 1 in a series connection. This type of system can be used in a variety of manufacturing sectors. All failure rates and repair rates are assumed to follow an exponential distribution. Linear differential equations' (LDEs) method was used to develop equation for evaluating reliability measures and statistical analysis such as mean time to system failure and system steady-state availability. Some particular cases have been evaluated by taking different values of the failure rate and repair rate. We also evaluated the effect of failure rate and repair rate on mean time to system failure and steady-state availability of the system. Outcome of the reliability measures have been verified through graphical and numerical illustration.

Keywords Reliability · Steady-state availability · Linear differential equation · Mean time to system failure (MTSF)

Introduction

System reliability is vital in almost all commercial industries that create numerous kinds of equipment and manufacture. Previously, many researchers and scientists have worked and developed numerous kinds of mathematical models and had given their validity under various reliability measures. The most important factors in increasing system dependability are redundancy and recovery. It contains

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methods for increasing effectiveness by reducing failure and restoring quotes, respectively. Researchers that looked into system protection and reliability measures for a repairable complex system focused on a few flaws and one repair policy. A system includes subsystems and the reliability of the entire system is predicated upon the reliability of subsystems and on its configuration. As we know that every subsystem must survive in every circumstance and evaluation of various reliability measures is essential for the system and subsystem for the survival.

In the present investigation, we have expressed the reliability measure like mean time to system failure and steady-state availability analysis of two subsystems with different configurations and realistic assumptions.

Many authors worked in the field of reliability have analyzed that reliability performance measures of interest such as steady-state probabilities, mean sojourn time, mean time to system failure, steady-state availability, expected busy period of the repairman, etc., are obtained using regenerative point technique with the Markov renewal process. But in this paper, we analyzed reliability performance measures by using first-order linear differential equation (LDE) technique which is an easier method compared to the regenerative point technique with the Markov renewal process which was used by many researchers. Also, it is convenient for computation with software such as MATLAB. El-said and El-Sherbeny [1] studied the reliability measure and availability analysis of two non-identical by using the linear differential equation. Gupta and Mittal [2] looked at the stochastic behavior of a two-unit warm standby system with two different types of repairmen and different amounts of patience time. Yusuf [3] considered two different repairable systems both with standby unit and requiring a supporting unit for their operations and developed explicit expressions for mean time to system failure (MTSF) and steady-state availability by using linear differential equations. Mokaddis et al. [4] used a linear differential equation to assess the reliability and availability of two dissimilar-unit cold standby systems with three modes, for which no cost-benefit analysis was performed. Ali et al. [5] investigated reliability analysis of a two dissimilar-unit cold standby systems with three modes and evaluated reliability and availability analysis using the Kolmogorov Forward equation method. Joshi et al. [6] discuss reliability and availability characteristics of a two-unit standby redundant system by linear differential equation solution technique. Singh and Ayagi [7] used copula to investigate the dependability of systems with two subsystems in a series architecture. Yusuf and Hussaini [8] discussed various measures of system effectiveness such as mean time to system failure (MTSF), steady state availability, busy period and profit function of a 2-out-of-3 repairable system with perfect repair are analyzed using Kolmogorov's forward equation method and showed perfect repair action plays a vital role on system performance. Yusuf and bala [9] considered a redundant air condition cooling system consisting of main unit and a warm standby reserved unit operating in different weather condition (High and low temperature). Using Kolmogorov's forward equations method, various measures of system performance are evaluated.

The goal of this research is to investigate a complex system with two subsystems, subsystems 1 and subsystem 2. Subsystem 1 has two identical units which are connected in a parallel arrangement, and subsystem 2 has a single unit that is connected to subsystem 1 in a series arrangement. Initially, in state $S_{0,}$ subsystem 1 and subsystem 2 are under the operational mode; after the non-success of any unit of subsystem 1, it gets repaired by a repairman, but the system is in working condition. If before the replacement of the first unit of subsystem 1, other unit of subsystem 1 also failed; then, the complete system becomes failed. If subsystem 2 fails, then the complete system becomes failed even though both units of subsystem 1 are in working condition. If one unit of subsystem 1 is in working condition and subsystem 2 fails, then the entire system will also failed.

In this paper, we study the reliability measures such as mean time to system failure and steady-state availability of the system by using linear differential equations and examine the effect of the failure rate and repair rate of subsystem 1 and subsystem 2 on these reliability measures. The paper is outlined as follows, the next section presents the set of assumptions taken throughout the study of the mathematical model, upon which the state of the system is determined. A graphical representation of these states is given by the "Transition Diagram" (Fig. 29.1), on which the transitional parameters (λ, λ_1 and μ_1, μ_2) are pointed out. In Sect. "Reliability Measures", mean time to system failure (MTSF) is evaluated for the system. The matrix differential calculus approach of a linear differential equation given by Magnus and Neudecker is used to get a closed-form of mean time to system failure and steady-state availability. In Sect. "Numerical Evaluation of System Behavior", we discussed the numerical evaluation of system behavior. In Sect. "Result and Discussion", results and discussion of various result obtained are specified.

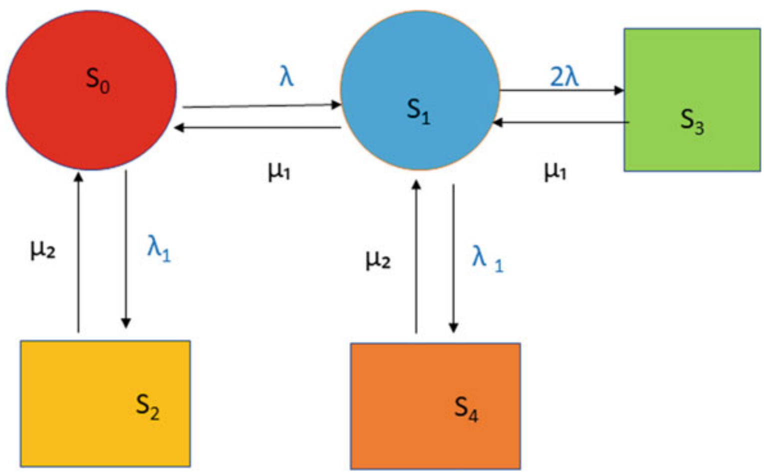

Fig. 29.1 State transition diagram

State Specification

S_0: In state $S_{0,}$ one and other units of subsystem 1 and one unit of subsystem 2 are operational. The system is in working condition.
S_1: In state $S_{1,}$ the first unit of subsystem 1 fails and the second units of subsystem 1 and subsystem 2 are operational.
S_2: In state $S_{2,}$ both units of subsystem 1 are operational and subsystem 2 fails. The system failed.
S_3: In state $S_{3,}$ the second unit of subsystem 1 fails and subsystem 2 is in working condition. The system failed.
S_4: In state S_4, the second unit of subsystem 1 is working and subsystem 2 fails. The system fails.

Assumption

Throughout the research of the mathematical model, the following assumption has been made:

- In the initial state, S_0, both subsystems are operational and fully functional.
- If any unit of subsystem 1 is operational in conjunction with subsystem 2, the system is operational.
- The system fails if subsystem 2 fails.
- Repair is as good as new (perfect repair)
- In one state, only one change is allowed at a time.
- All failure and repair rates are constant.
- Exponential distribution is followed by failure and repair rate.

Notation

- S_i: Transition state of the system, $i = 0, 1, 2, 3, 4$.
- λ: A failure rate of the first unit of subsystem 1.
- 2λ: A failure rate of the second unit of subsystem 1.
- λ_1: A failure rate of subsystem 2.
- μ_1: Repair rate of units of subsystem 1.
- μ_2: Repair rate of a unit of subsystem 2.

Reliability Measures

Mean Time to System Failure (MTSF)

The mean time to system failure (MTSF) for the proposed system will be calculated using the linear differential equation technique and the assumptions mentioned above. From Fig. 29.1, define $P_i(t)$ be the probability of the system at time t ($t \geq 0$) in state S_i. Let $P(t)$ be the probability row vector at time t, the initial conditions are:

$$P(0) = [P_0(0),\ P_1(0),\ P_2(0),\ P_3(0),\ P_4(0)] = [1,\ 0,\ 0,\ 0,\ 0] \tag{29.1}$$

We obtain the following differential equation:

$$\frac{\mathrm{d}p_0(t)}{\mathrm{d}t} = -(\lambda + \lambda_1)\, P_0(t)\ + \mu_1 P_1(t) +\ \mu_2 P_2(t) \tag{29.2}$$

$$\frac{\mathrm{d}p_1(t)}{\mathrm{d}t} = -(2\lambda + \lambda_1 +\ \mu_1)P_1(t) + \lambda P_0(t) + \mu_2 P_4(t) + \mu_1 P_3(t) \tag{29.3}$$

$$\frac{\mathrm{d}p_2(t)}{\mathrm{d}t} = -\mu_2 P_2(t) + \lambda_1 P_0(t) \tag{29.4}$$

$$\frac{\mathrm{d}p_3(t)}{\mathrm{d}t} = -\mu_1 P_3(t) + 2\lambda P_1(t) \tag{29.5}$$

$$\frac{\mathrm{d}p_4(t)}{\mathrm{d}t} = -\mu_2 P_4(t) + \lambda_1 P_1(t) \tag{29.6}$$

which can be written in the matrix form as:

$$\frac{\mathrm{d}p(t)}{\mathrm{d}t} = A \cdot P \tag{29.7}$$

where

$$A = \begin{bmatrix} -(\lambda + \lambda_1) & \mu_1 & \mu_2 & 0 & 0 \\ \lambda & -(2\lambda + \lambda_1 + \mu_1) & 0 & \mu_1 & \mu_2 \\ \lambda_1 & 0 & -\mu_2 & 0 & 0 \\ 0 & 2\lambda & 0 & -\mu_1 & 0 \\ 0 & \lambda_1 & 0 & 0 & -\mu_2 \end{bmatrix} \tag{29.8}$$

It is difficult to evaluate the transition solution; hence, we delete the rows and columns of the absorbing state of matrix A and take the transpose to produce a new matrix say $Q[1, 2, 3]$.

The expected time to reach an absorbing state is obtained from

$$E\left[T_{P(0)\to(\text{absorbing})}\right] = \mathrm{P}(0)\int_{0}^{\infty} \mathrm{e}^{Qt}\mathrm{d}t$$

and

$$\int_{0}^{\infty} \mathrm{e}^{Qt}\mathrm{d}t = Q^{-1}, \quad \text{since } Q^{-1} < 0$$

where

$$Q = \begin{bmatrix} -(\lambda + \lambda_1) & \lambda \\ \mu_1 & -(2\lambda + \lambda_1 + \mu_1) \end{bmatrix} \tag{29.9}$$

An explicit expression for the MTSF is given by:

$$E\left[T_{P(0)\to(\text{absorbing})}\right] = P(0)\left(-Q^{-1}\right)\begin{bmatrix} 1 \\ 1 \end{bmatrix} \tag{29.10}$$

$$\text{MTSF} = \frac{3\lambda + \lambda_1 + \mu_1}{2\lambda^2 + \lambda_1^2 + 3\lambda\lambda_1 + \mu_1\lambda_1} \tag{29.11}$$

Steady-State Availability Analysis A(∞)

For the availability analysis of the states of the system represented by Fig. 29.1. The initial condition for this problem is the same as for the reliability case.

$$P(0) = [P_0(0),\ P_1(0),\ P_2(0),\ P_3(0),\ P_4(0)] = [1,\ 0,\ 0,\ 0,\ 0]$$

The differential equation can be expressed as:

$$\dot{P} = A \cdot P$$

$$\begin{bmatrix} \dot{P}_0 \\ \dot{P}_1 \\ \dot{P}_2 \\ \dot{P}_3 \\ \dot{P}_4 \end{bmatrix} = \begin{bmatrix} -(\lambda+\lambda_1) & \mu_1 & \mu_2 & 0 & 0 \\ \lambda & -(2\lambda+\lambda_1+\mu_1) & 0 & \mu_1 & \mu_2 \\ \lambda_1 & 0 & -\mu_2 & 0 & 0 \\ 0 & 2\lambda & 0 & -\mu_1 & 0 \\ 0 & \lambda_1 & 0 & 0 & -\mu_2 \end{bmatrix} \begin{bmatrix} P_0(\infty) \\ P_1(\infty) \\ P_2(\infty) \\ P_3(\infty) \\ P_4(\infty) \end{bmatrix}$$

The steady-state availability is given by

$$A(\infty) = P_0(\infty) + P_1(\infty) \tag{29.12}$$

In the steady-state availability, the derivatives of the state probabilities become zero so that.

$$A \cdot P = 0$$

which in matrix form

$$\begin{bmatrix} -(\lambda+\lambda_1) & \mu_1 & \mu_2 & 0 & 0 \\ \lambda & -(2\lambda+\lambda_1+\mu_1) & 0 & \mu_1 & \mu_2 \\ \lambda_1 & 0 & -\mu_2 & 0 & 0 \\ 0 & 2\lambda & 0 & -\mu_1 & 0 \\ 0 & \lambda_1 & 0 & 0 & -\mu_2 \end{bmatrix} \begin{bmatrix} P_0(\infty) \\ P_1(\infty) \\ P_2(\infty) \\ P_3(\infty) \\ P_4(\infty) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{29.13}$$

To obtain $P_0(\infty)$ and $P_1(\infty)$, we solve the above using the following normalizing condition

$$P_0(\infty) + P_1(\infty) + P_2(\infty) + P_3(\infty) + P_4(\infty) = 1 \tag{29.14}$$

To obtain $P_0(\infty)$ and $P_1(\infty)$, we substitute (29.14) in one of the redundant rows of (29.13) and obtain the solution of the following system of linear equations in matrix form by using MATLAB.

$$\begin{bmatrix} -(\lambda+\lambda_1) & \mu_1 & \mu_2 & 0 & 0 \\ \lambda & -(2\lambda+\lambda_1+\mu_1) & 0 & \mu_1 & \mu_2 \\ \lambda_1 & 0 & -\mu_2 & 0 & 0 \\ 0 & 2\lambda & 0 & -\mu_1 & 0 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} P_0(\infty) \\ P_1(\infty) \\ P_2(\infty) \\ P_3(\infty) \\ P_4(\infty) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \tag{29.15}$$

The solution of (29.15) provides the steady-state probabilities in the availability case.

For Fig. 29.1, the explicit expression for $A(\infty)$

$$A(\infty) = P_0(\infty) + P_1(\infty)$$

$$A(\infty) = \frac{\mu_1\mu_2(\lambda+\mu_1)}{\mu_1(\lambda+\mu_1)(\mu_2+\lambda_1) + 2\lambda^2\mu_2} \tag{29.16}$$

Numerical Evaluation of System Behavior

Setting $\lambda/\lambda_1 = 0.25$, $\mu_1 = 0.25$, $\mu_2 = 0.5$ and varying λ/λ_1 from 0.1 to 1.0 in Eq. (29.11) and in Eq. (29.16), respectively, we obtain Table 29.1 demonstrate the variation of MTSF and the variation in the availability of the system with respect to failure rate λ/λ_1.

For the study of system behavior, we plot graph in Fig. 29.2 for MTSF and system availability, respectively, with respect to failure rate λ and λ_1.

Setting λ and $\lambda_1 = 0.25$, $\mu_1/\mu_2 = 0.5$, and varying μ_1/μ_2 from 0.1 to 1.0 in Eq. (29.11) and in Eq. (29.16), respectively, we obtain Table 29.2 whose column demonstrates the variation of MTSF and the variation in the availability of the system with respect to repair rate μ_1/μ_2.

Table 29.1 Values of MTSF and Availability for failure rate

λ/λ_1	MTSF (λ)	MTSF (λ_1)	Availability (λ)	Availability (λ_1)
0.1	3.636	4.681	0.579	0.455
0.2	3.098	3.288	0.452	0.417
0.3	2.642	2.524	0.356	0.385
0.4	2.282	2.044	0.288	0.357
0.5	2.000	1.714	0.240	0.333
0.6	1.776	1.474	0.205	0.313
0.7	1.595	1.293	0.178	0.294
0.8	1.446	1.150	0.157	0.278
0.9	1.322	1.035	0.140	0.263
1.0	1.217	0.941	0.127	0.250

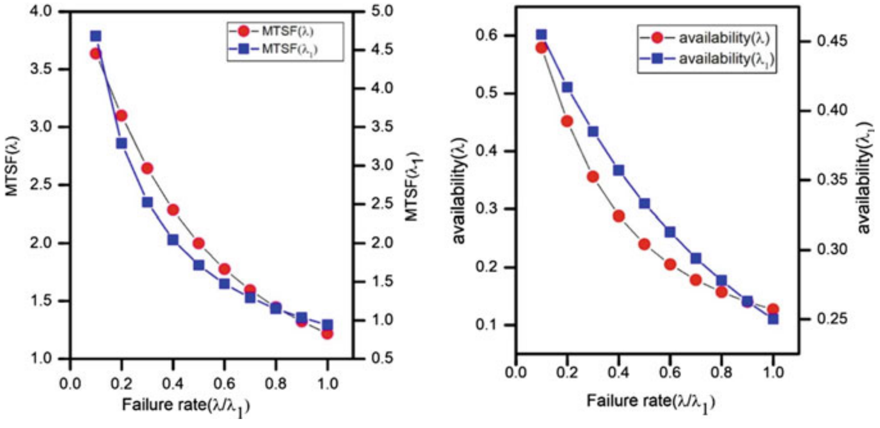

Fig. 29.2 Effect of failure rate on mean time to system failure (MTSF) and availability

Table 29.2 Values of MTSF and Availability for repair rate

μ_1/μ_2	MTSF (μ_1)	Availability (μ_1)	Availability (μ_2)
0.1	2.75	0.197	0.261
0.2	2.823	0.346	0.388
0.3	2.889	0.443	0.462
0.4	2.947	0.505	0.511
0.5	3.0	0.545	0.545
0.6	3.048	0.573	0.571
0.7	3.091	0.592	0.592
0.8	3.130	0.606	0.608
0.9	3.167	0.617	0.621
1.0	3.2	0.625	0.632

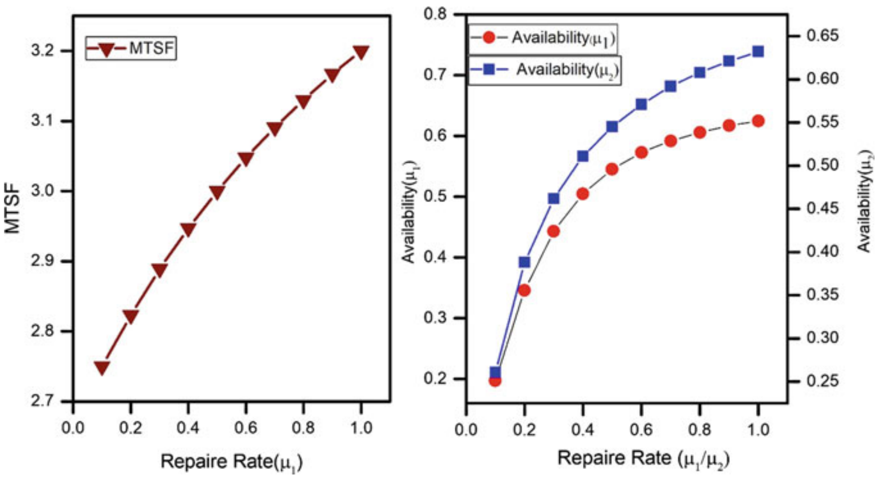

Fig. 29.3 Effect of repair rate on mean time to system failure (MTSF) and availability

For the study of system behavior, we plot graph in Fig. 29.3 for MTSF and system availability, respectively, with respect to repair rate μ_1 and μ_2.

Result and Discussion

Figure 29.2 shows that as λ/λ_1 increases, both mean time to system failure (MTSF) and availability decrease with it. When we compare mean time to system failure with respect to λ and λ_1, we observe that the value of mean time to system failure with respect to λ_1 is slightly more than the mean time to system failure with respect to λ, whereas availability of the system with respect to λ is slightly greater than the availability of system with respect to λ_1.

In Fig. 29.3 we can see that as μ_1/μ_2 increases, both mean times to system failure and availability increase with it. When we compare the availability of the system with respect to μ_1 and μ_2, it can be seen that availability of the system with respect to μ_2 is slightly greater than the availability with respect to μ_1.

Conclusion

In the present investigation, we have explored the reliability measures such as mean time to system failure and availability analysis of a system with two subsystems, subsystem 1 and subsystem 2 with a different configuration. For evaluating reliability measure, linear differential equation method is used with setting realistic assumptions. We construct the various differential equations of the proposed system with probability row vector with initial assumption and find expected mean to reach absorbing state for mean time to system failure and availability analysis. Numerical and graphical illustrations are discussed with result obtained for various reliability measures such as mean time to system failure and steady-state availability.

The reliability measures mean time to system failure and availability were compared with respect to the failure rate of subsystem 1 and subsystem 2 at a fixed value of $\lambda/\lambda_1 = 0.25$, $\mu_1 = 0.25$, $\mu_2 = 0.5$ and varying λ/λ_1 from 0.1 to 1.0. In this case, the result indicates that the measures mean time to system failure and availability are decreasing with the increase of failure rate. In other cases, mean time to system failure and availability of the system increase with the increase of repair rate of subsystem 1 and subsystem 2 at a fixed value of λ and $\lambda_1 = 0.25$, $\mu_1/\mu_2 = 0.5$, and varying μ_1/μ_2 from 0.1 to 1.0.

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Chapter 30
Equitable Allocation of COVID Vaccines to States in India: An Optimization Approach

Ronak Tiwari and R. Sridharan

Abstract This study explores an objective way to equitably allocate COVID-19 vaccines, based on vulnerability of states in India. A unique weighted rank approach is introduced to calculate the vulnerability index for each state. Population-adjusted vulnerability index is finally used to make vaccine allocations to all the states, given a limited supply. The resulting allocations are then compared with the allocations made by the Government of India. A comparison of results shows that the allocations made by the proposed model successfully capture the vulnerability aspect of a state and are closer to the actual allocation figures. Finally, a possible extension of the current approach using a nonlinear programming model, under capacity limitations, is discussed.

Keywords Vaccine equity · Equitable resource allocation · Vaccine inequity Gini coefficient · Weighted rank vulnerability index · Nonlinear programming · Optimization

Introduction

With the new variants of SARS-CoV-2 constantly emerging, ‘Vaccine Inequity’ has become a global concern. At the time, this paper is being written, a new variant of concern, ‘Omicron’, has emerged. One of the reasons, for this periodic upheaval of potentially dangerous mutations of the virus, is the inequitable access to vaccines around the world [1]. While rich nations are already administering the booster doses, population in low-income countries have barely received their first dose [2]. This great disparity in access to vaccines is only pushing the pandemic ahead. Fair distribution of vaccines is a multi-tier challenge. Initiatives like COVAX were set up in the beginning

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of the pandemic for risk pooling and leveraging the buying power of wealthy nations. But largely, such initiatives have not done so well, particularly because of bilateral agreements between the vaccine manufacturers and wealthy nations, that took place early on [3, 4]. The reason for such unequitable distribution is the hoarding behavior of some countries, while some do not have enough resources to buy vaccines. The situation of COVID is still uncertain in many parts of the world, and an answer to accelerate the end of this pandemic could be a consideration of vaccine equity at all tiers. Given the scale of the pandemic, the supply and logistics constraints for manufacturing, procurement, and distribution of vaccines are expected to be periodic. In situations, when there is a huge surge in cases, the equitable allocation of vaccines becomes an important policy decision [5]. This paper focuses on deriving a data-driven framework to equitably allocate vaccines to vulnerable groups, under supply constraints.

Literature Review

Literature on equitable vaccine distribution has been largely qualitative [6, 7], with only recent efforts being reported on use of simulation and optimization techniques for the same [8–10]. The literature does recognize the problem of equity superficially, for example, Enayati and Ôzaltin [11] consider equity in an implicit manner. Similarly, Duijzer et al. [12] focus on the same problem which is addressed in the present paper. Tavana et al. [8] look at the problem from a supply chain distribution point of view. In fact, the OR community has greatly contributed to solve a variety of decision problems relating to COVID [13–15]. However, there are almost no studies which focus on the problem of equitable allocation or from a policymaker's perspective. This paper attempts to fill this exact this research gap by proposing a rank-based framework, coupled with the Gini coefficient of inequality.

Analysis and Mathematical Model

The focus of this paper is equitable distribution of COVID vaccines, to states in India. This study proposes a unique weighted rank approach to quantify the vulnerability of each state and use the population-adjusted vulnerability index to allocate vaccines to states and union territories (UTs) in India, in case of supply constraints. To elaborate the practical significance of the problem dealt in this paper, the reader is referred to [5]. Although the paper is premised on COVID vaccine allocations, the approach exemplified in this paper is ubiquitous in the sense that it applies to all problem instances where resources need to be allocated equitably, and the factors causing vulnerability are known.

Weighted Rank Vulnerability Index (WRVI)

The motivation to use WRVI is to map the vulnerability of any region (states or UTs) to a number between 0 and 1. To do this, the present paper assumes that a region will be more vulnerable if it performs badly, on all the factors which cause populations to become more vulnerable. For example, if the comorbidity disease load for a state is more, compared to another state, then it is ranked higher. This study considers three important factors: (a) comorbidity and health Infrastructure conditions, (b) proportion of population untouched of COVID, and (c) demographics, that drive vulnerability toward COVID. Selection of these factors was done based on the WHO's guidelines for prioritization. Each Indian state, based on how it performs across these factors, is ranked. A weighted sum of ranks, on performance of all the states across the mentioned three factors, is then calculated. To calculate the vulnerability index, the weighted sum of ranks is again ranked to produce the vulnerability rankings. The vulnerability rankings (R_i) for each state i is then divided into equidistant partitions. This way, if a state was ranked 1, then it is regarded as the most vulnerable among the group of states under consideration (n) and is given a WRVI of 1. Similarly, a state which is least vulnerable is also given a vulnerability score which is not zero. The reason for this is explained in the next subsection.

$$\text{WRVI} = 1 - \frac{R_i - 1}{n} \tag{30.1}$$

Population-Adjusted Weighted Rank Vulnerability Index (PAWRVI)

Population and its size are very important factors which govern the allocation decisions. But, to also add an aspect of vulnerability, WRVI is calculated. However, WRVI is only an indication of vulnerability of each state and cannot independently suffice as a proportion for allocation decisions. It is for this reason: a population-adjusted weighted rank vulnerability index (PAWRVI) is conceptualized.

PAWRVI acts as a weighted sum of populations with vulnerability index as weights. It will give a number between 0 and 1 and represents the ideal allocations for equitable vaccine distribution, among all states. For a state i with population size P_i, the PAWRVI can be calculated as follows:

$$\text{PAWRVI}_i = \frac{\text{WRVI}_i \times \text{P}_i}{\sum_{i=1}^{n} \text{WRVI}_i \times \text{P}_i} \tag{30.2}$$

Ideal Allocations with PAWRVI

PAWRVI can be used as a measure of vulnerability-adjusted vaccine allocations in percentages. These allocations are subjected to weights and the perceived factors for population vulnerability. Assuming that the calculated vulnerability index is an ideal way to capture the COVID vulnerability of population, PAWRVI should provide most equitable allocations.

$$\text{Allocation percentage}_i = \text{PAWRVI}_i \times 100 \tag{30.3}$$

Gini Coefficient

Gini coefficient [16] is a widely accepted index for measuring inequality among groups. Its value lies between 0 and 1. A Gini coefficient of 1 implies that there is perfect inequality, while a 0 implies perfect equality in distribution among the groups. It was originally studied to measure income inequality in nations. Expression to calculate Gini coefficient (g) between allocations (A) within population groups (l) and (k), in present context, can be written as follows:

$$g = \sum_{k,l} |\text{PAWRVI}_l A_k - \text{PAWRVI}_k A_l| \tag{30.4}$$

Nonlinear Programming Model to Minimize Inequality

A further extension of the scope of the ideas presented in this paper could be to use the Gini coefficient as a measure of vulnerability inequality. An optimization model with an objective function to minimize the value of Gini coefficient is described as follows:

$$\begin{aligned}
\text{Minimize}: \quad & g = \sum_{k,l} |\text{PAWRVI}_l A_k - \text{PAWRVI}_k A_l|,\ k, l \in S \\
& \text{A}_\text{k} \le C_k, \quad k \in S \\
\text{Subject to}: \quad & \sum_{k \in S} \text{A}_k = \text{Doses Procured} \\
& A_k \ge 0, k \in S
\end{aligned}$$

C_k = Capacity limitation of kth state
S = States and UTs

When the supply is limited and there are capacity constraints within each state, this linear programming model can be used make equitable allocations to the states. In the present context, however, it is important to note that the model requires calculation of PAWRVI and WRVI values for each state based on the determined factors (see Sect. "Weighted Rank Vulnerability Index (WRVI))" that drive vulnerability.

Results and Discussion

Data from Indian Government repositories [5, 17–20] were collected, and an analysis was done to calculate the vulnerability rank for all the 28 states and 6 UTs. The weighted rank can be calculated as follows:

$$\text{Weighted rank} = 0.25 \times A + 0.25 \times B + 0.5 \times C \quad (30.5)$$

The paper does acknowledge the subjective nature of weights chosen for the analysis. But, the rationale behind the selection of weights emanates from the wide literature and sources for understanding population vulnerability against COVID. Nevertheless, the approach presented in the paper is only directive, and it is in the best interest of the health experts and policymakers to use data for understanding vulnerability (Table 30.1).

The vulnerability ranks are simple rankings given to the three factors. A ranking of 1 implies that the state is highly vulnerable. It is important to note that the factors *A*, *B*, and *C* are ranked via other measurable indicators of the same. For example, comorbidity and health infrastructure performance can be measured by the number of health centers in a state; similarly, comorbidity can be measured via the percentage of population diagnosed with comorbid conditions. The data for this meta-analysis were taken from Ministry of Statistics and Program Implementation (MOSPI). A similar equal weighted ranks' approach was used to arrive at these numbers in columns *A*, *B*, and *C*.

After calculation of the vulnerability rank, a vulnerability index is calculated using Eq. (30.1) as shown in Table 30.2.

Now, the last step is to convert the WRVI to PAWRVI using Eq. (30.2) which is given in Table 30.3. PAWRVI is a proportion measure, which the current study suggests. It modifies the population percentage values to incorporate the idea of vulnerability.

A comparison between the allocations made by the Government of India and made by the present study is made. However, because there are no proven or widely accepted measures for equitable allocations, a limitation of the current study is that there is no way to ascertain the effectiveness of the present approach mathematically. Therefore, it falls upon the policymakers and expert judgment to test the effectiveness of the approach empirically (Table 30.4).

Table 30.1 Calculation of vulnerability ranks

States	A	B	C	D	R_i
Andaman & Nicobar Islands	19	31	36	30.5	35
Andhra Pradesh	6	15	13	11.75	8
Arunachal Pradesh	36	16	22	24	29
Assam	23	5	7	10.5	6
Bihar	28	7	1	9.25	3
Chandigarh	5	34	19	19.25	19
Chhattisgarh	33	23	11	19.5	20
Dadra & Nagar Haveli and Daman & Diu	19	21	17	18.5	18
Delhi	8	33	19	19.75	21
Goa	2	34	32	25	30
Gujarat	17	24	10	15.25	14
Haryana	9	21	14	14.5	13
Himachal Pradesh	14	24	34	26.5	31
Jammu & Kashmir	16	18	23	20	23
Jharkhand	29	6	3	10.25	4
Karnataka	14	27	6	13.25	9
Kerala	1	24	27	19.75	21
Ladakh	27	32	32	30.75	36
Lakshadweep	11	36	35	29.25	33
Madhya Pradesh	32	20	5	15.5	16
Maharashtra	12	13	8	10.25	4
Manipur	22	10	16	16	17
Meghalaya	34	2	28	23	26
Mizoram	30	17	31	27.25	32
Nagaland	35	1	26	22	25
Odisha	26	9	11	14.25	12
Puducherry	4	29	25	20.75	24
Punjab	3	12	19	13.25	9
Rajasthan	24	19	9	15.25	14
Sikkim	31	27	30	29.5	34
Tamil Nadu	7	11	18	13.5	11
Telangana	12	3	14	10.75	7
Tripura	21	14	29	23.25	27
Uttar Pradesh	25	7	1	8.5	2
Uttarakhand	17	30	24	23.75	28
West Bengal	10	3	4	5.25	1

A = Comorbidity and health infrastructure rankings, B = Population untouched from COVID rankings, C = Demographics like population density, literacy rate, median age of population, D = Weighted ranks using Eq. (30.5), R_i = Vulnerability ranking of the state

Table 30.2 Calculating vulnerability index

States	Vulnerability rank (R_i)	WRVI
Andaman & Nicobar Islands	35	0.06
Andhra Pradesh	8	0.81
Arunachal Pradesh	29	0.22
Assam	6	0.86
Bihar	3	0.94
Chandigarh	19	0.50
Chhattisgarh	20	0.47
Dadra & Nagar Haveli and Daman & Diu	18	0.53
Delhi	21	0.44
Goa	30	0.19
Gujarat	14	0.64
Haryana	13	0.67
Himachal Pradesh	31	0.17
Jammu & Kashmir	23	0.39
Jharkhand	4	0.92
Karnataka	9	0.78
Kerala	21	0.44
Ladakh	36	0.03
Lakshadweep	33	0.11
Madhya Pradesh	16	0.58
Maharashtra	4	0.92
Manipur	17	0.56
Meghalaya	26	0.31
Mizoram	32	0.14
Nagaland	25	0.33
Odisha	12	0.69
Puducherry	24	0.36
Punjab	9	0.78
Rajasthan	14	0.64
Sikkim	34	0.08
Tamil Nadu	11	0.72
Telangana	7	0.83
Tripura	27	0.28
Uttar Pradesh	2	0.97
Uttarakhand	28	0.25
West Bengal	1	1.00

Table 30.3 Final allocations

States	PAWRVI	Allocations (%)
Andaman & Nicobar Islands	2.078E-05	0.002
Andhra Pradesh	0.0397162	3.97
Arunachal Pradesh	0.000317	0.03
Assam	0.0280832	2.81
Bihar	0.1079089	10.79
Chandigarh	0.0005615	0.06
Chhattisgarh	0.0129412	1.29
Dadra & Nagar Haveli and Daman & Diu	0.0005135	0.05
Delhi	0.0084648	0.85
Goa	0.000283	0.03
Gujarat	0.0414233	4.14
Haryana	0.0182549	1.83
Himachal Pradesh	0.0011501	0.12
Jammu & Kashmir	0.0048627	0.49
Jharkhand	0.0327493	3.27
Karnataka	0.0484781	4.85
Kerala	0.0147341	1.47
Ladakh	7.707E-06	0.0008
Lakshadweep	7.082E-06	0.0007
Madhya Pradesh	0.0457955	4.58
Maharashtra	0.1062654	10.63
Manipur	0.0016362	0.16
Meghalaya	0.0009348	0.09
Mizoram	0.0001571	0.02
Nagaland	0.0006802	0.07
Odisha	0.0295944	2.96
Puducherry	0.0005243	0.05
Punjab	0.0220017	2.20
Rajasthan	0.0470752	4.71
Sikkim	5.249E-05	0.01
Tamil Nadu	0.0515623	5.16
Telangana	0.0293198	2.93
Tripura	0.0010524	0.11
Uttar Pradesh	0.2086244	20.86
Uttarakhand	0.002651	0.27
West Bengal	0.0915954	9.16

Table 30.4 Comparison between allocations

States	E (%)	F (%)	G (%)
Andaman & Nicobar Islands	0.123	0.002	0.03
Andhra Pradesh	4.248	3.97	3.89
Arunachal Pradesh	0.225	0.03	0.11
Assam	1.563	2.81	2.57
Bihar	4.707	10.79	9.01
Chandigarh	0.198	0.06	0.09
Chhattisgarh	3.983	1.29	2.16
Dadra & Nagar Haveli and Daman & Diu	0.100	0.05	0.08
Delhi	2.300	0.85	1.50
Goa	0.336	0.03	0.11
Gujarat	7.684	4.14	5.11
Haryana	2.609	1.83	2.16
Himachal Pradesh	1.859	0.12	0.54
Jammu & Kashmir	2.367	0.49	0.99
Jharkhand	1.617	3.27	2.82
Karnataka	6.187	4.85	4.91
Kerala	4.210	1.47	2.61
Ladakh	0.094	0.0008	0.02
Lakshadweep	0.026	0.0007	0.01
Madhya Pradesh	5.362	4.58	6.19
Maharashtra	10.773	10.63	9.14
Manipur	0.194	0.16	0.23
Meghalaya	0.262	0.09	0.24
Mizoram	0.280	0.02	0.09
Nagaland	0.186	0.07	0.16
Odisha	3.619	2.96	3.36
Puducherry	0.184	0.05	0.11
Punjab	2.854	2.20	2.23
Rajasthan	7.954	4.71	5.81
Sikkim	0.194	0.01	0.05
Tamil Nadu	3.317	5.16	5.63
Telangana	3.869	2.93	2.77
Tripura	0.595	0.11	0.30
Uttar Pradesh	8.307	20.86	16.92
Uttarakhand	1.490	0.27	0.84
West Bengal	6.125	9.16	7.22

E = Vaccine allocation percentages by the Government of India, F = Allocation percentages according to the PAWRVI, G = Population percentages

Conclusion and Future Scope

The goal of this study was to explore an objective and data-driven technique to make equitable vaccine allocations to states and UTs in India. Two simple measures to quantify vulnerability of a group are presented, with a Gini coefficient-based optimization model formulation, in the end. The approach presented in this paper is one of the firsts to consider an equity-based objective function and a new data-driven vulnerability measurement technique. The optimization model formulated in the model can be well extended to include logistics and cold storage constraints at an aggregate level.

Apart from proofs of government facing similar problems in the past (see Annapurani [5]), to provide a contemporary basis for the practicality of the problem, we argue that supply is a function of demand, and in the event of emergence of a new deadly variant overthrowing vaccine immunity, there will be a critical need for re-immunization, which in common terminology would be a surge in demand. However, given the manufacturing capacity limitations, logistical challenges, wastages, hoarding behaviors, etc., supply will face an obvious shortage. On the other hand, it would be in the interest of governments and policymakers to leverage economies of scale in ordering, to avoid the situation of states vying for procuring vaccines, and effectively containing the rejuvenated spread, which gives rise to the need of effectively (equitably) allocating vaccines. At the same time, it is reasonable to assume that there will be cold-storage infrastructure limitations, logistical bottlenecks, etc., that indicate a constrained capacity. These things (i.e., supply limitations, capacity constraints, order size) evidently form the schema of the optimization model mentioned above and prove the practical applicability of the model. In addition to that, the approach promises a wide range of applicability in other areas of research like inventory prepositioning in humanitarian logistics, positioning of risk mitigation inventory for building resilience in supply chains, etc. Efforts in deriving suitable and simple operations' research techniques for equitable vaccine allocations can really enhance the policymaker's decision-making capability. The approaches and the ideas presented in this short paper are extendable to other similar resource allocation problems and promise a wide arena for applicability in general.

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Chapter 31
Consumers' Attitudes Toward Retail Markets: A Multi-criteria-Based Group Decision-Making Approach

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Abstract The present paper identifies the attributes that the buyers consider while they purchase in the three types of retail markets such as traditional, modern, and online using a pilot study in tune with findings of the extant literature. Next, for each such market, a group of buyers with varying demographic profiles were approached to order the attributes reflecting the fundamental psychology of the consumers. We use a consumers' opinion-based integrated framework wherein a widely popular scale-based measure such as Thurstone scaling (TS) is applied to find out the preferential order of the factors that manifest the attitudes of the customers toward different types of retail markets. Further, we prioritize the attributes for each market using the Full Consistency Method (FUCOM). Our study reveals that sales person's behavior, product quality, and options for touch and feel are given more importance by the consumers in traditional market. In case of modern markets, consumers select quality of the products, market location, and touch as dominant criteria. In this respect, the behavioral choices are quite similar in nature. For digital retail markets using online media, buyers look for quality, offers, and price of the products with more significance than others. Our findings therefore support the general notions about these three types of markets. We observe that product quality remains a unanimous choice for the consumers irrespective of the nature of the retail market. For all cases, the DFC value is significantly low suggesting the validity of the results obtained by using FUCOM. The results of TS and FUCOM are found to be consistent.

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Keywords Consumer behavior · Retail markets · Digital retailing · Multi-criteria decision-making (MCDM) · Full Consistency Method (FUCOM) · Psychological assessment · Thurstone scaling (TS)

Introduction

The retailing concept is essentially a customer-oriented, company-wide approach to developing and implementing a marketing strategy. It provides guidelines, which must be followed by all retailers irrespective of their size, channel design, and medium of selling [1]. The traditional pattern of business maneuver is used to operate in the form of bargain system. During the earlier days, the major trading and commercial activities are used to get conducted at some specific populated and socio-economically advanced places only because of non-availability of proper connectivity and networking pattern. The age-old exchange concept of goods and services were termed as 'haat' that was used to get organized once in a week on a fixed day and time at some specific locations well reachable nearby surrounding areas [2]. Gradually, 'mela', mandis or haat, peddlers, vegetable vendors, neighborhood kirana stores were considered as the common format of business activities. These formats were commonly comprised goods and services like agricultural products, food, grocery items, household accessories, daily necessities, and handicrafts along with the involvement of regional cultural activities of the respective destinations within the country. This is important to mention that public rationing system was introduced in the social structure during the British period in Mumbai, and as a result, food grains distribution is started as a part of public system management.

Over the years, the concept of traditional 'push'-based marketing and selling has undergone a paradigm shift toward 'pull'-oriented customized offerings. Since the last two decades, we have been witnessing rapid technological developments and also massive changes in the consumer psychology. Retail is one of the sectors that has been experiencing phenomenal transformations with the advent of modern 'mall' culture and boom in e-commerce segment [3]. The recent pandemic has supplemented to the drift in the consumer behaviors toward technology-driven retails with perspectives of changes in the social pattern [4]. In this context, this paper aims to delve into consumers' psychology manifested through their behavioral choices while they buy products in different retail markets based on their preferences due to addition of several value-added features. According to a recent report [5], there has been a phenomenal metamorphosis in retailing, and the newly popular e-commerce segment may touch base $79.5 billion by 2025. The rapid development in smartphones within the affordable price range and increasing internet users has attributed to the scale-up growth of e-commerce. Does that mean an end of traditional retail formats? Well, in the country like India where the market size is sufficiently large and demographic variety is very high, there is a co-existence of traditional, modern, and online or e-commerce retails apart from unorganized formats. Therefore, it is quite necessary to discern consumers' psychology while buying in different formats of organized retails.

The choice of retail format significantly influences the pre-purchase decision-making which eventually becomes an enabler of buying [6].

Retail business houses are operating with their product line, sales and distribution, and outlets. There is a large number of retail players in the market enjoying the basic needs of competition. A variety of products are available with different cost structures which are dependent on the buying power of the customers. The choice and preferences of consumers largely influence their buying style and consumption pattern [7, 8]. In order to be successful, there is a need to investigate which is the features need to be addressed with priorities in retail formats. In consequences, retailers have to approach the consumers with alternative strategies as a technique of marketing [9]. Over the last two decades, owing to massive developments in the technology and an increased use of internet, consumers have become able to gather more information than ever. Moreover, aftermath the outbreak of Covid-19, the entire nation is passing through an unprecedented social disruption. As a cumulative effect, there has been a considerable change in the buying choices and preferences and consumption pattern of the consumers [10]. Therefore, it is quite imperative to examine the factors that influence consumers' buying decisions.

The objective of this paper is to figure out the key factors that influence consumers' psychology manifested through their buying decisions in the three formats of retail market. We aim to find out whether there is any commonality. We surmise that there may not be any commonality in the priority order as the formats of the retail market are different in nature.

This paper contributes the growing literature in the following way. Firstly, within our limited search, we observe that scantiness of research concentrated on finding out the impact of demographic variables, behavioral factors, and retail and service attributes on buying decision in general or product wise (for example, [3, 6, 11, 12]) in a specific retail format. However, the comparative analysis of consumers' behavior in three different formats of the retail market in general seems to be not so well explored area. Secondly, in this paper, we use a new hybrid group decision making framework, wherein the first stage applies a well-established scaling method such as TS which is used in psychological analysis to prioritize the choice factors of the consumers, and in the second stage, a recently developed linear programming-based criteria weight finding algorithm such as FUCOM is utilized to validate the findings of TS-based analysis. It may be noted in this context that in this paper a considerably large group size is considered which is also quite rare in the extant literature on MCDM.

The rest of the paper proceeds in the following way. In Sect. "Related Work", we present a brief literature review. Section "Research Methodology" describes the research methodology. In Sect. "Findings and Discussion", findings are highlighted and necessary discussions are made. Some of implications of this research are mentioned in Sect. "Research Implications". Finally, Sect. "Conclusion and Future Scope" concludes this paper and sheds light on some of future scope of work.

Related Work

In this era of transition, it is justified to present the theme of the present study with the investigation of changes of retail market scenario from the traditional retail formats to today's modern and structured retail setups. In India, a numerous reputed retail contestants are enthused to consider the country as an attractive destination after the beginning of Liberalization, Privatization and Globalization (LPG) pattern of socio-economic growth. Though the retail concept is primitive, its existence at past was not so significant as there was no such organized attractive format of retail destinations. During the earlier days, the retail market used to be highly scattered with dispersed small outlets lacking in vigor and vitality. Since the availability of the products and their accessibility were limited in ancient times, the demand and consumption patterns were also different and unstructured in nature. Over the years, given massive developments in information and communication technology shift toward knowledge-based economy and society, proliferation of media and easy accessibility, transformational change in the socio-economic, political, cultural, and competitive environments, and retail operations have become more dynamic, structured, and strategically planned. The retailers are now focusing more on designing and delivering unique value propositions and experience to the customers to sustain in the intense competition. There has been introduction of a number of store types such as: departmental store, convenience store, hypermarket, shopping mall, specialty stores, super market [13, 14]. These retail variants are aiming at customers' convenience, comfort, and generation of satisfaction.

According to Mathipoorani and Suguna [14], retail marketing is comprised the series of activities related to peddling products directly to consumers through varied channels such as stores, malls, kiosks, vending machines, movable vans, or other designated places. In exchange of that, direct marketing is based on online attempts to complete a sale directly to the customers through social media and digital platform or through phone, email or website promotion, YouTube channels, etc. The effective execution of the components of the traditional marketing mix such as product, place, price, and promotion is essential for boom in retail marketing.

It has been realized that repeat buying behavior and customer loyalty are influenced by store attributes and customers' emotional attachment toward the features of the retail formats [15]. According to Walters and Knee [16], retail organizations are focusing on non-price attributes to differentiate themselves by adapting definite attributes of the retail stores across formats in line with the demand of the customers. In the view of the said context, consumers feel attraction toward buying preferred products based on the prioritized features that may vary across demographic profile of the customers. Buchanan et al. [17] have pointed out that consistency among the various marketing formats is essential in building brand image and equity as well as loyalty of the customers. Researchers identified that the consumer loyalty is the key element among the essential strategies of retail marketing. So, brand preference by customers is essentially significant in order to address the interest of the customers toward loyalty. The interesting idea was explored by Gonzalez-Benito et al. [18] that the competition in retail formats is gradually changing over the years to avail competitive advantages based on store types and its benefits to align with the customer's type and their preferences. A number of scholars investigated the features of the stores that are important such as facilities like offer, brand, quality, service, payment mode, pricing, comfort in markets, civic amenities.

Morschett et al. [19] explored other types of competitive factors within retail industry that is based on quality of performance, store atmosphere, service, convenience, price, etc. Indian market has various types of retailers at present; among them, small-sized retailers are high in number. These features are the instruments of advertisement to promote the stores through mass media and word of mouth to the middle-class families. As a result, the companies could increase the number of customers in a slow and steady way. Product differentiation with additional features may avail competitive advantages which in turn increase product attractiveness and market share too [20].

Yadav and Garg [3] worked on Indian retail market to discern the significant store and retail attributes that impact consumers' purchasing. Utilizing exploratory factor analysis (EFA), the authors reported that brand, variety of products and features, price, service, convenience, location of store, and salesman's behavior are some of attributes that garner consumers' attentions. Deka [12] added to the previous study and contended that store attributes like location, convenience, ambience, availability of the products, pricing, opportunity for credit purchase, availability of information, service, and offers influence the choice of retail shops by the customers. Moreover, demographic variables like age, profession, gender, and income also impact the decision of store selection. Kumar and Narayana [21] highlighted that the attitude of the customers has resulted in retail organizations to focus on maintaining balance between the pricing range and quality of the product along with other features to survive in competition. In a continuing study, Gupta and Shukla [6] incorporated the dimensions of risk behavior, norms, and behavioral control as manifests of attitude of the buyers that play substantial role in deciding the retail format. In a recent study [22], an attempt has been made to explore behavioral shift in choice and preferences of consumers in taking buying decisions in two markets such as traditional and modern types vis-à-vis income level, general awareness, and influence of media. The findings reflected that there are considerable dissimilarities across the formats. The study of [22] has motivated us to carry out the present research significantly. Moving further, the study of [11] concentrated on e-commerce or online retailing and noticed that wide variety of products with affordable discounted pricing and convenience outperform the traditional brick-and-mortar stores.

Research Methodology

The research methodology followed in this paper is schematically represented in Fig. 31.1. In this paper, we explore the factors that consumers give importance through the literature review and a pilot survey, wherein we interviewed several consumers.

After identification of the factors, we enquire the comparative priority of the factors from consumers' perspective through structured response. Consumers' psychology is a complex behavioral subject. Hence, we propose a consumers' opinion-based integrated framework, wherein a widely popular scale-based measure such as 'Thurstone Scaling' (TS) [23] is applied to find out the preferential order of the factors that manifest the attitudes of the customers toward different types of retail markets. We further apply an optimization-based criteria rating model such Full Consistency Method (FUCOM) to rank the factors and compare with the result of TS. In this way, we validate the result and reduce possible subjective bias in the decision-making to a reasonable extent. In our study, we consider consumers from varying age levels, educational backgrounds, genders, occupations, and income level for achieving a considerably wide perspective. Further, demographics significantly influence the choice factors of the consumers specifically in different market formats [1]. The respondents' profiles are summarized in Table 31.1. Table 31.2 summarizes the factors that influence consumers' decision-making process in three retail formats.

Thurstone Scaling (TS)

TS is a popular scaling method which uses pairwise comparison of the alternatives mostly in the domain of psychology. The concept of TS stems from the fundamental Thurstone's Law of Comparative Judgment (LCJ) theory [23]. According to this technique, the more is the preference of an alternative of a pair over the other alternatives, the greater is the likelihood of that particular alternative holds a greater scale weight. TS depends on probabilistic selection and focuses on obtaining group scale values [24]. An interval scale-based distance measure is used in TS as an instrument. In comparison with the LCJ-based TS scale method in quantitative set up for psychological assessment, the Likert scale (LS) often is associated with risk [25]. TS finds its applications in various problems. For instance, the study of Stepchenkova and Park [26] used TS method in developing a behavioral scale measure for examining the tourists' attitudes towards authenticity of the cultural and heritage products, attractions and places to travel. In a recent study [27], TS was used to gauge the efficacy of e-learning tools in meeting the learning outcomes of the students with diverse background and skill levels. In some other recent applications, TS has been used in exploring consumers' attitudes toward product attributes [28], highlighting key abilities of the teachers for using ICT tools [29] and identifying critical factors causing delay in the process vis-à-vis Ph.D scholars [30]. However, the use of TS is limited in applied research due to its assumption that randomness in the observations follows a normal distribution [31]. Also, there is an underlying assumption that the respondents reflect their true opinions on the central issue [32]. Nevertheless, TS is the first formal method of measuring attitude toward a particular issue by measuring and also comparing respondents' attitudes on generic issues applying the scale to a wide range of surveys. So, TS helps in mapping from ordinal scale space to interval

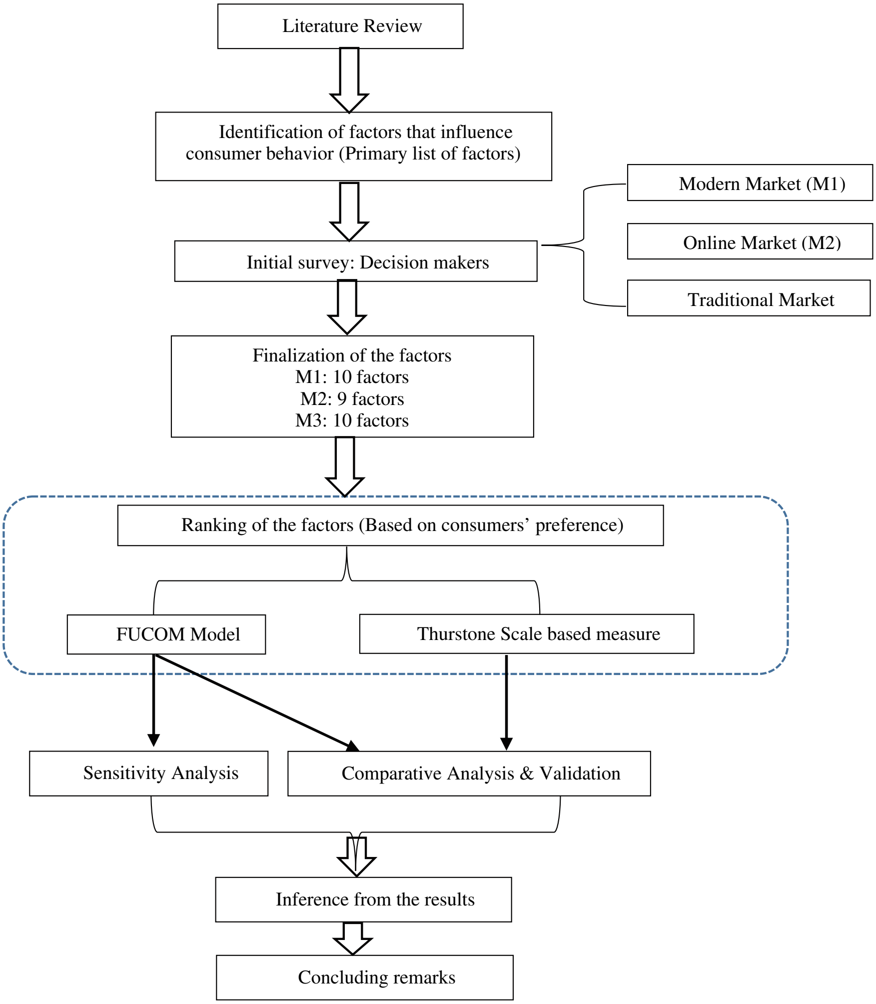

Fig. 31.1 Research framework

scale space to estimate the distance among the features or attitude of the customers toward features of retail formats.

FUCOM

In our paper, we use FUCOM method to calculate the weights of the factors related to consumers' attitude toward three different types of retail markets. These weights

Table 31.1 Respondents' profile

Modern market		Online market		Traditional market	
Category	Percentage	Category	Percentage	Category	Percentage
Gender		*Gender*		*Gender*	
Male	64.5	Male	47.4	Male	64.4
Female	35.5	Female	52.6	Female	35.6
Total	100	Total	100	Total	100
Age group (in years)		*Age group (in years)*		*Age group (in years)*	
Below 21	4	Below 21	6.4	Below 21	10
21–30	56	21–30	66.0	21–30	52.2
31–40	18	31–40	10.9	31–40	12.2
41–50	12	41–50	8.3	41–50	13.3
51–60	10	51–60	7.7	51–60	7.8
Above 60	0	Above 60	0.6	Above 60	4.4
Total	100	Total	100	Total	100
Total number	200		156		90

Table 31.2 Behavioral attributes of the consumers

Retail format	Attributes
Traditional	BP = Bargaining Power/Facility; DP = Display of Products; QP = Quality of Product; ML = Market Location; RS = Relationship with Seller; TF = Touch and Feel; PR = Pricing Range; PV = Product Variety; BSP = Behavior of the Sales Person; OH = Operating Hours
Modern	CA = Civic Amenities; QP = Quality of Product; ML = Market Location; RS = Relationship with Shop-Keeper; CS = Customer Service; CF = Credit Facility; TF = Touch and Feel; BSP = Behavior of the Sales Person; OA = Overall Ambience; CM = Comfort
Online	AB = Availability of Brand; PO = Promotional Offer; QP = Quality of Product; CS = Customer Service; MP = Mode of Payment; PR = Pricing Range; PV = Product Variety; OH = Operating Hours; CM = Comfort

manifest relative importance of the factors as preferred by the consumers. Accordingly, the attributes are ranked based on their weights. In this way, we validate the result and reduce possible subjective bias in the decision-making to a reasonable extent and cross-examine the results obtained by using the TS. For any MAGDM- or MCDM-related problems, criteria weights are of utmost importance [33]. Using of subjective opinions brings about a substantial amount of complexities and bias which compels the final solution to deviate from the level of desired accuracy and posits a lot of ambiguity [34, 35]. Particularly, in case of pairwise comparisons, the problem gets amplified. It is evident that more is the number of comparison, more is the likelihood of the inconsistency [36–38]. In this regard, the FUCOM method [39] provides a set of opportunities to the researchers over its counterparts such as AHP or BWM like following:

- A lesser number of pairwise comparison (for FUCOM, we need $(n-1)$ number of comparisons) that reduces the chance of inconsistency.
- Option to check the validity and consistency of the result by calculating the deviation from full consistency (DFC).

FUCOM has garnered attentions from the researchers in solving a lot of various types of complex problems, for example, comparative analysis of video conferencing platforms used in education [40], smartphone selection [41], finding out the determinants of investment in the cryptocurrency [42], mineral potential mapping [43], healthcare waste management [44] among others.

The algorithm of FUCOM is described below:

Step 1. Ranking of the criteria by the decision-makers according to their relative importance.

Suppose, $C = \{C_1, C_2, C_3, \ldots, C_n\}$ is the set of criteria and the following is the order of the criteria as per the preference of the decision-maker $C_j(1) \succ C_j(2) \succ C_j(3) \succ \ldots C_j(r)$, where r is the rank of the particular criterion. However, there may be possibility that any two criteria hold the same rank (in that case, an '$=$' may be used).

Step 2. Deriving comparative priority of the criteria.

The comparative priority (CP) of the criterion $C_j(r)$ as compared with $C_j(r+1)$ is given by $\Phi_{r/(r+1)}$.

The CP can be defined in two ways: (a) based on decision-maker's defined way and (b) based on a predetermined scale.

The first ranked criterion being the most significant will be compared with itself which leads to a total of $(n-1)$ number of comparisons.

Step 3. Calculation of the final values of the weight coefficients of the criteria.

The final weight values are calculated based on the following two conditions:

$$\text{(a)} \quad \frac{w_r}{w_{r+1}} = \Phi_{r/(r+1)} \tag{31.1}$$

$$\text{(b)} \quad \text{Mathematical transitivity}: \frac{w_r}{w_{r+2}} = \Phi_{r/(r+1)} \otimes \Phi_{(r+1)/(r+2)} \tag{31.2}$$

Step 4. Constructing the final model.

The full consistency or maximum possible consistency can be achieved if DFC (χ) is minimum subject to satisfaction of both the conditions as mentioned in the step 3. The final model is given by:

$$\text{Min}\chi$$

s.t.

$$\left| \frac{w_{j(r)}}{w_{j(r+1)}} - \Phi_{r/(r+1)} \right| \le \chi, \quad \forall j$$

$$\left| \frac{w_{j(r)}}{w_{j(r+2)}} - \Phi_{r/(r+1)} \otimes \Phi_{(r+1)/(r+2)} \right| \le \chi, \quad \forall j \tag{31.3}$$

$$\sum w_j = 1, \quad w_j \ge 0, \forall j$$

The calculations are carried out using MS Excel and Lingo 19.0 software.

Findings and Discussion

In this section, we present the findings of our two stage data analysis, where in the first step, we carry out the analysis using TS. Tables 31.3, 31.4 and 31.5 exhibit the relative importance of the attitudinal factors in three different markets.

Table 31.3 Distance of marketing features on modern market

Factor	CA	QP	ML	RS	CS	CF	TF	BSP	OA	CM
Distance	0.600	1.000	0.600	0.240	0.620	0.330	0.530	0.160	0.000	0.070
Rank	3	1	3	5	2	6	4	8	9	7

(CA = Civic Amenities; QP = Quality of Product; ML = Market Location; RS = Relationship with Shop-Keeper; CS = Customer Service; CF = Credit Facility; TF = Touch and Feel; BSP = Behavior of the Sales Person; OA = Overall Ambience; CM = Comfort)

Table 31.4 Distance of marketing features on online market

Factor	AB	PO	QP	CS	MP	PR	PV	OH	CM
Distance	0.520	0.840	1.000	0.390	0.190	0.830	0.790	0.000	0.140
Rank	5	2	1	6	7	3	4	9	8

(AB = Availability of Brand; PO = Promotional Offer; QP = Quality of Product; CS = Customer Service; MP = Mode of Payment; PR = Pricing Range; PV = Product Variety; OH = Operating Hours; CM = Comfort)

Table 31.5 Distance of marketing features on traditional market

Factor	BP	DP	QP	ML	RS	TF	PR	PV	BSP	OH
Distance	0.000	0.630	0.750	0.230	0.310	0.690	0.580	0.630	1.000	0.330
Rank	9	4	2	8	7	3	5	4	1	6

(BP = Bargaining Power/Facility; DP = Display of Products; QP = Quality of Product; ML = Market Location; RS = Relationship with Seller; TF = Touch and Feel; PR = Pricing Range; PV = Product Variety; BSP = Behavior of the Sales Person; OH = Operating Hours)

From Table 31.3, it is evident that product quality, service, amenities, and location play important role in bringing the customers to the modern market. The result reflects that people prefer prominent location for the modern retail markets (e.g., shopping malls) for easy access and convenient to travel. This quite understandable as nowadays elderly people also travel to modern markets for buying products in bulk quantity and improper distant location of the modern market makes it difficult for them [45]. We notice that amenities also significantly influencing the buying behavior of the consumers support the views expressed in earlier works. For example, Chebat et al. [46] mentioned about the role of wayfinding in terms of providing assistance and information to the consumers (particularly, female customers) in creating memorable shopping experience that in turn acts as an enabler. The quality of products and services always remains at top of the mind of the customers [47]. However, our study contradicts the views of Flacandji and Krey [48] as the findings indicate that overall ambience stands at the last position (see Table 31.3).

Table 31.4 exhibits the consumer behavior in the online market. In the context of present post-pandemic situation, online market has been experience an upsurge in customer enlisting as a result of social distancing, restrictions in travelling and inclination toward offers and discounts [49]. Not surprisingly, our results reveal that quality, offers and discounts, price, and variety of the products attract the consumers. Since the online markets are not restricted by time and physical visits, the factors like operating time and comfort score last. However, the quality is of utmost importance here since for online market, consumers cannot touch and feel the products.

Moving toward the traditional market, consumers believe that behavior of the sales person during the physical encounter, quality of the products, and touch and feel influence their buying decisions (see Table 31.5). The traditional markets are mostly driven by 'bazars' and 'kirana' shops. Hence, quite understandably location does not stand as a biggest barrier.

However, it is observed from Tables 31.3 and 31.5 that two factors hold same distances. Therefore, precise ordering of the factors cannot be done. We now move to find out the factor weights using FUCOM method following the procedural steps. The following Tables 31.6, 31.7 and 31.8 show the weight coefficients of the attitudinal factors in three different retail markets along with the respective DFC values.

It is observed from the Tables 31.6, 31.7 and 31.8 that the DFC values are negligible that suggest the validity of the results obtained by using FUCOM. Further, it is seen that the results obtained from both the frameworks are comparable.

Research Implications

This specific study has multiple implications from research point of view:

- The result implies that training on interpersonal relationship and talent acquisition are important criterion for traditional format to increase sales at large where as people are highly quality conscious in both modern and online formats.

Table 31.6 Weight coefficients (modern market)

Variable	Description	Value/weight	Rank
CHI	DFC	0.000083	
W1	CA	0.106420	4
W2	QP	0.123332	1
W3	ML	0.107706	2
W4	RS	0.087226	9
W5	CS	0.104231	5
W6	CF	0.095975	6
W7	TF	0.106429	3
W8	BSP	0.091580	8
W9	OA	0.085508	10
W10	CM	0.091587	7

Table 31.7 Weight coefficients (online market)

Variable	Description	Value/weight	Rank
CHI	DFC	0.0003339	
W1	AB	0.1084927	5
W2	PO	0.1383387	2
W3	QP	0.1589511	1
W4	CS	0.0935282	6
W5	MP	0.0801212	8
W6	PR	0.1363391	3
W7	PV	0.1262009	4
W8	OH	0.0755031	9
W9	CM	0.0825249	7

- The retailers of modern markets are to pay attention in civic amenities, whereas promotional offer is an important component to attract customers in online markets.
- In traditional market, customers feel attraction towards personal relationship in buying and they are comfortable with the features like display, touch and feel, and price range after bargaining.
- In modern formats customers are prone to a good location where facilities like parking and other entertainments are added features. Here, the retailers must pay attention to customer service and public relation.
- In online pattern, a different psychographic has found where customers are very much oriented to availability of brand and product variety within price range which is very interesting,

Table 31.8 Weight coefficients (traditional market)

Variable	Description	Value/weight	Rank
CHI	DFC	0.00037342	
W1	BP	0.0775016	10
W2	DP	0.1032923	4
W3	QP	0.1103959	2
W4	ML	0.0913552	9
W5	RS	0.0922346	8
W6	TF	0.1033308	3
W7	PR	0.0981036	6
W8	PV	0.0999563	5
W9	BSP	0.1311916	1
W10	OH	0.0926379	7

So, therefore, it has been found that three formats have a few homogeneities, but more heterogeneity exists. Consciousness towards product quality is almost common to all the formats as part of feature. It is needless to mention that retailers should concentrate more on quality product within a competitive price range. The fundamental research findings express the value of the study as the retailers have to employ new differentiation strategies rather than try to compete against mass merchandisers purely on the basis of price. This is an urgent need of the day to concentrate more on quality to compete with brands. Infrastructure development is needed. According to research results, personalized services are to be given to the customers and public relation to be maintained with the support of social and digital media. This research will assist the retailers to frame strategies by establishing a system of open communication between functional features and maintaining a balance between brand building and promotional schemes through proper consumer survey. This study will support to create a roadmap towards success because retailers need to compete for survival by accepting all the challenges for sustainable development.

Conclusion and Future Scope

The present study has attempted to discern the consumers' psychology manifested through their preferences and considerations while selecting a retail format and taking buying decisions. We have carried out a comparative analysis of the factors influencing consumers' choices in three formats of the retail market such as modern, online, and traditional. For each of the three formats, we have administered a structured questionnaire to a considerably sizeable group of customers. A widely used psychological scaling method such as TS has been applied to prioritize the factors in all three formats. To further validate the result and arrive at a conclusion, we have used a linear programming-based group decision-making framework such as FUCOM. Our initial surmise was that there may not be any commonalities in the choice factors. A total of nine factors in the online format and ten factors for both modern and traditional markets have been prioritized. We have seen some commonality in the sense that quality of the products and services remains as a top choice to the consumers. In line with the nature of the retail markets, the other key attributes include amenities, location (for modern market), offers, price, and variety (online market) and behavior of the sellers and touch and feel good factors (for traditional format).

However, the present study may be extended further with the some of the following scopes. Firstly, we have prioritized the consumers' opinions in the three retail formats. However, one future study may propose to analyze the impact of each such factor on pre-purchase, purchase, and post-purchase phases. Secondly, we have seen that quality is a unanimous choice by the consumers. But, in a future study, an attempt may be made to examine whether there is a difference in the degree of influence on the buying behavior in respective retail formats. Thirdly, we have not considered the factors such as push selling, impulsive buying behavior. A study may be undertaken to figure out the impact of these catalysts. Fourthly, in the present paper the mediation and moderation effect of the demographic variables have not been considered. A further work may be conducted in this regard. Fifth, the impact of Covid-19 on the choice factors in all three retail markets may be called for. In this regard, a comparative analysis of pre- and post-Covid consumer behaviors may also be initiated. Finally, in this paper, we have not used any fuzzy number-based analysis though we have done an opinion-based analysis. Therefore, one shall aim to test the efficacy of FUCOM method in a large group decision-making set up using fuzzy number-based calculation.

Nevertheless, the present paper sheds light on important dimensions of consumer behavior analysis in three types of retail formats using a novel integrated framework of TS-FUCOM which seems to be rare in the literature. Therefore, we are hopeful that this study may add value to growing literature and may attract more research in this respect.

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Chapter 32
Effect of COVID-19 on Stock Price: A Time Series-Based Analysis of FMCG and Consumer Durables Sector in India

Sanjib Biswas, Gautam Bandyopadhyay, and Banhi Guha

Abstract In recent time, Indian stock market (ISM) has been confronted with the effect of COVID-19. Since the last week of March 2020, the country was under complete lockdown other than essential services. Though, Government of India has taken several economic measures as a part of the risk mitigation plan, in tune with the global economic slump, ISM has witnessed volatility in the stock prices. In this backdrop, the present study attempts to analyze the immediate short run impact of the COVID-19 crisis on ISM focusing two sectors such as Fast-Moving Consumer Goods (FMCG) and Consumer Durables (CD) listed in BSE, India. Since, the products of these two sectors are of common use by the household, it is interesting to unveil the impact of COVID-19 on the stock prices of the constituent companies. The study period is from April 01, 2013 to March 31, 2021. We consider top 25 FMCG and 05 CD stocks based on their aggregated market capitalization (for the period FY 2013–2014 to FY 2020–2021). Monthly closing stock prices of were collected from the BSE database. The underlying objective is to uncover the nature of impact of COVID-19 on the stock prices of these companies. A time series analysis is carried out to discern the effect of the outbreak of COVID-19 on the closing stock price movements of the companies under study. We use autoregressive integrated moving average (ARIMA) model to predict the monthly closing prices for the period April 2020–March 2021 and calculate the deviation for all the stocks. It is seen that the companies got hit by COVID-19. The result contradicts the fact the companies which

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have multiple products and better reach in the market could stand. The study supports the general notion that luxury goods were not purchased during the crisis period.

Keywords COVID-19 · Bombay Stock Exchange · Stock price movement · Time series model · ARIMA

Introduction

In recent time, the world economy has been affected substantially with the outbreak of the fatal Coronavirus (COVID-19). Since the first instance of COVID-19 case appeared in Wuhan, China, on December 08, 2019, in no time, the deadly virus got spread in more than 170 countries across the globe while causing 216,303,376 number of infected patients and a death of 4,498,451 people as on August 30, 2021 (according to the update report by the World Health Organization). Several countries had been compelled to go for lockdown for a prolonged period. Amidst an uncertain environment where question of survival is a matter of great concern with the effect of this Public Health Emergency of International Concern (declared by WHO on 30 January 2020), the global economy has been facing a rapid downturn. No other contagious virus epidemic such as the Spanish Flu has had such a significant impact on the stock market as the Coronavirus outbreak. The initial stock market surges ascribed to COVID-19 expansions, which occurred in late February and early March, were largely responses to news of the pandemic's progress in the USA. Policy retorts to the outbreak, including news regarding real or potential fiscal and monetary policy moves, are also reflected in the hurdles later in March and until the end of April 2020. Former research finding has suggested Government constraints on specific movement and business activity, as well as voluntary social distancing have the significant impact in a service-oriented economy. These are foremost reasons for aggressive stock market reaction in the USA than it did to preceding pandemics in 1918–1919, 1957–1958, and 1968 [1].

The research based on the data from April 2019 to April 2020 proposed that variations in the number of cases and deaths in the USA and six another country hit hard by the COVID-19 disaster have no impact on US stock market returns, with the exception of the number of reported cases in China [2]. Nevertheless, the study of Onali suggested, there is indication of a beneficial impact on the conditional heteroscedasticity of the Dow Jones and S&P500 returns in several nations.

The research of [3] suggested that during the lockdown period, the market reacted favorably, with significantly positive Average Abnormal Returns (AAR), and investors expected the lockout and responded positively, whereas during the pre-lockdown time, investors freaked, as seen by negative AAR. Another present empirical research examines the pandemic influence on the performance of the Indian Stock Market based on BSE 500 and BSE Sensex and eight sectoral indices of Bombay Stock Exchange (BSE) of India. Then, the study compared the composite keys of India with three global indexes S&P 500, Nikkei 225, and FTSE 100 [4]. The result showed that the Indian Stock Market has a similar standard deviation to worldwide markets, but it has a larger negative skewness and larger positive kurtosis of returns, making it appear more volatile. The analysis of [5] demonstrates that the daily increase of COVID-19 cases has a negative influence on stock returns in Vietnam. The analysis also reveals that the stock market in Vietnam behaved differently before and after the nationwide lockdown. Though pandemic had a large negative impact on Vietnam's stock returns before to the shutdown, the lockdown period had a substantial favorable impact on the overall market and several economic sectors in Vietnam. Substantial connotation was found in between actions taken to avoid the expansion of Coronavirus like travel limitations, lockdown, etc., and discrete income. Such study by [6] showed that this preventive actions impacted savings and investment behavior. While both genders experienced a drop in investment (approx. 43%), the variance in percentage decline was statistically insignificant.

The study of Subramaniam and Chakraborty [7] discovers a high negative relationship between COVID fear and stock returns. Unlike other studies, this one shows that the relationship lasts for a long time. The findings also showed that Coronavirus fear has a significant impact on the stock market. Unlike the frequent times in the past, the emotion lingers for a long time and is not quickly overturned. The paper of Sahoo [8] explored the day-of-the-week result by closing daily data for Nifty 50, Nifty 50 Midcap, Nifty 100, Nifty 100 Midcap, Nifty 100 Smallcap, and Nifty 200 for before and during the COVID-19 disaster. For Mondays, a negative return was observed when it was done during COVID-19 health crisis which was in contrary preceding COVID-19 period. During the epidemic, the effect of Tuesday on index return was found statistically significant and positive for all indexes.

During initial three weeks after a country's initial COVID-19 case notification, one study has observed larger drops and higher volatility for stock markets in nations with lesser individualism and larger uncertainty prevention [9]. The majority of the sectors like pharmaceutical, FMCG, financial services, oil and gas, etc., did well and gained anomalous returns in the 21 days following the announcement. Several sectors regained their position when the market index fell [10]. The paper by [11] empirically studied the impact of Coronavirus on the instability of Indian stock prices based on the daily final prices of stock indices, Nifty, and Sensex from September 2019 to July 2020. According to reports, India's stock market witnessed instability during the pandemic. Their results finally discovered that the indices were lesser in the COVID-19 period as compared to pre-COVID-19 period. The event study approach [12] recommended that there have been substantial irregular returns and cumulative unusual returns in the sector like health care and pharmaceutical sector. However, while relating with further sectors using alternative econometric model, the returns are not statistically substantial and do not explicitly specify the alike. Another study [13] showed COVID-19 had a minor impact on the returns of India's sectoral stock indices. The study on Chinese stock market delivered indication of a substantial adverse effect on stock returns in all companies comprising in the Hang Seng Index and Shanghai Stock Exchange Composite Index for January to March 2020. Also, the study on Chinese stock market proposed that few sectors achieved improvement than others during Coronavirus outbreak precisely in IT and pharmaceutical sectors [14]. In a follow-up work by Verma et al. [15], the researchers had shown financial segments in India undergone worst where all other segments were impacted momentarily. Segments like FMCG, IT, and pharma had positive or partial impacts.

In Indian financial market, there was a negative influence of COVID-19 on the day-to-day returns. In India, sectors like pharma and FMCG had shown healthier result as compared to another stock. The research also suggested government must focus on strengthening sustainable healthiness and safety infrastructure for the populations [16]. The banking segment had effected harshly with the outbreak as lending, and EMIs were put on hold for a considerable period. Another study in Indian stock market found succeeding lockdown declaration and monetary easing tempted a near-normalization of skewness and kurtosis while volatility level continued raised [17]. The research [18] had done an experimentation with algorithms to discern the effect of Indian stock market news. In short run, COVID-19 had a negative impact on Indian sectors like automobile, FMCG, pharmaceuticals, and oil and gas [19]. In longer period, automobile, oil and gas, metals, and the banking segment have grieved extremely. Also, this former study revealed that no particular indices underachieved the domestic average other than NIFTY Auto. The study by Li et al. [20] and Onali [2] suggested that COVID-19 anxiety was the decisive reason pouring community consideration and stock market volatility. The study further had shown stock market performance and GDP development diminished suggestively through average growths during the pandemic.

In this context, the present study presents a comparative analysis of stock performance of pre- and post-COVID periods using time series model-based predictions. Essentially, we aim to capture the early effect of COVID-19 on the stock prices (that leads to return) for the sectors of FMCG and CD. FMCG and CD sectors are familiar and related to the common requirements of the household. The present paper contributes to the literature in the following ways. First, the study related to the early effect of COVID-19 on stock performance for the FMCG and CD sectors in Indian context which is not plentiful. Second, we consider a long period before COVID-19 for building the time series model using ARIMA method and use the model to capture the short run impact. The rest of the paper proceeds in the following way. In Sect. "Data and Methodology", we present the research methodology. In Sect. "Findings and Discussion", findings are highlighted and necessary discussions are made. Finally, Sect. "Conclusion and Future Scope" concludes this paper and sheds light on some of future scope of work.

Data and Methodology

Selection of Sample and Description of Data

In this paper, we consider the stocks pertaining to FMCG and CD sectors listed in BSE, India. We first consider all the stocks (from FMCG and CD sectors) that are enlisted in BSE since April 01, 2013 (Source: BSE website). Subsequently, we obtain a total of 60 stocks for FMCG and 09 stocks for CD. We carry out this first level of filtration because we allow a substantially longer period prior to COVID-19. After that, we collect the year wise market capitalization (MC) for each stocks for the period Fy 2013–2014 to Fy 2020–2021. We then aggregate MC using geometric mean. We use geometric mean to offset the presence of outlier, if any. Since, MC usually not equals to zero, our selection of geometric mean is justified. After, we get the aggregated MC for all stocks, and the values are sorted in descending order and top 25 stocks from FMCG sector and 05 stocks from CD sector have been selected as the final sample (see Tables 32.1 and 32.2).

We then collect the monthly closing price for all stocks pertaining to the final sample (for both FMCG and CD). We use ARIMA model, build on the monthly closing prices from Fy 2013–2014 to Fy 2019–2020, i.e., for the period of April 2013 to March 2020 (pre-COVID-19 phase) to forecast the monthly closing prices of the immediate aftershock period (April 2020 to December 2020). We collect the data from CMIE Prowess database. We use SPSS (version 25) and MS Excel for ARIMA-based analysis.

ARIMA

Autoregressive integrated moving average (ARIMA) is actually a class of models that 'explains' a given time series based on its own past values, that is, its own lags and the lagged forecast errors, so that equation can be used to forecast future values. Any 'non-seasonal' time series that exhibits patterns and is not a random white noise can be modeled with ARIMA models [21].

Table 32.1 List of stocks in the final sample (FMCG sector)

S/L	Company Name	Aggregated MC
FMCG 1	I T C Ltd.	2,841,812.94
FMCG 2	Hindustan Unilever Ltd.	2,693,994.71
FMCG 3	Nestle India Ltd.	841,979.24
FMCG 4	Dabur India Ltd.	561,526.32
FMCG 5	Godrej Consumer Products Ltd.	524,086.04
FMCG 6	Britannia Industries Ltd.	413,364.88
FMCG 7	Marico Ltd.	330,354.25
FMCG 8	Colgate-Palmolive (India) Ltd.	285,786.73
FMCG 9	United Breweries Ltd.	256,616.37
FMCG 10	Procter & Gamble Hygiene & Health Care Ltd.	254,697.15
FMCG 11	Emami Ltd.	173,939.00
FMCG 12	Gillette India Ltd.	153,719.42
FMCG 13	Tata Consumer Products Ltd.	146,047.65
FMCG 14	Hatsun Agro Products Ltd.	69,472.61
FMCG 15	Jyothy Labs Ltd.	51,708.25
FMCG 16	K R B L Ltd.	47,416.28
FMCG 17	Zydus Wellness Ltd.	47,167.60
FMCG 18	Bajaj Consumer Care Ltd.	45,124.72
FMCG 19	Godfrey Phillips India Ltd.	44,720.47
FMCG 20	Bombay Burmah Trdg. Corpn. Ltd.	41,816.76
FMCG 21	E I D-Parry (India) Ltd.	36,948.83
FMCG 22	Avanti Feeds Ltd.	29,084.56
FMCG 23	Future Consumer Ltd.	27,836.49
FMCG 24	Radico Khaitan Ltd.	27,551.10
FMCG 25	C C L Products (India) Ltd.	25,828.12

Table 32.2 List of stocks in the final sample (CD sector)

S/L	Company name	Aggregated MC
CD1	Titan Company Ltd.	559,898.5
CD2	Voltas Ltd.	138,882.3
CD3	Whirlpool Of India Ltd.	131,797.4
CD4	Rajesh Exports Ltd.	121,748.8
CD5	Symphony Ltd.	76,950.97

An ARIMA model is characterized by 3 terms: p, d, q. Here, 'p' is the order of the 'autoregressive' (AR) term. It refers to the number of lags of Y to be used as predictors; 'q' is the order of the 'moving average' (MA) term. It refers to the number of lagged forecast errors that should go into the ARIMA model. The value of d, therefore, is the minimum number of differencing needed to make the series stationary. If the time series is already stationary, then $d = 0$. If a time series has seasonal patterns, then you need to add seasonal terms and it becomes SARIMA, short for 'Seasonal ARIMA'.

An ARIMA model is one where the time series was differentiated at least once to make it stationary and you combine the AR and the MA terms.

$$Y_t = \alpha + \beta_1 Y_{t-1} + \beta_2 Y_{t-2} + \cdots + \beta_p Y_{t-p} \in_t + \phi_1 \in_{t-1} + \phi_2 \in_{t-2} + \cdots + \phi_q \in_{t-q} \tag{32.1}$$

Steps in ARIMA modeling

1. Check for the stationarity of data (if not use suitable differencing value to make it stationary).
2. Select AR and MA terms: Use the ACF and PACF to decide whether to include an AR term(s), MA term(s), or both.
3. Build the model.
4. Validate the model by comparing the predicted values to the actuals.

This means ARIMA modeling involves model identification (using AFC and PACF to get an idea of the amount of differencing and the size of the lag that will be required), parameter estimation by using a fitting procedure to find the coefficients of the regression model), and lastly, model checking. ARIMA has been used by several researchers (for example, [22–25]) in predicting the stock prices.

Findings and Discussion

In this section, we present the findings of the ARIMA model-based forecasting of stock prices. Tables 32.3, 32.4, 32.5, 32.6, 32.7, 32.8 and 32.9.

Clearly, from the results, it is observed that the ARIMA models show fit statistically as suggested by Ljung-Box test. We now move to use ARIMA models to forecast the stock prices for the early period April 2020 to December 2020. We are inquisitive to examine the difference between the actual values and forecasted values for the closing prices for all stocks under study and subsequently, explore the nature of changes (see Tables 32.10, 32.11, 32.12, 32.13, 32.14 and 32.15).

We find that despite the fetal impact of the COVID-19, majority of the stocks for FMCG and CD sector have experienced positive differences in the closing prices.

Table 32.3 ARIMA models for FMCG stocks (part 1)

Company	ARIMA (*p*, *d*, *q*)	Ljung-Box *Q* (18) [sig]	Constant	Sig	AR lag1	Sig
Bajaj Consumer Care Ltd.	(1, 0, 0)	0.251	266.35	0.003	0.974	0
Bombay Burmah Trdg. Corpn. Ltd.	(1, 0, 0)	0.259	615.864	0.038	0.958	0
Britannia Industries Ltd.	(1, 0, 0)	0.982	2283.599	0.032	0.973	0
C C L Products (India) Ltd.	(1, 0, 0)	0.871	217.406	0	0.925	0
Colgate-Palmolive (India) Ltd.	(1, 0, 0)	0.888	1266.634	0	0.908	0
E I D-Parry (India) Ltd.	(1, 0, 0)	0.986	189.561	0	0.954	0
Emami Ltd.	(1, 0, 0)	0.775	613.601	0.013	0.966	0
Gillette India Ltd.	(1, 0, 0)	0.997	4080.808	0.009	0.986	0
Godfrey Phillips India Ltd.	(1, 0, 0)	0.852	1497.993	0.001	0.932	0
Godrej Consumer Products Ltd.	(1, 0, 0)	0.624	961.048	0	0.913	0
Hatsun Agro Products Ltd.	(1, 0, 0)	0.963	392.836	0.011	0.972	0
I T C Ltd.	(1, 0, 0)	0.982	288.02	0	0.909	0
Jyothy Labs Ltd.	(1, 0, 0)	0.657	217.275	0	0.95	0
Marico Ltd.	(1, 0, 0)	0.692	294.707	0	0.909	0
Radico Khaitan Ltd.	(1, 0, 0)	0.425	198.697	0.006	0.958	0
Tata Consumer Products Ltd.	(1, 0, 0)	0.399	197.124	0	0.964	0
United Breweries Ltd.	(1, 0, 0)	0.698	931.127	0	0.926	0
Zydus Wellness Ltd.	(1, 0, 0)	0.388	915.107	0.001	0.97	0
Future Consumer Ltd.	(1, 0, 0) Without constant	0.477			0.984	0
Procter & Gamble Hygiene & Health Care Ltd.	(1, 0, 0) Without constant	0.885			0.998	0

Table 32.4 ARIMA models for FMCG stocks (part 2)

Company	ARIMA (*p*, *d*, *q*)	Ljung-Box *Q* (18) [sig]	Constant	Sig	AR lag1	Sig	AR lag2	Sig
Avanti Feeds Ltd.	(2, 0, 0)	0.823	862.995	0.003	1.14	0	− 0.26	0.016

Table 32.5 ARIMA models for FMCG stocks (part 3)

Company	ARIMA (*p*, *d*, *q*)	Ljung-Box *Q* (18) [sig]	Constant	Sig	AR lag1	Sig	MA lag1	Sig
Dabur India Ltd.	(1, 0, 1)	0.979	293.976	0.006	0.981	0	− 0.327	0.004

Conclusion and Future Scope

In the present paper, we aim to examine the impact of the COVID-19 on the stock returns. The stock return depends on the closing prices. Hence, we use ARIMA modeling to forecast the monthly closing prices of the selected stocks pertaining to FMCG and CD sectors. The models are built on the historical monthly closing prices spanning over a period of 7 years prior to COVID-19. We find the no. of the periods the (actual-forecast) closing prices are greater than zero, which suggests positive price movements, i.e., appreciation or indication of positive returns than expected. We observe that in Indian context, COVID-19 has not affected the FMCG and CD sectors. However, we notice that the result contradicts the fact the companies, which have multiple products and better reach in the market could stand. The present paper has some future scopes. For example, one future study may concentrate on using advanced machine learning algorithm such as LSTM with daily closing price data to forecast. Further, a future work may examine the impact of the COVID-19 considering short and medium terms. In the present study, we have only considered the closing prices. To examine the impact of COVID-19, it is important to assess the fundamental performance, which may be addressed by a future work. However, the present study assumes its importance as a rare work on assessing short run impact of COVID-19 on return of stock prices for FMCG and CD sectors.

Table 32.6 ARIMA models for FMCG stocks (part 4)

Company	ARIMA (*p*, *d*, *q*)	Ljung-Box *Q* (18) [sig]	Constant	Sig	AR lag1	Sig	AR lag2	Sig	MA lag1	Sig	MA lag2	Sig
Hindustan Unilever Ltd.	(2, 1, 2)	0.958	20.662	0.001	0.663	0	− 0.894	0	0.614	0.003	− 0.661	0.001

Table 32.7 ARIMA models for FMCG stocks (part 5)

Company	ARIMA (*p*, *d*, *q*)	Ljung-Box *Q* (18) [sig]	AR lag1	Sig	AR lag2	Sig	MA lag1	Sig	MA lag2	Sig
K R B L Ltd.	(2, 1, 2) without constant	0.338	− 0.703	0	− 0.863	0	− 0.609	0.008	− 0.723	0.002

Table 32.8 ARIMA models for FMCG stocks (part 6)

Company	ARIMA (*p*, *d*, *q*)	Ljung-Box *Q* (18) [sig]	AR lag1	Sig	MA lag1	Sig
Nestle India Ltd.	(1, 1, 1) without constant	0.346	0.99	0	0.946	0

Table 32.9 ARIMA models for CD stocks

Company	ARIMA (*p*, *d*, *q*)	Ljung-Box *Q* (18) [Sig]	Constant	Sig	AR lag1	Sig
Titan Company Ltd.	(1, 0, 0)	0.213	594.86	0.027	0.978	0
Voltas Ltd.	(1, 0, 0)	0.945	331.14	0.035	0.979	0
Symphony Ltd.	(1, 0, 0) Transformation -Sqrt	0.233	31.531	0	0.962	0
Whirlpool Of India Ltd.	(1, 0, 0) without constant	0.122	–	–	0.996	0
Rajesh Exports Ltd.	(1, 0, 0)	0.267	395.893	0.034	0.979	0

Table 32.10 Forecasting results for FMCG stocks (part 1)

Company name	Avanti Feeds Ltd.	Bajaj Consumer Care Ltd.	Bombay Burmah Trdg. Corpn. Ltd.	Britannia Industries Ltd.	C C L Products (India) Ltd.
Forecasted value					
Apr-20	319.53	136.27	727	2678.85	182.42
May-20	392.84	139.61	722.36	2668.35	185.03
Jun-20	469.72	142.86	717.9	Apr-07	187.44
Jul-20	538.13	146.03	713.64	2648.16	189.68
Aug-20	595.91	149.11	709.55	2638.47	191.74
Sep-20	643.83	152.12	705.63	2629.03	193.66
Oct-20	683.28	155.05	701.88	2619.85	195.43
Nov-20	715.67	157.9	698.28	2610.91	197.07
Dec-20	742.24	160.68	694.84	2602.21	198.58
Actual–Forecast					
Apr-20	112.82	8.73	176.6	485.95	10.53
May-20	21.01	− 7.26	186.14	711.85	27.07
Jun-20	34.68	3.34	332.2	944.38	47.41
Jul-20	− 91.63	29.52	543.86	1175.99	44.77
Aug-20	− 78.21	22.94	644.2	1085.93	55.16

(continued)

Table 32.10 (continued)

Company name	Avanti Feeds Ltd.	Bajaj Consumer Care Ltd.	Bombay Burmah Trdg. Corpn. Ltd.	Britannia Industries Ltd.	C C L Products (India) Ltd.
Sep-20	− 147.38	30.03	599.07	1166.92	71.59
Oct-20	− 201.58	30.95	507.62	854.8	45.97
Nov-20	− 193.72	46.95	618.22	1023.19	68.63
Dec-20	− 219.64	53.47	603.36	972.89	69.62
No. of periods with +ve difference	3	8	9	9	9
No. of period with −ve difference	6	1	0	0	0

Table 32.11 Forecasting results for FMCG stocks (part 2)

Company name	Colgate-Palmolive (India) Ltd.	Dabur India Ltd.	E I D-Parry (India) Ltd.	Emami Ltd.	Future Consumer Ltd.
Forecasted value					
Apr-20	1254.21	434.5	142.04	185.07	6.9
May-20	1255.36	431.8	144.22	199.64	6.79
Jun-20	1256.4	429.15	146.3	213.7	6.68
Jul-20	1257.34	426.56	148.29	227.29	6.57
Aug-20	1258.2	424.01	150.18	240.42	6.47
Sep-20	1258.97	421.51	151.99	253.1	6.37
Oct-20	1259.68	419.07	153.72	265.35	6.26
Nov-20	1260.32	416.66	155.36	277.18	6.16
Dec-20	1260.9	414.31	156.93	288.62	6.07
Actual–Forecast					
Apr-20	206.49	54.85	3.21	9.78	2.32
May-20	139.69	34.8	51.08	− 5.94	1.75
Jun-20	149	36.6	130.7	7.35	10.29
Jul-20	166.16	86.89	142.31	12.61	4.58
Aug-20	105.85	51.04	136.57	115.68	5.58
Sep-20	173.58	89.19	131.11	97.95	2.23
Oct-20	257.67	92.48	122.28	94.8	1.14
Nov-20	248.58	82.94	191.39	164.07	2.07
Dec-20	304.55	119.69	187.07	135.88	2.44
No. of period with + ve difference	9	9	9	8	9

(continued)

Table 32.11 (continued)

Company name	Colgate-Palmolive (India) Ltd.	Dabur India Ltd.	E I D-Parry (India) Ltd.	Emami Ltd.	Future Consumer Ltd.
No. of period with −ve difference	0	0	0	1	0

Table 32.12 Forecasting results for FMCG stocks (part 3)

Company name	Gillette India Ltd.	Godfrey Phillips India Ltd.	Godrej Consumer Products Ltd.	Hatsun Agro Products Ltd.	Hindustan Unilever Ltd.
Forecasted value					
Apr-20	5449.28	980.46	559.01	508.7	2257.9
May-20	5430.52	1015.82	594.04	505.49	2191.43
Jun-20	5412.01	1048.77	626.01	502.38	2208.77
Jul-20	5393.76	1079.46	655.2	499.34	2305.16
Aug-20	5375.76	1108.06	681.84	496.4	2379.03
Sep-20	5358.01	1134.71	706.16	493.53	2367.27
Oct-20	5340.5	1159.53	728.37	490.74	2318.82
Nov-20	5323.23	1182.66	748.64	488.03	2322.63
Dec-20	5306.2	1204.2	767.14	485.4	2393.92
Actual–Forecast					
Apr-20	− 118.08	25.34	− 16.21	− 7.3	− 62.2
May-20	− 594.47	− 28.42	39.86	18.96	− 134.53
Jun-20	− 432.91	− 30.82	64.69	153.97	− 28.02
Jul-20	− 370.91	− 163.21	36.7	160.86	− 92.26
Aug-20	49.74	− 142.91	−30.89	248.55	− 262.48
Sep-20	− 17.16	− 212.46	19.14	286.47	− 298.92
Oct-20	− 62.8	− 263.98	− 63.87	309.51	− 246.52
Nov-20	485.97	− 276.11	− 48.84	459.62	− 185.43
Dec-20	484.1	− 230.95	− 26.99	238.8	− 0.37
No. of period with + ve difference	3	1	4	8	0
No. of period with −ve difference	6	8	5	1	9

Table 32.13 Forecasting results for FMCG stocks (part 4)

Company name	I T C Ltd.	Jyothy Labs Ltd.	K R B L Ltd.	Marico Ltd.	Nestle India Ltd.
Forecasted value					
Apr-20	182.52	99.94	141.74	276.65	16,538.31
May-20	192.09	105.83	139.92	278.29	16,771.9
Jun-20	200.8	111.42	136.5	279.78	17,003.17
Jul-20	208.71	116.73	140.48	281.14	17,232.17
Aug-20	215.9	121.77	140.63	282.37	17,458.9
Sep-20	222.44	126.56	137.09	283.49	17,683.39
Oct-20	228.39	131.11	139.44	284.51	17,905.66
Nov-20	233.8	135.43	140.85	285.43	18,125.74
Dec-20	238.72	139.53	137.83	286.27	18,343.64
Actual–Forecast					
Apr-20	− 0.37	16.11	49.86	10.25	1367.79
May-20	5.36	0.07	47.43	65.81	761.95
Jun-20	− 6.2	6.03	109	72.02	174.83
Jul-20	− 14.61	6.62	130.07	82.21	− 705.72
Aug-20	− 24.65	20.38	121.57	86.78	− 1512.8
Sep-20	− 50.69	20.19	152.81	79.36	− 1758.69
Oct-20	− 63.14	4.59	116.71	72.04	− 742.86
Nov-20	− 40.1	5.42	116.85	79.52	− 468.14
Dec-20	− 29.72	6.82	107.12	116.13	48.71
No. of period with + ve difference	1	9	9	9	4
No. of period with −ve difference	8	0	0	0	5

Table 32.14 Forecasting results for FMCG stocks (part 5)

Company name	Procter & Gamble Hygiene & Health Care Ltd.	Radico Khaitan Ltd.	Tata Consumer Products Ltd.	United Breweries Ltd.	Zydus Wellness Ltd.
Forecasted value					
Apr-20	10,423.6	264.26	291.29	918.7	1285.5
May-20	10,399.95	261.49	287.91	919.62	1274.34
Jun-20	10,376.35	258.84	284.65	920.47	1263.52
Jul-20	10,352.8	256.31	281.51	921.26	1253.02

(continued)

Table 32.14 (continued)

Company name	Procter & Gamble Hygiene & Health Care Ltd.	Radico Khaitan Ltd.	Tata Consumer Products Ltd.	United Breweries Ltd.	Zydus Wellness Ltd.
Aug-20	10,329.31	253.87	278.48	921.99	1242.84
Sep-20	10,305.88	251.55	275.56	922.67	1232.97
Oct-20	10,282.49	249.32	272.75	923.3	1223.39
Nov-20	10,259.16	247.18	270.03	923.88	1214.1
Dec-20	10,235.89	245.13	267.41	924.42	1205.1
Actual–Forecast					
Apr-20	76.5	29.64	60.06	19.5	73.9
May-20	− 502.95	68.81	80.04	37.03	− 22.34
Jun-20	− 221.25	114.26	102.9	117.13	10.48
Jul-20	− 13.5	111.94	146.19	31.09	354.33
Aug-20	− 342.76	147.63	254.62	88.76	373.16
Sep-20	− 371.08	154.35	224.19	31.13	607.33
Oct-20	− 140.04	187.68	220.55	2.2	647.36
Nov-20	293.24	207.02	268.97	113.42	646.2
Dec-20	837.86	211.97	322.24	261.33	783.55
No. of period with + ve difference	3	9	9	9	8
No. of period with −ve difference	6	0	0	0	1

Table 32.15 Forecasting results for CD stocks

Company name	Symphony Ltd.	Titan Company Ltd.	Voltas Ltd.	Whirlpool Of India Ltd.	Rajesh Exports Ltd.
Forecasted value					
Apr-20	805.78	926.15	475.71	1801.17	539.13
May-20	819.67	918.96	472.64	1794.76	536.17
Jun-20	832.81	911.93	469.64	1788.38	533.28
Jul-20	845.26	905.04	466.7	1782.02	530.44
Aug-20	857.04	898.31	463.82	1775.68	527.67
Sep-20	868.21	891.73	461	1769.37	524.95
Oct-20	878.79	885.28	458.24	1763.07	522.28

(continued)

Table 32.15 (continued)

Company name	Symphony Ltd.	Titan Company Ltd.	Voltas Ltd.	Whirlpool Of India Ltd.	Rajesh Exports Ltd.
Nov-20	888.82	878.98	455.54	1756.8	519.67
Dec-20	898.34	872.81	452.9	1750.55	517.12
Actual–Forecast					
Apr-20	116.52	43.65	30.09	256.33	79.72
May-20	76.88	−28.36	7.36	144.19	− 47.22
Jun-20	43.34	38.12	76.51	284.07	− 67.53
Jul-20	0.14	138.21	130.65	287.93	− 73.89
Aug-20	− 4.64	200.89	171.53	346.52	− 80.42
Sep-20	82.69	308.77	219.8	435.48	− 77.6
Oct-20	− 33.04	279.62	248.16	356.48	− 60.48
Nov-20	− 43.77	478.87	350.66	424.6	− 68.82
Dec-20	127.76	694.69	373.1	875.9	− 26.62
No. of period with + ve difference	6	8	9	9	1
No. of period with −ve difference	3	1	0	0	8

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