Approximate Simulation for Transition Systems with Regular Expressions

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Abstract. Simulation is a well-established technique for verifying wheth-er the behaviors of one labeled transition system (LTS) can mimic all behaviors of another LTS. Transition systems with regular expressions (RE-TSs) are an extension of LTSs, which are used as semantic models in modal or temporal logics to solve model checking problems. This paper presents approximate simulation, an extension of simulation of an LTS by a RE-TS, by combining general simulation and metrics, and discusses its properties. First, the notion of approximate simulation is introduced. Then, we investigate properties and an equivalent formalism of approximate simulation. On the other hand, we propose two approaches of fixed point characterization for approximate simulation, and study the relationship between them.

Keywords: Approximate simulation · Regular expression · Metric · Graph pattern matching

1 Introduction

Analysis and verification of concurrent and reactive systems [1] is a well-established research field. Labeled transition systems (LTSs) [1,2] are typically used as models to describe the behaviors of concurrent and reactive systems. In order to compare the behaviors of LTSs, researchers proposed a variety of verification methods. Among them, simulations [1,3,4,30] have a wide range of applications in the analysis of LTSs.

Classical simulation verification techniques return a boolean answer that indicates whether one system can mimic all behaviors of another system. However, as pointed out in [8,11–13], these techniques are restrictive and not robust: Two systems either are simulated or are not simulated, regardless of how close the behaviors of two systems are. To overcome this limitation, the majority of existing works can be roughly grouped into two directions. One of them is based on the notion of metric, which assigns a non-negative real number to each pair

of states of systems (e.g., [7,8,13,18,25–27]). The other direction is to propose numerous approximate simulations (e.g., [10–12,14,28]), which characterize two almost similar states by a parameter δ.

In fact, in order to model the systems which are required to satisfy the requirements of different aspects, there are a large number of extensions of LTSs in existing literatures (e.g., [5–9,16,21,24]), such as transition systems with regular expressions (RE-TSs) (e.g., [15,17,19,20,22,23]), fuzzy transition systems (e.g., [31,33,34]) and probabilistic transition systems (e.g., [11,12]). RE-TSs have been used in classical modal and temporal logics as semantic models to express the properties of models of systems. For example, Bozzelli [17] investigated the model checking problem for interval temporal logic extended with regular expressions. Beer [20], Brazdil [23] and Mateescu [22] extended computation tree logic (CTL) by applying regular expressions so as to enhance the expression of CTL. From a different point of view, Fan [15] added regular expressions to pattern graphs, and used simulation to solve graph pattern matching which is a classical graph challenge, and considered as one of the most studied problems in the literature. It is regrettable that the simualtion of an LTS by a RE-TS has been ignored in the setting of approximate. To alleviate the aforementioned problem, we will use the notion of metrics to propose an approximate simulation, and study some properties about the approximate simulation.

The paper is organized as follows. Some preliminaries are given in Sect. 2. In Sect. 3, we give the notion of the approximate simualtion. We study some related properties about the approximate simualtion, and provide two fixed point characterizations in Sect. 4 and conclude the paper in Sect. 5.

2   Preliminaries

In this section, we recall some notations and definitions about regular expressions, metric spaces and transition systems with regular expressions.

We denote the sets of real numbers, non-negative reals, natural numbers and positive integers by $\mathbb{R}$, $\mathbb{R}^+$, $\mathbb{N}$, $\mathbb{Z}^+$, respectively. We use I to denote the set of indexes. Let Σ be a finite set. We denote the set of finite strings over Σ by Σ^*. We write $\mathcal{P}(\Sigma)$ for the power set of Σ. Let $\rho = a_1 \ldots a_n$, $\sigma = b_1 \ldots b_m \in \Sigma^*$ be two strings. Then, the concatenation of ρ and σ is the string $\rho\sigma = a_1 \ldots a_n b_1 \ldots b_m$. We also denote the i-th symbol of ρ and the length of ρ by ρ_i and $|\rho|$, respectively. Let $\mathcal{P}$, $\mathcal{Q} \subseteq \Sigma^*$. The concatenation of $\mathcal{P}$ and $\mathcal{Q}$ is $\mathcal{PQ} = \{\rho\sigma \in \Sigma^* : \rho \in \mathcal{P}, \sigma \in \mathcal{Q}\}$.

Regular expressions [29] ω over Σ are defined by the following grammar,

$$\omega ::= a|a^k|a^+|(\omega_1)|\omega_1\omega_2|\omega_1 + \omega_2,$$

where $a \in \Sigma$. The set of all regular expressions over Σ is written as $\varpi(\Sigma)$. The language $L(\omega) \subseteq \Sigma^*$ of a regular expression $\omega \in \varpi(\Sigma)$ is defined indutively by

(1) $L(a) = \{a\}$;
(2) $L(a^k) = \{\underbrace{a \cdots a}_{k}\}$;

(3) $L(a^+) = L(a) \cup L(a^2) \cup \cdots$;
(4) $L((\omega_1)) = L(\omega_1)$;
(5) $L(\omega_1\omega_2) = L(\omega_1)L(\omega_2)$;
(6) $L(\omega_1 + \omega_2) = L(\omega_1) \cup L(\omega_2)$.

We recall the definition of metrics taken from [32]. A function $d : \Sigma \times \Sigma \to \mathbb{R}$ is a metric over Σ if for all $x, y, z \in \Sigma$

(1) $d(x, y) \geq 0$, $d(x, y) = 0$ iff $x = y$;
(2) $d(x, y) = d(y, x)$;
(3) $d(x, z) \leq d(x, y) + d(y, z)$.

And, the pair (Σ, d) is called a metric space. When d is clear from context, we write Σ instead of (Σ, d).

Next, we review the definition of transition system with regular expressions [15,17,20,22,23]. A transition system with regular expressions (for short, RE-TS) is a tuple $\mathcal{RT} = (S, s_0, \varpi(\Sigma), \to)$ where S is a finite set of states, $s_0 \in S$ is a initial state, $\varpi(\Sigma)$ is the set of all regular expressions over Σ, and $\to \subseteq S \times \varpi(\Sigma) \times S$ is a set of transitions.

A labeled transition system (LTS) can be viewed as a special case of RE-TS, where $\varpi(\Sigma)$ is replaced with Σ. Let $\mathcal{RT} = (S, s_0, \varpi(\Sigma), \to)$ be a RE-TS. We write $s \xrightarrow{\omega} s'$ for $(s, \omega, s') \in \to$, where $s, s' \in S$. A trace σ is an infinite sequence of elements in $\varpi(\Sigma)$. For $j \geq 1$, let σ_j denote the jth element. A path from s in $\mathcal{RT}$ is an infinite sequence $\pi = s \xrightarrow{\omega_1} s' \xrightarrow{\omega_2} s'' \cdots$, and we denote by $\mathrm{tr}(\pi) = \omega_1\omega_2 \cdots$ the trace induced by it. For $s \in S$, we denote by $\mathrm{Path}(s)$ the set of paths from s and by $\mathrm{Trace}(s) = \{tr(\pi) : \pi \in \mathrm{Path}(s)\}$ the set of traces from s. We use $\mathcal{L}(\mathcal{RT})$ to denote the languages of $\mathcal{RT}$, where $\mathcal{L}(\mathcal{RT}) = \mathrm{Trace}(s_0)$. If $\mathcal{RT}$ is an LTS, $\to \subseteq S \times \Sigma \times S$ can be extended to $\to^* \subseteq S \times \Sigma^* \times S$. We write $s \xrightarrow{\rho}{}^* s'$ for $(s, \rho, s') \in \to^*$.

3   Approximate Simulation

In this section, we define an approximate simulation. Before formally defining approximate simulation, we will introduce the notion of exact simulation of an LTS by a RE-TS [15].

Definition 1. *Let $\mathcal{RT}_1 = (S_1, s_{1,0}, \Sigma, \to_1)$ be an LTS and $\mathcal{RT}_2 = (S_2, s_{2,0}, \varpi(\Sigma), \to_2)$ be a RE-TS. A relation $R \subseteq S_1 \times S_2$ is called a simulation if for any $(s_1, s_2) \in R$ and for each $s_1 \xrightarrow{\rho}{}^* s_1'$, there exists $s_2 \xrightarrow{\omega} s_2'$ such that $\rho \in L(\omega)$ and $(s_1', s_2') \in R$. We say that $\mathcal{RT}_2$ simulates $\mathcal{RT}_1$, denoted by $\mathcal{RT}_1 \preceq \mathcal{RT}_2$, if there exists a simulation R such that $(s_{1,0}, s_{2,0}) \in R$.*

Example 1. Consider an LTS $\mathcal{RT}_1 = (S_1, s_{1,0}, \Sigma, \to_1)$ and a RE-TS $\mathcal{RT}_2 = (S_2, s_{2,0}, \varpi(\Sigma), \to_2)$. The transition diagrams of $\mathcal{RT}_1$ and $\mathcal{RT}_2$ are depicted in Fig. 1. By Definition 2, we can find a relation $R = \{(s_{1,0}, s_{2,0}), (s_{1,1}, s_{2,1}), (s_{1,2}, s_{2,1}), (s_{1,3}, s_{2,2}), (s_{1,4}, s_{2,2})\}$ which is a simulation, and $(s_{1,0}, s_{2,0}) \in R$. Therefore, we can obtain $\mathcal{RT}_1 \preceq \mathcal{RT}_2$.
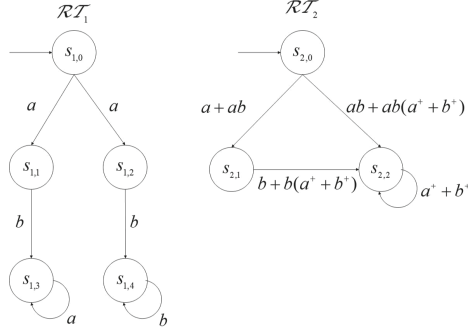
Fig. 1. An LTS and a RE-TS.

The following definitions and propositions will help us to define the approximate simulation.

The definition of string distance sd will be developed in a discounted version [7,8], in which distance of each step is decreased exponentially over time by a discounting factor $\alpha \in (0,1]$.

Let (Σ, d) be a metric space, and $\alpha \in (0,1]$. The string distance $sd : \Sigma^* \times \Sigma^* \to \mathbb{R} \cup \{+\infty\}$ is defined as

$$sd(\rho, \sigma) = \begin{cases} \max_{1 \le i \le |\rho|} \alpha^{i-1} d(\rho_i, \sigma_i) & \text{if } |\rho| = |\sigma|, \\ +\infty & \text{otherwise,} \end{cases}$$

for all strings $\rho, \sigma \in \Sigma^*$.

Proposition 1. *Let d be a metric over Σ. Then, sd is a metric on the set Σ^*.*

Proof. By the definition of metric, we need to prove the follow properties:

(1) $sd(\rho, \sigma) \ge 0$ and $sd(\rho, \sigma) = 0$ iff $\rho = \sigma$,
(2) $sd(\rho, \sigma) = sd(\sigma, \rho)$,
(3) $sd(\rho, \varsigma) + sd(\varsigma, \sigma) \ge sd(\rho, \sigma)$

where $\rho, \sigma, \varsigma \in \Sigma^*$.

The properties (1) and (2) are obvious.

For property (3): We have to discuss the following four cases.

The cases of $|\rho| \ne |\sigma| \ne |\varsigma|$, $|\rho| = |\sigma| \ne |\varsigma|$, and $|\rho| \ne |\sigma| = |\varsigma|$ follow immediately from the definition of sd. We only consider the case of $|\rho| = |\sigma| = |\varsigma| = n$.

From the definition of sd, we know that

$$sd(\rho, \varsigma) + sd(\varsigma, \sigma) = \max_{1 \le i \le n} \alpha^{i-1} d(\rho_i, \varsigma_i) + \max_{1 \le j \le n} \alpha^{j-1} d(\varsigma_j, \sigma_j)$$

and $sd(\rho, \sigma) = \max_{1 \le k \le n} \alpha^{k-1} d(\rho_k, \sigma_k)$. It is sufficient to show that there exists a $1 \le p \le n$ such that $\alpha^{p-1} d(\rho_p, \sigma_p) = sd(\rho, \sigma)$. Then, $d(\rho_p, \sigma_p) \le d(\rho_p, \varsigma_p) + d(\varsigma_p, \sigma_p)$.

Therefore,

$$
\begin{aligned}
sd(\rho, \sigma) &= \alpha^{p-1} d(\rho_p, \sigma_p) \\
&\leq \alpha^{p-1} d(\rho_p, \varsigma_p) + \alpha^{p-1} d(\varsigma_p, \sigma_p) \\
&\leq \max_{1 \leq i \leq n} \alpha^{i-1} d(\rho_i, \varsigma_i) + \max_{1 \leq j \leq n} \alpha^{j-1} d(\varsigma_j, \sigma_j) \\
&= sd(\rho, \varsigma) + sd(\varsigma, \sigma).
\end{aligned}
$$

Let (Σ, d) be a metric space. The distance between a string $\rho \in \Sigma^*$ and the language $L(\omega) \subseteq \Sigma^*$ of a regular expression $\omega \in \varpi(\Sigma)$ is defined as

$$
d^*(\rho, L(\omega)) = \inf_{\sigma \in L(\omega)} sd(\rho, \sigma).
$$

Proposition 2. *Let (Σ, d) be a metric space. Then, the following properties hold:*

(1) $d^*(\rho, L(\omega)) \geq 0$, for all $\rho \in \Sigma^*$ and $\omega \in \varpi(\Sigma)$.
(2) $\rho \in L(\omega)$ iff $d^*(\rho, L(\omega)) = 0$, where $\rho \in \Sigma^*$, $\omega \in \varpi(\Sigma)$.

Proof. For property (1): It follows immediately from Proposition 1.

For property (2): First, for the 'if' part, let $d^*(\rho, L(\omega)) = 0$. Without loss of generality, suppose $|\rho| = n$. Since Σ is a finite set, assume that there are m elements in Σ. According to the definition of sd, there are at most $nm + 1$ values. Then, there exists $\sigma \in L(\omega)$ such that $sd(\rho, \sigma) = 0$. So by Proposition 1 and the definition of sd, $\rho = \sigma$ holds. Hence $\rho \in L(\omega)$.

Second, for the 'only if' part, suppose that $\rho \in L(\omega)$. Then, there exists $\sigma \in L(\omega)$ such that $\rho = \sigma$. By Proposition 1 and the definition of sd, $sd(\rho, \sigma) = 0$ holds. Therefore, $\inf_{\sigma \in L(\omega)} sd(\rho, \sigma) = 0$, and $d^*(\rho, L(\omega)) = 0$.

By the above definitions and propositions, we will introduce the definition of approximate simulation.

Definition 2. *Let $\mathcal{RT}_1 = (S_1, s_{1,0}, \Sigma, \rightarrow_1)$ be an LTS, $\mathcal{RT}_2 = (S_2, s_{2,0}, \varpi(\Sigma), \rightarrow_2)$ be a RE-TS, (Σ, d) be a metric space, and $\delta \in \mathbb{R}^+$. A relation $R_\delta \subseteq S_1 \times S_2$ is called a δ-simulation if for any $(s_1, s_2) \in R$ and for each $s_1 \xrightarrow{\rho} {}^* s_1'$, there exists $s_2 \xrightarrow{\omega} s_2'$ such that $d^*(\rho, L(\omega)) \leq \delta$ and $(s_1', s_2') \in R$. We say that $\mathcal{RT}_2$ δ-simulates $\mathcal{RT}_1$, denoted by $\mathcal{RT}_1 \preceq_\delta \mathcal{RT}_2$, if there exists a δ-simulation R_δ such that $(s_{1,0}, s_{2,0}) \in R_\delta$.*

We replace $s_{1,0} \xrightarrow{a} s_{1,1}$ by $s_{1,0} \xrightarrow{c} s_{1,1}$ in Fig. 1, where $d(a, c) = 0.5$. Then, according to Definition 2, we can obtain that there exists a 0.5-simulation $R_{0.5}$ such that $(s_{1,0}, s_{2,0}) \in R_{0.5}$. Thus, $\mathcal{RT}_1 \preceq_{0.5} \mathcal{RT}_2$.

In above definition, when $\delta = 0$, we recover the established definition of exact simualtion.

4   Related Properties

In this section, we will introduce some properties related with the approximate simulation.

Given a precision parameter δ, which used to measure the degree of simulation, the following lemma ensures that the set of δ-simulation has a maximal element, denoted by $R_\delta^{\max}$. Moreover, by using $R_\delta^{\max}$, we give an equivalent expression for δ-simulation.

Lemma 1. *Let $\mathcal{RT}_1 = (S_1, s_{1,0}, \Sigma, \rightarrow_1)$ be an LTS, $\mathcal{RT}_2 = (S_2, s_{2,0}, \varpi(\Sigma), \rightarrow_2)$ be a RE-TS, (Σ, d) be a metric space, and $\delta \in \mathbb{R}^+$. Suppose that $\{R_\delta^i\}_{i\in I}$ is a family of δ-simulation of $\mathcal{RT}_1$ by $\mathcal{RT}_2$, and $R_\delta^{\max} = \bigcup_{i \in I} R_\delta^i$. Then, the following properties hold:*

(1) $R_\delta^{\max}$ is a δ-simulation of $\mathcal{RT}_1$ by $\mathcal{RT}_2$.
(2) $\mathcal{RT}_1 \preceq_\delta \mathcal{RT}_2$ iff $(s_{1,0}, s_{2,0}) \in R_\delta^{\max}$.

Proof. For property (1): Consider any $(s_1, s_2) \in R_\delta^{\max} = \bigcup_{i \in I} R_\delta^i$. It is enough to show that there exists a $i \in I$ such that $(s_1, s_2) \in R_\delta^i$. By Definition 2, we can know that for each $s_1 \xrightarrow{\rho} {}^* s_1'$ in $\mathcal{RT}_1$, there exists $s_2 \xrightarrow{\omega} s_2'$ in $\mathcal{RT}_2$ such that $d^*(\rho, L(\omega)) \leq \delta$ and

$$(s_1', s_2') \in R_\delta^i \subseteq \bigcup_{i \in I} R_\delta^i = R_\delta^{\max}.$$

Hence, $R_\delta^{\max}$ is a δ-simulation of $\mathcal{RT}_1$ by $\mathcal{RT}_2$.

For property (2): It follows immediately from Definition 2 and property (1).

Proposition 3. *Let $\mathcal{RT}_i = (S_i, s_{i,0}, \Sigma, \rightarrow_i)$, $i = 1, 2$, be two LTSs, $\mathcal{RT}_3 = (S_3, s_{3,0}, \varpi(\Sigma), \rightarrow_3)$ be a RE-TS, and (Σ, d) be a metric space. Then, the following properties hold:*

(1) For all $\delta \geq 0$, $\mathcal{RT}_1 \preceq_\delta \mathcal{RT}_1$.
(2) For all $\delta \geq 0$, if $\mathcal{RT}_1 \preceq_\delta \mathcal{RT}_3$, then for all $\delta' > \delta$, $\mathcal{RT}_1 \preceq_{\delta'} \mathcal{RT}_3$.
(3) For all $\delta, \delta' \geq 0$, if $\mathcal{RT}_1 \preceq_\delta \mathcal{RT}_2$ and $\mathcal{RT}_2 \preceq_{\delta'} \mathcal{RT}_3$, then $\mathcal{RT}_1 \preceq_{\delta+\delta'} \mathcal{RT}_3$.

Proof. For property (1): Let $R = \{(s, s) \in S_1 \times S_1 : s \in S_1\}$. Then, for each $(s, s) \in R$ and $s \xrightarrow{a} {}^* s'$, it is obvious that $s \xrightarrow{a} s'$, $d^*(a, L(a)) = 0 \leq \delta$ and $(s', s') \in R$. Therefore, R is a δ-simulation by Definition 2. Thus $\mathcal{RT}_1 \preceq_\delta \mathcal{RT}_1$ because $(s_{1,0}, s_{1,0}) \in R$.

For property (2): Suppose that $\mathcal{RT}_1 \preceq_\delta \mathcal{RT}_3$ and $\delta' > \delta \geq 0$. From Lemma 1, we can know that $(s_{1,0}, s_{3,0}) \in R_\delta^{\max} = \bigcup_{i \in I} R_\delta^i$. Therefore, there exists $i \in I$ such that $(s_{1,0}, s_{3,0}) \in R_\delta^i$. By Definition 2, we have that for each $(s_1, s_3) \in R_\delta^i$ and $s_1 \xrightarrow{\rho} {}^* s_1'$, there exists $s_3 \xrightarrow{\omega} s_3'$ such that $d^*(\rho, L(\omega)) \leq \delta < \delta'$ and $(s_1', s_3') \in R_\delta^i$. Hence, R_δ^i is a δ'-simulation. Thus, $\mathcal{RT}_1 \preceq_{\delta'} \mathcal{RT}_3$.

For property (3): Let $R = \{(s_1, s_3) \in S_1 \times S_3 : \exists s_2 \in S_2, (s_1, s_2) \in R_\delta^{\max}$ and $(s_2, s_3) \in R_{\delta'}^{\max}\}$. For each $(s_1, s_3) \in R$, suppose that

$$s_1 \xrightarrow{a_1} {}^* s_1' \xrightarrow{a_2} {}^* \cdots \xrightarrow{a_n} {}^* s_1''.$$

We know from Lemma 1 that there exists

$$s_2 \xrightarrow{a_1'} s_2' \xrightarrow{a_2'} \cdots \xrightarrow{a_n'} s_2''$$

such that $d^*(a_i, L(a_i')) \leq \delta$ and $(s_1'', s_2'') \in R_\delta^{\max}$, where $a_i, a_i' \in \Sigma$ for every $i \in \mathbb{N}$. Let $\rho = a_1 a_2 \cdots a_n$ and $\rho' = a_1' a_2' \cdots a_n'$. By the definition of sd, we have $sd(\rho, \rho') \leq \delta$. From Lemma 1, we know that there exists $s_3 \xrightarrow{\omega} s_3'$ such that $d^*(\rho', L(\omega)) \leq \delta'$ and $(s_2'', s_3') \in R_{\delta'}^{\max}$ for each $s_2 \xrightarrow{\rho'} {}^* s_2''$ and $(s_2, s_3) \in R_{\delta'}^{\max}$. Therefore, there exists $\sigma \in L(\omega)$ such that $sd(\rho', \sigma) \leq \delta'$. By Proposition 1,

$$sd(\rho, \sigma) \leq sd(\rho, \rho') + sd(\rho', \sigma) \leq \delta + \delta'.$$

Thus, $d^*(\rho, L(\omega)) \leq \delta + \delta'$ and $(s_1'', s_3') \in R$. Therefore, R is a $\delta + \delta'$-simulation, and $(s_{1,0}, s_{3,0}) \in R$ because $(s_{1,0}, s_{2,0}) \in R_\delta^{\max}$ and $(s_{2,0}, s_{3,0}) \in R_{\delta'}^{\max}$. Hence, $\mathcal{RT}_1 \preceq_{\delta+\delta'} \mathcal{RT}_3$.

Given an LTS $\mathcal{RT}_1 = (S_1, s_{1,0}, \Sigma, \rightarrow_1)$ and a RE-TS $\mathcal{RT}_2 = (S_2, s_{2,0}, \varpi(\Sigma), \rightarrow_2)$, we say that $\text{Trace}(s_1) \subseteq \text{Trace}(s_2)$ if for each $\sigma_1 = \rho_1 \rho_2 \cdots \in \text{Trace}(s_1)$, there exists $\sigma_2 = \omega_1 \omega_2 \cdots \in \text{Trace}(s_2)$ such that $\rho_i \in L(\omega_i)$ for all $i \in \mathbb{Z}^+$, where $(s_1, s_2) \in S_1 \times S_2$.

Let $\mathcal{RT}_1 = (S_1, s_{1,0}, \Sigma, \rightarrow_1)$ be an LTS, $\mathcal{RT}_2 = (S_2, s_{2,0}, \varpi(\Sigma), \rightarrow_2)$ be a RE-TS, (Σ, d) be a metric space, and $(s_1, s_2) \in S_1 \times S_2$. Given two traces $\sigma_1 = \rho_1 \rho_2 \cdots \in \text{Trace}(s_1)$ and $\sigma_2 = \omega_1 \omega_2 \cdots \in \text{Trace}(s_2)$, the trace distance between σ_1 and σ_2 is defined as

$$td(\sigma_1, \sigma_2) = \sup_{i \in \mathbb{Z}^+} d^*(\rho_i, \omega_i).$$

The trace distance between s_1 and s_2 is defined as

$$\mathcal{T}d(s_1, s_2) = \sup_{\sigma_1 \in \text{Trace}(s_1)} \inf_{\sigma_2 \in \text{Trace}(s_2)} td(\sigma_1, \sigma_2).$$

The language distance between $\mathcal{RT}_1$ and $\mathcal{RT}_2$ is defined as

$$\mathcal{L}d(\mathcal{RT}_1, \mathcal{RT}_2) = \sup_{\sigma_1 \in \text{Trace}(s_{1,0})} \inf_{\sigma_2 \in \text{Trace}(s_{2,0})} td(\sigma_1, \sigma_2).$$

Proposition 4. *Let* $\mathcal{RT}_1 = (S_1, s_{1,0}, \Sigma, \rightarrow_1)$ *be an LTS,* $\mathcal{RT}_2 = (S_2, s_{2,0}, \varpi(\Sigma), \rightarrow_2)$ *be a RE-TS, and* (Σ, d) *be a metric space. Then, the following properties hold:*

(1) $\text{Trace}(s_1) \subseteq \text{Trace}(s_2)$ *iff* $\mathcal{T}d(s_1, s_2) = 0$, *where* $s_1 \in S_1$ *and* $s_2 \in S_2$.
(2) $\mathcal{L}(\mathcal{RT}_1) \subseteq \mathcal{L}(\mathcal{RT}_2)$ *iff* $\mathcal{L}d(\mathcal{RT}_1, \mathcal{RT}_2) = 0$.

Proof. For property (1): First, for the 'if' part, suppose that $\mathcal{T}d(s_1, s_2) = 0$. It is sufficient to know from the definition of $\mathcal{T}d$ that $\inf_{\sigma_2 \in \text{Trace}(s_2)} td(\sigma_1, \sigma_2) = 0$ for each $\sigma_1 = \rho_1 \rho_2 \cdots \in \text{Trace}(s_1)$. Therefore, there exists $\sigma_2 = \omega_1 \omega_2 \cdots \in \text{Trace}(s_2)$

such that $td(\sigma_1, \sigma_2) = 0$. From the definition of td, we have $d^*(\rho_i, L(\omega_i)) = 0$ for all $i \in \mathbb{Z}^+$. From Proposition 2, there is $\rho_i \in L(\omega_i)$ for all $i \in \mathbb{Z}^+$. Hence, $\text{Trace}(s_1) \subseteq \text{Trace}(s_2)$.

Second, for the 'only if' part, let $\text{Trace}(s_1) \subseteq \text{Trace}(s_2)$. In other words, for each $\sigma_1 = \rho_1 \rho_2 \cdots \in \text{Trace}(s_1)$, there exists $\sigma_2 = \omega_1 \omega_2 \cdots \in \text{Trace}(s_2)$ such that $\rho_i \in L(\omega_i)$ for all $i \in \mathbb{Z}^+$. And, there is $d^*(\rho_i, L(\omega_i)) = 0$ from Proposition 2 for all $i \in \mathbb{Z}^+$. Moreover, we have $td(\sigma_1, \sigma_2) = 0$ from the definition of td. Hence, $\inf\limits_{\sigma_2 \in \text{Trace}(s_2)} td(\sigma_1, \sigma_2) = 0$ for each $\sigma_1 \in \text{Trace}(s_1)$. Therefore, $\mathcal{T}d(s_1, s_2) = 0$.

For property (2): For the 'if' part, suppose that $\mathcal{L}d(\mathcal{RT}_1, \mathcal{RT}_2) = 0$. It is sufficient to show that $\inf\limits_{\sigma_2 \in \mathcal{L}(\mathcal{RT}_2)} td(\sigma_1, \sigma_2) = 0$ for each $\sigma_1 = \rho_1 \rho_2 \cdots \in \mathcal{L}(\mathcal{RT}_1)$. Therefore, there exists $\sigma_2 = \omega_1 \omega_2 \cdots \in \mathcal{L}(\mathcal{RT}_2)$ such that $td(\sigma_1, \sigma_2) = 0$. From the definition of td, there is $d^*(\rho_i, L(\omega_i)) = 0$ for all $i \in \mathbb{Z}^+$. Then, we have $\rho_i \in L(\omega_i)$ by Proposition 2. Hence, $\mathcal{L}(\mathcal{RT}_1) \subseteq \mathcal{L}(\mathcal{RT}_2)$.

For the 'only if' part, consider that $\mathcal{L}(\mathcal{RT}_1) \subseteq \mathcal{L}(\mathcal{RT}_2)$. In other words, for each trace $\sigma_1 = \rho_1 \rho_2 \cdots$ in $\mathcal{RT}_1$, there exists a trace $\sigma_2 = \omega_1 \omega_2 \cdots$ in $\mathcal{RT}_2$ such that $\rho_i \in L(\omega_i)$ for all $i \in \mathbb{Z}^+$. We know from Proposition 2 that $d^*(\rho_i, L(\omega_i)) = 0$. By the definition of td, we have $td(\sigma_1, \sigma_2) = 0$. Hence, $\inf\limits_{\sigma_2 \in \mathcal{L}(\mathcal{RT}_2)} td(\sigma_1, \sigma_2) = 0$ for each $\sigma_1 = \rho_1 \rho_2 \ldots \in \mathcal{L}(\mathcal{RT}_1)$. Thus, we have $\mathcal{L}d(\mathcal{RT}_1, \mathcal{RT}_2) = 0$ from the definition of $\mathcal{L}d$.

Let $\mathcal{RT}_1 = (S_1, s_{1,0}, \Sigma, \rightarrow_1)$ be an LTS, $\mathcal{RT}_2 = (S_2, s_{2,0}, \varpi(\Sigma), \rightarrow_2)$ be a RE-TS, and (Σ, d) be a metric space. The simulation distance between $\mathcal{RT}_1$ and $\mathcal{RT}_2$ is defined as

$$\mathcal{S}d(\mathcal{RT}_1, \mathcal{RT}_2) = \inf\{\delta : \mathcal{RT}_1 \preceq_\delta \mathcal{RT}_2\}.$$

Lemma 2. *Let $\mathcal{RT}_1 = (S_1, s_{1,0}, \Sigma, \rightarrow_1)$ be an LTS and $\mathcal{RT}_2 = (S_2, s_{2,0}, \varpi(\Sigma), \rightarrow_2)$ be a RE-TS, and (Σ, d) be a metric space. Then, the following properties hold:*

(1) $\mathcal{S}d(\mathcal{RT}_1, \mathcal{RT}_2) \geq 0$.
(2) $\mathcal{S}d(\mathcal{RT}_1, \mathcal{RT}_2) = 0$ if $\mathcal{RT}_1 \preceq \mathcal{RT}_2$.

Proof. For property (1): It follows immediately from Proposition 2.

For property (2): Consider that $\mathcal{RT}_1 \preceq \mathcal{RT}_2$. By Definition 1, there exists a simulation $R \subset S_1 \times S_2$ such that: For each $(s_1, s_2) \in R$ and $s_1 \xrightarrow{\rho}{}^* s_1'$, there exists $s_2 \xrightarrow{\omega} s_2'$ such that $\rho \in L(\omega)$ and $(s_1', s_2') \in R$. And, we have $d^*(\rho, L(\omega)) = 0$ by Proposition 2. Then, there exists a $\delta = 0$ and R is a δ-simulation. By the definition of $\mathcal{S}d$ and property (1), $\mathcal{S}d(\mathcal{RT}_1, \mathcal{RT}_2) = 0$.

The relationship between the simulation distance and the language distance is captured by the following theorem.

Theorem 1. *Let $\mathcal{RT}_1 = (S_1, s_{1,0}, \Sigma, \rightarrow_1)$ be an LTS, $\mathcal{RT}_2 = (S_2, s_{2,0}, \varpi(\Sigma), \rightarrow_2)$ be a RE-TS, and (Σ, d) be a metric space. Then,*

$$\mathcal{L}d(\mathcal{RT}_1, \mathcal{RT}_2) \leq \mathcal{S}d(\mathcal{RT}_1, \mathcal{RT}_2).$$

Proof. Consider a $\delta \geq \mathcal{S}d(\mathcal{RT}_1, \mathcal{RT}_2)$. By Proposition 3, $\mathcal{RT}_1 \preceq_\delta \mathcal{RT}_2$ holds. Let $\sigma_1 = \rho_1\rho_2\cdots \in \mathcal{L}(\mathcal{RT}_1)$. Then, there exists a path

$$s_1 \xrightarrow{\rho_1}{}^* s_2 \xrightarrow{\rho_2}{}^* s_3 \xrightarrow{\rho_3}{}^* \cdots$$

in $\mathcal{RT}_1$ where $s_1 = s_{0,1}$. By Lemma 1, we know $(s_{0,1}, s_{0,2}) \in R_\delta^{\max}$. Therefore, there exists a path

$$s_1' \xrightarrow{\omega_1} s_2' \xrightarrow{\omega_2} s_3' \xrightarrow{\omega_3} \cdots$$

such that $(s_i, s_i') \in R_\delta^{\max}$ for all $i \in \mathbb{Z}^+$, where $s_1' = s_{0,2}$. Let $\sigma_2 = \omega_1\omega_2\cdots$. By Proposition 2 and the definition of td, it is obvious that $td(\sigma_1, \sigma_2) \leq \delta$. Hence, $\inf_{\sigma_2 \in \mathcal{L}(\mathcal{RT}_2)} td(\sigma_1, \sigma_2) \leq \delta$ for each $\sigma_1 \in \mathcal{L}(\mathcal{RT}_1)$. Thus $\mathcal{L}d(\mathcal{RT}_1, \mathcal{RT}_2) \leq \mathcal{S}d(\mathcal{RT}_1, \mathcal{RT}_2)$.

We next propose two approaches of fixed point characterization of approximate simulation.

We first give a fixed point characterization of maximal δ-simulation for a given δ.

Let $\mathcal{RT}_1 = (S_1, s_{1,0}, \Sigma, \rightarrow_1)$ be an LTS, $\mathcal{RT}_2 = (S_2, s_{2,0}, \varpi(\Sigma), \rightarrow_2)$ be a RE-TS, and (Σ, d) be a metric space. For a given $\delta \geq 0$, we define the following sequence $\{R_\delta^i\}_{i\in\mathbb{N}}$ of subsets of $S_1 \times S_2$:

$R_\delta^0 = S_1 \times S_2$;

$R_\delta^{i+1} = \{(s_1, s_2) \in R_\delta^i : \text{for } s_1 \xrightarrow{\rho}{}^* s_1', \text{there exists } s_2 \xrightarrow{\omega} s_2' \text{ such that } d^*(\rho, L(\omega)) \leq$
$\qquad \delta \text{ and } (s_1', s_2') \in R_\delta^i\}.$

Since S_1 and S_2 are finite, it is clear that $\{R_\delta^i\}_{i\in\mathbb{N}}$ reaches a fixed point in a finite number steps by the definition of $\{R_\delta^i\}_{i\in\mathbb{N}}$.

Lemma 3. *Let $\{R_\delta^i\}_{i\in\mathbb{N}}$ be the sequence of sets defined by definition of $\{R_\delta^i\}_{i\in\mathbb{N}}$. Then, the following properties hold:*

(1) $R_\delta^{i+1} \subseteq R_\delta^i$ for every $i \in \mathbb{N}$.
(2) For each $i \in \mathbb{N}$, $R_\delta^{\max} \subseteq R_\delta^i$.
(3) There exists some $n \in \mathbb{N}$ such that $R_\delta^{\max} = R_\delta^n$.

Proof. For property (1): It follows immediately from the definition of $\{R_\delta^i\}_{i\in\mathbb{N}}$.

For property (2): This will be proved by induction with regard to i.

The initial step is for $i = 1$. It follows from the definition of $\{R_\delta^i\}_{i\in\mathbb{N}}$.

The induction hypothesis is that (2) holds for $i = k$. We now show that (2) holds for $i = k + 1$, i.e., $R_\delta^{\max} \subseteq R_\delta^{k+1}$.

We have to discuss the following two cases.

The first case is that $R_\delta^{\max} = \emptyset$. Then, $R_\delta^{\max} = \emptyset \subseteq R_\delta^{k+1}$ obviously.

The second case is that $R_\delta^{\max} \neq \emptyset$. Then there exists some $(s_1, s_2) \in R_\delta^{\max}$. For each $(s_1, s_2) \in R_\delta^{\max} \subseteq R_\delta^k$, we know rom Lemma 1 that for each $s_1 \xrightarrow{\rho}{}^* s_1'$, there exists $s_2 \xrightarrow{\omega} s_2'$ such that $d^*(\rho, L(\omega)) \leq \delta$, and $(s_1', s_2') \in R_\delta^{\max}$. Then,

by the induction hypothesis, $(s_1', s_2') \in R_\delta^{\max} \subseteq R_\delta^k$. Hence, It follows from the definition of $\{R_\delta^i\}_{i \in \mathbb{N}}$ that $(s_1, s_2) \in R_\delta^{k+1}$.

Thus (2) holds for $i = k + 1$, which proves the property.

For property (3): By using propery (1) and the definition of $\{R_\delta^i\}_{i \in \mathbb{N}}$, we can obtain that there exists some $k \in \mathbb{N}$ such that $R_\delta^j = R_\delta^k$ for every $j \geq k$.

We have to discuss the following two cases.

The first case is that there exists a k such that $R_\delta^k = \emptyset$. By property (1) and (2), $R_\delta^j \subseteq R_\delta^k$ for every $j \geq k$, and $R_\delta^{\max} \subseteq R_\delta^k$. Hence, $R_\delta^{\max} \subseteq R_\delta^k = \emptyset$.

The second case is that $R_\delta^j \neq \emptyset$ for all $j \in \mathbb{N}$. Let $(s_1, s_2) \in R_\delta^{k+1}$. Then, for each $s_1 \xrightarrow{\rho}^* s_1'$, there exists $s_2 \xrightarrow{\omega} s_2'$ such that $d^*(\rho, L(\omega)) \leq \delta$ and $(s_1', s_2') \in R_\delta^k$. Since $R_\delta^{k+1} = R_\delta^k$, it is sufficient to show that R_δ^{k+1} is a δ-simulation and $R_\delta^k = R_\delta^{k+1} \subseteq R_\delta^{\max}$. By property (2), $R_\delta^{\max} \subseteq R_\delta^k$. Hence, $R_\delta^{\max} = R_\delta^k$.

We will introduce another approach, which characterizes the maximal δ-simualtion as the level sets of a function for a given δ.

Let $\mathcal{RT}_1 = (S_1, s_{1,0}, \Sigma, \rightarrow_1)$ be an LTS, $\mathcal{RT}_2 = (S_2, s_{2,0}, \varpi(\Sigma), \rightarrow_2)$ be a RE-TS, and (Σ, d) be a metric space. Define the following sequence $\{f^i\}_{i \in \mathbb{N}}$ of functions from $S_1 \times S_2$ to $\mathbb{R}^+ \cup \{+\infty\}$:

$$f^0(s_1, s_2) = 0;$$
$$f^{i+1}(s_1, s_2) = \sup_{s_1 \xrightarrow{\rho}^* s_1'} \inf_{s_2 \xrightarrow{\omega} s_2'} \max(d^*(\rho, L(\omega)), f^i(s_1', s_2')).$$

Lemma 4. *Let $\{f^i\}_{i \in \mathbb{N}}$ be the sequence of functions defined by definition of $\{f^i\}_{i \in \mathbb{N}}$. Then, the sequence $\{f^i(s_1, s_2)\}_{i \in \mathbb{N}}$ is non-decreasing for all $(s_1, s_2) \in S_1 \times S_2$.*

Proof. To prove the lemma, it suffices to verify that

$$f^i(s_1, s_2) \leq f^{i+1}(s_1, s_2)$$

for every $i \geq 0$.

This will be proved by induction on i.

The initial step is for $i = 0$. It follows from the definition of $\{f^i\}_{i \in \mathbb{N}}$ and Proposition 2.

The induction hypothesis is that the lemma holds for $i = k$. We now show that the lemma holds for $i = k + 1$, i.e., $f^{k+1}(s_1, s_2) \leq f^{k+2}(s_1, s_2)$ for each $(s_1, s_2) \in S_1 \times S_2$. Then,

$$f^{k+2}(s_1, s_2) = \sup_{s_1 \xrightarrow{\rho}^* s_1'} \inf_{s_2 \xrightarrow{\omega} s_2'} \max(d^*(\rho, L(\omega)), f^{k+1}(s_1', s_2'))$$
$$\geq \sup_{s_1 \xrightarrow{\rho}^* s_1'} \inf_{s_2 \xrightarrow{\omega} s_2'} \max(d^*(\rho, L(\omega)), f^k(s_1', s_2'))$$
$$= f^{k+1}(s_1, s_2).$$

This completes the proof of the lemma.

$\{f^i\}_{i \in \mathbb{N}}$ reaches a fixed point in a finite number of steps, which is shown in [8]. Then, for each $(s_1, s_2) \in S_1 \times S_2$, there exists a $k \in \mathbb{N}$ such that; for each $i \leq k$, $f^k(s_1, s_2) \geq f^i(s_1, s_2)$, and for each $j \geq k$, $f^k(s_1, s_2) = f^j(s_1, s_2)$. Let $f^{\min}$ be the branching distance between $\mathcal{RT}_1$ and $\mathcal{RT}_2$. Then, $f^{\min}(s_1, s_2) = f^k(s_1, s_2)$ for all $(s_1, s_2) \in S_1 \times S_2$ [9].

The following theorem will give the relationship between the two approaches.

Theorem 2. *Let $\delta \geq 0$, $\{R_\delta^i\}_{i \in \mathbb{N}}$ be the sequence defined by the definition of $\{R_\delta^i\}_{i \in \mathbb{N}}$, $\{f^i\}_{i \in \mathbb{N}}$ be the sequence defined by the definition of $\{f^i\}_{i \in \mathbb{N}}$, $R_\delta^{\max}$ be the maximal δ-simulation of $\mathcal{RT}_1$ by $\mathcal{RT}_2$, and $f^{\min}$ be the branching distance between $\mathcal{RT}_1$ and $\mathcal{RT}_2$. Then, the following assertions hold:*

(1) $R_\delta^i = \{(s_1, s_2) \in S_1 \times S_2 : f^i(s_1, s_2) \leq \delta\}$ for every $i \in \mathbb{N}$ and $\delta \geq 0$.
(2) $R_\delta^{\max} = \{(s_1, s_2) \in S_1 \times S_2 : f^{\min}(s_1, s_2) \leq \delta\}$ for every $\delta \geq 0$.

Proof. Assertions (1): This will be proved by induction with regard to i.

The initial step is for $i = 0$. Let $\delta \geq 0$, if $i = 0$, then the theorem states that $R_\delta^0 = \{(s_1, s_2) \in S_1 \times S_2 : f^0(s_1, s_2) \leq \delta\}$. By definitions of $\{R_\delta^i\}_{i \in \mathbb{N}}$ and $\{f^i\}_{i \in \mathbb{N}}$, $R_\delta^0 = \{(s_1, s_2) \in S_1 \times S_2 : f^0(s_1, s_2) \leq \delta\} = S_1 \times S_2$.

The induction hypothesis is that property (1) holds for $i = k$. We now show that property (1) holds for $i = k + 1$, i.e.,

$$R_\delta^{k+1} = \{(s_1, s_2) \in S_1 \times S_2 : f^{k+1}(s_1, s_2) \leq \delta\}.$$

First, assume $(s_1, s_2) \in R_\delta^{k+1}$. Then, it is sufficient to show that there exists $s_2 \xrightarrow{\omega} s_2'$ and $(s_1', s_2') \in R$ for each $s_1 \xrightarrow{\rho} {}^* s_1'$. In fact, by induction hypothesis we obtain

$$\sup_{s_1 \xrightarrow{\rho} {}^* s_1'} \inf_{s_2 \xrightarrow{\omega} s_2'} \max(d^*(\rho, L(\omega)), f^k(s_1', s_2')) \leq \delta.$$

Since $(s_1, s_2) \in R_\delta^{k+1} \subseteq R_\delta^k$,

$$\sup_{s_1 \xrightarrow{\rho} {}^* s_1'} \inf_{s_2 \xrightarrow{\omega} s_2'} d^*(\rho, L(\omega)) \leq f^k(s_1, s_2) \leq \delta.$$

Therefore, $f^{k+1}(s_1, s_2) \leq \delta$.

Second, let $f^{k+1}(s_1, s_2) \leq \delta$. Then, for each $s_1 \xrightarrow{\rho} {}^* s_1'$, there exists $s_2 \xrightarrow{\omega} s_2'$ such that $f^i(s_1', s_2') \leq \delta$. It follows from the definition of $\{R_\delta^i\}_{i \in \mathbb{N}}$, Lemma 4 and induction hypothesis that

$$f^k(s_1, s_2) \leq f^{k+1} \leq \delta.$$

Hence, $(s_1, s_2) \in R_\delta^{k+1}$. This completes the proof of the property (1).

Assertion (2): Consider $(s_1, s_2) \in S_1 \times S_2$. Suppose $f^{\min}(s_1, s_2) \leq \delta$. Then, there exists $f^i(s_1, s_2) \leq \delta$ for each $i \in \mathbb{N}$. We obtain by Lemma 3 and property (1) that $(s_1, s_2) \in R_\delta^{\max}$. On the contrary, let $(s_1, s_2) \in R_\delta^{\max}$. Then, $(s_1, s_2) \in R_\delta^i$ for each $i \in \mathbb{N}$. We obtain by property (1) that $f^i(s_1, s_2) \leq \delta$ for each $i \in \mathbb{N}$. By using Lemma 4, we have $f^{\min}(s_1, s_2) \leq \delta$.

5   Conclusion

In this paper, we have proposed an approximate simulation by using the notion of metrics to measure the behavioral closeness between a RE-TS and an LTS. The approximate simuation has some properties: First, it satisfies reflexivity, and satisfies transitivity with some limitations. Second, like exact simualtion, it has a maximal element. Then we study the relationship between trance disdance and trance inclusion, and the relationship between language distance and language inclusion. Moreover, we give the relationship between language distance and simualtion distance. Finally, we present two approaches to characterize approximate simualtion by using fixed point, and give the relationship between the two approaches.

As a future work, we will use the results of this paper to graph partten matching query, and consider logical characterizations of δ-simulation.

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