Novel q-Rung Orthopair Fuzzy Hamacher Dual Muirhead Mean Operator for Multi-attribute Decision-Making

Sukhwinder Singh Rawat and Komal

Abstract Real-life multi-attribute decision-making (MADM) has some major issues related to the space of the problem, inter-dependency among attributes, flexibility in the aggregation process, etc. So, our objective is to deal with these issues by adopting suitable tools and techniques like the q-rung orthopair fuzzy set (q-ROFS) for handling space-related difficulty. Dual Muirhead mean (DMM) is applied to address the inter-dependency among attributes, and for a flexible aggregation process, the Hamacher t-norm (TN) and t-conorm (TCN) are utilised. By fusing these approaches, this paper proposes two novel aggregation operators (AOs) named q-rung orthopair fuzzy Hamacher dual Muirhead mean (q-ROFHDMM) and q-rung orthopair fuzzy Hamacher weighted dual Muirhead mean (q-ROFHWDMM) operators. The essential properties of these AOs and special cases are explored as well. Finally, the q-ROFHWDMM operator has been used to construct a MADM method. The study also examines a practical example of selecting an enterprise resource planning (ERP) system, as well as sensitive and comparative analysis.

Keywords Dual Muirhead mean · Hamacher t-norm and t-conorm · Multi-attribute decision-making · q-Rung orthopair fuzzy set

1 Introduction

MADM is a prominent technique that is used to find the best option from a set of available options that depends on various attributes. Several MADM techniques exist in the literature to handle real-life MADM problems. Most of the real-life MADM problems have some common issues that need to be resolved for meaningful and realistic decision-making (DM). Among many, two major challenges faced by decision-makers are (i) expressing the assessment values of an alternative with respect to multiple attributes and (ii) considering the interactional behaviour of these attributes

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in the DM process. To address the first problem, Zadeh introduced the notion of the fuzzy set [1], which assigns a membership degree to every element in order to express the impreciseness and vagueness of that element in the set. But the Zadeh fuzzy set does not address the sense of dissatisfaction. As a result, Atanassov [2] introduced intuitionistic fuzzy set (IFS) in 1986, which used both membership degree(μ) and non-membership degree (ν) with conditions $\mu, \nu \in [0, 1], 0 \leq \mu + \nu \leq 1$. In 2013, Yager discovered that the condition $0 \leq \mu + \nu \leq 1$ (IFS) on μ and ν is violated in many real-life DM problems. To overcome this drawback, Yager [3] extended the space of intuitionistic fuzzy numbers and proposed the Pythagorean fuzzy set (PFS) by making use of the conditions $0 \leq \mu, \nu \leq 1;\ 0 \leq \mu^2 + \nu^2 \leq 1$. Further, it is observed by many researchers that there are still many real-life DM problems in which assessment values (μ, ν) violate the PFS condition. For example, if (0.8, 0.7) is the assessment data provided by the decision-maker, then we get $0.8^2 + 0.7^2 \geq 1$. Therefore, more extended decision space is required. To further extend the decision space of fuzzy information (μ and ν), a generalised orthopair fuzzy set, i.e. q-ROFS, has been introduced by Yager in 2017 [4] . Its membership and non-membership degrees satisfy the conditions $\mu, \nu \in [0, 1];\ 0 \leq \mu^q + \nu^q \leq 1;\ q \geq 1$. As the AOs-based MADM approaches provide both comprehensive values and ranking orders of the alternatives, and also the DM process of these approaches is more intuitive than the classical ones such as TOPSIS, AHP, TODIM, PROMETHEE, etc [5]. Several AOs and their utilisation provide various MADM methods for q-rung orthopair fuzzy numbers (q-ROFNs). For instant, Liu and Wang [6] developed weighted geometric (WG) and weighted average (WA) operators; Liu and Liu [7] proposed Bonferroni mean (BM) and geometric BM (GBM) operators; Wei et al. [8] introduced generalised Heronian mean and geometric Heronian mean operators; Wei et al. [9] developed Maclaurin symmetric mean (MSM) and geometric MSM (GMSM) operators. Rawat and Komal recently used Muirhead mean (MM), Hamacher TN and TCN for q-ROFNs and introduced some AOs as well as a MADM approach based on them. The MM and DMM are aggregation functions which address the inter-dependency of multiple attributes through the correlation of their arguments for every permutation [11]. Various well-known means, like arithmetic mean (AM), geometric mean (GM), GBM and GMSM, are some special cases of DMM [12]. Hamacher TN and TCN are conjunctive and disjunctive aggregation functions [13]. Also, they are strictly decreasing and increasing with parameter γ, respectively, which helps to model conjunction and disjunction among arguments and provides flexibility in the aggregation process [14]. Consequently, many researchers utilised Hamacher TN and TCN-based arithmetic operations to develop some AOs for various fuzzy numbers like intuitionistic fuzzy numbers (IFNs), Pythagorean fuzzy numbers (PFNs), complex IFNs and q-ROFNs [15–18].

The focus of this article is to develop some novel Hamacher TN and TCN-based DMM operators for generalised orthopair fuzzy numbers. This fusion of Hamacher norms and DMM operator provides both interrelationship among multiple attributes and flexible aggregation process due to the additional parameter γ in Hamacher norms. The structure of the paper is as follows: In Sect. 2, definitions of q-ROFS, Hamacher TN and TCN, MM and DMM operators are discussed briefly. Section

3 introduces the q-ROFHDMM and q-ROFHWDMM operators with their essential properties and special cases. Further, in Sect. 4, the q-ROFHWDMM operator-based MADM approach has been developed, and a real-life DM problem has been examined through this. This section also provides sensitive and comparative analyses. Finally, some concluding remarks are given in Sect. 5.

2 Preliminaries

2.1 q-Rung Orthopair Fuzzy Set (q-ROFS)

Definition 1 ([4]) *The q-ROFS $\Im$ on a universal set U is defined as*

$$\Im = \{\langle x, (\mu_\Im(x), \nu_\Im(x))\rangle \mid x \in U\} \tag{1}$$

where $\mu_\Im(x) : U \to [0, 1]$ is membership and $\nu_\Im(x) : U \to [0, 1]$ is non-membership functions that holds, $0 \le (\mu_\Im(x))^q + (\nu_\Im(x))^q \le 1$ for all $q \ge 1$. The degree of hesitancy of x in $\Im$ is defined as $\pi_\Im(x) = (1 - (\mu_\Im(x))^q - (\nu_\Im(x))^q)^{1/q}$ and the q-rung orthopair fuzzy number (q-ROFN) can be written as $(\mu_\Im, \nu_\Im)$.

Definition 2 ([6]) The basic arithmetic operations on any two q-ROFNs, $\aleph_1 = (\mu_1, \nu_1)$, and $\aleph_2 = (\mu_2, \nu_2)$, are as follows:

1. $\aleph_1 \oplus \aleph_2 = \big((\mu_1^q + \mu_2^q - \mu_1^q\mu_2^q)^{1/q}, \nu_1\nu_2\big)$,
2. $\aleph_1 \otimes \aleph_2 = \big(\mu_1\mu_2, (\nu_1^q + \nu_2^q - \nu_1^q\nu_2^q)^{1/q}\big)$,
3. $\lambda\aleph_1 = \big((1 - (1 - \mu_1^q)^\lambda)^{1/q}, \nu_1^\lambda\big)$,
4. $\aleph_1^\lambda = \big(\mu_1^\lambda, (1 - (1 - \nu_1^q)^\lambda)^{1/q}\big)$.

For comparing any two q-ROFNs, we have a score function (S) and an accuracy function (A) as follows:

Definition 3 ([6]) Let $\aleph = (\mu_\aleph, \nu_\aleph)$ be a q-ROFN, then the score value of $\aleph$ is obtained by $S(\aleph) \in [-1, 1]$ which is defined as

$$S(\aleph) = \mu_\aleph^q - \nu_\aleph^q \tag{2}$$

The accuracy value of $\aleph$ is obtained by $A(\aleph) \in [0, 1]$ which is defined as

$$A(\aleph) = \mu_\aleph^q + \nu_\aleph^q \tag{3}$$

Definition 4 For any two q-ROFNs say $\aleph = (\mu_\aleph, \nu_\aleph)$ and $\kappa = (\mu_\kappa, \nu_\kappa)$:

1. If $S(\aleph) > S(\kappa)$, then $\aleph \succ \kappa$
2. If $S(\aleph) = S(\kappa)$, then
   (a) If $A(\aleph) > A(\kappa)$, then $\aleph \succ \kappa$;
   (b) If $A(\aleph) = A(\kappa)$, then $\aleph = \kappa$.

2.2 Hamacher t-Norm (TN) and t-Conorm (TCN)

Hamacher TN (T) as product ($\otimes$) and Hamacher TCN (T^*) as sum ($\oplus$) are defined as follows [13]:

$$T(\imath, \jmath) = \imath \otimes \jmath = \frac{\imath \jmath}{\gamma + (1-\gamma)(\imath + \jmath - \imath \jmath)},$$
$$T^*(\imath, \jmath) = \imath \oplus \jmath = \frac{\imath + \jmath - \imath \jmath - (1-\gamma)\imath \jmath}{1-(1-\gamma)(\imath \jmath)};\ \gamma > 0.$$

For $\gamma = 1$, the Hamacher TN and TCN becomes algebraic TN and TCN:

$$T(\imath, \jmath) = \imath \otimes \jmath = \imath \jmath,\ \ T^*(\imath, \jmath) = \imath \oplus \jmath = \imath + \jmath - \imath \jmath.$$

Similarly, for $\gamma = 2$, the Hamacher TN and TCN becomes Einstein TN and TCN:

$$T(\imath, \jmath) = \imath \otimes \jmath = \frac{\imath \jmath}{1 + (1-\imath)(1-\jmath)},\ \ T^*(\imath, \jmath) = \imath \oplus \jmath = \frac{\imath + \jmath}{1 + \imath \jmath}.$$

2.3 Hamacher Operations for q-ROFNs

If $\aleph_1 = (\mu_1, \nu_1)$ and $\aleph_2 = (\mu_2, \nu_2)$ are any two q-ROFNs and $\gamma > 0$, then the following arithmetic operations for q-ROFNs are defined using Hamacher TN and TCN [19]:

$$\aleph_1 \oplus \aleph_2 = \left(\left(\frac{(\mu_1)^q + (\mu_2)^q - (\mu_1)^q(\mu_2)^q - (1-\gamma)(\mu_1)^q(\mu_2)^q}{1-(1-\gamma)(\mu_1)^q(\mu_2)^q} \right)^{1/q}, \right.$$
$$\left. \frac{\nu_1 \nu_2}{\left(\gamma + (1-\gamma)\left((\nu_1)^q + (\nu_2)^q - (\nu_1)^q(\nu_2)^q\right)\right)^{1/q}} \right)$$

$$\aleph_1 \otimes \aleph_2 = \left(\frac{\mu_1 \mu_2}{\left(\gamma + (1-\gamma)\left((\mu_1)^q + (\mu_2)^q - (\mu_1)^q(\mu_2)^q\right)\right)^{1/q}}, \right.$$
$$\left. \left(\frac{(\nu_1)^q + (\nu_2)^q - (\nu_1)^q(\nu_2)^q - (1-\gamma)(\nu_1)^q(\nu_2)^q}{1-(1-\gamma)(\nu_1)^q(\nu_2)^q} \right)^{1/q} \right)$$

$$\lambda\aleph_1 = \left(\left(\frac{(1+(\gamma-1)\mu_1^q)^\lambda - (1-\mu_1^q)^\lambda}{(1+(\gamma-1)\mu_1^q)^\lambda + (\gamma-1)(1-\mu_1^q)^\lambda}\right)^{1/q}, \frac{(\gamma)^{1/q}\nu_1^\lambda}{\left((1+(\gamma-1)(1-\nu_1^q))^\lambda + (\gamma-1)(\nu_1^q)^\lambda\right)^{1/q}}\right)$$

$$\aleph_1^\lambda = \left(\frac{(\gamma)^{1/q}\mu_1^\lambda}{\left((1+(\gamma-1)(1-\mu_1^q))^\lambda + (\gamma-1)(\mu_1^q)^\lambda\right)^{1/q}}, \left(\frac{(1+(\gamma-1)\nu_1^q)^\lambda - (1-\nu_1^q)^\lambda}{(1+(\gamma-1)\nu_1^q)^\lambda + (\gamma-1)(1-\nu_1^q)^\lambda}\right)^{1/q}\right)$$

For $\gamma = 1$ Hamacher operations becomes algebraic operations and for $\gamma = 2$ they changes to Einstein operations.

2.4 Muirhead Mean (MM)

Definition 5 ([11]) The MM operator for n numbers say $\varsigma_1, \varsigma_2, ..., \varsigma_n$ and a parameter vector $P = (p_1, p_2, ..., p_n) \in \Re^n$ is defined as

$$\mathrm{MM}^P(\varsigma_1, \varsigma_2, ..., \varsigma_n) = \left(\frac{1}{n!}\sum_{\pi\in S_n}\prod_{j=1}^n \varsigma_{\pi(j)}^{p_j}\right)^{\frac{1}{\sum_{j=1}^n p_j}} \tag{4}$$

where S_n is the symmetric group of degree n.

2.5 Dual Muirhead Mean (DMM)

Definition 6 ([11]) The DMM operator for n numbers say $\varsigma_1, \varsigma_2, ..., \varsigma_n$ and a parameter vector $P = (p_1, p_2, ..., p_n) \in \Re^n$ is defined as

$$\mathrm{DMM}^P(\varsigma_1, \varsigma_2, ..., \varsigma_n) = \frac{1}{\sum_{j=1}^n p_j}\left(\prod_{\pi\in S_n}\sum_{j=1}^n p_j\varsigma_{\pi(j)}\right)^{\frac{1}{n!}} \tag{5}$$

where S_n is the symmetric group of degree n. Some special cases of the DMM operator for different values of P are as follows [12]:

1. If $P = (1, 0, 0..., 0)$, then the DMM operator becomes the GM operator

$$\mathrm{DMM}^{P}(\varsigma_1, \varsigma_2, ..., \varsigma_n) = \left(\prod_{i=1}^{n} \varsigma_i\right)^{\frac{1}{n}}.$$

2. If $P = (1, 1, ..., 1)$ or $(1/n, 1/n, ..., 1/n)$, then the DMM operator becomes the AM operator

$$\mathrm{DMM}^{(1,0,0,...,0)}(\varsigma_1, \varsigma_2, ..., \varsigma_n) = \frac{1}{n}\sum_{i=1}^{n} \varsigma_i.$$

3. If $P = (p_1, p_2, 0, 0, ..., 0)$, then the DMM operator becomes the GBM operator

$$\mathrm{DMM}^{(p_1,p_2,0,0,...,0)}(\varsigma_1, \varsigma_2, ..., \varsigma_n) = \frac{1}{p_1 + p_2} \prod_{i,j=1\ i\neq j}^{n} (p_1\varsigma_i + p_2\varsigma_j)^{\frac{1}{n(n-1)}}.$$

4. If $P = (\overbrace{1, 1, ..., 1}^{k}, \overbrace{0, 0, ..., 0}^{n-k})$, then the DMM operator becomes the GMSM operator

$$\mathrm{DMM}^{(\overbrace{1, 1, ..., 1}^{k},\overbrace{0, 0, ..., 0}^{n-k})}(\varsigma_1, \varsigma_2, ..., \varsigma_n) = \frac{1}{k}\left(\prod_{1\leq i_1\leq ...\leq i_k\leq n} \sum_{j=1}^{k} \varsigma_{i_j}\right)^{\frac{1}{C_n^k}}.$$

3 q-Rung Orthopair Fuzzy Hamacher Dual Muirhead Mean Operators

3.1 The q-ROFHDMM Operator

Definition 7 Let $\varsigma_i = (\mu_i, \nu_i)$ be any q-ROFN and $P = (p_1, p_2, ..., p_n) \in \Re^n$ be a parameter vector such that $\sum_{j=1}^{n} p_j > 0$, then q-ROFHDMM operator on such n q-ROFNs is defined as

$$\text{q-ROFHDMM}^{P}(\varsigma_1, \varsigma_2, ..., \varsigma_n) = \frac{1}{\sum_{j=1}^{n} p_j}\left(\bigotimes_{\pi\in S_n} \bigoplus_{j=1}^{n} (p_j\varsigma_{\pi(j)})\right)^{\frac{1}{n!}} \tag{6}$$

where S_n is the symmetric group of degree n.

Theorem 1 *For any collection $\{\varsigma_1, \varsigma_2, ..., \varsigma_n\}$ of q-ROFNs, the aggregated value on applying the q-ROFHDMM operator is also a q-ROFN and it is defined as*

$$\left(\left(\frac{\left(\left(\prod_{\pi\in S_n}(\phi_2+(\gamma^2-1)\varphi_2)\right)^{\frac{1}{n!}}+(\gamma^2-1)\left(\prod_{\pi\in S_n}(\phi_2-\varphi_2)\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum_{j=1}^{n}p_j}}-\left(\left(\prod_{\pi\in S_n}(\phi_2+(\gamma^2-1)\varphi_2)\right)^{\frac{1}{n!}}-\left(\prod_{\pi\in S_n}(\phi_2-\varphi_2)\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum_{j=1}^{n}p_j}}}{\left(\left(\prod_{\pi\in S_n}(\phi_2+(\gamma^2-1)\varphi_2)\right)^{\frac{1}{n!}}+(\gamma^2-1)\left(\prod_{\pi\in S_n}(\phi_2-\varphi_2)\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum_{j=1}^{n}p_j}}+(\gamma-1)\left(\left(\prod_{\pi\in S_n}(\phi_2+(\gamma^2-1)\varphi_2)\right)^{\frac{1}{n!}}-\left(\prod_{\pi\in S_n}(\phi_2-\varphi_2)\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum_{j=1}^{n}p_j}}}\right)^{1/q},\right.$$
$$\left.\left(\frac{\gamma\left(\left(\prod_{\pi\in S_n}(\psi_2+(\gamma^2-1)\chi_2)\right)^{\frac{1}{n!}}-\left(\prod_{\pi\in S_n}(\psi_2-\chi_2)\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum_{j=1}^{n}p_j}}}{\left(\left(\prod_{\pi\in S_n}(\psi_2+(\gamma^2-1)\chi_2)\right)^{\frac{1}{n!}}+(\gamma^2-1)\left(\prod_{\pi\in S_n}(\psi_2-\chi_2)\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum_{j=1}^{n}p_j}}+(\gamma-1)\left(\left(\prod_{\pi\in S_n}(\psi_2+(\gamma^2-1)\chi_2)\right)^{\frac{1}{n!}}-\left(\prod_{\pi\in S_n}(\psi_2-\chi_2)\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum_{j=1}^{n}p_j}}}\right)^{1/q}\right) \quad (7)$$

where
$\phi_2=\prod_{j=1}^{n}(1+(\gamma-1)\mu_{\pi(j)}^{q})^{p_j}$,
$\varphi_2=\prod_{j=1}^{n}(1-\mu_{\pi(j)}^{q})^{p_j}$,
$\psi_2=\prod_{j=1}^{n}\left(1+(\gamma-1)(1-\nu_{\pi(j)}^{q})\right)^{p_j}$,
$\chi_2=\prod_{j=1}^{n}(\nu_{\pi(j)}^{q})^{p_j}$.

Proof The Eq. (7) is proved using mathematical induction and Hamacher operations of q-ROFNs, as discussed in Sect. 2.3:

$$p_j\varsigma_{\pi(j)}=\left(\left(\frac{(1+(\gamma-1)\mu_{\pi(j)}^{q})^{p_j}-(1-\mu_{\pi(j)}^{q})^{p_j}}{(1+(\gamma-1)\mu_{\pi(j)}^{q})^{p_j}+(\gamma-1)(1-\mu_{\pi(j)}^{q})^{p_j}}\right)^{1/q},\right.$$
$$\left.\frac{\gamma^{1/q}\nu_{\pi(j)}^{p_j}}{\left((1+(\gamma-1)(1-\nu_{\pi(j)}^{q}))^{p_j}+(\gamma-1)(\nu_{\pi(j)}^{q})^{p_j}\right)^{1/q}}\right)$$

Suppose we have two q-ROFNs $\varsigma_{\pi(1)}=(\mu_{\pi(1)},\nu_{\pi(1)})$ and $\varsigma_{\pi(2)}=(\mu_{\pi(2)},\nu_{\pi(2)})$, then

$$p_1\varsigma_{\pi(1)}\oplus p_2\varsigma_{\pi(2)}$$
$$=\left(\left(\frac{(1+(\gamma-1)\mu_{\pi(1)}^{q})^{p_1}-(1-\mu_{\pi(1)}^{q})^{p_1}}{(1+(\gamma-1)\mu_{\pi(1)}^{q})^{p_1}+(\gamma-1)(1-\mu_{\pi(1)}^{q})^{p_1}}\right)^{1/q},\right.$$
$$\left.\frac{\gamma^{1/q}\nu_{\pi(1)}^{p_1}}{\left((1+(\gamma-1)(1-\nu_{\pi(1)}^{q}))^{p_1}+(\gamma-1)(\nu_{\pi(1)}^{q})^{p_1}\right)^{1/q}}\right)$$
$$\oplus\left(\left(\frac{(1+(\gamma-1)\mu_{\pi(2)}^{q})^{p_2}-(1-\mu_{\pi(2)}^{q})^{p_2}}{(1+(\gamma-1)\mu_{\pi(2)}^{q})^{p_2}+(\gamma-1)(1-\mu_{\pi(2)}^{q})^{p_2}}\right)^{1/q},\right.$$

$$\left.\frac{\gamma^{1/q}\nu_{\pi(2)}^{p_2}}{\left((1+(\gamma-1)(1-\nu_{\pi(2)}^q))^{p_2}+(\gamma-1)(\nu_{\pi(2)}^q)^{p_2}\right)^{1/q}}\right)$$

$$=\left(\left(\frac{\prod_{j=1}^{2}(1+(\gamma-1)\mu_{\pi(j)}^q)^{p_j}-\prod_{j=1}^{2}(1-\mu_{\pi(j)}^q)^{p_j}}{\prod_{j=1}^{2}(1+(\gamma-1)\mu_{\pi(j)}^q)^{p_j}+(\gamma-1)\prod_{j=1}^{2}(1-\mu_{\pi(j)}^q)^{p_j}}\right)^{1/q},\right.$$

$$\left.\frac{\gamma^{1/q}\prod_{j=1}^{2}\nu_{\pi(j)}^{p_j}}{\left(\prod_{j=1}^{2}(1+(\gamma-1)(1-\nu_{\pi(j)}^q))^{p_j}+(\gamma-1)\prod_{j=1}^{2}(\nu_{\pi(j)}^q)^{p_j}\right)^{1/q}}\right)$$

Assuming that it is also true for $j=n-1$,

$$\sum_{j=1}^{n-1}p_j\varsigma_{\pi(j)}=\left(\left(\frac{\prod_{j=1}^{n-1}(1+(\gamma-1)\mu_{\pi(j)}^q)^{p_j}-\prod_{j=1}^{n-1}(1-\mu_{\pi(j)}^q)^{p_j}}{\prod_{j=1}^{n-1}(1+(\gamma-1)\mu_{\pi(j)}^q)^{p_j}+(\gamma-1)\prod_{j=1}^{n-1}(1-\mu_{\pi(j)}^q)^{p_j}}\right)^{1/q},\right.$$

$$\left.\frac{\gamma^{1/q}\prod_{j=1}^{n-1}\nu_{\pi(j)}^{p_j}}{\left(\prod_{j=1}^{n-1}(1+(\gamma-1)(1-\nu_{\pi(j)}^q))^{p_j}+(\gamma-1)\prod_{j=1}^{n-1}(\nu_{\pi(j)}^q)^{p_j}\right)^{1/q}}\right)$$

Now, the target is to show that this is also true for $j=n$.

$$\sum_{j=1}^{n-1}p_j\varsigma_{\pi(j)}\oplus p_n\varsigma_{\pi(n)}$$

$$=\left(\left(\frac{\prod_{j=1}^{n-1}(1+(\gamma-1)\mu_{\pi(j)}^q)^{p_j}-\prod_{j=1}^{n-1}(1-\mu_{\pi(j)}^q)^{p_j}}{\prod_{j=1}^{n-1}(1+(\gamma-1)\mu_{\pi(j)}^q)^{p_j}+(\gamma-1)\prod_{j=1}^{n-1}(1-\mu_{\pi(j)}^q)^{p_j}}\right)^{1/q},\right.$$

$$\left.\frac{\gamma^{1/q}\prod_{j=1}^{n-1}\nu_{\pi(j)}^{p_j}}{\left(\prod_{j=1}^{n-1}(1+(\gamma-1)(1-\nu_{\pi(j)}^q))^{p_j}+(\gamma-1)\prod_{j=1}^{n-1}(\nu_{\pi(j)}^q)^{p_j}\right)^{1/q}}\right)$$

$$\oplus\left(\left(\frac{(1+(\gamma-1)\mu_{\pi(n)}^q)^{p_n}-(1-\mu_{\pi(n)}^q)^{p_n}}{(1+(\gamma-1)\mu_{\pi(n)}^q)^{p_n}+(\gamma-1)(1-\mu_{\pi(n)}^q)^{p_n}}\right)^{1/q},\right.$$

$$\left.\frac{\gamma^{1/q}\nu_{\pi(n)}^{p_n}}{\left((1+(\gamma-1)(1-\nu_{\pi(n)}^q))^{p_n}+(\gamma-1)(\nu_{\pi(n)}^q)^{p_n}\right)^{1/q}}\right)$$

$$=\left(\left(\frac{\prod_{j=1}^{n}(1+(\gamma-1)\mu_{\pi(j)}^q)^{p_j}-\prod_{j=1}^{n}(1-\mu_{\pi(j)}^q)^{p_j}}{\prod_{j=1}^{n}(1+(\gamma-1)\mu_{\pi(j)}^q)^{p_j}+(\gamma-1)\prod_{j=1}^{n}(1-\mu_{\pi(j)}^q)^{p_j}}\right)^{1/q},\right.$$

$$\left.\frac{\gamma^{1/q}\prod_{j=1}^{n}\nu_{\pi(j)}^{p_j}}{\left(\prod_{j=1}^{n}(1+(\gamma-1)(1-\nu_{\pi(j)}^q))^{p_j}+(\gamma-1)\prod_{j=1}^{n}(\nu_{\pi(j)}^q)^{p_j}\right)^{1/q}}\right)$$

$$=\sum_{j=1}^{n}p_j\varsigma_{\pi(j)}$$

Then, taking the product of for all permutations, we get

$$\prod_{\pi \in S_n} \sum_{j=1}^{n} p_j \varsigma_{\pi(j)}$$
$$= \left(\left(\frac{\gamma \prod_{\pi \in S_n} (\phi_2 - \varphi_2)}{\prod_{\pi \in S_n} (\phi_2 + (\gamma^2 - 1)\varphi_2) + (\gamma - 1) \prod_{\pi \in S_n} (\phi_2 - \varphi_2)} \right)^{1/q}, \right.$$
$$\left. \left(\frac{\prod_{\pi \in S_n} (\psi_2 + (\gamma^2 - 1)\chi_2) - \prod_{\pi \in S_n} (\psi_2 - \chi_2)}{\prod_{\pi \in S_n} (\psi_2 + (\gamma^2 - 1)\chi_2) + (\gamma - 1) \prod_{\pi \in S_n} (\psi_2 - \chi_2)} \right)^{1/q} \right)$$

and

$$\left(\prod_{\pi \in S_n} \sum_{j=1}^{n} \varsigma_{\pi(j)}^{p_j} \right)^{\frac{1}{n!}}$$
$$= \left(\left(\frac{\gamma \left(\prod_{\pi \in S_n} (\phi_2 - \varphi_2) \right)^{\frac{1}{n!}}}{\left(\prod_{\pi \in S_n} (\phi_2 + (\gamma^2 - 1)\varphi_2) \right)^{\frac{1}{n!}} + (\gamma - 1) \left(\prod_{\pi \in S_n} (\phi_2 - \varphi_2) \right)^{\frac{1}{n!}}} \right)^{1/q}, \right.$$
$$\left. \left(\frac{\left(\prod_{\pi \in S_n} (\psi_2 + (\gamma^2 - 1)\chi_2) \right)^{\frac{1}{n!}} - \left(\prod_{\pi \in S_n} (\psi_2 - \chi_2) \right)^{\frac{1}{n!}}}{\left(\prod_{\pi \in S_n} (\psi_2 + (\gamma^2 - 1)\chi_2) \right)^{\frac{1}{n!}} + (\gamma - 1) \left(\prod_{\pi \in S_n} (\psi_2 - \chi_2) \right)^{\frac{1}{n!}}} \right)^{1/q} \right)$$

Finally,

$$\frac{1}{\sum_{j=1}^{n} p_j} \left(\prod_{\pi \in S_n} \sum_{j=1}^{n} (p_j \varsigma_{\pi(j)}) \right)^{\frac{1}{n!}}$$

$$
= \left(\begin{array}{c} \left(\dfrac{ \left(\left(\prod\limits_{\pi \in S_n} (\phi_2 + (\gamma^2-1)\varphi_2) \right)^{\frac{1}{n!}} + (\gamma^2-1) \left(\prod\limits_{\pi \in S_n} (\phi_2 - \varphi_2) \right)^{\frac{1}{n!}} \right)^{\frac{1}{\sum_{j=1}^{n} p_j}} - \left(\left(\prod\limits_{\pi \in S_n} (\phi_2 + (\gamma^2-1)\varphi_2) \right)^{\frac{1}{n!}} - \left(\prod\limits_{\pi \in S_n} (\phi_2 - \varphi_2) \right)^{\frac{1}{n!}} \right)^{\frac{1}{\sum_{j=1}^{n} p_j}} }{ \left(\left(\prod\limits_{\pi \in S_n} (\phi_2 + (\gamma^2-1)\varphi_2) \right)^{\frac{1}{n!}} + (\gamma^2-1) \left(\prod\limits_{\pi \in S_n} (\phi_2 - \varphi_2) \right)^{\frac{1}{n!}} \right)^{\frac{1}{\sum_{j=1}^{n} p_j}} + (\gamma-1) \left(\left(\prod\limits_{\pi \in S_n} (\phi_2 + (\gamma^2-1)\varphi_2) \right)^{\frac{1}{n!}} - \left(\prod\limits_{\pi \in S_n} (\phi_2 - \varphi_2) \right)^{\frac{1}{n!}} \right)^{\frac{1}{\sum_{j=1}^{n} p_j}} } \right)^{1/q}, \\ \left(\dfrac{ \gamma \left(\left(\prod\limits_{\pi \in S_n} (\psi_2 + (\gamma^2-1)\chi_2) \right)^{\frac{1}{n!}} - \left(\prod\limits_{\pi \in S_n} (\psi_2 - \chi_2) \right)^{\frac{1}{n!}} \right)^{\frac{1}{\sum_{j=1}^{n} p_j}} }{ \left(\left(\prod\limits_{\pi \in S_n} (\psi_2 + (\gamma^2-1)\chi_2) \right)^{\frac{1}{n!}} + (\gamma^2-1) \left(\prod\limits_{\pi \in S_n} (\psi_2 - \chi_2) \right)^{\frac{1}{n!}} \right)^{\frac{1}{\sum_{j=1}^{n} p_j}} + (\gamma-1) \left(\left(\prod\limits_{\pi \in S_n} (\psi_2 + (\gamma^2-1)\chi_2) \right)^{\frac{1}{n!}} - \left(\prod\limits_{\pi \in S_n} (\psi_2 - \chi_2) \right)^{\frac{1}{n!}} \right)^{\frac{1}{\sum_{j=1}^{n} p_j}} } \right)^{1/q} \end{array} \right) \tag{8}
$$

which illustrates that Eq. (7) holds.

Now, to show that Eq. (7) or (8) is a q-ROFN, we will prove the following:
(i) $0 \le \mu' \le 1$
(ii) $0 \le \nu' \le 1$
(iii) $0 \le (\mu')^q + (\nu')^q \le 1$
where μ' is the membership degree and ν' is the non-membership degree of Eq. (8).

Proof (i) and (ii). For any $\gamma > 0$, $q \ge 1$ and $P \in \Re^n$ s.t. $\sum_{j=1}^{n} p_j > 0$, we have $\phi_2,\ \varphi_2,\ \psi_2,\ \chi_2 \ge 0$ with $\phi_2 \ge \varphi_2$, $\psi_2 \ge \chi_2$ and the q-ROFN (μ', ν') can be written as $\left(\left(\frac{E^*-F^*}{E^*-F^*+\gamma F^*}\right)^{1/q}, \left(1 - \frac{G^*-H^*}{G^*-H^*+\gamma H^*}\right)^{1/q}\right)$. where

$E^* = \left(\left(\prod_{\pi \in S_n} (\phi_2 + (\gamma^2 - 1)\varphi_2)\right)^{\frac{1}{n!}} + (\gamma^2 - 1)\left(\prod_{\pi \in S_n} (\phi_2 - \varphi_2)\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum_{j=1}^{n} p_j}}$,

$F^* = \left(\left(\prod_{\pi \in S_n} (\phi_2 + (\gamma^2 - 1)\varphi_2)\right)^{\frac{1}{n!}} - \left(\prod_{\pi \in S_n} (\phi_2 - \varphi_2)\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum_{j=1}^{n} p_j}}$,

$G^* = \left(\left(\prod_{\pi \in S_n} (\psi_2 + (\gamma^2 - 1)\chi_2)\right)^{\frac{1}{n!}} + (\gamma^2 - 1)\left(\prod_{\pi \in S_n} (\psi_2 - \chi_2)\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum_{j=1}^{n} p_j}}$,

$H^* = \left(\left(\prod_{\pi \in S_n} (\psi_2 + (\gamma^2 - 1)\chi_2)\right)^{\frac{1}{n!}} - \left(\prod_{\pi \in S_n} (\psi_2 - \chi_2)\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum_{j=1}^{n} p_j}}$.

Since $E^*,\ F^*,\ G^*,\ H^* \ge 0$ s.t. $E^* \ge F^*$ and $G^* \ge H^*$. Therefore, it is easy to show that μ' and ν' satisfy the conditions (i) and (ii), respectively.

Proof (iii). Conditions (i) and (ii) $\Rightarrow 0 \le (\mu')^q + (\nu')^q$. For $(\mu')^q + (\nu')^q \le 1$, we know that $\mu^q_{\pi(j)} + \nu^q_{\pi(j)} \le 1$ or $\mu^q_{\pi(j)} \le 1 - \nu^q_{\pi(j)}$. Now by using $\mu^q_{\pi(j)} \le 1 - \nu^q_{\pi(j)}$ and Eq. (8) for μ' and ν', we will get

$$(\mu')^q + (\nu')^q \le 1. \qquad \text{Q.E.D}$$

Some important properties such as idempotency, monotonicity, boundedness and commutativity of the q-ROFHDMM operator are given below.

Property 1 (Idempotency) *If all the considered q-ROFNs are equal, that is,* $\varsigma_i = \varsigma = (\mu, \nu)$ *for all* $i = 1, 2, ..., n$*, then*

$$\text{q-ROFHDMM}^P(\varsigma_1, \varsigma_2, ..., \varsigma_n) = \varsigma = (\mu, \nu).$$

Property 2 (Monotonicity) *If* $\varsigma_i = (\mu_i, \nu_i)$ *and* $\varsigma_i' = (\mu_i', \nu_i')$ *for* $i = 1, 2, ..., n$ *are any two collection of q-ROFNs s.t.* $\mu_i \leq \mu_i'$, $\nu_i \geq \nu_i'$ *for all* i*, then*

$$\text{q-ROFHDMM}^P(\varsigma_1, \varsigma_2, ..., \varsigma_n) \leq \text{q-ROFHDMM}^P(\varsigma_1', \varsigma_2', ..., \varsigma_n').$$

Property 3 (Boundedness) *For any collection* $\varsigma_i = (\mu_i, \nu_i)$ *for* $i = 1, 2, ..., n$ *of q-ROFNs, if* $\varsigma^- = \left(\min_{i=1}^{n}(\mu_i), \max_{i=1}^{n}(\nu_i)\right)$ *and* $\varsigma^+ = \left(\max_{i=1}^{n}(\mu_i), \min_{i=1}^{n}(\nu_i)\right)$*, then*

$$\varsigma^- \leq \text{q-ROFHDMM}^P(\varsigma_1, \varsigma_2, ..., \varsigma_n) \leq \varsigma^+.$$

Property 4 (Commutativity) *For any permutation of* $\varsigma_i (i = 1, 2, ..., n)$ *say* $\varsigma_i' (i = 1, 2, ..., n)$*, the aggregated value remains unaffected. That is*

$$\text{q-ROFHDMM}^P(\varsigma_1', \varsigma_2', ..., \varsigma_n') = \text{q-ROFHDMM}^P(\varsigma_1, \varsigma_2, ..., \varsigma_n).$$

Now, some special cases of the q-ROFHDMM operator w.r.t γ *and* P *are discussed hereafter.*

1. For $\gamma = 1$, q-ROFHDMM operator becomes q-rung orthopair fuzzy dual Muirhead mean(q-ROFDMM) operator.
2. For $\gamma = 2$, q-ROFHDMM operator becomes q-rung orthopair fuzzy Einstein dual Muirhead mean(q-ROFEDMM) operator.
3. For $P = (1, 0, 0, ..., 0)$, q-ROFHDMM operator becomes q-rung orthopair fuzzy Hamacher geometric averaging(q-ROFHG) operator.
4. For $P = (1, 1, ..., 1)$ or $P = (1/n, 1/n, ..., 1/n)$, q-ROFHDMM operator becomes q-rung orthopair fuzzy Hamacher arithmetic averaging(q-ROFHA) operator.
5. For $P = (1, 1, 0, 0, ..., 0)$, q-ROFHDMM operator becomes q-rung orthopair fuzzy Hamacher geometric Bonferroni mean(q-ROFHGBM) operator.
6. For $P = (\overbrace{1, 1, ..., 1}^{k}, \overbrace{0, 0, ..., 0}^{n-k})$, q-ROFHDMM operator become q-rung orthopair fuzzy Hamacher geometric Maclaurin symmetric mean(q-ROFHGMSM) operator.

3.2 The q-ROFHWDMM Operator

Definition 8 Consider a set of q-ROFNs $\{\varsigma_1, \varsigma_2, ..., \varsigma_n\}$, a parameter vector $P = (p_1, p_2, ..., p_n) \in \Re^n$ such that $\sum_{j=1}^{n} p_j > 0$, and a weight vector $\omega = (\omega_1, \omega_2, ..., \omega_n)^T$, where $\omega_i \in [0, 1]$ corresponding to ς_i such that $\sum_{i=1}^{n} \omega_i = 1$. The q-ROFHWDMM operator is thus defined as

$$\text{q-ROFHWDMM}^{P}(\varsigma_1, \varsigma_2, ..., \varsigma_n) = \frac{1}{\sum_{j=1}^{n} p_j}\left(\bigotimes_{\pi\in S_n}\bigoplus_{j=1}^{n}\left(p_j\varsigma_{\pi(j)}^{nw_{\pi(j)}}\right)\right)^{\frac{1}{n!}}$$

where S_n is the symmetric group of degree n.

Theorem 2 *For any collection* $\{\varsigma_1, \varsigma_2, ..., \varsigma_n\}$ *of q-ROFNs, the aggregated value using q-ROFHWDMM operator is also a q-ROFN and it is defined as*

$$\text{q-ROFHWDMM}^{P}(\varsigma_1, \varsigma_2, ..., \varsigma_n) =$$

$$\left(\left(\frac{\left(\left(\prod\limits_{\pi\in S_n}(\phi_2'+(\gamma^2-1)\varphi_2')\right)^{\frac{1}{n!}}+(\gamma^2-1)\left(\prod\limits_{\pi\in S_n}(\phi_2'-\varphi_2')\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum_{j=1}^{n}p_j}}-\left(\left(\prod\limits_{\pi\in S_n}(\phi_2'+(\gamma^2-1)\varphi_2')\right)^{\frac{1}{n!}}-\left(\prod\limits_{\pi\in S_n}(\phi_2'-\varphi_2')\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum_{j=1}^{n}p_j}}}{\left(\left(\prod\limits_{\pi\in S_n}(\phi_2'+(\gamma^2-1)\varphi_2')\right)^{\frac{1}{n!}}+(\gamma^2-1)\left(\prod\limits_{\pi\in S_n}(\phi_2'-\varphi_2')\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum_{j=1}^{n}p_j}}+(\gamma-1)\left(\left(\prod\limits_{\pi\in S_n}(\phi_2'+(\gamma^2-1)\varphi_2')\right)^{\frac{1}{n!}}-\left(\prod\limits_{\pi\in S_n}(\phi_2'-\varphi_2')\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum_{j=1}^{n}p_j}}}\right)^{1/q},\right.$$
$$\left.\left(\frac{\gamma\left(\left(\prod\limits_{\pi\in S_n}(\psi_2'+(\gamma^2-1)\chi_2')\right)^{\frac{1}{n!}}-\left(\prod\limits_{\pi\in S_n}(\psi_2'-\chi_2')\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum_{j=1}^{n}p_j}}}{\left(\left(\prod\limits_{\pi\in S_n}(\psi_2'+(\gamma^2-1)\chi_2')\right)^{\frac{1}{n!}}+(\gamma^2-1)\left(\prod\limits_{\pi\in S_n}(\psi_2-\chi_2')\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum_{j=1}^{n}p_j}}+(\gamma-1)\left(\left(\prod\limits_{\pi\in S_n}(\psi_2'+(\gamma^2-1)\chi_2')\right)^{\frac{1}{n!}}-\left(\prod\limits_{\pi\in S_n}(\psi_2'-\chi_2')\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum_{j=1}^{n}p_j}}}\right)^{1/q}\right)$$

where

$$\phi_2' = \prod_{j=1}^{n}\left(\left(1+(\gamma-1)(1-\mu_{\pi(j)}^{q})\right)^{nw_{\pi(j)}}+(\gamma^2-1)\left(\mu_{\pi(j)}^{q}\right)^{nw_{\pi(j)}}\right)^{p_j},$$
$$\varphi_2' = \prod_{j=1}^{n}\left(\left(1+(\gamma-1)(1-\mu_{\pi(j)}^{q})\right)^{nw_{\pi(j)}}-\left(\mu_{\pi(j)}^{q}\right)^{nw_{\pi(j)}}\right)^{p_j},$$
$$\psi_2' = \prod_{j=1}^{n}\left(\left(1+(\gamma-1)\nu_{\pi(j)}^{q}\right)^{nw_{\pi(j)}}+(\gamma^2-1)\left(1-\nu_{\pi(j)}^{q}\right)^{nw_{\pi(j)}}\right)^{p_j},$$
$$\chi_2' = \prod_{j=1}^{n}\left(\left(1+(\gamma-1)\nu_{\pi(j)}^{q}\right)^{nw_{\pi(j)}}-\left(1-\nu_{\pi(j)}^{q}\right)^{nw_{\pi(j)}}\right)^{p_j}.$$

Corollary 1 *The q-ROFHDMM is a specific case of the q-ROFHWDMM operator. That is, for* $w = (1/n, 1/n, ..., 1/n)^T$*, the q-ROFHWDMM operator reduces to q-ROFHDMM operator.*

The two fundamental properties, monotonicity and boundedness, of the q-ROFHWDMM operator are discussed hereafter.

Property 5 (Monotonicity) *If* $\varsigma_i = (\mu_i, \nu_i)$ *and* $\varsigma_i' = (\mu_i', \nu_i')$ *for* $i = 1, 2, ..., n$ *are any two collection of q-ROFNs s.t.* $\mu_i \leq \mu_i'$, $\nu_i \geq \nu_i'$ *for all* i*, then*

$$\textit{q-ROFHWDMM}^{P}(\varsigma_1, \varsigma_2, ..., \varsigma_n) \leq \textit{q-ROFHWDMM}^{P}(\varsigma_1', \varsigma_2', ..., \varsigma_n').$$

Property 6 (Boundedness) *For any collection* $\varsigma_i = (\mu_i, \nu_i)$ *for* $i = 1, 2, ..., n$ *of q-ROFNs, if* $\varsigma^- = \left(\min\limits_{i=1}^{n}(\mu_i), \max\limits_{i=1}^{n}(\nu_i)\right)$ *and* $\varsigma^+ = \left(\max\limits_{i=1}^{n}(\mu_i), \min\limits_{i=1}^{n}(\nu_i)\right)$*, then*

$$\varsigma^- \leq \textit{q-ROFHWDMM}^{P}(\varsigma_1, \varsigma_2, ..., \varsigma_n) \leq \varsigma^+.$$

4 Application of the Proposed AOs on MADM

4.1 MADM Method Based on the q-ROFHWDMM Operator

Now we'll develop a MADM method that uses the q-ROFHWDMM operator. To implement this, let us take $\Im = \{\Im_1, \Im_2, ..., \Im_m\}$ be the set of all feasible alternatives, which are being evaluated on the basis of n-attributes $\{\zeta_1, \zeta_2, ..., \zeta_n\}$ with the weight vector $\omega = \{\omega_1, \omega_2, ..., \omega_n\}$ such that $\omega_j \in [0, 1]$ and $\sum_{j=1}^{w} \omega_j = 1$. Let $\hbar = (\aleph_{ij})_{m\times n}$ be the decision matrix, where $\aleph_{ij} = (\mu_{ij}, \nu_{ij})$ is an assessment value (as q-ROFN) of an alternative $\Im_i$ with respect to the attribute ζ_j.

The step-by-step approach of this generalised orthopair fuzzy MADM method is given hereafter.

Step 1. Normalisation of $\hbar$:
Generally, two types of attributes are involved in any decision matrix: cost and benefit types. To consider these attributes simultaneously, we need to normalise the decision matrix as follows:

$$\aleph_{ij} = (\mu_{ij}, \nu_{ij}) = \begin{cases} (\mu_{ij}, \nu_{ij}), & \text{for benefit attributes } \zeta_j \\ (\nu_{ij}, \mu_{ij}), & \text{for cost attributes } \zeta_j \end{cases}$$

Step 2. Evaluate comprehensive values:
To get a comprehensive value $\aleph_i$ for each alternative $\Im_i$, apply the proposed q-ROFHWDMM operator which aggregates the assessment values $\aleph_{ij} (j = 1, 2, ..., n)$.

$$\aleph_i = \text{q-ROFHWDMM}(\aleph_{i1}, \aleph_{i2}, ..., \aleph_{in})$$

Step 3. Find the score and accuracy values:
First, compute the $S(\aleph_i)$ for each $\aleph_i (i = 1, 2, ..., m)$. Now if any two or more score values match, then calculate their accuracy values $A(\aleph_i)$ according to the Eqs. (2) and (3), respectively.

Step 4. Rank the alternatives:
Now use definition 4 to rank the alternatives $(\Im_i)$ and choose the most appealing one.

4.2 An Illustrative Example

Now, a practical MADM problem adopted from [8] is presented to illustrate the applicability of the developed MADM technique. The target of this MADM problem is to help an organisation install an ERP system. For that, five viable ERP systems have been chosen by the project team. $\Im_i (i = 1, 2, 3, 4, 5)$ i.e. 5-alternatives and 4-attributes $\zeta_j (j = 1, 2, 3, 4)$ that are (1) function and technology ζ_1; (2) strategic fitness ζ_2; (3) vendor's ability ζ_3; (4) vendor's reputation ζ_4 and $\omega = (0.2, 0.1, 0.3, 0.4)$

Table 1 Decision matrix ($\hbar$) taken from [8]

Alternative	Attributes			
	ζ_1	ζ_2	ζ_3	ζ_4
$\Im_1$	(0.5, 0.8)	(0.6, 0.3)	(0.3, 0.6)	(0.5, 0.7)
$\Im_2$	(0.7, 0.5)	(0.7, 0.2)	(0.7, 0.2)	(0.4, 0.5)
$\Im_3$	(0.6, 0.4)	(0.5, 0.7)	(0.5, 0.3)	(0.6, 0.3)
$\Im_4$	(0.8, 0.1)	(0.6, 0.3)	(0.3, 0.4)	(0.5, 0.6)
$\Im_5$	(0.6, 0.4)	(0.4, 0.8)	(0.7, 0.6)	(0.5, 0.8)

Table 2 Final results of all $\Im_i$

Alternatives	Comprehensive values	Score values	Ranking
$\Im_1$	(0.6118, 0.5381)	0.0732	4
$\Im_2$	(0.7275, 0.3030)	0.3572	1
$\Im_3$	(0.6210, 0.3863)	0.1818	3
$\Im_4$	(0.7033, 0.2781)	0.3264	2
$\Im_5$	(0.6240, 0.6010)	0.0258	5

denotes the weight vector of these qualities. The associated information of these five alternative with respect to four attributes is given in the form of a decision matrix $\hbar = (\aleph_{ij})_{5\times4}$ of q-ROFNs as provided in the Table 1.

In order to achieve the most suitable alternative, we utilised the MADM method given in Sect. 4.1.

Step 1. Normalisation of $\hbar$:
Here, the given decision matrix ($\hbar$) does not need to be normalised, as all four ζ_j are benefit type.
Step 2. Evaluate comprehensive values:
Now apply q-ROFHWDMM operator and compute the comprehensive values $\aleph_i\,(i = 1, 2, 3, 4, 5)$ for all alternatives $\Im_i\,(i = 1, 2, 3, 4, 5)$ using decision matrix $\hbar$ (Table 1), for $q = 3$, $\gamma = 1$, and $P = (1, 1, 1, 1)$. The comprehensive values are presented in column 2 of Table 2.
Step 3. Find the score and accuracy values:
For each $\aleph_i\,(i = 1, 2, 3, 4, 5)$, compute score value $S(\aleph_i)$. Computed score values are presented in column 3 of Table 2.
Step 4. Ranking of alternatives:
Finally, based on the calculated $S(\aleph_i)$, rank the alternatives $\Im_i$ as discussed in step 4 of section 4.1 and result are presented in column 4 of Table 2. From Table 2, it's clear that alternative A_2 is the best alternative among possible potential ERP systems. The final choice of alternative may depend on the parameters' values $q,\ \gamma,\ P$ and AOs

applied. Therefore, it is obvious to investigate the efficiency of the proposed method corresponding to the parameters' values selected and AOs used. Therefore, sections 4.3 and 4.4 discusses sensitivity analysis and comparative analysis, respectively.

4.3 Sensitivity Analysis

To investigate flexibility and capability of the proposed MADM method, a sensitivity analysis has been carried out by changing the parameter q, γ and then P one by one. The effects on the final result due to these variations are analysed and discussed hereafter.

Table 3 shows the variation in score values by assigning different integer values to $q \in [2, 10]$ and fixing the values of $\gamma = 1$ and $P = (1, 1, 1, 1)$. Similarly in Table 4, γ varies from 1 to 10; however, the other two parameters q and P are fixed as 3 and $(1, 1, 1, 1)$, respectively. From Tables 3 and 4, it is observable that, on increasing the value of parameters q (Table 3) and γ (Table 4), the score values and ranking results of some alternatives changes accordingly, which reflects the influence of these two parameters (q and γ) on the final decision. The parameter q not just provides the larger assessment space but also influences the final results. Similarly, the γ parameter makes the aggregation process more flexible and affects the final results. However, for the studied MADM problem, the best alternative obtained through all considered variations is unanimously $\Im_2$. Further, to examine the effect of interrelationship among attributes, different values of the parameter vector P were analysed on fixing the values of parameters q and γ as 3 and 1 respectively, and evaluated score values and ranking results are shown in Table 5. In this case, Table 5 shows that, on considering the interdependency of multiple attributes, the ranking results are slightly different from those in the case of no interaction. But the best alternative for all the considered variations of P of multiple interrelationships is $\Im_2$.

4.4 Comparative Analysis

To demonstrate the compatibility of the developed AOs, this section compares six existing AOs, q-ROFWA and q-ROFWG [6], q-ROFWBM [7], q-ROFGWHM and q-ROFWGHM [8], q-ROFWMSM [9], and one proposed AO (q-ROFHWDMM) under same q-ROFNs environment with $q = 3$. The q-ROFWA and q-ROFWG has no additional parameter other than q [6]. The q-ROFWBM operator takes into account the correlation between any two attributes [7], and its additional parameters are set to $s = 1, t = 1$. The selected values of their extra parameters for applying q-ROFGWHM and q-ROFWGHM operators are $\phi = 1,\ \varphi = 1$, and they assess the

Table 3 Results by varying q in q-ROFHWDMM operator

q	Score values ($S(\aleph_i)$)					Ranking orders
2	$S(\aleph_1) = 0.0663$	$S(\aleph_2) = 0.4201$	$S(\aleph_3) = 0.2260$	$S(\aleph_4) = 0.3873$	$S(\aleph_5) = 0.0315$	$\Im_2 \succ \Im_4 \succ \Im_3 \succ \Im_1 \succ \Im_5$
3	$S(\aleph_1) = 0.0732$	$S(\aleph_2) = 0.3572$	$S(\aleph_3) = 0.1818$	$S(\aleph_4) = 0.3264$	$S(\aleph_5) = 0.0258$	$\Im_2 \succ \Im_4 \succ \Im_3 \succ \Im_1 \succ \Im_5$
4	$S(\aleph_1) = 0.0749$	$S(\aleph_2) = 0.2899$	$S(\aleph_3) = 0.1362$	$S(\aleph_4) = 0.2658$	$S(\aleph_5) = 0.0226$	$\Im_2 \succ \Im_4 \succ \Im_3 \succ \Im_1 \succ \Im_5$
5	$S(\aleph_1) = 0.0709$	$S(\aleph_2) = 0.2326$	$S(\aleph_3) = 0.0994$	$S(\aleph_4) = 0.2150$	$S(\aleph_5) = 0.0196$	$\Im_2 \succ \Im_4 \succ \Im_3 \succ \Im_1 \succ \Im_5$
6	$S(\aleph_1) = 0.0636$	$S(\aleph_2) = 0.1869$	$S(\aleph_3) = 0.0720$	$S(\aleph_4) = 0.1742$	$S(\aleph_5) = 0.0164$	$\Im_2 \succ \Im_4 \succ \Im_3 \succ \Im_1 \succ \Im_5$
7	$S(\aleph_1) = 0.0551$	$S(\aleph_2) = 0.1511$	$S(\aleph_3) = 0.0521$	$S(\aleph_4) = 0.1416$	$S(\aleph_5) = 0.0132$	$\Im_2 \succ \Im_4 \succ \Im_1 \succ \Im_3 \succ \Im_5$
8	$S(\aleph_1) = 0.0467$	$S(\aleph_2) = 0.1230$	$S(\aleph_3) = 0.0379$	$S(\aleph_4) = 0.1156$	$S(\aleph_5) = 0.0103$	$\Im_2 \succ \Im_4 \succ \Im_1 \succ \Im_3 \succ \Im_5$
9	$S(\aleph_1) = 0.0389$	$S(\aleph_2) = 0.1009$	$S(\aleph_3) = 0.0276$	$S(\aleph_4) = 0.0945$	$S(\aleph_5) = 0.0079$	$\Im_2 \succ \Im_4 \succ \Im_1 \succ \Im_3 \succ \Im_5$
10	$S(\aleph_1) = 0.0322$	$S(\aleph_2) = 0.0833$	$S(\aleph_3) = 0.0202$	$S(\aleph_4) = 0.0775$	$S(\aleph_5) = 0.0059$	$\Im_2 \succ \Im_4 \succ \Im_1 \succ \Im_3 \succ \Im_5$

correlation between any two attributes [8].The q-ROFWMSM operator takes into account interactions among any number of attributes [9], and its granularity parameter is set to $k = 2$, allowing it to consider correlation between two any attributes for that very same interactional behavior. To maintain the same operational behavior for the developed AO (q-ROFHWDMM) also, the selected values of γ and P are 1 and $(1, 1, 0, 0)$ respectively. Table 6 suggested that the best alternative and the worst alternative obtained from all the different operators under investigation are almost the same.

Table 4 Results by changing γ in q-ROFHWDMM operator

γ	Score values(S($\aleph_i$))					Ranking orders
1	$S(\aleph_1) =$ 0.0732	$S(\aleph_2) =$ 0.3572	$S(\aleph_3) =$ 0.1818	$S(\aleph_4) =$ 0.3264	$S(\aleph_5) =$ 0.0258	$\Im_2 \succ \Im_4 \succ \Im_3 \succ \Im_1 \succ \Im_5$
2	$S(\aleph_1) =$ 0.0606	$S(\aleph_2) =$ 0.3533	$S(\aleph_3) =$ 0.1929	$S(\aleph_4) =$ 0.3184	$S(\aleph_5) =$ 0.0288	$\Im_2 \succ \Im_4 \succ \Im_3 \succ \Im_1 \succ \Im_5$
3	$S(\aleph_1) =$ 0.0508	$S(\aleph_2) =$ 0.3490	$S(\aleph_3) =$ 0.1968	$S(\aleph_4) =$ 0.3099	$S(\aleph_5) =$ 0.0299	$\Im_2 \succ \Im_4 \succ \Im_3 \succ \Im_1 \succ \Im_5$
4	$S(\aleph_1) =$ 0.0425	$S(\aleph_2) =$ 0.3451	$S(\aleph_3) =$ 0.1980	$S(\aleph_4) =$ 0.3021	$S(\aleph_5) =$ 0.0297	$\Im_2 \succ \Im_4 \succ \Im_3 \succ \Im_1 \succ \Im_5$
5	$S(\aleph_1) =$ 0.0353	$S(\aleph_2) =$ 0.3415	$S(\aleph_3) =$ 0.1981	$S(\aleph_4) =$ 0.2951	$S(\aleph_5) =$ 0.0288	$\Im_2 \succ \Im_4 \succ \Im_3 \succ \Im_1 \succ \Im_5$
6	$S(\aleph_1) =$ 0.0290	$S(\aleph_2) =$ 0.3384	$S(\aleph_3) =$ 0.1976	$S(\aleph_4) =$ 0.2889	$S(\aleph_5) =$ 0.0275	$\Im_2 \succ \Im_4 \succ \Im_3 \succ \Im_1 \succ \Im_5$
7	$S(\aleph_1) =$ 0.0234	$S(\aleph_2) =$ 0.3355	$S(\aleph_3) =$ 0.1970	$S(\aleph_4) =$ 0.2833	$S(\aleph_5) =$ 0.0261	$\Im_2 \succ \Im_4 \succ \Im_3 \succ \Im_5 \succ \Im_1$
8	$S(\aleph_1) =$ 0.0183	$S(\aleph_2) =$ 0.3329	$S(\aleph_3) =$ 0.1962	$S(\aleph_4) =$ 0.2782	$S(\aleph_5) =$ 0.0247	$\Im_2 \succ \Im_4 \succ \Im_3 \succ \Im_5 \succ \Im_1$
9	$S(\aleph_1) =$ 0.0137	$S(\aleph_2) =$ 0.3305	$S(\aleph_3) =$ 0.1954	$S(\aleph_4) =$ 0.2736	$S(\aleph_5) =$ 0.0233	$\Im_2 \succ \Im_4 \succ \Im_3 \succ \Im_5 \succ \Im_1$
10	$S(\aleph_1) =$ 0.0095	$S(\aleph_2) =$ 0.3284	$S(\aleph_3) =$ 0.1945	$S(\aleph_4) =$ 0.2694	$S(\aleph_5) =$ 0.0218	$\Im_2 \succ \Im_4 \succ \Im_3 \succ \Im_5 \succ \Im_1$

5 Conclusions

In the light of the interrelationship between multiple attributes in MADM problems, this paper proposes two novel AOs that are q-ROFHDMM and q-ROFHWDMM operators. These are Hamacher TN and TCN-inspired DMM operators under the q-ROFN environment. The advantage of combining Hamacher TN and TCN-inspired arithmetic procedures with DMM in proposed AOs is that they can capture not only the correlation between multiple attributes but also provide a flexible aggregation process due to γ and P in AOs. Some essential properties of these AOs are also given in the paper. The generality of the developed AOs is investigated through some special cases. Further, utilising the proposed AO (q-ROFHWDMM), a MADM approach

Table 5 Results by altering P in q-ROFHWDMM operator

Parameter vector(P)	Score values (S($\aleph_i$))					Ranking results
(1, 0, 0, 0)	$S(\aleph_1) = -0.2377$	$S(\aleph_2) = 0.0953$	$S(\aleph_3) = 0.1019$	$S(\aleph_4) = -0.0027$	$S(\aleph_5) = -0.1826$	$\Im_3 \succ \Im_2 \succ \Im_4 \succ \Im_5 \succ \Im_1$
(2, 0, 0, 0)	$S(\aleph_1) = -0.2813$	$S(\aleph_2) = 0.0467$	$S(\aleph_3) = 0.0815$	$S(\aleph_4) = -0.0710$	$S(\aleph_5) = -0.2462$	$\Im_3 \succ \Im_2 \succ \Im_4 \succ \Im_5 \succ \Im_1$
(1, 1, 0, 0)	$S(\aleph_1) = -0.1366$	$S(\aleph_2) = 0.2682$	$S(\aleph_3) = 0.1456$	$S(\aleph_4) = 0.1570$	$S(\aleph_5) = -0.0531$	$\Im_2 \succ \Im_4 \succ \Im_3 \succ \Im_5 \succ \Im_1$
(1, 1, 1, 0)	$S(\aleph_1) = -0.0516$	$S(\aleph_2) = 0.3236$	$S(\aleph_3) = 0.1688$	$S(\aleph_4) = 0.2814$	$S(\aleph_5) = -0.0139$	$\Im_2 \succ \Im_4 \succ \Im_3 \succ \Im_5 \succ \Im_1$
(1, 1, 1, 1)	$S(\aleph_1) = 0.0732$	$S(\aleph_2) = 0.3572$	$S(\aleph_3) = 0.1818$	$S(\aleph_4) = 0.3264$	$S(\aleph_5) = 0.0258$	$\Im_2 \succ \Im_4 \succ \Im_3 \succ \Im_1 \succ \Im_5$
(2, 2, 2, 2)	$S(\aleph_1) = 0.0732$	$S(\aleph_2) = 0.3572$	$S(\aleph_3) = 0.1818$	$S(\aleph_4) = 0.3264$	$S(\aleph_5) = 0.0258$	$\Im_2 \succ \Im_4 \succ \Im_3 \succ \Im_1 \succ \Im_5$
(3, 3, 3, 3)	$S(\aleph_1) = 0.0732$	$S(\aleph_2) = 0.3850$	$S(\aleph_3) = 0.1819$	$S(\aleph_4) = 0.3479$	$S(\aleph_5) = 0.0258$	$\Im_2 \succ \Im_4 \succ \Im_3 \succ \Im_1 \succ \Im_5$
(4, 4, 4, 4)	$S(\aleph_1) = 0.0732$	$S(\aleph_2) = 0.3850$	$S(\aleph_3) = 0.2395$	$S(\aleph_4) = 0.3479$	$S(\aleph_5) = 0.0258$	$\Im_2 \succ \Im_4 \succ \Im_3 \succ \Im_1 \succ \Im_5$
(1, 2, 3, 4)	$S(\aleph_1) = -0.0316$	$S(\aleph_2) = 0.3114$	$S(\aleph_3) = 0.1650$	$S(\aleph_4) = 0.2693$	$S(\aleph_5) = -0.0279$	$\Im_2 \succ \Im_4 \succ \Im_3 \succ \Im_5 \succ \Im_1$

has been developed. To show the applicability of the proposed approach, a MADM problem related to the selection of an ERP system has been solved. Sensitivity analysis for different variations and comparative analysis with six existing AOs have also been done to demonstrate the efficiency and compatibility of the proposed AOs. Our analysis and results conclude that the developed AOs are more flexible and general and can solve a wide range of real-life MADM problems. In future research, the proposed AOs may be further extended in various directions, including changing the uncertain environment, considering the heterogeneous relationship among attributes and so on.

Table 6 Score and ranking results for different AOs

AOs	Score values (S($\aleph_i$))					Ranking order
$q-ROFWA$ [6]	$S(\aleph_1)=$ -0.1443	$S(\aleph_2)=$ 0.2015	$S(\aleph_3)=$ 0.1394	$S(\aleph_4)=$ 0.1635	$S(\aleph_5)=$ -0.0515	$\Im_2 \succ \Im_4 \succ \Im_3 \succ \Im_5 \succ \Im_1$
$q-ROFWG$ [6]	$S(\aleph_1)=$ -0.2377	$S(\aleph_2)=$ 0.0953	$S(\aleph_3)=$ 0.1019	$S(\aleph_4)=$ -0.0027	$S(\aleph_5)=$ -0.1826	$\Im_3 \succ \Im_2 \succ \Im_4 \succ \Im_5 \succ \Im_1$
$q-ROFWBM^{1,1}$ [7]	$S(\aleph_1)=$ -0.6917	$S(\aleph_2)=$ -0.4263	$S(\aleph_3)=$ -0.4687	$S(\aleph_4)=$ -0.4372	$S(\aleph_5)=$ -0.6853	$\Im_2 \succ \Im_4 \succ \Im_3 \succ \Im_5 \succ \Im_1$
$q-ROFGWHM^{1,1}$ [8]	$S(\aleph_1)=$ -0.3070	$S(\aleph_2)=$ 0.0635	$S(\aleph_3)=$ 0.0412	$S(\aleph_4)=$ 0.0055	$S(\aleph_5)=$ -0.2345	$\Im_2 \succ \Im_4 \succ \Im_3 \succ \Im_5 \succ \Im_1$
$q-ROFWGHM^{1,1}$ [8]	$S(\aleph_1)=$ -0.0821	$S(\aleph_2)=$ 0.2208	$S(\aleph_3)=$ 0.2241	$S(\aleph_4)=$ 0.1228	$S(\aleph_5)=$ -0.0044	$\Im_3 \succ \Im_2 \succ \Im_4 \succ \Im_5 \succ \Im_1$
$q-ROFWMSM^{k=2}$ [9]	$S(\aleph_1)=$ 0.4898	$S(\aleph_2)=$ 0.6936	$S(\aleph_3)=$ 0.6421	$S(\aleph_4)=$ 0.6254	$S(\aleph_5)=$ 0.5812	$\Im_2 \succ \Im_3 \succ \Im_4 \succ \Im_5 \succ \Im_1$
$q-ROFHWDMM^{(1,1,0,0)}$	$S(\aleph_1)=$ -0.1366	$S(\aleph_2)=$ 0.2682	$S(\aleph_3)=$ 0.1456	$S(\aleph_4)=$ 0.1570	$S(\aleph_5)=$ -0.0531	$\Im_2 \succ \Im_4 \succ \Im_3 \succ \Im_5 \succ \Im_1$

References

1. Zadeh, L.A.: Fuzzy sets. Inf. Control **8**, 338–356 (1965)
2. Atanassov, K.T.: Intuitionistic fuzzy sets. Fuzzy Sets Syst. **20**, 87–96 (1986)
3. Yager, R.R.: Pythagorean fuzzy subsets. In: Proceeding of The Joint IFSA World Congress and NAFIPS Annual Meeting, Edmonton, Canada pp. 57–61 (2013)
4. Yager, R.R.: Generalized orthopair fuzzy sets. IEEE Trans. Fuzzy Syst. **25**(5), 1222–1230 (2017)
5. Zhenghai, A., Xu, Z., Yager, R.R., Ye, J.: q-Rung orthopair fuzzy integrals in the frame of continuous Archimedean t-norms and t-conorms and their application. IEEE Trans. Fuzzy Syst. **29**(5), 996–1007 (2020)
6. Liu, P., Wang, P.: Some q-rung orthopair fuzzy aggregation Operators and their applications to multiple-attribute decision making. Int. J. Intell. Syst. **32**(2), 259–280 (2018)
7. Liu, P., Liu, J.: Some q-rung orthopair fuzzy Bonferroni mean operators and their application to multi-attribute group decision making. Int. J. Intell. Syst. **33**(2), 315–347 (2018)
8. Wei, G., Gao, H., Wei, Y.: Some q-rung orthopair fuzzy Heronian mean operators in multiple attribute decision making. Int. J. Intell. Syst. **33**(7), 1426–1458 (2018)
9. Wei, G., Wei, C., Wang, J., Gao, H., Wei, Y.: Some q-rung orthopair fuzzy Maclaurin symmetric mean operators and their applications to potential evaluation of emerging technology commercialization. Int. J. Intell. Syst. **34**(1), 50–81 (2019)
10. Rawat, S.S., Komal.: Multiple attribute decision making based on q-rung orthopair fuzzy Hamacher Muirhead mean operators. Soft Comput. **26**, 2465–2487 (2022)
11. Muirhead, R.F.: Some methods applicable to identities and inequalities of symmetric algebraic functions of *n* letters. Proc. Edinb. Math. Soc. **21**(3), 144–162 (1902)
12. Qin, J., Liu, X.: 2-tuple linguistic Muirhead mean operators for multiple attribute group decision making and its application to supplier selection. Kybernetes **45**(1), 2–29 (2016)

13. Hamacher, H.: Uber logische verknunpfungenn unssharfer Aussagen und deren Zugenhorige Bewertungsfunktione Trappl, Klir, Riccardi (Eds.), Progress in Cybernatics and Systems Research, vol. 3, pp. 276–288 (1978)
14. Batyrshin, I., Kaynak, O.: Parametric classes of generalized conjunction and disjunction operations for fuzzy modeling. IEEE Trans. Fuzzy Syst. **7**(5), 586–596 (1999)
15. Huang, J.Y.: Intuitionistic fuzzy Hamacher aggregation operators and their application to multiple attribute decision making. J. Intell. Fuzzy Syst. **27**, 505–513 (2014)
16. Wu, S.J., Wei, G.W.: Pythagorean fuzzy Hamacher aggregation operators and their application to multiple attribute decision making. Int. J. Inf. Technol. Decis. Making **21**(3), 189–201 (2017)
17. Akram, M., Peng, X., Sattar, A.: A new decision-making model using complex intuitionistic fuzzy Hamacher aggregation operators. Soft. Comput. **25**, 7059–7086 (2021)
18. Darko, A.P., Liang, D.: Some q-rung orthopair fuzzy Hamacher aggregation operators and their application to multiple attribute group decision making with modified EDAS method. Eng. Appl. Artif. Intell. **87**, 103259 (2020)
19. Liu, P., Wang, P.: Multiple-attribute decision-making based on Archimedean Bonferroni operators of q-rung orthopair fuzzy numbers. IEEE Trans. Fuzzy Syst. **27**(5), 834–848 (2018)