Blow-up Analysis and Global Existence of Solutions for a Fractional Reaction-Diffusion Equation

R. Saranya and N. Annapoorani

Abstract This paper is concerned with the blow-up phenomena and global existence of a fractional nonlinear reaction-diffusion equation with a non-local source term. Under sufficient conditions on the weight function $a(x)$ and when the initial data is small enough, the global existence of solutions is proved using the comparison principle. We establish a finite time blow-up of the solution with large initial data by converting the fractional PDE into a simple ordinary differential inequality using the differential inequality technique. Moreover, by solving the obtained ordinary differential inequality, an upper bound of the blow-up time is also deduced.

Keywords Blow-up · Global existence · Fractional partial differential equation

1 Introduction

In this paper, we consider the following fractional nonlinear reaction-diffusion equation

$$\begin{aligned} \frac{{}^C\partial^\alpha u(x,t)}{\partial t^\alpha} &= D \triangle u(x,t) + a(x) f(u) + u^p(x,t), && x \in \Omega,\ t \in (0,T), \\ u(x,0) &= u_0(x), && x \in \Omega, \\ u(x,t) &= 0, && x \in \partial\Omega, t \in (0,T), \end{aligned} \tag{1}$$

where Ω is a bounded convex region in $\mathcal{R}^n (n \geq 1)$ with smooth boundary $\partial\Omega$ and $D > 0$ is the diffusion coefficient. $\frac{{}^C\partial^\alpha u}{\partial t^\alpha}$ is the Caputo fractional derivative of order $0 < \alpha < 1$ which is defined with respect to the time variable as

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$$\frac{{}^{C}\partial^{\alpha} u}{\partial t^{\alpha}} = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} \frac{\partial u(x,s)}{\partial s} \,\mathrm{d}s. \tag{2}$$

Suppose that the nonlinearity $f(u) = u^l(x,t)\left(\int_{\Omega} u^{l+1}(x,t)\mathrm{d}x\right)^m$ is a non-negative continuous function. The nonlinear terms $a(x)f(u) + u^p$ represents the reaction-kinetics. The exponents $l \geq 0$, $m > 0$ and $p > 1$, and the weight function $a(x) \in C(\overline{\Omega})$ satisfies

(A1) $a(x) > 0, x \in \Omega.$
(A2) $0 \leq C_1 < a(x) < C_2 < \infty, \forall\, x \in \overline{\Omega}.$

Fractional derivative is an arbitrary order derivative which incorporates memory phenomena such that it concatenates both integral and differential operators. Fractional Calculus is once thought of esoteric in nature. But in recent few decades, it has been accustomed to model biological, physical and engineering processes. Besides, most visual phenomena of quantum mechanics, fluid dynamics, ecological systems and numerous models are controlled by fractional differential equations within their domain of existence.

Reaction-diffusion equation of type (1) emerges naturally in various mathematical models from dynamics of bio-reactors and bio-sensors, population dynamics, combustion theory, compressible reactant gas model and so on [2–4, 6, 11, 12]. In chemical systems, those equations illustrate the production of the material, by chemical reaction, which competes with the diffusion of that material. Systems of such equations generally comprise numerous interacting components as chemical reactions and are widely used to trace out the formation of patterns in a variety of processes in the applied sciences.

The nonlinear processes lead to the study of new problems in the areas of partial differential equations and analysis. The blow-up of the solution in the nonlinear evolution problem is one of the most remarkable properties that differs from the linear ones. The singularities that occur in linear problems are often known as fixed singularities whereas in nonlinear problems, they are known to be movable singularities as it depends on the initial data and other properties of the problem.

In recent decades, there are many works established which concerns the global existence and blow-up phenomena of local and non-local reaction-diffusion equation [9, 10, 17]. For $p = l = 0$, the blow-up phenomena of time-fractional diffusion equation (1) with variable exponents is discussed by Manimaran and Shangerganesh [15], where the non-local source term determines human-controlled distribution function. The global existence and lower and upper bounds of the blow-up time of the solution are obtained for (1) when $\alpha = 1$ by Ma and Fang [13]. Cao et al. [7] established a finite time blow-up and long-time behavior of the solution of a time-fractional diffusion equation with local source term. Pinasco [16] discussed the blow-up solution for the parabolic and hyperbolic problems with a non-local source term using Kaplan's eigenvalue method and established a local existence theory for the respective problems with a fixed-point technique. Ma et al. [14] investigated the blow-up phenomena of a reaction-diffusion equation with weighted exponential nonlinearity when $\alpha = 1$. Tao et al. [18] established a global existence by constructing suitable sub- and super-solutions and obtained lower bounds of the blow-up time by Kaplan's eigenvalue method.

Motivated by the above works, we analyze the blow-up phenomena of the problem (1) according to the conditions on the exponents l, m and p. Moreover, we prove the global existence of solutions to the problem (1). The outline of this paper is as follows: In Sect. 2, finite time blow-up of the solution is established for $0 \leq l < 1$ and $l > 1$. In Sect. 3, we establish a comparison principle for (1) by defining sub- and super-solutions. Also, global existence is proved using comparison principle for $l, m \geq 1$ and $p > 1$.

2 Blow-Up of Solutions in Finite Time

In this section, we derive the energy functionals of the problem (1) which is important to derive the blow-up phenomena of the solution. The positive initial energy obtained in Lemmas 2 and 3 leads to the blow-up of the solution obtained in Theorems 1 and 2, respectively. Using maximum principle and monotonicity condition as in Cao et al. [7], the Caputo fractional derivative of order $0 < \alpha < 1$ can be written as

$$\frac{{}^C\partial^\alpha u}{\partial t^\alpha} \leq \frac{t^{1-\alpha}}{\Gamma(2-\alpha)} \frac{\partial u}{\partial t}. \tag{3}$$

Lemma 1 (Jensen's Inequality [5]) *Suppose that* Φ *is a real valued function on* Ω *and let* χ *and* φ *be non-negative Riemann-integrable functions on* Ω. *Then,*

$$\Phi\left(\int_\Omega \chi(x)\varphi(x)\mathrm{d}x\right) \leq \left(\int_\Omega \Phi(\chi(x))\varphi(x)\mathrm{d}x\right),$$

where $\int_\Omega \varphi(x)\mathrm{d}x = 1$.

Lemma 2 *Let the assumptions* $(A1)-(A2)$ *hold true and the exponents* $l, m \geq 1$ *and* $p > 1$. *Define an energy function*

$$\mathcal{E}(t) := -D\int_\Omega |\nabla u|^2 \mathrm{d}x + \frac{C_2}{m+1}\left(\int_\Omega u^{l+1}\mathrm{d}x\right)^m + \int_\Omega u^{p+1}\mathrm{d}x. \tag{4}$$

Then for $u_0(x) \geq 0$, $\mathcal{E}'(t) > 0$ *which implies* $\mathcal{E}(t) > \mathcal{E}(0)$.

Proof Multiplying (1) by u_t and integrating over Ω, we use $(A1) - (A2)$ to get

$$\begin{aligned}\int_\Omega \frac{{}^C\partial^\alpha u}{\partial t^\alpha} u_t \mathrm{d}x &= D\int_\Omega \triangle u u_t \mathrm{d}x + \int_\Omega a(x) f(u) u_t \mathrm{d}x + \int_\Omega u^p u_t \mathrm{d}x \\ &\leq -D\int_\Omega \nabla u.\nabla u_t \mathrm{d}x + C_2\Big(\int_\Omega u^l u_t \mathrm{d}x\Big) \\ &\quad \Big(\int_\Omega u^{l+1}\mathrm{d}x\Big)^m + \int_\Omega u^p u_t \mathrm{d}x \\ &= -\frac{D}{2}\frac{d}{dt}\int_\Omega |\nabla u|^2 \mathrm{d}x + \frac{C_2}{l+1}\frac{d}{dt}\Big(\int_\Omega u^{l+1}\mathrm{d}x\Big) \\ &\quad \Big(\int_\Omega u^{l+1}\mathrm{d}x\Big)^m + \frac{1}{p+1}\frac{d}{dt}\int_\Omega u^{p+1}\mathrm{d}x \\ &= -\frac{D}{2}\frac{d}{dt}\int_\Omega |\nabla u|^2 \mathrm{d}x + \frac{C_2}{(l+1)(m+1)} \\ &\quad \frac{d}{dt}\Big(\int_\Omega u^{l+1}\mathrm{d}x\Big)^{m+1} + \frac{1}{p+1}\frac{d}{dt}\int_\Omega u^{p+1}\mathrm{d}x.\end{aligned}$$

Thus, we have

$$\begin{aligned}&\frac{\Gamma(2-\alpha)}{t^{1-\alpha}}\int_\Omega \left|\frac{{}^C\partial^\alpha u}{\partial t^\alpha}\right|^2 \mathrm{d}x \\ &\quad\leq \frac{1}{2}\frac{d}{dt}\Big(-D\int_\Omega |\nabla u|^2\mathrm{d}x + \frac{C_2}{m+1}\Big(\int_\Omega u^{l+1}\mathrm{d}x\Big)^{m+1} + \int_\Omega u^{p+1}\mathrm{d}x\Big).\end{aligned}$$

This implies from (4) that

$$\mathcal{E}'(t) \geq \frac{2\Gamma(2-\alpha)}{t^{1-\alpha}}\int_\Omega \left|\frac{{}^C\partial^\alpha u}{\partial t^\alpha}\right|^2 \mathrm{d}x > 0.$$

Thus for $u_0(x) \geq 0$, $\mathcal{E}(t) > \mathcal{E}(0) > 0$.

Theorem 1 *Let the assumptions* $(A1)-(A2)$ *hold true for the weight function* $a(x)$ *and the exponents* $l, m \geq 1$ *and* $p > 1$*. If* $u(x,t)$ *is a non-negative solution of the problem (1), then for sufficiently large initial data* $u_0(x) \geq 0$*, there exists a finite time* t_* *such that*

$$\lim_{t\to t_*} u(x,t) = \infty.$$

Define an auxiliary function

$$\Psi(t) = \int_\Omega u^2(x,t)\mathrm{d}x. \tag{5}$$

Then the upper bound for the blow-up time can be deduced as

$$t_* \leq \left(\frac{2\alpha \Psi^{\frac{2-(m+1)(l+1)}{2}}(0)}{K((m+1)(l+1)-2)} \right)^{\dfrac{1}{\alpha}}. \tag{6}$$

Proof Differentiating (5) with respect to t, we have

$$\Psi'(t) = 2\int_{\Omega} u u_t \mathrm{d}x.$$

Using the fact $\dfrac{1}{\Gamma(2-\alpha)} \leq 2$ and in view of (3),

$$\begin{aligned}
\Psi'(t) &\geq \frac{2\Gamma(2-\alpha)}{t^{1-\alpha}} \int_{\Omega} u \frac{{}^{C}\partial^{\alpha} u}{\partial t^{\alpha}} \mathrm{d}x \\
&\geq \frac{1}{t^{1-\alpha}} \int_{\Omega} u \Big(D \triangle u + a(x) u^{l}(x,t) \Big(\int_{\Omega} u^{l+1}(x,t) \mathrm{d}x \Big)^{m} + u^{p}(x,t) \Big) \mathrm{d}x \\
&\geq \frac{1}{t^{1-\alpha}} \Big(-D \int_{\Omega} |\nabla u|^2 \mathrm{d}x + C_1 \int_{\Omega} u^{l+1} \Big(\int_{\Omega} u^{l+1} \mathrm{d}x \Big)^{m} \mathrm{d}x + \int_{\Omega} u^{p+1} \mathrm{d}x \Big) \\
&= \frac{1}{t^{1-\alpha}} \Big(-D \int_{\Omega} |\nabla u|^2 \mathrm{d}x + C_1 \Big(\int_{\Omega} u^{l+1} \mathrm{d}x \Big)^{m+1} + \int_{\Omega} u^{p+1} \mathrm{d}x \Big) \\
&= \frac{1}{t^{1-\alpha}} \Big(-D \int_{\Omega} |\nabla u|^2 \mathrm{d}x + \frac{C_2(m+1)}{m+1} \Big(\int_{\Omega} u^{l+1} \mathrm{d}x \Big)^{m+1} + \int_{\Omega} u^{p+1} \mathrm{d}x \Big) \\
&\quad + \frac{C_1 - C_2}{t^{1-\alpha}} \Big(\int_{\Omega} u^{l+1} \mathrm{d}x \Big)^{m+1} \\
&= \frac{1}{t^{1-\alpha}} \Big(-D \int_{\Omega} |\nabla u|^2 \mathrm{d}x + \frac{C_2}{m+1} \Big(\int_{\Omega} u^{l+1} \mathrm{d}x \Big)^{m+1} + \int_{\Omega} u^{p+1} \mathrm{d}x \Big) \\
&\quad + \frac{mC_2}{(m+1)t^{1-\alpha}} \Big(\int_{\Omega} u^{l+1} \mathrm{d}x \Big)^{m+1} + \frac{C_1 - C_2}{t^{1-\alpha}} \Big(\int_{\Omega} u^{l+1} \mathrm{d}x \Big)^{m+1}.
\end{aligned}$$

Suppose that $\left(\dfrac{mC_2}{m+1} + (C_1 - C_2) \right) = K > 0$. Then from Lemma 2 and using Jensen's inequality, we have

$$\begin{aligned}
\Psi'(t) &\geq \frac{1}{t^{1-\alpha}} \Big(\mathcal{E}(t) + K \Big(\int_{\Omega} u^{l+1} \mathrm{d}x \Big)^{m+1} \Big) \\
&\geq \frac{K}{t^{1-\alpha}} \Big(\int_{\Omega} u^{2} \mathrm{d}x \Big)^{\dfrac{(m+1)(l+1)}{2}} \qquad (7) \\
\Psi'(t) &\geq \frac{K}{t^{1-\alpha}} \Psi^{\dfrac{(m+1)(l+1)}{2}}(t). \qquad (8)
\end{aligned}$$

From (7), we see that $\Psi'(t) > 0$ which implies that $\Psi(t) > \Psi(0)$. Now rearranging and integrating (8) with respect to the time variable from 0 to t, we get

$$\Psi^{1-\frac{(m+1)(l+1)}{2}}(0) - \Psi^{1-\frac{(m+1)(l+1)}{2}}(t) \geq \frac{K(m+1)(l+1)-2}{2\alpha} t^{\alpha}.$$

If $\lim_{t\to t_*} \Psi(t) = \infty$, then we obtain the upper bound of the blow-up time t_* as in (6). Thus, we see that the solution of the problem (1) becomes unbounded in L^2 norm.

Lemma 3 *Suppose that* $0 \leq l < 1, m, p > 1$ *and the assumptions* $(A1) - (A2)$ *hold true. Define an energy function*

$$\mathcal{F}(t) := -D\int_{\Omega} |\nabla u|^2 \mathrm{d}x + \frac{2C_2}{(l+1)(m+1)}\left(\int_{\Omega} u^{l+1}\mathrm{d}x\right)^m + \int_{\Omega} u^{p+1}\mathrm{d}x. \quad (9)$$

Then for $u_0(x) \geq 0$, $\mathcal{F}'(t) > 0$ *which implies that* $\mathcal{F}(t) > \mathcal{F}(0)$.

Proof Following the same procedure as in the proof of Lemma 2, we have

$$\begin{aligned}
\int_{\Omega} \frac{{}^{C}\partial^{\alpha} u}{\partial t^{\alpha}} u_t \mathrm{d}x \leq & -\frac{D}{2}\frac{d}{dt}\int_{\Omega} |\nabla u|^2 \mathrm{d}x + \frac{C_2}{(l+1)(m+1)}\frac{d}{dt}\left(\int_{\Omega} u^{l+1}\mathrm{d}x\right)^{m+1} \\
& + \frac{1}{p+1}\frac{d}{dt}\int_{\Omega} u^{p+1}\mathrm{d}x \\
= & \frac{1}{2}\frac{d}{dt}\Bigg(-D\int_{\Omega} |\nabla u|^2 \mathrm{d}x + \frac{2C_2}{(l+1)(m+1)}\left(\int_{\Omega} u^{l+1}\mathrm{d}x\right)^{m+1} \\
& + \int_{\Omega} u^{p+1}\mathrm{d}x\Bigg).
\end{aligned}$$

Thus from (9), we obtain $\mathcal{F}'(t) \geq 0$. This suggests $\mathcal{F}(t) > \mathcal{F}(0)$.

Theorem 2 *Let* $0 \leq l < 1$, $m, p > 1$ *and the assumptions* $(A1) - (A2)$ *hold true for the weight function* $a(x)$. *Then the solution of the problem (1) blows up in finite time* t_* *such that the upper bound can be deduced as*

$$t_* \leq \left(\frac{2\alpha\Psi^{\frac{2-(m+1)(l+1)}{2}}(0)}{M((m+1)(l+1)-2)}\right)^{\frac{1}{\alpha}}. \quad (10)$$

Proof Using the auxiliary function defined as in (5) and following the same procedure as in Theorem 1, we have

$$\begin{aligned}\Psi'(t) &\geq \frac{1}{t^{1-\alpha}}\bigg(-D\int_\Omega |\nabla u|^2 \mathrm{d}x + C_1\bigg(\int_\Omega u^{l+1}\mathrm{d}x\bigg)^{m+1} + \int_\Omega u^{p+1}\mathrm{d}x\bigg) \\ &= \frac{1}{t^{1-\alpha}}\bigg(-D\int_\Omega |\nabla u|^2 \mathrm{d}x + \frac{2C_2}{(l+1)(m+1)}\bigg(\int_\Omega u^{l+1}\mathrm{d}x\bigg)^{m+1} \\ &\quad + \int_\Omega u^{p+1}\mathrm{d}x\bigg) \\ &\quad + \frac{1}{t^{1-\alpha}}\bigg((C_1 - 2C_2)\bigg(\int_\Omega u^{l+1}\mathrm{d}x\bigg)^{m+1} + \frac{2C_2K}{K+1}\bigg(\int_\Omega u^{l+1}\mathrm{d}x\bigg)^{m+1}\bigg),\end{aligned}$$

where $K = lm + l + m$. Assuming for suitable values of C_1, C_2, l, m and p, the constant

$$\frac{C_1(K+1) - 2C_2}{K+1} = M > 0.$$

Now by Lemma 3 and Jensen's inequality, we have

$$\begin{aligned}\Psi'(t) &\geq \frac{M}{t^{1-\alpha}}\bigg(\int_\Omega u^2\mathrm{d}x\bigg)^{\frac{(l+1)(m+1)}{2}} \\ &= \frac{M}{t^{1-\alpha}}\Psi^{\frac{(l+1)(m+1)}{2}}(t).\end{aligned} \tag{11}$$

Inequality (11) shows that $\Psi(t) > \Psi(0)$. Also by appropriate choice of constants l and m, we see that $\dfrac{(m+1)(l+1)}{2} > 1$. Now, integrating (11) from 0 to t, we get

$$\Psi^{1-\frac{(m+1)(l+1)}{2}}(0) - \Psi^{1-\frac{(m+1)(l+1)}{2}}(t) \geq \frac{M(m+1)(l+1)-2}{2\alpha}t^\alpha.$$

If $\lim_{t\to t_*}\Psi(t) = \infty$, then the blow-up time of the solution of the problem (1) is obtained as in (10).

3 Global Existence

This section discusses the global existence by constructing appropriate upper and lower solutions to the problem (1). We prove the comparison principle by defining the sub- and super-solutions of the solution to the problem (1) as follows.

Definition 1 A smooth function $w(x, t)$ is called the super-solution to the problem (1) on $(0, T)$ provided

$$\begin{aligned} &\frac{{}^C\partial^\alpha w}{\partial t^\alpha} \geq D \triangle w(x,t) + a(x)w^l(x,t)\left(\int_\Omega w^{l+1}(x,t)\mathrm{d}x\right)^m + w^p(x,t), \\ &w(x,0) \geq w_0(x), \qquad\qquad x \in \Omega, \\ &w(x,t) \geq 0, \qquad\qquad x \in \partial\Omega, t \in (0,T). \end{aligned} \tag{12}$$

By reversing the inequalities (12), we can define the sub-solution $v(x,t)$ to the problem (1) similar to Definition 1. To prove the comparison principle, we are in need of the following lemmas from fractional calculus.

Lemma 4 ([1]) *Let X be a Hilbert Space and $u : [0,T] \to X$. Then for $0 < \alpha < 1$,*

$$2(u(t), {}_0^C D_t^\alpha u(t)) \geq {}_0^C D_t^\alpha |u(t)|^2.$$

Lemma 5 (Gronwall type lemma [8]) *Assume that $\alpha, T, \epsilon_1, \epsilon_2 \in \mathcal{R}_+$ and let $u : [0,T] \to \mathcal{R}$ is a continuous function satisfying the inequality*

$$|u(t)| \leq \epsilon_1 + \frac{\epsilon_2}{\Gamma(\alpha)} \int_0^T (t-s)^{\alpha-1} |u(s)| \mathrm{d}s$$

for all $t \in [0,T]$. Then,

$$|u(t)| \leq \epsilon_1 E_\alpha(\epsilon_2 t^\alpha), \qquad \forall t \in [0,T].$$

Here, E_α represents the Mittag-Leffler function of order α.

Next to prove the global existence, we present the comparison principle to the problem (1).

Theorem 3 *If $w(x,t)$ and $v(x,t)$ be the super- and sub-solutions to the problem (1), then for any $(x,t) \in \Omega \times (0,T)$, $w(x,t) \geq v(x,t)$.*

Proof Define $z(x,t) = v(x,t) - w(x,t)$. Suppose that for some $t_1 \in (0,T)$, $z(x,t_1) \geq 0$. We prove by contradiction that there does not exist such t_1 such that we prove $w \geq v$. Now using the definitions of $w(x,t)$ and $v(x,t)$, we can write

$$\frac{{}^C\partial^\alpha z}{\partial t^\alpha} \leq D \triangle z + a(x)\left(v^l\left(\int_\Omega v^{l+1}\mathrm{d}x\right)^m - w^l\left(\int_\Omega w^{l+1}\mathrm{d}x\right)^m\right) + (v^p - w^p). \tag{13}$$

Let $z^+ = \max(0, z(x,t))$. Multiply (13) by z^+ and integrate over Ω to get

$$\begin{aligned}\int_\Omega \frac{{}^C\partial^\alpha z^+}{\partial t^\alpha}.z^+\mathrm{d}x &\le D\int_\Omega \triangle z^+.z^+\mathrm{d}x + \int_\Omega a(x)\Bigg(v^l\Big(\int_\Omega v^{l+1}\mathrm{d}x\Big)^m \\ &\quad -w^l\Big(\int_\Omega w^{l+1}\mathrm{d}x\Big)^m\Bigg).z^+\mathrm{d}x + \int_\Omega (v^p - w^p).z^+\mathrm{d}x \\ &\le -D\int_\Omega |\nabla z^+|^2\mathrm{d}x + C_1\int_\Omega \Bigg(v^l\Big(\int_\Omega v^{l+1}\mathrm{d}x\Big)^m \\ &\quad -w^l\Big(\int_\Omega w^{l+1}\mathrm{d}x\Big)^m\Bigg).z^+\mathrm{d}x + \int_\Omega (v^p - w^p).z^+\mathrm{d}x. \quad (14)\end{aligned}$$

For $p > 1$, from [16], we write

$$v^p - w^p \le p\psi^{p-1}(v - w), \quad (15)$$

where $\psi \in \mathcal{R}^n$ is bounded in $\Omega \times (0, T)$. Using (15) in the last term of the inequality (14), we have

$$\int_\Omega (v^p - w^p).z^+\mathrm{d}x \le C_3\int_\Omega (z^+)^2\mathrm{d}x. \quad (16)$$

For $a, b, c, d > 0$, if $(ac - bd) > 0$, we have $(a + b)(c - d) \ge (bc - ad)$. Using (15) we write

$$\begin{aligned}\int_\Omega \Bigg(v^l\Big(\int_\Omega v^{l+1}\mathrm{d}x\Big)^m - w^l\Big(\int_\Omega w^{l+1}\mathrm{d}x\Big)^m\Bigg).z^+\mathrm{d}x &\le C_4\int_\Omega (v^l - w^l).z^+\mathrm{d}x \\ &\le C_5\int_\Omega (z^+)^2\mathrm{d}x, \quad (17)\end{aligned}$$

where $C_4 := \sup_{t\ge 0}\phi(t)$ and $\phi(t) := \Bigg(\Big(\int_\Omega v^{l+1}\mathrm{d}x\Big)^m + \Big(\int_\Omega w^{l+1}\mathrm{d}x\Big)^m\Bigg)$. Since the first term in the RHS of inequality (14) is strictly negative and inserting (16) and (17) in (14), we have

$$\frac{1}{2}\frac{{}^C\partial^\alpha}{\partial t^\alpha}\int_\Omega (z^+)^2\mathrm{d}x \le C_6\int_\Omega (z^+)^2\mathrm{d}x.$$

Taking I^α on both sides and using Gronwall type of lemma, we get

$$\int_\Omega (z^+)^2\mathrm{d}x \le \Big(\int_\Omega (z^+(x,0))^2\mathrm{d}x\Big)E_\alpha(C_6 t^\alpha).$$

The definitions of sub- and super-solution imply that $z^+(x, 0) \le 0$. Hence, $v(x, t) \le w(x, t)$.

Consider the eigenvalue problem

$$\begin{aligned} \triangle \chi + \lambda \chi &= 0, \qquad x \in \Omega \\ \chi &= 0, \qquad x \in \partial \Omega, \end{aligned} \tag{18}$$

where $\chi_1(x)$ is the first eigenfunction corresponding to the eigenvalue λ_1 and $\int_\Omega \chi_1(x)\mathrm{d}x = 1$. Next, we propose the main theorem of global existence of solutions to the problem (1).

Theorem 4 *Let the exponents $l, m \geq 1$ and the assumptions $(A1) - (A2)$ hold true. If the initial data $u_0(x) \leq \eta^{-\beta_1}\chi_1(x)$, where β is an arbitrary positive constant, $\chi_1(x) > 0$ and $\eta > 0$ is large enough, then the solution to the problem (1) exists for all $t > 0$.*

Proof We construct a function $w(x, t)$ as in [13]. Let $\eta_1 > 0$ and $\beta_1 > 0$ be the constants to be determined later such that

$$w(x, t) = (\eta_1 + t)^{-\beta_1}\chi_1(x). \tag{19}$$

We claim that $w(x, t)$ is the super-solution to the solution of the problem (1). Now, we compute

$$\begin{aligned} &\frac{{}^C\partial^\alpha w}{\partial t^\alpha} - D \triangle w - a(x)w^l\left(\int_\Omega w^{l+1}\mathrm{d}x\right)^m - w^p \\ &\geq \frac{t^{-\alpha}}{\Gamma(1-\alpha)}\left(\frac{1}{(\eta_1+t)^{\beta_1}} - \frac{1}{\eta_1^{\beta_1}}\right)\chi_1(x) + \frac{D\lambda_1\chi_1(x)}{(\eta_1+t)^{\beta_1}} \\ &\quad - \frac{C_2\chi_1^l}{(\eta_1+t)^{\beta_1 l(1+m)-\beta_1 m}}\int_\Omega\left(\chi_1^{l+1}\mathrm{d}x\right)^m - \frac{\chi_1^p}{(\eta_1+t)^{p\beta_1}} \\ &\geq \frac{\chi_1}{(\eta_1+t)^{\beta_1}}\left[\frac{t^{-\alpha}}{\Gamma(1-\alpha)}\left(1 - \frac{(\eta_1+t)^{\beta_1}}{\eta_1^{\beta_1}}\right) + D\lambda_1 - \frac{C_2}{(\eta_1+t)^{\beta_1(l+lm+m-1)}}\right. \\ &\qquad\left. - \frac{1}{(\eta_1+t)^{\beta_1(1-p)}}\right]. \end{aligned}$$

Choosing η_1 sufficiently large, we have

$$\frac{{}^C\partial^\alpha w}{\partial t^\alpha} - D \triangle w - a(x)w^l\left(\int_\Omega w^{l+1}\mathrm{d}x\right)^m - w^p \geq 0. \tag{20}$$

Also by the hypotheses, the initial data satisfies

$$w(x, 0) = \eta^{-\beta_1}\chi_1(x) \geq u_0(x). \tag{21}$$

Hence, the inequalities (20)–(21) show that $w(x,t)$ is a super-solution to the problem (1.1) and it exists globally. By Comparison Principle, $u(x,t)$ exists for all $t > 0$.

Remark If we set a function $v(x,t) = \eta_2 (T-t)^{-\beta_2} \chi_1(x)$ with the initial data $u_0(x) \geq \eta_2 T^{-\beta_2} \chi_1(x)$, where $\beta_2 > 0$ and $\eta > 0$ sufficiently large,

$$\begin{aligned} &\frac{{}^C\partial^\alpha v}{\partial t^\alpha} - D \triangle v - a(x) v^l \left(\int_\Omega v^{l+1} \mathrm{d}x \right)^m - v^p \\ &\leq \frac{\eta_2 \beta_2 \chi_1(x)}{\Gamma(1-\alpha)(\alpha+\beta_2) t^{\alpha+\beta_2}} + \frac{D \eta_2 \lambda_1 \chi_1}{(T-t)^{\beta_2}} - C_2 \frac{\eta_2^{l+lm+m} \chi_1^l}{(T-t)^{-\beta_2(l+lm+m)}} - \frac{\eta_2^p \chi_1^p}{(T-t)^{p\beta_2}} \\ &\leq \eta_2 \chi_1 \left[\frac{\beta_2}{\Gamma(1-\alpha)(\alpha+\beta_2) t^{\alpha+\beta_2}} + \frac{D\lambda_1}{(T-t)^{\beta_2}} - C_2 \frac{\eta_2^{K-1} \chi_1^{l-1}}{(T-t)^{-\beta_2 K}} - \frac{\eta_2^{p-1} \chi_1^{p-1}}{(T-t)^{p\beta_2}} \right]. \end{aligned}$$

For large values of η_2 and β_2, we see that

$$\frac{{}^C\partial^\alpha v}{\partial t^\alpha} - D \triangle v - a(x) v^l \left(\int_\Omega v^{l+1} \mathrm{d}x \right)^m - v^p \leq 0.$$

Hence with the choice of initial data taken, $v(x,t)$ is a sub-solution to the problem (1) and it blows up at finite time $t^* \leq T$.

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