Separation Axioms in Bipolar Fuzzy Topological Spaces

Manjeet Singh and Asha Gupta

Abstract In this paper, the definition of the bipolar fuzzy (bf) point has been generalized, and using this, the concept of separation axioms has been introduced in bipolar fuzzy settings. Moreover, the relation between these separation axioms has been established.

Keywords Bipolar fuzzy set · Bipolar fuzzy topology

1 Introduction

Fuzzy sets have been introduced by Zadeh [1]. After that, in every branch of science and technology, fuzzy sets have been used to generalize all the concepts. The concept of general topological space is generalized by using fuzzy sets to fuzzy topological space by Chang [2]. Further, a number of papers have been devoted to generalize almost all the concepts of general topology in fuzzy topological space(fts). Tripathy and Borgohain [3], Tripathy and Baruah [4, 5] have investigated Different classes in fuzzy numbers of sequence spaces. Tripathy and Ray [6] have studied mixed fts. The concept of fuzzy sets has been generalized to bipolar fuzzy (briefly bf) sets by Zhang [7]. After that, basic operations on bf sets have been defined by Lee [8, 9]. Moreover, regular bf graphs have been studied by Akram and Dudek [10] and bf topological spaces have been defined by Azhagappan and Kamaraj. Recently, bf point, a neighborhood system, the notion of compactness, and few other properties have been introduced in bf topological space by Kim et al. [11].

In the present work, the concept of bf point has been generalized of Kim et al. [11] and also observed that the notion of disjointness $K \cap L = \emptyset \Leftrightarrow K \subseteq coL$ (coL is the complement of set L) is no longer valid for bf sets. So there is a deviation

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from general topology to bf topology, only the implication $K \cap L = \emptyset \Rightarrow K \subseteq coL$ is valid. The concept of separation axioms in bf settings has been introduced by using the generalized bf point and the notion of disjointness. Moreover, the relation between these separation axioms has been established.

2 Preliminaries and Definitions

In this section, we summarize some definitions and results of bf topological space which is helpful in the following section.

Let X be a nonempty set. Then a pair $K = (K^+, K^-)$ is called a bf set in X, if $K^+ : X \to [0, 1]$ and $K^- : X \to [-1, 0]$ are mappings. For each $x \in X$, the positive membership degree $K^+(x)$ is used to denote the satisfaction degree of the element x to the property corresponding to the bf set K and the negative membership degree $K^-(x)$ is used to denote the satisfaction degree of the element x to some implicit counter-property corresponding to the bf set K. The empty bf set is denoted by $0_{bp} = (0^+, 0^-)$ and defined by $0^+(x) = 0 = 0^-(x)$ for all $x \in X$. Also, the whole bf set is denoted by $1_{bp} = (1^+, 1^-)$ and defined by $1^+(x) = 1$ and $1^-(x) = -1$ for all $x \in X$.

Definition 1 ([9]) Let X be a nonempty set and let K, L be two bf sets in X.

(i) We say that K is a subset of L, denoted by $K \subseteq L$, if for each $x \in X$,

$$K^+(x) \leq L^+(x) \, and \, K^-(x) \geq L^-(x).$$

(ii) We say that K is equal to L, denoted by $K = L$, if $K \subseteq L$ and $L \subseteq K$.

(iii) The complement of K, denoted by $K^c = ((K^c)^+, (K^c)^-)$, is a bf set in X defined as: for each $x \in X$, $K^c(x) = (1 - K^+(x), -1 - K^-(x))$, i.e.,

$$(K^c)^+(x) = 1 - K^+(x), (K^c)^-(x) = -1 - K^-(x).$$

(iv) The intersection of K and L, denoted by $K \cap L$, is a bf set in X defined as: for each $x \in X$,

$$(K \cap L)(x) = (K^+(x) \wedge L^+(x), K^-(x) \vee L^-(x)).$$

(v) The union of K and L, denoted by $K \cup L$, is a bf set in X defined as: for each $x \in X$,

$$(K \cup L)(x) = (K^+(x) \vee L^+(x), K^-(x) \wedge L^-(x)).$$

Definition 2 ([9]) Let X be a nonempty set and let $\{K_i : i \in I\}$ be a family of subsets of X.

(i) The intersection of $\{K_i : i \in I\}$, denoted by $\bigcap_{i \in I} K_i$ is a bf set in X defined by: for each $x \in X$,

$$(\bigcap_{i \in I} K_i)(x) = (\bigwedge_{i \in I} K_i^+(x), \bigvee_{i \in I} K_i^-(x)).$$

(ii) The union of $\{K_i : i \in I\}$, denoted by $\bigcup_{i \in I} K_i$ is a bf set in X defined by: for each $x \in X$,

$$(\bigcup_{i \in I} K_i)(x) = (\bigvee_{i \in I} K_i^+(x), \bigwedge_{i \in I} K_i^-(x)).$$

Definition 3 ([11]) Let X be a nonempty set. Suppose a collection of bf sets of X is τ, then τ is said to be bf topology on X, if the following axioms is satisfied:

(i) $0_{bp}, 1_{bp} \in \tau$.
(ii) if $K, L \in \tau$, then $K \cap L \in \tau$.
(iii) if $\{K_i : i \in I\} \subset \tau$, then $\bigcup_{i \in I} K_i \in \tau$.

In this case, a bf topological space is denoted by the pair (X, τ) and each element of τ is said to be an open bf set of X. The closed bf set is the complement of an open bf set.

Definition 4 ([11]) Let X and Y be nonempty sets, let $K \subseteq X$ and $L \subseteq Y$ and let $f : X \to Y$ be a mapping. Then

(i) The image of K under f, denoted by $f(K) = (f(K^+), f(K^-))$, is a bf set in Y defined as follows: for each $y \in Y$,

$$f(K^+)(y) = \begin{cases} \vee_{x \in f^{-1}(y)} K^+(x), & if \quad f^{-1}(y) \neq \emptyset; \\ 0, & otherwise. \end{cases}$$

and

$$f(K^-)(y) = \begin{cases} \wedge_{x \in f^{-1}(y)} K^-(x), & if \quad f^{-1}(y) \neq \emptyset; \\ 0, & otherwise. \end{cases}$$

(ii) The preimage of L under f, denoted by $f^{-1}(L) = (f^{-1}(L^+), f^{-1}(L^-))$, is a bf set in Y defined as follows: for each $x \in X$, $[f^{-1}(L^+)](x) = L^+ \circ f(x)$ and

$$[f^{-1}(L^-)](x) = L^- \circ f(x).$$

Definition 5 ([11]) Let (X, τ_1), (Y, τ_2) be two bf topological spaces. Then a mapping $f : (X, \tau_1) \to (Y, \tau_2)$ is said to be continuous if $f^{-1}(V) \in \tau_1$, for each $V \in \tau_2$.

3 Separation Axioms

In this section, firstly, we define the generalized form of bipolar fuzzy point and show some properties of general topology that is not valid in bf settings. Secondly, we define separation axioms in bf topology and discuss the relations between these separation axioms.

Definition 6 ([11]) Let $X \neq \emptyset$ be a set and x in X, $(\alpha, \beta) \in (0, 1] \times [-1, 0)$. Then $x_{(\alpha,\beta)}$ with the values (α, β) and the support x is said to be a bf point in X, if for every y in X,

$$x_{(\alpha,\beta)}(y) = \begin{cases} (\alpha, \beta), & if\ y = x \\ (0, 0), & otherwise \end{cases}$$

The bf point has been generalized in the following definition:

Definition 7 Let x in X, $(0, 0) \neq (\alpha, \beta) \in [0, 1] \times [-1, 0]$ and K a bf set of X. Then

(i) $x_{(\alpha,\beta)}$ with the values (α, β) and the support x is called a generalized bf point in X , if for every y in X,

$$x_{(\alpha,\beta)}(y) = \begin{cases} (\alpha, \beta), & if\ y = x \\ (0, 0), & otherwise \end{cases}$$

(ii) K contains $x_{(\alpha,\beta)}$ (i.e. $x_{(\alpha,\beta)} \in K$), if

$$K^-(x) \leq \beta\, and\ K^+(x) \geq \alpha$$

On the basis of the preceding definition, the following implications hold:

$$K = L \iff \text{for all } p \in X,\ p \in K \Leftrightarrow p \in L;$$
$$K \subseteq L \iff \text{for all } p \in X,\ p \in K \Rightarrow p \in L;$$
$$p \in K \cap L \iff \text{for all } p \in X, p \in K \wedge p \in L;$$

more generally,

$$p \in \bigcap_{i \in I} K_i \iff (p \in K_i,\ \forall\, i \in I),$$

I is any index set.

We remark

$$p \in K \cup L \Leftarrow \text{for all } p \in X,\ p \in K \vee p \in L,$$

holds, but the converse of this implication does not remain valid.

Example 1 Suppose two bf subsets K, L and $K^+(x) = \frac{3}{4}$, $K^-(x) = -\frac{1}{2}$ for each x in X and $L^+(x) = \frac{1}{2}$, $L^-(x) = -\frac{3}{4}$ for each x in X. Now $K \cup L \subseteq X$, then $(K \cup L)^+(x) = \frac{3}{4}$ and $(K \cup L)^-(x) = -\frac{3}{4}$ for each x in X. If p in $K \cup L$ such that $(K \cup L)^+(p) = \frac{3}{4}$ and $(K \cup L)^-(p) = -\frac{3}{4}$, then neither p in K nor p in L.

To introduce the bf separation axioms, we have to discuss the notion of disjointness. The equivalence of set theory

$$K \cap L = \emptyset \Leftrightarrow K \subseteq coL.$$

is not valid for bf set theory; indeed, the following implication is true.

$$K \cap L = \emptyset \Rightarrow K \subseteq coL.$$

The separation axioms in bf settings are defined by using notion of disjointness in bf settings. So, we get the deviation from general topology to bf topology:

Definition 8 A bf topological space is called:

1. BFT_0 if for each pair consisting of two different bf points (p, q) with supports x and y, there exists an open bf set R such that $p \in R$ and $q \cap R = 0$ $(i.e.\ R^+(y) = 0$ and $R^-(y) = 0)$ or $q \in R$ and $p \cap R = 0$ $(i.e.\ R^+(x) = 0$ and $R^-(x) = 0)$.
2. $BFT_{0\alpha}$ if for each pair consisting of two different bf points (p, q) with supports x and y, there exists an open bf set R such that $p \in R \subseteq coq$ or $q \in R \subseteq cop$.
3. BFT_1 if for each pair consisting of two different bf points (p, q) with supports x and y, there exist two open bf sets R and S such that $p \in R$, $q \cap R = 0$ $(i.e.\ R^+(y) = 0$ and $R^-(y) = 0)$ and $q \in S$, $p \cap S = 0$ $(i.e.\ S^+(x) = 0$ and $S^-(x) = 0)$.
4. $BFT_{1\alpha}$ if and only if for each pair consisting of two different bf points (p, q) with supports x and y, there exist two open bf sets R and S such that $p \in R \subseteq coq$ and $q \in S \subseteq cop$.
5. BFT_s (strong BFT_1) if every bf singleton is a closed bf set.
6. BFT_2 (BFT-Hausdorff) if for each pair consisting of two different bf points (p, q) with supports x and y, there exist two open bf sets R and S such that $p \in R$, $q \in S$ and $R \cap S = 0$.
7. $BFT_{2\alpha}$ (strong BFT-Hausdorff) if for each pair consisting of two different bf points (p, q) with supports x and y, there exist two open bf sets R and S such that $p \in R$, $q \in S$ and $R \subseteq coS$.
8. $BFT_{2\frac{1}{2}}$ if for each pair consisting of two different bf points (p, q) with supports x and y, there exist two open bf sets R and S such that $p \in R$, $q \in S$ and $clR \cap clS = 0$.
9. $BFT_{2\frac{1}{2}\alpha}$ if for each pair consisting of two different bf points (p, q) with supports x and y, there exist two open bf sets R and S such that $p \in R$, $q \in S$ and $clR \subseteq co(clS)$.

With the help of above definitions the following implications can be noticed:

1. (X, τ) is $BFT_i \implies (X, \tau)$ is $BFT_{i\alpha}$ $i = 0, 1, 2, 2\frac{1}{2}$
2. (X, τ) is $BFT_{2\frac{1}{2}} \implies (X, \tau)$ is $BFT_2 \implies (X, \tau)$ is $BFT_1 \implies (X, \tau)$ is BFT_0
3. (X, τ) is $BFT_{2\frac{1}{2}\alpha} \implies (X, \tau)$ is $BFT_{2\alpha} \implies (X, \tau)$ is $BFT_{1\alpha} \implies (X, \tau)$ is $BFT_{0\alpha}$
4. (X, τ) is $BFT_s \implies (X, \tau)$ is BFT_1

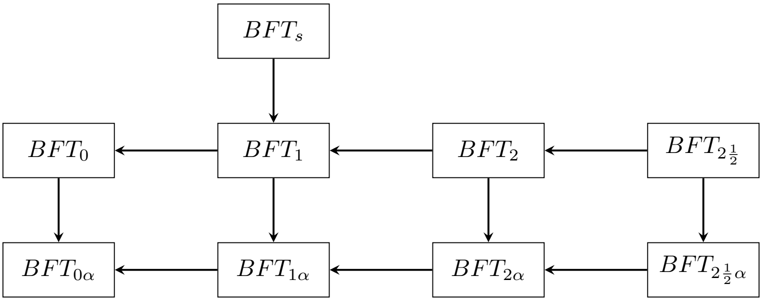

Fig. 1 Implication between separation axioms

Theorem 1 *A space is BFT$_1$ if and only if every bf singleton with $\alpha = 1$ and $\beta = -1$ is closed.*

Proof Let p_0 be an arbitrary bf singleton with support $x_0 \in X$, $\beta = p_0^-(x_0) = -1$ and $\alpha = p_0^+(x_0) = 1$. Suppose p is another bf point with support x, there exist O_0 and O_p open bf sets such that $p_0 \subseteq O_0 \subseteq cop$ and $p \subseteq O_p \subseteq cop_0$. Since every bf set is the union of all the bf singleton, it contains, i.e., $cop_0 = \cup_{p \subseteq cop_0} p$. From $cop_0^+(x_0) = 1 - p_0^+(x_0) = 0$ and $cop_0^-(x_0) = -1 - p_0^-(x_0) = 0$, we deduce $cop_0 = \cup_{p \subseteq cop_0} O_p$, and thus the cop_0 is open.

Conversely, consider p_1 and p_2 be a pair of two different bf points with support x_1 and x_2. Let q_1 and q_2 be another pair of bf points with support x_1 and x_2, respectively, such that $q_1^-(x_1) = q_2^-(x_2) = -1$ and $q_1^+(x_1) = q_2^+(x_2) = 1$. So, the bf sets coq_1 and coq_2 are open bf and satisfy the conditions $p_1 \subseteq coq_2 \subseteq cop_2$ and $p_2 \subseteq coq_1 \subseteq cop_1$.

Theorem 2 *A weakest bf topology τ exists for every X, such that (X, τ) is BFT_s.*

Proof Let X be any arbitrary set. Consider the collection τ of bf sets on X defined by

$$\tau = \{O : O \subseteq X, supp(coO)\ is\ finite\}.$$

We can easily prove that τ is a bf topology. Clearly, each bf point on X is bf closed, then (X, τ) is BFT_s. Now to prove τ is the weakest bf topology, suppose σ is any other bf topology which is also BFT_s. Let $R \in \tau$ be any set, then $supp(coR) = \{x_1, x_2, x_3, ..., x_n\}$. Consider the bf points p_i for every $i \in \{1, ..., n\}$ defined by

$$p_i^+(x_i) = coR^+(x_i) and p_i^-(x_i) = coR^-(x_i)$$

$$p_i^+(x) = 0\, and\; p_i^+(x) = 0\; for\; x \neq x_i$$

The family $\{p_i\}_{i=1}^n$ of σ-closed bf sets is a finite family. From $coR = \cup_{i=1}^n p_i$, we conclude that coR is σ-closed. Hence, $R \in \sigma$.

Definition 9 A bft space (X, τ) is called bf regular if for each pair having a bf point p and bf closed set F in X such that $p \in coF$, there exists a pair of open bf sets (R, S) such that p in R, F is subset of S and $R \cap S = \emptyset$. A bf regular which is also BFT_s is said to be BFT_3.

Definition 10 A bft space (X, τ) is called bf α-regular if for each pair having a bf point p and closed bf set F in X such that $p \in coF$, there exists a pair of open bf sets (R, S) such that p in R, F is subset of S and $R \subseteq coS$. A bf α-regular which is also BFT_s is said to be $BFT_{3\alpha}$.

Theorem 3 *A space (X, τ) is α-regular if and only if for each pair consisting of a bf open set R and a bf point p such that $p \in R$, there exists a bf open set S such that $p \in S \subseteq clS \subseteq R$.*

Proof Suppose p is a bf point and R is a bf open set in X such that $p \in R$. Since co$R$ is closed and $p \in co(coR)$, by α-regularity of space X that there exists bf open sets S_1 and S_2 such that $p \in S_1$, $coR \subseteq S_2$ and $S_1 \subseteq coS_2$. Since coS_2 is closed, $clS_1 \subseteq cl(coS_2) = coS_2 \subseteq R$. So $p \in S_1 \subseteq clS_1 \subseteq R$.

Conversely, let F be a bf closed set and p be any bf point in X such that $p \in coF$. By using the condition, there exists bf open set S_1 such that $p \in S_1 \subseteq clS_1 \subseteq coF$. Using the condition again, there exists bf open set S_2 such that $p \in S_2 \subseteq clS_2 \subseteq S_1$. To complete the proof, take $R_1 = S_2$ and $R_2 = co(clS_1)$ because $p \in R_1$, $F \subseteq R_2$ and $R_1 = S_2 \subseteq S_1 \subseteq clS_1 = co(co(clS_1)) = coR_2$.

Theorem 4 *Every bf $T_{3\alpha}$-space is also a bf $T_{2\frac{1}{2}\alpha}$-space.*

Proof Let (X, τ) be a bf $T_{3\alpha}$-space and p, q be two different bf points with support $x_p \neq x_q$ in X with values $(r_p, -r_p^{'})$ and $(r_q, -r_q^{'})$, respectively. Let p_1 be the crisp bf point with support x_p and value $(1, -1)$. By using the definition of $T_{3\alpha}$-space p_1 is a bf closed set and $q \in co(p_1)$. Since there exists bf open sets R and S such that $q \in R$, $p_1 \subseteq S$ and $R \subseteq coS$. Since $p^+(x_p) < 1 = p_1^+(x_p)$ and $p^-(x_p) > -1 = p_1^-(x_p)$, it follows that $p \in S$. Hence, (X, τ) is a bf $T_{2\frac{1}{2}\alpha}$-space.

Theorem 5 *Every bf α-regular $T_{0\alpha}$-space is a bf $T_{2\frac{1}{2}\alpha}$-space.*

Proof Let (X, τ) be a bf α-regular $T_{0\alpha}$-space and p, q be two different bf points with supports $x_p \neq x_q$ in X and values $(r_p, -r_p^{'})$, $(r_q, -r_q^{'})$ respectively. Let p_1, q_1 be bf points with supports x_p, x_q, respectively, and with values $p_1^+(x_p) = \frac{1}{2}(1 + r_p)$, $p_1^-(x_p) = -\frac{1}{2}(1 + r_p^{'})$ and $q_1^+(x_q) = \frac{1}{2}(1 + r_q)$, $q_1^-(x_q) = -\frac{1}{2}(1 + r_q^{'})$. Therefore, p_1, q_1 are two distinct bf points in X. There exists a bf open set R such that $p_1 \in$

$R \subseteq coq_1$ or $q_1 \in R \subseteq cop_1$ because X is a $T_{0\alpha}$-space. Firstly, if $p_1 \in R \subseteq coq_1$, there exists a bf open set S such that $p_1 \in S \subseteq clS \subseteq R$ because X is α-regular. Since $clS \subseteq R$ and $R \subseteq coq_1$, then $clS \subseteq coq_1$ that is $q_1 \subseteq co(clS)$. Now $q(x_q) = r_q < \frac{1}{2}(1 + r_q) = q_1(x_q)$ and $p(x_p) < p_{1(x_p)}$, then we get $q \in co(clS)$ and $p \in S$. Now let $O_1 = S$ and $O_2 = co(clS)$ are bf open sets in X and $S \subseteq co(co(clS))$. Therefore, there exists bf open sets O_1, O_2 in X such that $p \in O_1$, $q \in O_2$ and $O_1 \subseteq coO_2$. By using the previous theorem, there exist bf open sets O_3 and O_4 such that $p \in O_3 \subseteq clO_3 \subseteq O_1$ and $q \in O_4 \subseteq clO_4 \subseteq O_2$. Therefore, we get $p \in O_3$, $q \in O_4$ and $clO_3 \subseteq O_1 \subseteq coO_2 \subseteq co(clO_4)$. Secondly, if $q_1 \in R \subseteq cop_1$. We get the similar result. So, (X, τ) is $T_{2\frac{1}{2}\alpha}$.

Definition 11 A space (X, τ) is called bf normal if for every pair consisting of bf closed sets F_1, F_2 such that $F_1 \subseteq coF_2$, there exists a pair consisting of open fuzzy sets R, S such that $F_1 \subseteq R$, $F_2 \subseteq S$ and $R \cap S = \emptyset$. A bf normal which is also BFT_s is said to be BFT_4.

Definition 12 A space (X, τ) is called bf α-normal if every pair consisting of bf closed sets F_1, F_2 in X such that $F_1 \subseteq coF_2$, there exists a pair consisting of open fuzzy sets R, S such that $F_1 \subseteq R$, $F_2 \subseteq S$ and $R \subseteq coS$. A bf α-normal which is also BFT_s is said to be $BFT_{4\alpha}$.

Theorem 6 *Every bf $T_{4\alpha}$-space is also a bf $T_{3\alpha}$-space.*

Proof Let p be a bf closed point with support x_p and F a bf closed set in X such that $p \in coF$. Let p_1 be bf point with support x_p and with value $p_1^+(x_p) = \frac{1}{2}(p^+(x_p) + coF^+(x_p))$ and $p_1^-(x_p) = \frac{1}{2}(p^-(x_p) + coF^-(x_p))$. Therefore, $p \in p_1$, $p_1 \in coF$ and p_1 is closed because X is T_s-space. By using the α-normality of X and $p_1 \in coF$, there exist two bf open sets R, S with $p_1 \subseteq R$, $F \subseteq S$ and $R \subseteq coS$. Therefore, $p \in R$, $F \subseteq S$ and $R \subseteq coS$. So, X is bf $T_{3\alpha}$-space.

From the above results, we have observed the following implications for the separation axioms in bf topological spaces (Fig. 2).

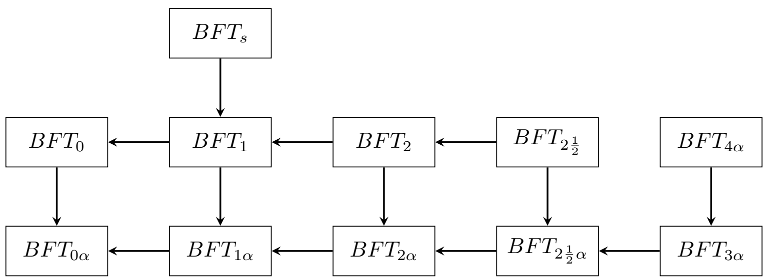

Fig. 2 Implication between separation axioms

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